SATELLITES AND CONCORDANCE OF KNOTS IN 3–MANIFOLDS

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ABSTRACT. Given a 3–manifold Y and a free homotopy class in $[S^1, Y]$, we investigate the set of topological concordance classes of knots in $Y \times [0, 1]$ representing the given homotopy class. The concordance group of knots in the 3–sphere acts on this set. We show in many cases that the action is not transitive, using two techniques. Our first technique uses Reidemeister torsion invariants, and the second uses linking numbers in covering spaces. In particular, we show using covering links that for the trivial homotopy class, and for any 3–manifold that is not the 3–sphere, the set of orbits is infinite. On the other hand, for the case that $Y = S^1 \times S^2$, we apply topological surgery theory to show that all knots with winding number one are concordant.

1. INTRODUCTION

In this paper we study the problem of concordance of knots in general 3–manifolds. Throughout the paper, embeddings are assumed to be locally flat unless specified otherwise. Let Y^3 be a closed, oriented 3–manifold and fix an orientation for S^1. An embedding $L \colon \bigsqcup_m S^1 \hookrightarrow Y^3$, considered up to ambient isotopy, is called an *m–component link* and a 1–component link is called a *knot*. We will sometimes write $L \subset Y$ as a shorthand for a link. Links L_0 and L_1 are *concordant* if there exists a proper embedding $A \colon \bigsqcup_m S^1 \times [0, 1] \hookrightarrow Y \times [0, 1]$ such that $L_0 = A|_{\bigsqcup_m S^1 \times \{0\}} \subset Y \times \{0\}$ and $L_1 = A|_{\bigsqcup_m S^1 \times \{1\}} \subset Y \times \{1\}$, in which case we say that A is a *concordance* between the links.

Denote the equivalence relation of concordance by $L_0 \sim L_1$ and the set of concordance equivalence classes of knots in Y by $\mathcal{C}(Y)$. For $Y = S^3$ we write $\mathcal{C} = \mathcal{C}(S^3)$. For topological spaces U, V, denote the set of *free* (i.e. unbased) homotopy classes of maps $U \to V$ by $[U, V]$. The composition of any continuous function A with projection to Y

$$S^1 \times [0, 1] \xrightarrow{A} Y \times [0, 1] \to Y$$

is a continuous function, so concordant knots have the same unbased homotopy class. We denote the set of concordance classes of knots which realise a given unbased homotopy class $x \in [S^1, Y]$ by $\mathcal{C}_x(Y)$, and we observe there is a partition of sets

$$\mathcal{C}(Y) = \bigsqcup_{x \in [S^1, Y]} \mathcal{C}_x(Y).$$

To study concordance of knots in 3–manifolds, we will fix a pair (Y, x) with $x \in [S^1, Y]$, and investigate the set $\mathcal{C}_x(Y)$.

1.1. **Almost-concordance.** The connected sum of knots $(S^3, J)\#(Y, K)$ defines a new knot $(Y, J\#K)$, which is freely homotopic to K in Y, since all knots in S^3 are freely null-homotopic.

Definition 1.1. For each pair (Y, x), the *local action* of the concordance group $\mathcal{C}$ on the set $\mathcal{C}_x(Y)$ is defined by

$$\mathcal{C} \times \mathcal{C}_x(Y) \to \mathcal{C}_x(Y) \qquad ([J], [K]) \mapsto [J \# K].$$

For $K \in \mathcal{C}_x(Y)$ the *almost-concordance* class of K is the orbit of K under the local action of $\mathcal{C}$ on $\mathcal{C}_x(Y)$.

The study of this local action can be traced back to Milnor's study of link homotopy using what are now called the Milnor's $\overline{\mu}$ invariants [Mil57]. These invariants of a link $L \subset S^3$ come from looking at quotients of $\pi_1(S^3 \setminus L)$ by the lower central subgroups, which has the intentional effect of making them blind to local knotting. Extensions of these invariants to *knots* in general 3–manifolds were studied by Miller [Mil95], Schneiderman [Sch03] and Heck [Hec11]. In particular, Miller produces an infinite family of knots in $Y = S^1 \times S^1 \times S^1$, each homotopic to $S^1 \times \{pt\} \times \{pt\}$, and his invariants can be used to show that these knots are mutually distinct in almost-concordance.

Celoria [Cel16], who coined the term 'almost-concordance', recently studied the case of the null-homotopic class in lens spaces $L(n, 1)$ with $n \geq 3$. Using a generalisation of the τ invariant from knot Floer homology, he showed the existence of an infinite family of knots mutually distinct in smooth almost-concordance. In [Cel16, Conjecture 44] it is conjectured that, when $Y \neq S^3$, within each free homotopy class in $[S^1, Y]$ there are infinitely many distinct almost-concordance classes. For $Y = S^3$ the local action is transitive.

We prove in the first section of this paper that, besides S^3, there is another much less obvious case which must be excluded from such a conjecture. In the case $Y = S^1 \times S^2$ and $x \in [S^1, Y]$ a primitive element, we determine that the set $\mathcal{C}_x(Y)$ consists of a single class. The proof of this fact uses topological surgery theory, see Theorem 2.1.

Theorem 1.2 (Concordance light bulb theorem)**.** *If a knot K in $S^1 \times S^2$ is freely homotopic to $S^1 \times \{pt\}$, then K is concordant to $S^1 \times \{pt\}$.*

So the following adjusted conjecture is a good starting point for studying almost-concordance, and is the central focus of this paper.

Conjecture 1.3. *Fix a closed 3–manifold Y and a free homotopy class $x \in [S^1, Y]$. Unless $Y = S^3$, or $(Y, x) = (S^1 \times S^2, [S^1 \times \{pt\}])$, there are infinitely many distinct almost-concordance classes within the set $\mathcal{C}_x(Y)$.*

In this paper we have two main sets of results towards proving Conjecture 1.3. The results employ very different techniques and are each effective under different circumstances.

As our first main result, we prove the following statement in Theorem 6.3, which generates many infinite families of examples confirming Conjecture 1.3.

Theorem 1.4. *Let Y be a closed, orientable 3–manifold and $x \in [S^1, Y]$ a free homotopy class. Denote its homology class with $[x] \in H_1(Y; \mathbb{Z})$. If $[x] = 2u$ for a primitive class u of infinite order, then there are infinitely many distinct almost-concordance classes within the set $\mathcal{C}_x(Y)$.*

We expect that one could push the methods of this paper further, using suitably cunning calculations, to deal with the case that $[x] = nu$ for a primitive class u of infinite order and any $n \geq 2$. The case $n = 1$ must be excluded as a consequence of Theorem 2.1.

The theorem is proved using twisted Reidemeister torsion combined with a satellite construction. The actual topological almost-concordance invariants we obtain are rather technical to state, so we leave their precise formulation to the body of

the paper in Corollary 5.5. The principle is roughly as follows. The local action of $\mathcal{C}$ can only affect the twisted Reidemeister torsion of a knot K in a 3–manifold by multiplication with the Alexander polynomial of a knot in S^3, where the variable of the polynomial corresponds to the meridian of the knot $K \subset Y$. If we can modify K in Y so that we introduce more drastic changes to the twisted torsion, but do not change the free homotopy class of the knot, then we can potentially change the almost-concordance class in a detectable way. The technical difficulty we have overcome is in choosing free homotopy classes of knots, and a coefficient system for the homology, so that the twisted Reidemeister torsion is both non-trivial and a concordance invariant.

Our second main technique for distinguishing almost-concordance classes comes from analysing linking numbers of covering links. In Section 7 we give a self-contained and elementary proof of the following.

Proposition 1.5. *Let Y be a spherical space form and let x be the null-homotopic free homotopy class. Then if $Y \neq S^3$, the set $\mathcal{C}_x(Y)$ contains infinitely many almost-concordance classes.*

The proof exhibits knots that lift to links in S^3 with different pairwise linking numbers. We observe that almost-concordant knots lift to links for which the set of pairwise linking numbers coincide, and the result follows. We remark that the examples exhibited by Celoria [Cel16] also lift to links in S^3 with different pairwise linking numbers. So in particular these examples are distinct in topological almost-concordance, a fact which cannot be detected by the τ invariant.

We proceed to generalise this idea into a more systematic approach, which can be applied to any 3–manifold $Y \neq S^3$ to obtain the second main theorem of this paper.

Theorem 1.6. *For any closed orientable 3–manifold $Y \neq S^3$ and x the null-homotopic class there are infinitely many distinct almost-concordance classes within the set $\mathcal{C}_x(Y)$.*

We note that it seems likely this theorem could also be proved using Schneiderman's concordance invariant from [Sch03].

In fact, the covering link obstruction we develop in the final section of the paper works even more generally, in the case where $x \in [S^1, Y]$ is *any* torsion class. So even stronger versions of the above statement are proven in Corollary 8.4 and Theorem 8.5.

1.2. **Almost-concordance and piecewise linear I–equivalence.** The roots of almost-concordance in the smooth category go back to the 1960s. Stallings [Sta65] defined links L_0 and L_1 to be *I–equivalent* if there exists a proper (not necessarily locally flat) embedding $A \colon \bigsqcup_m S^1 \times [0,1] \hookrightarrow Y \times [0,1]$ such that $L_0 = A|_{\bigsqcup_m S^1 \times \{0\}} \subset Y \times \{0\}$ and $L_1 = A|_{\bigsqcup_m S^1 \times \{1\}} \subset Y \times \{1\}$. Observe that a concordance is then a locally flat I–equivalence and a smooth concordance is a smooth I–equivalence. The intermediate notion of a piecewise linear I–equivalence turns out to be highly relevant to our current discussion. Precisely, given a closed 3–manifold Y it is the main result of Rolfsen [Rol85] that the smooth almost-concordance class of a knot is exactly the PL I–equivalence class of the knot. There does not appear to be a similar interpretation of topological almost-concordance directly in terms of some kind of I–equivalence. For discussions of classical invariants of PL I–equivalence see Rolfsen [Rol85] and Hillman [Hil12, §1.5], also compare Goldsmith [Gol78].

1.3. **Almost-concordance and homology surgery.** Cappell and Shaneson [CS74] developed powerful tools to analyse knot concordance via homology surgery. We

now briefly discuss how the subtleties of their setup relate to almost-concordance and to our invariants.

To apply their method to study the concordance set $\mathcal{C}_x(Y)$, one must first pick a knot K representing x as a target knot. A knot $J \subset Y$ is then called K–*characteristic* if there exists a degree 1 normal map (of pairs) from the exterior of J to the exterior of K, which is the identity on the boundary. A concordance from J to J' is called K–*characteristic* if there exists a degree 1 normal map (of triads) from the exterior of the concordance to the exterior of the trivial concordance from K to itself, restricting to the identity on the interior boundary and restricting to degree 1 normal maps on the respective exteriors of J and J'. Write $\mathrm{Char}_K(Y)$ for the set of equivalence classes of K–characteristic knots in Y modulo K–characteristic concordance.

If $Y = S^3$ and U is the unknot, every knot and concordance is U–characteristic in a natural way. Local knotting defines an action of the group $\mathrm{Char}_U(S^3) \cong \mathcal{C}$ on $\mathrm{Char}_K(Y)$, which intertwines with the action of $\mathcal{C}$ on $\mathcal{C}_x(Y)$ under the natural map $i_{K,Y} \colon \mathrm{Char}_K(Y) \to \mathcal{C}_x(Y)$. Consequently, almost-concordance invariants are invariant on the orbits of the action of $\mathcal{C}$ on $\mathrm{Char}_K(Y)$, and could potentially also determine whether a knot is K–characteristic. However, the maps $i_{K,Y}$ are not known to be injective in general, so statements in the reverse direction are less clear. We observe that the question of injectivity of these maps is closely analogous to the question of whether slice boundary links in S^3 are moreover boundary slice.

However, in practice, historical examples of characteristic concordance obstructions turn out to be almost-concordance invariants. In particular, the invariants of Miller [Mil95], mentioned above, which were not originally intended as almost-concordance invariants, turn out to be so. We note that Goldsmith [Gol78] has related Milnor's invariants for classical links to linking numbers in covering links. This suggests that our covering link invariants may be closely related to Miller's knot invariants.

To place our two approaches in the context of Cappell-Shaneson's theory, our covering link invariants could also be used to obstruct a knot being characteristic, whereas our Reidemeister torsion invariants can be thought of as detecting non-triviality in a quotient of Cappell-Shaneson homology surgery obstruction groups $\Gamma_4(\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}] \to \mathbb{Z})/\Gamma_4(\mathbb{Z}[\mathbb{Z}] \to \mathbb{Z})$.

1.4. **Further questions.** As the study of PL I–equivalence was largely conducted before the seminal work of Freedman, and so before the current appreciation of the difference between smooth and topological concordance, the aforementioned classical PL I–equivalence invariants are really invariants of topological almost-concordance. This suggests the following question, which has not been classically studied (and which we do not address in this present work).

Question 1.7. *Are there knots $K, K' \subset Y$ which are topologically but not smoothly almost-concordant to one another?*

This paper is focussed on the whether the local action of Definition 1.1 is transitive, but one can ask about other properties of this action.

Question 1.8. *Given a 3–manifold Y and class $x \in [S^1, Y]$, when is the local action $\mathcal{C} \times \mathcal{C}_x(Y) \to \mathcal{C}_x(Y)$ free? When is it faithful?*

For the question of whether this action is free, compare with the action considered in a high-dimensional setting by Cappell–Shaneson [CS74, §6].

2. A CASE WHEN ALMOST-CONCORDANCE IS TRIVIAL

For Z a submanifold of a manifold X, we introduce the notation $\nu Z \subset X$ for some open tubular neighbourhood of Z in X.

In this section, let $Y = S^1 \times S^2$ and let x be the free homotopy class of a generator $J := S^1 \times \{\mathrm{pt}\} \subset S^1 \times S^2$ of $\pi_1(S^1 \times S^2) \cong \mathbb{Z}$. Note that altering J by a local knot does not alter the isotopy class, as we can change the crossings of the local knot arbitrarily by isotopies in $S^1 \times S^2$. We may consider other knots in the same free homotopy class that are not isotopic to J, but in fact up to concordance, we now show there is no difference.

Theorem 2.1 (Concordance light bulb theorem)**.** *Suppose that a knot $K \subset S^1 \times S^2$ lies in the free homotopy class of x. Then K is concordant to J.*

Proof. Let $X_K := Y \setminus \nu K$ and let $X_J := Y \setminus \nu J$, noting that $X_J \cong S^1 \times D^2$. Let $M_K := X_K \cup X_J$, joined along the boundary tori with meridian mapping to meridian and longitude mapping to longitude. Note that if K and J are concordant, then the boundary of the exterior of the concordance is M_K.

Claim. The homology $H_1(Y \setminus \nu K; \mathbb{Z}) \cong \mathbb{Z}$.

The $\mathbb{Z}$–coefficient Mayer–Vietoris sequence for the decomposition $Y = (Y \setminus \nu K) \cup_{S^1 \times S^1} \nu K$ yields:

$$H_2(Y) \to H_1(S^1 \times S^1) \to H_1(Y \setminus \nu K) \oplus H_1(\nu K) \to H_1(Y) \to 0.$$

The generator of $H_2(Y) \cong \mathbb{Z}$ maps to the meridian of K in $H_1(S^1 \times S^1)$. The longitude maps onto $H_1(\nu K)$. It follows that $H_1(Y \setminus \nu K) \cong H_1(Y) \cong \mathbb{Z}$ as claimed.

Claim. The homology $H_1(Y \setminus \nu K; \mathbb{Z}[\mathbb{Z}]) = 0$.

To see this, note that we can understand $Y \setminus \nu K$ as a Kirby diagram by considering a 2–component link $L = L_1 \cup L_2 \subset S^3$ with linking number one, where we take L_2 to be unknotted and marked with a zero, and L_1 is defined as the knot that becomes K after 0–surgery on L_2. Under the abelianisation $\pi_1(X_L) \to \mathbb{Z}$, the meridian of L_1 is sent to zero, while the meridian of L_2 is sent to a generator. The Alexander polynomial of a 2–component link with linking number one satisfies $\Delta_L(1, t) = \Delta_{L_2}(t)$, by the Torres condition [Hil12, Section 5.1]. But L_2 is unknotted, so $\Delta_{L_2}(t) = 1$ and therefore $H_1(X_L; \mathbb{Z}[\mathbb{Z}]) = 0$. Glue in the surgery solid torus to the boundary of νL_2, to obtain $Y \setminus \nu K$. This solid torus also has $H_1(S^1 \times D^2; \mathbb{Z}[\mathbb{Z}]) = 0$, so it follows that $H_1(Y \setminus \nu K; \mathbb{Z}[\mathbb{Z}]) = 0$ as claimed.

Claim. There exists a choice of framing on M_K such that it is null bordant over S^1, in other words represents the 0 class in $\Omega_3^{fr}(S^1)$.

By the Atiyah–Hirzebruch spectral sequence we have $\Omega_3^{fr}(S^1) \cong \Omega_3^{fr} \oplus \Omega_2^{fr}$. As in Davis [Dav06], Cha–Powell [CP14], we can choose any framing to start, and then alter it in a neighbourhood of a point until the framing gives the zero element of $\Omega_3^{fr} \cong \mathbb{Z}/24\mathbb{Z}$. This procedure is possible because the J–homomorphism $\pi_3(O) \to \pi_3^S \cong \Omega_3^{fr}$ is onto. The element in $\Omega_2^{fr} \cong \Omega_2^{spin} \cong \mathbb{Z}/2\mathbb{Z}$ represented by

M_K is trivial: it is not too hard to see that the surface produced by transversality is a sphere. In X_K, the inverse image of this sphere is a sphere punctured by the knot K, potentially in several places. Each of the punctures is bounded by a meridian of K, which is identified with a meridian of J. But then a meridian of J bounds an embedded disc in X_J. This completes the proof of the claim.

Now follow the standard procedure from Freedman–Quinn [FQ90], Hillman [Hil12, §7.6], Davis [Dav06]. The fact that M_K is framed null bordant gives rise to a degree one normal map $W \to S^1 \times D^3$ which is a $\mathbb{Z}[\mathbb{Z}]$–homology equivalence on the boundary. The surgery obstruction to changing $W \to S^1 \times D^3$ into a homotopy equivalence lies in $L_4(\mathbb{Z}[\mathbb{Z}])$. We have

$$L_4(\mathbb{Z}[\mathbb{Z}]) \cong L_4(\mathbb{Z}) \oplus L_3(\mathbb{Z}) \cong L_4(\mathbb{Z}) \cong L_0(\mathbb{Z}) \cong 8\mathbb{Z}$$

so that the surgery obstruction may be calculated as the signature of W. Hence we can kill the obstruction by taking W connected sum with the E_8 manifold, with appropriate orientations, sufficiently many times. As $\mathbb{Z}$ is a 'good' group (in the sense of [FQ90]), we may now do surgery on our normal map to get a homotopy equivalence $W' \to S^1 \times D^3$, where $\partial W'$ is still M_K.

Glue in $S^1 \times D^2 \times D^1$ to part of the boundary, namely a thickening $S^1 \times S^1 \times D^1$ in $M_K = X_K \cup_{S^1 \times S^1} X_J$ of the gluing torus $S^1 \times S^1$, to obtain a concordance from K to J in a 4–manifold $V = W' \cup_{S^1 \times S^1 \times D^1} S^1 \times D^2 \times D^1$. Note that V and $S^1 \times S^2 \times I$ have the same fundamental group and the same homology over $\mathbb{Z}[\mathbb{Z}]$, so by the Hurewicz Theorem they have the same homotopy groups.

Claim. The 4–manifold V is homeomorphic to $S^1 \times S^2 \times I$.

Cap off V on the top and bottom boundaries with copies of $S^1 \times D^3$. This creates a 4–manifold Z that has the same homotopy groups as $S^1 \times S^3$. Then Z is homeomorphic to $S^1 \times S^3$ [FQ90, Theorem 10.7A]. Now remove the images of the two caps $S^1 \times D^3$ in $S^1 \times S^3$. These are isotopic to standard embeddings, so the outcome is $S^1 \times S^2 \times I$ as claimed.

This means that the concordance of K to J in V is in fact a concordance of K to J in $S^1 \times S^2 \times I = Y \times I$ as required, which completes the proof of the theorem. □

Corollary 2.2. *If x is the free homotopy class of $S^1 \times \{\mathrm{pt}\}$ then the set $\mathcal{C}_x(S^1 \times S^2)$ contains exactly one almost-concordance class.*

3. Almost-concordance and satellites

We will recast almost-concordance as the orbit relation of a satellite action. The construction of satellite knots in S^3 can be described in many equivalent ways – here is a generalisation for one of these satellite constructions to knots in a general 3–manifold. A *knot framing* ψ of K is an embedding $\psi \colon S^1 \times D^2 \to Y$ such that $\psi(S^1 \times \{0\})$ is K. Associated to a framing is the longitude $\lambda_\psi = \psi(S^1 \times \{\mathrm{pt}\})$.

A framed knot (P, ψ) with $P \subset \mathrm{int}(S^1 \times D^2)$ is called a *framed pattern*. For a framed knot (K, ψ_K) in Y and a framed pattern (P, ψ_P), the associated *satellite knot* is the framed knot $P(K) = \psi_K(P) \subset Y$ with framing $\psi_K \circ \psi_P$. The set $\mathcal{P}$ of framed patterns with the operation $P \cdot Q := P(Q)$, which is called *satellite action*, is a monoid. This monoid $\mathcal{P}$ acts on the set of framed knots $\mathrm{FrKnots}(Y)$ via

$$\mathcal{P} \times \mathrm{FrKnots}(Y) \to \mathrm{FrKnots}(Y); \qquad (P, K) \mapsto P(K).$$

As before, we denote the set of framed knots in the homotopy class $x \in [S^1, Y]$ by $\mathrm{FrKnots}_x(Y)$.

The *winding number* of a pattern is the unique $n \in \mathbb{Z}$ such that the knot represents n times the positive generator of $H_1(S^1 \times D^2; \mathbb{Z})$. Patterns with winding number 1 form a submonoid of $\mathcal{P}$. Suppose that P is such a pattern. Then $P(K)$ is

always freely homotopic to K. This follows from the observation that, for patterns with winding number 1, P is a generator of $\pi_1(S^1 \times D^2)$.

Given any knot J in S^3, form a pattern $P_J \subset \text{int}(S^1 \times D^2)$ by removing from S^3 a small open tubular neighbourhood of the meridian to J. (We identify the exterior $S^3 \setminus \nu U$ of any unknot $U \subset S^3$ with $S^1 \times D^2$ by mapping the meridian of U to $S^1 \times \{\text{pt}\}$ and a 0–framed longitude of U to $\{\text{pt}\} \times \partial D^2$.) The pattern P_J is taken to be canonically framed using the 0–framing of J in S^3. A pattern P obtained in this way has winding number 1.

Proposition 3.1. *Let Y be a 3–manifold and fix $x \in [S^1, Y]$. Then the underlying unframed knot of $P_J(K)$ is $K \# J$ and is thus independent of the framing of K. Moreover, the group action*

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{C}_x(Y) & \to & \mathcal{C}_x(Y) \\ (J, K) & \mapsto & P_J(K). \end{array}$$

is well-defined and agrees with the local action of Definition 1.1.

Proof. It is enough to show that for J in S^3 and K in Y, with any framing on K, we have $P_J(K) = K \# J$. The *disc knot* of J is $\Delta_J \colon D^1 \hookrightarrow D^3$, obtained as the knotted embedding of D^1 in the exterior of small open 3–ball around a point $\text{pt} \in J \subset S^3$. As such $J = \Delta_J \cup \Delta_U$ for U the unknot in S^3. But now it is clear that the knotted part of P_J can be forced into a small ball $D^3 \subset S^1 \times D^2$ and hence $P_J = P_U \# J$. So regardless of the framing of K, the construction of $P_J(K)$ yields $K \# J$. $\qquad \square$

For the convenience of the reader, we recall the following lemma.

Lemma 3.2. *Let $L \subset S^3$ be a 2–component link with linking number 1. Then for each boundary component T of $S^3 \setminus \nu L$, and for all $k \in \mathbb{Z}$, the induced map*

$$H_k(T; \mathbb{Z}) \xrightarrow{\cong} H_k(S^3 \setminus \nu L; \mathbb{Z})$$

is an isomorphism. Consequently, $S^3 \setminus \nu L$ is a homology bordism.

Proof. Let L_0, and L_1 be the two components. The claim follows from Mayer–Vietoris sequences of the decomposition $S^3 \setminus \nu L_0 = S^3 \setminus \nu L \cup \nu L_1$ and $S^3 \setminus \nu L_1 = S^3 \setminus \nu L \cup \nu L_0$. $\qquad \square$

Let $P \subset S^1 \times D^2$ be a framed pattern with winding number 1. We say the pattern is *well-framed* if the longitude λ_P is homologous to $S^1 \times \{\text{pt}\} \subset S^1 \times \partial D^2$ in $S^1 \times D^2 \setminus P$. By Lemma 3.2 the manifold $S^1 \times D^2 \setminus \nu P$ is a homology bordism, so there always exists a well-framing.

Lemma 3.3. *Let $K \in \text{FrKnots}(Y)$ be a framed knot in a 3–manifold Y and $P \in \mathcal{P}$ a winding number 1 pattern which is well-framed. Pick tubular neighborhoods νK and $\nu P(K)$ such that $\nu P(K) \subset \nu K$. Then the following statements hold:*

(1) *$[P(K)] = [K] \in H_1(Y; \mathbb{Z})$.*
(2) *the meridian μ_K is homologous in $\nu K \setminus \nu P(K)$ to the meridian $\mu_{P(K)}$, and*
(3) *the longitude λ_K is homologous in $\nu K \setminus \nu P(K)$ to the longitude $\lambda_{P(K)}$.*

Proof. By assumption P is a winding number 1 pattern, so $P(K)$ is homologous to K already in νK and the first statement follows.

Consider the 3–manifold $M = \nu K \setminus \nu P(K)$. By Lemma 3.2, we can compose the isomorphisms induced by the inclusions

$$H_1(\partial \nu K; \mathbb{Z}) \xrightarrow{\cong} H_1(M; \mathbb{Z}) \xleftarrow{\cong} H_1(\partial \nu P(K); \mathbb{Z})$$

and obtain an isomorphism $\Phi \colon H_1(\partial \nu P(K); \mathbb{Z}) \to H_1(\partial \nu K; \mathbb{Z})$.

For the statement concerning meridians, we consider the diagram of maps induced by inclusions:

$$\begin{array}{ccc} H_1(\partial \nu K; \mathbb{Z}) & \longrightarrow & H_1(\nu K; \mathbb{Z}) \\ \uparrow \scriptstyle\Phi & & \uparrow \scriptstyle\cong \\ H_1(\partial \nu P(K); \mathbb{Z}) & \longrightarrow & H_1(\nu P(K); \mathbb{Z}) \end{array}$$

We see that Φ restricts to an isomorphism between the kernels of the horizontal maps. Recall that the meridian up to a sign is characterised by the kernel of the respective horizontal map and so Φ maps the meridian of $P(K)$ to the meridian of K.

The longitude of K is homologous to the longitude of $P(K)$ as the pattern P is well-framed. $\square$

4. Twisted Reidemeister torsion and satellites

To talk precisely about Reidemeister torsion, we establish some algebraic conventions and notation.

Let R be a ring R with unit and an involution $r \mapsto \bar{r}$. One example to keep in mind is the the group ring $\mathbb{Z}[\pi]$ for a group π which carries the involution $\Sigma_g n_g g \mapsto \Sigma_g n_g g^{-1}$.

Given a left R–module A, let A^t denote the right R–module defined by the action $a \cdot r := \bar{r} \cdot a$ for $a \in A$ and $r \in R$. Similarly, we may switch right R–modules to left ones, and if S is another ring with involution we may switch (S, R)–bimodules to (R, S)–bimodules. For an (R, S)–bimodule B and a left R–module A, the abelian group $\text{Hom}_R(A, B)$ has a natural right S–module structure. Using the natural (R, R)–bimodule structure on R a left R–module A determines a right R–module $A^\vee := \text{Hom}_R(A, R)$. A chain complex of left R–modules C determines the dual chain complex of right R–modules $C^{-*} := \text{Hom}_R(C_*, R)$. Here, recall that $d^{-r} = (-1)^{r+1} d_r^\vee : C^{-r} \to C^{-r+1}$.

A group homomorphism $\varphi \colon \pi \to R^\times$ into the units of the ring R is called a *representation*. It is called *unitary* if $\varphi(g^{-1}) = \overline{\varphi(g)}$ for all $g \in \pi$. A unitary representation φ induces a homomorphism $\varphi \colon \mathbb{Z}[\pi] \to R$ of rings with involution. With this homomorphism, we can give R the structure of a $(\mathbb{Z}[\pi], R)$–bimodule.

The following is straightforward to prove and is left to the reader.

Lemma 4.1. *Let* $\varphi \colon \pi \to R^\times$ *be a unitary representation. Let* A *be a left* $\mathbb{Z}[\pi]$–*module. Then the following map is well-defined and an isomorphism of left* R–*modules.*

$$\begin{array}{ccc} \text{Hom}_{\mathbb{Z}[\pi]}(A, R)^t & \to & \text{Hom}_R(A^t \otimes_\varphi R, R), \\ f & \mapsto & \left(a \otimes b \mapsto \overline{f(a)} \cdot b \right). \end{array}$$

For a CW pair (X, Y) and $p \colon \widetilde{X} \to X$ the universal cover, write $\widetilde{Y} = p^{-1}(Y)$. Then setting $\pi = \pi_1(X)$ and writing $C = C_*(\widetilde{X}, \widetilde{Y}; \mathbb{Z})$ for the chain complex of left $\mathbb{Z}[\pi]$–modules, we write $C^{-*}(\widetilde{X}, \widetilde{Y}; \mathbb{Z}) := C^{-*}$ for the dual chain complex. Given a unitary representation $\varphi \colon \pi \to R^\times$, we define the following left R–modules

$$\begin{array}{rcl} H_r(X, Y; \varphi) & := & H_r\big((C_*(\widetilde{X}, \widetilde{Y}; \mathbb{Z})^t \otimes_\varphi R)^t\big), \\ H^r(X, Y; \varphi) & := & H_{-r}\big((C^{-*}(\widetilde{X}, \widetilde{Y}; \mathbb{Z}) \otimes_\varphi R)^t\big) \cong H_r(\text{Hom}_{\mathbb{Z}[\pi]}(C_*(\widetilde{X}, \widetilde{Y}; \mathbb{Z}), R)^t) \end{array}$$

Suppose (X, Y) is an n–dimensional Poincaré pair. For $R = Q$ a field, we may apply twisted Poincaré–Lefschetz duality, then Lemma 4.1 and finally the Universal

Coefficient Theorem to obtain

$$
\begin{aligned}
H_r(X;\varphi) &\cong H^{n-r}(X,Y;\varphi) \\
&\cong H_{n-r}(\mathrm{Hom}_Q(C_*(\widetilde{X},\widetilde{Y};\mathbb{Z})^t \otimes_\varphi Q, Q)) \\
&\cong \mathrm{Hom}_Q(H_{n-r}(X,Y;\varphi)^t, Q).
\end{aligned}
$$

When $H_*(Y;\varphi) = 0$, we have $H_*(X;\varphi) \cong H_*(X,Y;\varphi)$, so in this case we obtain further a Poincaré duality of Q–vector spaces of the form:

$$
H_r(X;\varphi) \cong (H_{n-r}(X;\varphi)^t)^\vee. \tag{$\dagger$}
$$

4.1. **Self-dual based torsion.** We recall the algebraic setup for torsion invariants. Suppose C is a based chain complex over a field Q and $\mathcal{B} = \{\mathcal{B}_*\}$ is a basis for $H_*(C)$, i.e. $\mathcal{B}_i$ is a basis of the Q–vector space $H_i(C)$. The *torsion* $\tau(C;\mathcal{B}) \in Q^\times$ is defined as in [Tur01, Definition I.3.1]. If $H_*(C)$ is identically zero, then we will just write $\tau(C) \in Q^\times$ for the torsion.

Let (X,Y) be a finite CW pair with $\pi = \pi_1(X)$ and let $\varphi\colon \pi \to Q \setminus \{0\}$ be a representation to a field Q. Let $\mathcal{B} = \{\mathcal{B}_*\}$ be a basis of $H_*(X,Y;\varphi)$. The universal cover $(\widetilde{X}, \widetilde{Y})$ has a natural cell structure, and the chain complex $C_*(\widetilde{X}, \widetilde{Y};\mathbb{Z})$ can be based over $\mathbb{Z}[\pi]$ by choosing a lift of each cell of (X,Y) and orienting it. This gives rise to a basing of $C_*(\widetilde{X}, \widetilde{Y};\mathbb{Z})^t \otimes_\varphi Q$ over Q. We can then define the *twisted torsion*

$$
\tau^\varphi(X,Y;\mathcal{B}) \in Q^\times
$$

to be the torsion of $C_*(\widetilde{X}, \widetilde{Y};\mathbb{Z})^t \otimes_\varphi Q$ with respect to $\mathcal{B}$. We will drop $\mathcal{B}$ from the notation if $H_*(X,Y;\varphi) = 0$.

Remark. The element $\tau^\varphi(X,Y;\mathcal{B})$ is well-defined up to multiplication by an element in $\pm\varphi(\pi)$, and is invariant under simple homotopy preserving $\mathcal{B}$ [Tur01, Section II.6.1 and Corollary II.9.2]. By Chapman's theorem [Cha74] the invariant $\tau^\varphi(X,Y;\mathcal{B})$ only depends on the homeomorphism type of (X,Y) and the basis $\mathcal{B}$. In particular, when (M,N) is a manifold pair, we can define $\tau^\varphi(M,N;\mathcal{B})$ by picking any finite CW structure for (M,N).

Now we consider a special case of this construction and explain how to deal with the choice of basis $\mathcal{B}$. Let $(M,\partial M)$ be a 3–manifold with boundary. We will focus in a rather special kind of representation obtained as follows. Let F be a free abelian group. Furthermore, assume we have two group homomorphisms $\rho\colon \pi_1(M) \to \{\pm 1\} \subset \mathbb{Q}^\times$ and $\alpha\colon \pi_1(M) \to F$. Denote the quotient field of $\mathbb{Q}[F]$ by $\mathbb{Q}(F)$. One can check directly that the homomorphism $\rho \otimes \alpha\colon \pi_1(M) \to \mathbb{Q}(F)$ is a representation, which we write as φ to save notation.

Definition 4.2. A representation $\varphi\colon \pi_1(M) \to \mathbb{Q}(F)$ obtained by the construction above is called *sign-twisted*.

Suppose $\varphi\colon \pi_1(Y) \to \mathbb{Q}(F)$ is a sign-twisted representation such that $H_*(\partial Y;\varphi) = 0$. Since M is odd-dimensional, we can pick a basis $\mathcal{B} = \{\mathcal{B}_*\}$ for $H_*(M;\varphi)$ with the following property: for each r, $\mathcal{B}_r$ is the dual basis of $\mathcal{B}_{n-r}$ via the Poincaré duality isomorphism ($\dagger$). We call such a basis $\mathcal{B} = \{\mathcal{B}_*\}$ a *self-dual basis* for $H_*(M;\varphi)$.

Definition 4.3. The *norm subgroup* $N(F)$ is

$$
N(F) := \{r \cdot f \cdot q \cdot \bar{q} \mid r \in \mathbb{Q}^\times, f \in F \text{ and } q \in \mathbb{Q}(F) \setminus \{0\}\}.
$$

The *self-dual based torsion* $\tau^\varphi(M)$ is the following element in the quotient

$$
\tau^\varphi(M;\mathcal{B}) \in \mathbb{Q}(F)^\times / N(F).
$$

Remark. We note the following about the preceding definition.

(1) The self-dual based torsion is indeed well-defined. Switching from one self-dual basis to another changes the torsion by an element of the form $\pm q\bar{q}$ with $q \in \mathbb{Q}(F) \setminus \{0\}$ [CF13, Lemma 2.3]. Furthermore, different choices of lifts of the cells change the torsion only by $\pm \alpha(\pi_1(M))$ [Tur01, Section II.6.1].

(2) Note that $N(F)$ is *not* just the group of "norms" $q \cdot \bar{q}$, but also their products with all elements of F and also all non-zero rational numbers. The fact that the rational numbers are also contained plays a rôle later on, in Proposition 5.4.

Now we provide a way to distinguish elements in the quotient $\mathbb{Q}(F)^\times / N(F)$ by constructing epimorphisms to $\mathbb{Z}/2\mathbb{Z}$. Recall that $\mathbb{Z}[F]$ is a unique factorisation domain. It follows that given any irreducible polynomial g over $\mathbb{Z}[F]$ we have a well-defined monoid homomorphism

$$\begin{aligned} \Phi_g' \colon \mathbb{Z}[F]^\times &\to \mathbb{N}_0 \\ q &\mapsto \text{maximal } n \text{ such that } g^n \text{ divides } q. \end{aligned}$$

This extends to an epimorphism

$$\begin{aligned} \Phi_g' \colon \mathbb{Q}(F)^\times &\to \mathbb{Z} \\ rs^{-1} &\mapsto \Phi_g(r) - \Phi_g(s). \end{aligned}$$

We call a polyomial $g \in \mathbb{Z}[F]$ *symmetric*, if there exist a unit $a \in \mathbb{Z}[F]^\times$ such that $g = a\bar{g}$. If g is symmetric, then for any $q \in \mathbb{Z}[F]^\times$ we have $\Phi_g(\bar{q}) = \Phi_{\bar{g}}(q) = \Phi_g(q)$. Thus we see that Φ_g descends to an epimorphism

$$\begin{aligned} \Phi_g \colon \mathbb{Q}(F)^\times / N(F) &\to \mathbb{Z}/2\mathbb{Z} \\ [rs^{-1}] &\mapsto \Phi_g(r) - \Phi_g(s) \mod 2. \end{aligned}$$

Definition 4.4. For an irreducible and symmetric polynomial $g \in \mathbb{Z}[F]$, we call $\Phi_g \colon \mathbb{Q}(F)^\times / N(F) \to \mathbb{Z}/2\mathbb{Z}$ the *parity homomorphism*.

4.2. **Alexander polynomial of a pattern.** Let $P \subset \operatorname{int}(S^1 \times D^2)$ be a winding number 1 pattern which is well-framed. We denote the meridian of $P \subset S^1 \times D^2$ by s. By Lemma 3.3, the homology class of s agrees with the class $[\{pt\} \times \partial D^2] \in H_1(S^1 \times D^2 \setminus \nu P; \mathbb{Z})$. As P is well-framed, the homology class t of the longitude of P is homologous in $S^1 \times D^2 \setminus P$ to that of the curve $S^1 \times \{pt\}$.

The meridian s and the longitude t determine a preferred isomorphism $H_1(S^1 \times D^2 \setminus \nu P; \mathbb{Z}) \cong \mathbb{Z}\langle s, t \rangle$, where $\mathbb{Z}\langle s, t \rangle$ denotes the free abelian group on the generators s and t. As usual, we consider the Alexander module $H_1(S^1 \times D^2 \setminus \nu P; \mathbb{Z}[s^{\pm 1}, t^{\pm 1}])$ of the pattern. This is a module over $\mathbb{Z}[s^{\pm 1}, t^{\pm 1}]$, and thus we can consider its order. (We refer to [Tur01, I.4.2] for the definition of the order of a $\mathbb{Z}[s^{\pm 1}, t^{\pm 1}]$–module.) We denote the order of the Alexander module by

$$\Delta_P(s, t) \in \mathbb{Z}[H_1(S^1 \times D^2 \setminus \nu P; \mathbb{Z})] = \mathbb{Z}[s^{\pm 1}, t^{\pm 1}],$$

and we refer to this as the *Alexander polynomial* of P.

Consider the standard embedding $S^1 \times D^2 \subset S^3$. The closure of its complement is also a solid torus with core c. With this embedding, we can associate to the pattern P the two-component link $L(P) = P \cup c \subset S^3$.

We have a diffeomorphism between the complements of P and $L(P)$

$$(S^1 \times D^2) \setminus \nu P \xrightarrow{\cong} S^3 \setminus \nu L(P),$$

which is isotopic to the inclusion. Correspondingly, we can also relate the Alexander polynomial of P to the Alexander polynomial of $L(P)$.

Lemma 4.5. *Let P be a winding number one pattern. Then the following holds:*

(1) *Under the identification* $(S^1 \times D^2) \setminus \nu P = S^3 \setminus \nu L(P)$*, the meridian of the pattern corresponds to the meridian of the first component of* $L(P)$*, while the longitude of the pattern corresponds to the meridian of the second component of* $L(P)$*.*

(2) *We have* $\Delta_P(s,t) = \Delta_{L(P)}(s,t)$*, where* $\Delta_{L(P)}(s,t)$ *denotes the usual two-variable Alexander polynomial of the 2–component link* $L(P)$*.*

Proof. The lemma follows immediately from the fact that the Alexander polynomial of a link in S^3 only depends on the complement together with the meridians.   □

For local patterns, the Alexander polynomial has the structure described in the next lemma.

Lemma 4.6. *Let* J *be an oriented knot in* S^3 *and let* μ *be a meridian of* J*. We denote the pattern that is given by removing a small open tubular neighbourhood of* μ *from* S^3 *by* P_J*. Then* $L(P_J) = J \cup \mu$ *and*

$$\Delta_{P_J}(s,t) \doteq \Delta_{L(P_J)}(s,t) \doteq \Delta_J(s),$$

where $\doteq$ *denotes equality up to units in* $\mathbb{Z}[s^{\pm 1}, t^{\pm 1}]$*.*

Proof. See e.g. [FK08, Proposition 5.1].   □

After this detour on Alexander polynomials, we proceed by calculating the self-dual based torsion of a satellite. Fix a 3–manifold Y^3, a class $x \in [S^1, Y]$ and pick a knot K representing x. Also pick a framing $\psi \colon S^1 \times D^2 \hookrightarrow Y$ of K. Note that the complement $Y \setminus \nu P(K)$ is glued from two pieces along a 2–torus

$$Y \setminus \nu P(K) = Y \setminus \psi(S^1 \times D^2) \cup \psi\left(S^1 \times D^2 \setminus \nu P\right) \cong Y \setminus \nu K \cup_{T^2} \left(S^1 \times D^2 \setminus \nu P\right).$$

The glueing formula for torsion allows us to express the torsion of the satellite in terms of the torsion of K and the torsion of the pattern P.

Proposition 4.7. *Let* Y *be a closed oriented 3–manifold. Let* P *be a pattern with winding number* 1 *and let*

$$\varphi \colon \pi_1(Y \setminus P(K)) \to \mathbb{Q}(F)$$

be a sign-twisted representation. Suppose that $\varphi \not\equiv 1$ *and that it is non-trivial when restricted to* $\partial \nu P(K)$*. Then*

$$\tau^\varphi(Y \setminus \nu P(K)) = \tau^\varphi(Y \setminus \nu K) \cdot \tau^\varphi(S^1 \times D^2 \setminus \nu P) \in \mathbb{Q}(F)^\times / N(F).$$

Proof. Using coefficients determined by φ, consider the short exact Mayer–Vietoris sequence of chain groups of $\mathbb{Q}(F)$–vector spaces

$$0 \to C_*(T^2) \to C_*(Y \setminus \nu K) \oplus C_*(S^1 \times D^2 \setminus \nu P) \to C_*(Y \setminus \nu P(K)) \to 0,$$

and choose cell bases for the chain groups. Furthermore, choose bases $\mathcal{B}'''$, $\mathcal{B}$, $\mathcal{B}'$, $\mathcal{B}''$ for the respective homologies of these chain complexes reading left to right.

As $\varphi \not\equiv 1$ and it is non-trivial when restricted to T^2, we obtain that $H_*(T^2; \varphi) = 0$ and $\tau^\varphi(T^2) = 1$ [Tur01, Lemma II.11.11], and so $\mathcal{B}'''$ is empty. From [CF13, Theorem 2.2(4)], we deduce that

$$\tau^\varphi(Y \setminus \nu P(K); \mathcal{B}'') = \tau^\varphi(Y \setminus \nu K; \mathcal{B}) \cdot \tau^\varphi(S^1 \times D^2 \setminus P; \mathcal{B}') \cdot \tau(H) \in \mathbb{Q}(H)^\times / N(F),$$

where H is the Mayer–Vietoris sequence in homology, with coefficients determined by φ. Here H is thought of as an acyclic chain complex and $\tau(H)$ is calculated using the chain basis determined by $\mathcal{B}, \mathcal{B}', \mathcal{B}''$.

Claim. The homology $H_*(S^1 \times D^2 \setminus P; \varphi) = 0$.

As P has winding number 1, the inclusion $\nu P \hookrightarrow S^1 \times D^2$ is a homotopy equivalence. So the Mayer–Vietoris sequence of $S^1 \times D^2 = \nu P \cup_{\partial \nu P} (S^1 \times D^2 \setminus \nu P)$ with (φ)–coefficients determines isomorphisms $H_*(\partial \nu P; \varphi) \cong H_*(S^1 \times D^2 \setminus \nu P; \varphi)$. By assumption we have $\varphi \not\equiv 1$, and considering the composition

$$H_1(\partial \nu P; \mathbb{Z}) \twoheadrightarrow H_1(\nu P; \mathbb{Z}) \xrightarrow{\cong} H_1(S^1 \times D^2; \mathbb{Z})$$

which is induced by inclusions, we also see that φ restricts to a non-trivial representation on $\partial \nu P$. As $\partial \nu P$ is a 2–torus, this implies that $H_*(\partial \nu P; \varphi) = 0$ [Tur01, Lemma II.11.11] and the claim follows.

We have seen that the Mayer–Vietoris sequence H is non-zero only in degree $r = 1, 2$ and so consists of based isomorphisms $A_r \colon H_r(Y \setminus \psi(S^1 \times D^2); \varphi) \xrightarrow{\cong} H_r(Y \setminus \nu P(K); \varphi)$. The proposition now follows from the next claim.

Claim. The torsion $\tau(H) \in N(F)$.

By definition we have that the torsion equals $\tau(H) = \det(A_1) \cdot \det(A_2)^{-1}$. But consider the commutative diagram, where we use the Poincaré duality isomorphisms already observed in Equation (†):

$$
\begin{array}{ccc}
H_2(Y \setminus \psi(S^1 \times D^2); \varphi) & \xrightarrow[\cong]{A_2} & H_2(Y \setminus \nu P(K); \varphi) \\
\cong \downarrow {\scriptstyle \mathrm{Hom}_Q(-^t, Q)} & & \cong \downarrow {\scriptstyle \mathrm{Hom}_Q(-^t, Q)} \\
(H_2(Y \setminus \psi(S^1 \times D^2); \varphi)^t)^\vee & \xleftarrow[\cong]{(A_2)^\vee} & (H_2(Y \setminus \nu P(K); \varphi)^t)^\vee \\
\cong \uparrow {\scriptstyle \mathrm{PD}} & & \cong \uparrow {\scriptstyle \mathrm{PD}} \\
H_1(Y \setminus \psi(S^1 \times D^2); \varphi) & \xrightarrow[\cong]{A_1} & H_1(Y \setminus \nu P(K); \varphi)
\end{array}
$$

As $\mathcal{B}$ and $\mathcal{B}''$ are each self-dual bases, the Poincaré duality arrows are given by the identity matrix in this basis. The matrix for $(A_2)^\vee$ is the transpose dual matrix $\overline{A_2{}^t}$, so from the bottom square we deduce that $\overline{A_2{}^t} A_1 = \mathrm{id}$, whence $\det(A_2)^{-1} = \det(A_2{}^t)^{-1} = \overline{\det(A_1)}$, and so $\tau(H)$ is a norm as required. This completes the proof of the claim and therefore of the proposition. $\qquad\square$

We can express the factor $\tau^\varphi(S^1 \times D^2 \setminus \nu P)$ in terms of the Alexander polynomial of the link $L(P)$ introduced earlier in this section.

Proposition 4.8. *Let P be a pattern with winding number 1 and a sign-twisted representation $\varphi \colon \pi_1(S^1 \times D^2 \setminus \nu P) \to \mathbb{Q}(F)$ with associated map $h \colon H_1(S^1 \times D^2 \setminus \nu P; \mathbb{Z}) \to \mathbb{Q}(F)$. Then*

$$\tau^\varphi(S^1 \times D^2 \setminus \nu P) = \Delta_{L(P)}(h(s), h(t)) \in \mathbb{Q}(F)/N(F),$$

where s is the meridian of the pattern and t is the longitude of $S^1 \times D^2$.

Proof. The space $S^1 \times D^2 \setminus \nu P$ is homeomorphic to the exterior of the 2–component link L in S^3, consisting of the embedded pattern P in S^3 together with an embedded loop $\{\mathrm{pt}\} \times \partial D^2$ for any choice $\mathrm{pt} \in S^1$. Note that $H_1(S^1 \times D^2 \setminus \nu P; \mathbb{Z}) =: H$ is free abelian. We already showed in the proof of Proposition 4.7 that $H_*(S^1 \times D^2 \setminus P; \varphi) = 0$, so in fact this torsion can be calculated using torsion results in the acyclic chain complex setting. We write $\psi \colon \pi_1(S^1 \times D^2 \setminus \nu P) \to H \subset \mathbb{Q}(H)^\times$ for the abelianisation map.

The torsion can be expressed in terms of generators of the order ideals [Tur01, Theorem 4.7] as follows

$$\tau^\psi(S^1 \times D^2 \setminus \nu P) = \prod_{i=0}^{2} \mathrm{ord}(H_i(S^3 \setminus L; \psi))^{(-1)^{i+1}} \in \mathbb{Q}(H)^\times / N(H).$$

(Strictly speaking this equality only holds if the right-hand side is non-zero, but we will see in a few lines that this is the case.) But as $S^3 \setminus \nu L$ is a 3–manifold with non-empty boundary and $\mathrm{rk}\, H > 1$ we conclude that $\mathrm{ord}(H_0(S^3 \setminus L; \psi)) = \mathrm{ord}(H_2(S^3 \setminus L; \psi)) = 1$ [FV11, Proposition 3.2 (5) and 3.2 (6)]. We have

$$\tau^\psi(S^1 \times D^2 \setminus \nu P) = \mathrm{ord}(H_1(S^3 \setminus L; \psi)) = \Delta_{L(P)}(s, t) \in \mathbb{Q}(H)^\times.$$

But h induces a map $h : \mathbb{Q}(H) \to \mathbb{Q}(F)$ and under this map $h(\tau^\psi(S^1 \times D^2 \setminus \nu P)) = \tau^\varphi(S^1 \times D^2 \setminus \nu P)$ [Tur01, Proposition I.3.6]. Hence we have

$$\tau^\varphi(S^1 \times D^2 \setminus \nu P) = h(\Delta_{L(P)}(s, t)) = \Delta_{L(P)}(h(s), h(t)) \in \mathbb{Q}(F)^\times / N(F).$$

By the Torres condition [Hil12, Section 5.1], $\Delta_{L(P)}(1, 1)$ is equal to the linking number of $L(P)$, so in particular the right-hand side is non-zero. Now the proof is complete. $\qquad\square$

5. Topological almost-concordance invariants

In this section, we describe how self-dual based torsion gives rise to an almost-concordance invariant. In the 3–sphere S^3, the meridian of a knot K always defines a non-torsion class in $H_1(S^3 \setminus \nu K; \mathbb{Z})$. As we will see in the following proposition, in a general 3–manifold Y, this might not be the case.

Proposition 5.1. *Let Y be a closed oriented 3–manifold. Let $x \in [S^1, Y]$ be a free homotopy class. Let K be a knot in the homotopy class x. Suppose that $[x]$ has infinite order in $H_1(Y; \mathbb{Z})$. Then the following holds:*

(1) *The meridian μ of K represents a torsion element in $H_1(Y \setminus \nu K; \mathbb{Z})$.*
(2) *If $[x] \in H_1(Y; \mathbb{Z})$ equals pa for some prime p and $a \in H_1(Y; \mathbb{Z})$, then the meridian represents a non-zero element in $H_1(Y \setminus \nu K; \mathbb{Z}_p)$.*

This proposition is surely well-known to the experts, but we include a proof for the convenience of the reader.

Proof. Let $[x] \neq 0 \in H_1(Y; \mathbb{Z})$ be of the form $[x] = n \cdot u$ where u is a primitive element of $H_1(Y; \mathbb{Z})$ of infinite order. Let K be a knot representing x. Let μ be its meridian and pick a longitude λ. By a slight abuse of notation, we denote the corresponding elements in the various homology groups by the same symbol. In the following we identify the boundary torus of $Y \setminus \nu K$ with the product $\mu \times \lambda$.

We first consider the Mayer–Vietoris sequence with $\mathbb{Z}$–coefficients

$$\ldots \to H_1(\mu \times \lambda; \mathbb{Z}) \to H_1(Y \setminus \nu K; \mathbb{Z}) \oplus H_1(\nu K; \mathbb{Z}) \to H_1(Y; \mathbb{Z}) \to 0.$$

Since λ has infinite order in $H_1(Y; \mathbb{Z})$, it also has infinite order in $H_1(Y \setminus \nu K; \mathbb{Z})$. Also note that $\mu = 0$ in $H_1(Y; \mathbb{Z})$ since the meridian bounds a disc in Y.

Recall that the half-live-half-die lemma [Lic97, Lemma 8.15] says that for any orientable 3–manifold Z the kernel of $H_1(\partial Z) \to H_1(Z)$ has rank one-half the first betti number of ∂Z. From this lemma it follows that $H_1(\mu \times \lambda; \mathbb{Z}) \to H_1(Y \setminus \nu K; \mathbb{Z})$ has a kernel of rank one. Therefore the kernel is generated by an element of the form $a[\mu] + b[\lambda]$ with $(a, b) \neq (0, 0)$. Since $\mu = 0$ and $\lambda \neq 0$ in $H_1(Y; \mathbb{Z})$ we have $b = 0$. Thus we have shown that μ is torsion in $H_1(Y \setminus \nu K)$. This completes the proof of (1).

Now, to prove (2), suppose that p is a prime number such that $[x] = pa \in H_1(Y; \mathbb{Z})$ for some $a \in H_1(Y; \mathbb{Z})$. We consider the same Mayer–Vietoris sequence as above, but now with coefficients in $\mathbb{Z}/p\mathbb{Z} =: \mathbb{Z}_p$. We obtain:

$$H_2(Y; \mathbb{Z}_p) \to H_1(\mu \times \lambda; \mathbb{Z}_p) \to H_1(Y \setminus \nu K; \mathbb{Z}_p) \oplus H_1(\nu K; \mathbb{Z}_p) \to H_1(Y; \mathbb{Z}_p) \to 0.$$

Since $H_1(\nu K; \mathbb{Z}_p) = H_1(\lambda; \mathbb{Z}_p)$, this sequence simplifies to

$$H_2(Y; \mathbb{Z}_p) \to \mathbb{Z}_p\langle \mu \rangle \to H_1(Y \setminus \nu K; \mathbb{Z}_p) \to H_1(Y; \mathbb{Z}_p) \to 0,$$

where we recall that $\mathbb{Z}_p\langle\mu\rangle$ denotes the free $\mathbb{Z}_p$–module generated by μ. By our hypothesis, $\lambda = 0 \in H_1(Y;\mathbb{Z}_p)$. By the exactness of the sequence, there exists a $k \in \mathbb{Z}_p$ such that $\lambda' := k \cdot \mu + \lambda$ is zero in $H_1(Y \setminus \nu K;\mathbb{Z}_p)$. Note that μ and λ' also form a basis for $H_1(\mu \times \lambda;\mathbb{Z}_p)$. Since $\lambda' = 0$ in $H_1(Y \setminus \nu K;\mathbb{Z}_p)$, it follows from the aforementioned half-live-half-die lemma that $\mathbb{Z}_p\langle\mu\rangle \to H_1(Y \setminus \nu K;\mathbb{Z}_p)$ is injective, hence μ is a non-zero element in $H_1(Y \setminus \nu K;\mathbb{Z}_p)$. $\square$

For an abelian group H, let FH denote the maximal free abelian quotient of H. We will always view FH as a multiplicative group. We consider the following knot invariant.

Definition 5.2. Let K be an oriented knot in a 3–manifold Y.
   (1) A homomorphism $\rho\colon H_1(Y \setminus \nu K;\mathbb{Z}/2\mathbb{Z}) \to \{\pm 1\}$ which is non-trivial on the meridian is called a *meridional character*. Denote the set of meridional characters by $\mathfrak{C}(K)$.
   (2) Abbreviate $F := FH_1(Y;\mathbb{Z})$. For a knot K consider the representation

$$\alpha\colon H_1(Y \setminus K;\mathbb{Z}) \xrightarrow{i} H_1(Y;\mathbb{Z}) \to F,$$

which is induced by the inclusion i. Let $\rho\colon H_1(Y \setminus \nu K;\mathbb{Z}/2\mathbb{Z}) \to \{\pm 1\}$ be a meridional character. Define the *self-dual torsion* of K to be

$$\tau_\rho(K) := \tau^{\alpha\otimes\rho}(Y \setminus \nu K) \in \mathbb{Q}(F)/N(F).$$

Proposition 5.3. *Let Y be a closed oriented 3–manifold. Let K_0, K_1 be two concordant knots in Y. Then there exists an isomorphism $\varphi\colon H_1(Y \setminus \nu K_1;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\cong} H_1(Y \setminus \nu K_0;\mathbb{Z}/2\mathbb{Z})$ which sends the meridian of K_1 to the meridian of K_0 such that for any homomorphism $\rho\colon H_1(Y \setminus \nu K_0;\mathbb{Z}/2\mathbb{Z}) \to \{\pm 1\}$ the equality below holds:*

$$\tau_\rho(K_0) = \tau_{\rho\circ\varphi}(K_1) \in \mathbb{Q}(F)^\times/N(F).$$

Proof. Let $K_j \subset Y$, $j = 0, 1$, be two concordant knots and let $A \subset Y \times I$ be an annulus witnessing this. For any concordance A between K_0 and K_1, a Mayer–Vietoris argument shows that the inclusion induced map $H_1(Y \setminus \nu K_j;\mathbb{Z}) \xrightarrow{\cong} H_1(Y \times [0,1] \setminus \nu A;\mathbb{Z})$ is an isomorphism.

Denote $Y \times [0,1] \setminus \nu A$ by W. We obtain the following commutative diagram

where φ is defined by composition of the given two vertical isomorphisms. Note that the two inclusion maps send the meridian of the knot to the meridian of the annulus, in particular φ sends the meridian of K_1 to the meridian of K_0.

The representation $\alpha \otimes \rho\colon \pi_1(Y \setminus \nu K_0) \to \mathbb{Q}(F)$ can be extended to W as follows: define $\alpha_W\colon H_1(W;\mathbb{Z}) \to FH_1(Y \times I;\mathbb{Z}) = F$ induced by filling in the annulus A. With the isomorphism $H_1(Y \setminus \nu K_0;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\cong} H_1(W;\mathbb{Z}/2\mathbb{Z})$, we extend ρ over W to $\rho_W\colon H_1(W;\mathbb{Z}/2\mathbb{Z}) \to \{\pm 1\}$. By the diagram, it restricts to $\rho \circ \varphi$ on $Y \setminus \nu K_1$.

The boundary of W is $\partial W = Y \setminus \nu K_0 \cup_{T^2} Y \setminus \nu K_1$. Note that both inclusions $Y \setminus \nu K_i \subset W$ induce homology equivalences. As our representation ρ_W is to a 2–group, we may use [CF13, Lemma 3.3] to conclude that the (equivariant) intersection form of W, with $(\alpha_W \otimes \rho_W)$–coefficients vanishes (indeed, the underlying module

is trivial). This claim allows us to use [CF13, Theorem 2.4] to conclude that the torsion $\tau^{\alpha_W \otimes \rho_W}(\partial W) \in N(F)$ is contained in the norm subgroup.

Use the multiplicativity of Reidemeister torsion corresponding to decompositions of spaces and the fact that $\tau_\rho(T^2)$ vanishes as the representation ρ_W is non-trivial on the meridian of A [Tur01, Lemma 11.11], to obtain the equation:

$$\tau^{\alpha_W \otimes \rho_W}(\partial W) = \tau_\rho(Y \setminus \nu K_0) \cdot \tau_{\rho \circ \varphi}(Y \setminus \nu K_1).$$

Multiply the equation above with $\tau_{\rho \circ \varphi}(Y \setminus \nu K_1)^{-1} \overline{\tau_{\rho \circ \varphi}(Y \setminus \nu K_1)^{-1}}$. We then use the fact that $\tau_{\rho \circ \varphi}(Y \setminus \nu K_1) = \overline{\tau_{\rho \circ \varphi}(Y \setminus \nu K_1)}$ — [Tur01, Corollary II.14.2] and [CF13, Lemma 2.5] — to obtain the desired equality

$$\tau_\rho(Y \setminus \nu K_0) \; = \; \tau_{\rho \circ \varphi}(Y \setminus \nu K_1) \in \mathbb{Q}(F)/N(F).$$

$\square$

The self-dual torsion also behaves well with respect to local knotting. Recall that to a knot J in S^3, we can associate a well-framed winding number one pattern by removing a meridian such that $P_J(K) = K \# J$, see Proposition 3.1.

Proposition 5.4. *Let Y be a closed oriented 3–manifold. Denote $F := FH_1(Y; \mathbb{Z})$. Let K be a knot in Y such that $[K] \in H_1(Y; \mathbb{Z})$ is non-torsion and J a knot in S^3 with corresponding pattern P. Pick neighbourhoods $\nu P(K) \subset \nu K$. Let*

$$\rho \colon H_1(Y \setminus \nu(K \# J)) = H_1(Y \setminus \nu P(K); \mathbb{Z}/2\mathbb{Z}) \to \{\pm 1\}$$

be a homomorphism which is non-trivial on the meridian. Denote

$$\rho' \colon H_1(Y \setminus \nu K; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\cong} H_1(Y \setminus \nu P(K); \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\rho} \mathbb{Z}/2\mathbb{Z},$$

which is induced by the inclusion. Then

$$\tau_\rho(K \# J) \; = \; \tau_{\rho'}(K) \in \mathbb{Q}(F)^\times/N(F).$$

Proof. The inclusion induced map $H_1(Y \setminus \nu K; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\cong} H_1(Y \setminus \nu P(K); \mathbb{Z}/2\mathbb{Z})$ is an isomorphism as $\nu K \setminus \nu P(K)$ is a homology bordism, as shown in Lemma 3.2.

Let $\alpha \colon H_1(Y \setminus \nu P(K); \mathbb{Z}) \to F$ be the homomorphism induced by the inclusion. Let s be the meridian of $P(K)$. We compute

$$\begin{aligned} \tau_\rho(Y \setminus \nu P(K)) &= \tau_{\rho'}(Y \setminus \nu K) \cdot \tau_\rho(\nu K \setminus \nu P(K)) \\ &= \tau^{\alpha \otimes \rho'}(Y \setminus \nu K) \cdot \Delta_J((\alpha \otimes \rho)(s)) \in \mathbb{Q}(F)^\times/N(F), \end{aligned}$$

where the first equality follows from Proposition 4.7 and the second from Lemma 4.6. We make the following observations:

(1) The character ρ takes values in $\pm 1 \in \mathbb{Q}$ and the map α takes values in F, thus $\alpha \otimes (\rho' \circ \varphi)(s)$ is of the form $\pm f$ with $f \in F$.

(2) By Lemma 3.3, the homology class of the meridian s in $H_1(Y \setminus \nu P(K); \mathbb{Z})$ equals to the one of the meridian of K. We had assumed that $[K]$ has infinite order in homology. It follows from Proposition 5.1 that $\alpha(s)$ is trivial in F. In particular, we see that $(\alpha \otimes \rho)(s) = \pm 1$. It is well-known that for the Alexander polynomial of a knot J, the integer $\Delta_J(\pm 1)$ is odd, in particular non-zero.

Summarising, $\Delta_J(\alpha \otimes \rho)(s) \in \mathbb{Q}^\times \subset N(F)$. This concludes the proof of the proposition. $\square$

Corollary 5.5. *Let K be a knot in a closed, oriented 3–manifold Y such that $[K] \in H_1(Y; \mathbb{Z})$ is non-torsion. Let $\mathfrak{C}(K)$ be the set of homomorphisms $H_1(Y \setminus \nu K; \mathbb{Z}/2\mathbb{Z}) \to \{\pm 1\}$ which are non-trivial on the meridian. Then the set of self-dual torsions*

$$I_K = \{\tau_\rho(K) \mid \rho \in \mathfrak{C}(K)\} \subset \mathbb{Q}(F)/N(F)$$

is an almost-concordance invariant.

6. CHANGING THE ALMOST-CONCORDANCE CLASS USING SATELLITES

In this section we apply our almost-concordance invariants; using a satellite construction to modify certain knots within their free homotopy classes, we will produce infinite families of examples that serve to confirm Conjecture 1.3 in many cases. For our particular satellite constructions, we will make use of the patterns P_n, $n \geq 1$, with winding number one, shown in Figure 1.
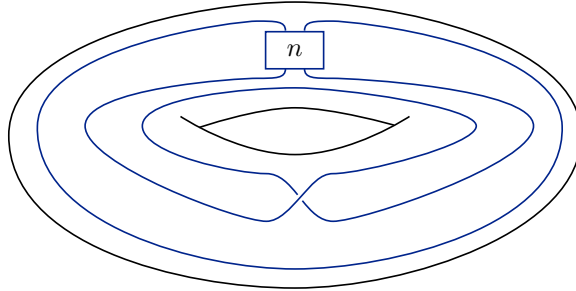

FIGURE 1. The Mazur patterns P_n, where the n–box denotes n full right-handed twists.

Following Cooper [Coo82] and Cimasoni–Florens [CF08], we compute the multivariable Alexander polynomials of P_n using C–complexes. Recall that a C–complex for an m–component link L consists of a choice of Seifert surface F_j for each link component L_j, where the surfaces are allowed to intersect one another, but only in clasp singularities as depicted in Figure 2. In particular, there are no triple
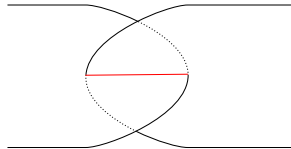

FIGURE 2. Clasp singularities between two surfaces.

intersections. The Seifert form of the C–complex is defined to have underlying $\mathbb{Z}$–module $H_1(F; \mathbb{Z})$, where F is the union of the surfaces F_j. To describe the Seifert pairing on this module, we will first pick a normal direction for each component F_j. The ways an embedded curve in F can be pushed into the complement $S^3 \setminus F$ are then encoded as a choice of function $\varepsilon \colon \{1, \ldots, N\} \to \{0, 1\}$, where $\varepsilon(j) = 0$ and $\varepsilon(j) = 1$ stand for negative and positive push-offs respectively from the component F_j. Denote the resulting push-off for an embedded curve $x \subset F$, by $i_\varepsilon x \subset S^3 \setminus F$. Now define a pairing on $H_1(F; \mathbb{Z})$ via the formula

$$\beta(x, y) := \sum_\varepsilon (-1)^{|\varepsilon|} \operatorname{lk}(i_\varepsilon x, y) X^\varepsilon \in \mathbb{Z}[X_1, \ldots, X_N],$$

where $|\varepsilon| := \sum_j \varepsilon(j)$ and $X^\varepsilon := \prod_j X_j^{\varepsilon(j)}$.

The complement of a standard solid torus in S^3 is a neighbourhood of an unknot c, so when considering the 2–component link $L(P)$ in S^3 arising from a winding number 1 pattern $P \subset S^1 \times D^2$ and this unknot c, we always write s for the meridian of P and t for the meridian of c, see Section 4.2.

Proposition 6.1. *For $n \geq 1$, the Mazur pattern P_n has multivariable Alexander polynomial*

$$\Delta_{L(P_n)}(s,t) = n(s^2t + st^2 - s^2 - t^2 + s + t) - (2n-1)st \in \mathbb{Z}[s^{\pm 1}, t^{\pm 1}].$$

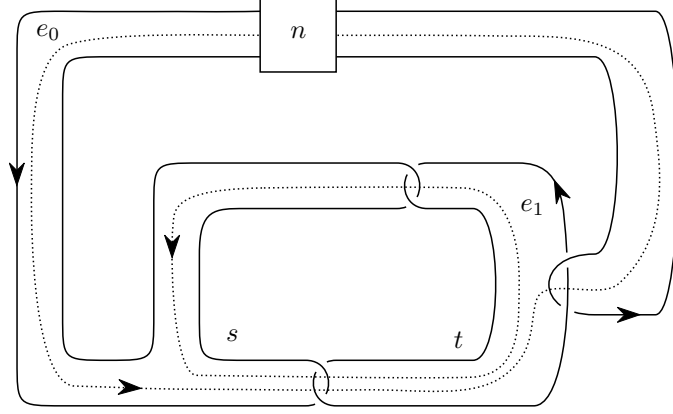*Proof.* Consider the C–complex with the generators of $H_1(F; \mathbb{Z})$ as sketched in Figure 3.

FIGURE 3. Diagram for C–complex of the Mazur pattern P_n.

For the self-intersection, we read off the contributions and obtain:

$$\beta(e_0, e_0) = n(1-t)(1-s) + \left(-\tfrac{1}{2}\right)(s+t) + \tfrac{1}{2}(s+t)$$
$$= n(1-t)(1-s),$$

and

$$\beta(e_1, e_1) = \left(-\tfrac{1}{2}\right)(s+t) + \left(-\tfrac{1}{2}\right)(s+t)$$
$$= -(s+t).$$

The value of $\beta(e_0, e_1)$ can be computed as follows

$$\beta(e_0, e_1) = \tfrac{1}{2}(1-s) - \tfrac{1}{2}(s+t) + \left(-\tfrac{1}{2}\right)(1-t) = -s.$$

Consequently, we also obtain $\beta(e_1, e_0) = -t$. We can compute the multivariable Alexander polynomial $\Delta_{L(P_n)}(s,t)$ in the ring $\mathbb{Z}[s^{\pm 1}, t^{\pm 1}, (1-s)^{-1}, (1-t)^{-1}]$ as the determinant [CF08, Corollary 3.6]

$$\det \begin{pmatrix} n(1-t)(1-s) & -s \\ -t & -(s+t) \end{pmatrix} = -n(s^2t + st^2 - s^2 - t^2 + s + t) + (2n-1)st.$$

We must now show that the equality holds moreover in $\mathbb{Z}[s^{\pm 1}, t^{\pm 1}]$. The link $L(P_n) \subset S^3$ has two components and we denote the component corresponding to P_n with $\overline{P}$. From Lemma 4.6, we deduce that $|\Delta_{L(P_n)}(1,1)| = |\Delta_{\overline{P}}(1)| = 1$. As also

$$|-n(1+1-1-1+1+1) + (2n-1)| = 1,$$

we obtain that in $\mathbb{Z}[s^{\pm 1}, t^{\pm 1}]$ we have

$$\Delta_{L(P_n)}(s,t) = -n(s^2t + st^2 - s^2 - t^2 + s + t) + (2n-1)st.$$

$\square$

For F a finitely generated torsion-free abelian group, we call two non-zero $g, h \in \mathbb{Z}[F]$ *associates* if $g = f \cdot h$ for a unit $f \in \mathbb{Z}[F]^\times$.

Lemma 6.2. *Let F be a finitely generated torsion-free abelian group and let u be a primitive element in F. Then for any prime $p \neq 2$ the polynomial $\Delta_{P_p}(-1, u^2)$ is irreducible in $\mathbb{Z}[F]$. Furthermore for two different primes p and q, the elements $\Delta_{P_p}(-1, u^2)$ and $\Delta_{P_q}(-1, u^2)$ in $\mathbb{Z}[F]$ are non-associates.*

Proof. We can extend $\{u\}$ to a basis $\{u, v_1, \ldots, v_n\}$ for the torsion-free abelian group F. As before we use multiplicative notation for F. By Proposition 6.1 we have
$$\Delta_{P_p}(s, t) \;=\; p(s^2 t + s t^2 - s^2 - t^2 + s + t) - (2p - 1)st.$$
Thus we have
$$\Delta_{P_p}(-1, u^2) \;=\; p(u^2 - u^4 - 1 - u^4 - 1 + u^2) + (2p-1)u^2 = -2p(u^4 + 1) + (4p - 1)u^2.$$
Since units $\mathbb{Z}[F]^\times$ are of the form $\pm f$ for $f \in F$, we obtain that for two different primes p and q the elements $\Delta_{P_p}(-1, u^2)$ and $\Delta_{P_q}(-1, u^2)$ in $\mathbb{Z}[F]$ are non-associates.

Now we show that $g(u) := -2p(u^4 + 1) + (4p - 1)u^2$ is irreducible in $\mathbb{Z}[u^{\pm 1}]$. The reader may verify that the polynomial $g(u)$ has no real roots. In particular, it cannot have an irreducible factor over $\mathbb{Z}[u^{\pm 1}]$ of degree 1. Suppose, for a contradiction, that $g(u) = r_1(u) \cdot r_2(u)$ for polynomials $r_i(u) = a_i u^2 + b_i u + c_i$, $i = 1, 2$.

Claim. We have $a_i = \varepsilon c_i$, for $i = 1, 2$ and some $\varepsilon \in \{\pm 1\}$.

To see the claim, multiply out the product $r_1(u) \cdot r_2(u)$ and compare coefficients with $g(u)$. We obtain $a_1 a_2 = -2p = c_1 c_2$ from the coefficients of u^4 and 1. Observe that this implies that none of the a_i nor the c_i can be zero. We also obtain $b_1 a_2 + a_1 b_2 = 0 = b_1 c_2 + b_2 c_1$, from the coefficients of u and u^3, so that either $b_1 = 0$ or $a_2/a_1 = -b_2/b_1 = c_2/c_1$. If $b_1 = 0$, then since $a_1 \neq 0$, we also have $b_2 = 0$. But if $b_1 = b_2 = 0$, then from comparing coefficients of u^2 we have $a_2 c_1 + a_1 c_2 = 4p - 1$, and since p is prime one can check the possibilities for the a_i and the c_i that satisfy all of $a_2 c_1 + a_1 c_2 = 4p - 1$, $a_1 a_2 = -2p$ and $c_1 c_2 = -2p$, to see that this set of equations has no solutions. Thus $b_1 \neq 0$, and so we have the equations $a_1 a_2 = c_1 c_2$ and $a_2/a_1 = c_2/c_1$. Rearrange the first equation to yield $a_1/c_1 = c_2/a_2$. Rearrange the second equation and substitute, to give $a_2 = c_2 a_1/c_1 = c_2 c_2/a_2$, so $c_2^2 = a_2^2$. Thus $c_2 = \varepsilon a_2$ for some $\varepsilon \in \{\pm 1\}$. It follows that similarly we have $c_1 = \varepsilon a_1$. This completes the proof of the claim.

By the claim, we may rearrange $r_i = a_i(u^2 + \varepsilon) + b_i u$. Considering that $a_1 a_2 = -2p$, we will assume (after relabelling) that p divides a_1, and as $p \neq 2$, this means p does not divide a_2. The coefficient of u^3 in g is $a_1 b_2 + a_2 b_1 = 0$ and hence p divides b_1. This is now a contradiction as we have p dividing r_1 and hence g, but on the other hand the residue of g in $\mathbb{Z}_p[u]$ is u^2. We conclude from this that $g(u)$ is irreducible in $\mathbb{Z}[u]$. Hence also in $\mathbb{Z}[u^{\pm 1}]$.

It is elementary to show that this implies that $g(u)$ is also irreducible in $\mathbb{Z}[F] = \mathbb{Z}[u^{\pm 1}][v_1^{\pm 1}, \ldots, v_n^{\pm 1}]$. To wit, given R a unique factorisation domain, and some irreducible $q \in R$, to show q is also irreducible over $R[x^{\pm 1}]$ suppose that $q = r(x) \cdot s(x)$. Then by looking at the degrees we see that $r(x)$ and $s(x)$ are of the form ax^n, bx^{-n}, respectively, for some $a, b \in R$. But then $q = a \cdot b \in R$ and so one of a, b is a unit in R, therefore one of r, s is a unit in $R[x^{\pm 1}]$.   □

Recall the parity homomorphism $\Phi_g \colon \mathbb{Q}(F)^\times/N(F) \to \mathbb{Z}/2\mathbb{Z}$ for an irreducible, symmetric and non-constant polynomial $g \in \mathbb{Z}[F]$ from Definition 4.4.

Theorem 6.3. *Fix (Y, x) and let K be a knot representing x. Suppose that $[x] = 2u \in H_1(Y; \mathbb{Z})$, for some primitive homology class u of infinite order. Then there exists an infinite set $I \subset \mathbb{N}$ such that for any $i \neq j$ in I the knots $P_i(K)$ and $P_j(K)$ belong to distinct almost-concordance classes within the set $\mathcal{C}_x(Y)$.*

Proof. Fix a framing $\psi\colon S^1 \times D^2 \hookrightarrow Y$ of K, and define the knots $P_i(K)$, $i \in \mathbb{N}$. Write $F := FH_1(Y;\mathbb{Z})$, and let $\alpha\colon H_1(Y \setminus \nu K;\mathbb{Z}) \to F$ be the inclusion induced map. Define

$$G := \prod_{\rho \in \mathfrak{C}(K)} \tau^{\alpha \otimes \rho}(Y \setminus \nu K) \in \mathbb{Z}[F],$$

where we take the product over all meridional representations $H_1(Y \setminus \nu K;\mathbb{Z}/2\mathbb{Z}) \to \{\pm 1\}$ of K. Note that there are only finitely many such representations and hence this is a finite product. Define

$$I := \{\text{odd primes } p \,|\, \Delta_{P_p}(-1, u^2) \in \mathbb{Z}[F] \text{ does not divide } G \in \mathbb{Z}[F]\}.$$

Note that I is an infinite set since the $\Delta_{P_p}(-1, u^2)$ in $\mathbb{Z}[F]$ are pairwise non-associates. Given $i \in I$, we write $g_i := \Delta_{P_i}(-1, u^2)$.

Recall that the set $I_n := \{\tau_\rho(P_n(K)) \,|\, \rho \in \mathfrak{C}(K)\}$ is an almost-concordance invariant, see Corollary 5.5. As a result the theorem immediately follows from the next claim.

Claim. For $n, m \in I$ the sets I_n and I_m only intersect if $n = m$.

To prove the claim, consider an element $\tau_\rho(P_n(K)) \in I_n$. We compute $\tau_\rho(P_n(K))$ in terms of $\tau_\rho(P_n(K))$ and the Alexander polynomial Δ_{P_n}. By Lemma 3.3 and since $[P_n(K)] = [K] = u^2 \in F$, we can use Mayer–Vietoris for torsion to compute the following equalities in $\mathbb{Q}(F)^\times / N(F)$:

$$\tau_\rho(P_n(K)) = \tau_\rho(K) \cdot \Delta_{P_n}(-1, u^2)$$

Note that the parity homomorphism $\Phi_{g_m}\colon \mathbb{Q}(F)^\times / N(F) \to \mathbb{Z}/2\mathbb{Z}$, see Definition 4.4, vanishes on $\tau_\rho(K)$. Consequently, we obtain

$$\begin{aligned}
\Phi_{g_m}(\tau_\rho(P_n(K))) &= \Phi_{g_m}(\tau_\rho(K)) + \Phi_{g_m}(\Delta_{P_n}(-1, u^2)) \\
&= \Phi_{g_m}(\Delta_{P_n}(-1, u^2)) \\
&= \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}
\end{aligned}$$

where in the last line we used Lemma 6.2. This shows the claim, which completes the proof of the theorem. $\qquad\square$

## 7. The null-homotopic class for spherical space forms

In Section 8 we will investigate linking numbers in covering spaces, to give obstructions to almost-concordance. Before embarking on the general version, we give a more easily digestible version for a special case, as a warm up. The next result also follows fairly easily from Schneiderman's concordance invariant [Sch03].

Recall that a spherical space form is a compact 3-manifold Y with universal cover S^3. For example, lens spaces $L(r, s)$ are spherical space forms.

Proposition 7.1. *Let Y be a spherical space form and let x be the null-homotopic free homotopy class. Then if $Y \neq S^3$, the set $\mathcal{C}_x(Y)$ contains infinitely many almost-concordance classes.*

Proof. Choose a non-trivial element $g \in \pi_1(Y)$, and fix an embedded circle representing g. In a solid torus neighbourhood of g, insert the knot K_n shown in Figure 4. Note that K_n represents x in Y.

Let $p\colon S^3 \to Y$ be the universal cover and consider the link given by the preimage $L_n := p^{-1}(K_n) \subset S^3$. Linking number between components of L_n is not affected by local knotting of K_n in Y. Also, a concordance of K_n in $Y \times I$ lifts to a concordance of L_n in $S^3 \times I$, and linking number is a concordance invariant for
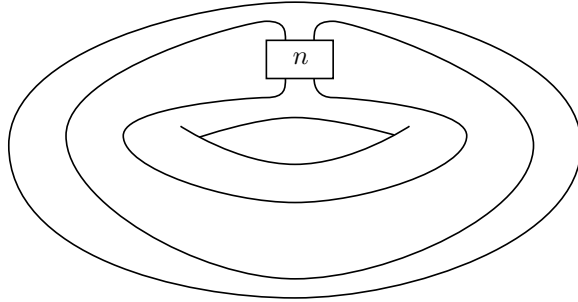
FIGURE 4. The knot K_n where the n–box represents n full twists.

links in S^3. Hence to show the K_n are pairwise non almost-concordant, it suffices to show that for $m \neq n$, the linking numbers between the components of L_n are different from those of L_m.

To see this, observe that the knot K_n bounds an immersed disc inside the solid torus, so is null-homotopic. Moreover we can lift this disc to see that each component of the covering link L_n bounds an embedded disc in S^3. The link L_n has $|\pi_1(Y)|$ components. If the order of g is not equal to two, then different components link with linking numbers either n or 0, as can be computed by lifting the aforementioned disc to S^3, and counting intersections of the other components of the lift with the lifted disc. Specifically, $\mathrm{lk}(\widetilde{K}_n, g^{\pm 1} \cdot \widetilde{K}_n) = n$, and $\mathrm{lk}(\widetilde{K}_n, h \cdot \widetilde{K}_n) = 0$ for $h \neq g, g^{-1}$. If the order of g is two, then the linking number between different lifts is either $2n$ or 0; similarly to the generic case, $\mathrm{lk}(\widetilde{K}_n, g \cdot \widetilde{K}_n) = 2n$, and $\mathrm{lk}(\widetilde{K}_n, h \cdot \widetilde{K}_n) = 0$ for $h \neq g$. In particular, the nonzero linking number is realised between at least two components of L_n. It follows that the set of pairwise linking numbers of L_n is different from the corresponding set for L_m, as required. $\square$

8. THE TORSION CASE WHEN Y IS NOT THE 3–SPHERE

In this section, let Y be a closed oriented 3–manifold, with $Y \neq S^3$, and $x \in [S^1, Y]$ be *torsion*, that is for any choice of basepoint and basing path, x is finite order in $\pi_1(Y)$. We will now construct a family of pairwise non almost-concordant knots in the torsion class x. As in Section 6, a satellite construction will be used to build the family, but now the almost-concordance invariant we use to distinguish the knots in the family will be based on the idea of covering links. Recall that, given a knot $K \subset Y$ and a finite covering space $p \colon \widetilde{Y} \to Y$, the associated *covering link* is the inverse image $p^{-1}(K)$.

Observe that for each $m > 0$, the local action from Definition 1.1 extends to an obvious action of the m–fold product $\mathcal{C} \times \cdots \times \mathcal{C}$ on the set of concordance classes of m–component links in a 3–manifold. We call the orbit of a link L under this action the *almost-concordance* class of L.

Lemma 8.1. *Let K, K' be two almost-concordant knots in Y and let $\widetilde{Y} \to Y$ be a finite cover. Then also the associated covering links are almost-concordant.*

Proof. Combine the fact that a concordance between K and K' lifts to a concordance in $\widetilde{Y} \times I$ between the covering links, and the fact that connected sum with local knots lifts to connected sums with local knots. $\square$

To take advantage of this observation, we study a notion of linking numbers for links in general 3–manifolds. Let N be a compact oriented 3–manifold. Later on, we will specialise this theory to $N = \widetilde{Y}$. For two knots $K, J \subset N$ representing torsion homology classes, we define their *linking number* $\mathrm{lk}(K, J)$ as follows: pick

a class $C \in H_2(N, \nu K; \mathbb{Q})$ with $\partial C = [K] \in H_1(\nu K; \mathbb{Q})$. The relative intersection pairing

$$H_2(N, \nu K; \mathbb{Q}) \times H_1(N \setminus \nu K; \mathbb{Q}) \to \mathbb{Q}$$

allows us to define $\mathrm{lk}(K, J) := C \cdot [J]$.

Claim. The number $\mathrm{lk}(K, J)$ is well-defined. That is, it does not depend on the choice of C.

Let C, C' be distinct choices. Then there exists a class a class $S \in H_2(N; \mathbb{Q})$ with $S = C - C' \in H_2(N, \nu K; \mathbb{Q})$. By the commutativity of the diagram

$$H_2(N, \nu K; \mathbb{Q}) \times H_1(N \setminus \nu K; \mathbb{Q})$$

$$H_2(N; \mathbb{Q}) \times H_1(N \setminus \nu K; \mathbb{Q}) \longrightarrow \mathbb{Q}$$

$$H_2(N; \mathbb{Q}) \times H_1(N; \mathbb{Q})$$

and the fact that $[J]$ gets sent to $0 \in H_1(N; \mathbb{Q})$, we deduce that $C \cdot [J] - C' \cdot [J] = S \cdot [J] = 0$, which completes the proof of the claim.

Lemma 8.2. *Let L, L' be two almost-concordant links in N, whose components are torsion in $H_1(N; \mathbb{Z})$. Then the linking numbers between the components of L agree with the ones between the components of L'.*

Proof. Let $L = L_1 \cup \cdots \cup L_m \subset N \times \{0\}$, let $L' = L_1 \cup \cdots L'_m \subset N \times \{1\}$, and let $A = A_1 \cup \cdots \cup A_n \subset N \times I$ be a concordance between L and L'.

Fix $i \neq j \in \{1, \ldots, m\}$. The annulus A_i determines a class in $H_2(N \times I, N \times \{0, 1\}; \mathbb{Q})$. Pick a class $C_j \in H_2(N \times \{0\}, L_j; \mathbb{Q})$ with $\partial C_j = [L_j] \in H_1(L_j; \mathbb{Q})$, and choose a class $C'_j \in H_2(N \times \{1\}, L'_j; \mathbb{Q})$ with $\partial C'_j = [L'_j] \in H_1(L'_j; \mathbb{Q})$. Lift C_j and C'_j to 2–chains in $C_2(N \times \{k\}; \mathbb{Q})$, for $k = 0, 1$ respectively. The sum of chains $D_j := C_j + A_j + C'_j$ represents a class in $H_2(N \times I; \mathbb{Q})$. Since $A_i \cap A_j = \varnothing$, we have that the intersection between D_j and A_j is contained in the boundary $N \times \{0, 1\}$, and so $[D_j] \cdot [A_i] = \mathrm{lk}(L_i, L_j) - \mathrm{lk}(L'_i, L'_j)$.

Next we argue that A_i is zero in $H_2(N \times I, N \times \partial I; \mathbb{Q})$. Consider the long exact sequence of a pair:

$$H_2(N \times \partial I; \mathbb{Q}) \to H_2(N \times I; \mathbb{Q}) \to H_2(N \times I, N \times \partial I; \mathbb{Q}) \to H_1(N \times \partial I; \mathbb{Q}).$$

The class $A_i \in H_2(N \times I, N \times \partial I; \mathbb{Q})$ lies in the image of $H_2(N \times I; \mathbb{Q})$. But since $N \times I$ is a product, the map

$$H_2(N \times \partial I; \mathbb{Q}) \cong H_2(N; \mathbb{Q}) \oplus H_2(N; \mathbb{Q}) \to H_2(N \times I; \mathbb{Q}) \cong H_2(N; \mathbb{Q})$$

is surjective, and therefore the map $H_2(N \times I; \mathbb{Q}) \to H_2(N \times I, N \times \partial I; \mathbb{Q})$ is the zero map. It follows that $A_i = 0 \in H_2(N \times I, N \times \partial I; \mathbb{Q})$ as desired.

The intersection pairing $\lambda \colon H_2(N \times I; \mathbb{Q}) \times H_2(N \times I, N \times \partial I; \mathbb{Q}) \to \mathbb{Q}$ is (by definition) adjoint to the composition of the two isomorphisms

$$H_2(N \times I; \mathbb{Q}) \xrightarrow{\cong} H^2(N \times I, N \times \partial I; \mathbb{Q}) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Q}}(H_2(N \times I, N \times \partial I; \mathbb{Q}), \mathbb{Q}),$$

given by Poincaré duality and the Universal Coefficient Theorem. Since such a pairing can be computed by geometric intersections, we compute $[D_j] \cdot [A_i]$ as $\lambda([D_j], [A_i]) = \lambda([D_j], 0) = 0$. Therefore $\mathrm{lk}(L_i, L_j) - \mathrm{lk}(L'_i, L'_j) = 0$ as required. $\square$

Theorem 8.3. *Let $Y \neq S^3$ be a closed, oriented 3–manifold. Let $x \in [S^1, Y]$ be torsion and denote the normal closure of x by $H := \langle\langle x \rangle\rangle \subset \pi_1(Y)$. If the quotient $\pi_1(Y)/H$ is non-trivial, then $\mathcal{C}_x(Y)$ contains infinitely many distinct almost-concordance classes.*

Proof. First we need the following fact, which uses general results about fundamental groups of 3–manifolds.

Claim. There exists an epimorphism $\varphi \colon \pi_1(Y)/H \twoheadrightarrow G$ to some non-trivial finite group G.

By the Prime Decomposition Theorem and the Geometrisation Theorem, $Y \cong N_1 \# \ldots \# N_r$ for some closed, oriented 3–manifolds N_i, where for all i, either $A_i := \pi_1(N_i)$ is finite or A_i is torsion free. See [AFW15, C.3 and §§1.2, 1.7] for details and references. As $x \in \pi_1(Y) \cong A_1 * \cdots * A_r$ is a torsion element, it must be conjugate to an element $a \in A_i$ for some i [Ser03, Cor. I.1.1]. By reordering, assume $i = 1$. There is now an obvious epimorphism

$$\Phi \colon A_1 * \cdots * A_r / \langle\langle x \rangle\rangle \twoheadrightarrow A_2 * \cdots * A_r.$$

For the case $r \neq 1$, consider that the codomain of Φ is the fundamental group of a 3–manifold and is therefore residually finite, by the Geometrisation Theorem. Hence the required epimorphism φ exists. For the case $r = 1$, we have that A_1 is finite and we may take φ as the identity map. This completes the proof of the claim.

Pick such an epimorphism $\varphi \colon \pi_1(Y)/H \twoheadrightarrow G$ and compose with the quotient map to obtain an epimorphism $\varphi \colon \pi_1(Y) \twoheadrightarrow G$, that vanishes on H. Pick a knot η with $\varphi(\eta) \neq 0$ nontrivial and another knot α representing x, and disjoint from η. Pick a genus 2 handlebody $B \subset Y$ whose cores are η and α. Let $K_n \subset B$ be the knot described in Figure 5. The homotopy class of K_n agrees with the one of α and therefore K_n represents the class x.
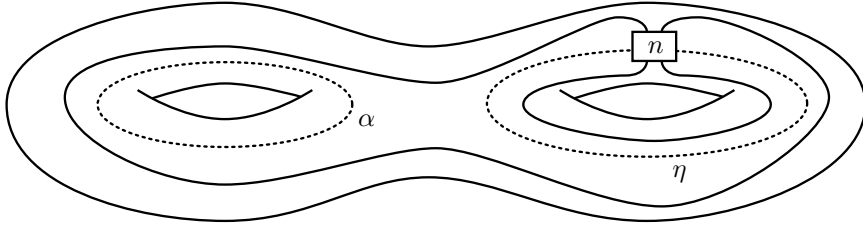

FIGURE 5. The knot K_n where the n–box represents n full twists.

We show that the set of knots $\{K_n \,|\, n > 1\}$ contains infinitely many pairwise not almost-concordant elements. Consider the finite cover $p \colon \widetilde{Y} \to Y$ associated to the kernel of $\varphi \colon \pi_1(Y) \to G$, and denote the covering link of K_n by L_n, that is $L_n := p^{-1}(K_n)$. As $\varphi(\alpha) = 0$, the restriction of φ to $\pi_1(B)$ is an epimorphism to the group generated by $\varphi(\eta)$ in G. As G is finite, and $\varphi(\eta) \neq 0$, so $\varphi(\eta)$ must generate a finite cyclic group C_k for $k > 1$. In other words, the cover of B induced by φ is determined by an epimorphism $\pi_1(B) \twoheadrightarrow C_k$, and thus we obtain a cover which in each component contains components of the link L_n as depicted in Figure 6. Let $S_n := \{\mathrm{lk}(C, D) \,|\, C, D \text{ components of } L_n\} \subset \mathbb{Z}$ be the set of linking numbers.

Claim. The maximal integer in the set S_n becomes arbitrarily large as $n \to \infty$.

Suppose that η does not map to a 2–torsion element in C_k. In the case that η maps to 2–torsion, the argument is similar, as in the proof of Proposition 7.1. Pick a
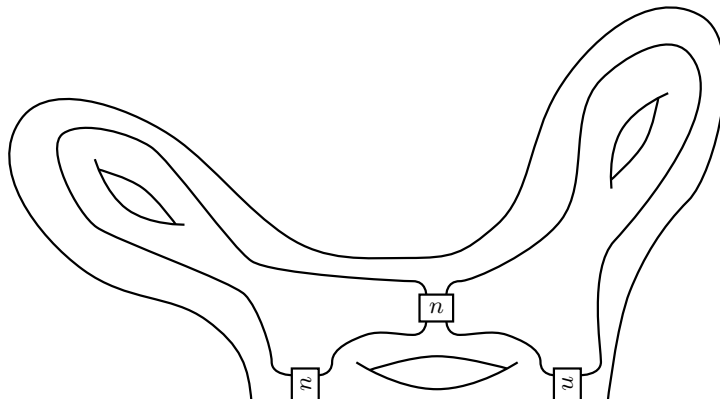
FIGURE 6. Part of a component of the preimage of the handle-body B, with some components of the link L_n.

connected component of the preimage of the handlebody B in $\widetilde{Y}$. Furthermore, pick two link components C, D of L_n in B, which are related by an n–twist box and hence a single n–twist box as η is not 2–torsion in C_k. Note that in the complement of D in Y, the homology class of D decomposes as $[D] = [D_{\text{dist}}] + [D_{\text{box}}] \in H_1(\widetilde{Y} \setminus \nu C; \mathbb{Q})$, where the curves D_{dist} and D_{box} are described in Figure 7. As D_{box} is contained
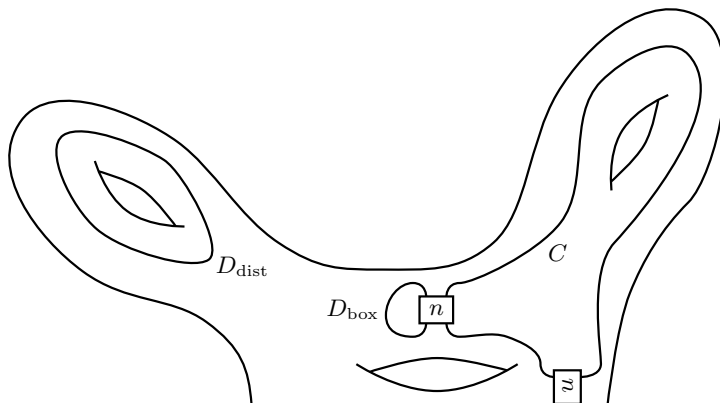

FIGURE 7. The two contributions to the linking number $\text{lk}(C, D)$.

in a 3–ball, we may compute the linking number $\text{lk}(C, D_{\text{box}}) = n$. Consequently, we get
$$\text{lk}(C, D) = \text{lk}(C, D_{\text{dist}}) + \text{lk}(C, D_{\text{box}}) = \text{lk}(C, D_{\text{dist}}) + n.$$
As the number $\text{lk}(C, D_{\text{dist}})$ is independent of n, this proves the claim.

By Lemma 8.2, the set S_n is an almost-concordance invariant and therefore the set $\{L_n \,|\, n > 1\}$ contains infinitely many distinct almost-concordance classes. By Lemma 8.1, so does the set $\{K_n \,|\, n > 1\}$. □

Corollary 8.4.

(1) *Let Y' be a spherical 3–manifold, and let $Y := Y' \# Z$ for some $Z \neq S^3$. Then any class in $x \in \pi_1(Y')$ is torsion, and since $\pi_1(Z) \neq 1$, we can apply the theorem to see that $\mathcal{C}_x(Y)$ contains infinitely many almost-concordance classes.*

(2) *For any 3–manifold $Y \neq S^3$, the null-homotopic class x contains infinitely many almost-concordance classes.*

The next theorem is not quite a corollary of Theorem 8.3 because the class x in question is not torsion in homotopy, however the same ideas as in that theorem also work in the following case.

Theorem 8.5. *Let Y be a closed oriented 3–manifold. Suppose that Y has a non-separating embedded oriented surface Σ, i.e. $\operatorname{rk} H_1(Y) \geq 1$. Suppose $x = [\alpha]$ for a separating curve α on Σ. Then $\mathcal{C}_x(Y)$ contains infinitely many distinct almost-concordance classes.*

Proof. For some $k > 1$, consider the cover $\overline{Y}$ associated to the kernel $\ker \varphi$ of the map

$$\varphi \colon \pi_1(Y) \to \mathbb{Z}/k\mathbb{Z}$$
$$g \mapsto [g] \cdot_Y [\Sigma]$$

given by the intersection number, in Y, with the surface Σ. Note that $[\alpha] \in \ker \varphi$. Furthermore, pick a surface $S \subset \Sigma$ bounding α. For this surface S, we have $\pi_1(S) \subset \ker \varphi$ and so S lifts along the cover $\overline{Y} \to Y$.

Pick a curve $\eta \subset Y$ not intersecting S such that $\varphi(\eta) = 1$, and pick a genus two handlebody B whose two cores are the curves η and α. Just as in Figure 5, consider the knots $K_n \subset B$ and also the covering links L_n of K_n, now corresponding to our cover $\overline{Y} \to Y$. The computation of linking numbers is in fact much easier now than in the previous theorem, as we can use S to build a Seifert surface for L_n. As shown in Figure 8, the link $\alpha \cup K_n$ has a Seifert surface in B, to which we attach
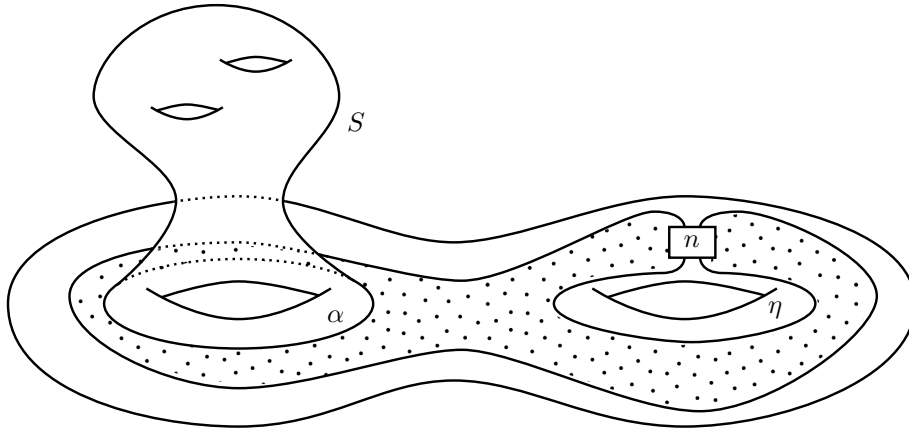

FIGURE 8. The Seifert surface for K_n.

the surface S along α, resulting in a Seifert surface for K_n. This surface clearly lifts to give a Seifert surface for each component of the covering link L_n.

Suppose $\varphi(\eta)$ is not 2–torsion. In the other case the argument is similar. Take C, D to be link components of L_n related by an n–twist box, and assume we have lifted our Seifert surface for K_n to a Seifert surface for D. Decomposing D as in Figure 7, we see that $\operatorname{lk}(C, D) = n + \operatorname{lk}(C, D_{\text{dist}})$. But now the algebraic intersection $[\overline{S}] \cdot [D_{\text{dist}}] = \operatorname{lk}(C, D_{\text{dist}})$ where $\overline{S}$ is the lift of S in our lifted Seifert surface. But D_{dist} maps to the boundary of S in Y, so we must have geometric intersection $\overline{S} \cdot D_{\text{dist}} = 0$.

Thus, any two components of L_n link exactly 0 or n times. By Lemma 8.2 we see that the links L_n lie in distinct almost-concordance classes, and so $\{K_n \,|\, n > 1\}$ represents a set of distinct almost-concordance classes in $\mathcal{C}_x(Y)$. $\square$

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