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HOMEOMORPHISMS OF COMPACT 3-MANIFOLDS

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IN THIS paper, we study the possibility of extending some of Wladhausen's results[13], to compact irreducible 3-manifolds which are not P^2-irreducible. We say that a compact irreducible 3-manifold M is sufficiently large if M can be reduced to 3-cells and $P^2\times I$'s by splitting along incompressible surfaces. For many M, M is sufficiently large if and only if its orientation cover is a connected sum of sufficiently large 3-manifolds (in the sense of Waldhausen) and $S^1\times S^2$'s. Also a large class of P^2-reducible manifolds are sufficiently large in our sense (see §4). The main result of this paper says that two sufficiently large 3-manifolds are homeomorphic if and only if they have isomorphic group systems (see §3). However, every isomorphism of group systems need not even be induced by a map (see §2).

One of the results needed in proving homeomorphism theorems in the orientable case is that homotopy equivalences can be split along incompressible surfaces (see [1, 2] and [8]). Since P^2-reducible 3-manifolds are not aspherical, it is hard to carry out this kind of proof. We get around this difficulty by generalizing the invariant of [9] to all compact 3-manifolds. That is, to every compact 3-manifold M, we associate an invariant $\tau(M)$ in a certain homology group of the group system of M with coefficients in the twisted integers $\tilde{\mathbf{Z}}$. The theorem of §1 says that an isomorphism Θ of the group systems of two manifolds M and N is induced by a homotopy equivalence if and only $\Theta_*\tau(M)=\pm\tau(N)$. In §2 we show that for P^2-reducible and irreducible manifolds M and N, if there is an isomorphism of group systems, then one can find an isomorphism which carries the invariant to the invariant. This together with the naturality of the invariant τ and the theorem of §1 help to prove the homeomorphism theorem in §3. In §4 we give criteria for an irreducible manifold to be sufficiently large.

The results of §1 and §2 in the closed case are obtained earlier (in [11]) and are proved more generally here. However, we will use heavily the results of [10].

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§1. AN INVARIANT FOR THE HOMOTOPY TYPE OF COMPACT 3-MANIFOLDS

In this section, we construct an invariant which characterizes the homotopy type of compact 3-manifolds. This was done in [9] and [11] for closed 3-manifolds, and we follow the same pattern using the homology of group systems considered by H. Trotter[12]. By a *group system* $\{G, G_i, \varphi_i\}$, we mean a group G and finite number of homomorphisms $\varphi_i\colon G_i\to G$. A map of group systems $\Phi\colon\{G, G_i, \varphi_i\}\to\{G', G_i', \varphi_i'\}$ consists of homomorphisms $\varphi\colon G\to G'$, $\varphi_i\colon G_i\to G'_{j(i)}$ (where for each i, $G'_{j(i)}$ may be any of the G_i''s) such that $f\circ\varphi_i=\varphi'_{j(i)}\circ f_i$. We say that Φ is an isomorphism, if φ, φ_i are all isomorphisms, the number of G_i's and G_i''s is the same and j is a permutation. Let $\{G, G_i, \varphi_i\}$ and $\{G, G_i, \varphi_i'\}$ be two group systems involving the same groups but possibly different maps. We say they are *equivalent* if for each i there is an element g_i of G such that $\varphi_i'(x)=g_i\varphi_i(x)g_i^{-1}$ for each $x\in G_i$. Composition of an isomorphism and an equivalence will be called a *conjugate isomorphism* or briefly a c-isomorphism. Trotter shows in [12] how to associate homology and cohomology to a group system $\{G, G_i, \varphi_i\}$ with coefficients in G-modules. He also shows that equivalent group systems possess naturally isomorphic homologies. We make the following identification for studying the homologies:

Let K be a $K(G,1)$-space and K_i be $K(G_i,1)$-spaces. For each i, we can construct a map $f_i\colon K_i\to K$ inducing φ. Form the $K(G,1)$-space, again denoted by K, from the Union $K_i\times[0,1]\cup K$ by identifying $(x,1)$ with $f_i(x)$ for each $x\in K_i$. Thus we can consider K_i as subcomplexes of K. We make the convention that G acts on the left of $\tilde{K}$ the universal cover of

K. Then for any right G-module A and any left G-module B, we identify $H_k(\{\varphi_i\}; A)$ with $H_k(K, \cup K_i; A)$ and $H^k(\{\varphi_i\}; B)$ with $H^k(K, \cup K_i; B)$. Here we have abbreviated the group system as $\{\varphi_i\}$.

Let M be any compact 3-manifold and let m be a base point in the interior of M. We fix a local orientation at m. Then (M, m) defines a class of equivalent group systems which we denote by $\mathfrak{G}(M, m)$ or $\mathfrak{G}(M)$. We form $(K, \cup K_i)$ as above for $\mathfrak{G}(M)$. Then we have a map $\alpha: (M, \partial M) \to (K, \cup K_i)$ inducing isomorphisms in the fundamental groups of various spaces. We usually consider the $\pi_1(M, m)$ module $\tilde{\mathbf{Z}}$, that is integers on which $\pi_1(M, m)$ action is given by orientation. We define $\tau(M, m)$ to be the image of the fundamental cycle by the map $\alpha_*: H_3(M, \partial M; \tilde{\mathbf{Z}}) \to H_3(K, \cup K_i; \tilde{\mathbf{Z}})$. Thus $\tau(M, m)$ is an element of $H_3(\mathfrak{G}(M, m); \tilde{\mathbf{Z}})$. We sometines abbreviate it as $\tau(M)$.

1.1 Theorem. *Let $(M, \partial M)$ and $(N, \partial N)$ be compact 3-manifolds and let $\Phi: \mathfrak{G}(M, m) \to \mathfrak{G}(N, n)$ be a orientation-true c-isomorphism of their group systems. There is an oriented homotopy equivalence $f: (M, \partial M, m) \to (N, \partial N, n)$ with $f_* = \Phi$ if and only if $\Phi_* \tau(M, m) = \tau(N, n)$.*

1.2 *Remark.* This is an extension of the results of [9] and [11] and the methods there can be adopted to prove 1.1. However, we will use the results of Hendriks in [4], where he gave a short proof of the theorem in the closed case.

We will first recall Hendrik's results: For any compact 3-manifold $(M, \partial M, m)$, Hendriks associates an element $\Lambda(M, m)$ in $H^3(M, \partial M; \pi_2(M - B^0))$. Here B is a 3-cell in the interior of M and B^0 is the interior of B. The invariant $\Lambda(M, m)$ is the obstruction to retracting M onto $M - B^0$. Let $B(M, m) = \Lambda(M, m) \cap [M]$, the dual of $\Lambda(M, m)$. Thus $B(M, m)$ is an element of $H_0(M, \pi_2(M - B^0) \otimes \tilde{\mathbf{Z}})$. Here the tensor product is taken over integers and the action of π_1 is the diagonal action. Let M^i and N^i denote the i-skeletons of M and N respectively. Suppose that we are given a map $f^1: M^1 \cup \partial M \to N$ such that $f^1(\partial M) \subset \partial N$ and f induces an isomorphism in the fundamental groups. Let $\varphi^2: M - B^0 \to N - D^0$ be an extension of f'; where D is a cell in the interior of N. Then, Hendriks proves by obstruction theory

1.3 Theorem. *There is a map $f: (M, \partial M) \to (N, \partial N)$ extending f^1 if and only if $f_*(\varphi_\#^2 \otimes 1) B(M, m) = dB(N, n)$ for some integer d. Moreover, if the condition is satisfied one can realize a f of degree d.*

Identifying $H_0(M; \pi_2(M - B^0) \otimes \tilde{\mathbf{Z}})$ with $\tilde{\mathbf{Z}} \underset{\pi_1(M,m)}{\otimes} \pi_2(M - B^0, m)$ and assuming that M is closed, Hendriks shows that there is an injection $h_{M,m}: H_3(\pi_1(M, m); \tilde{\mathbf{Z}}) \to \tilde{\mathbf{Z}} \underset{\pi_1(M,m)}{\otimes} \pi_2(M - B^0, m)$ such that $h_{M,m}(\tau(M, m)) = B(M, m)$.

We will now prove 1.1 using 1.3. Necessity of the condition is clear, so we have only to prove the sufficiency. Firstly by filling up with discs we may assume that none of the boundary components of M and N is a 2-sphere. We form a $(K, \cup \partial K_i)$ from $(M, \partial M)$ as follows. If a component of ∂M is a projective plane, we adjoin an infinite projective space P^∞ to M along the particular boundary component. The components which are not projective planes are left unchanged. Then we add cells of dimension ≥ 3 away from the boundary to M to embed it in a $K(\pi, 1)$-space K. Let K_i denote the boundary components of M which are not projective planes and the infinite projective planes added along the boundary. We will denote $\cup K_1$ by ∂K and we have an embedding $\alpha: (M, \partial M) \to (K, \partial K)$ which induces isomorphisms on the fundamental groups of various spaces involved. Moreover the 2-skeleton M^2 of M coincides with the 2-skeleton K^2 of K. We make a similar construction for $(N, \partial N)$ and obtain an embedding $\beta: (N, \partial N) \to (L, \partial L)$. We use α (resp. β) to identify the group systems of $(K, \partial K)$ and $(M, \partial M)$ (resp. $(L, \partial L)$ and $(N, \partial N)$). Thus we have a c-isomorphism Φ of the group systems of $(K, \partial K)$ and $(L, \partial L)$.

Reorder the components L_i of ∂L so that Φ maps $\pi_1(K_i)$ to $\pi_1(L_i)$; let φ_i be the induced isomorphism. If K_i is not a P^∞ we take any map $\bar{f}_i$ inducing φ_i. If K_1 is a P^∞, let K_i^2 and L_i^2 denote the 2-skeletons of K_i and L_i; K_i^2, L_i^2 are boundary components of M and N respectively. Consider the induced orientations on K_i^2, L_i^2 and let $\bar{f}_i^2$ be any map of K_i^2 to L_i^2 which induces an isomorphism in the fundamental groups and which is of degree one. We take $\bar{f}_i$ to be any extension of $\bar{f}_i^2$. We next take a map $\bar{f}: K \to L$ inducing φ. Since Φ is a c-isomorphism, $\bar{f}|K_i$ and $\bar{f}_i$ are homotopic. Hence by homotopy extension Theorem we may assume that $\bar{f}|K_i = \bar{f}_i$. We will further homotope $\bar{f}$ to make it cellular. The restriction of $\bar{f}$ gives us a map $M^2 \to N^2$ such that ∂M is mapped to ∂N. We will denote f^1 the restriction of $\bar{f}$ to $\partial M \cup M^1$ and by f^2 the restriction of $\bar{f}$ to M^2.

Let $\bar{\tau}(M, m)$ and $\bar{\tau}(N, n)$ denote the images of the fundamental cycles of M and N in $H_3(K, \partial M; \tilde{\mathbf{Z}})$ and $H_3(L, \partial N; \tilde{\mathbf{Z}})$ respectively. Consider the following commutative diagram, where all the vertical maps are induced by $\bar{f}$.

$$\begin{array}{ccccccc}
0 \longrightarrow H_3(K, \partial M; \tilde{\mathbf{Z}}) & \xrightarrow{e} & H_3(K, M - B^0; \tilde{\mathbf{Z}}) & \xleftarrow{h} & \tilde{\mathbf{Z}} \underset{\pi_1(M,m)}{\otimes} \pi_3(K, M - B^0) & \xleftarrow{1 \otimes \partial} & \tilde{\mathbf{Z}} \underset{\pi_1(M,m)}{\otimes} \pi_2(M - B^0) \\
\downarrow \bar{f}_* & & \downarrow \bar{f}_* & & \downarrow 1 \otimes \bar{f}_* & & \downarrow 1 \otimes \bar{f}_* \\
0 \longrightarrow H_3(L, \partial N; \tilde{\mathbf{Z}}) & \longrightarrow & H_3(L, N - D^0; \tilde{\mathbf{Z}}) & \longleftarrow & \tilde{\mathbf{Z}} \underset{\pi_1(N,n)}{\otimes} \pi_3(L, N - D^0) & \xleftarrow{1 \otimes \partial} & \tilde{\mathbf{Z}} \underset{\pi_1(N,n)}{\otimes} \pi_2(N - D^0)
\end{array}$$

Here (in both horizontal sequences) $1 \otimes \partial$ is an isomorphism since $M - B^0$ (resp. $N - D^0$) contains the 2-skeleton of K. The map h is the Herewicz isomorphism with local coefficients $\tilde{\mathbf{Z}}$ and e is injective since the previous term is $H_3(M - B^0, \partial M; \tilde{\mathbf{Z}})$ (resp. $H_3(N - D^0, \partial M; \tilde{\mathbf{Z}})$ and up to homotopy there are no extra 3-cells. Identifying the last terms with $H_0(M; \pi_2(M - B^0) \otimes \tilde{\mathbf{Z}})$ and $H_0(N - D^0) \otimes \tilde{\mathbf{Z}})$, inspection shows that $h_{M,m}(\bar{\tau}(M, m)) = B(M, m)$ and $h_{N,n}(\bar{\tau}(N, n)) = B(N, n)$, where $h_{M,m}$ (resp. $h_{N,n}$) is the composite $(1 \otimes \partial)^{-1} \circ h \circ e$. Since $h_{M,m}$ and $h_{N,n}$ are injective, we see that $\bar{f}_*(\bar{\tau}(M, m)) = \bar{\tau}(N, n)$ if and only if $(1 \otimes \bar{f}_*)B(M, m) = B(N, n)$. Next consider the diagram

$$\begin{array}{ccccc}
\longrightarrow H_3(\partial K, \partial M; \tilde{\mathbf{Z}}) & \xrightarrow{j_M} & H_3(K, \partial M; \tilde{\mathbf{Z}}) & \xrightarrow{i_M} & H_3(K, \partial K; \tilde{\mathbf{Z}}) \\
\downarrow \bar{f}_* & & \downarrow \bar{f}_* & & \downarrow \bar{f}_* \; . \\
\longrightarrow H_3(\partial L, \partial N; \tilde{\mathbf{Z}}) & \xrightarrow{j_N} & H_3(L, \partial N; \tilde{\mathbf{Z}}) & \xrightarrow{i_N} & H_3(L, \partial L; \tilde{\mathbf{Z}}) \\
\downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
H_2(\partial N; \tilde{\mathbf{Z}}) & \xrightarrow{=} & H_2(\partial N; \tilde{\mathbf{Z}}) & \xrightarrow{i} & H_2(\partial L, \tilde{\mathbf{Z}})
\end{array}$$

Let $y = \bar{f}_*(\bar{\tau}(M, m)) = -\bar{\tau}(N, n)$. Then $i_N(y) = \bar{f}_*(\tau(M, m)) - \tau(N, n) = 0$. Hence $y = j_N(x)$ for some x. Moreover $i\partial(y) = 0$. If L_i is a component of ∂N which is not a projective plane, then $H_2(L_i; \tilde{\mathbf{Z}})$ is a direct summand of both $H_2(\partial N; \tilde{\mathbf{Z}})$ and $H_2(\partial L; \tilde{\mathbf{Z}})$. This shows that the component of ∂y in $H_2(L_i; \tilde{\mathbf{Z}})$ is zero if L_i is not the infinite projective space. If L_i is an infinite projective space, then $\bar{f}_i$ is a degree one map (by construction) on the particular component of ∂M. If N_i denotes the corresponding component of ∂N, then we have that the component of ∂y in $H_2(N_i; \tilde{\mathbf{Z}})$ is zero. Thus $\partial y = 0$. This implies that $\partial x = 0$. Now $H_3(\partial L, \partial N; \tilde{\mathbf{Z}})$ is isomorphic to a direct sum $\bigoplus H_3(P^\infty, P^2; \tilde{\mathbf{Z}})$ and the boundary map corresponds to the direct sum boundary maps $H_3(P^\infty, P^2; \tilde{\mathbf{Z}}) \to H_2(P^2; \tilde{\mathbf{Z}})$. Since this map is injective we conclude that $x = 0$. Thus $y = 0$ or $\bar{f}_*(\bar{\tau}(M, m)) = \bar{\tau}(N, n)$. By the previous argument, this implies that $B(M, m)$ goes to $B(N, n)$. Now, Theorem 1.3 gives a degree one map $f: (M, \partial M) \to (N, \partial N)$ inducing. Such a map is automatically homotopy equivalence. This completes the proof of Theorem 1.3.

The invariant clearly has naturality properties with respect splitting along closed surfaces. We study one of these properties in §3. The proof of 1.1 shows:

1.4 *Addendum to* 1.1. If in 1.1 we are given an oriented homotopy equivalence $g: \partial M \to \partial N$ compatible with Φ, then we can find a f extending g iff $\Phi_*(\tau(M)) = \tau(N)$.

§2. THE INVARIANT IN THE CASE OF IRREDUCIBLE 3-MANIFOLDS

Irreducible 3-manifolds which are P^2-irreducible are either aspherical or orientable and have a homotopy sphere as their universal cover. Hence the invariant is easily understood in these case. So we will study it in the case of compact irreducible 3-manifolds which contain 2-sided projective planes. If the manifold is homotopy equivalent to $P^2 \times I$ (or $P^2 \times S^1$), the invariant is easily calculated and is seen to be the generator of group isomorphic to $\tilde{\mathbf{Z}}_2$ (or $\emptyset$). For the rest of the section, we will assume that M satisfies the following hypothesis:

H1. M is irreducible, ∂M is incompressible, M contains two sided projective planes and M is not homotopy equivalent to $P^2 \times I$ or $P^2 \times S^1$.

For such a (M, m), we showed in [10] (see proposition 4.1 and 4.2 of [10]) that by splitting M along a finite number of two sided projective planes, we obtain components $N_1, \ldots, N_k$ $(k \geq 1)$ such that

(a) $\partial N_i \neq 0$, N_i is not a homotopy $P^2 \times I$ and any projective plane in $N_i - \partial N_i$ is pseudo-parallel to the boundary.

(b) If $\tilde{N}_i$ denotes the orientation cover of N_i, then $\pi_1(\tilde{N}_i)$ is indecomposable, torsion-free and is not isomorphic to $\mathbf{Z}$, and

(c) If $\hat{N}_i$ denotes the manifold obtained by capping the boundary spheres in $\tilde{N}_i$, then $\hat{N}_i$ is aspherical.

If $\pi_1(\tilde{M}, \tilde{m})$, the fundamental group of the orientation cover has a free decomposition of length n, then the number of projective planes (along which M is split) is exactly $n-1$. We may arrange the N_i so that there are connected submanifolds M_i, $1 \leq i \leq n$, of M with

(2.1) $N_1 = M_1 \subset M_2 \ldots \subset M_n = m$, and

(2.2) For $1 < i \leq k$, M_i is obtained by identifying a component of ∂M_{i-1} with a component of ∂N_i. For $k < i \leq n$, M_i is obtained by identifying two distinct components of ∂M_{i-1}.

Let the images of the above projective planes in M $M_2, \ldots, P_n$. We will denote by P_1 the projective plane in ∂N_1 which contains the base point m. Of course, $P_1 = P_i$ for some i, $1 < i \leq n$.

If we attach infinite projective spaces to N_i along the boundary components which are projective planes, the resulting manifolds $\bar{N}_i$ are easily seen to be aspherical. In the orientation cover $\tilde{N}_i$, this corresponds adding infinite spheres along sphere boundary components; the result we will still denote by $\hat{N}_i$. Similarly, the space $\bar{M}$ obtained from M by attaching infinite projective spaces along $P_2, \ldots, P_n$ is aspherical. We will denote its double cover by $\hat{M}$. Note that $\bar{M}$ is built from $\bar{N}_i$, just as M is built from N_i. Similar remark holds for $\hat{M}$ and $\hat{N}_i$. The subspaces corresponding to M_i will be denoted by $\bar{M}_i, \tilde{M}_i, \hat{M}_i, \ldots$. If X is any one of these spaces $\bar{N}_i, \hat{N}_i, \bar{M}_i, \hat{M}_i$, ∂X will denote the following : Suppose X is obtained from a submanifold Y of M (resp. $\tilde{M}$). Then ∂X will be the union of ∂Y and the infinite projective spaces (resp. infinite spheres) added along the components of ∂Y. Thus $\tau(Y)$ can be identified with the image of the fundamental cycle of Y in $H_3(X, \partial X; \tilde{\mathbf{Z}})$.

By excision, $H_3(\bar{N}_i, \partial\bar{N}_i; \tilde{\mathbf{Z}}) \approx H_3(N_i, \partial N_i; \tilde{\mathbf{Z}}) \approx \mathbf{Z}$. For $1 < i \leq k$, we have exact sequences:

$$0 \longrightarrow H_3(\bar{M}_i, \partial\bar{M}_i; \tilde{\mathbf{Z}}) \longrightarrow H_3(\bar{M}_i, \partial\bar{M}_{i-1} \cup \partial\bar{N}_i; \tilde{\mathbf{Z}}) \longrightarrow H_2(\partial\bar{M}_{i-1} \cup \partial\bar{N}_i, \partial\bar{M}_i; \tilde{\mathbf{Z}}) \longrightarrow H_2(\bar{M}_i, \partial\bar{M}_i; \tilde{\mathbf{Z}}) \longrightarrow$$

Next,

$$H_3(\bar{M}_i, \partial\bar{M}_{i-1} \cup \partial\bar{N}_i; \tilde{\mathbf{Z}}) \approx H_3(\bar{M}_{i-1}, \partial\bar{M}_{i-1}; \tilde{\mathbf{Z}}) \oplus H_3(\bar{N}_i, \partial\bar{N}_i; \tilde{\mathbf{Z}}),$$

and

$$H_2(\partial\bar{M}_{i-1} \cup \partial\bar{N}_i, \partial\bar{M}_i; \tilde{\mathbf{Z}}) \approx H_2(\partial\bar{M}_{i-1} \cap \partial\bar{N}_i; \tilde{\mathbf{Z}}) \approx H_2(P^\infty; \tilde{\mathbf{Z}}) \approx \mathbf{Z}_2.$$

Moreover, the fundamental cycle of M_{i-1} (or N_i) shows that this last term is homologous to the rest of the boundary components of M_{i-1} (or N_i). Thus, we have short exact sequences:

$$0 \longrightarrow H_3(M_i, \partial M_i; \tilde{\mathbf{Z}}) \longrightarrow H_3(M_{i-1}, \partial M_{i-1}; \tilde{\mathbf{Z}}) \oplus H_3(N_i, \partial N_i; \tilde{\mathbf{Z}}) \longrightarrow \mathbf{Z}_2 \longrightarrow 0, \qquad 1 < i \leq k. \tag{2.3}$$

This shows that $H_3(\bar{M}_i, \partial\bar{M}_i; \tilde{\mathbf{Z}})$ is a free abelian group of rank i. Comparison of this with the exact sequence

$$0 \longrightarrow H_3(\hat{M}_i, \partial\hat{M}_i; \mathbf{Z}) \longrightarrow H_3(\hat{M}_{i-1}, \partial\hat{M}_{i-1}; \mathbf{Z}) \oplus H_3(\hat{N}_i, \partial\hat{N}_i; \mathbf{Z}) \longrightarrow 0 \tag{2.3'}$$

shows that, we have exact sequences

$$0 \longrightarrow H_3(\hat{M}_i, \partial\hat{M}_i; \mathbf{Z}) \xrightarrow{\ i\ } H_3(\bar{M}_i, \partial\bar{M}_i; \tilde{\mathbf{Z}}) \longrightarrow \mathbf{Z}_2 \longrightarrow 0 \tag{2.4}$$

and that

$$j_*(\tau(\tilde{M}_i)) = 2\tau(M_i).$$

For i with $k < i \leq n$, we have isomorphisms

$$H_3(\tilde{M}_i, \partial\tilde{M}_i; \tilde{\mathbf{Z}}) \longrightarrow H_3(\tilde{M}_{i-1}, \partial\tilde{M}_{i-1}; \tilde{\mathbf{Z}})$$

since the extra projective plane has connected complement in M_i. Again, comparison with $\hat{M}$ shows that (2.4) is valid for all i, $1 \leq i \leq n$. Thus, we have

2.5 Lemma. *If M satisfies* H1, *$H_3(\mathfrak{G}(M, m); \tilde{\mathbf{Z}})$ is a free abelian group of rank k. The inclusion i: $\mathfrak{G}(\tilde{M}, \tilde{m}) \to \mathfrak{G}(M, m)$ yields an exact sequence*

$$0 \longrightarrow H_3(\mathfrak{G}(\tilde{M}, \tilde{m}); \mathbf{Z}) \xrightarrow{i_*} H_3(\mathfrak{G}(M, m); \tilde{\mathbf{Z}}) \longrightarrow \mathbf{Z}_2 \longrightarrow 0$$

such that $i_*(\tau(\tilde{M}, \tilde{m})) = 2\tau(M, m)$.

We will use this to prove:

2.6 Proposition. *Let (M^1, m^1) and (M^2, m^2) satisfy* H1, *and let Θ: $\mathfrak{G}(M^1, m^1) \to \mathfrak{G}(M^2, m^2)$ be a c-isomorphism of their group systems. Then there is a c-isomorphism Φ: $\mathfrak{G}(M^1, m^1) \to \mathfrak{G}(M^2, m^2)$ such that $\Phi_*(\tau(M^1, m^1)) = \tau(M^2, m^2)$.*

Proof. We will first consider a M as in the discussion preceding 2.5. Identify $H_3(\mathfrak{G}(\tilde{M}); \mathbf{Z})$ with $\bigoplus H_3(\mathfrak{G}(\tilde{N}_i); \mathbf{Z})$. This is a free abelian group on k generators and we may assume that $\tau(\tilde{M}, \tilde{m})$ is represented by $(1, 1, \ldots, 1)$. Let ϵ_l denote the element of $H_3(\mathfrak{G}(\tilde{M}); \tilde{\mathbf{Z}})$ which has -1 in lth place and 1 elsewhere. We will first show that there is an orientation-true c-automorphism Ψ of $\mathfrak{G}(M)$ such that $\tilde{\Psi}_*(\tau(M)) = \epsilon_l$. Here $\tilde{\Psi}$ denotes the c-automorphism of $\mathfrak{G}(\tilde{M})$ induced by Ψ.

Let g_j be the element of $\pi_1(M, m)$ represented by P_j. We identify $\pi_1(N_i)$, $\pi_1(M_i)$, with subgroups of $\pi_1(M, m)$. By choosing paths from P_j to m suitably, we may assume that g_j are in $\pi_1(M_k, m)$. If P_j^1 and P_j^2 are the components of ∂M_k corresponding to $P_j (j > k)$, we may assume that P_j^1 represents g_j with respect to a path in M_k from P_j^1 to m. Let P_j^2 represent $\bar{g}_j$ with respect to a path from P_j^2 to m in M_k. The composite path from P_j^2 to P_j^1 gives a generator t_j in $\pi_1(M, m)$ such that $g_j = t_j\bar{g}_jt_j^{-1}$.

Let M_l^1 and M_l^2 be the two parts into which $P_j (2 \leq j \leq k)$ divides M_k, with M_l^1 containing N_j. We define a c-automorphism Φ_1 of $\mathfrak{G}(M_k, m)$ by defining φ_1: $\pi_1(M_k, m) \to \pi_1(M_k, m)$ as follows:

$$\varphi_1(g) = g \quad \text{if} \quad g \in \pi_1(M_l^2)$$
$$\varphi_1(g) = g_lgg_l \quad \text{if} \quad g \in \pi_1(M_l^1).$$

Let $X_1, \ldots, X_t$ be the components of the closure of $M_l^1 - N_l$. Let $P_{j(s)} = N \cap X_s$, $1 \leq s \leq t$. We define Φ_2 by defining φ_2: $\pi_1(M_k, m) \to \pi_1(M_k, m)$ as follows:

$$\varphi_2(g) = g \quad \text{if} \quad g \in \pi_1(M_l^2 \cup N_l)$$

and

$$\varphi_2(g) = g_{j(s)}gg_{j(s)} \quad \text{if} \quad g \in \pi_1(X_s), \quad 1 \leq s \leq t.$$

Let $\Phi = \Phi_2 \circ \Phi_1$ and $\varphi = \varphi_2 \circ \varphi_1$. φ carries g_j, $\bar{g}_j$ to their conjugates. Let $\varphi(g_j) = h_jg_jh_j^{-1}$ and $\varphi(\bar{g}_j) = \bar{h}_j\bar{g}_j\bar{h}_j^{-1}$. The elements $h_jt_j(\bar{h}_j)^{-1}$ and $h_jg_jt_j(\bar{h}_j)^{-1}$ have opposite parities. We map t_j onto the one which has the same parity as t_j. Thus Φ extends to an orientation-true c-isomorphism Ψ of $\mathfrak{G}(M, m)$. And clearly Ψ has the desired property.

We next show that with the hypotheses of 2.6, Θ can be replaced by an orientation-true c-isomorphism. This together with 2.5 and the last statement of the above paragraph will complete the proof of 2.6. We first split M^2 as above. Then by 4.3 of [10], M^1 can be split exactly as M^2 is split. We will denote the subspaces of $M^j (j = 1, 2)$ corresponding $N_i, M_i, P_i, \ldots$ by $N_i^j, M_i^j, P_i^j, \ldots$. Since M^1 is split exactly as M^2 is split, $\mathfrak{G}(N_i^1), \mathfrak{G}(M_i^1), \ldots$ are carried to $\mathfrak{G}(N_i^2), \mathfrak{G}(M_i^2), \ldots$ by Θ. Moreover if t_i^2 (resp. t_i^1) is represented by a loop formed of a path joining $P_i^{2,2}$ (resp. $P_i^{2,1}$) to $P_i^{1,2}$ (resp. $P_i^{1,1}$) and $\theta(t_i^1) = t_i^2$. All this can be achieved by taking a map $(M^1, m^1) \to (\tilde{M}^2, m^2)$ inducing Θ and splitting M^1 exactly as $\tilde{M}^2$ is split (see [10]). We next observe that the induced maps $\mathfrak{G}(N_i^1)\to\mathfrak{G}(N_i^2)$ are automatically orientation-true. This is seen as follows. Let θ_i be the induced isomorphism $\pi_1(N_i^1)\to\pi_1(N_i^2)$. Consider $\tilde{N}_i^2$ and the cover L_i of N_i^1 corresponding to the subgroup $\theta_i^{-1}(\pi_1(\tilde{N}_i^2))$. Since $\pi_1(\tilde{N}_i^2)$ is torsion-free, L_i does not have any two-sided projective planes in it. Let $\hat{L}_i$ be the manifold obtained by capping of the boundary spheres in L_i. Then, we have an c-isomorphism $\tilde{\Theta}_i$: $\mathfrak{G}(\hat{L}_i)\to\mathfrak{G}(\hat{N}_i^2)$, and $\hat{N}_i^2$ is aspherical. $\pi_1(\hat{L}_I)$ is indecomposable, infinite and is not isomorphic to $\mathbf{Z}$ and ∂L_I is incompressible; all these properties follow from those of $\hat{N}_i^2$. Hence $\hat{L}_i$ is also aspherical and $\tilde{\Theta}_i$ is induced by a homotopy equivalence $(\hat{L}_i, \partial\hat{L}_i)\to(\hat{N}_i^2, \partial\hat{N}_i^2)$. Hence $\hat{L}_i$ and L_i are orientable and thus $\pi_1(L_i)=\pi_1(\tilde{N}_i^1)$ in $\pi_1(N_i^1)$. Hence θ_i is orientation-true. Then, it follows that the map induced by Θ from $\mathfrak{G}(M_k^1)\to\mathfrak{G}(M_k^2)$ is also orientation-true. Thus, the only thing that can go wrong is that for some $j>k$, t_j^1 and t_j^2 may have opposite parities. We change Θ, by mapping t_j^1 to $g_j^2t_j^2$ for such j. It is east to verify that we obtain an orientation-true c-isomorphism. This completes the proof of 2.6.

2.7 *Remark.* The above proof shows that even for M^1, M^2 satisfying H1, there are c-isomorphisms Θ: $\mathfrak{G}(M^1)\to\mathfrak{G}(M^2)$ which are not orientation-true and even if they are orientation-true, they need not be induced by maps. On the other hand, it is immediate from 2.5, 2.6 and 1.1:

2.8 COROLLARY. *Let M^1 and M^2 satisfy* H1. *If $\mathfrak{G}(M^1)$ and $\mathfrak{G}(M^2)$ are c-isomorphic, then $(M^1, \partial M^1)$ and $(M^2, \partial M^2)$ are homotopy equivalent.*

2.9 *Remark.* In the next section we show that 2.6 and 2.8 are more generally true. Similarly 2.5 can be extended, but we do not need it for the homeomorphism problem.

§3. CONSTRUCTION OF HOMEOMORPHISMS

In this section, we consider the homeomorphism problem for a class of irreducible 3-manifolds which contain 2-sided projective planes. We first extend Waldhausen's notion of "sufficiently large" (see [13] and [3]). Let M_1 be a compact irreducible 3-manifold. A *hierarchy* for M_i (of length n) is a sequence of triples.

M_j, $F_j\subset M_j$, $\cup(F_j)\subset M_j$, M_{j+1} = closure of $(M_j-\cup(F_j))$ in M_j where j ranges from 1 to n, such that

(a) F_j is incompressible and two-sided in M_j, $\cup(F_j)$ is a regular neighbourhood of F_j in M_j, and

(b) each component of M_n either a 3-cell or homeomorphic to $P^2\times I$.

We say that M is *sufficiently large* if M has a hierarchy. In §4, we show that for many M, M is sufficiently large if and only if its orientation cover is connected sum of $S^1\times S^2$'s and sufficiently large 3-manifolds (in the sense of Waldhausen). We note that the 3-cell is also sufficiently large.

We first have to consider the splittings of 3-manifolds along incompressible surfaces and see how the associated invariants behave (we will only consider splittings along connected incompressible surfaces). Let S be a property embedded two-sided incompressible surface in a 3-manifold N. Let N_1, N_2 (one of them may be vacuous) be the manifolds obtained by splitting N along S. If T is a component of ∂N and T' a component of ∂N_1 or ∂N_2 we say that $T'<T$ if $T'\cap T\neq\emptyset$. Let M be another 3-manifold, and let Φ: $\mathfrak{G}(M,m)\to\mathfrak{G}(N,n)$ be a c-isomorphism of their group systems. Let T be a component of ∂M and L a component of ∂N. We say that T and L are Φ-related, if one of the φ_i carries $\pi_1(T)$ to $\pi_1(L)$. The main tool of the homeomorphism theorem is the following:

3.1 PROPOSITION. *Let M and N be compact irreducible 3-manifolds and let Θ: $\mathfrak{G}(M)\to\mathfrak{G}(N)$ be a c-isomorphism of their group systems. Let S be a property embedded, connected, two-sided incompressible surface in N and let N_1, N_2 be the manifolds obtained by splitting N along S. If the complement of S is connected, we assume that $N_2=\emptyset$. Then*

(a) *There is a connected, two-sided, properly embedded incompressible surface R homeomorphic to S in M dividing M into M_1, $M_2 (M_2=\emptyset$ if $N_2=\emptyset)$ and there are c-isomorphisms Θ_j: $\mathfrak{G}(M_j)\to\mathfrak{G}(N_j)$, $j=1,2$ such that*

$$\Theta/\pi_1(M_j)=\Theta_j/\pi_1(M_j),\qquad j=1,2.$$

(b) *If $L_1, L_2<T, L_1', L_2'<T'$ (one of the L_j and one of the L_j' may be vacuous) with $L_j\subset\partial M_j$, $L_j'\subset\partial N_j$ and if T, T' are Θ-related, then L_j, L_j' are Θ_j-related, $j=1,2$. If $L_1, L_2\subset\partial M_1$*

(this happens only when $N_2 = \emptyset$), and if T, T' are Θ-related then we can reorder the L'_j such that (L_j, L'_j), $j = 1, 2$ are Θ_1-related.

(c) *If $\Theta_*(\tau(M)) = \tau(N)$, then $(\Theta_j)_*(\tau(M_j)) = \tau(N_j)$, $j = 1, 2$.*

3.2 *Addendum.* In 3.1, if there is a oriented homotopy equivalence $f: (M, \partial M) \to (N, \partial N)$ with $f_* = \Theta$ and $f/\partial M$ is a homeomorphism, then we can find oriented homotopy equivalences $f_j: (M_j, \partial M_j) \to (N_j, \partial M_j)$ with $(f_j)_* = \Theta_j$, $f_j/\partial M_j$ is a homeomorphism and $f_j|(\partial M \cap \partial M_j) = f|(\partial M \cap \partial M_j)$, $j = 1, 2$.

3.3 *Remark.* In the P^2-irreducible case of 3.1, the manifolds are aspherical and stronger results than 3.1 are valid (see [1], [2] and [8]). At two points below, we appeal to the arguments of these papers.

Proof of 3.1. We first construct a system of $K(\pi, 1)$-spaces from N as in §1 and §2 and use it to split M. We saw in §2, that for irreducible 3-manifolds with incompressible boundary, we can construct $K(\pi, 1)$-spaces in a natural way by adjoining infinite projective spaces along a finite number of two-sided projective planes. The same result is easily seen to be valid for any compact irreducible 3-manifold, since we can first split along discs to obtain boundary-incompressible manifolds. Since S is incompressible, we may assume that none of these projective planes intersects S and for the same reason no component of ∂N intersecting S is a projective plane. We first add P^∞'s along the components of ∂N which are projective planes and then add a finite number of P^∞'s along two-sided projective planes in the criterion of N away from S. Thus we obtain a system of $K(\pi, 1)$-space, K, K_i and an inclusion $\alpha: (N, \partial N) \to (K, \cup K_i)$ which induces an isomorphism of the group systems. The 2-skeleton of K is the same as the 2-skeleton of N and α is a homeomorphism on the components of ∂N which are not projective planes. As before we denote $\cup K_i$ by ∂K. Since K, K_i are $K(\pi, 1)$-spaces, we can find a map $f: (M, \partial M) \to (K, \partial K)$ with $f_* = \Theta$. We may assume that f is a homeomorphism on the components of ∂M which are not projective planes. By a further homotopy constant on ∂m we may make f transverse to S. Now, we use Loop Theorem (see [1], [2] and [8]) to make $f^{-1}(S)$ incompressible. Then for each component R of $f^{-1}(S)$, $(f/R)_*$ maps $\pi_1(R)$ injectively into $\pi_1(S)$. If $\partial S \neq \emptyset$, $\pi_1(S)$ is free and thus R cannot be closed. Moreover $f/\partial R$ maps ∂R homeomorphically into ∂S. Thus the degree of f/R is not zero. Hence $f/\partial R$ is a homeomorphism onto ∂S. Hence $\partial(f^{-1}(S)) \subset \partial R$. Since $f^{-1}(S)$ has no closed components, we conclude that $f^{-1}(S) = R$. If S is closed $f^{-1}(S)$, even when incompressible may have several components. In this case, [1], [2], [8] show how to reduce $f^{-1}(S)$ to one component without changing f on ∂M if M is P^2-irreducible. However, the same arguments are valid in P^2-reducible case using the Theorems 4.1, 5.2 and 5.4 of [10]. Thus in either case, we can make $R = f^{-1}(S)$ connected and incompressible and $f/R: R \to S$ induces isomorphisms in the fundamental groups. From the construction 3.1(a) and 3.1(b) are clear. In 3.2 we can take $f = \alpha \circ g$. If $\partial R \neq \emptyset$, we homotope f/R to a homeomorphism of R onto S without changing $f/\partial R$. Since the 2-skeleton of K is the same as the 2-skeleton of N, this gives us a map f' of M^2 to N^2 (where M^2 and N^2 are the 2-skeletons of M and N) such that $f'/\partial M_1 \cup \partial M_2$ is a homeomorphism and extends $f/\partial M$. Moreover the induced maps $\mathfrak{G}(M_1) \to \mathfrak{G}(N_1)$ and $\mathfrak{G}(M_2) \to \mathfrak{G}(N_2)$ are the same as those given by 3.1(a) and hence if we prove 3.1(c), 3.2 follows. It remains to prove 3.1(c).

We first note that we may consider N as the union $N_1 \underset{S\times\{-1\}}{\cup} S \times [-1, +1] \underset{S\times\{+1\}}{\cup} N_2$ and we may consider M as the union $M_1 \underset{R\times\{-1\}}{\cup} R \times [-1, +1] \underset{R\times\{+1\}}{\cup} M_2$. Split K along S and write similarly $K = K_1 \underset{S\times\{-1\}}{\cup} S \times [-1, +1] \underset{S\times\{+1\}}{\cup} K_2$. If one of the N_i is empty, one has to make appropriate changes which we will omit. Let ∂K_i, $i = 1, 2$ denote ∂N_i union the infinite projective planes added to ∂N along the components of ∂N which are in ∂N_i. Then $(K_i, \partial K_i)$ may not form a $K(\pi, 1)$-system for $(N_i, \partial N_i)$, if S is a disc and the resulting component of ∂N_i is a projective plane. In this case, we adjoin an infinite projective space to ∂K_i along the projective plane and then add further cells of dimension ≥ 3 to obtain $K(\pi, 1)$-system $(K'_i, \partial K'_i)$, $i = 1, 2$. In any case, we have $(N_i, \partial N_i) \subset (K_i, \partial K_i) \subset (K'_i, \partial K'_i)$, $i = 1, 2$

$$\text{and} \quad \partial K'_1 \cap S \times [-1, +1] = \partial K_1 \cap S \times [-1, +1] = S \times \{-1\}$$

and (3.4)

$$\partial K'_2 \cap S \times [-1, +1] = \partial K_2 \cap S \times [-1, +1] = S \times \{+1\}.$$

And f induces maps of $(M_i, \partial M_i)$ to $(K'_i, \partial K'_i)$, $i = 1, 2$. Consider the following diagram:

$$\begin{array}{ccccccc}
\longrightarrow & H_3(N, \partial N) & \xrightarrow{j_N} & H_3(N, \partial N_1 \cup \partial N_2) & \xleftarrow{i_N} & H_3(N_1, \partial N_1) \oplus H_3(N_2, \partial N_2) \oplus H_3(S \times I, S \times \partial I) & \longleftarrow 0 \\
 & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle (\alpha_{1*}, \alpha_{2*})} & \\
\longrightarrow & H_3(K, \partial K) & \xrightarrow{j_K} & H_3(K', \partial K'_1 \cup \partial K'_2 \cup S \times \partial I) & \xleftarrow{i_K} & H_3(K'_1, \partial K'_1) \oplus H_3(K'_2, \partial K'_2) \oplus H_3(S \times I, S \times \partial I) & \longleftarrow 0 \\
 & \uparrow{\scriptstyle f_*} & & \uparrow{\scriptstyle f_*} & & \uparrow{\scriptstyle (f_{1*}, f_{2*})} & \\
\longrightarrow & H_3(M, \partial M) & \xrightarrow{j_M} & H_3(M, \partial M_1 \cup \partial M_2) & \xleftarrow{i_M} & H_3(M_1, \partial M_1) \oplus H_3(M_2, \partial M_2) \oplus H_3(R \times I, R \times \partial I) & \longleftarrow 0
\end{array} \tag{3.5}$$

In the diagram (3.5), the maps in the horizontal sequences are induced by inclusions and so also the vertical maps in the upper squares. The vertical maps in the lower squares are induced by f and its restrictions $f_1, f_2, \ldots$. All the homologies are with respect to the coefficients $\tilde{\mathbf{Z}}$. Since we are assuming that $\Theta_*(\tau(M)) = \tau(N)$, Θ is orientation-true and all the maps are well defined. The map i_K is injective by 3.4. Let r and s denote the fundamental cycles of $R \times I$ and $S \times I$ respectively. We have

$$\begin{aligned}
j_N([N]) &= i_N([N_1], [N_2], s) \\
j_M([M]) &= i_M([M_1], [M_2], r) \\
\alpha_*[N] &= \tau(N), (\alpha_j)_*([N_j]) = \tau(N_j), \qquad j = 1, 2 \\
f_*[M] &= \tau(M), (f_j)_*([M_j]) = \tau(M_j), \qquad j = 1, 2.
\end{aligned}$$

By hypothesis,

$$\alpha_*[N] = f_*[M] = \tau(N) = \tau(M).$$

Hence

$$j_K \alpha_*[N] = j_K f_*[M].$$

But

$$\begin{aligned}
j_K \alpha_*[N] &= \alpha_* j_N[N] \\
&= \alpha_* i_N([N_1], [N_2], s) \\
&= i_K(\tau(N_1), \tau(N_2), s).
\end{aligned}$$

Similarly

$$j_K f_*[M] = i_K(\tau(M_1), \tau(M_2), r).$$

Since i_K is injective, we conclude that

$$\tau(M_i) = \tau(N_i) \quad \text{in} \quad H_3(K'_i, \partial K'_i; \tilde{\mathbf{Z}}), \qquad i = 1, 2.$$

This completes the proof of 3.1(c).

3.6 COROLLARY. 2.6 *and* 2.8 *are valid without the assumption that ∂M is incompressible. That is, it is enough to assume that M^1 and M^2 are non-orientable and irreducible.*

Proof. We first observe that in 2.6, given Θ, the Φ we constructed has the property that if T, T', $T \subset \partial M^1$, $T' \subset \partial M^2$ are Θ-related, then T, T' are Φ-related. Similarly in 2.8 given c-isomorphism $\Theta: \mathfrak{G}(M^1) \to \mathfrak{G}(M^2)$, there is a oriented homotopy equivalence $f: (M^1, \partial M^1) \to (M^2, \partial M^2)$ such that if T, T' are Θ-related, then $f(T) \subset T'$. Also 2.6 and 2.8 are easily seen to be valid when M^1, M^2 are homotopy equivalent to $P^2 \times S^1$ or $P^2 \times I$.

We now extend these results, by induction on the number of discs needed to reduce M^2 to boundary incompressible manifolds. Suppose that this number is n for M^2 and we have extended 2.6 and 2.8 when the number is $\leq n - 1$. Let $\Theta: \mathfrak{G}(M^1) \to \mathfrak{G}(M^2)$ be a c-isomorphism and let S be an incompressible disc in M^2. Let S split M^2 into $M^{2,1}$ and $M^{2,2}$. We apply 3.1 to obtain $R, M^{1,1}$ and $M^{1,2}$ and c-isomorphisms $\Theta_i: \mathfrak{G}(M^{1,j}) \to \mathfrak{G}(M^{2,j})$, $j = 1, 2$ satisfying 3.1(b). By induction we can find $\Phi j: \mathfrak{G}(M^{1,j}) \to \mathfrak{G}(M^{2,j})$, $j = 1, 2$ such that $(\Phi_j)_*(\tau(M^{1,j})) = \tau(M^{2,j})$ and Θ_j-related components are Φ_j-related. Applying 2.8, we find oriented homotopy equivalences $f_j: M^{1,j} \to M^{2,j}$ such that $f_j(T) \subset T'$ if T and T' are Φ_j-related, $j = 1, 2$. By 3.1(b) and the relation of Θ_j and Φ_j, we can patch them up to obtain an oriented homotopy equivalence $f: (M^1, \partial M^1) \to (M^2, \partial M^2)$. This proves 2.8. Taking the induced map in the group systems, we see that 2.6 is valid too.

Now let M^1 and M^2 be two irreducible 3-manifolds which are sufficiently large and let $\Theta: \mathfrak{G}(M^1) \to \mathfrak{G}(M^2)$ be a c-isomorphism. By 3.6 we can find a c-isomorphism $\Psi: \mathfrak{G}(M^1) \to \mathfrak{G}(M^2)$ with $\Psi_*(\tau(M^1)) = \tau(M^2)$. By 2.8, we can find a map $f: (M^1, \partial M^1) \to (M^2, \partial M^2)$ such that $f_* = \Psi$ and $f|\partial M^1$ is a homeomorphism. We take a hierachy for M^2 and split M^2 along the first incompressible surface S to obtain $M^{2,1}$ and $M^{2,2}$. By 3.1 and 3.2 we can split M^1 into $M^{1,1}$ and $M^{1,2}$ and find maps $f_j: (M^{1,j}, \partial M^{1,j}) \to (M^{2,j}, \partial M^{2,j})$ satisfying the conclusions of 3.1 and 3.2. In particular $f_1 \cup f_2|\partial M = f|\partial M$. Now $M^{2,j}$ have hierarchies of length less than the hierarchy of M^2. Using 3.1(c), we can easily construct homeomorphisms when $M^{2,j}$ are 3-cells or $P^2 \times I$'s. Thus by induction we have

3.7 (Homeomorphism Theorem). *Let M^1 and M^2 be two irreducible, sufficiently large 3-manifolds. If there is a c-isomorphism $\Theta: \mathfrak{G}(M^1) \to \mathfrak{G}(M^2)$, then $(M^1, \partial M^1)$ and $(M^2, \partial M^2)$ are homeomorphic. If $\Theta_*(\tau(M^1)) = \tau(M^2)$, then we can find a homeomorphism $f: (M^1, \partial M^1) \to (M^2, \partial M^2)$ with $f_* = \Theta$.*

§4. CRITERIA FOR SUFFICIENTLY LARGENESS

In the case of a 3-manifold M which is irreducible and P^2-irreducible, good criteria are known for existence of hierarchies (see [13] and [3]). Note that in our terminology "sufficiently large" is equivalent to "having a hierarchy". We say that an irreducible 3-manifold satisfies *Property H* if it satisfies H1 and H2 below:

H1) ∂M is incompressible and any two sided projective plane in M is parallel to the boundary.

H2) If ∂M consists entirely of projective planes, then their number is not two.

We say that a 3-manifold M satisfies *Property H'* if by splitting M along projective planes and incompressible discs, M can be reduced to manifolds satisfying Property H. It can be seen that the manifolds constructed by Jaco in [5] satisfy Property H and the manifolds constructed by Row in [7] satisfy Property H'. The main result of this section is the following Theorem which is proved in collaboration with F. Waldhausen.

4.1 Theorem. *Let M be a non-orientable irreducible 3-manifold which satisfies Property H'. Then M is sufficiently large.*

We will first show that the theorem follows from the lemma 4.2 below. We denote by $\hat{M}$, the manifold obtained by capping the boundary spheres in $\tilde{M}$.

4.2 Lemma. *If M is non-orientable and satisfies H and if $\hat{M}$ is sufficiently large, then there exists a two-sided incompressible surface $(S, \partial S) \subset (M, \partial M)$ such that S represents a non-trivial element of $H_2(M, \partial M; \tilde{\mathbf{Z}})$ or $H_2(M, \partial M; \tilde{\mathbf{Z}})$.*

If we prove this lemma, the proof of 4.1 can be completed as in [13], pp. 61–64. Waldhausen needs that if $\partial M \neq \emptyset$, then ∂S should represent a non-trivial element of $H_2(\partial M; \mathbf{Z})$. However, the only implication of this that he uses is that S is not parallel to the boundary. Clearly, if $(S, \partial S)$ represents a non-trivial element of $H_2(M, \partial M; \tilde{\mathbf{Z}})$ or $H_2(M, \partial M; \mathbf{Z})$ then S cannot be parallel to the boundary. Thus, it remains to prove Lemma 4.2.

Proof of 4.2. The lemma is proved in [3] if $\partial M = \emptyset$, since if M satisfies H and $\partial M = \emptyset$, then M is P^2-irreducible. Thus in what follows we may assume that $\partial M \neq \emptyset$. However, $\partial \hat{M}$ may be empty.

Let $\mathbf{Z}[\mathbf{Z}_2]$ denote the group ring of the cyclic group of order 2. Then, we have an exact sequence:

$$0 \longrightarrow \tilde{\mathbf{Z}} \longrightarrow \mathbf{Z}[\mathbf{Z}_2] \longrightarrow \mathbf{Z} \longrightarrow 0. \tag{4.3}$$

Identifying $\mathbf{Z}_2$ with the deck transformation group of $\tilde{M}$, we see that $H_i(M;\mathbf{Z}[\mathbf{Z}_2]) \approx H_i(\tilde{M};\mathbf{Z})$ and $H^i(M;\mathbf{Z}[\mathbf{Z}_2]) \approx H^i(\tilde{M};\mathbf{Z})$. Since M is non-orientable $H_3(M,\partial M;\mathbf{Z}) = 0$. Then, from (4.3), we obtain an exact sequence:

$$0 \longrightarrow H_2(M,\partial M;\tilde{\mathbf{Z}}) \longrightarrow H_2(\tilde{M},\partial\tilde{M};\mathbf{Z}) \xrightarrow{i} H_2(M,\partial M;\mathbf{Z}) \longrightarrow \cdots$$

If $p:\tilde{M}\to M$ is the covering projection, then the map i can be identified with p_*. If $H_2(M,\partial M;\tilde{\mathbf{Z}}) \neq 0$, then its dual $H^1(M;\mathbf{Z}) \neq 0$ and as in [14], we can map M to a circle and obtain an incompressible surface $(S,\partial S)$ in $(M,\partial M)$ representing a non-trivial element of $H_2(M,\partial M;\tilde{\mathbf{Z}})$. Thus we have to consider the case when $H_2(M,\partial M;\tilde{\mathbf{Z}}) = 0$. In this case the map $i: H_2(\tilde{M},\partial\tilde{M};\mathbf{Z}) \to H_2(M,\partial M;\mathbf{Z})$ is injective. We claim that $H_2(\tilde{M},\partial\tilde{M};\mathbf{Z})$ is a non-trivial torsion-free group. That there is no torsion is clear by duality because $H^1(\tilde{M};\mathbf{Z})$ is isomorphic to $\mathrm{Hom}(H_1(\tilde{M};\mathbf{Z});\mathbf{Z})$ which is torsion-free. To show that it is non-trivial, it is enough to show that $H_2(\hat{M},\partial\hat{M};\mathbf{Z})$ is non-trivial. Again if $\partial\hat{M} \neq \emptyset$, the statement follows from [14]. If $\partial\hat{M} = \emptyset$, then by our assumption, ∂M has at least 4 projective planes in the boundary. Then extending the involution on $\tilde{M}$ to $\hat{M}$, we get an orientation reversing involution on $\hat{M}$ with at least 4 fixed points. An application of Lefshetz' fixed point formula shows that $H_2(\hat{M};\mathbf{Z}) \neq 0$. Since $\partial\hat{M} = \emptyset$ in this case, we have in general $H_2(\hat{M},\partial\hat{M};\mathbf{Z}) \neq 0$ and therefore $H_2(\tilde{M},\partial\tilde{M};\mathbf{Z}) \neq 0$. Now, by [14], we can find a two-sided incompressible, properly embedded surface $(R,\partial R)$ in $(\hat{M},\partial\hat{M})$ representing a non-trivial element $H_2(\hat{M},\partial\hat{M};\mathbf{Z})$. By pushing it off a finite number of discs, we can move it into $\tilde{M}$. Again $(R,\partial R)$ represents a non-trivial element of $H_2(\tilde{M},\partial\tilde{M};\mathbf{Z})$. Now 4.2 will follow from 4.4 and the Loop Theorem.

4.4 Sublemma. *If there is a properly embedded surface $(R,\partial R) \subset (\tilde{M},\partial\tilde{M})$ representing a non-trivial element r of $H_2(\tilde{M},\partial\tilde{M};\mathbf{Z})$ then there is a properly embedded two-sided orientable surface $(S,\partial S) \subset (M,\partial M)$ representing either $i_*(r)$ or $2(i_*(r))$ in $H_2(M,\partial M;\mathbf{Z})$.*

Proof of 4.4. We will use the methods of cuts (see C. D. Papakyriokopoulos's paper in *Annals of Math.* 66 (1957) or J. R. Stallings, *ibid* 72 (1960)) to prove 4.4. We may first assume that $p(R)$ intersects transversally. The self-intersections of $p(R)$ consists of segments or circles. The inverse of image of a segment ω in R consists of two disjoint segments ω_1, ω_2 which are properly embedded in R and which map homeomorphically onto ω under p. The inverse image of circle ω will consist of (i) either two disjoint circles ω_1, ω_2 in the interior of R each of which maps homeomorphically onto ω under p, or (ii) one circle ω_1 in the interior of R such that $P|\omega_2: \omega_1 \to \omega$ in a double cover. Figures 1a and 1b represent the cases when ω is segment or ω is circle whose inverse image consists of two circles. If ω is segment, 1a (resp. 1b) represents a regular neighbourhood of ω in M. The two discs at the edges are in ∂M. If ω is a circle identify the two discs such that the segment a_1c_1 is identified with b_1d_1 and a_2c_2 is identified with b_2d_2. The resulting solid torus represents a neighbourhood of ω in $M-\partial M$ (ω is necessarily an orientation preserving loop). The arrows in the boundary represent the following. We fix orientation on R and choose the induced orientation on the pieces of R (rectangles or annuli) which map into the neighbourhood of ω, say T, represented in the figures. In the pieces in T, we choose the orientation given by these via p and the induced orientations on the boundaries are represented by arrows. Thus 1a and 1b represent two possibilities. By reversing the arrows we get the other two possible cases. Figure 2 corresponds to the case when $p^{-1}(\omega)$ is connected. In this case ω is necessarily orientation reversing. Identify the two discs at the edges so that the segments a_1c_1 is identified with b_2d_2 and a_2c_2 with b_1d_1. The results is a solid Klein bottle which represents a neighbourhood of ω in $M-\partial M$. The arrows again represent the orientation chosen as above. By reversing all the arrows, we get the other possible case.

We next get an orientable surface S' in $(M,\partial M)$ such that $[S'] = p[R]$. Outside the neighbourhoods of ω chosen as above we take S to be the part of $p(R)$ outside the neighbourhoods. And we take the induced orientation. If ω is segment, in case 1a, we take two disjoint rectangular pieces in the 3-cell such that one of them has c_2d_2 and c_1d_1 in its boundary, intersects the boundary of the 3-cell transversally and meets the annulus in the figure exactly along the segments c_2d_2 and c_1d_1. The other is chosen similarly with respect a_1b_1 and a_2b_2. If ω is a circle, in case 1a, we further choose the rectangle spanning c_2d_2, c_1d_1 so that the two extra pieces in the boundary match when we identify the edge cells. In either case we choose an

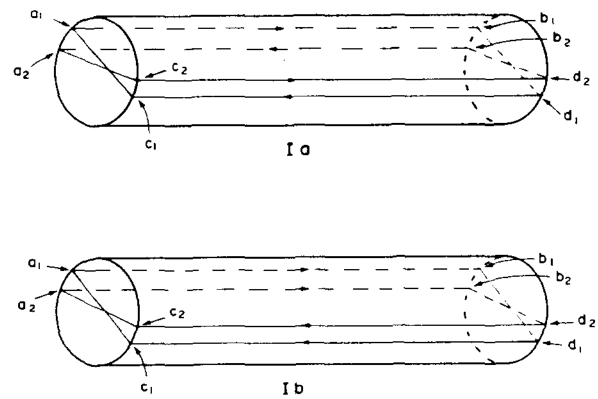

Fig. 1. In 1a, 1b, if ω is a circle a_1c_1, b_1d_1 and a_2c_2, b_2d_2 are identified.

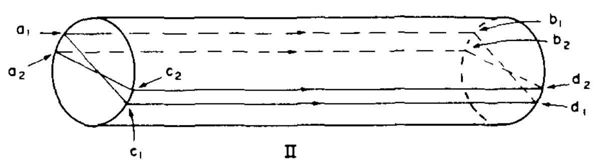

Fig. 2. In 2, a_1c_1, b_2d_2 and a_2c_2, b_1d_1 are identified.

orientation on the rectangle so that the orientation induced on the pieces c_2d_2 and c_1d_1 is as shown in Fig. 1a. Case 1b is handled similarly. In case 2 (Fig. 2), we take two disjoint rectangles spanning a_1b_1, c_2d_2 and a_2b_2 and a_2b_2, c_1d_1 such that in the identification of the edge cells, the extra pieces in the boundaries of the rectangles match properly. Thus we get one annulus and we choose the orientation which induces the orientation on the pieces $a_1b_1, \ldots$ as indicated in the figure. We adjoin these rectangles or annuli to the image of R outside the neighbourhoods and call it S'. Clearly, we get an orientable surface with $p[R] = [S']$. In general S' is not two-sided. If S' is two-sided we take $S = S'$. Otherwise, we take a regular neighbourhood V of S' and take $S =$ the closure of $\partial V - (\partial V \cap \partial M)$. It is clear that $[S]$ represents $2[S']$ in $H_2(M, \partial M; \mathbf{Z})$. This proves 4.3 and therefore 4.2 and 4.1.

4.5 *Remark.* In the proof of 4.2, what we mainly needed was that either $H_2(M, \partial M; \tilde{\mathbf{Z}}) \neq 0$ or $H_2(\tilde{M}, \partial \tilde{M}; \mathbf{Z}) \neq 0$. Even when ∂M consists of two projective planes, one of these conditions may be satisfied and in that case M is sufficiently large. However, if $\hat{M}$ is closed and $H_2(\hat{M}; \mathbf{Z}) = 0$, even if $\hat{M}$ is sufficiently large, we have not been able to show that M is sufficiently large.

The above proofs show that for M satisfying H1 even if ∂M consists of two projective planes, if we can find an incompressible surface in M not parallel to the boundary, then M is sufficiently large. Thus as in [13] and [14], we have

4.6 PROPOSITION. *If M satisfies* H1, *M is sufficiently large if either*

(1) *$\pi_1(M)$ is non-trivial amalgamated free product or*

(2) *$H_1(M; \mathbf{Z})$ is a non-trivial infinite group.*

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