ZEEMAN'S WORK ON UNKNOTTING

MARK POWELL

Abstract. I survey Zeeman's work on knot theory, in particular his unknotting theorems. These are the notes from the talk I gave at the Zeeman centenary conference at the University of Warwick, on 17th December 2025.

Contents

1. Introduction

According to Zeeman, there are three fundamental problems of geometric topology:

(1) Homeomorphism; whether two given manifolds are homeomorphic, and whether we can classify a sub-collection up to homeomorphism.
(2) Embedding; given two manifolds, whether there exists an embedding of one in the other.
(3) Isotopy; given two embeddings, whether they are isotopic, and whether we can classify embeddings up to isotopy.

Zeeman made important, highly nontrivial contributions to all of these problems. His work on the Poincaré conjecture [Zee61, Zee62a] contributed to (1). His work with Penrose and Whitehead [PWZ61] gave new embeddings of manifolds in Euclidean space, contributing to (2). However his most celebrated contributions were to (3), and in particular to unknotting theorems. I will focus on these.

In this talk, a knot is an embedding $S^{n-k} \subseteq S^n$. The sphere S^m is a smooth manifold, but we can consider it as a PL (Piecewise-Linear) manifold, namely a manifold with a triangulation such that the links of all vertices are spheres. We can also just consider the underlying topological manifold. Corresponding to each of these manifold categories, we consider embeddings that are smooth, PL locally flat, or topologically locally flat.

Here, an embedding $M^{n-k} \hookrightarrow N^n$ is locally flat if for every point x of the image $M \subseteq N$, there is an open set $U_x \subseteq N$ containing x, and a homeomorphism of pairs $(U_x, U_x \cap M) \cong (\mathbb{R}^n, \mathbb{R}^{n-k})$ to the image of the standard embedding of $\mathbb{R}^{n-k}$ in $\mathbb{R}^n$, that sends x to the origin.

Definition 1.1. A (smooth/PL/Top) knot $S^{n-k} \subseteq S^n$ is (smoothly/PL/topologically) *unknotted* if it bounds a (smooth/PL/topological) disc $D^{n-k+1} \subseteq S^n$.

In fact any two unknots are isotopic, and so we may speak of *the* unknot.

2. There are nontrivial codimension two knots in all dimensions

I will shortly be talking about unknotting theorems, namely results that characterise the unknot. First, to put these results in the proper context, I would like to convince you that nontrivial knots are ubiquitous, and indeed there exist nontrivial knots in all dimensions at least three.

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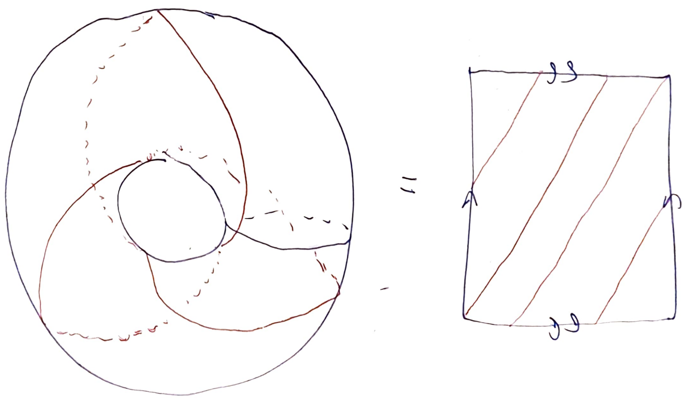

FIGURE 1. The trefoil knot $K := T_{2,3}$ on the surface of the standard torus $T^2 \subseteq S^3$. In homology, it travels 2 times around the longitude of T^2 and 3 times around the meridian, whence the name $T_{2,3}$. On the right we recall that T^2 can be represented as a quotient of $I \times I$, identifying opposite edges, and we draw the same knot. In the right hand picture it is easier to see that the complement $T^2 \setminus K$ is homeomorphic to an annulus $S^1 \times (0, 1)$.

2.1. **Nontrivial knots via spinning.**

Theorem 2.1 (Artin, [Art25]). *In each dimension $n \geq 3$, there exists a nontrivial knot $S^{n-2} \subseteq S^n$.*

To prove this, we start with $S^1 \subseteq S^3$. Figure 1 shows a knot $S^1 \subseteq S^3$. To see it, consider a standard unknotted torus $T^2 \subseteq S^3$. On its surface, consider a simple closed curve that wraps two times around the longitude and three times around the meridian. This gives the torus knot $K := T_{2,3}$, otherwise known as a trefoil. In fact, for any coprime integers $p, q > 1$, we have an analogous torus knot $T_{p,q} \subseteq S^3$.

The complement of the torus knot is homeomorphic to a union

$$S^3 \setminus K \cong S^1 \times D^2 \cup_{S^1 \times (0,1)} D^2 \times S^1.$$

Up to homotopy, the curve $S^1 \times \{1/2\}$ in the annulus $S^1 \times (0, 1)$ wraps two times around the S^1 factor of $S^1 \times D^2$ and three times around the S^1 factor of $D^2 \times S^1$. We can therefore use the Seifert-van Kampen theorem to see that the fundamental group is

$$\pi_1(S^3 \setminus K) \cong \langle s, t \mid s^2 = t^3 \rangle.$$

This group is not isomorphic to $\mathbb{Z}$, because it admits a surjective homomorphism to the symmetric group S_3, sending $s \mapsto (12)$ and $t \mapsto (123)$. Thus the trefoil $K = T_{2,3}$ is a nontrivial knot.

We can now introduce Artin's *spinning* construction. This takes an input a knotted arc $A \cong D^1$ in the half-space $H := \mathbb{R}^3_{x_1 \geq 0}$, with endpoints on $P := \{x_1 = 0\} \subseteq \mathbb{R}^3$, and produces a 2-knot $S^2 \subseteq S^4$. Such a knotted arc $A \subseteq H$ is shown in Figure 2, with the property that closing it up with an arc on P yields a trefoil..

In 3-space, a rotation has a 1-dimensional axis, for example the x_3 axis. Imagine an arc in the half-plane $\{x_1 \geq 0, x_2 = 0\} \subseteq \mathbb{R}^3$. If we rotate this plane through 2π about the z axis, the arc traces out a 2-sphere S^2.

Similarly, if we rotated the half-space H in 4-space around the plane P, the arc A again traces out a 2-sphere, which will be the result of the spinning construction. More precisely, we consider

$$S^4 \supseteq \mathbb{R}^4 \cong (S^1 \times H/\sim) \supseteq (S^1 \times A/\sim) \cong S^2,$$

where $(x, y) \sim (x', y')$ if and only if ($x = x'$ and $y = y'$) or $y = y' \in P$. We obtain a 2-knot $S^2 \subseteq S^4$.

Proposition 2.2. $\pi_1(S^4 \setminus \mathrm{Spin}(A)) \cong \pi_1(H \setminus A) \cong \pi_1(S^3 \setminus T_{2,3})$.

Since $\pi_1(S^3 \setminus T_{2,3}) \not\cong \mathbb{Z}$, it follows that $\mathrm{Spin}(A)$ is not the unknot. Indeed, the complement of the trivial 2-knot is homeomorphic to $S^1 \times \mathbb{R}^3$, and so has fundamental group $\mathbb{Z}$.

I will not give all the details of the proof of the proposition. The idea is as follows. The complement is obtained as a quotient of $S^1 \times (H \setminus A)$, which has fundamental group $\mathbb{Z} \times \pi_1(H \setminus A)$. Identifying $S^1 \times \{y\}$ to a point, for each $y \in P$, has the effect of killing the extra $\mathbb{Z}$ factor.

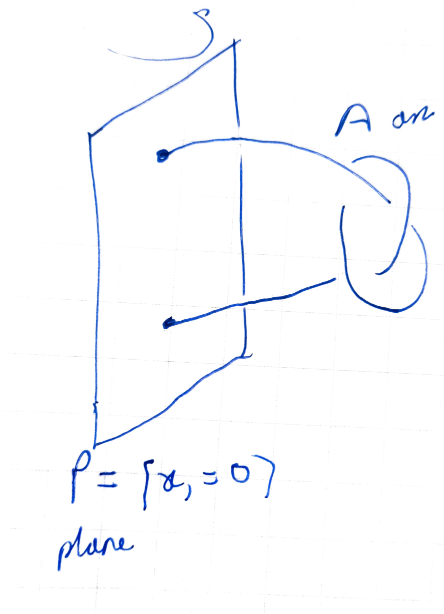

FIGURE 2. A knotted arc A in the half-space H, with endpoints on the plane P, such that closing it up with an arc on P yields a trefoil.

We have now seen that there are nontrivial 2-knots. Next, the spinning procedure generalises to higher dimensions, giving knots

$$S^n \supseteq \mathbb{R}^n \cong (S^{n-3} \times H/\sim) \supseteq (S^{n-3} \times A/\sim) \cong S^{n-2},$$

with the equivalence relation the same as before. We obtain a knot $S^{n-2} \subseteq S^n$, and again we have that $\pi_1(S^n \setminus S^{n-2}) \cong \pi_1(H \setminus A)$. Since the fundamental group of $H \setminus A$ is not cyclic, it follows that the spun knot is nontrivial, as before. This completes my discussion of the proof of Theorem 2.1, which stated that there are nontrivial knots of codimension two in all dimensions $n \geq 3$.

2.2. **Nontrivial linking.** There is also linking. To make this interesting, and not just an immediate consequence of Theorem 2.1, one can require that the components be unknotted. This brings me to the first theorem of Zeeman I would like to mention.

Theorem 2.3 (Zeeman [Zee60a]). *For all $n \geq 3$, there exists a nontrivial link $S^{n-2} \sqcup S^{n-2} \subseteq S^n$ with unknotted components.*

Proof. Consider the pairs of arcs $A_1 \sqcup A_2$ in H shown in Figure 3. Each arc is individually trivial. However Zeeman computed that $\pi_1(H \setminus (A_1 \sqcup A_2))$ is not isomorphic to the free group on two generators, F_2. This is a relatively straightforward computation, using the standard Wirtinger presentation of the fundamental group of a link complement. Spin the arcs by S^{n-3}, as above, to obtain a link $S^{n-2} \sqcup S^{n-2} \subseteq S^n$, such that the fundamental group of the complement is again not isomorphic to F_2. Since the original arcs A_1 and A_2 are individually trivial, the components of the spun link are trivial. □

2.3. **Twist spinning.** I should mention one more result of Zeeman on spinning. There is a generalisation of spinning called twist spinning. As we rotate a knotted arc around P, we can add k full twists to the entire arc. This is rather like the Euler coordinates of the Lagrangian top in classical mechanics. The first coordinate is precession. But the top can also spin on its own axis while it precesses, and that is what happens with twist spinning. Zeeman [Zee65] showed that k-twist spun 2-knots have exteriors that fibre over S^1, and he described the fibre as the k-fold branched cover of S^3, branched over the knot-closure of the original arc, punctured. A consequence is the following beautiful result.

Theorem 2.4 (Zeeman, [Zee65]). *The 1-twist spin of any arc in H is unknotted.*

It follows that every knot of the form $K\#r\overline{K}$, where r denotes reversed string orientation and $\overline{K}$ is the mirror image of K, appears as an equatorial cross-section $J \cap S^3$ of an unknotted 2-knot $J \cong S^2 \subseteq S^4$.

Twist spinning is great for constructing examples. For instance, it was used to find examples of distinct 2-knots whose complements have isomorphic fundamental groups, but distinct second homotopy groups.

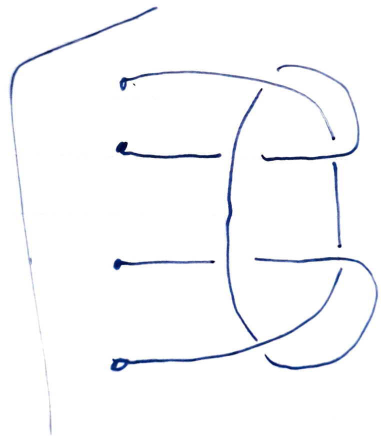

FIGURE 3. A pair of arcs $A_1 \sqcup A_2$ in the half-space H, with endpoints on the plane P. Each arc is individually trivial, but they are linked in an interesting way.

3. Knots in other codimensions

So far, all the examples I have presented have been in codimension two. This is with good reason, as I shall now explain.

3.1. **Codimension one.** In codimension one, we have the following result, known as the Schoenflies theorem.

Theorem 3.1. *For $n \neq 4$, every smooth/PL./Top knot is unknotted. For $n = 4$, this is true in the topological category.*

For $n = 4$, the smooth and PL versions remain open questions. In high dimension, the theorem follows from the h-cobordism theorem in the smooth and PL cases. The topological case is due to Mazur [Maz59], Morse [Mor60], and Brown [Bro60]. Thus there are no known nontrivial knots in codimension one, and the only case in which they could possibly arise is $S^3 \subseteq S^4$.

3.2. **Codimension at least three.** On the other hand, in codimension at least three we have the following theorem, which is one of Zeeman's most well-known results. Zeeman proved the PL case, and Stallings proved the topological case. Their proofs were obtained independently, and more or less simultaneously. Neither implies the other.

Theorem 3.2 (Zeeman [Zee63] (PL), Stallings [Sta63] (Top))**.** *Suppose $k \geq 3$. Every Top/PL knot $S^{n-k} \subseteq S^n$ is Top/PL unknotted, respectively.*

Zeeman's publications on Theorem 3.2 are in fact as follows.

- 1960 – $S^2 \subseteq S^5$, half a page in the Bulletin of the AMS [Zee60c].
- 1960 – $S^{n-k} \subseteq S^n$ for $k > (n/3) + 1$. Annals of Mathematics [Zee60b].
- 1962 – $S^3 \subseteq S^6$. Proc. AMS [Zee62b].
- 1963 – $S^{n-k} \subseteq S^n$ for $k \geq 3$. Again, in the Annals of Mathematics [Zee63].

The first article contains a short, beautiful proof that every knot $S^2 \subseteq S^5$ is trivial. Zeeman was so proud of it that it appears in the background of his portrait in Hertford College, Oxford; see Figure 4. The proof in the 1960 Annals paper [Zee60b] is essentially this proof, but made more rigorous.

The third article is also worthy of mention. When he wrote it, Zeeman already had the proof that appeared in the Annals in 1963 [Zee63]. However this proof involves a long induction, that is not easy to read. In [Zee60c], Zeeman had conjectured that knots of $S^3 \subseteq S^6$ can be nontrivial. In fact this is true in the smooth category, as I shall discuss below. But in the PL world, all knots $S^3 \subseteq S^6$ are trivial. Since this case is of particular interest, and out of a desire to write a proof that can be understood by a wider audience, Zeeman decided to write up a shorter and more readable version of his general proof, just in the case of 3-spheres in the 6-sphere. One can already see the importance he placed on communicating mathematics, even during the time he focused on pure mathematics.

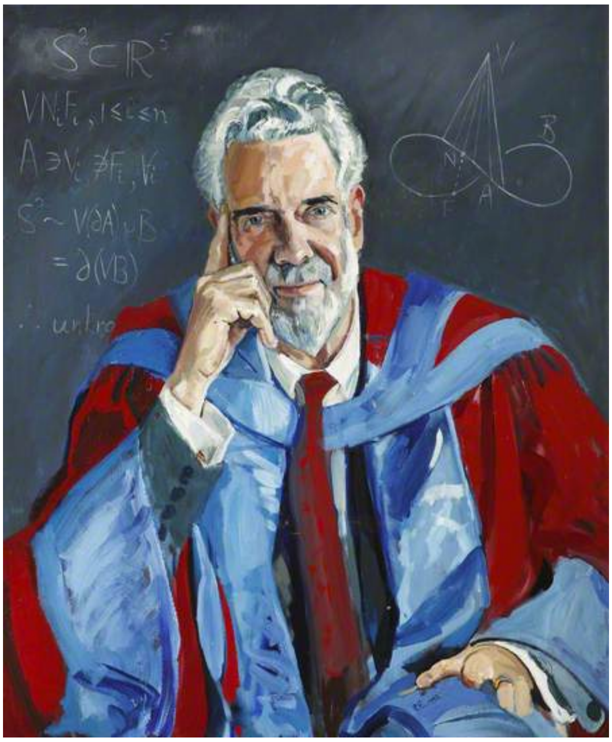

FIGURE 4. Zeeman's portrait from Hertford College, Oxford. The blackboard in the background shows a five line version of his proof that every 2-sphere in S^5 is unknotted. He was particularly proud of this proof, and so when the painter suggested to have some mathematics on the blackboard behind him, this was what Zeeman provided.

4. ZEEMAN'S PROOF FOR 2-SPHERES IN THE 5-SPHERE

Now I want to present Zeeman's short and elegant proof from [Zee60c] that every 2-sphere in S^5 is unknotted.

The first step is to choose a vertex v, in some triangulation of S^5, that is far away from S^2, and in general position with respect to S^2. Consider the cone $v \cdot S^2$, consisting of all straight arcs between the points of S^2 and v. By general position, there are finitely many singular lines. The cone is three dimensional, and the ambient dimension is five, so we expect one dimensional singularities.

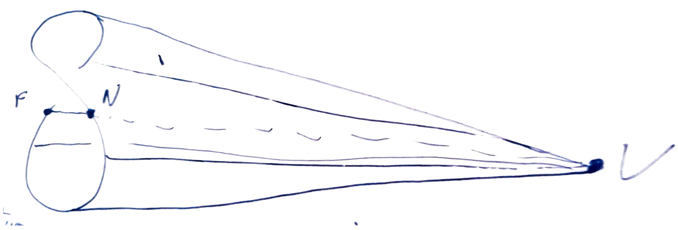

FIGURE 5. The cone $v \cdot S^2$, with a single near and a single far point.

These singularities occur when there are two points in S^2 that lie on the same ray from v. By general position, there are no other singularities of $v \cdot S^2$. For each pair of points that lie on the same ray from v, we call the point on S^2 closest to v a *near* point, and the point farthest from v the *far* point. See Figure 5

Thus, on S^2, there are two collections of points, the near points and the far points, of the same (finite) cardinality. Choose a circle $C \subseteq S^2$ separating the near points and the far points. Let $A \subseteq S^2$ be the closed disc containing the near points and let B be the closed disc containing the far points. So S^2 decomposes as $B \cup_C A$. See Figure 6.

Now comes the key observation:

$$v \cdot C \cup_C A = \partial(v \cdot A).$$

That is, the cone $v \cdot A$ on A is a 3-ball, whose boundary is $v \cdot C \cup_C A$. Pushing A across $v \cdot A$ yields a PL isotopy from $B \cup_C A$ to $B \cup_C v \cdot C$. There is no obstruction to performing the isotopy because we only

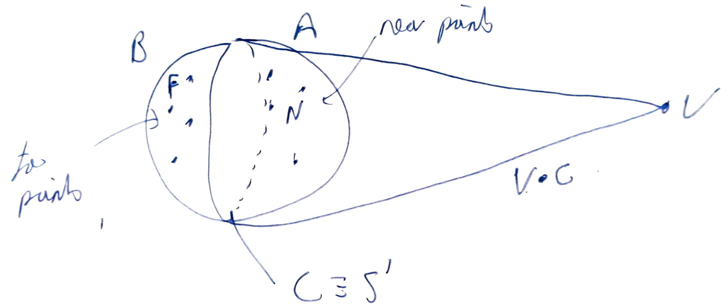

FIGURE 6. The cone $v \cdot S^2$, and the sphere S^2 divided into $B \cup_C A$, where the disc A contains the near points and the disc B contains the far points.

push the hemisphere containing the near points, and we push them closer to v. It remains to observe that $B \cup_C v \cdot C = \partial(v \cdot B)$, because we no longer have any of the near points, and so arcs from the far points can reach v unobstructed. In summary, we have

$$B \cup_C A \sim B \cup_C v \cdot C = \partial(v \cdot B).$$

In words, $B \cup_C A$ is PL isotopic to $B \cup_C v \cdot C$, and $B \cup_C v \cdot C$ bounds a PL ball, and hence is unknotted. Since $B \cup_C A$ is isotopic to an unknotted sphere, it is itself unknotted (by isotopy extension [HZ64]).

This proof generalises to show unknotting whenever $k > n/3 + 1$, and this is what Zeeman's first Annals paper on unknotting achieved [Zee60b]. It is worth noting that this proof does not work for $S^3 \subseteq S^6$. The singular arcs will give rise to circles in S^3, which might link each other. It would then not be possible to separate the near circles from the far circles with an S^2. This led Zeeman to initially think that there could be a knotted $S^3 \subseteq S^6$. His later proof in [Zee62b] is significantly more complicated than the one I just explained, with a preliminary step to separate the circles.

Haelfiger's work on nontrivial smooth knots $S^3 \subseteq S^6$, which I will discuss later, showed that the question is indeed extremely subtle. The proofs of Zeeman and Stallings can only work in the PL and Top cases, and must fail in the smooth category.

5. Comparison with codimension two

I should also mention the status of unknotting in codimension two.

5.1. **Characterising the codimension two unknot.** We have seen that there are nontrivial knots in codimension two. Nevertheless, we can characterise the unknot as follows.

Theorem 5.1 (Stallings [Sta63], Levine [Lev65], Papakyriakopoulos [Pap57], Freedman-Quinn [FQ90])**.** *Suppose that $n \neq 4$. Then a smooth/PL/Top knot $S^{n-2} \subseteq S^n$ is smoothly/PL/topologically unknotted if and only if $S^n \setminus S^{n-2} \simeq S^1$. For $n = 4$, a topological 2-knot $S^2 \subseteq S^4$ is topologically unknotted if and only if $S^4 \setminus S^2 \simeq S^1$.*

For $n = 4$, the homotopy equivalence $S^4 \setminus S^2 \simeq S^1$ holds if and only if $\pi_1(S^4 \setminus S^2) \cong \mathbb{Z}$. It is open in the smooth and PL cases whether there is a nontrivial 2-knot in S^4 with $\pi_1(S^4 \setminus S^2) \cong \mathbb{Z}$.

5.2. **Knotted surfaces in the 4-sphere.** This discussion gives me the occasion to mention some of my work on unknotting. With Conway, we considered closed, orientable surfaces Σ_g of genus g in the 4-sphere.

Theorem 5.2 (Conway-P. [CP23])**.** *For $g \geq 3$, $\Sigma_g \subseteq S^4$ is topologically unknotted (bounds a handlebody) if and only if $\pi_1(S^4 \setminus \Sigma_g) \cong \mathbb{Z}$.*

The theorem was proven for $g = 0$ be Freedman and Quinn, as noted above. It remains open for $g = 1, 2$. In the smooth and PL categories, it is open for all g whether there are nontrivial knotted surfaces with $\pi_1(S^4 \setminus \Sigma_g) \cong \mathbb{Z}$.

Later, with Conway and Orson, we proved analogous results for nonorientable surfaces F in S^4 with $\pi_1(S^4 \setminus F) \cong \mathbb{Z}/2$, in the majority of cases [COP23]. Here unknotted means isotopic to a standard embedding, and there a multiple standard embeddings, depending on the Euler class of the normal bundle. We proved that such surfaces are standard provided the normal Euler number is non-extremal in the permitted range, which depends on the first Betti number of the surface F. The case of $\mathbb{RP}^2$ was already proven by Lawson [Law84]. The smooth version is not true: there are examples of nonorientable surfaces that are topologically but not smoothly isotopic to the standard surface. The most striking result in this direction is rather recent, due to Miyazawa [Miy23], who showed that the standard $\mathbb{RP}^2$ connected sum with a certain 2-knot (a roll-spun pretzel knot) yields an exotic embedding $\mathbb{RP}^2 \subseteq S^4$.

6. Comparison with the smooth category

6.1. **Nontrivial smooth knots in arbitrarily high codimensions.** Theorem 3.2 contrasts starkly with Haefliger's work on smooth knots in codimension at least three. As mentioned before, this shows that the proof of Theorem 3.2 must be rather subtle.

Theorem 6.1 (Haefliger, [Hae62b]). *For every $k \geq 1$, there is a smoothly nontrivial knot $S^{4k-1} \subseteq S^{6k}$.*

In fact, this result was improved to $(2\ell - 1)$ knots, for every $\ell \geq 2$. Moreover, if we consider the set of oriented knots $S^{2\ell-1} \subseteq S^{3\ell}$, up to isotopy, they form a group under the operation of connected sum (the fact that there are inverses uses Theorem 7.1 below), which is isomorphic to $\mathbb{Z}$ if ℓ is even and $\ell \geq 2$, and is isomorphic to $\mathbb{Z}/2$ if ℓ is odd and $\ell \geq 3$.

For each ℓ, a generator of this group is called the *Haefliger trefoil*. In the classical dimension, if we take a copy of the Borromean rings and band the components together to form a knot, in a standard way, we obtain a trefoil, as shown in Figure 7.

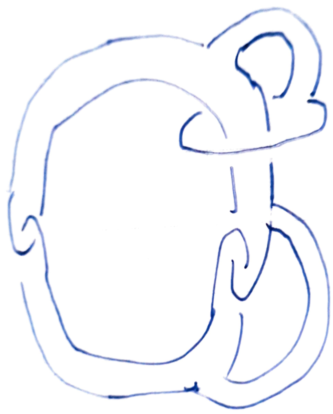

Figure 7. The Borromean rings, which is a 3-component link, with the components banded together to form a knot.

The Haefliger trefoil in $S^{2\ell-1} \subseteq S^{3\ell}$ arises by taking a higher-dimensional version of the Borromean rings, consisting of a link $S^{2\ell-1} \sqcup S^{2\ell-1} \sqcup S^{2\ell-1} \subseteq S^{3\ell}$, and then tubing the components together. It is called a trefoil because of the analogue with the classical dimension. To generalise, first we consider the ordinary Borromean rings in standard form, as ellipses living in the coordinate planes, as shown in Figure 8:

$$\{2x^2 + y^2 = 1, z = 0\} \cup \{2z^2 + x^2 = 1, y = 0\} \cup \{2y^2 + z^2 = 1, x = 0\} \subseteq \mathbb{R}^3 \subseteq S^3.$$

Replacing each of x, y, z with vectors

$$\underline{x} := (x_1, \ldots, x_\ell),\ \underline{y} := (y_1, \ldots, y_\ell),\ \underline{z} := (z_1, \ldots, z_\ell) \in \mathbb{R}^\ell,$$

we can express the $(2\ell - 1)$-dimensional Borromean rings as

$$\{2|\underline{x}|^2 + |\underline{y}|^2 = 1, \underline{z} = 0\} \cup \{2|\underline{z}|^2 + |\underline{x}|^2 = 1, \underline{y} = 0\} \cup \{2|\underline{y}|^2 + |\underline{z}|^2 = 1, \underline{x} = 0\} \subseteq \mathbb{R}^{3\ell} \subseteq S^{3\ell}.$$

Tubing the components together yields the Haefliger trefoil $S^{2\ell-1} \subseteq S^{3\ell}$. This knot is smoothly nontrivial but is unknotted if we consider it as a PL or a Top knot, by Theorem 3.2.

Let me briefly define Haefliger's invariant, in the case that $\ell = 2$, so $S^3 \subseteq S^6$. First, he showed that every $S^3 \subseteq S^6$ bounds a framed 4-manifold $V \subseteq D^7$, i.e. a 4-manifold whose tangent bundle is trivial. He defined a homomorphism

$$\begin{aligned} \lambda \colon H_2(V, \partial V) &\to \mathbb{Z} \\ x &\mapsto \mathrm{lk}(x^+, V). \end{aligned}$$

Figure 8. The Borromean rings, represented as three ellipses in the coordinate planes.

That is, we can push x off V, and then compute its linking number with V. This yields a cohomology class

$$\lambda \in H^2(V, \partial V; \mathbb{Z}) \cong \mathrm{Hom}(H_2(V, \partial V), \mathbb{Z}).$$

Then Haefliger considered

$$\lambda^2 \in H^4(V, \partial V; \mathbb{Z}) \cong H_0(V; \mathbb{Z}) \cong \mathbb{Z}.$$

Haefliger proved that this integer is a knot invariant and computed that it is nonzero on the Haeflger trefoil.

6.2. **Smooth knots are trivial in Haefliger's stable range.** In contrast to the above beautiful examples, smoothly nontrivial knots cannot occur for arbitrary (n, k).

Theorem 6.2 (Haefliger [Hae62a])**.** *Suppose that $k > (n/3) + 1$. Then every smooth knot $S^{n-k} \subseteq S^n$ is smoothly trivial.*

Note that this is the same as the range in Zeeman's 1960 Annals paper [Zee60b]. It seems likely that the PL proof of Zeeman that I explained, using near and far points, can be adapted to work in the smooth category. As already noted, Haefliger's examples show that Zeeman's and Stallings' 1963 proofs cannot in general work smoothly in the range $(n/3) + 1 \geq k \geq 3$.

7. Concordance implies isotopy

I want to move on to discussing results that were inspired by Zeeman's work. His unknotting theorems are beautiful in their own right, but the mathematics they have inspired makes them even more impressive.

7.1. **Statement.** In Zeeman's main unknotting theorem for codimension at least three (Theorem 3.2), he introduced the method of 'sunny collapsing'. This method was used to great effect by his student Hudson, who proved the following theorem in the smooth and PL categories. The topological adaptation was done later by Pedersen.

Theorem 7.1 (Concordance implies isotopy, Hudson [Hud70], Pedersen [Ped77])**.** *Fix $k \geq 3$, and let $M_0^{n-k}, M_1^{n-k} \subseteq N^n$ be smooth/PL/Top submanifolds such that $M_0 \cong M_1$. Then M_0 and M_0 are smooth/PL/Top concordant if and only if they are smooth/PL/Top isotopic.*

Here, M_0 and M_1 are *concordant* if there is an embedding $M \times I \subseteq N \times I$ such that

$$(M \times I) \cap (N \times \{i\}) = M_i$$

for $i = 0, 1$.

7.2. **An application.** This theorem can be used retrospectively to give an alternative proof of Theorem 3.2. Here is a nice application of this theorem to links.

Example 7.2. I will sketch a proof that two links $L_0, L_1 \colon S^3 \sqcup S^3 \hookrightarrow S^7$ are isotopic if and only if their linking numbers are equal, i.e. $\mathrm{lk}(L_0) = \mathrm{lk}(L_1) \in \mathbb{Z}$.

The linking number of L_i is defined as follows. The first component bounds an orientable 4-manifold W in S^7 (in fact, a 4-disc, since all knots $S^3 \subseteq S^7$ are in Haefliger/Zeeman's stable range). Count the intersections, with sign, of the second component S^3 with W. The resulting intersection count is the linking number $\mathrm{lk}(L_i)$. This is certainly an isotopy invariant, and so the 'only if' direction holds.

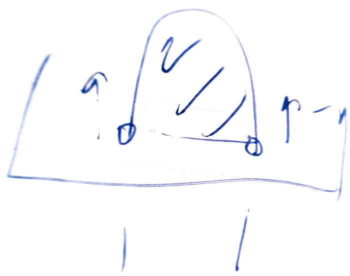

Figure 9. A dimensionally-reduced version picture of a Whitney disc. The disc guides an isotopy, called the Whitney move, that removed two algebraically cancelling double points.

For the 'if' direction, first we note that, since $\pi_3(S^7) = 0$, we have that L_0 and L_1 are homotopic. By general position, we can assume that the homotopy gives rise to a generic immersion

$$(S^3 \sqcup S^3) \times I \looparrowright S^7 \times I,$$

whose singularities are isolated double points. Since $\mathrm{lk}(L_0) = \mathrm{lk}(L_1)$, these double points can be assumed to cancel algebraically, and hence they can be paired up by embedded, framed Whitney discs, as shown in Figure 9. The Whitney discs guide an series of isotopies, called Whitney moves, that remove all the double points. We are left with a concordance between L_0 and L_1. Then, using the concordance implies isotopy theorem, we deduce that L_0 and L_1 are isotopic.

8. Disjunction theorems

In the modern context, Zeeman's original questions about existence and uniqueness of embeddings can be thought of as the starting point of the investigation of *spaces of embeddings* $\mathrm{Emb}(M, N)$, where M and N are manifolds and $\dim M \leq \dim N$. The existence question asks whether this space is nonempty, and the isotopy question asks how many path components it has. What are the higher homotopy or homology groups? What do generators look like, and can they be detected by computable invariants?

8.1. **Embedding spaces.** Concordance implies isotopy turns out to be an extremely useful statement for investigating embedding spaces. It is a key ingredient for the proofs of so-called *disjunction theorems*. Goodwillie, Klein, and Weiss [Wei99, GW99, GKW01, GK15] used disjunction theorems to develop *embedding calculus*, which, especially for codimension at least three (the same range in which Theorems 7.1 and 3.2 apply), is a powerful tool for investigating the homotopy types of the spaces $\mathrm{Emb}(M, N)$.

I will mention two examples of results that are known about embedding spaces using these methods. Let X be a 4-manifold with nonempty boundary, and consider the embedding space $\mathrm{Emb}(D^1, X)$ of embeddings of the arc D^1 into X, where the embedding on S^0 is fixed to be the same two points in ∂X.

(i) If X is simply-connected, then $\pi_1\mathrm{Emb}(D^1, X) \cong \pi_2(X)$ [Kos24]
(ii) If X is aspherical, then $\pi_1\mathrm{Emb}(D^1, X) \cong \mathbb{Z}[\pi_1(X) \setminus \{1\}]$ [Gab21].

8.2. **Spaces of smooth structures.** Another famous disjunction theorem is the Morlet lemma of disjunction, a comprehensive treatment of which was given in [BLR75]. The Morlet lemma is a crucial part of modern efforts to understand the space of smooth structures on a given manifold (of dimension at least five). It states that, for $n = 5$,

$$\Omega^n_0 \frac{\mathrm{Top}(n)}{\mathrm{O}(n)} \simeq \mathrm{BDiff}_\partial(D^n).$$

The right hand side is the classifying space of the topological group of boundary-fixing diffeomorphisms of the n-disc. One can sometimes obtain information about this space. For example, the fundamental group can be identified with group of homotopy spheres θ_{n+1}. Recent impressive work on the higher homotopy groups includes [Wat09, KRW21, KRW25].

The left hand side is the (identity path-component of the) n-fold loop space of the homotopy fibre of the map $\mathrm{BO}(n) \to \mathrm{BTop}(n)$, and is therefore the space that controls the space of lifts of the topological tangent bundle $X \to \mathrm{BTop}(n)$, of a topological n-manifold X to a vector bundle $X \to \mathrm{BO}(n)$. The main theorem of Kirby and Siebenman's smoothing theory [KS77] states that this space of lifts is homotopy equivalent to the space of smooth structures on X. So understanding the homotopy type of the universal space $\mathrm{Top}(n)/\mathrm{O}(n)$ can have wide ranging consequences for spaces of smooth structures on all n-manifolds.

Again, the proof of the Morlet lemma makes crucial use of the concordance implies isotopy theorem, proved first by Hudson, who was inspired by Zeeman's unknotting theorems.

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School of Mathematics and Statistics, University of Glasgow, United Kingdom
Email address: `mark.powell@glasgow.ac.uk`