Symmetries of simply-connected four-manifolds, especially algebraic surfaces

by Steven H. Weintraub *

In [W1] we investigated cyclic group actions on closed highly-connected 4k-manifolds, $k > 1$. For $k = 1$, that paper yielded a construction of actions of cyclic groups $\mathbb{Z}/n$ on simply-connected 4-manifolds with homology sphere boundaries, the action being free on the boundary, a not terribly striking result. However, at that time, over a decade ago, technology in dimension 4 was not sufficient to close off the boundary. Since then there has been great progress in dimension 4. Kwasik and Vogel [KV2] (and also Ruberman (unpublished)) proved that, for $n = 2$, any free $\mathbb{Z}/n$-action on a homology 3-sphere extends to an action on a contractible 4 manifold with exactly one fixed point. Combining this result with [W1], as they observed, immediately yields $\mathbb{Z}/2$ actions on closed 4-manifolds. There are in general not locally smoothable [KV1]. This result was generalized by Edmonds [E1] to the case of arbitrary prime n, thereby obtaining actions of these cyclic groups. In fact, he replaced the plumbing method of [W1] by a linking method, which has technical advantages in dimension 4, and so was able to show that every simply-connected 4-manifold M has an action of the cyclic group $\mathbb{Z}/n$, for $n = 2$ or n a prime greater than 3, with isolated fixed points, and furthermore that this action may be constructed to be locally smooth if either n is odd or $n = 2$ and the Kirby-Siebenmann invariant of

* Partially supported by the N.S.F. and the DFG Schwerpunktprogramm "Komplexe Mannigfaltigkeiten".

M is zero. This latter condition had been shown to be necessary by [KV1]. These actions are all trivial on homology, and the structure of such actions (when semi-free) is described by [W1]. Thus in this situation, i.e., actions which are trivial on homology and have isolated fixed points, the situation is reasonably well-understood. (Note that the above discussion all takes place in the topological category).

In section 1 of this paper we use some relatively simple constructions in algebraic geometry to produce examples of algebraic actions of a product of cyclic groups $G = \mathbb{Z}/n_1 \times \mathbb{Z}/n_2$ on simply-connected algebraic surfaces (theorem 1.5). As they are algebraic, they are a fortiori smooth. We further investigate the restrictions of these actions to various cyclic subgroups of G (Corollaries 1.8, 1.10, and 1.11). We see that we obtain a rich collection of actions, all of which are non-trivial on homology and which have a variety of fixed-point sets, sometimes with algebraic curves as components, sometimes with only isolated points. We make no pretense of classification, and indeed the richness of the structure of these examples, and of others that may be similarly constructed (remarks 1.7 and 1.14) show that even a topological classification of actions, without the restriction that they be trivial on homology, will be no mean feat.

The existence of topological actions on M which are non-trivial on $V = H_2(M;\mathbb{Z})$ leads to obvious questions about automorphisms of V. Since V admits a non-singular symmetric bilinear form $\varphi : V \otimes V \to \mathbb{Z}$, the intersection form on M, these also raise questions about automorphisms of φ. We discuss these questions, again presenting some examples, in section 2. (A wealth of information, from a different viewpoint, is obtained in [A], which we commend to the reader).

We close this introduction by noting that section 1 takes place in the algebraic and section 2 in the topological category, so what is referred to as an algebraic curve (resp. algebraic surface) in section 1 is referred to as a Riemann surface (resp. 4-manifold) in section 2.

Section 1. Construction of algebraic actions.

We begin by constructing some algebraic surfaces:

Definition 1.1 For non-negative integers g_1, g_2 let M_{g_1,g_2} be an algebraic surface constructed as follows: Let C_{g_i} be a hyperelliptic algebraic curve of genus g_i with hyperelliptic involution j_i having fixed points $p_{i,1},\ldots,p_{i,2g_i+2}$, for $i = 1,2$. Let $\widetilde{C_{g_1} \times C_{g_2}}$ be the product of these two curves blown up at the $(2g_1+2)$ $(2g_2+2)$ points $\{p_{1,*} \times p_{2,*}\}$. Let $\tilde{j}$ be the lift to $\widetilde{C_{g_1} \times C_{g_2}}$ of the product involution $j = j_1 \times j_2$ on $C_{g_1} \times C_{g_2}$. Then M_{g_1,g_2} is defined to be the quotient

$$M_{g_1,g_2} = \widetilde{C_{g_1} \times C_{g_2}} / \tilde{j}.$$

Theorem 1.2 M_{g_1,g_2} is a non-singular algebraic surface with the following properties:

1) M_{g_1,g_2} is simply-connected
2) The homology of M_{g_1,g_2} is torsion-free
3) The index of M_{g_1,g_2} is $I = -(2g_1+2)(2g_2+2)$
4) The rank of $H_2(M_{g_1,g_2})$ is $|I| + 4g_1g_2+2$
5) The intersection form on M_{g_1,g_2} is an even form

if g_1 and g_2 are both odd; otherwise it is an odd form.

<u>Proof.</u> Except perhaps for part 5), this is well-known (cf. [K, section 6]).

We describe the construction a bit more carefully, and then outline the proof.

Recall what it means to blow up a point: We blow up $\mathbb{C}^2$ at the origin. For any $z \neq 0$ in $\mathbb{C}^2$ let $[z]$ denote the line it generates. Then $z \to [z]$ gives $\mathbb{C}^2-\{0\}$ the structure of a bundle over $\mathbb{P}^1$ with fiber $\mathbb{C}^1-\{0\}$. Fill in the 0-section to get a bundle B over $E = \mathbb{P}^1$. Then B is the blow-up, and the self-intersection number of E (i.e. the Euler class of B) is -1.

Note that any smooth action of any group on $\mathbb{C}^2$ fixing 0 lifts to B, as the differential of the action of any element is a linear map, so takes subspaces to subspaces, and any relation between group elements is also a relation between their differentials.

Now consider our case. Let $E = E_{k_1,k_2}$ be the blow up of $p_{1,k_1} \times p_{2,k_2}$. Then $\tilde{j}$ gives a map on E, which is the trivial map, as the differential of j at this fixed point is multiplication by -1, which leaves every linear subspace invariant. Then the quotient is still smooth at (the image of) E_1 but in the quotient E has self-intersection number -2.

To prove part 1), note that $\pi_1(M_{g_1,g_2}) = \pi_1(Q)$ where $Q = C_{g_1} \times C_{g_2}/j$. Observe that Q is the union of a C_{g_2}-bundle over $(C_{g_1}-\{p_{1,*}\})/j_1 = \mathbb{P}^1-\{2g_1+2 \text{ points}\}$ with a neighborhood of $2g_1+2$ copies of $C_{g_2}/j_2 = \mathbb{P}^1$ and then apply van Kampen's theorem. Then part 2) is immediate. Given this, we may compute homology with rational coefficients. The map $\tilde{j}$ acts trivially on $H_2(\widetilde{C_{g_1} \times C_{g_2}})$, and parts 3) and 4) obviously hold for $\widetilde{C_{g_1} \times C_{g_2}}$, hence for M_{g_1,g_2} as well. Part 5) can be proved by writing down an explicit basis for $H_2(M_{g_1,g_2})$ consisting of immersed 2-spheres and then geometrically calculating intersection numbers (though there should be a better way); as this is quite lengthy we shall omit it.

Now we construct actions on these surfaces:

Notation For a cyclic group $\mathbb{Z}/n$, let T, R, and A denote the following indecomposable $\mathbb{Z}[\mathbb{Z}/n]$-modules:

1) $T = \mathbb{Z}$ with $\mathbb{Z}/n$ acting trivially
2) $R = \mathbb{Z}[\mathbb{Z}/n]$, the group ring
3) $A \subset R$ the augmentation ideal

Over $\mathbb{Z}$ these modules have rank 1, n, and n-1 respectively.
Matrix representatives for the action of a generator of $\mathbb{Z}/n$ are given by the companion matrices of the polynomials $x-1$, x^n-1, and $(x^n-1)/(x-1)$ respectively.

Notation If C is a curve with hyperelliptic involution j, we call the fixed points of j the w-points of C. (If C has genus at least 2, these are the Weierstrass points of C.)

Lemma 1.3 Under the following conditions, there is a hyperelliptic curve C of genus g admitting a semi-free algebraic action of $\mathbb{Z}/n$, commuting with the hyperelliptic involution, with the properties stated:

Case 0) $2g + 2 = kn$, k even. The action has 4 fixed points, none of which is a w-point. The induced action on $H_1(C)$ is $(k-2)R \oplus 2A$.

Case 1) $2g + 2 = kn+1$, k and n necessarily both odd. The action has 3 fixed points, one of which is a w-point. The induced action on $H_1(C)$ is $(k-1)R \oplus A$.

Case 2) $2g + 2 = kn+2$, n odd (and so k even). The action has 2 fixed points, both of which are w-points. The induced action on $H_1(C)$ is kR.

Proof. In each case we exhibit C explicitly. We then verify the properties claimed in case 1); the others are similar and omitted.

Case 0): C is given by $y^2 = \dfrac{(1-x^n)(3-x^n)\cdots((k-1)-x^n)}{(2-x^n)(4-x^n)\cdots(k-x^n)}$ and the action of a generator of $\mathbb{Z}/n$ is $x\to\exp(2\pi i/n)x$, $y\to y$.

Case 1): C is given by $y^2 = (1-x^n)(2-x^n)\cdots(k-x^n)$ and the action of a generator of $\mathbb{Z}/n$ is $x \to \exp(2\pi i/n)x$, $y \to y$.

Case 2): C is given by $y^2 = x(1-x^n)\cdots(k-x^n)$ and the action of a generator of $\mathbb{Z}/n$ is $x \to \exp(2\pi i/n)x$, $y \to \exp(2\pi i(n+1)/2n)y$.

Analysis of case 1): The fixed points are $x = \infty$, $y = \infty$, a w-point, and $x = 0$, $y = \pm \sqrt{k!}$, two non-w-points. Observe that our claim about the action on homology is consistent with the Lefschetz fixed point formula, as the trace of the action of a non-trivial element on H_1 is -1, and there are 1-(-1)+1=3 fixed points. Indeed, if n is a prime and k = 1, by theorem 2.1 below this already gives the claim. In general we prove the claim by writing down a $\mathbb{Z}/n$ invariant set of branch cuts and cycles for the given curve.
We illustrate in the case $n = 3$, $k = 3$, which is entirely typical. See figure I. Note that a_i and b_i are closed curves which lift to closed curves in C, and the arcs b_i' are branch cuts so also lift to closed curves in C. We denote the lifts by the same letters. Clearly rotation by $2\pi/3$ sends a_i to a_{i+1} mod 3. But $a_1+a_2+a_3 = 0$, as it is homologous to a curve bounding a disc around the lift of each of 3 branch points in C, so the action here is exactly A. The action on $\{b_i\}$ and $\{b_i'\}$ gives 2R. To see that all these closed curves are non-zero, and indeed form a basis of $H_1(C;\mathbb{Z})$, one can explicitly write down the intersection matrix of these curves, using figure I, and verify that it is unimodular.

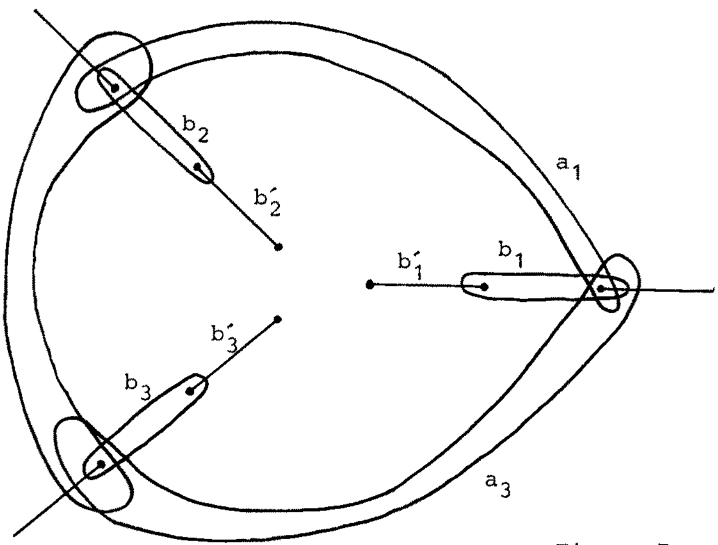

Figure I

(It is no accident that the trivial representation T never appears in these actions [A, proposition 3.3]).

We record a lemma for future use:

Lemma 1.4 Let $g = \mathrm{Diag}(\zeta_1,\zeta_2) \in GL_2(\mathbb{C})$.

Blow up $\mathbb{C}^2$ at 0 to B, and let $\tilde{g}$ be the lift of g to an action on B.

Let E be the inverse image of 0.

If $1 \neq \zeta_1 \neq \zeta_2 \neq 1$, $\tilde{g}$ has two fixed points on B, contained in E.

If $1 = \zeta_1 \neq \zeta_2$, $\tilde{g}$ fixes $\mathbb{C}$, the fiber over a point in E, and an additional isolated point in E.

If $1 \neq \zeta_1 = \zeta_2$, $\tilde{g}$ fixes E.

Proof. The action of $\tilde{g}$ on B-E is the same as that of g on $\mathbb{C}^2$-{0}, and the action of $\tilde{g}$ on E is induced by the action of g on lines in $\mathbb{C}^2$. The lemma follows immediately.

Notation. We let W denote the subgroup of $H_2(M_{g_1,g_2}:\mathbb{Z})$ of rank $(2g_1+2)(2g_2+2)$ generated by the clases $\{E_{*,*}\}$, each of which is a $\mathbb{P}^1$ arising as the image in M_{g_1,g_2} of a blowing of a point $p_{1,*} \times p_{2,*}$.

In the statement below we denote by $(m_o,\dots,m_3)$ the representation $m_oT \oplus m_1R_1 \oplus m_2R_2 \oplus m_3R$ of $\mathbb{Z}/n_1 \times \mathbb{Z}/n_2$, where R_i is the regular representation of the i-th factor, the other factor acting trivially ($i = 1,2$), T the one-dimensional trivial representation, and R the regular representation of this group.

Theorem 1.5 Suppose that, for $i = 1$ and 2, the equation $2g_i + 2 = k_i n_i + e_i$, $e_i = 0,1$, or 2, holds, with k_i and n_i satisfying the parity conditions of lemma 1.3.
Then an algebraic surface M_{g_1,g_2} as in definition 1.1 admits an algebraic action of $\mathbb{Z}/n_1 \times \mathbb{Z}/n_2$ with the following properties:

Fixed point set

	$e_1 = 0$	$e_1 = 1$	$e_1 = 2$
$e_2 = 0$	8 isolated points	6 isolated points	4 isolated points
$e_2 = 1$		6 isolated points	6 isolated points
$e_2 = 2$			8 isolated points

Induced action on W

	$e_1 = 0$	$e_1 = 1$	$e_1 = 2$
$e_2 = 0$	$(0,0,0,k_1k_2)$	$(0,0,k_2,k_1k_2)$	$(0,0,k_2,k_1k_2)$
$e_2 = 1$		$(1,k_1,k_2,k_1k_2)$	$(2,k_1,2k_2,k_1k_2)$
$e_2 = 0$			$(4,2k_1,2k_2,k_1k_2)$

Remark 1.6 Note that the information given in the theorem determines $H_2(M_{g_1,g_2}; \mathbb{Z}[\frac{1}{2}])$ as a $\mathbb{Z}/n_1 \times \mathbb{Z}/n_2$ representation space, and hence, if $n_1 n_2$ is odd, $H_2(M_{g_1,g_2};\mathbb{Z})$. This is because $Q_{g_1,g_2} = C_{g_1} \times C_{g_2}/j$ is a $\mathbb{Z}/2$-homology manifold, and so $H_2(M_{g_1,g_2};\mathbb{Z}[\frac{1}{2}])$ decomposes into $H_2(Q_{g_1,g_2};\mathbb{Z}[\frac{1}{2}]) \oplus W \otimes \mathbb{Z}[\frac{1}{2}]$. Then, by [CR, 23.12, 23.13, and 25.12] the functor "$\otimes\ \mathbb{Z}[\frac{1}{2}]$" gives an isomorphism between lattices of $\mathbb{Z}/n_1 \times \mathbb{Z}/n_2$ over $\mathbb{Z}$ and over $\mathbb{Z}[\frac{1}{2}]$. (Of course, the action on the

first summand is immediate from 1.3 and the Kunneth formula.)

Proof. Given $e_i = 0,1$, or 2, we have an action Φ_i of $\mathbb{Z}/n_i$ on C_{g_i} given by case e_i of lemma 1.3.

This gives a product action $\Phi = \Phi_1 \times \Phi_2$ on $C_{g_1} \times C_{g_2}$ commuting with $j_1 \times j_2$. We must show that Φ lifts to an action $\tilde{\Phi}$ on $\widetilde{C_{g_1} \times C_{g_2}}$. The only possible difficulty than can arise is when a point p, blown up to a $\mathbb{P}^1$ E, is the fixed set of a non-trivial subgroup of $\mathbb{Z}/n_1 \times \mathbb{Z}/n_2$. But here, by the proof of theorem 1.2, the action lifts as well. Finally, as $\tilde{J}$ acts trivially on E, $\tilde{\Phi}$ commutes with $\tilde{J}$, thus giving an action Ψ on M_{g_1,g_2}, as claimed.

Now we investigate the action. There are 6 cases; we do one and leave the others for the reader. We consider the case $e_1 = e_2 = 1$. Denote the w-points of C_{g_1} by $p_o,\ldots,p_{k_1n_1}$ and the fixed points of $\mathbb{Z}/n_1$ by $r_o = p_o, r_1, r_2$, and similarly for C_{g_2}, with p replaced by q and r by s.

The action of Ψ on W is the same as that of Φ on $\{p_a \times q_b\}$. The point $\{p_o \times q_o\}$ is fixed, giving T; the points $\{p_o \times q_b,\ b \neq 0\}$ give k_2R_2; the points $\{p_a \times q_o,\ a \neq 0\}$ give k_1R_1; the points $\{p_a \times q_b,\ a,b \neq 0\}$ give k_1k_2R.

A fixed point for Ψ not on a $\mathbb{P}^1$ arising via a blow-up is the image of a fixed point of $\tilde{\Phi}$ not on such a $\mathbb{P}^1$. There are 8 of these, $\{r_a \times s_b, (a,b) \neq (0,0)\}$ and they are identified in pairs by $\tilde{J}$, giving 4 fixed points on M_{g_1,g_2}.

A fixed point for Ψ lying on such a $\mathbb{P}^1$ must lie on an invariant $\mathbb{P}^1$. As we have just seen, there is exactly one of these, arising from the blow-up of $r_o \times s_o$, and in the obvious coordinate system there the elements of $\mathbb{Z}/n_1 \times \mathbb{Z}/n_2$ act as $\mathrm{Diag}(\zeta_1,\zeta_2)$, where ζ_1 (resp. ζ_2) runs over all n_1-st (resp. n_2-nd) roots of unity. By lemma 1.4, this gives 2 fixed points in $\widetilde{C_{g_1} \times C_{g_2}}$ and hence 2 in M_{g_1,g_2}, for a total of 6.

Remark 1.7 By beginning with even more special curves than in lemma 1.3, we may find actions of the dihedral group D_{2n}. For example, in case 0) the curve $y^2 = (2-x^n)/(1-2x^n)$ also has the automorphism $(x,y) \to (1/x,1/y)$, and in case 2) the curve $y^2 = x(2-x^n)(1-2x^n)$ also has the automorphism $(x,y) \to (1/x,y/x^{n+1})$. Then the construction in theorem 1.5 goes through unchanged, producing examples of M_{g_1,g_2} with actions of $D_{2n_1} \times \mathbb{Z}/n_2$ (or vice-versa) or $D_{2n_1} \times D_{2n_2}$.

The coarse analysis in the proof of his theorem does not reveal the true nature of these actions. To see it more closely, we investigate the action of cyclic subgroups $G = \mathbb{Z}/n$ contained in $\mathbb{Z}/n_1 \times \mathbb{Z}/n_2$. We restrict ourselves to n odd so that, by remark 1.6, we may have both a convenient and a definitive statement.

Corollary 1.8 Let $n = n_1$ and let G be included in $\mathbb{Z}/n_1 \times \mathbb{Z}/n_2$ as the first factor. Then G acts semi-freely on M_{g_1,g_2} with the following properties:

$e_1 = 0$: The fixed point set is 2 copies of C_{g_2}.

The induced action on H_2 is

$$2T \oplus 4g_2A \oplus (4k_1g_2 + 2k_1-4g_2)R$$

$e_1 = 1$: The fixed point set is $C_{g_2} \cup \mathbb{P}^1 \cup 2g_2+2$ isolated points.

The induced action on H_2 is

$$(2g_2+4)\ T \oplus 2g_2A \oplus (4k_1g_2 + 2k_1-2g_2)R$$

$e_1 = 2$: The fixed point set is (2 copies of $\mathbb{P}^1$) $\cup\ 4g_2+4$ isolated points

The induced action on H_2 is

$$(4g_2+6)\ T \oplus (4k_1g_2+ 2k_1)R$$

Remark 1.9 We obtain such an action for any value of g_2.

Proof. Again, we consider a single case, $e_1 = 1$.

We take homology with coefficients in $\mathbb{Z}\ [\frac{1}{2}]$. Then, by remark 1.6, $H_2(M_{g_1,g_2}) = H_2(Q) \oplus W$, where $Q = C_{g_1} \times C_{g_2}/j$. By the Kunneth formula and the consequent fact that j acts trivially on $H_2(Q)$, we obtain, by lemma 1.3,

$$H_2(Q) = 2T \oplus ((k_1-1)R \oplus A)(2g_2).$$

As for W, following the notation in the proof of 1.5, the $(2g_2+2)$ copies of $\mathbb{P}^1$ arising from blowing up the fiber over p_o are left fixed, while the points $p_1,\dots,p_{k_1n}$ are moved freely, hence so are the $k_1n(2g_2+2)$ copies of $\mathbb{P}^1$ in the fibers over them, giving $W = (2g_2+2)T \oplus k_1(2g_2+2)R$. Again, by remark 1.6, this gives the representation on integral homology as well.

Now for the fixed points. The fixed-point set of Ψ is again the image of the fixed point set of Φ. Since the action on the second coordinate is trivial, this latter comes from "fibers" over the 3 fixed points r_o,r_1,r_2. To be more precise, the fibers over r_1 and r_2 are each a copy of C_{g_2}, and $j(r_1) = r_2$, so these two fibers have image one copy of C_{g_2} in the quotient, which is fixed. This gives one component of the fixed point set in M_{g_1,g_2}. On the other hand, r_o is left fixed, and the quotient of the fiber over it in $C_{g_1} \times C_{g_2}$ by j is $C_{g_2}/j = \mathbb{P}^1$ in M_{g_1,g_2}.However, this fiber contains $(2g_2+ 2)$ w-points. Each is blown up to $\mathbb{P}^1$. At each point a generator of $\mathbb{Z}/n$ acts locally like $\mathrm{Diag}(1,\exp(-2\pi i/n))$, so by lemma 1.4, we obtain an extra isolated fixed point on the $\mathbb{P}^1$ arising from the blow-up of this point in $\widetilde{C_{g_1} \times C_{g_2}}$, and these all have distinct images in M_{g_1,g_2} so we obtain also $\mathbb{P}^1 \cup 2g_2+ 2$ isolated fixed points in M_{g_1,g_2}, as claimed.

Corollary 1.10 Let $n = n_1 = n_2$ and let G be included in $\mathbb{Z}/n_1 \times \mathbb{Z}/n_2$ as the diagonal. Then G acts semi-freely on M_{g_1,g_2} with the following properties:

$e_1 = 0$: The fixed point set is 8 isolated points.

The induced action on H_2 is

$$6\,T \oplus (2k_1k_2n - (2k_1+2k_2))R$$

$e_1 = 1$: The fixed point set is $\mathbb{P}^1 \cup 4$ isolated points.

The induced action on H_2 is

$$4\,T \oplus 2k_1k_2nR$$

$e_1 = 2$: The fixed point set is (2 copies of $\mathbb{P}^1$) $\cup$ 4 isolated points.

The induced action on H_2 is

$$6\,T \oplus (2k_1k_2n + (2k_1+ 2k_2))R$$

Proof. Again we consider the case e_1 $(= e_2) = 1$.
We again take homology with coefficients in $\mathbb{Z}[\frac{1}{2}]$ and note that the representation we obtain is also the representation over $\mathbb{Z}$.
We have $H_2(M_{g_1,g_2}) = H_2(Q) \oplus W$. from theorem 1.5 it is immediate that $W = T \oplus (k_1 + k_2+ k_1k_2n)R$.

As in the last corollary, using lemma 1.3,

$$H_2(Q) = 2\,T \oplus (k_1-1)R \oplus A) \otimes ((k_2-1)R \oplus A).$$

Then, using the facts that $A \otimes R = (n-1)R$ and $A \otimes A = T \oplus (n-2)R$ the claim about the homology follows.

Now for the fixed-point set. As in the proof of the theorem, the only candidates for fixed points are $\{r_a \times s_b,\ (a,b) \neq (0,0)\}$ and the blow-up of $r_o \times s_o$. The first of these is a set of 8 points in $\widetilde{C_{g_1} \times C_{g_2}}$, identified in pairs by $\mathfrak{J}$, giving 4 isolated points. As for $r_o \times s_o$, in the obvious local coordinate system there a generator of $\mathbb{Z}/n$ acts by Diag(exp($-2\pi i/n$),exp($-2\pi i/n$)); since these two diagonal entries are the same, by lemma 1.4 the entire blown-up $\mathbb{P}^1$ in $\widetilde{C_{g_1} \times C_{g_2}}$ is fixed as is its image in M_{g_1,g_2}, giving the fixed-point set claimed.

Corollary 1.11 Let $n = n_1 = n_2$ and let G be included in $\mathbb{Z}/n_1 \times \mathbb{Z}/n_2$ as the subgroup generated by the element $(1,m)$, where m and n are relatively prime. In any case the representation of G on $H_2(M_{g_1,g_2})$ is the same as in corollary 1.10.

If $e_1 = 0$, then for any m the fixed-point set is 8 points.

If $e_1 = 1$, then for $m \not\equiv 1(n)$ the fixed-point set is 6 points.

If $e_1 = 2$, then for $m \not\equiv \pm 1(n)$ the fixed-point set is 8 points.

Proof. Again take $e_1 = 1$. The proof is exactly the same as that of the preceding corollary, except at the end we find a generator of $\mathbb{Z}/n$ acting by Diag(exp ($-2\pi i/n$),exp($-2\pi im/n$)); since these two diagonal entries are distinct, by lemma 1.4 we obtain two isolated fixed points instead of a fixed $\mathbb{P}^1$.

Remark. 1.12 In the classification of algebraic surfaces [BPV, VI.1] our examples fit in as follows: $M_{o,g}$ is $\mathbb{P}^2$ blown up at $4g + 5$ points, so is rational. $M_{1,1}$ is by definition a Kummer surface and is hence K3. $M_{1,g}$ for $g > 1$ is properly elliptic, and M_{g_1,g_2} for $g_1,g_2 \geq 2$ is of general type.

Remark. 1.13 Let Aut(M) denote the group of algebraic (biregular) automorphisms of the algebra surface M. For a general K3 surface M, Aut(M) may be infinite, but for a Kummer surface Aut(M) is always finite [PS, section 7]. It is a theorem of Painlevé that for a surface M of general type, Aut(M) is always finite. In our case, from Hirzebruch's index theorem [BPV, I.3.1] we may calculate that M_{g_1,g_2} has chern number $c_1^2 = 4(g_1-1)(g_2-1)$. Miyaoka's inequality [BPV, VII.4] states that for a surface of general type, $3c_2 \geq c_1^2$, so that c_1^2 is a good parameter to choose. Our examples of groups acting on M_{g_1,g_2} were easy to construct, and their orders grow (asymptotically) linearly with c_1^2. There are not known group actions on families of surfaces of general type where the order of the group grows like $(c_1^2)^{1+\delta}$ for any $\delta > 0$. (Of course, if we were just interested in this latter question we could take product actions on $C_{g_1} \times C_{g_2}$, giving non-simply connected examples with the same rate of growth.)

Remark. 1.14 The reader will undoubtedly have noticed that we never used any special property of hyperelliptic curves, and that we could produce entirely analogous examples beginning with m-gonal curves, i.e. curves of the form $y^m = f(x)$. We leave this as a final exercise for the reader.

Section 2. Automorphisms of lattices and of forms.

In this section p will always denote a prime.

By [CR,34.31] the following is a complete list of indecomposable integral representations of $\mathbb{Z}/p$:

1) T, of rank 1
2) R, of rank p
3) A, of rank p-1 (Note that A is isomorphic to the ring of algebraic integers in $\mathbb{Q}(\exp(2\pi i/p))$.

For each non-principal ideal class in $\mathbb{Q}(\exp(2\pi i/p))$ (assuming there are any):

4) I, a representative of that class, of rank $p-1$

5) $I \otimes_A R$, of rank p

However, the representations that can arise from group actions are much more specialized. Taking the special cases of the union of [W2, theorem 1.6 and following remark] and [HR, proposition 10] which are relevant to us here, we have

Theorem 2.1 Let $\mathbb{Z}/p$ act smoothly or simplicially in an orientation -preserving way on X, where X is either a Riemann surface ($d = 1$) or a simply-connected 4-manifold ($d = 2$). Then as a representation space of $\mathbb{Z}/p$, $H_d(X:\mathbb{Z})$ is isomorphic to $tT \oplus aA \oplus rR$ for some non-negative integers t,a,r. Furthermore, if $d = 1$ then t and r are even, and if $p = 2$ then a is also even. If $d = 2$ and p is odd then a is even.

Kwasik and Schultz show in [KS] that this theorem continues to hold for locally linear actions, but not for topological ones. (They remark there that this latter fact had earlier been shown by Ruberman and Weinberger.)

Let V be a free $\mathbb{Z}$-module as above equipped with a non-singular symmetric bilinear form φ invariant under the action of $\mathbb{Z}/p$. (We are thinking of V as $H_2(X;\mathbb{Z})$, X a simply-connected 4-manifold with a $\mathbb{Z}/p$-action, and φ the intersection form on V.) There is no reason to expect that the above splitting on V need be an orthogonal splitting with respect to φ, and indeed it need not be. Here are two examples (see also [HR, example 8]):

Example 2.2 ([W2, proposition 1.7]) Let $V = (\mathbb{Z})^{2p-2}$, p odd, and let φ be the bilinear form whose matrix $H = (h_{ij})$ is given by:

a) $h_{i,j} = 1$ if $i \neq j$, $1 \leq i,j \leq p + 1$

b) $h_{i,j} = 1$ if $(i-j) = \pm 1$

c) $h_{i,i} = 2$ for $i = 1,\ldots,p$

d) $h_{i,i} = 2$ for $i = p+1,\ldots,2p-2$

e) $h_{i,j} = 0$ otherwise.

Let $\mathbb{Z}/p$ act on V by permuting the first p coordinates cyclically, so V splits as $R \oplus (p-2)T$. Then φ is a non-singular form on V, invariant under the action of $\mathbb{Z}/p$, but V has no φ-orthogonal splitting. (φ is an even form of index 0).

Proof. The form φ is non-singular as $\det(H) = 1$ [W2, 1.7] and is clearly invariant. However, V cannot admit any φ-orthogonal splitting as both R and $(p-2)T$ are odd-dimensional and hence cannot admit a non-singular even form. To compute the index of φ, note that -1 is an eigenvalue of H of multiplicity $p-1$. On the other hand, by making a change-of-basis (over $\mathbb{Q}$) involving only the first p rows, we obtain a matrix H' similar to H whose lower right hand corner (of size $(p-2) \times (p-2)$) is identical to that of H, is orthogonal to the upper left-hand corner, and is positive definite, so H has at least $p-2$, and hence $p-1$ positive eigenvalues.

Here is a smaller (indeed, minimal) example with a more interesting form:

Example 2.3 Let $V = (\mathbb{Z})^{p+1}$, p congruent to 1 modulo 4, and let φ be the bilinear form whose matrix $H = (h_{ij})$ is given by:

a) $h_{i,i} = 1$ if $i \neq j$, $1 \leq i,j \leq p$

b) $h_{i,j} = 0$ for $1 \leq i \leq p$

c) $h_{i,p+1} = h_{p+1,i} = (p+1)/2$ for $1 \leq i \leq p$

d) $h_{p+1,p+1} = (p^2+3p+4)/4$

Let $\mathbb{Z}/p$ act on V by permuting the first p coordinates cyclically, so V splits as $R \oplus T$. Then φ is a non-singular form on V, invariant under the action of $\mathbb{Z}/p$, but V has no φ-orthogonal splitting. (φ is a form of index $1-p$).

Proof. It is easy to check that $\det(H) = -1$, so φ is non-singular and obviously invariant. If $p \equiv 1$ (8), then φ is an even form, so V cannot split (as above). If $p \equiv 5$ (8), one can check that φ cannot represent ± 1 on the space of vectors invariant under $\mathbb{Z}/p$ (vectors of the form $(x,\ldots x,y)$) so V cannot split here either. To compute the index of φ, note that -1 is again an eigenvalue of H of multiplicity p-1, and then the remaining 2 eigenvalues must have opposite signs.

We also have the following illuminating theorem, proved (independently) in [E2, proposition 2.4] and [A, proposition 4.3 and theorem 4.14]:

Theorem 2.4 Let $\mathbb{Z}/p$ act in an orientation-preserving way on a simply-connected 4-manifold X, with fixed-point set F and induced action on $H_2(X)$ given by $tT \oplus aA \oplus rR$. If the action is free, then $t = 0$ and $a = 2$. Otherwise, $a = \mathrm{rank}_{\mathbb{Z}}\, H_1(F)$.

This theorem points out that we made a mistake in [W2, theorem 2.4] where we claimed to construct actions with isolated fixed points where the induced action on homology had $a \neq 0$. (The error is on page 270 where we claim, in effect, to construct copies of $A \oplus T$. In fact the extension is non-trivial, so we have constructed copies of R.) However, the proof of this theorem does show that if φ is an even, unimodular form on $V = tT \oplus rR$ invariant under $\mathbb{Z}/p$, it may be realized by an action on a manifold-with-boundary, and hence, by [E1, 3.1], by an (definitely not smooth, in general) action on a closed 4-manifold.

Since we are interested in realizing actions which are non-trivial on homology, it is in any case natural to ask what their effect on homology can possibly be.

This is clearly equivalent to the following question: Let $\varphi : V \otimes V \to \mathbb{Z}$ be a non-singular symmetric bilinear form on a free $\mathbb{Z}$-lattice V. What is the automorphism group $Aut(\varphi)$ and in particular what are its finite subgroups?

First, a construction:

Lemma 2.5 Let $a-b \equiv 0\ (4)$, $a,b \geq 0$. Let $\mathbb{R}^{a+b}$ have the bilinear form given by the matrix $Diag(1,\ldots,1,-1,\ldots,-1)$ with a 1's and b -1's, with respect to the standard basis $\{e_i\}$. Let $E_{a,b}$ be the lattice spanned by $\{e_i + e_j\}$ and by $(e_1+\ldots+e_{a+b})/2$. Then the restriction φ of the above form to $E_{a,b}$ is a non-singular symmetric bilinear form with index a-b. This form is odd (even) if $(a-b)/4$ is odd (even).

Proof. Exactly the same as in [S,V.1.4] for the case $E_a = E_{a,o}$.

We denote the form φ by $E_{a,b}$. Note that $\Sigma_a \times \Sigma_b$ (Σ the symmetric group) operates on $E_{a,b}$ by permuting coordinates, leaving φ invariant, and any cyclic subgroup theorem operates as the representation $tT \oplus rR$ for some t and r, so can be realized by an action on a simply-connected 4-manifold. For example, $E_{3,19}$ is the intersection form of the Kummer surface, so this has an action of $\mathbb{Z}/19$.

Definition 2.6 Let $Aut_o(E_{a,b}) = (\Sigma_a \times \Sigma_b) \times (\mathbb{Z}/2)^{(a+b-1)} \subset Aut(E_{a,b})$ consist of the automorphisms generated by the above permutations and by changing the signs of any even number of basis vectors e_i.

In general this subgroup is far from being the full automorphism group. The following automorphism groups are known ([S, V. 2.3]): $Aut_o(E_{16}) = Aut(E_{16})$. Also, $Aut(E_8 \oplus E_8) = (Aut(E_8) \oplus Aut(E_8)) \times \mathbb{Z}/2$, with $\mathbb{Z}/2$ acting by switching the copies. However, $Aut_o(E_8) \neq Aut(E_8)$, and indeed, $[Aut(E_8) : Aut_o(E_8)] = 3^3 \cdot 5$.

We conclude with an example to show that the "extra" automorphism of order 5 of E_8 gives the representation 2A. (For order 3 we get 4A). This example was shown to us by M. Kneser.

Example 2.7 Let $\mathbb{R}^5$ have the usual inner product, given by the identity matrix in the standard basis $\{e_i\}$. Let $\mathbb{Z}/5$ operate on $\mathbb{R}^5$ by cyclically permuting coordinates.

Let $A_4 = \{v = \Sigma v_i e_i \in \mathbb{Z}^5 \mid v_1 + \ldots + v_5 = 0\}$.

Note that A_4 is a lattice in $\mathbb{R}^5$ and the action of $\mathbb{Z}/5$ on A_4 is exactly the representation A. Let A be the dual lattice of A,

$A^{\#} = \{w = \Sigma w_i e_i \in \mathbb{R}^5 \mid \Sigma w_i v_i \in \mathbb{Z}$ for all $v = \Sigma v_i e_i \in A\}$.

Note that $[A^{\#} : A] = 5$ and that $A^{\#}/A$ is generated either by $w_o = (1/5, 1/5, 1/5, 1/5, -4/5)$

or by $w_{oo} = (2/5, 2/5, 2/5, -3/5, -3/5)$

Let $w = (w_o, w_{oo})$. If $\oplus$ denotes here orthogonal direct sum, let E be the lattice in $\mathbb{R}^5 \oplus \mathbb{R}^5$ generated by $A_4 \oplus A_4$ and w.

Let φ be the obvious inner product on E.

Then $A_4 \oplus A_4 \subset E \subset A_4^{\#} \oplus A_4^{\#}$, and $[A_4^{\#} \oplus A_4^{\#} : A_4 \oplus A_4] = 25$.

Further, $[E : A_4 \oplus A_4] = [A_4^{\#} \oplus A_4^{\#} : E] = 5$.

Let $E^{\#}$ be the dual lattice of E.

Note that $\varphi(w,w) = 2 \in \mathbb{Z}$, so $E^{\#} \supset E$ and hence $E^{\#} = E$.

Clearly φ is positive definite, and indeed even, so E must be isomorphic to E_8. Also, as a $\mathbb{Z}/5$ representation space E is isomorphic to 2A (as $E \otimes \mathbb{Q}$ is clearly isomorphic to $2A \otimes \mathbb{Q}$).

This example raises the possibility that the Kummer surface, say, has an action of $\mathbb{Z}/5$ realizing $14T \oplus 2A$, or $6T \oplus 4A$, as the induced action on H_2. We do not know whether such an action exists.

References

[A] Assadi, A.: Integral representations of finite transformation groups III: Simply-connected four-manifolds, to appear.

[BPV] Barth, W., Peters, C., and van de Ven, A.: Compact Complex Surfaces. Springer, 1984.

[CR] Curtis, C.W. and Reiner, I.: Methods of Representation Theory. Wiley, 1981.

[E1] Edmonds, A.: Construction of group actions on four-manifolds. Trans. Amer. Math. Soc. 299 (1987), 155-170.

[E2] ———: Aspects of group actions on four-manifolds, to appear in Topology Appl.

[HR] Hambleton, I. and Riehm,C.: Splitting of hermitian forms over group rings. Invent. Math. 49 (1987), 19-33.

[K] Kas, A.: Ordinary double points and obstructed surfaces. Topology 16 (1977), 51-64.

[KS] Kwasik, S. and Schultz, R.: Homological properties of periodic homeomorphisms of 4-manifolds, to appear in Duke J. Math.

[KV1] Kwasik, S. and Vogel, P.: Asymmetric 4-dimensional manifolds. Duke J. Math. 53 (1986), 759-764.

[KV2] ——————: Non-locally smoothable topological symmetries of 4-manifolds. Duke J. Math. 53 (1986), 765-770.

[PS] Piatetski-Shapiro, I.I. and Shafarevich, I.R.: A Torelli theorem for algebraic surfaces of type K3. Math. USSR Izvestija 5 (1971), 547-588.

[S] Serre, J.-P.: Cours d'arithmétique. Presses universitaires de France, 1970

[W1] Weintraub, S. H.: Semi-free $\mathbb{Z}_p$-actions on highly-connected manifolds. Math. Z. 145 (1975), 163-185.

[W2] ——————: Topological realization of equivariant intersection forms. Pacific J. Math. 73 (1977),257-280.

Mathematisches Institut der Universität Bayreuth
Postfach 10 12 51
D-8580 Bayreuth
Fed. Rep. Germany