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SURVEY ARTICLE

C.T.C. Wall's 1964 articles on 4-manifolds

Mark Powell

School of Mathematics and Statistics, University of Glasgow, University Place, Glasgow, UK

Correspondence
Mark Powell, School of Mathematics and Statistics, University of Glasgow, University Place, Glasgow G12 8QQ, UK.
Email: mark.powell@glasgow.ac.uk

Abstract

I survey C. T. C. Wall's influential papers, 'Diffeomorphisms of 4-manifolds' and 'On simply-connected 4-manifolds', published in 1964 on pp. 131–149 of volume 39 of the *Journal of the London Mathematical Society*.

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1 | INTRODUCTION

This is a survey of C. T. C. Wall's influential 1964 papers [87, 88] on 4-manifold topology. Wall's papers were published consecutively on 19 pp. of issue 39 of the *Journal of the London Mathematical Society*.

Both papers primarily concern smooth, closed, oriented, simply-connected 4-dimensional manifolds M, hereafter known as *scosc 4-manifolds*. An important invariant of such manifolds is the *intersection form*

$$\begin{aligned} Q_M : H_2(M) \times H_2(M) &\to \mathbb{Z} \\ (x, y) &\mapsto \langle \mathrm{PD}^{-1}(y), x\rangle. \end{aligned}$$

Here, $\mathrm{PD}^{-1}(y) \in H^2(M;\mathbb{Z})$ is the Poincaré dual of y, which we may evaluate using the Kronecker pairing $H^2(M;\mathbb{Z}) \times H_2(M) \to \mathbb{Z}$ on x. The intersection form is bilinear, symmetric, and nonsingular. The terminology 'intersection form' comes from the following geometric interpretation. Any two classes $x, y \in H_2(M)$ can be represented by immersions $\overline{x}, \overline{y} : S^2 \looparrowright M$ that intersect each other at finitely many transverse double points. At each double point $p \in \overline{x}(S^2) \pitchfork \overline{y}(S^2)$, the

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orientation of S^2 induces an orientation of the tangent space T_pM. Comparing this with the given orientation of M yields a sign $\varepsilon_p \in \{\pm 1\}$. Summing over all double points yields $\sum_p \varepsilon_p = Q_M(x, y)$.

Prior to Wall's work in [87, 88], there were few theorems known in 4-manifold topology. Early successes were as follows. Whitehead [91] and Milnor [67] had proven that two closed, simply-connected 4-manifolds with isometric intersection form are homotopy equivalent, Rokhlin [75] (see [38, pp. 17–21] for a French translation from Russian) proved that the signature of a closed, smooth, spin 4-manifold is divisible by 16, and Kervaire–Milnor applied this to obstruct smooth embeddings of spheres [51].

Wall's contributions quickly entered the canon of core 4-manifolds knowledge (see, e.g., [53, 81]), and surely helped inspire the advances of Cappell–Shaneson [14, 15] in the 1970s and the spectacular results of Freedman and Quinn [30, 33] and Donaldson [25, 26] in the 1980s. Wall was awarded the prestigious 1965 Berwick prize of the London Mathematical Society for these papers. The paper [87] has been cited 86 times and [88] has been cited 78 times, according to Mathscinet.

I will summarise the contents of the papers, explain Wall's key insights, and I will describe the various directions in which these ideas have been developed since.

1.1 | Diffeomorphisms of 4-manifolds

In the first of the two articles, [87], Wall considered isometries of Q_M, that is, $f\colon H_2(M) \xrightarrow{\cong} H_2(M)$ with

$$Q_M(f(x), f(y)) = Q_M(x, y).$$

He showed how to realise isometries by diffeomorphisms $F\colon M \xrightarrow{\cong} M$; here 'realise' means that $F_* = f$. If M is of the form $N\#(S^2 \times S^2)$, then in many cases Wall showed that the 'induced isomorphism' map

$$\mathcal{I}_M\colon \pi_0 \operatorname{Diff}^+(M) \to \operatorname{Aut}(H_2(M), Q_M); \quad F \mapsto F_*$$

is surjective. Here, $\pi_0 \operatorname{Diff}^+(M)$ is the *mapping class group* of M, consisting of isotopy classes of orientation-preserving self-diffeomorphisms of M, and we write $\operatorname{Aut}(H_2(M), Q_M)$ for the group of isometries of Q_M. Here is the precise statement of Wall's theorem.

Theorem A [87, Theorem 2]. *Let N be a scosc 4-manifold and suppose that Q_N is indefinite or that the rank of $H_2(N)$ is at most 8. Set*

$$M := N\#(S^2 \times S^2).$$

Then, $\mathcal{I}_M$ is surjective. That is, every isometry of Q_M is induced by a self-diffeomorphism of M.

Recently, Ruberman–Strle [79] extended Theorem A to remove the hypothesis that Q_N be indefinite or $\operatorname{rk} H_2(N) \leqslant 8$, at the expense of only realising elements in the image of the stabilisation map $\operatorname{Aut}(H_2(N), Q_N) \to \operatorname{Aut}(H_2(M), Q_M)$ that extends an isomorphism by the identity on $H_2(S^2 \times S^2)$. I will describe further extensions of Theorem A in Section 3.

1.2 | On simply-connected 4-manifolds

In the second article, [88], Wall applied [87] to improve on Whitehead's theorem, by giving classifications of scosc 4-manifolds up to h-cobordism and up to stable diffeomorphism.

An *h-cobordism* $(W; M, M')$ is a compact, smooth 5-dimensional cobordism W between M and M' such that the inclusion maps $M \to W$ and $M' \to W$ are both homotopy equivalences. Shortly before Wall's work, Smale [83] (see also [68]) had proven that for $n \geqslant 6$ every n-dimensional h-cobordism is diffeomorphic to the product $M \times [0,1]$. This theorem formed the basis of surgery theory and its successes in the classification of high-dimensional manifolds, for example, [8, 52, 85, 89]. It was unknown at the time whether five-dimensional h-cobordisms are smoothly products, and in fact this was shown in the 1980s to be false in general by Donaldson [26]. However in 1964 Wall was able to show the following.

Theorem B [88, Theorem 2]. *Two scosc 4-manifolds with isometric intersection forms are h-cobordant.*

This was extremely useful, as it meant that once Freedman [30] had established the 5-dimensional h-cobordism theorem in the topological category, the homeomorphism classification of scosc 4-manifolds followed immediately. Namely, two such 4-manifolds are homeomorphic if and only if their intersection forms are isometric.

Analysing the failure of Wall's h-cobordisms to be products led to the discovery of exotic pairs of 4-manifolds, that is, 4-manifolds that are homeomorphic but not diffeomorphic. In particular, it led to one of the constructions of exotic structures on $\mathbb{R}^4$ [37, Theorem 9.3.1].

Moreover, Wall's h-cobordisms between scosc 4-manifolds led Matveyev [66] and Curtis–Freedman–Hsiang–Stong [23] to the celebrated cork theorem, which states that every such h-cobordism W can be decomposed into $(X \times I) \cup_{\partial X \times I} V$, the union of a product cobordism $X \times I$ and a contractible h-cobordism V. They then deduced that any pair of such 4-manifolds M_1 and M_2 admits a *cork* [1], namely a contractible submanifold $C \subseteq M_1$ with an involution $\tau : \partial C \xrightarrow{\cong} \partial C$ such that $M_2 \cong (M_1 \setminus \mathring{C}) \cup_\tau C$. This implies that all exoticness of scosc 4-manifolds can be localised to contractible submanifolds.

I must also mention Kreck's result [62], refining Wall's, that the natural map, from the set of diffeomorphism classes rel. boundary of smooth h-cobordisms between scosc 4-manifolds M_1 and M_2 to the set of isometries between the intersection forms of M_1 and M_2, is an isomorphism. Theorem B is equivalent to the statement that each of these sets is nonempty if and only if the other is.

Next, we say that two 4-manifolds M_1 and M_2 are *stably diffeomorphic* if, for some k, we have that

$$M_1 \# k(S^2 \times S^2) \cong M_2 \# k(S^2 \times S^2). \tag{1.1}$$

By mimicking Smale's high-dimensional proof of the h-cobordism theorem as far as possible, Wall proved the following result.

Theorem C [88, Theorem 3]. *Two h-cobordant scosc 4-manifolds are stably diffeomorphic.*

Indefinite, symmetric, nonsingular, bilinear forms are classified up to isometry by their rank, signature, and parity [69]. Note that all symmetric, bilinear forms become indefinite after orthogonal sum with $Q_{S^2\times S^2}$, which is represented by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Combining this withB and C, we deduce that two scosc 4-manifolds are stably diffeomorphic if and only if their intersection forms have the same rank, signature, and parity. I will survey generalisations of this result in Section 6.

For stably diffeomorphic but not diffeomorphic 4-manifolds, how large must k be in (1.1)? It remains a famous open question whether there are h-cobordant scosc 4-manifolds for which one must take $k > 1$. For many examples, it has been shown that one $S^2 \times S^2$ summand suffices [3, 9, 19]. Recently, Sungkyung Kang [43] announced examples of pairs of compact, contractible 4-manifolds (which have nonempty boundary) where $k = 2$ is needed.

Motivated by Wall's result, topologists now study analogous questions on stable phenomena for diffeomorphisms of 4-manifolds and surfaces in 4-manifolds; see, for example, [4, 5, 18, 35, 54, 55, 65, 72].

1.3 | Outline

Here is a summary of the content of the rest of this survey.

- Section 2: key ideas in the proof of Theorem A.
- Section 3: extensions of Theorem A and related results.
- Section 4: key ideas in the proof of Theorem B.
- Section 5: key ideas in the proof of Theorem C.
- Section 6: extensions of Theorem C and related results.

2 | CONSTRUCTING DIFFEOMORPHISMS OF 4-MANIFOLDS

To perform *surgery* on an embedding $S^1 \times D^3 \hookrightarrow N$, we remove the interior $S^1 \times \mathring{D}^3$ and glue in $D^2 \times S^2$ in its place. An initial basic but fundamental observation is that for a scosc 4-manifold N, performing such a surgery yields either $N\#(S^2 \times S^2)$ or $N\#(S^2 \widetilde{\times} S^2)$, depending on which identification of $S^1 \times S^2$ boundary is used for gluing in $D^2 \times S^2$. Here, the manifold $S^2 \widetilde{\times} S^2$ is the S^2-bundle over S^2 obtained by gluing two copies of $S^2 \times D^2$ together using the Gluck twist $G : S^2 \times S^1 \xrightarrow{\cong} S^2 \times S^1$; this diffeomorphism rotates $S^2 \times \{e^{i\theta}\}$ through angle θ about a fixed axis, with $\theta \in [0, 2\pi)$. Bundles over S^2 with fibre S^2 and structure group $\mathrm{BDiff}^+ S^2$ are classified up to isomorphism by

$$[S^2, \mathrm{BDiff}^+ S^2] \cong \pi_2 \mathrm{BDiff}^+(S^2) \cong \pi_1 \operatorname{Diff}^+(S^2) \cong \pi_1 SO(3) \cong \mathbb{Z}/2,$$

hence there are exactly two such bundles, with total spaces $S^2 \times S^2$ and $S^2 \widetilde{\times} S^2$.

Since N is simply-connected, every embedded circle $S^1 \subseteq N$ is null-homotopic and hence isotopic to a trivially embedded circle. Let us fix a circle γ in N and a smoothly embedded disc D in N with boundary γ. Performing surgery using a framing $\overline{\nu}\gamma \cong S^1 \times D^3$ compatible with a normal bundle of D, that is, that extends to a stable framing of νD, yields $N\#(S^2 \times S^2)$, while a framing incompatible with νD yields $N\#(S^2 \widetilde{\times} S^2)$. If Q_N is odd, that is, if there exists some $x \in H_2(N)$ with $Q_N(x, x) \equiv 1 \mod 2$, then a change in the choice of disc can make either framing compatible,

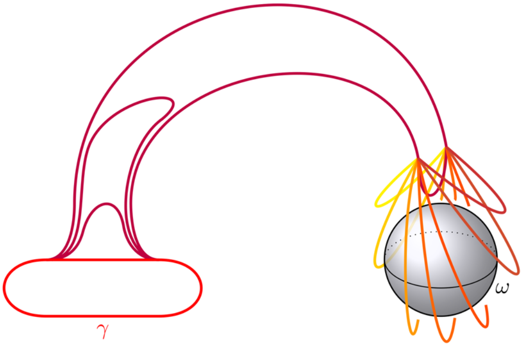

FIGURE 1 An isotopy of γ during which a sub-arc lassoes a sphere ω and then returns to its original position. Picture by Danica Kosanović.

and indeed this is consistent with the existence of a diffeomorphism $N\#(S^2 \times S^2) \cong N\#(S^2 \widetilde{\times} S^2)$. From now on, we will frame $\overline{\nu}\gamma$ in the compatible fashion.

Now, we proceed to describe Wall's construction of diffeomorphisms in [87]. Given a manifold N and a submanifold $P \subseteq N$, let $\mathrm{Diff}_P(N)$ denote the topological group of diffeomorphisms that fix some neighbourhood of P pointwise. Given a manifold L, let $\mathrm{Emb}(L, N)$ denote the space of smooth embeddings of L in N.

Let $c : S^1 \times D^3 \to N$ be an embedding of the thickened circle, with $\mathrm{im}\, c|_{S^1\times\{0\}} = \gamma$, so that $\overline{\nu}\gamma := c(S^1 \times D^3)$ is a tubular neighbourhood of γ. We have a fibration sequence

$$\mathrm{Diff}_{\overline{\nu}\gamma}(N) \to \mathrm{Diff}(N) \xrightarrow{F\mapsto F\circ c} \mathrm{Emb}(S^1 \times D^3, N). \tag{2.1}$$

The fibre of c, as shown, is $\mathrm{Diff}_{\overline{\nu}\gamma}(N)$, the group of diffeomorphisms of N that fix $\overline{\nu}\gamma$ pointwise. Isotopy extension and then restriction give rise to maps

$$\pi_1 \mathrm{Emb}(S^1 \times D^3, N) \to \pi_0 \mathrm{Diff}_{\overline{\nu}\gamma}(N) \to \pi_0 \mathrm{Diff}_\partial(N \backslash c(S^1 \times \mathring{D}^3)).$$

The first map is the connecting homomorphism in the long exact sequence in homotopy groups associated with the fibration (2.1). The idea is that we take the framed circle γ, and isotope it around in N until it returns to its original position. The framing can a priori be different from the original framing. An important example of such an isotopy 'swings' an arc of the circle over an immersed 2-sphere $\overline{\omega}$ in N. One can imagine that the arc 'lassoes' the sphere, as shown in Figure 1. More precisely, one should think of a generator of $\pi_1(\Omega S^2) \cong \pi_2(S^2) \cong \mathbb{Z}$, mapped into N via the immersion corresponding to $\overline{\omega}$. That is, we can decompose a 2-sphere into a union of arcs, all with the same two points as endpoints. This gives rise to a loop of arcs for a sub-arc of γ to be isotoped over. The rest of γ stays fixed. By general position we assume we obtain an embedded circle in N for all time.

Let $\omega := [\overline{\omega}] \in H_2(N)$ denote the homology class of $\overline{\omega}$, and suppose that moreover $Q_N(\omega, \omega)$ is even. Then, the framing of the circle after its journey agrees up to isotopy with the original framing. We can therefore obtain a loop of embeddings of $S^1 \times D^3$ as desired. Isotopy extension gives us a diffeomorphism of N, that fixes $\overline{\nu}\gamma$ and that is isotopic to the identity. However, as a diffeomorphism of $N \backslash c(S^1 \times \mathring{D}^3)$, it need not be isotopic to the identity.

Since we have a diffeomorphism in $\pi_0 \operatorname{Diff}_\partial \left(N \backslash c(S^1 \times \mathring{D}^3)\right)$ that fixes the boundary pointwise, we may extend by the identity over $D^2 \times S^2$, to obtain an element of

$$\pi_0 \operatorname{Diff}^+ \left((N \backslash c(S^1 \times \mathring{D}^3)) \cup D^2 \times S^2\right) \cong \pi_0 \operatorname{Diff}^+(N\#(S^2 \times S^2)).$$

As in the statement of Theorem A, let $M := N\#(S^2 \times S^2)$ and let $f_\omega : M \xrightarrow{\cong} M$ be the diffeomorphism just constructed. Let $x = [S^2 \times \mathrm{pt}]$ and $y = [\mathrm{pt} \times S^2]$. Then, if $Q_N(\omega, \omega) = 2s$, Wall showed that f_ω sends

$$\xi \mapsto \xi - Q_N(\xi, \omega) y; \quad x \mapsto x + \omega - sy; \quad y \mapsto y, \tag{2.2}$$

where $\xi \in H_2(N) \subseteq H_2(N) \oplus H_2(S^2 \times S^2) \cong H_2(M)$. This suffices to determine the effect of the constructed diffeomorphism on homology.

Wall's earlier impressive algebraic results from [86], together with a short additional argument in [87], showed that under the hypotheses of Theorem A on N, the group of isometries of Q_M is generated by the following.

(i) Isometries of the form (2.2). In an sosc 4-manifold N, every homology class in N can be represented by an immersed 2-sphere, so we can perform Wall's construction for every $\omega \in H_2(N)$ with $Q_N(\omega, \omega)$ even, to obtain a diffeomorphism f_ω realising the isometry (2.2).
(ii) Isometries of the hyperbolic summand $Q_{S^2 \times S^2}$, which can all be realised by diffeomorphisms;
(iii) In the case that Q_N is odd, $M = N\#(S^2 \times S^2) \cong N\#\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ and so there is a decomposition $H_2(M) \cong H_2(N) \oplus H_2(\mathbb{CP}^2) \oplus H_2(\overline{\mathbb{CP}}^2)$. Then, we need the isometry $(\mathrm{Id}, -\mathrm{Id}, \mathrm{Id})$ of Q_M, which can be smoothly realised via complex conjugation acting on the $\mathbb{CP}^2$ summand.

Thus, every isometry of Q_M can be smoothly realised.

In fact this does not just hold for closed M, but also whenever $H_1(\partial M) = 0$, because in that case the intersection form is still nonsingular, so the algebraic input from [86] continues to apply. Moreover, the construction of diffeomorphisms above did not use that M was closed. In the case of $\partial M \neq \emptyset$, the diffeomorphisms constructed can be assumed to restrict to the identity on ∂M.

3 | EXTENSIONS AND IMPROVEMENTS OF THEOREM A

Recall the map $\mathcal{I}_M : \pi_0 \operatorname{Diff}^+(M) \to \operatorname{Aut}(H_2(M), Q_M)$. Theorem A states that this map is surjective for many scosc 4-manifolds. Is $\mathcal{I}_M$ surjective in general? Is it injective? What happens in the topological category? There has been tremendous progress on these questions since Wall's work, even though much more remains to be done. I will survey some of this progress here.

First, as mentioned in the introduction, Ruberman–Strle [79] extended Wall's result to prove the following. Letting $M = N\#(S^2 \times S^2)$ as in Theorem A, let

$$s : \operatorname{Aut}(H_2(N), Q_N) \to \operatorname{Aut}(H_2(M), Q_M)$$

be the stabilisation map that extends an isomorphism of $H_2(N)$ to an isomorphism of $H_2(M) \cong H_2(N) \oplus H_2(S^2 \times S^2)$ by the identity on $H_2(S^2 \times S^2)$.

Theorem 3.1 (Ruberman–Strle). *Let N be a scosc 4-manifold and let $M := N\#(S^2 \times S^2)$. Then,* $\operatorname{im} s \subseteq \operatorname{im} \mathcal{I}_M$.

Wall made some preliminary observations about the non-simply-connected case in [87, Section 5]. This was generalised by Cappell–Shaneson in [14], where, inspired by Wall's work they developed stable surgery theory for 4-manifolds (see also [32]). In the stable setting, a great deal of the high-dimensional theory can be reproduced. I will return to this theme later, in Section 6.

In general, $\mathcal{I}_M$ is not surjective. Examples due to Donaldson [27, Section VI] for the K_3 surface, and Friedman–Morgan [31, Theorem 6] for exotic copies of $\mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$, show that there are isometries of Q_M not realisable by any diffeomorphism. Examples are now known to be abundant: Seiberg–Witten theory produces a set of distinguished classes in second homology that must be preserved by any diffeomorphism, and so any isometry of Q_M that does not preserve this set cannot be realised by a diffeomorphism. For instance, Donaldson's argument for the K_3 surface extends to any homotopy K_3 surface using work of Morgan–Szabó [70], as explained in [6, Remark 7.7]. One elegant construction of homotopy K_3 surfaces proceeds via Fintushel and Stern's knot surgery operation [29].

The map $\mathcal{I}_M$ is in general far from being injective. The first examples of diffeomorphisms acting as the identity on second homology but not isotopic to the identity were produced by Ruberman in [77, 78]. I will describe a variation on Ruberman's example next, due to Baraglia–Konno [7]. Let

$$M := K_3\#(S^2 \times S^2).$$

Let $\mathcal{K}$ be a smooth, closed 4-manifold that is homeomorphic to K_3 but not diffeomorphic to it, for example, arising from knot surgery [29]. Suppose also that

$$M' := \mathcal{K}\#(S^2 \times S^2)$$

is diffeomorphic to M. Let $\phi : M \to M'$ be such a diffeomorphism. Such a choice of $\mathcal{K}$ exists, and its action respects the decomposition of H_2 and acts as the identity on $H_2(S^2 \times S^2)$. Let $r : S^2 \times S^2 \to S^2 \times S^2$ be the composition of symplectic Dehn twists [2, 82] in the spheres representing (1,1) and $(1, -1)$ in $H_2(S^2 \times S^2) \cong \mathbb{Z} \oplus \mathbb{Z}$. We consider the diffeomorphism

$$(\mathrm{Id}_{K_3}\#r)\circ\phi^{-1}\circ(\mathrm{Id}_{\mathcal{K}}\#r)\circ\phi : M \to M.$$

This acts trivially on $H_2(M)$. If the Seiberg–Witten invariants of K_3 and $\mathcal{K}$ differ, then it is not isotopic to the identity [7].

In contrast to the wild behaviour of diffeomorphisms revealed by gauge theory, in the topological category the picture is much cleaner, and the topological mapping class group has been entirely reduced to algebra.

Theorem 3.2. *For every topological, closed, oriented, simply-connected 4-manifold M, the map*

$$\mathcal{I}_M^{\mathrm{Top}} : \pi_0\,\mathrm{Homeo}^+(M) \xrightarrow{\cong} \mathrm{Aut}(H_2(M), Q_M),$$

taking the induced isomorphism of $H_2(M)$, is an isomorphism.

Surjectivity is due to Freedman [30]. Injectivity follows from combining work of Kreck [59], Perron [73], and Quinn [74] (the latter with a recent correction from [34]). Thus, the work of Donaldson [27] and Friedman–Morgan [31] implies the existence of self-homeomorphisms of scosc 4-manifolds not isotopic to a diffeomorphism, and hence of smooth structures that are diffeomorphic but not isotopic. Similarly, Ruberman's diffeomorphisms are topologically but not smoothly isotopic to the identity; they are *exotic diffeomorphisms*.

One can extend to simply-connected, compact 4-manifolds with nonempty boundary. Saeki [80] studied a stable version, and combining Saeki's work with that of Perron [73], Quinn [74], Boyer [10, 11], and Orson–Powell [71] completes the computation of the topological mapping class groups of all topological, compact, simply-connected 4-manifolds.

The idea of using fibrations similar to (2.1) to construct diffeomorphisms has been increasingly exploited in recent years. Budney–Gabai [12, 13] used such a construction to define *barbell diffeomorphisms*, obtaining interesting self-diffeomorphisms of $S^1 \times D^3$, and 3-balls in S^4 that are knotted rel. boundary. This has been reformulated by Kosanović and collaborators [28, 56–58], through a families version of Habiro's claspers. It remains an interesting question to determine precisely which of these give rise to nontrivial diffeomorphisms.

4 | CONSTRUCTING *h*-COBORDISMS

Let M_1 and M_2 be scosc 4-manifolds with isometric intersection pairings. To obtain an h-cobordism, one could apply Whitehead's theorem to obtain a homotopy equivalence $f : M_1 \to M_2$, consider this as an element in the structure set $\mathcal{S}(M_2)$, and then apply an argument inspired by surgery theory to obtain an h-cobordism. Even though the surgery sequence is not defined or exact in this dimension, one can still aim to produce a 5-dimensional cobordism and then surger that to an h-cobordism. In 1964, some of the surgery technology, in particular Sullivan's ideas for computing normal invariants, were not yet available. But this approach was certainly known to Wall by 1970 [90, Remark after Theorem 16.5].

Instead, in 1964 Wall instead gave a direct and elementary argument, that became a prototype for theorems classifying 4-manifolds via surgery methods. In particular, results coming from Kreck's modified surgery theory [61] use ideas directly analogous to Wall's, and presumably inspired by them, that bypass the homotopy classification, instead relying on bordism theory and the algebra of Lagrangians of intersection forms to surger a cobordism to an h-cobordism.

Let me give an outline of the key ideas in Wall's proof. We consider

$$N := M_1 \# -M_2,$$

a scosc 4-manifold with zero signature $\sigma(N) = 0$, since $Q_{M_1} \cong Q_{M_2}$ implies $\sigma(M_1) = \sigma(M_2)$ and since signature is additive. Bordism theory implies that N bounds a compact 5-manifold W, that can be assumed spin if N is. Next, by surgery on circles and spheres in W, one can arrange for W to be homotopy equivalent to a wedge of 2-spheres. Since W is 5-dimensional, by general position we can represent generators for $H_2(W)$ by an embedding of a boundary connected sum

$$V := \natural k(S^2 \times D^3) \natural \ell (S^2 \widetilde{\times} D^3),$$

where $S^2 \widetilde{\times} D^3$ is the D^3-bundle over S^2 with boundary $S^2 \widetilde{\times} S^2$. Wall then checked that $W \setminus \mathring{V}$ is an h-cobordism from N to $\partial V \cong \# k(S^2 \times S^2) \# \ell (S^2 \widetilde{\times} S^2)$.

The h-cobordism $W \backslash \mathring{V}$ induces an isomorphism $H_2(\partial V) \cong H_2(N)$. Let $\alpha : H_2(M_1) \xrightarrow{\cong} H_2(M_2)$ be the hypothesised isometry, and let

$$K := \{(x, \alpha(x)) \mid x \in H_2(M_1)\} \subseteq H_2(M_1) \oplus H_2(M_2) \cong H_2(N) \cong H_2(\partial V),$$

the 'diagonal' Lagrangian of $Q_{\partial V}$. Let

$$L := \ker(H_2(\partial V) \to H_2(V)).$$

As mentioned, both K and L are *Lagrangians*, meaning they are direct summands of $H_2(\partial V)$ and that $K = K^{\perp}$ and $L = L^{\perp}$. Here, for example,

$$L^{\perp} := \left\{ y \in H_2(\partial V) \mid Q_{\partial V}(x, y) = 0 \text{ for all } x \in L \right\}.$$

Wall showed that there is an isometry of T of $(H_2(\partial V), Q_{\partial V})$ such that $T(L) = K$. In terms of the later-developed formalism of surgery theory, this is related to the vanishing of the surgery obstruction group $L_5(\mathbb{Z})$.

Next, Wall applied his result on the existence of diffeomorphisms realising automorphisms, Theorem A from [87], to realise T as being induced by a diffeomorphism $\tau : \partial V \xrightarrow{\cong} \partial V$. Cut out the interior of V from W and form the union

$$W \backslash \mathring{V} \cup_{\tau} V.$$

This manifold has boundary

$$N = M_1 \# - M_2 \cong (M_1 \backslash \mathring{D}^4) \cup (S^3 \times I) \cup (M_2 \backslash \mathring{D}^4).$$

Glue $D^4 \times I$ to the $S^3 \times I$ part of the boundary to obtain a cobordism from M_1 to M_2. Wall concluded the proof by showing that the resulting cobordism is an h-cobordism, as required.

5 | *h*-COBORDISM IMPLIES STABLE DIFFEOMORPHISM

Although Wall worked with simply-connected 4-manifolds, the following generalisation of Theorem C is now known [64], with essentially the same proof as Wall's.

Theorem 5.1. *Let M_1 and M_2 be smooth, compact 4-manifolds with a diffeomorphism $\partial M_1 \cong \partial M_2$, and suppose that M_1 and M_2 are h-cobordant rel. boundary. Then, M_1 and M_2 are stably diffeomorphic rel. boundary.*

Here is an outline of the proof, which uses ideas from Smale's proof of the h-cobordism theorem [68, 83]. Let W be an h-cobordism, and consider a handle decomposition of W. The methods of proof of the h-cobordism theorem in high dimensions allow us to trade the handles of index 0, 1, 4, and 5 for 2- and 3-handles. So, we are left with a cobordism consisting only of 2- and 3-handles. The middle level $M_{1/2}$ between the 2- and 3-handles is obtained by adding 2-handles to M_1 and also by adding 2-handles to M_2. Since the maps $M_1 \to W$ and $M_2 \to W$ are in particular injective on fundamental groups, the 2-handles are always attached by circles that are null-homotopic, and hence are isotopically trivial. It follows that the middle level $M_{1/2}$ is obtained from M_1 (respectively M_2) by taking connected sum with copies of $S^2 \times S^2$ and $S^2 \widetilde{\times} S^2$.

If there are copies of $S^2 \widetilde{\times} S^2$ in $M_{1/2}$, we obtain an embedding of a 2-sphere in W with odd normal Euler number. Projecting this to M_1 via the homotopy inverse to the inclusion $M_1 \to W$ and applying general position, we obtain an immersion of a 2-sphere in M_1 with odd normal Euler number. It follows that the universal cover of M_1 is not spin, and in this case $M_1 \#(S^2 \widetilde{\times} S^2) \cong M_1 \cong (S^2 \times S^2)$. So, we may in fact assume that there are no copies of $S^2 \widetilde{\times} S^2$, and so the middle level is the sought for common stabilisation of M_1 and M_2.

As mentioned in the introduction, Theorem C can be rephrased in the following way, using the classification of symmetric, bilinear, unimodular forms.

Theorem 5.2. *Two scosc 4-manifolds M_1 and M_2 are stably diffeomorphic if and only if $\chi(M_1) = \chi(M_2)$, $\sigma(M_1) = \sigma(M_2)$, and they are either both spin or both not spin.*

Here, $\chi(M_i)$ denotes the Euler characteristic, $\sigma(M_i)$ denotes the signature of the intersection form, and recall that M_i is spin if and only if Q_{M_i} is even, i.e. $Q_{M_i}(x,x) \equiv 0 \mod 2$ for all $x \in H_2(M_i)$. Note that M_1 and M_2 do not have to be h-cobordant to deduce stable diffeomorphism. I will discuss generalisations of this result next, where we will see what the minimal assumptions on a bordism are to deduce stable diffeomorphism.

6 | EXTENSIONS AND IMPROVEMENTS OF THEOREMS B AND C

Theorem C was the prototype for a host of results on stable diffeomorphisms of 4-manifolds. I already mentioned that Cappell–Shaneson [14] developed a stable surgery sequence. However, Kreck's work [61] on modified surgery took the idea to another level. Kreck's theory leads to a simplified way to approach the stable classification, reducing it to computations of bordism groups, which can be approached via spectral sequences. Moreover, modified surgery theory allows one to go further and attempt to de-stabilise, in nice cases obtaining homeomorphisms. I will present some of the results obtained via these approaches below.

Beforehand, however, I want to mention the difference in the smooth and topological categories from the stable point of view. Gompf [36] showed that for orientable 4-manifolds, there is no difference.

Theorem 6.1 (Gompf). *Two smooth, compact, oriented 4-manifolds are stably diffeomorphic if and only if they are stably homeomorphic.*

However in the nonorientable case there can be differences. For example, Kreck [60] showed that

$$\mathbb{RP}^4 \# K_3 \text{ and } \mathbb{RP}^4 \# 11(S^2 \times S^2)$$

are (stably) homeomorphic but not stably diffeomorphic. I believe that these were the first published examples of closed, exotic 4-manifolds (Cappell–Shaneson's 4-manifolds homotopy equivalent but not diffeomorphic to $\mathbb{RP}^4$ [15] were observed slightly later [76, p. 221] to be homeomorphic to $\mathbb{RP}^4$). It is also worth remarking that one does not need gauge theory to detect these exotica; Kreck's obstruction is based on Rochlin's theorem, and is similar to Cappell–Shaneson's invariant from [15]. Also, observe that Kreck's exotica do not contradict Theorem 5.1, because these 4-manifolds are not even h-cobordant.

Now, let me return to Kreck's method. I will mostly focus on the case of spin 4-manifolds, to avoid technicalities. Kreck proved that if two closed, smooth, spin 4-manifolds M_1 and M_2 with the same fundamental group π are spin bordant over the classifying space $B\pi$, then they are stably diffeomorphic. Here, $B\pi$ can also be thought of as the Eilenberg–Maclane space $K(\pi, 1)$, and there are 2-connected maps $c_i : M_i \to B\pi$ classifying the universal covers. To obtain a complete stable classification one, has to factor out by the choice of spin structure and the choice of maps c_i.

In the case of universal cover non-spin, the situation is similar: one uses oriented bordism in place of spin bordism. The third case of non-spin 4-manifolds with spin universal covering uses twisted spin bordism groups, details of which I shall omit here; they were first worked out in [84].

Kreck's proof proceeds roughly as follows. Given a (spin) bordism over $B\pi$, we can surger it to an h-cobordism between $M_1\#k(S^2 \times S^2)$ and $M_2\#k(S^2 \times S^2)$ for some k, and then apply the argument in Section 5 to deduce that M_1 and M_2 are stably diffeomorphic.

Then, bordism groups can be computed with Atiyah–Hirzebruch spectral sequences, or a twisted version thereof defined by Teichner [84]. A sequence of recent papers by Kasprowski, Teichner, and their collaborators connect the invariants arising from the spectral sequences with more standard algebraic topological invariants, for example, arising from $\pi_2(M_i)$ or the equivariant intersection form [44–49]. See also [24].

A sample result generalising Theorem C is as follows, proven by Cavicchioli–Hegenbarth and Repovš [16, 17] in the spin case. This, and the non-spin case, follow in a fairly straightforward way from Kreck's bordism approach.

Theorem 6.2. *Two smooth, closed, oriented 4-manifolds M_1 and M_2 with free fundamental group are stably diffeomorphic if and only if $\chi(M_1) = \chi(M_2)$, $\sigma(M_1) = \sigma(M_2)$, and they are either both spin or both not spin.*

Once one has a solid stable classification, one can attempt to de-stabilise, to try to obtain homeomorphism results. This has been carried out successfully in the topological category, notably in the work of Hambleton and Kreck [39–41] on closed 4-manifolds with finite fundamental group, and later by Khan [50] for some infinite fundamental groups. As remarked upon, the proof method is a sophisticated generalisation of Wall's method for surgering a bordism to an h-cobordism that I outlined in Section 4. The approach was also used by Hambleton–Kreck–Teichner for classifying nonorientable 4-manifolds with fundamental group of order two, and by Conway–Powell [21] and Conway–Orson–Powell [22] for studying surfaces embedded in S^4 whose complements have cyclic fundamental groups.

On the other hand, it is possible for one stable equivalence class to contain many manifolds. Kreck–Schafer [63] showed that there can be distinct closed 4-manifolds in the same stable class, even up to homotopy equivalence. Hambleton–Nicholson [42] extended this work to find arbitrarily large families of homotopy equivalence classes of 4-manifolds in the same stable class, while Conway–Crowley–Powell [20] showed that there can even be infinitely many manifolds in the same stable class, if one allows nonempty boundary.

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ORCID

Mark Powell https://orcid.org/0000-0002-4086-8758

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