ON HOMOTOPY TORI AND THE ANNULUS THEOREM

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The results stated below were obtained in 1967, and will be more fully described in a forthcoming paper [8] (at present in preprint form). This announcement is being published now in view of their application by R. C. Kirby [4] (see also below) to the proof of the annulus theorem. I understand that the same results have recently been obtained by W.-C. Hsiang and J. Shaneson. My proof has also been recast in more geometrical form by Andrew Casson.

Write T^n for the product of n copies of the circle S^1. Consider pairs (M,f), M a closed PL-manifold and $f: M \to T^n$ a homotopy equivalence; set

$$(M,f) \sim (M',f')$$

if there is a PL-homeomorphism $h: M \to M'$ with $f' \circ h \simeq f$. Denote the set of equivalence classes by $\mathscr{S}(T^n)$. Similarly, using pairs (M,f) with M a compact PL-manifold and $f:(M,\partial M) \to (T^n \times D^k, T^n \times \partial D^k)$ a homotopy equivalence inducing a PL-homeomorphism on the boundary, we define $\mathscr{S}(T^n \times D^k, T^n \times \partial D^k)$.

THEOREM. *There are bijections*

$$H^3(T^n; \mathbb{Z}_2) \to \mathscr{S}(T^n) \quad (n \geqslant 5),$$
$$H^{3-k}(T^n; \mathbb{Z}_2) \to \mathscr{S}(T^n \times D^k, T^n \times \partial D^k), \quad (n+k \geqslant 5),$$

which are natural for covering maps $T^n \to T^n$.

COROLLARY. *For any* $f: M \to T^n$ *as above, there is a finite covering* $\tilde{T} \to T$ *such that the induced* $\tilde{f}: \tilde{M} \to \tilde{T}$ *is homotopic to a PL-homeomorphism. Similarly for* $f:(M,\partial M) \to (T^n \times D^k, T^n \times \partial D^k)$, *with homotopies rel the boundary, provided* $n+k \geqslant 5$ *and* $k \neq 3$.

The proof uses an exact sequence

$$\ldots L_{n+k+1}(\pi_1(T^n)) \xrightarrow{\partial} \mathscr{S}(T^n \times D^k, T^n \times \partial D^k) \xrightarrow{\eta} [S^k(T^n), G/PL] \xrightarrow{\theta} L_{n+k}(\pi_1(T^n)) \to \ldots,$$

in which $S^k(T^n)$ is the k-fold suspension of T^n, G/PL is the space studied by Sullivan [6], and the functors L_{n+k} are introduced in [8]. In [8; (12, 6)], a formula is obtained permitting the inductive computation of $L_i(\pi)$ for π free abelian; (Shaneson has announced [5] the same computation). Given a short exact sequence

$$0 \to K \to \pi \to \mathbb{Z} \to 0$$

with π free abelian, there is an isomorphism

$$L_i(\pi) \to L_i(K) \oplus L_{i-1}(K)$$

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whose first component is induced by a splitting, and the second by mapping into $K(\mathbb{Z}, 1) = S^1$, making transversal at a point of S^1, and taking the preimage. The proof was inspired by Farrell's Ph.D. thesis (Yale, 1967).

Write $n\mathbb{Z}$ for the direct sum of n copies of $\mathbb{Z}$ $\left(= \pi_1(T^n)\right)$. The above leads to a natural isomorphism

$$L_r(n\mathbb{Z}) = \bigoplus_{0 \leqslant i \leqslant n} \operatorname{Hom}\left(H^i(T^n; \mathbb{Z}), L_{r-i}(1)\right).$$

But $L_j(1)$ is well known (0 for j odd, $\mathbb{Z}_2$ for $j \equiv 2 \pmod 4$ and $\mathbb{Z}$ for $j \equiv 0 \pmod 4$), and Sullivan has given [6] a formula for surgery obstructions in the simply connected case. This leads to a formula here: if N^r is a compact PL manifold, and $f : N \to T^n = K(n\mathbb{Z}, 1)$ induces an isomorphism of fundamental groups, then the surgery obstruction corresponding to $g : (N, \partial N) \to (G/PL, *)$ is given by

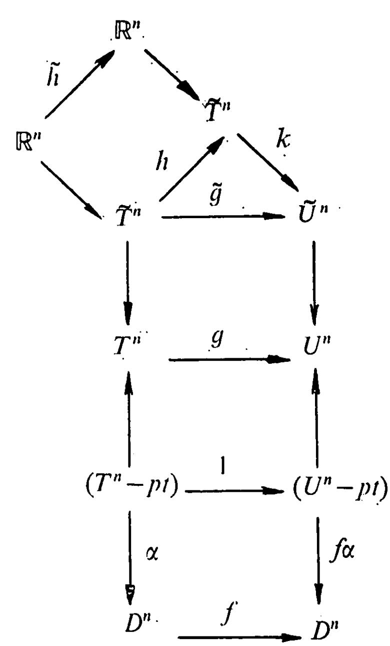
$$\theta(N, f, g)(x) = \begin{cases} 0 & r-i \text{ odd} \\ f^*(x)\, g^*(k_{r-i})\, [N, \partial N] & r-i \equiv 2 \pmod 4 \\ f^*(x)\, g^*(l_{r-i})\, [N, \partial N] & r-i \equiv 0 \pmod 4, \end{cases}$$

where $x \in H^i(T^n; \mathbb{Z})$ and

$$k_j \in H^j(G/PL; \mathbb{Z}_2) \quad \text{for } j \equiv 2 \pmod 4$$
$$l_j \in H^j(G/PL; Q) \quad \text{for } j \equiv 0 \pmod 4$$

are the classes defined by Sullivan.

Using Sullivan's determination [6] of the homotopy type of G/PL, the evaluation of $[S^k(T^n), G/PL]$ and of θ are now simple exercises; θ is injective (hence $\eta = 0$



and ∂ is surjective) and is an isomorphism except on the summand corresponding to l_4. The result follows; naturality for covering maps is obtained using the above formulae.

THEOREM (proved by Kirby **[4]** using the above). *Let $f: D^n \to \operatorname{Int} D^n$ be an orientation-preserving embedding with $f(S^{n-1})$ locally flat. Let $n \neq 4$. Then f extends to a homeomorphism $F: 2D^n \to D^n$, with $F(x) = \frac{1}{2}x$ on the boundary.*

Here, $2D^n$ denotes the disc of radius 2. For $n \leqslant 2$, the result is classical; for $n = 3$ it follows from triangulability of 3-manifolds. For $n \geqslant 5$, the proof is based on the diagram on the previous page.

Outline of Kirby's proof

Here, α is a PL-immersion (which exists since $T^n - pt$ is open and parallelisable), $(U^n - pt)$ is $(T^n - pt)$ with the PL structure induced by the immersion $f\alpha$. Since any two PL structures on $\mathbb{R}^n - pt$ $(= S^{n-1} \times \mathbb{R})$ are equivalent (this is due to Browder **[1]** for $n \geqslant 6$, and to me [7] for $n = 5$), this extends to a PL structure U^n on the torus. By the Corollary above, there is a covering $\tilde{g}$ of g homotopic to a PL-homeomorphism k: write $\tilde{g} = kh$, with $h \simeq 1$. Then the universal cover $\tilde{h}$ of h commutes with translations of $\mathbb{R}^n$ by the integer lattice, so $\|\tilde{h}(x) - x\|$ is uniformly bounded. By a result of Connell [3], $\tilde{h}$ is stable in the sense of Brown and Gluck **[2]**; as the other outside maps in the diagram are all PL, it follows that f is stable, which is equivalent to the statement given.

I am grateful to Larry Siebenmann for drawing my attention to **[4]** and for suggesting the use of finite coverings of tori and the extension to the case $n = 5$.

Added in Proof. The cases $k > 0$ of the theorem have been applied in a similar way: see the announcement **[9]** and the preprint **[10]**.

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