MAT8232 ALGEBRAIC TOPOLOGY II: HOMOTOPY THEORY

MARK POWELL

1. Introduction: the homotopy category

Homotopy theory is the study of continuous maps between topological spaces up to homotopy. Roughly, two maps $f, g\colon X \to Y$ are homotopic if there is a continuous family of maps $F_t\colon X \to Y$, for $0 \leq t \leq 1$, with $F_0 = f$ and $F_1 = g$. The set of homotopy classes of maps between spaces X and Y is denoted $[X, Y]$. The goal of this course is to understand these sets, and introduce many of the techniques that have been introduced for their study. We will primarily follow the book "A concise course in algebraic topology" by Peter May [May]. Most of the rest of the material comes from "Lecture notes in algebraic topology" by Jim Davis and Paul Kirk [DK].

In this section we introduce the special types of spaces that we will work with in order to prove theorems in homotopy theory, and we recall/introduce some of the basic notions and constructions that we will be using. This section was typed by Mathieu Gaudreau.

Conventions. A space X is a topological space with a choice of basepoint that we shall denote by $*_X$. Such a space is called a based space, but we shall abuse of notation and simply call it a space. Otherwise, we will specifically say that the space is unbased.

Given a subspace $A \subset X$, we shall always assume $*_X \in A$, unless otherwise stated. Also, all spaces considered are supposed path connected.

Finally, by a *map* $f\colon (X, *_X) \to (Y, *_Y)$ between based space, we shall mean a continuous map $f\colon X \to Y$ preserving the basepoints (i.e. such that $f(*_X) = *_Y$), unless otherwise stated. Note that when it is clear from the context, we may refer to the basepoint as simply $*$.

1.1. **Basic constructions.** In this subsection we remember some constructions on topological spaces that we will use later.

Cartesian Product. Define the product of the based spaces $(X, *_X)$ and $(Y, *_Y)$ by $(X \times Y, *_{X\times Y})$, where $X \times Y$ has the product topology and $*_{X\times Y} := (*_X, *_Y)$. Throughout these notes, we will denote the projection on the ith component (i.e. the map

$$\begin{array}{rcl} X_1 \times \cdots \times X_n & \to & X_i \\ (x_1, \cdots, x_n) & \mapsto & x_i \end{array}$$

We also have the diagonal map

$$\begin{array}{rcl}\Delta\colon X & \to & X \times X \\ x & \mapsto & (x,x).\end{array}$$

Moreover, given two continuous maps $f\colon X \to Y$ and $g\colon X' \to Y'$, define their product $f \times g\colon X \times Y \to X' \times Y'$ by $f \times g(x,y) := (f(x), g(y))$. Note that if f and g are continuous, so is $f \times g$ (indeed, we have that $(f \times g)^{-1}(U \times V) = f^{-1}(U) \times g^{-1}(V)$ is open if U and V are, because f and g are continuous).

Remark 1.1. Given two spaces X and Y, we can see the product $X \times Y$ as the unique space such that the following diagram commutes and such that the following property holds: for all pairs of continuous map $f\colon A \to X$, $g\colon A \to Y$, where A is some space, there exists a unique continuous function $(f,g)\colon A \to X \times Y$ such that $p_1 \circ (f,g) = f$ and $p_2 \circ (f,g) = g$ (the map is defined by $(f,g) := f \times g \circ \Delta$). $X \times Y$ can therefore be seen as the pull-back of the constant functions $p_*\colon X \to \{*\}$ and $p_*\colon Y \to \{*\}$, a notion that we will define later. This is an example of a universal property defining the space $X \times Y$, which for the product $X \times Y$ is a strange perspective, but this point of view is the one which generalises.

$$\begin{tikzcd}
A \arrow[dr, dotted, "{(f,g)}"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& X \times Y \arrow[r, "p_1"] \arrow[d, "p_2"] & X \arrow[d, "p_*"] \\
& Y \arrow[r, "p_*"] & \{*\}
\end{tikzcd}$$

Wedge Product. Given $(X, *_X)$ and $(Y, *_Y)$ two based spaces, we define their wedge product by

$$(X \vee Y, *_{X \vee Y}) := X \sqcup Y/(*_X \sim *_Y, [*_X]),$$

where $X \sqcup Y/*_X \sim *_Y$ has the quotient topology induced by the canonical projection $q\colon X \sqcup Y \to X \sqcup Y/*_X \sim *_Y$, and $[*_X]$ denotes $q(*_X)$ ($= q(*_Y)$). Because $X \vee Y$ has the quotient topology and by the factorization theorem, there exist a unique injective continuous map $i_1\colon X \to X \vee Y$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r] \arrow[dr, "i_1"] & X \sqcup Y \arrow[d, "q"] \\
& X \vee Y
\end{tikzcd}$$

Namely, i_1 is the map defined by $i_1(p) = q(p)$. Similarly, there exist a unique injective map $i_2\colon Y \to X \vee Y$.

Remark 1.2. Given two spaces X and Y, we can see the wedge product $X \vee Y$ as the unique space such that the following diagram commutes and such that the following property holds: for all pairs of continuous map $f\colon X \to B$, $g\colon Y \to B$, where B is a space, there exists a unique continuous function $\{f, g\}\colon X \vee Y \to B$ such that $\{f, g\} \circ i_1 = f$ and $\{f, g\} \circ i_2 = g$.

$$\begin{tikzcd}
\{*\} \arrow[r] \arrow[d] & X \arrow[d, "i_1"] \arrow[ddr, bend left, "f"] & \\
Y \arrow[r, "i_2"] \arrow[drr, bend right, "g"'] & X \vee Y \arrow[dr, dotted, "{\{f,g\}}"] & \\
& & B
\end{tikzcd}$$

$X \vee Y$ can therefore be seen as a push-out, a notion that we will also define later.

We call the map $\nabla\colon X \vee Y \to X$ defined by $\nabla = \{\mathrm{Id}, \mathrm{Id}\}$ the *fold map*. Note also that there exists an inclusion map $j\colon X \vee Y \to X \times Y$ defined by

$$j([p]) := \begin{cases} (p, *_Y), & p \in X; \\ (*_X, p), & p \in Y \end{cases}$$

Smash Product. Given two based spaces $(X, *_X)$ and $(Y, *_Y)$, we define their smash product by

$$(X \wedge Y, *_{X \wedge Y}) := ((X \times Y)/j(X \vee Y), j(q(*))),$$

where $X \wedge Y$ has the quotient topology induced by the projection.

1.2. **Homotopy.**

Definition 1.3. Given maps $f, g\colon X \to Y$, we write $f \simeq g$, meaning f is homotopic to g if there exist a continuous map $H\colon X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$, and $H(*, t) = *$, for all t. Moreover, we call H a (based) homotopy from f to g. We will say it is an unbased homotopy if we allow $*$ to move. Finally, we will denote the set of homotopy class of continuous maps from X to Y by $[X, Y]$.

Definition 1.4. A category $\mathcal{C}$ is a collection of objects and a set of morphisms $\mathrm{Mor}(A, B)$ between any two objects $A, B \in \mathcal{C}$ such that there is an identity morphism $\mathrm{Id}_A \in \mathrm{Mor}(A, B)$ and a composition law $\circ\colon \mathrm{Mor}(C, B) \times \mathrm{Mor}(A, B) \to \mathrm{Mor}(A, C)$, for each triple of objects $A, B, C \in \mathcal{C}$, satisfying:

(1) $h \circ (g \circ f) = (h \circ g) \circ f$, whenever it is defined (associativity);
(2) $\mathrm{Id} \circ f = f$, $f \circ \mathrm{Id} = f$ (neutrality of Id).

We say that $\mathcal{C}$ is small if the class of objects is a set.

Lemma 1.5.

(1) (Based Spaces, Based Maps) is a category called Top_*.
(2) (Based Spaces, Based Homotopy Classes of Maps) is a category called $hTop_*$.

The proof is left as an exercise.

Definition 1.6. Let (X, A) be a pair of spaces (i.e. $* \in A \subset X$) and $i\colon A \to X$ be the inclusion map.

(1) We call a continuous map $r\colon X \to A$ such that $r(a) = a$, for all $a \in A$, a *retraction*, and when such an r exist, we say that A is a *retract* of X.

(2) Moreover, if $i \circ r \simeq \mathrm{Id}_X$, we call r a *deformation retraction* and we say that A is a *deformation retract* of X.

(3) If the homotopy in (2) is fixed on A, we call r a *strong* deformation retraction and say that A is a *strong* deformation retract of X.

Example 1.7.

(1) Consider the inclusion $D^n \to \mathbb{R}^n$ and define $r\colon \mathbb{R}^n \to D^n$ by
$$r(x) := \begin{cases} x, & x \in D^n; \\ \frac{x}{\|x\|}, & x \notin D^n. \end{cases}$$
Exercise: Show that this is a strong deformation retraction.

(2) Consider the inclusion $S^{n-1} \to D^n\backslash\{0\}$ and let $r\colon D^n\backslash\{0\} \to S^{n-1}$ be the map defined by $r(x) := \frac{x}{\|x\|}$.
Exercise: Show that this is a (strong) deformation retraction.

(3) On the other hand, S^{n-1} is not a retract of D^n. Indeed, we know that $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$ and $H_{n-1}(D^n) = \{0\}$. If there were a retraction $r\colon D^n \to S^{n-1}$, we would have the following commutative diagram:
$$\begin{tikzcd} H_{n-1}(S^{n-1}) \arrow[r, "\mathrm{Id}"] \arrow[d, "i_*"] & H_{n-1}(S^{n-1}) \\ H_{n-1}(D^n) \arrow[ur, "r_*"] & \end{tikzcd}$$
where Id is the identity and $i_* = 0$ is the zero map. This give a contradiction.

Definition 1.8. Given two spaces X and Y, we say that X is homotopy equivalent to Y and write $X \simeq Y$ if there exist continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $f \circ g \simeq \mathrm{Id}_Y$ and $g \circ f \simeq \mathrm{Id}_X$. Moreover, we say that a space X is contractible when it is homotopy equivalent to the basepoint. Note that contractible can depend on the choice of basepoint. There exist spaces contractible (as unbased space), but not with any basepoint.

Example 1.9. Given a pair of spaces (A, X) such that A is a deformation retract of X with deformation retract r, if i denotes the inclusion, then $r \circ i = \mathrm{Id}_A$ and $i \circ r \simeq \mathrm{Id}_X$. Therefore, we have that A is homotopy equivalent to X.

2. Compactly generated spaces

This section was typed by Mathieu Gaudreau and Francis Beauchemin-Côté. The reader who wants to learn about compactly generated spaces and their properties in more detail than described in these notes is referred to Steenrod's well written original source [Ste].

Given a space Y, let Y^I denote the set of unbased continuous maps $f\colon I \to Y$. If we define $* \in Y^I$ as the path $*(t) := *_Y$ and put a topology on Y^I we have a based space $(Y^I, *)$. It is possible to reinterpret the notion of homotopy $F\colon f \sim g$ as a map

$$\begin{array}{rcl} F'\colon X & \to & Y^I \\ x & \mapsto & \gamma\colon t \mapsto F(x,t). \end{array}$$

In fact, if we put the right topology on Y^I we can show that this F' associated to each homotopy F is continuous. Our purpose in the present subsection is to define a topology on Y^I giving us just that. Before continuing in this direction, let us introduce two more basic constructions that we will need later.
Given a space X, we call the *cone* on X, written CX, the space

$$X \times I / X \times \{0\}.$$

We call the *reduced* cone of X, written $\widetilde{CX}$, the space

$$X \times I / X \times \{0\} \cup \{*\} \times I.$$

Finally, we call the *path space* of X, the space PX given by

$$\{\gamma \in X^I | \gamma(0) = *_Y\}.$$

We will topologize the path space with the subspace topology of X^I (which will make sense once we put a topology on X^I).

Lemma 2.1. *For any spaces X, Y, the cone CX and the path space PY are contractible.*

Proof. A homotopy between the identity on CX and the map to the basepoint is given by:

$$\begin{array}{rcl} F\colon CX \times I & \to & CX \\ ([x,t],s) & \mapsto & [x,(1-s)t]. \end{array}$$

A homotopy between the identity on PY and the map to the basepoint (the constant path) is given by:

$$\begin{array}{rcl} F\colon PY \times I & \to & PY \\ (\gamma, s) & \mapsto & \big(t \mapsto p((1-s)t)\big). \end{array}$$

$\square$

Now, let us formalise what we want. Given three spaces X, Y and Z, we would like to topologize the sets of morphisms of Top_* (i.e. the sets $\mathcal{C}(X,Y)$ of continuous maps from X to Y, for all spaces X and Y), such that $f\colon X \times Y \to Z$ is continuous if and only if the adjoint of f, defined by

$$\begin{array}{rcl} \overline{f}\colon X & \to & \mathcal{C}(Y,Z) \\ x & \mapsto & (y \mapsto f(x,y)) \end{array},$$

is continuous, and moreover, we would like the adjoint to induce a homeomorphism $\mathcal{C}(X \times Y, Z) \to \mathcal{C}(X, \mathcal{C}(Y,Z))$.

Definition 2.2. A topological space X is said to be compactly generated if:

(1) X is Hausdorff (i.e. for all pairs of points $x \neq y$ in X there exist disjoint open neighbourhoods of each);
(2) $A \subset X$ is closed if and only if $A \cap K$ is closed for every compact set $K \subset X$.

We denote the category of all compactly generated spaces by $\mathcal{K}$.

Lemma 2.3. *The following spaces are compactly supported:*

(1) locally compact Hausdorff spaces (e.g. manifolds);
(2) metric spaces;
(3) CW complexes.

Proof (of (1)). First remember that a space is said to be locally compact if every point admits a compact neighbourhood.
To show the lemma, we have to see part (2) of Definition 2.2. In one direction, if A is closed in X and K is a compact set of X, then we have that K is closed, because compact subsets of a Hausdorff space are closed. It follows that $A \cap K$ is closed. Therefore, all closed subsets in the topology of X are closed in the compactly generated topology. In the other direction, suppose $A \subset X$ is such that $A \cap K$ is closed for every compact set $K \subset X$ (i.e. suppose A is closed in the compactly generated topology). We want to show that A is closed. To show this, we will show that $X \backslash A$ is open in X. Let $x \in X \backslash A$. By local compactness, there exist a compact neighbourhood of x, say K_x. Let U_x be an open neighbourhood of x such that $x \in U_x \subset K_x$. Because $K_x \cap A$ is closed by hypothesis, we have that $X \backslash K_x \cap A$ is open. Therefore, $(X \backslash K_x \cap A) \cap U_x = U_x \backslash A =: V_x$ is an open neighbourhood of x missing A. By generality of $x \in X \backslash A$, we have $X \backslash A = \bigcup_{x \in X \backslash A} V_x$ and therefore $X \backslash A$ is open. □

Definition 2.4. Let X be a Hausdorff space. Define $k(X)$ as the same underlying set X with the compactly generated topology (i.e. declare a set A in $k(X)$ to be closed if its intersection $A \cap K$ with every compact subset $K \subset X$ is closed in X).

Definition 2.5. In category theory, given two categories $\mathcal{C}$ and $\mathcal{D}$, a *covariant functor* is an assignment $\mathcal{F} \colon \mathcal{C} \to \mathcal{D}$ sending each object A of $\mathcal{C}$ to an object $F(A)$ of $\mathcal{D}$ and each morphism f of $\mathrm{Mor}(A, B)$ to a morphism $F(f)$ of $\mathrm{Mor}(F(A), F(B))$ such that $F(\mathrm{Id}_A) = \mathrm{Id}_B$ and $F(f \circ g) = F(f) \circ F(g)$, whenever defined.

It can be shown that $k(X)$ is a covariant functor from (Hausdorff spaces, continuous maps) to (compactly generated spaces, continuous maps).

Lemma 2.6.

(1) If $X \in \mathcal{K}$, $k(X) = X$ and for all Hausdorff spaces Y the identity $k(Y) \to Y$ is continuous.
(2) Let X and Y be two Hausdorff spaces. If $f \colon X \to Y$ is continuous, then $k(f) \colon k(X) \to k(Y)$ also is.
(3) If $X \in \mathcal{K}$, then $k_* \colon \mathcal{C}(X, k(Y)) \to C(X, Y)$ is a bijection.

Proof. (1) *First, let us show that if $X \in \mathcal{K}$, then $k(X) = X$.* If A is closed in X, since X is compactly generated, $A \cap K$ is closed in X, for all compact subset $K \subset X$. Therefore A is closed in $k(X)$. Conversely, if $A \subset X$ is closed in $k(X)$,

$A \cap K$ is closed in X, for all compact subset $K \subset X$. Hence, A is closed in X, since $X \in \mathcal{K}$.

Now, let Y *be a Hausdorff space and let us show that* $\mathrm{Id}\colon k(Y) \to Y$ *is continuous* (i.e. that the topology of $k(Y)$ is finer or equal to the initial topology on Y). If $A \subset Y$ is closed, since every compact set in Y is closed (because Y is Hausdorff), we have that $A \cap K$ is closed for all compact subset $K \subset Y$. Therefore, A is closed in $k(Y)$. Thus the map $\mathrm{Id}\colon k(X) \to X$ is continuous is desired.

(2) Suppose that $f\colon X \to Y$ is continuous. Let $A \subset Y$ be a closed subset of $k(Y)$ and $C \subset X$ be a compact subset of X. Since f is continuous, $f(C)$ is compact. Hence, by definition of $k(Y)$, $A \cap f(C)$ is closed in Y. Therefore, by continuity of f we have that $f^{-1}(A \cap f(C))$ is closed in X. Since we have that $f^{-1}(A) \cap f^{-1}(f(C)) = f^{-1}(A \cap f(C))$ is closed and that C is closed (being a compact subset of a Hausdorff space), we have that their intersection is closed. Thus the following is closed

$$f^{-1}(A) \cap C = (f^{-1}(A) \cap f^{-1}(f(C))) \cap C.$$

By generality of $C \subset X$ compact, we have that $f^{-1}(A)$ is closed in $k(X)$.

(3) By (1), we have that k_*, which is defined by $k_*(f) = \mathrm{Id}_{\mathcal{K}} \circ f$, where $\mathrm{Id}_{\mathcal{K}}\colon k(Y) \to Y$ is the identity, is well-defined (i.e. it sends continuous map to continuous map). We want to show that this is a bijection. The injectivity is trivial. We will therefore focus on showing that for all continuous maps $f\colon X \to Y$, $f\colon X \to k(Y)$ is continuous or, in other words, that k_* is surjective. Let A be a closed subset of $k(Y)$. By hypothesis, we have that $X \in \mathcal{K}$ and consequently it suffices to show that $f^{-1}(A) \cap C$ is closed for every compact subset $C \subset X$. Let C be a compact subset of X. Easily, we have $f^{-1}(A \cap f(C)) \cap C = f^{-1}(A) \cap C$. The fact that A is a closed set in $k(Y)$ means in particular that $A \cap f(C)$ is closed in Y (because $f(C)$ is a compact set of Y, f being continuous by hypothesis). Therefore, by continuity of $f\colon X \to Y$, we have that $f^{-1}(A \cap f(C))$ is closed in X and consequently that $f^{-1}(A) \cap C$ is closed (because compact subspace of a Hausdorff space is closed and the intersection of closed sets is closed). By generality of the compact subset C and since X is compactly generated, this means that $f^{-1}(A)$ is closed in X.

$\square$

Remark 2.7. By (3) of the preceding lemma, we have that the set of singular chains on X and $k(X)$ are the same, since Δ_n (the standard n-simplex) is compactly generated (being compact and Hausdorff). Therefore, applying k does not change the singular homology of an Hausdorff space.

Remark 2.8. $X \times Y$ need not be compactly generated if X and Y are, but if X is a locally compact Hausdorff space and Y is compactly generated, then $X \times Y$ is compactly generated.

Proposition 2.9. *If* X *is compactly generated and if* $\pi\colon X \to Y$ *is quotient by a closed relation* $R \subseteq X \times X$ *then* Y *is compactly generated.*

Definition 2.10 (Colimits)**.** Let $\{X_i\}_{i\in\mathbb{N}}$ with $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} X_4 \to \dots$ be such that each X_i is compactly generated and $X_i \xrightarrow{f_i} X_{i+1}$ has closed image. Then the *colimit*, which can be constructed by

$$\operatorname{colim}\{X_i\} := \amalg X_i \ /x_i \sim f_i(x_i),$$

is compactly generated.

3. Function spaces and their topology

The section was typed by Francis Beauchemin-Côté.

Definition 3.1 (Subbasis)**.** A subbasis B for a topology T is a collection of sets such that every open set of T is a union of finite intersections of elements in B. In other words, B generates the topology T, i.e. $T = T(B)$ is the smallest topology containing B.

We want to give $C(X,Y) = \{f\colon X \to Y \mid f \text{ is continuous}\}$ a topology. For this, we specify a subbasis

$$W(K,U) := \{f\colon X \to Y \mid f(K) \subseteq U\}$$

where $K \subseteq X$ is compact and $U \subseteq Y$ is open. We then give $C(X,Y)$ the compact-open topology, which is $T(W(K,U))$.

With the Compact-Open topology on $C(X,Y)$, we can now define the topological space

$$\operatorname{Map}(X,Y) = Y^X := k(C(X,Y)).$$

Theorem 3.2 (Adjoint theorem)**.** *Let X,Y and Z be compact generated spaces. Then the map $\varphi\colon \operatorname{Map}(X\times Y,Z) \to \operatorname{Map}(X,\operatorname{Map}(Y,Z))$ defined by*

$$f \mapsto \varphi(f)\colon x \mapsto (y \mapsto f(x,y))$$

is a homeomorphism. We write $Z^{X\times Y} = (Z^Y)^X$.

Remark 3.3. This implies that $\cdot \times Y$ and $\operatorname{Map}(Y,\cdot)$ are adjoint functors.

Definition 3.4 (Adjoint functors)**.** Functors $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{D} \to \mathscr{C}$ are adjoint if there is a bijection

$$\mathscr{D}(F(A),B) \longleftrightarrow \mathscr{C}(A,G(B))$$

natural in A,B for all $A \in \operatorname{Ob}\mathscr{C}$ and $B \in \operatorname{Ob}\mathscr{D}$.

The reader should draw the commuting diagrams that are invoked by the word natural in the preceding definition. While we are thinking of the meaning of the word natural, here is its principal meaning, in the context of natural transformations.

Definition 3.5 (Natural transformation)**.** A natural transformation $\theta\colon F \to G$ between functors $F, G\colon \mathscr{C} \to \mathscr{D}$ is a morphism $\theta(A)\colon F(A) \to G(A)$ for all objects $A \in \operatorname{Ob}\mathscr{C}$ such that the following diagram commutes

$$\begin{tikzcd} F(A) \arrow[r, "\theta(A)"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\theta(A)"'] & G(B) \end{tikzcd}$$

for all $f \in \mathscr{C}(A, B)$.

Proposition 3.6. *Let X,Y and Z be compactly generated spaces.*

(1) The evaluation map $e\colon \operatorname{Map}(X,Y) \times X \to Y$ defined by $(f,x) \mapsto f(x)$ is continuous.
(2) The obvious map $\operatorname{Map}(X, Y \times Z) \cong \operatorname{Map}(X,Y) \times \operatorname{Map}(X,Z)$, $(f,g) \mapsto (f,g)$ is a homeomorphism.
(3) The map $\operatorname{Map}(X,Y) \times \operatorname{Map}(Y,Z) \to \operatorname{Map}(X,Z)$ defined by $(f,g) \mapsto g \circ f$ is continuous.

Proposition 3.7. *The topological spaces X and $k(X)$ have the same compact sets.*

Proof. Since $k(X) \to X$ is continuous, the compact sets in $k(X)$ are compact in X.

Now, let $C \subseteq X$ be a compact set and C' be C with subspace topology in $k(X)$. Let B be a closed subset of C'. Then $\exists A \subseteq k(X)$ closed such that $B = A \cap C'$. By definition, $A \cap C = A \cap C' = B$ is closed in X since C is compact. Hence, $\mathrm{id}\colon C \to C'$ is continuous, so C' is compact. □

Lemma 3.8.

(1) The map $e\colon C(X,Y) \times X \to Y$ defined by $(f,x) \mapsto f(x)$ is continuous on compact sets.
(2) If X, Y are compactly generated, then $e\colon \operatorname{Map}(X,Y) \times X \to Y$ si continuous.

Proof.

(1) It suffices to check continuity on sets of form $F \times A$ where $F \subseteq C(X,Y)$ compact and $A \subseteq X$ compact.
Let $(f_0, x_0) \in F \times A$ and let $U \subseteq Y$ be an open set containing $f_0(x_0)$. Since every Hausdorff compact space is normal and f_0 is continuous, then there exists $N \ni x_0$ an open neighbourhood $N \subseteq A$ with $f_0(\overline{N}) \subseteq U$. So $(W(\overline{N}, U) \cap F) \times N$ is open and contains (f_0, x_0).
(2) We apply k to $e\colon C(X,Y) \times X \to Y$.
If $g\colon A \to B$ is continuous on all compact sets, then $k(g)\colon k(A) \to k(B)$ is continuous. Hence $k(e)\colon \operatorname{Map}(X,Y) \times X \to Y$ is continuous (X and Y are compactly generated).

□

4. Cofibrations

This section was typed by Francis Beauchemin-Côté.

Definition 4.1. A map $i : A \to X$ is a *cofibration* if it satisfies the homotopy extension property (HEP):

For any space Y, for any map $f\colon X \to Y$ and for any homotopy $h\colon A \times I \to Y$ that starts with $f \circ i\colon A \to Y$, i.e. $h(a, 0) = f \circ i(a)$ for all $a \in A$, this can be extended to a homotopy $\tilde{h}\colon X \times I \to Y$ starting from $f\colon X \to Y$. That is, if for every Y, f and h there exists a homotopy $\tilde{h}\colon X \times I \to Y$ such that the following diagram commutes

$$\begin{tikzcd}
A \arrow[rr, "i"] \arrow[dd, "i_0"'] & & X \arrow[dl, "f"'] \arrow[dd, "i_0"] \\
& Y & \\
A \times I \arrow[ur, "h"] \arrow[rr, "i\times\mathrm{id}"'] & & X \times I \arrow[ul, dotted, "\tilde{h}"']
\end{tikzcd}$$

where $i_0\colon x \mapsto (x, 0)$.

In general, a diagram like that in the definition above represents a *problem*, and a resolution of the problem is a dotted map that makes the diagram commute. We will consider such problems often in this course.

Proposition 4.2. *If $A \subseteq X$ is a deformation retract, then $i\colon A \to X$ is a cofibration.*

This will follow easily from Theorem 4.11 below, which characterises cofibrations, but the reader could try to prove it directly now.

Definition 4.3.

- $Q := X/i(A)$ is called the cofibre of i (cofibration).
- $A \xrightarrow{i} X \xrightarrow{a} Q$ is called a cofibre sequence.

Definition 4.4 (Pushout)**.** Let $f\colon A \to B$ and $g\colon A \to C$ be two morphisms. The pushout of f and g, denoted $B \cup_A C$, is the unique space (up to isomorphism) satisfying the following universal property:

For any space Y and maps $i_B\colon B \to Y$, $i_C\colon C \to Y$ satisfying $i_B \circ f = i_C \circ g$, there exists a unique map $\theta\colon D \to Y$ such that the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "j_B"] \arrow[ddr, bend left, "i_B"] & \\
C \arrow[r, "j_C"'] \arrow[drr, bend right, "i_C"'] & B \cup_A C \arrow[dr, dashed, "\theta"] & \\
& & Y
\end{tikzcd}$$

commutes.

Lemma 4.5. *Push outs of cofibrations are cofibrations. That is, if $i\colon A \to X$ is a cofibration and $g\colon A \to B$ is any map, then $B \to B \cup_g X$ is a cofibration:*

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "g"'] & X \arrow[d] \\
B \arrow[r, "cofib^n"'] & B \cup_g X
\end{tikzcd}$$

Proof of Lemma 4.5. First, we see that $(B \cup_g X) \times I \cong (B \times I) \cup_{g\times \mathrm{id}} (X \times I)$. We want to find $\tilde{h}$ such that the following diagram commutes

$$\begin{tikzcd}
B \arrow[rr, "i_0"] \arrow[dd] & & B \times I \arrow[dl, "h"'] \arrow[dd] \\
& Y & \\
B \cup_g X \arrow[ur, "f"] \arrow[rr, "i_0"'] & & (B \cup_g X) \times I. \arrow[ul, dotted, "\tilde{h}"']
\end{tikzcd}$$

Consider the diagram

$$\begin{tikzcd}
A \arrow[rrrr, bend left, "i_0"] \arrow[dr, "g"] \arrow[dddd, bend right, "i"'] & & & & A \times I \arrow[dl, "g\times\mathrm{id}"] \arrow[dddd, bend left, "i\times\mathrm{id}"] \\
& B \arrow[rr, "i_0"] \arrow[dd] & & B \times I \arrow[dl, "h"'] \arrow[dd] & \\
\text{pushout} & & Y & & \text{pushout} \\
& B \cup_g X \arrow[ur, "f"] \arrow[rr, "i_0"'] & & (B \cup_g X) \times I \arrow[ul, dashed, "\tilde{h}"'] & \\
X \arrow[ur] \arrow[rrrr, bend right, "i_0"'] & & & & X \times I. \arrow[ul] \arrow[uull, dotted, "\bar{h}"]
\end{tikzcd}$$

If we ignore B and $(B \cup_g X) \times I$, since $i\colon A \to X$ is a cofibration, then there exists a homotopy $\bar{h}\colon X \times I \to Y$. Then, the universal property of the right-hand pushout give a homotopy $\tilde{h}$ such that

$$\begin{tikzcd}
 & B \times I \arrow[d] \arrow[ddl, bend right, "h"'] & A \times I \arrow[l, "g\times \mathrm{id}"'] \arrow[d] \\
 & (B \times I) \cup_g (X \times I) \arrow[dl, dashed, "\tilde{h}"] & X \times I \arrow[l] \arrow[dll, bend left, "\bar{h}"] \\
Y & &
\end{tikzcd}$$

commutes. Then $\tilde{h}$ is the required map. □

4.1. **A universal test space.**

Proposition 4.6. *We can replace Y in the test diagram of cofibration by the universal test space*

$$M_i := X \cup_{A\times\{0\}} A \times I = X \cup_i A \times I$$

where $i\colon A \to X$ is cofibration.

The space M_i is called the *mapping cylinder* of i.

Proof. Suppose the problem is soluble for M_i, that is there exists a map r such that

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, "i"'] & & A \times I \arrow[dl] \arrow[dd] \\
 & M_i & \\
X \arrow[ur] \arrow[rr, "i_0"'] & & X \times I \arrow[ul, dotted, "r"']
\end{tikzcd}$$

commutes. Since M_i is the pushout of

$$\begin{tikzcd}
A \arrow[r, "i_0"] \arrow[d, "i"'] & A \times I \arrow[d] \\
X \arrow[r] & M_i
\end{tikzcd}$$

for any maps $f\colon X \to Y$ and $h\colon A \times I \to Y$, there exists a unique map $\theta\colon M_i \to Y$ such that

$$\begin{tikzcd}
A \arrow[rrr, "i_1"] \arrow[ddd, "i"'] & & & A \times I \arrow[dll, "h"'] \arrow[ddl, "\mathrm{incl}"] \arrow[ddd, "i\times\mathrm{id}"] \\
& Y & & \\
& & M_i \arrow[ul, dashed, "\theta"'] & \\
X \arrow[uur, "f"] \arrow[urr, "\mathrm{incl}"'] \arrow[rrr, "i_1"'] & & & X \times I \arrow[ul, dotted, "r"']
\end{tikzcd}$$

commutes. Then $\theta \circ r$ is the desired map. □

Lemma 4.7. *If $(X \times \{0\}) \cup (i(A) \times I)$ is a retract of $X \times I$, then $i\colon A \to X$ is a cofibration. On the other hand, if a solution to the HEP exists for all Y, then certainly it exists for $Y = M_i$. This completes the proof of the proposition.*

Remark 4.8. The map r in the last proof satisfies $r \circ j$ where $j\colon M_i \to X \times I$ is defined by $(a,t) \mapsto (i(a),t)$ and $(x,0) \mapsto (x,0)$ by definition of retraction. This implies that a cofibration is injective with closed image.

4.2. **Replacing a map by a cofibration.** Any map $f\colon X \to Y$ factors as

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\
& M_f \arrow[ur, "r"'] &
\end{tikzcd}$$

where $M_f := Y \cup_{X\times\{1\}} X \times I$ is the mapping cylinder of f and $r\colon (x,t) \mapsto f(x)$ is a retraction of M_f onto Y.

If $j\colon Y \to M_f$ is the inclusion, then we have $r \circ j = \mathrm{id}_Y$ and $j \circ r \simeq \mathrm{id}_{M_f}$ by the homotopy $M_f \times I \to M_f$ defined by

$$\begin{aligned}
(y,s) &\mapsto y \\
((x,t),s) &\mapsto (x,(1-s)t).
\end{aligned}$$

Therefore $M_f \simeq Y$ and using next theorem we can show that $i\colon X \to M_f$ is a cofibration.

Remark 4.9. Hence, up to homotopy equivalence of the codomain, we can replace any map by a cofibration.

4.3. **Criteria for a map to be cofibration.**

Definition 4.10. A pair (X,A) is an NDR-pair (neighbourhood deformation retract) if there is a map $u\colon X \to I$ with $u^{-1}(0) = A$ and a homotopy $h\colon X \times I \to X$

with

$$\begin{aligned} &h(x,0) = x, \ \forall x \in X \\ &h(a,t) = a, \ \forall a \in A, \ t \in I \text{ and} \\ &h(x,1) \in A \text{ if } u(x) < 1. \end{aligned}$$

If moreover $u(x) < 1$ for all $x \in X$, (X, A) is a DR-pair (A is deformation retract of X).

The following theorem is due to Steenrod [Ste].

Theorem 4.11 (Characterisation of cofibrations.)**.** *Let A be a closed subspace of X. Then the following are equivalent:*

i) (X, A) *is an NDR-pair.*
ii) $(X \times I, X \times \{0\} \cup A \times I)$ *is a DR-pair.*
iii) $X \times \{0\} \cup A \times I$ *is a retract of* $X \times I$*.*
iv) $i\colon A \to X$ *is a cofibration.*

Lemma 4.12. *Suppose (X, A) and (Y, B) are NDR-pairs via (h, u) and (j, v) respectively. Then $(X \times Y, X \times B \cup A \times Y)$ is an NDR-pair via $w\colon (x, y) \mapsto \min(u(x), v(y))$ and*

$$k(x,y,t) = \begin{cases} \left(h(x,t), j\left(y, t\frac{u(x)}{v(y)}\right)\right) & \text{if } u(y) \geq u(x) \\ \left(h\left(x, t\frac{v(y)}{u(x)}\right), j(y,t)\right) & \text{if } u(y) \leq u(x). \end{cases}$$

If (X, A) or (X, B) is a DR-pair, then so is $(X \times Y, X \times B \cup A \times Y)$.

Proof of Theorem 4.11. Since $(I, \{0\})$ is a DR-pair, by lemma 4.12, $(i) \Rightarrow (ii)$, $(ii) \Rightarrow (iii)$ is trivial and we saw earlier in mapping cylinder that $(iii) \Leftrightarrow (iv)$. We need to show that $(iii) \Rightarrow (i)$.

Let $r\colon X \times I \to X \times \{0\} \cup A \times I$ be a retraction and define $u\colon X \to I$ by

$$u(x) = \sup\{t - p_2 \circ r(*, t)\colon t \in I\}$$

and $h\colon X \times I \to X$ by

$$h(x,y) = p_1 \circ r(x,t)$$

where $p_1\colon X \times I \to X$ and $p_2\colon X \times I \to I$ are projections.

Since $r(a,t) = (a,t)$ and $p_2(r(a,t)) = t$, we have $t - p_2 \circ r(a,t) = 0$ $\forall a \in A$, hence $A \subseteq u^{-1}(0)$.

Suppose now $u(x) = 0$ for $x \in X \backslash A$. Then $t \leq p_2 \circ r(x,t)$ $\forall t \in I \Rightarrow$ for $t \neq 0$, $r(x,t) \in A \times I$. Since A is closed, there exists an open set $U \ni x$, $U \subseteq X \backslash A$ with $r^{-1}(U) = U \times \{0\} \subseteq X \times I$ but this is not open, which contradicts the fact that r is continuous. Therefore, $u^{-1}(0) \subseteq A \Rightarrow u^{-1}(0) = A$. We also have

$$\begin{aligned} h(a,t) &= p_1(r(a,t)) = p_1(a,t) = a \\ h(x,0) &= p_1(r(x,0)) = p_1(x,0) = x \end{aligned}$$

and $u(x) = 1$ occurs only if $r(x,1) \in X \times \{0\} \subseteq X \times I$. If $u(x) < 1$, $h(x,1) = p_2 \circ r(x,1) = p_1(a,t) = a \in A$ for some $a \in A$, $t \in I$. So (h, u) present (X, A) as an NDR-pair as claimed. $\square$

4.4. **More properties of cofibrations.**

Proposition 4.13.

(i) Let $i\colon A \to X$ be a map. There exists a homotopy equivalence $h\colon M_i \to X$ such that

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[dr, "a\mapsto(a,1)"] & \\ X & & M_f \arrow[ll, "h"'] \end{tikzcd}$$

commutes. (This is called a homotopy equivalence under A).

(ii) Suppose moreover that i is a cofibration. Then h is a homotopy equivalence rel. A.

Proof. To prove (i), define $h\colon M_f \to X$ by $h(a,t) = i(a)$ for $a \in A$ and $t \in I$ and $h(x) = x$ for $x \in X$. Note that this fits into the diagram above. To show that h is a homotopy equivalence, we define the map $g\colon X \to M_f$ to be the inclusion, and we note that $h \circ g = \mathrm{Id}_X$. There is a homotopy $F\colon g \circ h \sim \mathrm{Id}_{M_f}$ defined by $F((a,t),s) = (a, st)$ and $F(x) = x$. This completes the proof of (i).

To prove (ii), we start by recalling that i being a cofibration implies that there is a retraction

$$r\colon X \times I \to X \times \{0\} \cup i(A) \times I,$$

expressing the mapping cylinder as a subset of $X \times I$. Define maps

$$\begin{array}{rcl} g\colon X & \to & M_i \\ x & \mapsto & r(x,1) \end{array}$$

and

$$\begin{array}{rcl} h\colon M_f & \to & X \\ (a,t) & \mapsto & i(a) \\ x & \mapsto & x \end{array}.$$

We claim that g and h are homotopy inverses relative to $A \times \{1\}$. To see this, first define $H = h \circ r\colon X \times I \to X$. This satisfies $H(x,1) = h \circ g(x)$, $H(x,0) = x$ and $H(i(a),t) = i(a)$ for all $t \in I$ and for all $a \in A$. So $H\colon h \circ g \sim \mathrm{Id}_X$. Next, define

$$\begin{array}{rcl} G\colon M_i \times I & \to & M_i \\ (x,t) & \mapsto & r(x,t) \\ ((a,s),t) & \mapsto & r(i(a),st). \end{array}$$

This satisfies $G(x,1) = r(x,1) = g \circ h(x)$, $G(x,0) = r(x,0) = x$ for all $x \in M_i$, and $G(i(a),t) = i(a)$ for all $a \in A$ and for all $t \in I$. So $G\colon g \circ h \sim \mathrm{Id}_{M_i}$. Thus indeed g, h are homotopy equivalences rel. A. □

Since $M_i \simeq X$ rel. A, we may quotient by A and obtain a homotopy equivalence

$$M_i/A = C_i \simeq X/A.$$

Here $C_i := M_i/(A \times \{1\})$ is the mapping cone of i.

Here is another useful property of cofibrations:

Proposition 4.14. *Let (X,A) be an NDR pair. Then $\widetilde{H}_*(X/A) \cong H_*(X,A)$.*

Proof. Let $W := u^{-1}([0,1))$. Then (W,A) is a DR pair and $W/A \simeq *$. Thus we have

$$\begin{aligned}\widetilde{H}_*(X/A) &\cong H_*(X/A,*) \cong H_*(X/A, W/A)\\ &\cong H_*(X-A, W-A) \cong H_*(X,W)\\ &\cong H_*(X,A)\end{aligned}$$

Here the isomorphisms are given respectively by the definition, by homotopy equivalence, by excision, by excision, and finally by homotopy equivalence again. □

Overall, one should think of cofibrations as nice inclusions, which are sufficiently general that most inclusions you would consider naturally are cofibrations, which satisfy many useful properties, and such that any map can be replaced by a cofibration up to a homotopy equivalence.

5. Fibrations

Fibrations, as suggested by the removal of the prefix "co," are in some sense dual to cofibrations. Whereas cofibrations are nice inclusions, fibrations are nice projections. They can be thought of as generalisations of fibre bundles. For a fibre bundle the fibre is well-defined up to homeomorphism, but in a fibration the fibre is only well-defined up to homotopy equivalence.

In this section we will write X^I for the space of free paths $\gamma\colon I \to X$ in X, and we will use the map $p_0\colon X^I \to X$ that sends $\gamma \mapsto \gamma(0)$.

Definition 5.1 (Fibration)**.** A surjective map $p\colon E \to B$ is a *fibration* if it satisfies the Covering Homotopy Property (CHP), also called the Homotopy Lifting Property (HLP) in the literature. This property is that, for any space Y and for any maps $f\colon Y \to E$ and $h\colon Y \to B^I$ with $p \circ f = p_0 \circ h$, there is a lift $\widetilde{h}\colon Y \to E^I$ such that the diagram

$$\begin{tikzcd} E \arrow[d, "p"'] & & E^I \arrow[ll, "p_0"'] \arrow[d, "p^I"] \\ & Y \arrow[ul, "f"] \arrow[ur, dashed, "\widetilde{h}"] \arrow[dr, "h"] & \\ B & & B^I \arrow[ll, "p_0"] \end{tikzcd}$$

commutes. Here $p^I\colon E^I \to B^I$ is the map induced from p by post-composition. An equivalent formulation, which explains the HLP terminology is asking for a solution to any diagram:

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "h"'] \arrow[ur, dashed, "\widetilde{h}"] & B. \end{tikzcd}$$

As a basic example, consider a product $E = F \times B$ and the projection $p = p_2\colon F \times B \to B$. This map is a fibration. To see this, consider the diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "i_0"'] & F \times B \arrow[d, "p_2"] \\ Y \times I \arrow[r, "h"'] \arrow[ur, dashed, "\widetilde{h}"] & B. \end{tikzcd}$$

Write $f(y) = (f_1(y), f_2(y)) \in F \times B$. Define

$$\begin{array}{rcl} \widetilde{h}\colon Y \times I & \to & F \times B \\ (y,t) & \mapsto & (f_1(y), h(y,t)). \end{array}$$

The fact that $p_2(f(y)) = f_2(y) = h(y, 0)$ implies that the diagram commutes as required.

In the literature the fibrations we have defined are sometimes called Hurewicz fibrations. You may also see Serre fibrations: these are weaker, only requiring the CHP to hold for $Y = I^n$. This can be sufficient when working with CW complexes, but we will not make use of this notion in this course.

Lemma 5.2 (Pullback of a fibration is a fibration)**.** *Suppose that $p\colon E \to B$ is a fibration, and $g\colon A \to B$ is a map. Consider the pullback*

$$\begin{tikzcd} A \times_g E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ A \arrow[r, "g"] & B. \end{tikzcd}$$

The pull back $A \times_g E$ can be thought of as a subset of $A \times E$: $\{(a, e) \in A \times E \mid p(e) = g(a) \in B\}$. We have that the induced projection $A \times_g E \to A$ is a fibration for any map $g\colon A \to B$.

To prove this, dualise the proof of the corresponding fact for cofibrations. This is left as an exercise. We actually know many examples already.

Theorem 5.3. *Let $p\colon E \to B$ be continuous. Suppose B is paracompact and there exists an open cover $\{U_\alpha\}$ of B for which $p|\colon p^{-1}(U_\alpha) \to U_\alpha$ is a fibration for each U_α. Then $p\colon E \to B$ is a fibration.*

Since B is Hausdorff, recall that B is paracompact if and only if it admits a partition of unity subordinate to any open cover. We will prove a more general version of this theorem next time. The idea is to lift the homotopies on each open set, and then patch the lifts together in a cunning fashion. For now, we just remark that this means that any fibre bundle with paracompact base is a fibration. In particular any covering map is a fibration.

Definition 5.4. For a fibration $p\colon E \to B$, let $F := p^{-1}(*)$ be the *fibre*, where $* \in B$ is the basepoint. Write $F \to E \to B$. This is a *fibre sequence*.

5.1. **Path space fibration.**

Example 5.5 (Path space fibration)**.** The path space fibration is a very important example. Let X be a space. Then recall that $PX = \{\gamma\colon I \to X \mid \gamma(0) = *_X\}$ is the space of based paths. The map

$$\begin{array}{rcl} e_1\colon PX & \to & X \\ \gamma & \mapsto & \gamma(1) \end{array}$$

is a fibration. We need to solve the diagram:

$$\begin{tikzcd} PX \arrow[d, "e_1"'] & & PX^I \arrow[ll, "p_0"'] \arrow[d, "e_1^I"] \\ & Y \arrow[ul, "f"] \arrow[ur, dashed, "\widetilde{h}"'] \arrow[dr, "h"] & \\ B & & B^I \arrow[ll, "p_0"] \end{tikzcd}$$

The idea is as follows. For each $y \in Y$ we are given two paths, $f(y)$ and $h(y)$, such that the endpoint of $f(y)$ is the start of $h(y)$. We need a path of paths that interpolates between $f(y)$ and the concatenation $f(y) \cdot h(y)$. This is given by, for example:

$$\widetilde{h}(y)(s) = \begin{cases} t \mapsto f(y)(t/(1-s/2)) & 0 \le t \le 1-s/2 \\ t \mapsto h(y)(2(t-1+s/2)) & 1 = s/2 \le t \le 1. \end{cases}$$

Since $h(y)(0) = f(y)(1)$, this is well-defined.

The fibre of $PX \to X$ is $e_1^{-1}(*) = \{\gamma\colon I \to X \mid \gamma(0) = \gamma(1)\} =: \Omega X$. The space ΩX is called the loop space of X and will be very important in future. If we want to emphasise the basepoint we might write $\Omega_* X$. The path space fibration is then written

$$\Omega X \to PX \to X.$$

5.2. **Universal test space.** There is a universal test space for fibrations (analogous to the mapping cylinder for cofibrations). Instead of general Y in the test diagram, we can let Y be the mapping path space

$$Np := E \times_p B^I = \{(e, \gamma) \in E \times B^I \mid p(e) = \gamma(0)\}.$$

That is, Np is the pullback in the square

$$\begin{tikzcd} Np \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ B^I \arrow[r, "p_0"'] & B. \end{tikzcd}$$

The maps in the test diagram are the projections $p_1\colon Np \to E$ and $p_2\colon Np \to B^I$. Consider the test diagram:

$$\begin{tikzcd}
E \arrow[ddd, "p"'] & & & E^I \arrow[lll, "p_0"'] \arrow[ddd, "p^I"] \\
& & Np \arrow[ull, "p_1"'] \arrow[ur, dashed, "s"] \arrow[ddr, "p_2"] & \\
& Y \arrow[uul, "f"] \arrow[ur, "g"] \arrow[dr, "h"] & & \\
B & & & B^I \arrow[lll, "p_0"']
\end{tikzcd}$$

Suppose that a solution exists for Np. Then the map s exists as in the diagram. But then by the universal property of pullbacks, for any Y, f, h as in the diagram, there exists a map $g\colon Y \to Np$ such that the diagram commutes. Then $s \circ g\colon Y \to E^I$ solves the problem.

The map $s\colon Np \to E^I$ that solves the problem is called a *path lifting function.* This is a function such that $k \circ s = \mathrm{Id}$, where $k = (p_0, p^I)\colon E^I \to Np$. In other words we require that $s(e,\gamma)(0) = e$ and $p \circ (s(e,\gamma)) = \gamma$.

5.3. **Relationship between fibrations and cofibrations.** Here is an appearance of both notions in one lemma.

Lemma 5.6. *Let $i\colon A \to X$ be a cofibration, and let B be a space. Then*

$$p := B^i\colon B^X \to B^A$$

is a fibration.

Proof. First, we have

$$B^{M_i} = B^{X\times\{0\}\cup A\times I} \cong B^X \times_p (B^A)^I = Np.$$

The central homeomorphism here follows from $(B^A)^I \cong B^{A\times I}$. Next, the fact that i is a cofibration implies that there is a relation $r\colon X \times I \to M_i$. Then

$$s = B^r\colon Np = B^{M_i} \to B^{X\times I} \cong (B^X)^I$$

is a path lifting function. $\square$

5.4. **Replacing a map by a fibration.** Let $f\colon X \to Y$ be a map. We can factor f as a homotopy equivalence followed by a fibration. Let $Nf = X \times_f Y^I$ as before. The map f coincides with the composition

$$X \xrightarrow{\nu} Nf \xrightarrow{\rho} Y$$

where $\nu(x) = (x, c_{f(x)})$, with $c_y\colon I \to Y$ the constant path at $y \in Y$, and $\rho(x,\gamma) = \gamma(1)$. We claim that ν is a homotopy equivalence and that ρ is a fibration.

To see that ν is a homotopy equivalence, let $p_1 \colon Nf \to X$ be the projection. Then $p_1 \circ \nu = \mathrm{Id}_X$ and we have a homotopy

$$\begin{array}{rcl} h\colon Nf \times I & \to & Nf \\ ((x,\gamma),t) & \mapsto & (x, s \mapsto \gamma((1-t)s)) \end{array}$$

from $\nu \circ \rho$ to Id_{Nf}.

Next, to see that ρ is a fibration, we need to solve the following diagram, for any space A and any maps g, h.

$$\begin{tikzcd}
Nf \arrow[dd, "\rho"'] & & Nf^I \arrow[ll, "p_0"'] \arrow[dd, "\rho^I"] \\
& A \arrow[ul, "f"'] \arrow[ur, dashed, "\widetilde{h}"] \arrow[dr, "h"] & \\
Y & & Y^I \arrow[ll, "p_0"]
\end{tikzcd}$$

Write $g(a) = (g_1(a), g_2(a)) \in X \times_f Y^I = Nf$, for $a \in A$. Then define $\widetilde{h}(a) = \big(t \mapsto (g_1(a), j(a,t))\big)$ with $g_1(a) \in X$ and $j(a,t) \in Y^I$ given by

$$\begin{array}{rcl} j(a,t)\colon I & \to & Y \\ s & \mapsto & \begin{cases} g_2(a)(s+st) & 0 \le s \le \frac{1}{1+t} \\ h(a, s+ts-1) & \frac{1}{1+t} \le s \le 1. \end{cases} \end{array}$$

This map $\widetilde{h}$ solves the problem, so ρ is a fibration as claimed.

5.5. **Criterion for a map to be a fibration.** Let $\mathscr{U}$ be an open cover of a space B. We say that $\mathscr{U}$ is *numerable* if there are maps $\lambda_U \colon B \to I$ for each $U \in \mathscr{U}$ such that $\lambda_U^{-1}((0,1]) = U$, and the cover is *locally finite*, that is for each $b \in B$ there is a neighbourhood $V_b \ni b$ such that $V_b \cap U \neq \emptyset$ for at most finitely many $U \in \mathscr{U}$.

Theorem 5.7. *Let $p\colon E \to B$ be a map and let $\mathscr{U}$ be a numerable open cover of B. Then p is a fibration if and only if $p|\colon p^{-1}(U) \to U$ is a fibration for every $U \in \mathscr{U}$.*

In particular, this implies that fibre bundles with paracompact base spaces are fibrations.

Proof. First, pullbacks of fibrations are fibrations, so if $E \to B$ is a fibration then the pullback along the inclusion $U \to B$ is a fibration for any subset U. This proves the only if direction.

So from now on, let $p\colon E \to B$ be a map such that $p|\colon p^{-1}(U) \to U$ is a fibration for every $U \in \mathscr{U}$. Our aim is to construct a path lifting function $s\colon Np = E \times_p B^I \to E^I$ by patching together the path lifting functions

$$s_U \colon p^{-1}(U) \times_p U^I \to p^{-1}(U)^I$$

that exist by hypothesis for each U. To do this consistently, we need some amount of set up. In particular, we need a special open cover of the path space B^I.

Let $\lambda_U\colon B \to I$ be maps such that $\lambda_U^{-1}((0,1]) = U$, that are given to us by the assumption that $\mathscr{U}$ is numerable. For $T = \{U_1, \dots, U_n\}$ a finite ordered subset of $\mathscr{U}$, write $c(T) = n$, and define functions

$$\begin{array}{rcl} \lambda_T\colon B^I & \to & I \\ \beta & \mapsto & \inf\{\lambda_{U_i} \circ \beta(t) \mid (i-1)/n \leq t \leq i/n, 1 \leq i \leq n\}. \end{array}$$

This is nonzero as long as β lies in U_i during the required time interval $[(i-1)/n, i/n]$. Next define subsets of the path space B^I as follows:

$$W_T := \lambda_T^{-1}((0,1]) = \{\beta \mid \beta(t) \in U_i \text{ if } t \in [(i-1)/n, i/n]\}.$$

We assert that $\{W_T\}$ is an open cover of B^I.

Next we need to improve $\{W_T\}$ to a locally finite cover. However $\{W_T \mid c(T) = n\}$ is locally finite for each n. We will use this observation to construct a suitable covering of B^I. Suppose that $c(T) = n$. Define a function

$$\begin{array}{rcl} \gamma_T\colon B^I & \to & I \\ \beta & \mapsto & \max\{0, \lambda_T(\beta) - n\sum_{c(S)<n} \lambda_S(\beta)\} \end{array}$$

and then define the sets

$$V_T := \{\beta \in B^I \mid \gamma_T > 0\}.$$

We assert that $\{V_T\}$ is a locally finite open cover of B^I.

Next, choose a total ordering of all the finite ordered subsets $T \subseteq \mathscr{U}$. Since $p|_U$ is a fibration, there are path lifting functions

$$s_U\colon p^{-1}(U) \times_p U^I \to (p^{-1}(U))^I$$

for each $U \in \mathscr{U}$. Recall that our aim is to piece them together to get a global path lifting function.

Fix $T = \{U_1, \dots, U_n\}$, and let $\beta \in V_T$. Define the path $\beta[u,v] := \beta|_{[u,v]}\colon [u,v] \to B$, the restriction of β to the interval $[u,v]$, where $0 \leq u \leq v \leq 1$. Suppose that $u \in [(i-1)/n, i/n]$ and $v \in [(j-1)/n, j/n]$, where $0 \leq i \leq j \leq n$. Suppose that $e \in p^{-1}(\beta(u))$.

Let $s_T(e, \beta[u,v])\colon [u,v] \to E$ be the path starting at e and covering $\beta[u,v]$ (that is, $p \circ s_T(t) = \beta(t)$ for all $t \in [u,v]$, obtained by using:

- s_{U_i} to lift over $[u, i/n]$;
- $s_{U_{i+k}}$ to lift over $[(i+k-1)/n, (i+k)/n]$;
- s_{U_j} to lift over $[(j-1)/n, v]$.

In order to do this, we need to rescale, since each s_U is for paths $I \to B$ but we lift on partial intervals only. Now define the lift we seek as follows. Define

$$s(e, \beta) \in E^I$$

by concatenating the paths $s_{T_j}(e_{j-1}, \beta[u_{j-1}, u_j])$, for $1 \leq j \leq q$, where:

- T_i, for $i = 1, \dots, q$ are the sets of subsets of $\mathscr{U}$, in order, for which $\beta \in V_T$;
- $u_j := \sum_{i=1}^j \gamma_{T_i}(\beta)$, $1 \leq j \leq q$;
- $e_0 = e$;
- e_j is the endpoint of $s_{T_j}(\beta[u_{j-1}, u_j])$, for $1 \leq j < q$.

We have that $s(e,\beta)(0) = e$ and $(p \circ s(e, \beta) = \beta$. Thus s is a path lifting function as required. $\square$

5.6. **Spaces over B and fibre homotopy equivalences.** A *space over B* is a map $p\colon E \to B$. A map of spaces over B is a diagram:

$$\begin{tikzcd} D \arrow[rr, "f"] \arrow[dr, "p"'] & & E \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

A homotopy over B is a map

$$\begin{tikzcd} D \times I \arrow[rr, "H"] \arrow[dr, "p"'] & & E \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

where

$$\begin{tikzcd} D \times \{t\} \arrow[rr, "H|"] \arrow[dr, "p|"'] & & E \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

is a map over B for all $t \in I$. Thus $f\colon D \to E$ is homotopy equivalent over B if there exists $g\colon E \to D$ over B with $f \circ g, g \circ f \sim \mathrm{Id}$ over B. The maps f and g are called *fibre homotopy equivalences.*

Proposition 5.8. *Let $p\colon D \to B$, $q\colon E \to B$ be fibrations and let $f\colon D \to E$ be a map over B. Suppose that f is a homotopy equivalence. Then f is a fibre homotopy homotopy equivalence.*

We will omit the proof, unfortunately.

5.7. **Change of fibre.** Let $p\colon E \to B$ be a fibration. Write $F_b := p^{-1}(b)$ for $b \in B$. For $b, b' \in B$ we have $F_b \simeq F_{b'}$. That is, for fibrations, all fibres are homotopy equivalent. (By contrast, for fibre bundles all the fibres are homeomorphic.)

Theorem 5.9. *Let $p\colon E \to B$ be a fibration, and suppose that B is path connected. Any two fibres of p are homotopy equivalent. In general, a path lifting function along a homotopy class rel. boundary of paths between b and b' determines a homotopy class of maps $F_b \to F_{b'}$. Applied to loops, we get a homomorphism $\pi_1(B, b) \to \pi_0(\mathrm{Aut}(F_b))$.*

Proof. Let $b, b' \in B$ be $F_b = p^{-1}(b)$. Let $i_b\colon F_b \to E$ be the inclusion. Let $\beta\colon I \to B$ be such that $\beta(0) = b$ and $\beta(1) = b'$. The HLP implies that there is a $\widetilde{\beta}$ that fits into the following diagram:

$$\begin{tikzcd} F_b \times \{0\} \arrow[rr, "i_b"] \arrow[d] & & E \arrow[d, "p"] \\ F_b \times I \arrow[r] \arrow[urr, dashed, "\widetilde{\beta}"] & I \arrow[r, "\beta"] & B \end{tikzcd}$$

For each $t \in I$, we get a map $\widetilde{\beta}_t \colon F_b \times \{t\} \to F_{\beta(t)}$. In particular we obtain a map

$$\widetilde{\beta}_1 \colon F_b \to F_{b'}.$$

We claim that whenever $\beta \sim \beta'$ rel. boundary, we have $\widetilde{\beta}_1 \sim \widetilde{\beta}'_1$. Given the claim, using the fact $\widetilde{(c_b)}_1 = \mathrm{Id}$ and that $\widetilde{(\beta \cdot \gamma)}_1 = \widetilde{\beta}_1 \circ \widetilde{\gamma}_1$, we see that $\widetilde{(\beta^{-1})}_1$ is a homotopy inverse $\widetilde{\beta}_1$.

It now remains to prove the claim. Let $h \colon I \times I \to B$ be a homotopy from $\beta \sim \beta'$ that fixes b, b', that is $h(1, s) = b'$ and $h(0, s) = b$ for all $s \in I$. Let $\widetilde{\beta}' \colon F_b \times I \to E$ cover $\beta' \circ p_2$.

Write $J^2 = I \times \partial I \cup \{0\} \times I \subset I^2 = I \times I$. Note that there is a homeomorphism of pairs $(I^2, J^2) \cong (I^2, I \times \{0\})$. Thus with test space $Y = F_b \times I$, we can apply the HLP with the pair $F_b \times J^2 \to F_b \times I^2$ instead of $F_b \times I \times \{0\} \to F_b \times I^2$. Define a map

$$\begin{aligned} f \colon F_b \times J^2 &\to E \\ (x, t, s) &\mapsto \begin{cases} \widetilde{\beta}(x, t) & s = 0 \\ \widetilde{\beta}' & s = 1 \\ i_b(x) & t = 0 \end{cases} \end{aligned}$$

Then apply the HLP to the following diagram:

$$\begin{tikzcd} F_b \times J^2 \arrow[rr, "f"] \arrow[d] & & E \arrow[d, "p"] \\ F_b \times I^2 \arrow[r, "p_2"] \arrow[urr, dashed, "\widetilde{h}"] & I^2 \arrow[r, "h"] & B \end{tikzcd}$$

We obtain a map $\widetilde{h} \colon F_b \times I^2 \to E$. The restriction $\widetilde{h}|_{F_b \times \{1\} \times I} \colon F_b \times I \to E$ is a homotopy $\widetilde{\beta}_1 \sim \widetilde{\beta}'_1$, as required. $\square$

5.8. **Examples: Hopf fibrations and homogeneous spaces.** Here are some important examples of fibrations. First, the Hopf fibrations are:

$$S^0 \to S^n \to \mathbb{RP}^n$$

$$S^1 \to S^{2n+1} \to \mathbb{CP}^n$$

$$S^3 \to S^{4n+3} \to \mathbb{HP}^n$$

These are given by expression the spheres as elements of $\mathbb{F}^{n+1}$ with norm one, and then considering a point in these coordinates as the same numbers in homogeneous coordinates for the projective space. For $n = 1$, the Hopf fibrations reduce to:

$$S^0 \to S^1 \to S^1$$

$$S^1 \to S^3 \to S^2$$

$$S^3 \to S^7 \to S^4$$

There is also an octonian fibration $S^7 \to S^{15} \to S^8$, but there are no higher octonian versions of the Hopf fibrations.

Now we consider homogeneous spaces, which can produce fibre bundles.

Definition 5.10. A map $p \to B$ has a local section at $b \in B$ if there is $U \ni b$ open and $s\colon U \to E$ with $p \circ s = i$ where $i\colon U \to B$ is the inclusion.

Any fibre bundle has a local section for all $b \in B$. Let G be a topological group, and let $H \leq G$ be a closed subgroup. We will consider the left cosets G/H. This coset space is sometimes called a homogeneous space.

Lemma 5.11. *If $p\colon G \to G/H$ has a local section at e, then it has a local section for all points $gH \in G/H$.*

Proof. Let $e \in U$ be the open set and let $s\colon U \to G$ be the local section. Given $x = gH \in G/H$, the translate gU is an open set containing x. Define

$$\begin{array}{rcl} s\colon gU & \to & G \\ gg'H & \mapsto & gs(g'H) \end{array}$$

where $g'H \in U$. This defines the desired local section. □

Now we see that a local section at the identity is in fact enough to prove that a map is a fibre bundle.

Lemma 5.12. *Let G be a topological group, let $K \leq H \leq G$ be closed subgroups. Suppose that $G \to G/H$ has a local section. Then*

$$H/K \to G/K \xrightarrow{p'} G/H$$

is a fibre bundle, where $gK \mapsto gH$ for all $g \in G$.

Proof. Let $\{U_\alpha\}$ be an open cover of G/H. By assumption we have a local section $s_\alpha\colon U_\alpha \to G$ for every α.

Define maps

$$\begin{array}{rcl} \psi_\alpha\colon U_\alpha \times H/K & \to & (p')^{-1}(U_\alpha) \subseteq G/K \\ (gH, hK) & \mapsto & s_\alpha(gH)hK \end{array}$$

and

$$\begin{array}{rcl} \theta_\alpha\colon (p')^{-1}(U_\alpha) & \to & U_\alpha \times H/K \\ gK & \mapsto & (gH, (s_\alpha(gH))^{-1}gK) \end{array}$$

We claim that these are continuous maps inverse to one another, and therefore are homeomorphisms. □

Now we move on to considering concrete examples. Recall that $O(n)$ denotes the orthogonal group of $n \times n$ matrices A such that $AA^T = A^TA = \mathrm{Id}$. Such matrices can be considered as living in $\mathbb{R}^{n^2}$, and with the subspace topology $O(n)$ is in fact a compact manifold and a topological group. There is an inclusion $O(k) \subset O(n)$, with $k < n$, where $A \mapsto \begin{pmatrix} A & 0 \\ 0 & I_{n-k} \end{pmatrix}$ where I_{n-k} denotes the size $n-k$ identity matrix. We will show that $O(k) \to O(n) \to O(n)/O(k)$ is a fibre bundle. In fact we will prove a more general statement.

Definition 5.13. A *k-frame* in $\mathbb{R}^n$ is an ordered orthonormal set of k-vectors $\{v_1, \ldots, v_k\} \subset \mathbb{R}^n$ (i.e. $v_i \cdot v_j = \delta_{ij}$.) Define $V_{k,n} = \{k\text{-frames in } \mathbb{R}^n\}$. This can be considered as a subset of $\mathbb{R}^{nk}$, and with the subspace topology this becomes a compact manifold, called the Stiefel manifold $V_{k,n}$.

Lemma 5.14. *There is a homeomorphism* $O(n)/O(n-k) \xrightarrow{\simeq} V_{k,n}$.

Proof. Define $\theta\colon A \mapsto \{v_1 = Ae_{n-k+1}, \dots, v_k = Ae_n\}$, where e_i is the ith standard basis vector. Since A is orthogonal,$\{v_1, dots, v_k\}$ is an orthonormal set. Moreover

$$\theta\left(\begin{pmatrix} B & 0 \\ 0 & I_k \end{pmatrix} A\right) = \theta(A)$$

for any $B \in O(n-k)$. Therefore θ descends to a well-defined map $\theta\colon O(n)/O(n-k) \to V_{n,k}$. If A, B have the same last k columns, then

$$A^{-1}B = A^T B = \begin{pmatrix} C & 0 \\ 0 & I_k \end{pmatrix},$$

which implies that θ is injective. Also note that θ is surjective. A continuous bijection from a compact space to a Hausdorff space is a homeomorphism. □

Proposition 5.15. *For* $0 \leq k \leq l \leq n$

$$O(n-k)/O(n-l) \to O(n)/O(n-l) \xrightarrow{p} O(n)/O(n-k),$$

or equivalently

$$V_{l-k,n-k} \to V_{l,n} \to V_{k,n},$$

is a fibre bundle.

Proof. We will show that there is a local section of $V_{n,n} \to V_{k,n}$, where the map sends an n-frame to the last k-vectors, at $e = O(n-k)$. That is, we need an open set U around e and a map $s\colon U \to O(n)$ such that $p \circ s = \mathrm{Id}$. Define $r((w_1, \dots, w_k)) = (e_1, \dots, e_{n-k}, w_1, \dots, w_k)$. Note that $r(e_{n-k+1}, \cdots, e_n) = (e_1, \dots, e_n)$. There exists an open set $U \subseteq V_{k,n}$ around $(e_{n-k+1}, \dots, e_n)$ with $r(u)$ nondegenerate for all $u \in U$. Now for $(v_1, \dots, v_k) \in U$, take $r(v_1, \dots, v_k) = e_1, \dots, e_{n-k}, v_1, \dots, v_k$, and apply the Gram-Schmidt process to $(v_k, \dots, v_1, e_{n-k+1}, \dots, e_1)$, to obtain an orthonormal set $v_k, \dots, v_1, e'_{n-k+1}, \dots, e'_1$. This gives an element $(e'_1, \dots, e'_{n-k+1}, v_1, \dots, v_k) \in V_{n,n}$. This completes the construction of the desired local section. □

Some key examples of Stiefel manifolds are $V_{n,n} \cong O(n)$ and $V_{1,n} \cong S^{n-1}$. Thus as special cases we have fibre bundles

$$O(m) \to O(m+1) \to S^m$$

and

$$S^{n-\ell} \cong V_{1,n-\ell+1} \to V_{\ell,n} \to V_{\ell-1,n}.$$

We will use these fibre bundles for homotopy computations later.

One more interesting example involved the Grassmannian. The Grassmannian $G_{n,k}$ is the set of k-dimensional vector subspaces of $\mathbb{R}^n$. This can also be topologised and becomes a manifold. There is a forgetful mapping $\pi\colon V_{k,n} \to G_{k,n}$, which is also a fibre bundle with fibre $V_{k,k}$. This fibre bundle is an exercise.

6. Exact sequences in homotopy sets and homotopy groups

This section was typed by Nima Hoda.

6.1. **Notation review/taking basepoints seriously.** We recall that the spaces we consider are, in general, based and compactly generated. We base the interval $(I, *) = ([0,1], \{1\})$ at the point 1. The reduced cone of a space X, then, is the smash product $X \wedge I$. The *reduced suspension* ΣX of a space X is $X \wedge S^1$. For a space Y, we let Y_+ denote the space $(Y \sqcup \{*\}, *)$. Note the identities

$$X \wedge Y_+ = (X \times Y)/(* \times Y)$$

and

$$X_+ \wedge Y_+ = (X \times Y)_+.$$

We call $X \wedge I_+$ the *reduced cylinder* of X. Note that a map $X \wedge I_+ \to Y$ is a based homotopy.

For a based map $f\colon X \to Y$, we redefine N_f as the pullback of f and the map $Y^I \to Y$ sending γ to $\gamma(1)$ (rather than $\gamma(0)$). That is,

$$N_f = \{(x, \gamma) \in X \times Y^I \mid f(x) = \gamma(1)\}.$$

Henceforth all cofibrations and fibrations are based and all basepoints of spaces are nondegenerate.

Definition 6.1. We say that the basepoint of X is *non-degenerate* if $* \to X$ is a cofibration.

Remark 6.2. The map $p\colon E \to B$ is a fibration if and only if it is an unbased fibration and $p(*_E) = *_B$.

Remark 6.3. The map $i\colon A \to X$ is a cofibration if and only if it is an unbased fibration and $p(*_E) = *_B$.

Remark 6.4. The map $i\colon A \to X$ is a cofibration if and only if $M_i = X \cup_i (A \wedge I_+)$ is a retract of $X \wedge I_+$.

From now on we will work without comment in the category $\mathcal{K}_*$ of compactly generated spaces with nondegenerate basepoints.

6.2. **Exact sequences of mapping sets.**

Definition 6.5. A sequence of functions of based sets

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is *exact* if $f(A) = g^{-1}(*_C)$.

Theorem 6.6. *Let $p\colon E \to B$ be a fibration, where B is path connected. Let $F = p^{-1}(*_B)$ be the fibre. Let Y be any space. Based homotopy classes of maps induce an exact sequence*

$$[Y, F] \xrightarrow{i_*} [Y, F] \xrightarrow{p_*} [Y, B].$$

Proof. Take $[g] \in [Y, F]$. Then

$$\begin{aligned} p_* \circ i_*([g]) = p \circ i \circ g\colon Y &\to B \\ y &\mapsto *_B \end{aligned}$$

and so $i_*([Y, F]) \subseteq p_*^{-1}([c_{*_B}])$.

Now, take $[f] \in p_*^{-1}([c_{*_B}])$. So $f\colon Y \to E$ and $p_*([f]) = [p \circ f] = [c_{*_B}]$, i.e., $p \circ f\colon Y \to B$ is homotopic to c_{*_B}. Let $G\colon Y \times I \to B$ be a homotopy witnessing $p \circ f \simeq c_{*_B}$. Now, define $H\colon Y \times I \to E$ via the homotopy lifting property, as in the following commutative diagram.

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "f"] \arrow[d, "i_0"'] & F \times B \arrow[d, "p"] \\ Y \times I \arrow[r, "G"'] \arrow[ur, dashed, "H"] & B \end{tikzcd}$$

Then $p \circ H(y,1) = G(y,1) = *_B$ so $H(Y,1) \subseteq F$. So $y \mapsto H(y,1)$ can be restricted to a map $f'\colon Y \to F$. But $H(y,0) = f(y)$ so we have $f \simeq i \circ f'$, i.e., $[f] = i_*([f'])$ and so $[f] \in i_*([Y,F])$. □

Theorem 6.7. *Let $i\colon A \to X$ be a cofibration and let $q\colon X \to X/A$ be the quotient map to the cofibre. If Y is path connected then*

$$[X/A, Y] \xrightarrow{q^*} [X,Y] \xrightarrow{i^*} [A,Y]$$

is exact.

Proof. We have $i^*q^*([g]) = [g \circ q \circ i] = [c_*]$ and so $q^*([X/A,Y]) \subseteq (i^*)^{-1}([c_*])$. Now, suppose $f\colon X \to Y$ is such that $f|_A = f \circ i\colon A \to Y$ is nullhomotopic, i.e., $[f] \in (i^*)^{-1}([c_*])$. Let $G\colon A \times I \to Y$ be nullhomotopy showing $f|_A \simeq c_*$ and extend it to $H\colon X \times I \to Y$ using the homotopy extension property as in the following commutative diagram.

$$\begin{tikzcd} A \arrow[rr, "i"] \arrow[dd, "i_0"'] & & X \arrow[dl, "f"'] \arrow[dd, "i_0"] \\ & Y & \\ A \times I \arrow[ur, "G"] \arrow[rr, "i \times \mathrm{id}"'] & & X \times I \arrow[ul, dashed, "H"'] \end{tikzcd}$$

The map $g\colon X \to Y$ given by $g(x) = H(x,1)$ satistfies $g(a) = H(a,1) = G(a,1) = *$ for all $a \in A$ and so descends to a map $g'\colon X/A \to Y$. But $H(x,0) = f(x)$ and so $[f] = [g] = q^*([g']) \in q^*([X/A,Y])$. □

6.3. **Fibration and cofibration exact sequences.** Any map $f\colon X \to Y$ factors through the fibration $\nu\colon N_f \to Y\colon (x,\gamma) \mapsto f(x)$ via the homotopy equivalence $X \xrightarrow{\simeq} N_f\colon x \mapsto (x, c_{f(x)})$. Letting F_f be the fibre of ν, we may similarly turn $F_f \to N_f$ into a fibration and continue in this way to obtain a sequence associated to $f\colon X \to Y$.

Definition 6.8. For a map $f\colon X \to Y$, the fibre F_f of $N_f \to Y$ is called the *homotopy fibre* of f.

Remark 6.9. We may replace $N_f \to Y$ in the definition of homotopy fibre with any fibration $Z \to Y$ through which f factors via a homotopy equivalence. The resulting fibre F of $Z \to Y$ will be homotopy equivalent to F_f. To see this, note that $N_f \to Y$ factors through f via the homotopy inverse $p_1 \colon N_f \to X$ of ν. So, by Proposition 5.8, $Z \to Y$ and $N_f \to Y$ are fibre homotopy equivalent over Y and it follows that F and F_f homotopy equivalent.

Proposition 6.10. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibre sequence. Then the homotopy fibre F_i of i is homotopy equivalent to ΩB.*

With cofibrations we can do something similar. The *homotopy cofibre* of a map $f \colon A \to X$ is the cofibre of $A \to M_f$, i.e., the mapping cone C_f of f.

Proposition 6.11. *Let $A \xrightarrow{i} X \xrightarrow{q} X/A$ be a cofibre sequence. Then the homotopy cofibre C_q of q is homotopy equivalent to ΣA.*

Proof of Proposition 6.10. Let $\nu \colon E \xrightarrow{\simeq} N_p$ be the map $e \mapsto (e, c_{p(e)})$, where

$$N_p = E \times_p B^I = \{(e,\gamma) \in E \times B^I \mid \gamma(1) = p(x)\}.$$

Let $\rho \colon N_p \to B$ be given by $\rho(e,\gamma) = \gamma(0)$. Then ν is a homotopy equivalence, ρ is a fibration and $\rho \circ \nu = p$. The fibre of ρ is

$$F_p = \{(e,\gamma) \mid \gamma(0) = *_B, \gamma(1) = p(e)\}.$$

Recall the path space fibre sequence $\Omega B \to PB \xrightarrow{\gamma \mapsto \gamma(1)} B$ and note that

$$\begin{tikzcd}
F_p \arrow[r, "p_2"] \arrow[d, "p_1"'] & PB \arrow[d, "\gamma \mapsto \gamma(1)"] \\
E \arrow[r, "p"] & B
\end{tikzcd}$$

is a pull back square. Then, by Lemma 5.2, $p_1 \colon F_p \to E$ is a fibration and we see that its fibre is ΩB.

Now, ν is a map over B from the fibration p to the fibration ρ, so by Proposition 5.8, ν is a fibre homotopy equivalence over B. It follows that $\nu|_{F \to F_p}$ is a homotopy equivalence. We have the following commutative diagram

$$\begin{tikzcd}
F_i \arrow[r] \arrow[dd] & N_i \arrow[d, "p_1"', "\simeq"] \arrow[dr, "(f,\gamma) \mapsto \gamma(1)"] & \\
& F \arrow[r, "i"] \arrow[d, "\nu|_F"', "\simeq"] & E \\
\Omega B \arrow[r] & F_p \arrow[ur] &
\end{tikzcd}$$

where $N_i \to E$ and $F_p \to E$ are fibrations. We again apply Proposition 5.8 to obtain that the map $F_i \to \Omega B$ between their fibres is a homotopy equivalence. $\square$

In the following diagram the right triangle commutes and the left triangle commutes up to homotopy.

$$\begin{tikzcd}
\Omega X \arrow[r, "-\Omega f"] \arrow[dr, "\chi \to (*,\chi)"'] & \Omega Y \arrow[r] \arrow[d, "\simeq"'] & F_f \arrow[r, "p_1"] & X \arrow[r, "f"] & Y \\
& F_{p_1} \arrow[ur, "\pi"'] & & &
\end{tikzcd}$$

Applying the loop functor to this sequence or extending it to the left we obtain two equivalent sequences as seen in the commutative diagram

$$\begin{tikzcd}
\Omega^2 Y \arrow[r] \arrow[d, "\tau"'] & \Omega F_f \arrow[r] \arrow[d, "\cong"] & \Omega X \arrow[r, "\Omega f"] \arrow[d, equal] & \Omega Y \arrow[d, equal] \\
\Omega^2 Y \arrow[r] & F_{\Omega f} \arrow[r] & \Omega X \arrow[r, "\Omega f"] & \Omega Y
\end{tikzcd}$$

where τ switches the loop coordinates $(s,t) \mapsto (t,s)$.

This enables us to iterate the following procedure: (1) take homotopy fibre; (2) Show the homotopy fibre is homotopy equivalent, with an appropriate homotopy commutative diagram, to a space that is an iterated loop space of X, Y of F_f. We construct the fibre sequence.

Definition 6.12. So, for a map $f\colon X \to Y$, we obtain a sequence of maps

$$\cdots \to \Omega^2 F_f \xrightarrow{\Omega^2 p_1} \Omega^2 X \xrightarrow{\Omega^2 f} \Omega^2 Y \xrightarrow{\Omega\iota} \Omega F_f \xrightarrow{\Omega p_1} \Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{\iota} F_f \xrightarrow{p_1} X \xrightarrow{f} Y$$

called the *fibre sequence generated* by f. Here

$$F_f = \{(x,\gamma) \in X \times_f PY \mid f(x) = \gamma(1), \gamma(0) = *_Y\}$$

and

$$(-\Omega f)(\gamma)(t) = (f \circ \gamma)(1-t).$$

For each pair of adjacent maps, the first is the inclusion of the homotopy fibre of the next, up to homotopy equivalence. Furthermore, any such sequence of maps ending with f is homotopy equivalent to the fibre sequence. That is, the fibre sequence is unique up to homotopy equivalence.

Proposition 6.13. *For any (based) space Z,*

(1) $[X, \Omega X] = [\Sigma Z, X]$ is a group.
(2) $[Z, \Omega^2 X] = [\Sigma^2 Z, X]$ is an abelian group.

Theorem 6.14. *For any based space Z, the fibre sequence induces an $f\colon X \to Y$ induces an exact sequence*

$$\begin{aligned}
\cdots \to [Z, \Omega^2 F_f] \to [Z, \Omega^2 X] \to [Z, \Omega^2 Y] \to \\
\to [Z, \Omega F_f] \to [Z, \Omega X] \to [Z, \Omega Y] \to \\
\to [Z, F_f] \to [Z, X] \to [Z, Y]
\end{aligned}$$

of based sets and of groups left of $[Z, \Omega Y]$ and of abelian groups left of $[Z, \Omega^2 Y]$.

Proof. This follows from Theorem 6.6 and the fibre sequence. $\square$

6.3.1. *Cofibration version.* Let $f\colon X \to Y$ be a based map. The homotopy cofibre of f is the (based) mapping cone C_f. The inclusion $i\colon Y \to C_f$ is a cofibration so we have the cofibre sequence

$$Y \xrightarrow{i} C_f \to C_f/Y \cong \Sigma X.$$

Since C_i is the homotopy cofibre of i, we have $C_i \simeq C_f/Y \cong \Sigma X$. Moreover, $C_f \to C_i$ is a cofibration with cofibre $C_i/C_f \cong \Sigma Y$. We get the following diagram

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "i"] & C_f \arrow[r] \arrow[dr] & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y \\
 & & & C_i \arrow[u, "\cong"'] \arrow[ur] &
\end{tikzcd}$$

where $(-\Sigma f)(x \wedge t) = f(x) \wedge (1-t)$ and where the left triangle commutes and the right triangle commutes up to homotopy.

We may continue this process starting from Σf rather than f. We may also apply the functor Σ to this sequence but the two are equivalent up to $\tau\colon \Sigma^2 X \to \Sigma^2 X$ which switches the coordinates of Σ^2.

$$\begin{tikzcd}
\Sigma X \arrow[r, "\Sigma f"] \arrow[d, equal] & \Sigma Y \arrow[r] \arrow[d] & C_{\Sigma f} \arrow[r] \arrow[d, "\cong"] & \Sigma^X \arrow[d, "\tau"] \\
\Sigma X \arrow[r] & \Sigma Y \arrow[r] & \Sigma C_f \arrow[r] & \Sigma^2 X
\end{tikzcd}$$

Definition 6.15. Iterating this process we get the *cofibre sequence*

$$X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{\pi} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma i} \Sigma C_f \xrightarrow{-\Sigma \pi} \Sigma^2 X \xrightarrow{\Sigma^2 f} \Sigma^2 Y \xrightarrow{\Sigma^2 i} \Sigma^2 C_f \xrightarrow{\Sigma^2 \pi} \cdots$$

generated by $f\colon X \to Y$.

Theorem 6.16. *For any based space Z,*

$$\begin{aligned}
\cdots \to [\Sigma^2 C_f, Z] \to [\Sigma^2 Y, Z] \to [\Sigma^2 X, Z] \to \\
\to [\Sigma C_f, Z] \to [\Sigma Y, Z] \to [\Sigma X, Z] \to \\
\to [C_f, Z] \to [Y, Z] \to [X, Z]
\end{aligned}$$

is an exact sequence of based sets, of groups left of $[\Sigma X, Z]$ and of abelian groups left of $[\Sigma^2 X, Z]$.

Definition 6.17. Let X be a (based) space and let $n \geq 0$. The nth *homotopy group* of X is

$$\pi_n(X) = [S^n, X],$$

where S^n is the (based) n-sphere. We have $\pi_n(X) = \pi_{n-1}(\Omega X) = \pi_0(\Omega^n X)$. When $n = 0$, $\pi_n(X)$ is just a set.

6.3.2. *Relative homotopy groups.* Let $A \subseteq X$ (based subspace). Then the homotopy fibre of the inclusion $A \to X$ is

$$P(X; *, A) = \{\gamma \in PX \mid \gamma(1) \in A\}.$$

Definition 6.18. The nth *relative homotopy group* of (X, A) is

$$\partial_n(X, A) = \pi_{n-1}(P(X; *, A), c_*) = \pi_0(\Omega^{n-1}P(X; *, A)).$$

This is a group if $n \geq 2$ and an abelian group if $n \geq 3$.

One can think of $\pi_n(X, A)$ as the set of (based) homotopy classes $[(D^n, S^{n-1}), (X, A)]$.

6.4. **Long exact sequence in homotopy groups.** Let $A \subseteq X$ be a (based) subspace and $i\colon A \to X$ its inclusion map. Then, as $F_i = P(X; *, A)$, the fibre sequence generated by i is

$$\cdots \to \Omega^2 A \xrightarrow{\Omega^2 i} \Omega^2 X \xrightarrow{\Omega\iota} \Omega P(X; *, A) \xrightarrow{\Omega e_1} \Omega A \xrightarrow{\Omega i} \Omega X \xrightarrow{\iota} P(X; *, A) \xrightarrow{e_1} A \xrightarrow{i} X.$$

So applying the functor $[S^0, -]$ we obtain an exact sequence

$$\cdots \to \pi_2(A) \to \pi_2(X) \to \pi_2(X, A) \xrightarrow{\partial} \pi_1(A) \to \pi_1(X) \to \pi_1(X, A) \xrightarrow{\partial} \pi_0(A) \to \pi_0(X)$$

where ∂ restricts (based map of pairs) $(D^n, S^{n-1}) \to (X, A)$ to $S^{n-1} \to A$.

6.5. **Long exact sequence of a fibration.** Now, let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibre sequence with B path connected. Let $\phi\colon \xrightarrow{\simeq} F_p$ be the homotopy equivalence over E given by $\phi(e) = (e, c_*) \in F_p$, where $F_p = E \times_p PB$ is the homotopy fibre of p.

We learnt in the last section that applying the $[S^0, -]$ functor to the fibre sequence

$$\cdots \to \Omega^2 F \to \Omega^2 E \to \Omega F_i \to \Omega F \to \Omega E \to F_i \to F \to E$$

generated by the inclusion $F_i \to F$ we obtain an exact sequence

$$\cdots \to \pi_2(F) \to \pi_2(E) \to \pi_2(E, F) \xrightarrow{\partial} \pi_1(F) \to \pi_1(E) \to \pi_1(E, F) \xrightarrow{\partial} \pi_0(F) \to \pi_0(E).$$

On the other hand, applying $[S^0, -]$ to the fibre sequence

$$\cdots \to \Omega^2 F_p \to \Omega^2 E \to \Omega^2 B \to \Omega F_p \to \Omega E \to \Omega B \to F_p \to E$$

generated by p we have the exact sequence

$$\cdots \to \pi_2(F_p) \to \pi_2(E) \to \pi_2(B) \to \pi_1(F_p) \to \pi_1(E) \to \pi_1(B) \to \pi_0(F_p) \to \pi_0(E).$$

However, the two fibre sequences are equivalent up to homotopy as the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega^2 E \arrow[r] \arrow[d] & \Omega F_i \arrow[r] \arrow[d, "-\Omega p"] & \Omega F \arrow[r] \arrow[d, "\Omega\phi"] & \Omega E \arrow[r] \arrow[d, equal] & F_i \arrow[r] \arrow[d, "-p"] & F \arrow[r] \arrow[d, "\simeq"] & E \arrow[d, equal] \\
\cdots \arrow[r] & \Omega^2 E \arrow[r] & \Omega^2 B \arrow[r] & \Omega F_p \arrow[r] & \Omega E \arrow[r] & \Omega B \arrow[r] & F_p \arrow[r] & E,
\end{tikzcd}$$

where $p(\gamma) = p \cdot \gamma$, commutes up to homotopy. So, applying $[S^0, -]$ we get

$$\begin{tikzcd}
\pi_{n+1}(F) \arrow[r] \arrow[d, "\cong"] & \pi_{n+1}(E) \arrow[r] \arrow[d, equal] & \pi_n(F_i) \arrow[r] \arrow[d] & \pi_n(F) \arrow[r] \arrow[d, "\cong"] & \pi_n(E) \arrow[d, equal] \\
\pi_{n+1}(F_p) \arrow[r] & \pi_{n+1}(E) \arrow[r] & \pi_{n+1}(B) \arrow[r] & \pi_n(F_p) \arrow[r] & \pi_n(E)
\end{tikzcd}$$

to which we may apply the 5-lemma to conclude that the middle arrow is also an isomorphism.

So, we may replace $\pi_n(F_i)$ with $\pi_{n+1}(B)$ in the top sequence to obtain the long exact sequence

$$\cdots \to \pi_2(F) \to \pi_2(E) \to \pi_2(B) \to \pi_1(F) \to \pi_1(E) \to \pi_1(B) \to \pi_0(F) \to \pi_0(E) \to \pi_0(B) = \{*\}$$

of the fibration $p \colon E \to B$.

Example 6.19. Recall the Hopf fibre sequence $S^1 \to S^3 \to S^2$. Then we obtain the long exact sequence

$$\cdots \to \pi_2(S^1) \to \pi_2(S^3) \to \pi_2(S^2) \to \pi_1(S^1) \to \pi_1(S^3) \to \pi_1(S^2) \to \{*\}.$$

Using the fact that $\pi_1(S^1) \cong \mathbb{Z}$ and that $\pi_n(S^1) = 0$ for $n > 1$ we may use this long exact sequence to show that $\pi_2(S^2) = \mathbb{Z}$ and that $\pi_n(S^3) = \pi_n(S^2)$ for $n > 2$.

6.6. **Example: orthogonal groups.** Recall that we have a fibration

$$O(m) \to O(m+1) \to S^m$$

Since $\pi_i(S^m) = 0$ for $0 < i < m$, the long exact sequence in homotopy groups of the fibration implies that $\pi_i(O(m)) \cong \pi_i(O(m+1))$ for $i < m-1$, and that $\pi_{m-1}(O(m)) \to \pi_{m-1}(O(m+1))$ is surjective. Thus, for i fixed, $\pi_i(O(m))$ is constant for m sufficiently large. Define

$$O := \operatorname{colim}_m O(m).$$

The homotopy groups of $O(m)$ stabilise, so the homotopy groups of the colimit are the homotopy groups of $O(m)$ for m sufficiently large. In fact, the homotopy groups of O are 8-periodic, $\Omega^8 O \simeq O$, by the Bott periodicity theorem, and these homotopy groups are given by: $\mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}$. The nonzero groups are related to the Hopf bundles. These homotopy groups are also related to the classification of vector bundles on spheres.

6.7. **Aside on H-spaces.** Write $j \colon Y \to Y \times Y$, for $i = 1, 2$, $y \mapsto (y, *)$ and $y \mapsto (*, y)$ respectively.

Suppose that there is a map

$$m \colon Y \times Y \to Y$$

such that

$$mj_1 \sim \mathrm{Id} \sim mj_2 \colon Y \to Y.$$

Then (Y, m) is an *H-space.*

If in addition

$$m \circ (m \times \mathrm{Id}) \sim m \circ (\mathrm{Id} \times m) \colon Y \times Y \times Y \to Y$$

then (Y, m) is a *homotopy associative H-space.* If moreover there is a map

$$i\colon Y \to Y$$

such that

$$m \circ (\mathrm{Id}, i) \sim * \sim m(i, \mathrm{Id})\colon Y \to Y$$

i.e. $y \mapsto m(y, i(y))$. Then (Y, m, i) is a *grouplike space.*

(Also, (Y, m) is homotopy commutative if

$$\begin{array}{rcl} Y \times Y & \to & Y \\ (x, y) & \mapsto & m(y, x) \end{array}$$

is homotopic to m.

If (Y, m) is a grouplike space, then $[X, Y]$ has a group structure. For any space X, ΩX is a grouplike space. We will see later that recognising a space a loop space of something else can be extremely useful in homotopy theory.

6.8. **Change of basepoint.** Note that one can consider elements of $\pi_n(X, x)$ as homotopy classes of maps $[(S^n, *), (X, x)]$. Since the inclusion $* \to S^n$ is a cofibration, by Lemma 5.6 we have a fibration

$$p\colon X^{S^n} \to X$$

given by evaluation at the basepoint. The fibre consists of based maps, and we can identify $\pi_0(F_x) = \pi_n(X, x)$, since a path in F_x corresponds to a based homotopy of maps $S^n \to X$. Now let $\xi\colon I \to X$ be a path in X with $\xi(0) = x$ and $\xi(1) = x'$. Since $X^{S^n} \to X$ is a fibration, the path lifting function induces a homotopy equivalence $\widetilde{\xi}_1\colon F_x \to F_{x'}$. This map induces a bijection:

$$\pi_0(F_x) \leftrightarrow \pi_0(F_{x'}).$$

We claim that, using the identifications with $\pi_n(X, x)$ and $\pi_n(X, x')$, this is an isomorphism of groups. We have to see that this map is a homomorphism.

For based maps $f, g\colon S^n \to X$, we can consider the composition

$$f + g\colon S^n \to S^n \vee S^n \xrightarrow{f \vee g} X \vee X \xrightarrow{\nabla} X,$$

with the first map given by pinching the equator to a point. This composition defined the addition on $\pi_n(X, x)$. Since $* \to S^n \vee S^n$ is a cofibration, as above we have a fibration $X^{S^n \vee S^n} \to X$. We have a map of fibrations

$$\begin{tikzcd} X^{S^n \vee S^n} \arrow[r] \arrow[d] & X^{S^n} \arrow[d] \\ X \arrow[r, "="] & X. \end{tikzcd}$$

The fibre of the left fibration is $F_x \times F_x$. The diagram induces a map of fibres $F_x \times F_x \to F_x$, which on π_0 induces the addition $m\colon \pi_n(X, x) \times \pi_n(X, x) \to \pi_n(X, x)$.

Moreover, the map on fibres is natural, so we have a commutative square

$$\begin{tikzcd} F_x \times F_x \arrow[r] \arrow[d] & F_x \arrow[d] \\ F_{x'} \times F_{x'} \arrow[r] & F_{x'}. \end{tikzcd}$$

On π_0, this induced the desired property that the change of basepoint map is a homomorphism. Since it is a bijection, it is an isomorphism.

With a little more work, we could prove the following theorem.

Theorem 6.20. *Let $f\colon (X,A) \to (Y,B)$ be a map of pairs and let $\alpha\colon I \to A$ be a path with $\alpha(0) = a$ and $\alpha(1) = a'$. Then we have a commutative diagram with vertical isomorphisms:*

$$\begin{tikzcd} \pi_n(X,A,a) \arrow[r, "f_*"] \arrow[d, "\widetilde{\alpha}_1"'] & \pi_n(Y,B,f(a)) \arrow[d, "\widetilde{f\circ\alpha}_1"] \\ \pi_n(X,A,a') \arrow[r, "f_*"] & \pi_n(Y,B,f(a')) \end{tikzcd}$$

Suppose moreover that $h\colon f \sim f'$ is a homotopy of maps of pairs $f\colon (X,A) \to (Y,B)$. Let $h(a)\colon I \to Y$ be the path given by $h(a)(t) = h(a,t)$. Then there is a commutative diagram with a horizontal isomorphism:

$$\begin{tikzcd} & \pi_n(X,A,a) \arrow[dl, "f_*"'] \arrow[dr, "f'_*"] & \\ \pi_n(Y,B,f(a)) \arrow[rr, "\widetilde{h(a)}_1"] & & \pi_n(Y,B,f'(a)). \end{tikzcd}$$

Corollary 6.21. *A homotopy equivalence of spaces/pairs induces isomorphisms on all homotopy groups (even if not a based homotopy equivalence).*

In the next section, we will prove a remarkable converse to this statement for CW complexes.

6.9. **Action of fundamental group on higher homotopy groups.** We saw above that a path $\xi\colon I \to X$, with $\gamma(0) = x$ and $\gamma(1) = x'$ induces a map on the fibres F_x, $F_{x'}$ of the fibration $X^{S^n} \to X$.

$$\widetilde{\xi}_1\colon \pi_0(F_x) \to \pi_0(F_{x'}).$$

With $x = x'$, ξ represents an element of $\pi_1(X,x)$, and we get a map

$$\widetilde{\xi}_1\colon \pi_n(X,x) \to \pi_n(X,x).$$

This induces a map

$$\pi_1(X,x) \to \mathrm{Aut}(\pi_n(X,x)).$$

With this action extended linearly, for $n \geq 2$, $\pi_n(X,x)$ becomes a module over the group ring $\mathbb{Z}[\pi_1(X,x)]$. When applied with $n = 1$, this does not extend linearly, and gives the conjugation action.

Recall that for $n > 2$, $\pi_n(X,x) \cong \pi_n(\widetilde{X},\widetilde{x})$. One can think of the action of $\pi_1(X,x)$ as the action of the deck transformations on the universal covering space $\widetilde{X}$.

For example, for the space $S^1 \vee S^2$, we have $\pi_2(S^1 \vee S^2) \cong \mathbb{Z}[\mathbb{Z}]$. It is finitely generated as a module over $\mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[\pi_1(S^1 \vee S^2)] = \mathbb{Z}[\pi_1(S^1)]$, but infinitely generated as an abelian group.

7. The HELP lemma and its consequences

The main goal of this section is to prove Whitehead's theorem.

Theorem 7.1. *Let $f\colon X \to Y$ be a map between CW complexes X, Y that induces isomorphisms $f_*\colon \pi_i(X) \to \pi_i(Y)$ for every $i \geq 0$. Then f is a homotopy equivalence.*

Consider the spaces $S^2 \times \mathbb{RP}^3$ and $S^3 \times \mathbb{RP}^2$. These spaces have isomorphic homotopy groups for all i. However this isomorphism cannot be induced by any map. One could try to see this directly, or if one believes Whitehead's theorem, then there cannot exist such a map, for the spaces are not homotopy equivalent, as can be seen by computing that the second integral homology groups differ: $H_2(S^2 \times \mathbb{RP}^3; \mathbb{Z}) \cong Z$ whereas $H_2(S^3 \times \mathbb{RP}^2; \mathbb{Z}) = 0$.

7.1. The HELP lemma.

Whitehead's theorem will follow quite easily once we have established the following technical lemma, and the HELP (homotopy extension and lifting property) lemma that follows quite easily from this technical lemma.

Definition 7.2 (n-equivalence)**.** We say that a map $e\colon Y \to Z$ is n-connected if $e_*\colon \pi_q(Y,y) \to \pi_q(Z,e(y))$ is an isomorphism for $q < n$ and a surjection for $q = n$. The map e is said to be a *weak equivalence* if e is an n-equivalence for all n.

In the following lemma and its proof, we will consider the *unreduced cone* $CX = X \times I / X \times \{1\}$. Also, let $f, f'\colon (X,A) \to (Y,B)$ be maps of pairs such that $f = f'$ on A. We say that f is homotopic to f' rel. A if there is a homotopy $h\colon X \times I \to Y$ such that $h(a,t) = f(a) = f'(a)$ for all $a \in A, t \in I$. Then we can consider the homotopy groups $\pi_{n+1}(X,x)$ as relative homotopy classes of maps $(CS^n, S^n) \to (X,x)$.

Lemma 7.3 (Technical lemma)**.** *Let $e\colon Y \to Z$ be a map. The following are equivalent.*

(i) For any $y \in Y$, the map $e_*\colon \pi_q(Y,y) \to \pi_q(Z,e(y))$ is an injection for $q = n$ and a surjection for $q = n+1$.
(ii) Given $f\colon CS^n \to Z$, $g\colon S^n \to Y$ and $h\colon S^n \times I \to Z$, such that $f|_{S^n} = h \circ i_0$ and $e \circ g = h \circ i_1$, there exist maps $\widetilde{g}$, $\widetilde{h}$ as in the diagram below making it

commute:

$$
\begin{tikzcd}
S^n \arrow[rr, "i_0"] \arrow[dd] & & S^n \times I \arrow[dl, "h"'] \arrow[dd] & & S^n \arrow[ll, "i_1"'] \arrow[dl, "g"] \arrow[dd] \\
& Z & & Y \arrow[ll, "e"] & \\
CS^n \arrow[rr, "i_0"'] \arrow[ur, "f"] & & CS^n \times I \arrow[ul, dashed, "\widetilde{h}"] & & CS^n \arrow[ll, "i_1"] \arrow[ul, dashed, "\widetilde{g}"]
\end{tikzcd}
$$

(iii) The conclusion of (ii) holds for $f|_{S^n} = e \circ g$ and h the constant homotopy.

Proof. (ii) implies (iii) trivially. Next we prove that (iii) implies (i). First consider the case that $n = 0$ as a warm up. There just one map $S^0 \to Z$, which appears as the restriction of f, $e \circ g$, and h restricted to each time t. Here is the relevant diagram:

$$
\begin{tikzcd}
S^0 \arrow[rr, "i_0"] \arrow[dd] & & S^0 \times I \arrow[dl, "h"'] \arrow[dd] & & S^0 \arrow[ll, "i_1"'] \arrow[dl, "g"] \arrow[dd] \\
& Z & & Y \arrow[ll, "e"] & \\
CS^0 \arrow[rr, "i_0"'] \arrow[ur, "f"] & & CS^0 \times I \arrow[ul, dashed, "\widetilde{h}"] & & CS^0 \arrow[ll, "i_1"] \arrow[ul, dashed, "\widetilde{g}"]
\end{tikzcd}
$$

Write $g(-1) = y$ and $g(1) = y'$. Whenever $e(y)$ and $e(y')$ can be connected by a path in Z, we have a map $f\colon CS^0 = D^1 \to Z$. The map $\widetilde{g}\colon CS^0 = D^1 \to Y$. Thus the map $e_*\colon \pi_0(Y, y) \to \pi_0(Z, e(y))$ is injective. Now consider general n. Let $g\colon S^n \to Y$ be a map based at $y \in Y$. This represents an element of $\pi_n(Y, y)$. The map f says that $e_*(g) \in \pi_n(Z, e(y))$ is null homotopic. Then the existence of the map $\widetilde{g}$ gives a null homotopy of g. This shows that

$$e_*\colon \pi_n(Y, y) \to \pi_n(Z, e(y))$$

is injective. Next we want to show the required surjectivity. Let g be the constant map $g\colon S^n \to \{y\} \to Y$. Then

$$f\colon (CS^n, S^n) \to (Z, e(y))$$

is an element of $\pi_{n+1}(Z, e(y))$. The map

$$\widetilde{g}\colon (CS^n, S^n) \to (Y, y)$$

is an element of $\pi_{n+1}(Y, y)$ as remarked at before the statement of the lemma. This shows that $e_*\colon \pi_{n+1}(Y, y) \to \pi_{n+1}(Z, e(y))$ is surjective, which completes the proof of (i) given (iii).

It now remains to prove that (i) implies (ii). So assume (i), namely that $\pi_n(Y) \to \pi_n(Z)$ is injective and $\pi_{n+1}(Y) \to \pi_{n+1}(Z)$ is surjective. The strategy of the proof is as follows. Suppose we are given $f\colon CS^n \to Z$, $g\colon S^n \to Y$, and $h\colon S^n \times I \to Z$. First we will show that $\pi_n(F(e)) = 0$. Then we will construct a map $S^n \to F(e)$

using f, g and h. Then, since this map is null homotopic, we can use the null homotopy to show the existence of $\widetilde{g}$ and $\widetilde{h}$.

We begin by choosing basepoints carefully.

(i) Let $* \in S^n$ be a basepoint.
(ii) Let $\diamond \in CS^n$ be the cone point.
(iii) Let $y_1 := g(*)$.
(iv) Let $z_1 := e(y_1)$.
(v) Let $z_0 := f(*, 0)$.
(vi) Let $z_{-1} := f(\diamond)$.

For $x \in S^n$, let $f_x\colon I \to Z$ be a path from $f(x,0) = h(x,0)$ to z_{-1}. Let $h_x\colon I \to Z$ be a path from $h(x,0)$ to $h(x,1) = e \circ g(x)$.

Recall that the homotopy fibre of e is

$$F(e, y_1) = \{(y, \xi) \in Y \times Z^I \mid \xi(0) = z_1 = e(y_1) \text{ and } e(y) = \xi(1)\}.$$

The basepoint of $F(e, y_1)$ is $w_1 := (y_1, c_{z_1})$. The fibration sequence is

$$\pi_{n+1}(Y, y_1) \xrightarrow{e_*} \pi_{(n+1)}(Z, z_1) \to \pi_n(F(e, y_1), w_1) \to \pi_n(Y, y_1) \xrightarrow{e_*} \pi_n(Z, z_1).$$

The assumptions (i) and exactness imply that $\pi_n(F(e, y_1), w_1) = 0$. Next, define a map

$$\begin{array}{rcl} k_0\colon S^n & \to & F(e, y_1) \\ x & \mapsto & \big(g(x), h_x \cdot f_x^{-1} \cdot f_* \cdot h_*^{-1}\big) \in Y \times Z^I. \end{array}$$

The reader should check that the given path is indeed a path from $e(g(x))$ to $z_1 = e(y_1)$. Note that $k_0(*)$ is not the basepoint w_1, so k_0 is not a based map. However $h_* \cdot f_*^{-1} \cdot f_* \cdot h_*^{-1}$ is homotopic to a constant map, so $k_0(*)$ is connected to the basepoint by a path in $F(e)$. Then the HEP for $* \to S^n$ implies that k_0 is homotopic to a based map:

$$\begin{tikzcd} S^n \arrow[r, "k_0"] \arrow[dr, dashed] & F(e, y_1) \\ * \arrow[u] \arrow[r] & F(e, y_1)^I \arrow[u, "p_0"'] \end{tikzcd}$$

Thus k_0 is homotopic to a based map $\widetilde{k_0} \in \pi_n(F(e, y_1), w_1) = 0$. Since $\widetilde{k_0}$ is null homotopic. Let

$$G\colon S^n \times I \to F(e, y_1)$$

be a homotopy from $\widetilde{k_0}$ to the constant map c_{w_1}. Write

$$G(x, t) = (\widetilde{g}(x, t), \xi(x, t));$$

this defines $\widetilde{g}$ and ξ. Note that $\widetilde{g}(x, 1) = y_1$ for all x. Define $j\colon S^n \times I \times I \to Z$ via $j(x, t, s) = \xi(x, t)(s)$. For each $x \in S^n$, the map $j(x, -, -)$ is given by

$$\begin{tikzcd} \cdot \arrow[r, no head, "c_{z_1}"] \arrow[d, no head, "c_{z_1}"'] & \cdot \arrow[d, no head, "e \circ \widetilde{g}_x"] \\ \cdot \arrow[r, no head, "h_x \cdot f_x^{-1} \cdot f_* \cdot h_*^{-1}"'] & \cdot \end{tikzcd}$$

We want a map $\widetilde{h}\colon S^n \times I \times I$ that behaves as:

$$\begin{tikzcd} \cdot \arrow[r, no head, "e\circ\widetilde{g}_x"] \arrow[d, no head, "h_x"'] & \cdot \arrow[d, no head, "h_*\cdot f_*^{-1}"] \\ \cdot \arrow[r, no head, "f_x"'] & \cdot \end{tikzcd}$$

This can be achieved by a reparametrisation of the square. Choose a suitable map $\Theta\colon I^2 \to I^2$, and then

$$\widetilde{h} = j \circ (\mathrm{Id} \times \Theta)\colon S^n \times I^2 \to Z$$

gives the required homotopy $\widetilde{h}$. $\square$

The technical lemma will now be used to prove the HELP theorem.

Theorem 7.4 (HELP)**.** *Let (X, A) be a relative CW complex (start with a space A, add cells to get the space X) of dimension $\leq n$, let $e\colon Y \to Z$ be an n-equivalence. Given $f\colon X \to Z$, $g\colon A \to Y$, and $h\colon A \times I \to Z$ as in the diagram making it commute, there are maps $\widetilde{g}$, $\widetilde{h}$ making the diagram commutes.*

$$\begin{tikzcd} A \arrow[rr, "i_0"] \arrow[dd] & & A \times I \arrow[dl, "h"'] \arrow[dd] & & A \arrow[ll, "i_1"'] \arrow[dl, "g"] \arrow[dd] \\ & Z & & Y \arrow[ll, "e"] & \\ X \arrow[ur, "f"] \arrow[rr, "i_0"'] & & X \times I \arrow[ul, dashed, "\widetilde{h}"] & & X \arrow[ll, "i_1"] \arrow[ul, dashed, "\widetilde{g}"'] \end{tikzcd}$$

Proof. If $e\colon Y \to Z$ is an n-equivalence then $\pi_q(Y) \to \pi_q(Z)$ is injective and $\pi_{q+1}(Y) \to \pi_{q+1}(Z)$ is surjective. Thus we can apply the technical lemma by inducting on the cells of (X, A). Order the i cells for each i, and work in increasing dimension of cells.

Let e^{q+1} be a $(q+1)$-cell of (X, A). Suppose that the maps $\widetilde{g}$, $\widetilde{h}$ have been defined on all previous cells in our ordering. Note that $(e^{q+1}, \partial e^q) = (CS^q, S^q)$. Then let $f|_{e^{q+1}}$ be the f in the technical lemma, let $\widetilde{h}|_{\partial e^{q+1}}$ be the h from the technical lemma, and let $\widetilde{g}|_{\partial e^{q+1}}$ be the g from the technical lemma.

$$\begin{tikzcd} S^n \arrow[rr, "i_0"] \arrow[dd] & & S^n \times I \arrow[dl, "\widetilde{h}|_{\partial e^{q+1}}"'] \arrow[dd] & & S^n \arrow[ll, "i_1"'] \arrow[dl, "\widetilde{g}|_{\partial e^{q+1}}"] \arrow[dd] \\ & Z & & Y \arrow[ll, "e"] & \\ CS^n \arrow[ur, "f|_{e^{q+1}}"] \arrow[rr, "i_0"'] & & CS^n \times I \arrow[ul, dashed, "\widetilde{h}"] & & CS^n \arrow[ll, "i_1"] \arrow[ul, dashed, "\widetilde{g}"'] \end{tikzcd}$$

Then there exist maps $\widetilde{h}$ and $\widetilde{g}$ as in the diagram, extending these maps to the cell e^{q+1}. This completes the induction step and therefore the proof. $\square$

7.2. **Whitehead theorems.** Whitehead's theorems on CW complexes will now follow relatively quickly from the HELP lemma. In the next theorem $[-,-]$ denotes unbased homotopy classes of maps.

Theorem 7.5 (Whitehead I)**.** *If X is a CW complex and $e\colon Y \to Z$ is an n-equivalence, then*

$$e_*\colon [X,Y] \to [X,Z]$$

is a bijection if $\dim X < n$ *and is a surjection if* $\dim X = n$.

Proof. To show surjectivity, take $(X,\emptyset)$ in the HELP theorem. That is, $A = \emptyset$. The diagram of HELP reduces to:

$$\begin{tikzcd}
& Z & & Y \arrow[ll, "e"'] & \\
X \arrow[ur, "f"] \arrow[rr, "i_0"'] & & X \times I \arrow[ul, dashed, "\widetilde{h}"] & & X \arrow[ll, "i_1"] \arrow[ul, dashed, "\widetilde{g}"']
\end{tikzcd}$$

That is, given $f\colon X \to Z$, there is a map $\widetilde{g}\colon X \to Y$ and a homotopy from f to $e \circ \widetilde{g} = e_*(\widetilde{g})$.

Next, to show injectivity, we apply HELP to $(X \times I, X \times \partial I)$, with h the constant homotopy. Since the dimension of $X \times I$ is one more than that of X, we see why injectivity holds only for $\dim X < n$. The HELP diagram becomes

$$\begin{tikzcd}
X \times \partial I \arrow[rr, "i_0"] \arrow[dd] & & X \times \partial I \times I \arrow[dl, "h"'] \arrow[dd] & & X \times \partial I \arrow[ll, "i_1"'] \arrow[dl, "g"] \arrow[dd] \\
& Z & & Y \arrow[ll, "e"] & \\
X \times I \arrow[ur, "f"] \arrow[rr, "i_0"'] & & X \times I \times I \arrow[ul, dashed, "\widetilde{h}"] & & X \times I \arrow[ll, "i_1"] \arrow[ul, dashed, "\widetilde{g}"']
\end{tikzcd}$$

That is, g represents two maps $g_0, g_1\colon X \to Y$, and f is a homotopy between $e \circ g_0$ and $e \circ g_1$. Then $\widetilde{g}$ is a homotopy between g_0 and g_1. $\square$

Now we are ready to prove the most well-known Whitehead theorem.

Theorem 7.6 (Whitehead II)**.** *An n-equivalence between CW complexes of dimension less than n is a homotopy equivalence. A weak equivalence between CW complexes is a homotopy equivalence.*

Proof. Suppose that $e\colon Y \to Z$ is either an n-equivalence for dimension of Y and Z less than n, or e is an n equivalence for all n. Then $e_*\colon [Z,Y] \to [Z,Z]$ and $e_*\colon [Y,Y] \to [Y,Z]$ are both bijective. Start with $\mathrm{Id} \in [Z,Z]$. Then by surjectivity of the first e_* there is a map $f\colon Z \to Y$ with $e \circ f \sim \mathrm{Id}$. Then this implies that $e \circ f \circ e \sim e\colon Y \to Z$. By injectivity of the second e_*, this means that $f \circ e \sim \mathrm{Id}$. Thus f and e are homotopy inverses, and e is a homotopy equivalence as claimed. $\square$

7.3. **Cellular approximation.** As another pay off for the work in establishing the HELP lemma, we can also prove that maps between CW complexes can be approximated by cellular maps.

Definition 7.7 (Cellular)**.** A map $f\colon X \to Y$ between CW complexes X, Y is *cellular* if $f(X^{(n)} \subset Y^{(n)}$ for every n.

Recall that a pair (X, A) is said to be n-connected if the inclusion map $A \to X$ is an n-equivalence. A space X is n-connected if $(X, *)$ is n-connected.

Lemma 7.8. *A relative CW complex (X, A) with no m cells for $m \leq n$ is n-connected. In particular $(X, X^{(n)})$ is n-connected for any CW complex X.*

Proof. Let $f\colon (I^q, \partial I^q, J^q) \to (X, A, a)$ represent an element of $\pi_q(X, A, a)$, for $q \leq n$. The image of f is compact so it hits finitely many cells. Induct on the cells with decreasing dimension. Homotope f so that I^q misses the centre of each cell e^r, with $r > n$, which can be achieved by smooth or simplicial approximation and general position. Then homotopy f off e^r. By induction we arrange that $f(I^q) \subset A$. $\square$

Theorem 7.9 (Cellular approximation)**.** *Let $f\colon (X, A) \to (Y, B)$ be a map between relative CW complexes X, Y. Then f is homotopic to a cellular map.*

Proof. The proof is again an induction proof. We induct on the skeleta $X^{(k)}$ of X, for increasing k. First, the base case. Points of $f(X^{(0)} - A)$ are connected to $Y^{(0)}$ by paths. This gives a homotopy from $f|_{X^{(0)}}$ to a map $g_0\colon X^{(0)} \to Y^{(0)}$. For the induction step, suppose that we have defined a map $g_n\colon X^{(n)} \to Y^{(n)}$ and a homotopy $h_n\colon f|_{X^{(n)}} \sim \iota_n \circ g_n\colon X^{(n)} \to Y$, where $\iota_n\colon Y^{(n)} \to Y^{(n+1)}$ is the inclusion of the n-skeleton.

We want to extend this to a cellular map $g_{n+1}\colon X^{(n+1)} \to Y^{(n+1)}$ with a homotopy $h_{n+1}\colon f|_{X^{(n+1)}} \sim \iota_{n+1} \sim g_{n+1}$. We do this one cell at a time. Let $j\colon S^n \to X^{(n)}$ be the attaching map of a $\widetilde{j}\colon D^{n+1} \to X$, an $(n+1)$-cell. By Lemma 7.8, ι_{n+1} is an $(n+1)$-equivalence. We apply the HELP theorem:

$$\begin{tikzcd}
S^n \arrow[rr, "i_0"] \arrow[dd] & & S^n \times I \arrow[dl, "h"'] \arrow[dd] & & S^n \arrow[ll, "i_1"'] \arrow[dl, "g_n \circ j"] \arrow[dd] \\
& Z & & Y \arrow[ll, "\iota_{n+1}"] & \\
D^{n+1} \arrow[rr, "i_0"'] \arrow[ur, "f \circ \widetilde{j}"] & & D^{n+1} \times I \arrow[ul, dashed, "h_{n+1}"] & & D^{n+1} \arrow[ll, "i_1"] \arrow[ul, dashed, "g_{n+1}"]
\end{tikzcd}$$

The HELP theorem yields the new g_{n+1} and h_{n+1} maps as required, for the new cell. Inducting on the $(n+1)$-cells of X yields the extension of the cellular map to $X^{(n+1)}$, together with a homotopy h_{n+1} between g_{n+1} and the original map f restricted to the $(n+1)$-skeleton of X. This completes the proof of the cellular approximation theorem. $\square$

8. Approximation by CW complexes

It will turn out to be very useful for proving several theorems in the near future, to be able to approximate any space, or indeed pairs an triads, up to weak equivalence, by a CW complex.

Theorem 8.1. *For any space X, there is a CW complex ΓX and a weak equivalence $\gamma\colon \Gamma X \to X$. For any $f\colon X \to Y$, there is a map $\Gamma f\colon \Gamma X \to \Gamma Y$ such that*

$$\begin{tikzcd}
\Gamma X \arrow[r, "\Gamma f"] \arrow[d, "\gamma"] & \Gamma Y \arrow[d, "\gamma"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

commutes. If X is n-connected, then ΓX can be chosen so that there is one 0-cell and no q-cells for $1 \leq q \leq n$.

The proof is a "big construction," so we would not hope that the resulting CW complex is going to be easier to work with explicitly. However the approximation by CW complexes will enable us to prove results about homotopy and homology groups of spaces by proving them for weak equivalent CW complexes.

Proof. We want to construct ΓX as a colimit

$$\begin{tikzcd}
X_1 \arrow[r, "i_1"] \arrow[dr, "\gamma_1"'] & X_2 \arrow[r, "i_2"] \arrow[d, "\gamma_2"] & X_3 \arrow[r, "i_3"] \arrow[dl, "\gamma_3"] & \cdots \arrow[r, "i_{n-1}"] & X_n \arrow[r, "i_n"] \arrow[dlll, "\gamma_n"] & X_{n+1} \arrow[r] \arrow[dllll] & \cdots \\
& X & & & & &
\end{tikzcd}$$

with $i_n\colon X_n \to X_{n+1}$ a cellular inclusion.

Assume that X is path connected, since we can repeat the construction given below for each path component separately. The base case is $X_1 := \bigvee_{(q,j)} S^q$, $q \geq 1$, one sphere for each pair (j, q), where for a fixed natural number q, $j\colon S^q \to X$ runs over a generating set for $\pi_q(X)$. The maps j determine a map $\gamma_1\colon X_1 \to X$, which induces surjections on all homotopy groups. Inductively, suppose we have CW complexes X_m such that $i_{m-1}\colon X_{m-1} \to X_m$, $\gamma_m\colon X_m \to X$ with $\gamma_m \circ i_{m-1} = \gamma_{m-1}$ for $m \leq n$, such that $(\gamma_m)_*\colon \pi_q(X_m) \to \pi_q(X_m)$ is surjective for all q and $(\gamma_m)_*$ is a bijection for $q < m$.

Define

$$X_{n+1} := X_n \cup \Big(\bigvee_{(f,g)} (S^n \wedge I_+) \Big)$$

where the wedge ranges over cellular maps $(f, g)\colon S^n \to X_n$ representing all possible homotopy classes $[f], [g] \in \pi_n(X_n)$ with $[f] \neq [g]$ but $[\gamma_n \circ f] = [\gamma_n \circ g]$. Recall that a based homotopy is the same as a map from $S^n \wedge I_+$. We identify $(s, 0) \in S^n \wedge I_+$ with $f(s)$ and $(s, 1) \in S^n \wedge I_+$ with $g(s)$. We have an inclusion map

$$i_n\colon X_n \to X_{n+1}$$

of CW complexes. We have that $(i_n)_*([f]) = (i_n)_*([g])$. We can therefore define a new map $\gamma_{n+1}\colon X_{n+1} \to X$, extending γ_n on X_n using the homotopies $h\colon \gamma_n \circ f \sim$

$\gamma_n \circ g$ to extend over the corresponding copy of $S^n \wedge I_+$. Then

$$(\gamma_{n+1})_* \colon \pi_q(X_{n+1}) \to \pi_q(X)$$

is surjective for all q, and is bijective for $q \leq n$, since we extended the previous map, but we made all n-dimensional homotopy classes of X_n equal in X_{n+1}, if they are equal in X. The n-skeleton is unchanged, so $\pi_q(X_{n+1}) = \pi_q(X_n)$ for all $q < n$.

Recall that we define $\Gamma X := \operatorname{colim} X_n$. This is also a CW complex since all the maps in the colimit construction are cellular. If X is n-connected, then the construction, as promised, did not use any cells of dimension less that n.

Finally, we need to see existence and uniqueness of the map Γf.

$$[\Gamma X, \Gamma Y] \leftrightarrow [\Gamma X, Y]$$

is a bijection by the first Whitehead theorem. There is a map

$$\Gamma X \xrightarrow{\gamma_X} X \xrightarrow{f} Y$$

and thus we obtain $\Gamma f \colon \Gamma X \to \Gamma Y$ by the bijection above. This Γf is unique up to homotopy. $\square$

There is also a relative version:

Theorem 8.2. *Let (X, A) be a pair of spaces, and let $\gamma_A \colon \Gamma A \to A$ be a choice of weakly equivalent CW complex. There exists a CW complex ΓX with a weak equivalence $\gamma_X \colon \Gamma X \to X$, such that $\Gamma A \subseteq \Gamma X$ is a subcomplex, and the restriction of γ_X to ΓA coincides with γ_A. Moreover, for any map of pairs $f \colon (X, A) \to (Y, B)$, there is an induced map*

$$\begin{tikzcd}
(\Gamma X, \Gamma A) \arrow[r, "\Gamma f"] \arrow[d, "(\gamma_X, \gamma_A)"] & (\Gamma Y, \Gamma B) \arrow[d, "(\gamma_Y, \gamma_B)"] \\
(X, A) \arrow[r, "f"] & (Y, B)
\end{tikzcd}$$

We will omit the proof of this theorem. The construction is again adding cells, in a similar manner to the proof of the previous theorem.

We will want to prove a version of excision for homotopy groups, at least in a certain range where it holds. This will be a key statement in our development of the theory. A key first step in the homotopy excision theorem is the approximation of excisive triads by CW triads, which we will do next.

Definition 8.3. An *excisive triad* $(X; A, B)$ is a space X with subspaces $A, B \subseteq X$ such that $X = \operatorname{Int} A \cup \operatorname{Int} B$. A CW triad $(X; A, B)$ is a CW complex X with subcomplexes A, B such that $X = A \cup B$.

Theorem 8.4. *Let $(X; A, B)$ be an excisive triad. Let $C = A \cap B$. There is a CW triad $(\Gamma X; \Gamma A, \Gamma B)$ and a map $\gamma \colon (\Gamma X; \Gamma A, \Gamma B) \to (X; A, B)$ such that with $\Gamma C = \Gamma A \cap \Gamma B$, we have that*

$$\Gamma C \to C,\ \Gamma A \to A,\ \Gamma B \to B,\ \Gamma X \to X$$

are each weak equivalences. If (A, C) is n-connected, then $(\Gamma A, \Gamma C)$ can be chosen to have no q-cells for $q \leq n$. Similarly for (B, C). As before, Γ is functorial and γ is natural.

Proof. Start with $\gamma \colon \Gamma C \to C$, and extend it to $(\Gamma A, \Gamma C) \to (A, C)$ and $(\Gamma B, \Gamma C) \to (B, C)$. Let $\Gamma X = \Gamma A \cup_{\Gamma C} \Gamma B \xrightarrow{\gamma} X$; the map to X exists by the universal property of push-outs. We need to show that $\gamma \colon \Gamma X \to X$ is a weak equivalence.

First, consider any two maps $j \colon C \to B$ and $i \colon C \to A$. Form the double mapping cylinder

$$M(i, j) = A \cup C \times I \cup B.$$

Lemma 8.5. *Suppose that $i \colon C \to A$ is a cofibration and $j \colon C \to B$ is any map. Then the collapse map $M(i, j) \to A \cup_C B$ is a homotopy equivalence.*

To see the lemma, first we know that the collapse map $Mi \to A$ is a cofibre homotopy equivalence under C.

$$\begin{tikzcd}
& C \arrow[dl] \arrow[dr] & \\
Mi \arrow[rr] & & A
\end{tikzcd}$$

Then the universal property of the pushout $A \cup_C B$ gives a map to $M(i, j)$.

$$\begin{tikzcd}
& C \arrow[r] \arrow[d] \arrow[ddl] & B \arrow[d] \arrow[ddr] & \\
& A \arrow[r] & A \cup_C B \arrow[dr] & \\
Mi \arrow[ur, "\simeq"] \arrow[rrr] & & & M(i, j) = M(i) \cup_C B
\end{tikzcd}$$

The fact that the homotopy equivalence in the diagram is under C implies that the induced map on the push outs in the diagram is a homotopy equivalence. This completes the proof of the lemma.

Using the lemma, we can replace the CW triad $(\Gamma X; \Gamma A, \Gamma B)$ by an excisive triad (as $\Gamma A, \Gamma B$ are subcomplexes, they are closed in ΓX so it is not an excisive triad). Denote the subcomplex inclusions $i \colon \Gamma C \to \Gamma A$, $j \colon \Gamma C \to \Gamma B$, $\Gamma X = \Gamma A \cup_{\Gamma C} \Gamma B$. Take the double mapping cylinder as in the lemma, and then

$$\big(M(i, j); \Gamma A \cup (\Gamma C \times [0, 2/3)), (C \times (1/3, 1] \cup \Gamma B)\big) \to (\Gamma X; \Gamma A, \Gamma B)$$

is a homotopy equivalence of triads. Now, $(\Gamma X; \Gamma A, \Gamma B) \to (X; A, B)$ is a weak equivalence of triads by the next more general theorem. The proof of approximation of excisive triads is now complete, modulo the next theorem. $\square$

Theorem 8.6. *Suppose that $(X; A, B) \to (X'; A', B')$ is a map of excisive triads, with $C = A \cap B$ and $C' = A' \cap B'$, such that the maps induced by e, $C \to C'$, $A \to A'$ and $B \to B'$ are all weak equivalences. Then $X \to X'$ is a weak equivalence.*

Proof. We need to solve the following diagram:

$$\begin{tikzcd} X \arrow[r, "e"] & X' \\ S^n \arrow[u, "g"] \arrow[r] & D^{n+1} \arrow[u, "f"'] \arrow[ul, dashed, "\widetilde{g}"] \end{tikzcd}$$

where the top right triangle commutes up to homotopy. That is, we need a lift $\widetilde{g}\colon D^{n+1} \to X$ whose restriction to S^n is equal to g, and $f \sim e \circ \widetilde{g}$ relative to S^n. With g the trivial map, this solves surjectivity on π_{n+1}. When $g \in \ker \pi_q(e)$, this solving this problem gives injectivity.

We may assume that there is an open subset $S^n \subset U \subset D^{n+1}$ such that $g\colon S^n \to X$ is the restriction of a function $\hat{g}\colon U \to X$ with $f|_U = e \circ \hat{g}$. To see this, define

$$\begin{aligned} d\colon D^{n+1} \times I &\to D^{n+1} \\ (x,t) &\mapsto \begin{cases} \frac{2x}{2-t} & |x| \leq \frac{2-t}{2} \\ \frac{x}{|x|} & |x| \geq \frac{2-t}{2} \end{cases} \end{aligned}$$

Note that $d_0 = \mathrm{Id}$ and $d_1(U) = S^n$. Define $U := \{x \mid |x| > 1/2\}$, $\hat{g} = g \circ d_1$ and $f' = f \circ d_1$. Then f' and $\hat{g}$ satisfy the properties that $f'|_U = e \circ \hat{g}$, $\hat{g}|_{S^n} = g$, and $f' \sim f$. Therefore we can replace f by f'. This completes the proof that we may assume that there is an open subset $S^n \subset U \subset D^{n+1}$ such that $g\colon S^n \to X$ is the restriction of a function $\hat{g}\colon U \to X$ with $f|_U = e \circ \hat{g}$.

Next, write

$$c_A := g^{-1}(X \smallsetminus \operatorname{Int} A) \cup \overline{f^{-1}(X' \smallsetminus A')}$$

and

$$c_B := g^{-1}(X \smallsetminus \operatorname{Int} B) \cup \overline{f^{-1}(X' \smallsetminus B')}$$

We have $c_A \cap c_B = \emptyset$. To see this, first replace g by $\hat{g}$ in the definitions to obtain:

$$\hat{c}_A := \hat{g}^{-1}(X \smallsetminus \operatorname{Int} A) \cup \overline{f^{-1}(X' \smallsetminus A')} \subseteq C_A$$

and

$$\hat{c}_B := \hat{g}^{-1}(X \smallsetminus \operatorname{Int} B) \cup \overline{f^{-1}(X' \smallsetminus B')} \subseteq C_B$$

We *claim* that $\hat{c}_A \cap \hat{c}_B = \emptyset$. This claim implies that $c_A \cap c_B = \emptyset$. Now we prove the claim.

First $\hat{g}^{-1}(X \smallsetminus \operatorname{Int} A) \cap \hat{g}^{-1}(X \smallsetminus \operatorname{Int} B)$ consists of points of S^n that map to

$$(X \smallsetminus \operatorname{Int} A) \cap (X \smallsetminus \operatorname{Int} B) = X \smallsetminus (\operatorname{Int} A \cup \operatorname{Int} B) = X \smallsetminus X = \emptyset.$$

Therefore $\hat{g}^{-1}(X \smallsetminus \operatorname{Int} A) \cap \hat{g}^{-1}(X \smallsetminus \operatorname{Int} B) = \emptyset$. Similarly,

$$(X' \smallsetminus \operatorname{Int} A') \cap (X' \smallsetminus \operatorname{Int} B') = X' \smallsetminus (\operatorname{Int} A' \cup \operatorname{Int} B') = X' \smallsetminus X' = \emptyset.$$

Therefore

$$f^{-1}(X' \smallsetminus \operatorname{Int} A') \cap f^{-1}(X' \smallsetminus \operatorname{Int} B') = \emptyset.$$

Since $\overline{f^{-1}(X' \smallsetminus A')} \subseteq f^{-1}(X' \smallsetminus \operatorname{Int} A')$, and similarly for B', we see that

$$\overline{f^{-1}(X' \smallsetminus A')} \cap \overline{f^{-1}(X' \smallsetminus B')} = \emptyset.$$

It remains, without loss of generality, to investigate

$$\hat{g}^{-1}(X \smallsetminus \operatorname{Int} A) \cap \overline{f^{-1}(X' \smallsetminus B')} \subset \hat{g}^{-1}(\operatorname{Int} B) \cap \overline{f^{-1}(X' \smallsetminus B')}.$$

We will show that the latter set is empty, to complete the proof of the claim. Recall that we have the open set $U \subset D^{n+1}$ contains S^n, and we have the map $\hat{g}\colon U \to X$. Note that $\hat{g}^{-1}(\int B)$ is open. Suppose for a contradiction that $\hat{g}^{-1}(\operatorname{Int} B) \cap \overline{f^{-1}(X' \smallsetminus B')}$ is nonempty. We can therefore start with an element $v \in \hat{g}^{-1}(\operatorname{Int} B) \cap \overline{f^{-1}(X' \smallsetminus B')}$, and then consider a small open set V containing v that is contained in $\hat{g}^{-1}(\int B)$. Since v lies in the closure of $f^{-1}(X' \smallsetminus B')$, there is an element $u \in V$ in

$$\hat{g}^{-1}(\operatorname{Int} B) \cap f^{-1}(X' \smallsetminus B').$$

Then $\hat{g}(u) \in \operatorname{Int} B \subset B$ and $f(u) \notin B'$. But $f|_U = e \circ \hat{g}$ and e is a map of triads therefore restricts to a map $e\colon B \to B'$. Thus $f(u) = e(\hat{g}(u))$ and so $f(u) \in B'$. This gives rise to a contradiction, thus $\hat{g}^{-1}(\operatorname{Int} B) \cap \overline{f^{-1}(X' \smallsetminus B')} = \emptyset$ as desired. This completes the proof of the claim that $\hat{c}_A \cap \hat{c}_B = \emptyset$ and therefore that $c_A \cap c_B = \emptyset$.

By subdivision of D^{n+1} cells, and then further subdivision into small enough cells, we can assume that no cell intersects both c_A and c_B. Define

$$K_A := \{\text{cells } \sigma \mid g(\sigma \cap S^n) \subseteq \operatorname{Int} A,\ f(\sigma) \subseteq \operatorname{Int} A'\}$$

and

$$K_B := \{\text{cells } \sigma \mid g(\sigma \cap S^n) \subseteq \operatorname{Int} B,\ f(\sigma) \subseteq \operatorname{Int} B'\}.$$

Note that $D^{n+1} = K_A \cup K_B$. To see this, if σ does not intersect c_A, then $\sigma \subset K_A$, while if σ does not intersect c_B, then $\sigma \subset K_B$. Therefore, since we subdivided so that no cell intersects both c_A and c_B, it follows that every cell lies in at least one of K_A, K_B, as required. Consider the following problem:

$$\begin{tikzcd}
A \cap B \arrow[rr, "e"] & & A' \cap B' \\
S^n \cap (K_A \cap K_B) \arrow[u, "g"] \arrow[rr] & & K_A \cap K_B \arrow[u, "f"'] \arrow[ull, dashed, "\overline{g}"]
\end{tikzcd}$$

The HELP theorem implies that there exists a map $\overline{g}\colon K_A \cap K_B \to A \cap B$ as in the diagram, and a homotopy

$$\overline{h}\colon K_A \cap K_B \times I \to A' \cap B'$$

with $\overline{h}\colon f \sim e \circ \overline{g}$ a homotopy rel. $S^n \cap K_A \cap K_B$.

Now define

$$\overline{g}_A\colon K_A \cap (S^n \cup K_B) \to A$$

by g on $K_A \cap S^n$ and by $\overline{g}$ on $K_A \cap K_B$. These agree on the intersection by the construction of $\overline{g}$. Also define

$$f = e \circ g\colon K_A \cap S^n \to A'$$

By restricting we have a homotopy

$$\overline{h}_A\colon f|_{K_A \cap S^n \cup K_B} \sim e \circ g_A$$

rel. $S^n \cap K_A$. Then we apply the HELP theorem to the diagram:

$$\begin{tikzcd}
K_A \cap (S^n \cup K_B) \arrow[rr, "i_0"] \arrow[dd] & & (K_A \cap (S^n \cup K_B)) \times I \arrow[dl, "\overline{h}_A"'] \arrow[dd] & & K_A \cap (S^n \cup K_B) \arrow[ll, "i_1"'] \arrow[dl, "\overline{g}_A"'] \arrow[dd] \\
& A' & & A \arrow[ll, "e"] & \\
K_A \arrow[ur, "f"] \arrow[rr, "i_0"'] & & K_A \times I \arrow[ul, dashed, "\widetilde{h}_A"] & & K_A \arrow[ll, "i_1"] \arrow[ul, dashed, "\widetilde{g}_A"]
\end{tikzcd}$$

to obtain a map $\widetilde{g}_A \colon K_A \to A$. Here we use that $e \colon A \to A'$ is a weak equivalence. Similarly, we obtain a map $\widetilde{g}_B \colon K_B \to B$. These maps agree by construction on $K_A \cap K_B$, and therefore induce maps

$$\widetilde{g}_A \cup \widetilde{g}_B = \widetilde{g} \colon D^{n+1} = K_A \cup K_B \to X$$

and a homotopy $\widetilde{h}_A \cup \widetilde{h}_B \colon f \sim e \circ \widetilde{g}$ rel. S^n. This is the required maps to solve the problems that show $e \colon X \to X$ is a weak equivalence, as desired. □

9. Homotopy excision

Definition 9.1. A map $f \colon (A, C) \to (X, B)$ of pairs is an n-equivalence if

$$(f_*)^{-1}\big(\operatorname{im}(\pi_0(B) \to \pi_0(X))\big) = \operatorname{im}(\pi_0(C) \to \pi_0(A))$$

and

$$f_* \colon \pi_q(A, C) \to \pi_q(X, B)$$

is a bijection for $q < n$ and a surjection for $q = n$, for all basepoints $* \in C$.

This section is about the homotopy excision, or Blakers-Massey theorem.

Theorem 9.2 (Homotopy excision)**.** *Let $(X; A, B)$ be an excisive triad with $C = A \cap B$. Suppose that (A, C) is $(m-1)$-connected and (B, C) is $(n-1)$-connected, with $m \geq 2$ and $n \geq 1$. Then $(A, C) \to (X, B)$ is an $(m+n-2)$-equivalence.*

9.1. Consequences of homotopy excision.

Before we prove the homotopy excision theorem, we investigate some of its main consequences.

Theorem 9.3. *Let $f \colon X \to Y$ be an $(n-1)$-equivalence between $(n-2)$-connected spaces, with $n \leq 2$. The quotient map*

$$q \colon (Mf, X) \to (Cf, *)$$

is a $(2n-2)$-equivalence. In particular, Cf is $(n-1)$-connected.

Proof. We define an excisive triad $(Cf; A, B)$ by taking $A = Y \cup (X \times [0, 2/3])$ and $B = (X \times [1/3, 1])/X \times \{1\}$. Then $C = A \cap B = X \times [1/3, 2/3]$. The map q is homotopic to the following sequence of maps:

$$(Mf, X) \xrightarrow{\simeq} (A, C) \xrightarrow{\text{inc}} (Cf, B) \xrightarrow{\simeq} (Cf, *)$$

The first and last maps are homotopy equivalences of pairs. We need to see that the map inc is a $(2n-2)$-equivalence. To see this, we will use homotopy excision. First, the long exact sequence in homotopy groups of the pair (Mf, X), together with the fact that X is $(n-2)$-connected, yields that (Mf, X) is an $(n-1)$-connected pair. Therefore (A, C) is $(n-1)$-connected. On the other hand, the cone CX is contractible, and X is $(n-2)$-connected, so the long exact sequence in homotopy groups of (CX, X) yields that (CX, X) is $(n-1)$-connected. Then $(CX, X) \simeq (B, C)$, so (B, C) is $(n-1)$-connected. Then homotopy excision implies that the map $(A, C) \to (Cf, B)$ is an $(2n-2)$-equivalence as required. □

Corollary 9.4. *Let $i\colon A \to X$ be a cofibration that is an $(n-1)$-equivalence between $(n-2)$-connected spaces. Then $(X, A) \to (X/A, *)$ is a $(2n-2)$-equivalence.*

Proof. This follows from the previous theorem and the following diagram. Recall that for cofibrations the vertical maps are homotopy equivalences.

$$\begin{tikzcd} (Mi, A) \arrow[r, "q"] \arrow[d, "r"', "\simeq"] & (Ci, *) \arrow[d, "\simeq"] \\ (X, A) \arrow[r] & (X/A, *) \end{tikzcd}$$

□

Now we come to a key result in homotopy theory, the Freudenthal suspension theorem. Define the suspension homomorphism

$$\begin{array}{rcl} \Sigma\colon \pi_q(X) & \to & \pi_{q+1}(\Sigma X) \\ f & \mapsto & f \wedge \mathrm{Id}\colon S^{q+1} = S^q \wedge S^1 \to X \wedge S^1 = \Sigma X \end{array}$$

Theorem 9.5 (Freudenthal suspension theorem)**.** *Let X be an $(n-1)$-connected space, with $n \geq 1$. Then Σ is a bijection for $q < 2n-1$ and a surjection for $q = 2n-1$.*

Proof. Write $C'X := X \wedge I$ with I the pair $(I, \{0\})$, i.e. $\{0\}$ as the basepoint of I. Thus

$$C'X = X \times I/(X \times \{0\} \cup * \times I).$$

Represent a homotopy class in $\pi_q(X)$ by $f\colon (I^q, \partial I^q) \to (X, *)$. Then $f \times \mathrm{Id}\colon I^{q+1} \to X \times I$ induces a map

$$(I^{q+1}, \partial I^{q+1}, J^q) \to (C'X, X, *)$$

Restricting to $I^q \times \{1\}$ gives f. If we quotient out by $X \times \{1\}$, we obtain Σf. We get a commutative diagram:

$$\begin{tikzcd} \pi_{q+1}(C'X, X, *) \arrow[d, "\partial"', "\cong"] \arrow[dr, "\rho"] & \\ \pi_q(X) \arrow[r, "\Sigma"] & \pi_{q+1}(\Sigma X) \end{tikzcd}$$

where the diagonal map ρ is induced by the quotient map that factors out $X \times \{1\}$. Since $C'X \simeq \mathrm{pt}$, we have that the vertical map ∂ is an isomorphism by the long exact sequence of a pair. Next $X \to C'X$ is a cofibration and an n-equivalence between $(n-1)$-connected spaces. Therefore ρ is a $2n$-equivalence by the corollary above. It follows that Σ is a bijection for $q+1 < 2n$ and a surjection for $q+1 = 2n$, as desired. □

Corollary 9.6. *For all $n \geq 1$, $\pi_n(S^n) \cong \mathbb{Z}$, with the suspension map $\Sigma\colon \pi_n(S^n) \to \pi_{n+1}(S^{n+1})$ an isomorphism.*

Proof. First recall from the Hopf fibration $S^1 \to S^3 \to S^2$ and the associated long exact sequence of a fibration that

$$0 = \pi_2(S^3) \to \pi_2(S^2) \to \pi_1(S^1) \to \pi_1(S^3) = 0$$

$\pi_2(S^2) \cong \pi_1(S^1) \cong \mathbb{Z}$. Then consider $\Sigma\colon \pi_2(S^2) \to \pi_3(S^3)$. Here $X = S^2$ is 1-connected so $n = 2$ in the Freudenthal theorem. Thus

$$\Sigma\colon \pi_q(S^2) \to \pi_{q+1}(S^3)$$

is an isomorphism for $q < 2 \cdot 2 - 1 = 3$ and is a surjection for $q = 3$. Actually $\pi_3(S^2) \to \pi_4(S^3)$ is a surjection $\mathbb{Z} \to \mathbb{Z}/2$, but we do not yet have the machinery to prove that $\pi_{n+1}(S^n)$ is nontrivial for $n > 2$. We will return to this later.

For higher spheres the isomorphism falls more easily within the range of the Freudenthal theorem. □

The Freudenthal theorem allows us to define

$$\pi_q^S(X) = \operatorname{colim} \pi_{q+n}(\Sigma^n X)$$

The stable homotopy groups of X. The Freudenthal theorem guarantees that the groups in the colimit eventually stabilise.

9.2. **Proof of homotopy excision.** Now we start on the proof of the homotopy excision theorem. First we introduce triples of spaces (X, Y, Z). Note that this means $Z \subseteq Y \subseteq X$, and is *not* a triad $(X; A, B)$, which means $X = \operatorname{Int} A \cup \operatorname{Int} B$ and so only in pathological situations is $B \subseteq A$.

Proposition 9.7. *Let (X, Y, Z) be a triple of spaces. Then there is a long exact sequence in homotopy groups*

$$\cdots \to \pi_q(Y, Z) \to \pi_q(X, Z) \to \pi_q(X, Y) \to \pi_{q-1}(Y, Z) \to \cdots$$

Proof. Use the long exact sequences of the various pairs involved and chase diagrams heroically. All the rows and columns except for the middle row are exact, and the diagram commutes. It follows from a diagram chase that the middle row is exact.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \pi_q(Z) \arrow[r] \arrow[d] & \pi_q(Z) \arrow[r] \arrow[d] & 0 \arrow[d] \\
\pi_{q+1}(X,Y) \arrow[r] \arrow[d] & \pi_q(Y) \arrow[r] \arrow[d] & \pi_q(X) \arrow[r] \arrow[d] & \pi_q(X,Y) \arrow[d] \\
\pi_{q+1}(X,Y) \arrow[r] \arrow[d] & \pi_q(Y,Z) \arrow[r] \arrow[d] & \pi_q(X,Z) \arrow[r] \arrow[d] & \pi_q(X,Y) \arrow[d] \\
0 \arrow[r] \arrow[d] & \pi_{q-1}(Z) \arrow[r] \arrow[d] & \pi_{q-1}(Z) \arrow[r] \arrow[d] & 0 \arrow[d] \\
\pi_q(X,Y) \arrow[r] \arrow[d] & \pi_{q-1}(Y) \arrow[r] \arrow[d] & \pi_{q-1}(X) \arrow[r] \arrow[d] & \pi_{q-1}(X,Y) \arrow[d] \\
\pi_q(X,Y) \arrow[r] & \pi_{q-1}(Y,Z) \arrow[r] & \pi_{q-1}(X,Z) \arrow[r] & \pi_{q-1}(X,Y)
\end{tikzcd}$$

□

Definition 9.8. Let $(X;A,B)$ be a triad. Define the triad homotopy groups

$$\pi_q(X;A,B) = \pi_{q-1}(P(X,*,B),P(A,*,C))$$

for $q \geq 0$.

The long exact sequence of a pair yields an exact sequence

$$\pi_{q+1}(X;A,B) \to \pi_q(A,C) \to \pi_q(X,B) \to \pi_q(X;A,B).$$

Thus in order to prove homotopy excision, we have to show that $\pi_q(X;A,B) = 0$ for $2 \leq q \leq m+n-2$. To understand elements of $\pi_q(X;A,B)$ better, note that we can represent them by maps of triads

$$(I^q; I^{q-2}\times\{1\}\times I, I^{q-1}\times\{1\}, J^{q-2}\times I \cup I^{q-1}\times\{0\}) \to (X;A,B,*)$$

The interior of the q-cube maps to X, and various subsets of its boundary map to $A, B, C = A\cap B$ and the basepoint $*$.

To prove the homotopy excision theorem, first approximate $(X;A,B)$ by a CW triad up to weak equivalence. Since we need to show that certain homotopy groups vanish, working with a weakly equivalent space is sufficient. Thus from now on we will assume that $(X;A,B)$ is a CW triad, that (A,C) has no relative q-cells for $q < m$, and (B,C) has not relative q-cells for $q < n$. Moreover we assume that X has finitely many cells, since the image of the compact set I^q intersects at most finitely many cells.

Claim. *It suffices to prove the vanishing of $\pi_q(X;A,B)$ for $2 \leq q \leq m+n-2$ when (A,C) has a single cell.*

To see the claim, let $A' \subset A$ be a subcomplex of A with $C \subseteq A'$, such that A can be obtained from A' by attaching a single cell. Let $X' = A' \cup B$. Then suppose for the induction hypothesis that the result holds for the triads $(X'; A', B)$ and $(X; A, X')$. The latter triad has $(A, C) = (A, A \cap X') = (A, A')$, consisting of a single cell, therefore we also assume for now (we will prove it presently) that homotopy excision holds for this triad. Then apply the five lemma to the diagram:

$$\begin{tikzcd}
\pi_{q+1}(A, A') \arrow[r] \arrow[d] & \pi_q(A', C) \arrow[r] \arrow[d] & \pi_q(A, C) \arrow[r] \arrow[d] & \pi_q(A, A') \arrow[r] \arrow[d] & \pi_{q-1}(A', C) \arrow[d] \\
\pi_{q+1}(X, X') \arrow[r] & \pi_q(X', B) \arrow[r] & \pi_q(X, B) \arrow[r] & \pi_q(X, X') \arrow[r] & \pi_{q-1}(X', B)
\end{tikzcd}$$

The top row is the long exact sequence associated to the triple (A, A', C) and the bottom row is associated to the triple (X, X', B). The vertical maps are isomorphisms by the assumption that homotopy excision holds for the triads $(X'; A', B)$ and $(X; A, X')$. This completes the proof of the claim.

Claim. *It suffices to prove the vanishing of $\pi_q(X; A, B)$ for $2 \le q \le m + n - 2$ when (B, C) has a single cell.*

Let $B' \subset B$ be a subcomplex with $C \subseteq B'$ such that B is obtained from B' by adding a single cell. Write $X' := A \cup B'$. Suppose that homotopy excision holds for $(X'; A, B')$ and $(X; X', B)$. The map $(A, C) \to (X, B)$ factors as $(A, C) \to (X', B') \to (X, B)$, and so homotopy excision also holds for $(X; A, B)$. This completes the proof of the claim.

From now on we assume that $A = C \cup D^m$ and $B = C \cup D^n$, with $m \ge 2$ and $n \ge 1$. We are given a map

$$f\colon (I^q; I^{q-2} \times \{1\} \times I, I^{q-1} \times \{1\}, J^{q-2} \times I \cup I^{q-1} \times \{0\}) \to (X; A, B, *)$$

and we want to show that this map is null-homotopic as a map of triads. Let $x \in D^m$ and $y \in D^n$ be interior points. Consider the sequence of inclusions of triads:

$$(A; A, A \setminus \{x\}) \subset (X \setminus \{y\}; A, X \setminus \{x, y\}) \subset (X; A, X \setminus \{x\}) \supset (X; A, B).$$

The first and last inclusions induce isomorphisms on triad homotopy groups, since we can homotope maps off D^n and D^m respectively, once we know that there is at least one interior point that is not in the image. Also, $\pi_q(A; A, A') = 0$ for any $A' \subset A$. Therefore the left-most group vanishes, and so it suffices to show that the middle inclusion induces an isomorphism on triad homotopy groups. That is, we have a map f into $(X; A, X \setminus \{x\})$ and we need to show that it is homotopic to a map into $(X \setminus \{y\}; A, X \setminus \{x, y\})$. That is, we have to miss a point y in the interior of D^n with a map of I^q for $2 \le q \le n + m - 2$. This will use simplicial approximation in the argument.

Let $D^m_{1/2} \subset D^m$ and $D^n_{1/2} \subset D^n$ be subdiscs of radius $1/2$. Subdivide I^q into small enough subcubes I^q_α such that each subcube has the property that $f(I^q_\alpha) \subset \mathrm{Int}(D^m)$ if $f(I^q_\alpha)$ intersects $D^m_{1/2}$, and holds the same for D^n. By simplicial approximation, the map f is homotopic to a map $g\colon I^q \to X$ such that the restriction of g to the

$(n-1)$-skeleton of I^q, $g|_{(I^q)^{(n-1)}}$, does not cover all of $D^n_{1/2}$, and similarly $g|_{(I^q)^{(m-1)}}$, does not cover all of $D^m_{1/2}$. Here we use the skeleta of the subdivided I^q. Moreover, simplicial approximation enables us to arrange that $\dim g^{-1}(y)$ is at most $q-n$, for some $y \in D^n_{1/2}$ that is not in the image of the $(n-1)$-skeleton of I^q. Here dimension needs to be correctly interpreted, and we skip over the precise details of how we use transversality. We could also use smooth approximation. Let $\pi\colon I^q \to I^{q-1}$ be projection on the the first $q-1$ coordinated. Define

$$K := \pi^{-1}(\pi(g^{-1}(y))).$$

This is a prism. The dimension is no more that the dimension of $g^{-1}(y)$ plus one, so

$$\dim K \leq q - n + 1 \leq m - 1.$$

Therefore $g(K)$ cannot cover $D^m_{1/2}$. Choose $x \in D^m_{1/2}$ that does not lie in $g(K)$. Since

$$g(\partial I^{q-1} \times I) \subseteq A,$$

we see that

(a) $\pi(g^{-1}(x)) \cup \partial I^{q-1}$ and
(b) $\pi(g^{-1}(y))$

are disjoint. We can therefore define a homotopy of g as desired. Let $v\colon I^{q-1} \to I$ be a function such that v is zero on (a) and is one on (b). Such a function exists by the Uryssohn lemma. Define a function $h\colon I^{q+1} \to I^q$ by

$$h(r, s, t) = (r, s - st \cdot v(r))$$

where $s, t \in I$ and $r \in I^{q-1}$. Define

$$f' = g \circ h_1$$

where $h_1(r, s) = h(r, s, 1)$. Note that

$$h(r, s, 0) = (r, s) \text{ and } h(r, 0, t) = (r, 0)$$

and

$$h(r, s, t) = (r, s)$$

if $t \in \partial I^{q-1}$. Thus $g \circ h_t$ defines a homotopy of maps of triads. Then observe that

$$h(r, s, t) = (r, s)$$

if $h(r, s, t) \in g^{-1}(x)$ since $r \in \pi(g^{-1}(x))$ implies that $v(r) = 0$, and

$$h(r, s, t) = (r, s - st)$$

if $h(r, s, t) \in g^{-1}(y)$, since $r \in \pi(g^{-1}(y))$ means that $v(r) = 1$. Thus f' has image in

$$(X \smallsetminus \{y\}; A, X \smallsetminus \{x, y\})$$

i.e. the image has been moved off y. This completes the proof that the inclusion of triads $(X \smallsetminus \{y\}; A, X \smallsetminus \{x, y\}) \subset (X; A, X \smallsetminus \{x\})$ induces an isomorphism of triad homotopy groups, and thus completes the proof that $\pi_q(X; A, B) = 0$ for $2 \leq q \leq m + n - 2$. This completes the proof of homotopy excision.

9.3. **Truncated long exact sequence in homotopy groups of a cofibration.**

Theorem 9.9. *Let $A \to X \to Q$ be a cofibre sequence, with $A \to X$ a cofibration and $Q = X/A$. Suppose that A is r-connected and Q is s-connected, with $r, s \geq 1$. Then there is a homomorphism $\partial_k \colon \pi_k(Q) \to \pi_{k-1}(A)$ such that*

$$\pi_{r+s}(A) \to \pi_{r+s}(X) \to \pi_{r+s}(Q) \xrightarrow{\partial_{r+s}} \cdots \to \pi_2(A) \to \pi_2(X) \to \pi_2(Q) \to 0$$

is exact.

Here is an example. Consider the Hopf map $\eta \colon S^3 \to S^2$, and replace S^2 by its mapping cylinder $M\eta$. Then $S^3 \to M\eta$ is a cofibration and the quotient is $\mathbb{CP}^2$. Since S^3 is 2-connected and S^2 is 1-connected, we have an exact sequence

$$\pi_3(S^3) \xrightarrow{\cong} \pi_3(S^2) \to \pi_3(\mathbb{CP}^2) \to \pi_2(S^3) \to \pi_2(S^2) \to \pi_2(\mathbb{CP}^2) \to 0$$

which implies that $\pi_3(\mathbb{CP}^2) = 0$ and $\pi_2(\mathbb{CP}^2) \cong \mathbb{Z}$.

Proof. Let $i \colon A \to X$ be the inclusion. We have a long exact sequence in homotopy groups associated to the map i:

$$\pi_n(Fi) \to \pi_n(A) \to \pi_n(X) \to \pi_{n-1}(Fi) \to \cdots$$

We want to replace $\pi_n(Fi)$ by $\pi_{n+1}(Ci)$ in the desired range.

Lemma 9.10. *Let $f \colon Y \to X$ be a map with Y m-connected and f an n-equivalence. Then $p \colon (Mf, Y) \to (Cf, *)$ induces an isomorphism $p_* \colon \pi_q(Mf, Y) \to \pi_q(Cf, *)$ for $2 \leq q \leq m + n$ and a surjection for $q = m + n + 1$.*

Now we prove the lemma. This is a slight variation on a theorem above. We repeat the very similar argument to be careful. Define an excisive triad $(Cf; A, B)$ by taking $A = X \cup (Y \times [0, 2/3])$ and $B = (Y \times [1/3, 1])/Y \times \{1\}$. Then $C = A \cap B = Y \times [1/3, 2/3]$. As above, the map p is homotopic to the following sequence of maps:

$$(Mf, Y) \xrightarrow{\simeq} (A, C) \xrightarrow{\text{inc}} (Cf, B) \xrightarrow{\simeq} (Cf, *)$$

The first and last maps are homotopy equivalences of pairs. We need to see that the map inc is a $(m + n + 1)$-equivalence. Since CY is contractible, the long exact sequence of a pair yields that $\pi_q(CY, Y) = \pi_q(B, C) = \pi_{q-1}(Y) = 0$ for $q - 1 \leq m$ so for $q \leq m_1$. By assumption $\pi_q(A, C) = \pi_q(Mf, Y) = 0$ for $q \leq n$. Then homotopy excision says that $\pi_q(Mf, X) \cong \pi_q(A, C) \to \pi_q(X, B) \cong \pi_q(Cf, *)$ is an isomorphism for $2 \leq q \leq m + n$ and a surjection for $q = m + n + 1$. This completes the proof of the lemma.

Let $f \colon X \to Y$ be a map as in the lemma. Recall that the homotopy fibre $Ff = X \times_f PY$, contains pairs consisting of a point $x \in X$ together with a path in Y from the basepoint of Y to $f(x)$. Now define

$$\begin{aligned} \eta \colon Ff &\to \Omega Cf \\ (x, \gamma) &\mapsto t \mapsto \begin{cases} \gamma(2t) & 0 \leq t \leq 1/2 \\ (x, 2t - 1) & 1/2 \leq t \leq 1 \end{cases} \end{aligned}$$

The map η induces a map on homotopy groups $\pi_{n-1}(Ff) \to \pi_n(Cf)$. We need to show that this map is an isomorphism in the required range.

Lemma 9.11. *Let $f\colon X \to Y$ be a map, let $j\colon X \to Mf$ be the inclusion, let $r\colon Mf \to Y$ be the retraction, and we have a quotient map $Mf \to Cf$. The following diagram commutes up to homotopy, where the map π is the canonical map induced by the quotient $Mf \to Cf$.*

$$\begin{tikzcd}
Fj = X \times_j P(Mf) \arrow[rr, "Fr = \mathrm{Id} \times Pr"] \arrow[dr, "\pi"'] & & X \times_f PY = Ff \arrow[dl, "\eta"] \\
& \Omega Cf &
\end{tikzcd}$$

It is a straightforward exercise to prove the lemma. We now have the following diagram.

$$\begin{tikzcd}
\pi_{q+1}(X) \arrow[r] \arrow[d, "="] & \pi_{q+1}(Mf) \arrow[r] \arrow[d, "\cong"] & \pi_{q+1}(Mf, X) = \pi_q(Fj) \arrow[r] \arrow[d, "Fr"] & \pi_q(X)^{=} \arrow[r] \arrow[d] & \pi_q(Mf) \arrow[r] \arrow[d, "\cong"] & \cdots \\
\pi_{q+1}(X) \arrow[r] & \pi_{q+1}(Y) \arrow[r] & \pi_q(Ff) \arrow[r] \arrow[d, "\eta_*"] & \pi_q(X) \arrow[r, "f_*"] & \pi_q(Y) \arrow[r] \arrow[d] & \cdots \\
& & \pi_q(\Omega Cf) = \pi_{q+1}(Cf) & & {} &
\end{tikzcd}$$

The top row is the long exact sequence of the pair (Mf, X). This sequence was proven using the exact sequence for a fibration obtained from the map $j\colon X \to Mf$. Two out of three vertical maps are isomorphisms, so the map Fr is an isomorphism by the five lemma. In the range of homotopy excision, namely for $q+1 \leq m+n$ the composition $\pi_{q+1}(Mf, X) \to \pi_{q+1}(Cf)$ is an isomorphism. It follows from the commutativity in Lemma 9.11 that $\eta_*\colon \pi_q(Ff) \to \pi_{q+1}(Cf)$ is an isomorphism.

We may therefore replace $\pi_q(Ff)$ with $\pi_{q+1}(Cf)$ in the bottom row, for $q+1 \leq m_n$. Now let apply this with $i\colon A \to X$ replacing $f\colon Y \to X$, with $r = m$ and $s = n$. Since i is a cofibration, $Ci \simeq X/A = Q$, so we obtain the desired truncated long exact sequence. □

10. Homology theories

This section was typed by Robert Graham. There are many different distinctions that are made in the literature. We can talk about homology theories versus cohomology theories, generalised (co)homology theories versus ordinary (co)homology theories and reduced (co)homology theories vs unreduced. For example DeRham cohomology is an ordinary unreduced cohomology theory, whereas cobordism theory, K theory and the stable homotopy functor are examples of generalised (un)reduced homology theories. The homology theories we are most familiar with, singular, cellular and simplicial are ordinary unreduced.

Let us explain these distinctions. Homology is covariant. So given $f\colon (X, A) \to (Y, B)$ we have a map

$$E_q(f)\colon E_q(X, A) \to E_q(Y, B)$$

On the other hand cohomology is contravariant (the notation will be explained below in detail).

An ordinary homology theory satisfies the dimension axiom, which states

$$E_0(*) = \pi$$
$$E_q(*) = 0,\ q \neq 0$$

where π is some abelian group, known as the coefficient group. Generalised theories need not satisfy this axiom.

The difference between reduced and unreduced is more substantial (at least at first glance). We will see more about this later, but briefly, reduced theories work well for based spaces, whereas unreduced theories do not require basepoints.

We will now define what a homology theory is.

Definition 10.1. Let π be an abelian group. An *ordinary homology theory* over π is a collection of functors E_q from the homotopy category of pairs of spaces (X, A) to abelian groups, together with natural transformations

$$\partial\colon E_q(X, A) \to E_{q-1}(A, \emptyset)$$

We write $E_q(X) := E_{q-1}(X, \emptyset)$. The following axioms must hold:

(1) Dimension Axiom. $E_0(*) = \pi$ and $E_q(*) = 0,\ i \neq 0$.
(2) Exactness Axiom. Given a pair (X, A) we have maps
$$(A, \emptyset) \to (X, \emptyset) \to (X, A)$$
Thus by applying E_q and using ∂ we have the following sequence
$$\ldots \to E_q(A) \to E_q(X) \to E_q(X, A) \to E_{q-1}(A) \to \ldots$$
The axiom says this is exact.
(3) Excision Axiom. Given an excisive triad $(X; A, B)$, the natural map
$$E_q(A, A \cap B) \to E_q(X, B)$$
is an equivalence.
(4) Additivity Axiom. Given $(X, A) = \coprod_i (X_i, A_i)$ then the map
$$\bigoplus_i E_q(X_i, A_i) \to E_q(X, A)$$
induced by $(X_i, A_i) \to (X, A)$ is an equivalence.
(5) Weak Equivalence Axiom. E_q sends weak equivalences to group isomorphisms.

Regular cellular homology is a homology theory as defined above with one caveat, it is defined only for CW complexes. The next theorem makes this precise.

Theorem 10.2. *Cellular homology is a collection of functors $H_q(-; \pi)$ from CW pairs (X, A) to abelian groups with natural transformations*

$$\partial\colon H_q(X, A; \pi) \to H_{q-1}(A; \pi)$$

It satisfies and is determined by the dimension, exactness, excision and additivity axioms. Moreover this theory is determined by and determines a theory E_q as defined above.

Proof. We shall not prove this, however we give one step. Fix a CW approximation functor Γ. Then we can define $E_q(X, A) := H_q(\Gamma X, \Gamma A; \pi)$. Different choices of Γ give rise to isomorphic, but not identical theories. $\square$

A very cool fact is that a particularly nice choice of CW approximation functor arises from the geometric realisation of the underlying simplicial set of a space X. This CW approximation functor gives rise to the singular homology of X.

Definition 10.3. A *generalised reduced homology theory* is a collection of functors $\widetilde{E}_q$ from the homotopy category of nondegenerately based spaces to abelian groups that satisfy the following:

(1) Exactness Axiom. For any cofibration $A \to X$ we have the following is exact
$$\widetilde{E}_q(A) \to \widetilde{E}_q(X) \to \widetilde{E}_q(X/A)$$
(2) Suspension Axiom. There are natural isomorphisms
$$\Sigma : \widetilde{E}_q \simeq \widetilde{E}_{q+1}(\Sigma X)$$
(3) Additivity Axiom. Given $X = \bigvee_i X_i$ the maps $\bigoplus_i \widetilde{E}_q(X_i) \to \widetilde{E}_q(X)$ induced by $X_i \to X$ are equivalences.
(4) Weak Equivalence Axiom. $\widetilde{E}_q$ sends weak equivalences to group isomorphisms.

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Of course this becomes an ordinary reduced homology theory if we add the dimension axiom.

Theorem 10.4. *A (generalized/ordinary) unreduced homology theory* (E_q, ∂) *determines and is determined by a (generalized/ordinary) reduced homology theory* $(\widetilde{E}_q, \Sigma)$

Note that the axioms of a reduced theory seem at first glance weaker than the axioms of an unreduced theory, so one direction of this theorem is particularly interesting.

Proof. First suppose we have an unreduced theory E_q. Define $\widetilde{E}_q(X) := E_q(X, *)$. We need to show this satisfies all the axioms. The weak equivalence axiom and the dimension axiom (if appropriate) are clear. To show exactness first note
$$E_q(A) \to E_q(X) \to E_q(X, A)$$
is exact. However $E_q(X, A) \simeq E_q(X/A, *)$ by excision, moreover by exactness $E_q(A) \simeq \widetilde{E}_q(A) \oplus E_q(*)$ and similarly $E_q(X) = \widetilde{E}_q(X) \oplus E_q(*)$. Therefore
$$\widetilde{E}_q(A) \oplus E_q(*) \to \widetilde{E}_q(X) \oplus E_q(*) \to \widetilde{E}_q(X/A)$$
is exact, which implies the axiom.

To show suspension note $\Sigma X = CX/X$. Then by exactness of E, we have an exact sequence
$$\widetilde{E}_q(X) \to \widetilde{E}_q(CX) \to \widetilde{E}_q(CX/X) \to \widetilde{E}_{q-1}(CX).$$

The extremal terms vanish, so we get the required isomorphism (it is simply ∂^{-1}).

Finally, to show additivity we compute:

$$\begin{aligned}\widetilde{E}_q(\bigvee_i X_i) &= \widetilde{E}_q(\coprod X_i/\coprod *_i)\\ &\simeq E_q(\coprod X_i, \coprod *_i)\\ &\simeq \bigoplus E_q(X_i, *_i)\\ &\simeq \widetilde{E}_q(X_i)\end{aligned}$$

Now we show the other direction. Suppose we have a reduced homology theory $(\widetilde{E}_\star, \Sigma)$. We define $E_q(X) := \widetilde{E}_q(X_+)$. More generally $E_q(X,A) := \widetilde{E}_q(C_{i_+})$, where $i_+ \colon A_+ \to X_+$ is defined by mapping the basepoint to the basepoint. Now $C_{i_+} \simeq M(i_+)/A_+$ and so from the cofibre sequence of i_+ we get an induced map $\widetilde{E}_q(C_{i_+}) \to \widetilde{E}_q(\Sigma A_+)$. By composing with Σ^{-1} we then get a map

$$\widetilde{E}_q(M(i_+)/A_+) \to \widetilde{E}_{q-1}(A_+)$$

which will serve as our ∂.

The weak equivalence and dimension (if appropriate) axioms are clear. Exactness for E_q follows immediately from exactness for $\widetilde{E}_q$.

For excision consider a triad $(X;A,B)$ and let $(\Gamma X;\Gamma A,\Gamma B)$ be a CW triad approximation. Denote $C = A\cap B$ and $\Gamma C = \Gamma A \cap \Gamma B$. We then have the following diagram.

$$\begin{tikzcd}
(\Gamma A)_+/(\Gamma C)_+ \arrow[r, "\simeq"] \arrow[d, "\text{w.e}"] & (\Gamma X)_+/(\Gamma B)_+ \arrow[d, "\text{w.e}"] \\
Mi_+/C_+ = Mi/C & Mi^B_+/B_+ = Mi^B/B
\end{tikzcd}$$

From which we conclude (by the weak equivalence axiom) $\widetilde{E}_q(Mi/C) \simeq \widetilde{E}_q(Mi^B/B)$. The resulting

Finally to show the additivity axiom we proceed as follows:

$$\begin{aligned}E_q(\coprod_i X_i) &= \widetilde{E}_q((\coprod_i X_i)_+)\\ &= \widetilde{E}_q(\bigvee_i (X_i)_+)\\ &= \bigoplus_i \widetilde{E}_q(X_i)_+\\ &= \bigoplus_i E_q(X_i)\end{aligned}$$

$\square$

Theorem 10.5. *Let E_q be a general or ordinary homology theory. Then E_q commutes with colimits. That is,* $\operatorname{colim}_i E_q(X_i) \simeq E_q(\operatorname{colim}_i X_i)$.

Theorem 10.6 (Mayer Vietoris theorem). *Given a triad $(X; A, B)$ with $C = A \cap B$ and $i\colon C \to A$, $j\colon C \to B$, $k\colon A \to X$ and $\ell\colon B \to X$. Then we have the following long exact sequence*

$$\dots E_q(C) \xrightarrow{(i_* \, j_*)} E_q(A) \oplus E_q(B) \xrightarrow{k_* - \ell_*} E_q(X) \xrightarrow{\Delta} E_{q-1}(C) \dots$$

where Δ is given by

$$E_q(X) \longrightarrow E_q(X, B) \xrightarrow{exc} E_q(A, C) \xrightarrow{\partial} E_{q-1}(C)$$

The Mayer-Vietoris theorem can be deduced from the axioms of generalised homology theories. Weave the long exact sequences of the various pairs into a braid of interlocking exact sequences.

11. The Hurewicz theorem

In this section we return to ordinary homology (i.e. cellular or singular). For every $n \in \mathbb{N}$ we have $\widetilde{H}_n(S^n) = \mathbb{Z}$. Let i_n be the generator of $\widetilde{H}_n(S^n)$. Now for a based space X we can define the *Hurewicz map* $h\colon \pi_n(X) \to \widetilde{H}_n(X)$ by sending $f\colon S^n \to X$ to $\widetilde{H}_n(f)(i_n)$.

Before we get to the main theorem we will prove some basics results about this map. First, h is a homomorphism. To show this let $f, g\colon S^n \to X$, then $f + g$ is defined by

$$S^n \xrightarrow{\text{collapse}} S^n \vee S^n \xrightarrow{f \vee g} X \vee X \xrightarrow{\nabla} X$$

Therefore

$$h(f+g) = \widetilde{H}_n(f+g)(i_n) = \widetilde{H}_n(\mathrm{id}, \mathrm{id}) \circ \widetilde{H}_n(f \vee g) \circ \widetilde{H}_n(\nabla)(i_n)$$

which maps

$$i_n \mapsto (i_n, i_n) \mapsto (\widetilde{H}_n(f)(i_n), \widetilde{H}_n(g)(i_n)) \mapsto \widetilde{H}_n(f)(i_n) + \widetilde{H}_n(g)(i_n)$$

so $h(f+g) = h(f) + h(g)$ as required.

We also claim that the Hurewicz map respects suspension, by which we mean the following commutes.

$$\begin{tikzcd}
\pi_n(X) \arrow[r, "h"] \arrow[d, "\Sigma"] & \widetilde{H}_n(X) \arrow[d, "\Sigma"] \\
\pi_{n+1}(\Sigma X) \arrow[r, "h"] & \widetilde{H}_{n+1}(\Sigma X)
\end{tikzcd}$$

This is proven by the following computation. Let $[f] \in \pi_n(X)$. Then

$$\begin{aligned} h \circ \Sigma([f]) &= h(\Sigma f) \\ &= H_n(\Sigma f)(i_{n+1}) \\ &= H_n(\Sigma f)(\Sigma i_n) \\ &= \Sigma(H_n(f)(i_n)) \\ &= \Sigma(h(f)) \\ &= \Sigma \circ h([f]) \end{aligned}$$

We will now prove a special case of Hurewicz theorem when X is a wedge of circles.

Lemma 11.1. *Let $X = \bigvee_\alpha S^n$. Then*

$$h \colon \pi_n(X) \to \widetilde{H}_n(X)$$

is an isomorphism for $n > 1$ and is the abelianisation for $n = 1$.

Proof. Next suppose $X = \bigvee_\alpha S^n$, here $\pi_1(\bigvee_\alpha S^n) = F(x_\alpha)$ the free group with generators x_α. whereas for $n > 1$ we have $\pi_n(\bigvee_\alpha (S^n)) = \bigoplus_\alpha \pi_n(S^n) = \bigoplus_\alpha \mathbb{Z}$. In both cases we have $\pi_n(S^n)$ is generated by the maps $i_\alpha \colon S^n \to \bigvee_\alpha S^n$, but $\widetilde{H}_n(\bigvee_\alpha S^n) = \bigoplus_\alpha \mathbb{Z}$ is generated by elements e_α and $i_\alpha(i_n) = e_\alpha$. □

We end this section with the full Hurewicz theorem.

Theorem 11.2 (Hurewicz theorem)**.** *Let X be an $n-1$ connected based space. Then the Hurewicz map h is an isomorphism for $n > 1$ and is the abelianisation homomorphism for $n = 1$.*

Proof. By CW approximation, X is a CW complex with one 0-cell and no m-cells for $1 \leq m < n$.

The inclusion map $X^{(n+1)} \to X$ induces an isomorphism $\pi_n(X^{(n+1)}) \to \pi_n(X)$ and another isomorphism $\widetilde{H}_n(X^{(n+1)}) \to \widetilde{H}_n(X)$.

Therefore it suffices to consider the case $X = X^{(n+1)}$. In this case X is the homotopy cofibre of some

$$\bigvee_\beta S^{n+1} \to \bigvee_\alpha S^n \to X$$

Denote $K = \bigvee_\beta S^{n+1}$ and $L = \bigvee_\alpha S^n$.We have the following two exact sequences connected by Hurewicz maps as pictured:

$$\begin{tikzcd} \pi_n(K) \arrow[r] \arrow[d] & \pi_n(L) \arrow[r] \arrow[d] & \pi_n(X) \arrow[r] \arrow[d] & 0 \\ \widetilde{H}_n(K) \arrow[r] & \widetilde{H}_n(L) \arrow[r] & \widetilde{H}_n(X) \arrow[r] & 0 \end{tikzcd}$$

For $n > 1$, since the first two downward arrows are isomorphism by the previous lemma, we see that the final arrow is an isomorphism. For $n = 1$ a similar argument holds where we replace the top row with its abelianisation. □

12. Homology via homotopy theory

In this section we return to generalised homology theories, and explain their connection to homotopy theory. Consider an Eilenberg-MacLane space $K(\pi, n)$ modelled by a CW complex. We have a homotopy equivalence

$$\widetilde{\sigma}\colon K(\pi, n) \to \Omega K(\pi, n+1)$$

since $\Omega K(\pi, n+1)$ is homotopy equivalent to a CW complex by a result of Milnor, and then using Whitehead's theorem. The map $\widetilde{\sigma}$ is adjoint to a map $\sigma\colon \Sigma K(\pi, n) \to K(\pi, n+1)$. There is a map

$$\pi_{q+n}(X \wedge K(\pi, n)) \to \pi_{q+n+1}(\Sigma(X \wedge K(\pi, n))) = \pi_{q+n+1}(X \wedge \Sigma K(\pi, n)) \xrightarrow{\mathrm{Id} \wedge \sigma} \pi_{q+n+1}(X \wedge K(\pi, n+1))$$

This enables us to make the following definition.

Theorem 12.1. *Let X be a CW complex and let π be an abelian group, and let $n \geq 0$.*

$$\widetilde{H}_q(X; \pi) \xrightarrow{\cong} \operatorname{colim}_n \pi_{q+n}(X \wedge K(\pi, n)).$$

Therefore

$$H_q(X; \pi) \cong \operatorname{colim} \pi_{q+n}(X_+ \wedge K(\pi, n)).$$

Proof. We need to check that the right hand side satisfies the axioms of a homology theory. Since ordinary homology is determined by the dimension axiom, this will also show that the theories coincide.

First we claim that $X \wedge K(\pi, 3q)$ is $(2q+1)$-connected. This follows easily by using a CW structure for $K(\pi, 3q)$ with no cells of dimension less than $3q$, together with cellular approximation of maps. The cofibration sequence for a cofibration in homotopy theory shows that

$$\pi_{4q}(A \wedge K(\pi, 2q)) \to \pi_{4q}(X \wedge K(\pi, 2q)) \to \pi_{4q}(X/A \wedge K(\pi, 2q)).$$

Then since colimits preserve exactness, we obtain the corresponding exact sequence for the colimits.

Next,

$$\pi_{q+n}(S^0 \wedge K(\pi, n)) = \pi_{q+n}(K(\pi, n)) = \begin{cases} \pi & q = 0 \\ 0 & q > 0 \end{cases}$$

which shows the dimension axiom.

To see the suspension axiom,

$$\pi_{q+n}(X \wedge K(\pi, n)) \xrightarrow{\Sigma} \pi_{q+n+1}(\Sigma(X \wedge K(\pi, n))) = \pi_{(q+1)+n}(\Sigma X \wedge K(\pi, n))$$

The first map is an isomorphism by the suspension theorem, provided n is high enough. Therefore on passage to colimits, we obtain the suspension axiom.

Finally,

$$\pi_{q+n}((\bigvee X_i) \wedge K(\pi, n)) = \pi_{q+n}(\bigvee (X_i \wedge K(\pi, n)))$$

$$= \sum \pi_{q+n}(X_i \wedge K(\pi, n)) \oplus \sum \pi_{q+n+j}(\prod_{i=i_1}^{i_j} X_i \wedge K(\pi, n), \bigvee X_i \wedge K(\pi, n))$$

The isomorphism arises from the decomposition of the homotopy groups of a wedge as the homotopy groups of the product and a series of relative terms. The j can be arbitrarily high, so the full colimit is required for these groups to vanish. They are generated by Whitehead products. There are no q-cells of $(X_i \wedge K(\pi, n)) \times (X_i \wedge K(\pi, n))$ for $0 < q \leq 2n-1$. Provided n is large enough, $q+n+j < 2n-1$, so that by cellular approximation, the homotopy groups vanish, and we are left with the sum $\sum_i \pi_{q+n}(X_i \wedge K(\pi, n))$. This completes the proof of additivity, and therefore completes the proof of the verification of the axioms. □

Now we show how to generalise this idea.

Definition 12.2. A *spectrum* is a sequence of spaces $\{T_n\}$, $n \geq 0$, and based maps

$$\Sigma T_n \to T_{n+1}$$

Theorem 12.3. *Let $\{T_n\}$ be a spectrum with T_n, an $(n-1)$-connected space with T_n homotopy equivalent to a CW complex for all n. Define $\widetilde{E}_q(X) = \operatorname{colim} \pi_{q+n}(X \wedge T_n)$. This is the colimit over the maps*

$$\pi_{q+n}(X \wedge T_n) \xrightarrow{\Sigma} \pi_{q+n+1}(\Sigma(X \wedge T_n)) \cong \pi_{q+n+1}(X \wedge \Sigma T_n) \xrightarrow{\mathrm{Id} \wedge \sigma} \pi_{q+n+1}(X \wedge T_{n+1}).$$

$\widetilde{E}_q(-)$ is a reduced (generalised) homology theory on based CW complexes (and therefore determines a theory on all nondegenerately based compactly generated spaces.)

We note that the dimension axiom is not required. An unreduced theory is then given by $\operatorname{colim} \pi_{q+n}(X_+ \wedge T_n)$. The proof is analogous to the proof of the previous theorem.

13. The Hilton-Milnor theorem

The product of two spheres $S^p \times S^q = e^0 \cup e^p \cup e^q \cup e^{p+q}$.

$$\pi_{p+q}(S^p \times S^q, S^p \vee S^q) = \pi_{p+q}(S^{p+q}) = \mathbb{Z} \to \partial \pi_{p+q-1}(S^p \vee S^q).$$

Take the attaching map of e^{p+q} generated the first group. Consider

$$\partial(1) = \theta \colon S^{p+q-1} \to S^p \vee S^q.$$

We write $\theta = [\iota_p, \iota_q]$, where ι_p and ι_q are generators of $\pi_p(S^p)$ and $\pi_q(S^q)$. We can generalise this to $X \vee Y$. Let $x \in \pi_p(X)$ and let $y \in \pi_q(Y)$. Write

$$[x, y] \colon S^{p+q-1} \xrightarrow{[\iota_p, \iota_q]} S^p \vee S^q \xrightarrow{x \vee y} X \vee Y.$$

These are called *Whitehead products.*

Theorem 13.1 (Hilton-Milnor theorem)**.**

$$\Omega(\Sigma X \vee \Sigma Y) \simeq \prod_{w \in W} \Omega\Sigma(X^{\{w_x\}} \wedge Y^{\{w_y\}})$$

where W is a basis for the free Lie algebra on x, y, the free algebra generated by x and y with the Lie bracket $[-,-]$ that satisfies $[a,b] = -[b,a]$. Also

$$X^{\{i\}} = X \wedge \cdots \wedge X$$

is the i-fold smash product of X,

$$Y^{\{i\}} = Y \wedge \cdots \wedge Y$$

is the i-fold smash product of Y, w_x and w_y are the number of appearances of x, y in bracket w.

This theorem helps explain the homotopy type of a wedge. We will unfortunately not provide a proof. To understand this let us do an example with $X = S^{p-1}$ and $Y = S^{q-1}$. Then

$$\begin{aligned}
\pi_n(S^p \vee S^q) &= \pi_n(\Sigma X \vee \Sigma Y) \\
&= \pi_{n+1}(\Omega(\Sigma X \vee \Sigma Y)) \\
&= \pi_{n+1}(\prod_{w \in W} \Omega\Sigma(X^{\{w_x\}} \wedge Y^{\{w_y\}})) \\
&= \bigoplus_{w \in W} \pi_n(\Sigma(X^{\{w_x\}} \wedge Y^{\{w_y\}})) \\
&= \bigoplus_{w \in W} \pi_n(\Sigma(S^{w_x(p-1)} \wedge S^{w_y(q-1)})) \\
&= \bigoplus_{w \in W} \pi_n(S^{w_x(p-1)+w_y(q-1)+1})
\end{aligned}$$

For example

$$\pi_3(S^2 \vee S^2) = \pi_3(S^2) \oplus \pi_3(S^2) \oplus \pi_3(S^3)$$

corresponding to the generators x, y and $[x, y]$ of the free Lie algebra. Also,

$$\pi_4(S^2 \vee S^2) = \pi_4(S^2) \oplus \pi_4(S^2) \oplus \pi_4(S^3) \oplus \pi_4(S^4) \oplus \pi_4(S^4)$$

corresponding to the generators $x, y, [x, y], [x, [x, y]]$ and $[y, [x, y]]$. Outside of wedges of spheres, the theorem is harder to apply, but it is certainly possible.

14. Cohomology and universal coefficients

We saw how to define cohomology using homotopy classes of maps in an exercise sheet. We should also see how to define cohomology using homological algebra.

Let R be a commutative ring. Let (C_*, ∂_i) be a chain complex of R-modules, that is we have R-modules C_i for $i \in \mathbb{Z}$, and R-module homomorphisms $\partial_i \colon C_i \to C_{i-1}$ with $\partial_{i-1} \circ \partial_i = 0$. The associated *cochain complex* is

$$\begin{array}{rcl}
\partial^i \colon C^{i-1} := \operatorname{Hom}_R(C_{i-1}, R) & \to & C^i := \operatorname{Hom}_R(C_i, R) \\
f & \mapsto & (-1)^{i-1} f \circ \partial_i.
\end{array}$$

Note that $\partial^{i+1} \circ \partial^i = 0$. The sign is not strictly necessary at this point, but makes certain diagrams commute later on, so technically should be included in the definition now.

Define

$$H^i(C_*) := H_i(C^*, \partial^*) = \ker \partial^{i+1} / \operatorname{im} \partial^i.$$

Let $C_* = C_*(X;R)$ be the singular/cellular chain complex of a space X. Then $H^i(X;R) := H^i(C_*)$. For example, $H^i(S^n;R)$ is equal to R for $i = 0, n$ and zero otherwise. Cohomology is of course closely related to homology. We aim to understand the relationship next.

14.1. **Universal coefficient theorem for cohomology.**

Definition 14.1 (Left and right exact)**.**

(1) A covariant functor $F\colon R\text{−mod} \to R\text{−mod}$ is called left exact if for any short exact sequence
$$0 \to A \to B \to C \to 0$$
we have that
$$0 \to F(A) \to F(B) \to F(C)$$
is exact.

(2) A contravariant functor $F\colon R\text{−mod} \to R\text{−mod}$ is called left exact if for any short exact sequence
$$0 \to A \to B \to C \to 0$$
we have that
$$0 \to F(C) \to F(B) \to F(A)$$
is exact.

(3) A covariant functor $F\colon R\text{−mod} \to R\text{−mod}$ is called right exact if for any short exact sequence
$$0 \to A \to B \to C \to 0$$
we have that
$$F(A) \to F(B) \to F(C) \to 0$$
is exact.

(4) A contravariant functor $F\colon R\text{−mod} \to R\text{−mod}$ is called left exact if for any short exact sequence
$$0 \to A \to B \to C \to 0$$
we have that
$$F(C) \to F(B) \to F(A) \to 0$$
is exact.

A functor that is both left and right exact is called exact.

For example, let N be an R-bimodule. Then:

(1) The functor $M \mapsto \mathrm{Hom}_R(M,N)$ is left exact contravariant.
(2) The functor $M \mapsto \mathrm{Hom}_R(N,M)$ is left exact covariant.
(3) The functor $M \mapsto N \otimes_R M$ is right exact covariant.

As an explicit example, consider the chain complex:
$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0.$$
Tensor this with $\mathbb{Z}/2$, to obtain
$$\mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{\cong} \mathbb{Z}/2 \to 0.$$

Tensoring is right exact. On the other hand applying $\mathrm{Hom}(-,\mathbb{Z})$ yields

$$0 \to \mathrm{Hom}(\mathbb{Z}/2,\mathbb{Z}) = 0 \to \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \xrightarrow{f\mapsto 2f} \mathrm{Hom}(\mathbb{Z},\mathbb{Z}).$$

This is a left exact functor. We will focus on the case of Hom to begin with.

Definition 14.2. An R module I is said to be injective if the diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d] & N \arrow[dl, dashed, "\exists"] \\ & I & \end{tikzcd}$$

for any R modules M and N.

Theorem 14.3. *Any divisible module over a PID is injective.*

A good exercise is to prove this for divisible abelian groups, since $\mathbb{Z}$ is a PID. Here a module M is divisible if for any $m \in M$ and for every $n \in \mathbb{Z}\setminus\{0\}$ there exists an $m' \in M$ such that $nm' = m$.

Proposition 14.4. *Let P be a projective module, let I be an injective module, and let $0 \to A \to B \to C \to 0$ be a short exact sequence. Then*

$$0 \to \mathrm{Hom}(C, I) \to \mathrm{Hom}(B, I) \to \mathrm{Hom}(A, I) \to 0$$

and

$$0 \to \mathrm{Hom}(P, A) \to \mathrm{Hom}(P, B) \to \mathrm{Hom}(P, C) \to 0$$

are exact.

The proposition follows immediately from the definitions. Often in applications the modules in question will not be projective or injective as required, and we want a way to understand the failure of the previous proposition to hold in these cases. For this we use Ext groups, which we will now work towards defining.

Definition 14.5. Given an R-module M, a *projective resolution* is an exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0,$$

where P_i is a projective R-module for all $i \in \mathbb{N}\cup\{0\}$.

An *injective resolution of M* is an exact sequence

$$0 \to M \to I_0 \to I_1 \to I_2 \to \cdots$$

where I_i is an injective R-module for all $i \geq 0$.

The deleted resolutions are

$$P_* = \cdots \to P_2 \to P_1 \to P_0$$

and

$$I_* = I_0 \to I_1 \to I_2 \to \cdots$$

which are exact apart from at P_0 and I_0.

We will focus to begin with on projective resolutions and the functor between R-modules $M \mapsto \mathrm{Hom}_R(M, N)$. This has what is called a "derived functor" called Ext. We will obtain R-modules $\mathrm{Ext}^i_R(M, N)$, with $i \geq 0$.

Proposition 14.6. *Let $f\colon M \to N$ be an R-module homomorphism. Then there is a unique chain homotopy class of chain maps $P_*^M \to P_*^N$ induces the given map $f\colon M \to N$.*

This uses what is called the fundamental lemma of homological algebra.

Lemma 14.7 (Fundamental lemma of homological algebra)**.** *Let P_* be a projective R-module chain complex, i.e. P_i is a projective module for all i, and let C_* be an acyclic R-module complex, that is $H_i(C_*) = 0$ for all $i > 0$. Both are assumed to be nonnegative. Let $\varphi\colon H_0(P_*) \to H_0(C_*)$ be a homomorphism. Then:*

(1) there is a chain map $f_i\colon P_i \to C_i$ ($\partial_C f_{i+1} = f_i \partial_P$) such that f_0 induces φ on H_0;

(2) any two such chain maps f and g are chain homotopic $f \sim g$, that is there exists a chain homotopy $h_i\colon P_i \to C_{i+1}$ such that $\partial_C h_i + h_{i-1}\partial_P = f_i - g_i$.

Proof. We give an outline of the proof. Let $M := H_0(P_*)$ and $M' := H_0(C_*)$. First construct, using the idea of the previous lemma, the vertical maps (apart from the far right vertical map, which is given), in the following diagram, using the fact that P_i is projective for all i and that the bottom row is exact, now it has been augmented with M'.

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "\partial_P"] \arrow[d, dashed] & P_1 \arrow[r, "\partial_P"] \arrow[d, dashed] & P_0 \arrow[r] \arrow[d, dashed] & M \arrow[r] \arrow[d, "\varphi"'] & 0 \\
\cdots \arrow[r] & C_2 \arrow[r, "\partial_C"] & C_1 \arrow[r, "\partial_C"] & C_0 \arrow[r] & M' \arrow[r] & 0
\end{tikzcd}$$

This shows (i). Now let f and g be two such chain maps as in (ii). Construct a chain homotopy h, again using the idea of the proof of the lemma above, fitting into the diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "\partial_P"] \arrow[d, "f_2-g_2"] \arrow[dl, dashed, "h_2"'] & P_1 \arrow[r, "\partial_P"] \arrow[d, "f_1-g_1"] \arrow[dl, dashed, "h_1"'] & P_0 \arrow[r] \arrow[d, "f_0-g_0"] \arrow[dl, dashed, "h_0"'] & M \arrow[r] \arrow[d, "\varphi-\varphi=0"] & 0 \\
\cdots \arrow[r] & C_2 \arrow[r, "\partial_C"] & C_1 \arrow[r, "\partial_C"] & C_0 \arrow[r] & M' \arrow[r] & 0
\end{tikzcd}$$

To do this one needs the following computation:

$$\begin{aligned}
&\partial_C((f_n - g_n) - h_{n-1}\partial_P) = (f_{n-1} - g_{n-1})\partial_P - \partial_C h_{n-1}\partial_P \\
=&f_{n-1}\partial_P - g_{n-1}\partial_P - h_{n-2}\partial_P^2 - f_{n-1}\partial_P + g_{n-1}\partial_P = 0.
\end{aligned}$$

We leave the details to the reader. They can be found, for example, in chapter 2 of [DK]. □

Now we define the Ext groups that are needed for the statement of the universal coefficient theorem.

Definition 14.8 (Ext_R^n)**.** Let M and N be R-modules and let $P_* \to M \to 0$ be an R-module projective resolution, with P_* the deleted resolution. Form

$\mathrm{Hom}_R(P_*, N)$. Then

$$\mathrm{Ext}^n_R(M,N) := H_n(\mathrm{Hom}_R(P_*,N)).$$

Equivalently, let $0 \to N \to I_*$ be an injective resolution of N, with I_* the deleted resolution. Then

$$\mathrm{Ext}^n_R(M,N) := H_n(\mathrm{Hom}_R(M,I_*))$$

It turns out that the definitions are equivalent with a projective resolution of the first argument or an injective resolution of the second argument. Here are some straightforward remarks.

(i) $\mathrm{Ext}^0_R(M,N) = \ker(\mathrm{Hom}_R(P_0,N) \to \mathrm{Hom}_R(P_1,N) = \mathrm{Hom}_R(M,N)$.
(ii) If M is projective then $\mathrm{Ext}^i_R(M,N) = 0$ for all $i > 0$.
(iii) If N is injective then $\mathrm{Ext}^i_R(M,N) = 0$ for all $i > 0$.

Now let $0 \to A \to B \to C \to 0$ be a short exact sequence of chain complexes. Each of these modules has a projective resolution, and the short exact sequence lifts to a short exact sequence of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & P_*^A \arrow[r] \arrow[d] & P_*^B \arrow[r] \arrow[d] & P_*^C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

by the fundamental lemma of homological algebra. Apply $\mathrm{Hom}(N,-)$ to the top row to obtain

$$0 \to \mathrm{Hom}_R(N, P_*^C) \to \mathrm{Hom}_R(N,P_*^B) \to \mathrm{Hom}_R(N,P_*^A) \to 0.$$

This is a short exact sequence of cochain complexes, which induces a long exact sequence in cohomology:

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Hom}_R(N,C) & \to & \mathrm{Hom}_R(N,B) & \to & \mathrm{Hom}_R(N,A) & \to & \\
 & \to & \mathrm{Ext}^1_R(N,C) & \to & \mathrm{Ext}^1_R(N,B) & \to & \mathrm{Ext}^1_R(N,A) & \to & \\
 & \to & \mathrm{Ext}^2_R(N,C) & \to & \mathrm{Ext}^2_R(N,B) & \to & \mathrm{Ext}^2_R(N,A) & \to & \cdots
\end{array}$$

A similar argument with injective resolutions gives rise to the long exact sequence

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Hom}_R(C,N) & \to & \mathrm{Hom}_R(B,N) & \to & \mathrm{Hom}_R(A,N) & \to & \\
 & \to & \mathrm{Ext}^1_R(C,N) & \to & \mathrm{Ext}^1_R(B,N) & \to & \mathrm{Ext}^1_R(A,N) & \to & \\
 & \to & \mathrm{Ext}^2_R(C,N) & \to & \mathrm{Ext}^2_R(B,N) & \to & \mathrm{Ext}^2_R(A,N) & \to & \cdots
\end{array}$$

Here are some examples of the Ext groups. The Ext^0 groups are equal to the corresponding Hom groups, so we omit the discussion of them.

(1) $\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/p) = 0$ for $n > 0$.
(2) $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}) = \mathbb{Z}/n$, and they Ext^i groups vanish for $i > 1$. In general, $\mathrm{Ext}^1_{\mathbb{Z}}(A,\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$, which picks up the torsion subgroup of A.
(3) If R is a field, $\mathrm{Ext}^i_R(M,N) = 0$ for $i > 0$.
(4) If R is a PID, then $\mathrm{Ext}^i_R(M,N) = 0$ for $i > 1$.
(5) $\mathrm{Ext}^n_R(\oplus_\alpha A_\alpha, B) = \prod_\alpha \mathrm{Ext}^n_R(A_\alpha, B)$.
(6) $\mathrm{Ext}^n_R(A, \oplus_\alpha B_\alpha) = \prod_\alpha \mathrm{Ext}^n_R(A, B_\alpha)$.

Now we state and prove the universal coefficient theorem for cohomology, in the case that R is a PID. In more generality for rings of homological dimension greater than one, there is a universal coefficient spectral sequence. But we won't cover that here.

Theorem 14.9 (The universal coefficient theorem)**.** *Let R be a PID, let M be an R-module and let (C_*, ∂) be a f.g. free R-module chain complex. Then*

$$0 \to \mathrm{Ext}^1_R(H_{r-1}(C_*), M) \xrightarrow{\alpha} H^r(C_*; M) \xrightarrow{\beta} \mathrm{Hom}_R(H_r(C_*), M) \to 0$$

is an exact sequence of abelian groups, which is natural in chain maps of $C_* \to C'_*$, and which splits, but the splitting is not natural. The map β sends $[f] \mapsto ([c] \mapsto f(c))$. If M is an (R,S)-bimodule, then this is an exact sequence of S-modules.

Proof. This proof is essentially from [Br]. Note that the map β is well-defined since f is a cocycle and c is a cycle.

Recall that for R a PID, any submodule of a free module is free. Also recall that for any chain complex C_* we let

$$Z_p := \ker(\partial_p \colon C_p \to C_{p-1})$$

the p-cycles, and

$$B_p = \mathrm{im}(\partial_{p+1} \colon C_{p+1} \to C_p),$$

the p-boundaries. Of course $H_p(C_*) = Z_p/B_p$. There are two exact sequences of R-modules, for each p.

(1) $0 \to Z_p \xrightarrow{\chi} C_p \xrightarrow{\theta} B_{p-1} \to 0$. The submodule B_{p-1} is free, whence projective, so the sequence splits. Let $\phi\colon C_p \to Z_p$ be a splitting.
(2) $0 \to B_p \xrightarrow{\gamma} Z_p \xrightarrow{q} H_p(C_*) \to 0$.

The proof will follow from the next diagram, and some fun diagram chasing.

$$\begin{tikzcd}
 & & & 0 & \\
0 \arrow[r] & \mathrm{Hom}(B_p, M) \arrow[r, "\theta^*"] & \mathrm{Hom}(C_{p+1}, M) & \mathrm{Ext}^1(H_{p-1}(C_*), M) \arrow[u] & \\
0 & \mathrm{Hom}(Z_p, M) \arrow[l] \arrow[u, "\gamma^*"] \arrow[r, leftrightarrow, "\chi^*", "\phi^*"'] & \mathrm{Hom}(C_p, M) \arrow[u, "\delta"] & \mathrm{Hom}(B_{p-1}, M) \arrow[l, "\theta^*"'] \arrow[u] & 0 \arrow[l] \\
 & \mathrm{Hom}(H_p(C_*), M) \arrow[u, "q^*"] & \mathrm{Hom}(C_{p-1}, M) \arrow[u, "\delta"'] \arrow[r, "\chi^*"] & \mathrm{Hom}(Z_{p-1}, M) \arrow[u, "\gamma^*"] \arrow[r] & 0 \\
 & 0 \arrow[u] & & &
\end{tikzcd}$$

Here is some explanation of the diagram. The left 3 terms of the top row come from the dual of (1), and $\mathrm{Hom}(-, M)$ is left exact so the part shown is exact. The middle row is also the dual of (1). Here Z_p is a submodule of a free module and hence is free, since R is a PID. Thus $\mathrm{Ext}^1_R(Z_p, M) = 0$, so the middle row is exact.

The left column is the dual of (2), and is left exact. The right three terms of the bottom row also come from the dual of (1), which is exact as described above. The middle column is the dual of the chain complex C_*. This is not exact, but $\delta^2 = 0$. The right hand column is part of the long exact sequence associated to the dual of (2). The two squares commute.

Now the proof is a diagram chase. Let $f \in \mathrm{Hom}(C_p, M)$, with $f \in \ker \delta$. Go left and up to get an element of $\mathrm{Hom}(B_p, M)$. By commutativity of the top left square, and injectivity of θ^*, this is the zero element. Let $g = \chi^*(f) \in \mathrm{Hom}(Z_p, M)$. Then there is an $h \in \mathrm{Hom}(H_p, M)$ with $g = q^*h$. We define $\beta(f) := h$. To see that this is well defined note that if we replace f with $f + \delta k$, then by commutativity of the bottom right square $\delta k \in \mathrm{im}(\theta^*)$, so maps to zero in $\mathrm{Hom}(Z_p, M)$, and therefore does not change the element of $\mathrm{Hom}(H_p, M)$ by injectivity of q^*.

The composition $\phi^* \circ q^*$ induces the splitting. This also shows surjectivity.

The map $\mathrm{Ext}^1(H_{p-1}, M) \to \mathrm{Hom}(C_p, M)$ is defined by lifting $x \in \mathrm{Ext}^1(H_{p-1}, M)$ to an element $y \in \mathrm{Hom}(B_{p-1}, M)$, then taking $\theta^*(y)$. More diagram chases show that this is well defined and injective.

It remains to show exactness at $H^p(C_*)$. This is also a straightforward diagram chase that is left to the reader. □

14.2. **Universal coefficient theorem for homology.** There is an analogous derived functor Tor for the tensor product. We give a less detailed treatment, but give the main statements here. Given R-modules M and N let P_*^M and P_*^N be projective resolutions. Then

$$\mathrm{Tor}_n^R(M, N) := H_n(P_*^M \otimes_R N)$$

or

$$\mathrm{Tor}_n^R(M, N) := H_n(M \otimes_R P_*^N).$$

Note that $\mathrm{Tor}_0^R(M, N) = M \otimes_R N$. Given an exact sequence $0 \to A \to B \to C \to 0$, we have a long exact sequence

$$\begin{array}{ccccccccc} \cdots & \to & \mathrm{Tor}_2^R(A, N) & \to & \mathrm{Tor}_2^R(B, N) & \to & \mathrm{Tor}_2^R(C, N) & \to & \\ & \to & \mathrm{Tor}_1^R(A, N) & \to & \mathrm{Tor}_1^R(B, N) & \to & \mathrm{Tor}_1^R(C, N) & \to & \\ & \to & A \otimes_R N & \to & B \otimes_R N & \to & C \otimes_R N & \to & 0 \end{array}$$

Theorem 14.10 (Universal coefficient theorem for homology)**.** *Let R be a PID, let C_* be a f.g. free R-module chain complex, and let M be an R-module. Then there is a split natural short exact sequence of abelian groups*

$$0 \to H_r(C_*) \otimes_R M \to H_r(C_* \otimes_R M) \to \mathrm{Tor}_1^R(H_{r-1}(C_*), M) \to 0.$$

The splitting is not natural. If M is an (R, S)-bimodule, this is a split exact sequence of S-modules.

Remark 14.11. Both of the universal coefficient theorems are special cases of corresponding universal coefficient spectral sequences. In fact, there are more general Künneth spectral sequences that imply the universal coefficient spectral sequences and imply the ordinary Künneth theorem.

Here is an application. Let W be a simply connected closed 4-manifold. Then $H_3(M;\mathbb{Z}) = 0 = H_1(M;\mathbb{Z})$ and $H_2(M;\mathbb{Z})$ is torsion-free. (We use Poincaré duality here too. To see this we have $H_3(M) \cong H^1(M) \cong \mathrm{Hom}(H_1(M),\mathbb{Z}) = 0$. The first isomorphism is by Poincaré duality and the second is from the universal coefficient theorem, since $\mathrm{Ext}^1(H_0(M),\mathbb{Z}) = 0$ as $H_0(M)$ is torsion-free. Then $H_1(M) = 0$ since $\pi_1(M) = 0$. Next, $H_3(M) = 0$ implies that $\mathrm{Ext}^1_{\mathbb{Z}}(H_2(M),\mathbb{Z}) \cong H^3(M)$, but $H^3(M) \cong H_1(M) = 0$, so $\mathrm{Ext}^1(H_2(M),\mathbb{Z}) = 0$, and $H_2(M)$ is torsion free as claimed.

14.3. **The Künneth theorem.** An R module M is flat if $M \otimes_R -$ is an exact functor. A chain complex is flat if every chain group is flat.

Let (C, ∂_C) and (D, ∂_D) be two chain complexes. The tensor product chain complex $C \otimes_D$ has

$$(C \otimes D)_n := \bigoplus_{p+q=n} C_p \otimes D_q$$

with

$$\partial_\otimes(c_p \otimes d_q) = \partial_C(c) \otimes d + (-1)^p c \otimes \partial(d_q).$$

Note that for X and Y CW complexes, we have $C_*(X \times Y) \cong C_*(X) \otimes C_*(Y)$. To see this, note that the product $e^p \times e^q$ of two cells is homeomorphic to a $(p+q)$-cell e^{p+q}, and the boundary of a product is $\partial(e^p \times e^q) = \partial e^p \times e^q \cup e^p \times \partial e^q$.

Theorem 14.12. *Let R be a PID, let C_* be a flat chain complex. Let D_* be any chain complex. Then there is a short exact sequence*

$$0 \to \bigoplus_{p+q=n} H_p(C) \otimes_R H_q(D) \to H_n(C \otimes_R D) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(C), H_q(D)) \to 0.$$

Take D_* to be a chain complex that is nonzero only in degree 0, with $D_0 = M$, to obtain the universal coefficient theorem for homology. We can use the Künneth theorem to compute the homology of the product of two spaces.

15. Cup products

15.1. **Algebraic definition.** We have a diagonal map $\Delta\colon X \to X \times X$ that send $x \mapsto (x, x)$.

Given two chain complexes C, D, the tensor product chain complex $C \otimes D$ has chain groups

$$(C \otimes D)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

with boundary map

$$\partial_{C\otimes D}(c \otimes d) = \partial_C(c) \otimes d + (-1)^{\deg c} c \otimes \partial_D(d).$$

Note that $C_*(X \times Y) = C(X) \otimes C(Y)$. To see this observe that the product $e^p \times e^q = e^{p+q}$ of two cells is a cell, and the boundary is

$$\partial e^p \times e^q \cup e^p \times (-1)^p \partial e^q.$$

Let π and π' be abelian groups. We have a map

$$\begin{array}{rcl} w\colon \operatorname{Hom}(C_*(X),\pi)\otimes\operatorname{Hom}(C_*(Y),\pi') & \to & \operatorname{Hom}(C_*(X)\otimes C^*(Y),\pi\otimes\pi') \\ f\otimes f' & \mapsto & \big(x\otimes x'\mapsto(-1)^{\deg f'\deg x}f(x)\otimes f'(x')\big) \end{array}$$

The composition of w with the identification

$$\operatorname{Hom}(C_*(X)\otimes C^*(Y),\pi\otimes\pi')\cong\operatorname{Hom}(C_*(X\times Y),\pi\otimes\pi')$$

gives rise to a map on cohomology

$$H^*(X;\pi)\times H^*(Y;\pi')\to H^*(X\times Y;\pi\otimes\pi').$$

Now we let $X=Y$ and $\pi=\pi'=R$ a commutative ring. Then we obtain the cup product maps

$$\cup\colon H^p(X;R)\times H^q(X;R)\to H^{p+q}(X\times X;R)\xrightarrow{\Delta^*}H^{p+q}(X;R)$$

by using the pullback of the diagonal map on cohomology. The cup product maps make the cohomology $H^*(X;R)$ into a graded *ring*, as well as a graded R-module. The cohomology ring is unital, associative, and graded commutative, in the sense that $x\cup y=(-1)^{\deg x\deg y}y\cup x$.

These properties can be seen by passing to cohomology from the following diagrams. First, the proof of unital uses this diagram:

$$\begin{tikzcd} & X \arrow[dl] \arrow[d] \arrow[dr] & \\ X\times * & X\times X \arrow[l] \arrow[r] & *\times X. \end{tikzcd}$$

The proof of associativity uses this diagram:

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d, "\Delta"] & X\times X \arrow[d, "\Delta\times\mathrm{Id}"] \\ X\times X \arrow[r, "\mathrm{Id}\times\Delta"] & X\times X\times X. \end{tikzcd}$$

Finally graded commutativity follows from this diagram:

$$\begin{tikzcd} & X \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ X\times X \arrow[rr, "t"] & & X\times X, \end{tikzcd}$$

where $t\colon X\times X\to X\times X$ switched the coordinates, that is $t(x,x')=(x',x)$. The signs difference in commutativity arises from the signs in the definition of w.

Here are some examples of cohomology rings.

(a) The cohomology ring of the torus is

$$H^*(S^1\times S^1;\mathbb{Z})\cong\mathbb{Z}[x,y]/(x^2,y^2,xy=-yx),$$

where $\deg x=\deg y=1$.

(b) The cohomology ring of $S^2 \times S^2$ is

$$H^*(S^2 \times S^2; \mathbb{Z}) \cong \mathbb{Z}[x, y]/(x^2, y^2, xy = yx),$$

where $\deg x = \deg y = 2$.

(c) The cohomology ring of $\mathbb{CP}^n = \mathbb{Z}[x]/(x^{n+1})$, with $\deg x = 2$. We will provide a spectral sequences computation of the cohomology ring of $\mathbb{CP}^n$ in the final chapter of these notes.

(d) The cohomology ring of $\mathbb{RP}^n = \mathbb{Z}/2[x]/(x^{n+1})$, with $\deg x = 1$.

In singular cohomology, let x be a sum of singular n-simplices x_i. Let p, q be integers such that $p + q = n$. A singular n-simplex $x_i \colon \Delta^n \to X$ has a front p-face

$${}_p\lfloor x \colon \Delta^p \to \Delta^{p+q} \xrightarrow{x} X,$$

where the inclusion map $\Delta^p \to \Delta^{p+q}$ maps to the first p coordinates. There is also a back q-face

$$x\rfloor_q \colon \Delta^q \to \Delta^{p+q} \xrightarrow{x} X,$$

where now the inclusion map goes to the last q coordinates. We then define

$$(f \cup g)(x) = f\big({}_p\lfloor x\big)g\big(x\rfloor_q\big).$$

The front p-face and back q-face are part of the Alexander-Whitney diagonal approximation chain map. The problem is that the diagonal map is not cellular, so one has to make a choice of cellular map that approximates it up to homotopy. Sending a simplex

$$x \mapsto \sum_{p+q=n} {}_p\lfloor x \otimes x\rfloor_q.$$

15.2. **Axiomatic treatment of cohomology and spectra.** Cohomology theories also have axiomatic treatment. That is we have functors $(X, A) \to E^q(X, A)$, and coboundary maps $\delta \colon E^q(X, A) \to E^{q+1}(X, A)$. There are reduced/unreduced versions, and generalised/ordinary depending on whether one includes the dimension axiom. The unreduced theory has axioms: long exact sequence, additivity, weak equivalence and excision, analogous to the homology versions. A reduced cohomology theory has exactness, suspension, additivity and weak equivalence axioms. An ordinary cohomology theory is determined by a theory on CW complexes and this is determined by the axioms (if dimension is included.)

We saw in the exercise sheet that

$$\widetilde{H}^n(X; \pi) = [X, K(\pi, n)]$$

is a reduced ordinary cohomology theory. To obtain unreduced cohomology, we use:

$$H^n(X; \pi) = [X_+, K(\pi, n)].$$

These coincide with singular or cellular cohomology because they both satisfy the dimension axiom, and such theories are determined up to natural isomorphism.

Definition 15.1. An Ω-spectrum is a sequence of based space $\{T_n\}$ with weak equivalences $\widetilde{\sigma} \colon T_n \to \Omega T_{n+1}$.

This is stronger than the normal definition of spectra, since the maps are not required to be weak equivalences. Note that whether $\widetilde{\sigma}\colon T_n \to \Omega T_{n+1}$ is a weak equivalence has rarely anything to do with whether the adjoint map $\sigma\colon T_n \to T_{n+1}$ is a weak equivalence (like for suspension spectra).

Theorem 15.2 (Brown representability theorem)**.** *Every generalised cohomology theory is represented by an Ω-spectrum, and every Ω-spectrum $\{T_n\}$ gives rise to a cohomology theory $\widetilde{E}^q(X) = [X, T_q]$.*

The situation for homology is similar but a little more complicated, and we will not go into it.

While homology commuted with colimits, it is not true in general that cohomology commutes with limits. Given a sequence of spaces $X_0 \subseteq X_1 \subseteq X_2 \cdots$ with $\bigcup X_i = X$ there is a surjective map $\widetilde{E}^q(X) \to \lim \widetilde{E}^q(X_i)$. The kernel is measured by the derived $\lim^1$ functor. We will not go into details here.

15.3. **Homotopy theory definition of cap products.** Let X and Y be spaces and let A, B be abelian groups. We have a map

$$H^p(X;A) \otimes H^q(Y,B) = [X, K(A,p)] \otimes [Y, K(B,q)] \to [X \wedge Y, K(A,p) \wedge K(B,q)].$$

If we can find a map

$$\phi_{p,q}\colon K(A,p) \wedge K(B,q) \to K(A \otimes B, p+q),$$

then we can compose with this to get an element of

$$[X \wedge Y, K(A \otimes B, p+q)].$$

This will enable us to define the cup product with $X = Y$ via the diagonal map $X \to X \wedge X$ and:

$$\begin{aligned} [X, K(A,p)] \otimes [X, K(B,q)] &\to [X, K(A \otimes B, p+q)] \\ f \otimes g &\mapsto \phi_{p,q} \circ (f \wedge g) \circ \Delta. \end{aligned}$$

To find the maps $\phi_{p,q}$, note that such a map is an element of $\widetilde{H}^{p+q}(K(A,p) \wedge K(B,q); A \otimes B)$, and consider the sequence of natural isomorphisms

$$\begin{aligned} &\widetilde{H}^{p+q}(K(A,p) \wedge K(B,q); A \otimes B) \\ \cong\,& \mathrm{Hom}(\widetilde{H}_{p+q}(K(A,p) \wedge K(B,q)), A \otimes B) \\ \cong\,& \mathrm{Hom}(\widetilde{H}_p(K(A,p)) \otimes \widetilde{H}_q(K(B,q)), A \otimes B) \\ \cong\,& \mathrm{Hom}(\pi_p(K(A,p)) \otimes \pi_q(K(B,q)), A \otimes B) \\ \cong\,& \mathrm{Hom}(A \otimes B, A \otimes B) \end{aligned}$$

The first three isomorphisms follow from the universal coefficient theorem, the Künneth theorem and the Hurewicz theorem. To see that these theorems apply, observe that $K(A,p)$ is $(p-1)$-connected, $K(B,q)$ is $(q-1)$-connected, and $K(A,p) \wedge K(B,q)$ is $(p+q-1)$-connected. Define the map $\phi_{p,q}\colon K(A,p) \wedge K(B,q) \to K(A \otimes B, p+q)$ to be the map associated to the cohomology class that is the preimage of the identity map $\mathrm{Id}_{A\otimes B} \in \mathrm{Hom}(A \otimes B, A \otimes B)$ under the above sequence of isomorphisms.

The properties of cup products, that they are have a unit, are associative and graded commutative, can also be proven from the homotopy theory definition. The algebraic and more homotopy theoretic definitions coincide. To see this, it suffices to convince oneself, by naturality, that they coincide on the Eilenberg Maclane spaces.

16. Cap products

Another important product in cohomology is called the cap product. Cap product with the fundamental class of a manifold M gives rise to the Poincaré duality maps from cohomology to homology. Recall that

$$[X, Y] = \pi_0(\mathrm{Map}(X, Y)).$$

We have the evaluation map

$$\varepsilon\colon \mathrm{Map}(X, Y) \wedge X \to Y,$$

and we saw long ago that this is a continuous map. Let A, B be coefficient abelian groups. The *cap product* will be a map

$$\cap\colon \widetilde{H}^p(X; A) \otimes \widetilde{H}_n(X; B) \to \widetilde{H}_{n-p}(X; A \otimes B)$$

In the case that $X = M$ is an n-dimensional manifold, let $[M] \in H_n(M; \mathbb{Z})$ be the fundamental class. Then

$$x \cap [M] \in \widetilde{H}_{n-p}(M; A)$$

is the Poincaré dual of x. Now we construct the cap product using homotopy theory. Using our definition of homology and cohomology in terms of homotopy groups, we will obtain a definition that easily generalises to homology and cohomology theories defined using spectra. The disadvantage of this approach is that we need our coefficients to be abelian groups, but cap products can also be defined for twisted coefficients over a $\mathbb{Z}[\pi_1(X)]$-module. We want a map

$$\pi_0(\mathrm{Map}(X, K(A, p))) \otimes \mathrm{colim}_q\, \pi_q(X \wedge K(B, q)) \to \mathrm{colim}_r\, \pi_{n-p+r}(X \wedge K(A \otimes B, r)).$$

Tensor products commute with colimits, so in fact we want a map

$$\mathrm{colim}_q\, \pi_0(\mathrm{Map}(X, K(A, p))) \otimes \pi_q(X \wedge K(B, q)) \to \mathrm{colim}_q\, \pi_{n+q}(X \wedge K(A \otimes B, p+q)),$$

by setting $r = p + q$. So we define maps without the colimits, and then we will simply pass to the colimit, to get the desired product. Here is a glorious sequence of maps that gives the map we want.

$$\begin{aligned}
&\pi_0(\mathrm{Map}(X, K(A, p))) \otimes \pi_q(X \wedge K(B, q))\\
&\xrightarrow{\wedge} \pi_{n+q}(\mathrm{Map}(X, K(A, p)) \wedge X \wedge K(B, q))\\
&\xrightarrow{\mathrm{Id}\wedge\Delta\wedge\mathrm{Id}} \pi_{n+q}(\mathrm{Map}(X, K(A, p)) \wedge X \wedge X \wedge K(B, q))\\
&\xrightarrow{\varepsilon\wedge\mathrm{Id}} \pi_{n+q}(K(A, p) \wedge X \wedge K(B, q))\\
&\xrightarrow{t\wedge\mathrm{Id}} \pi_{n+q}(X \wedge K(A, p) \wedge K(B, q))\\
&\xrightarrow{\mathrm{Id}\wedge\phi} \pi_{n+q}(X \wedge K(A \otimes B, p + q)).
\end{aligned}$$

In singular homology, we can define the cap product as follows. Let $y \in H_n(X;B)$ and $x \in H^p(X;A)$. Apply a diagonal chain approximation map

$$\Delta(y) = \sum_i b^i y_p^i \otimes y_{n-p}^i + \sum_{r \neq p} b^i y_r^i \otimes y_{n-r}^i$$

where $b^j \in B$ and $y = \sum b^j f^j$, with $f^j \colon \Delta^n \to Y$ a singular n-simplex, and the diagonal map on an n simplex $\Delta(f^j) = \sum y_r^i \otimes y_{n-r}^i$. Then

$$x \cap y = \sum_i (x(y_p^i) \otimes b^i) \cdot y_{n-p}^i.$$

Here $x(y_p^i) \in A$. Explicitly, one can use the Alexander-Whitney diagonal approximation with

$$\Delta(f^j) = \sum_{i=0}^n \left({}_i\lfloor f^j\right) \otimes \left(f^j\rfloor_i\right).$$

The key property of cap and cup products is

$$\langle \alpha \cup \beta, x \rangle = \langle \beta, \alpha \cap x \rangle,$$

where $\alpha \in \widetilde{H}^p(X), \beta \in \widetilde{H}^q(X)$ and $x \in \widetilde{H}_{p+q}(X)$. This is often very useful in the study of Poincaré duality.

Here is the cool thing about the construction of cap and cup products that we gave. If $\{T_n\}$ is an Ω-spectrum, with a sequence of maps $\phi_{p,q} \colon T_p \wedge T_q \to T_{p+q}$, then the associated generalised homology/cohomology theories have cap and cup product maps, defined in exactly analogous ways.

17. Cohomology operations

We say that a contravariant functor k from spaces to sets is *represented* if there is a space Z such that $k(X) = [X, Z]$ for all X. Let k be a represented functor and let k' be another contravariant functor from spaces to sets.

Lemma 17.1 (Yoneda lemma)**.** *There is a bijection between the set of natural transformations $\Phi \colon k \to k'$ and $\phi \in k'(Z)$.*

Proof. Given Φ, let $\phi = \Phi(\mathrm{Id}) \in k'(Z)$, where $\mathrm{Id} \in k(Z) = [Z, Z]$. Given $\phi \in k'(Z)$, define $\Phi \colon k(X) = [X, Z] \to k'(X)$ by $f \mapsto f^*(\phi)$. Note that $f^* \colon k'(Z) \to k'(X)$. □

Corollary 17.2. *Suppose that both functors are represented. Then there is a bijection between natural transformations $\Phi \colon [-, Z] \to [-, Z']$ and $\phi \in [Z, Z']$.*

Definition 17.3. A *cohomology operation* of type q and degree n between cohomology theories $\widetilde{E}^*$ and $\widetilde{F}^*$ is a natural transformation $\widetilde{E}^q \to \widetilde{F}^{q+n}$.

Definition 17.4. A *stable cohomology operation* of degree n is a sequence $\{\Phi^q \colon \widetilde{E}^q \to \widetilde{E}^{q+n}\}$ of cohomology operations of type q and degree n, such that for each based space X we have $\Sigma \circ \Phi^q = \Phi^{q+1} \circ \Sigma \colon \widetilde{E}^q(X) \to \widetilde{E}^{q+1+n}(\Sigma X)$.

Unlike the cup product, cohomology operations can survive stabilisation.

Theorem 17.5. *Cohomology operations*

$$\widetilde{H}^q(-,\pi) \to \widetilde{H}^{q+n}(-,\pi')$$

are in canonical bijective correspondence with $\widetilde{H}^{q+n}(K(\pi,q),\pi')$.

Proof. Cohomology operations are natural transformations from $[-,K(\pi,q)]$ to $[-,K(\pi',n+q)]$, which by the Yoneda lemma correspond to $[K(\pi,q),K(\pi',q+n)]$. □

The Steenrod operations

$$\mathrm{Sq}^n\colon H^p(X;\mathbb{Z}/2) \to H^{p+n}(X;\mathbb{Z}/2)$$

are stable cohomology operations such that
(i) $\mathrm{Sq}^0 = \mathrm{Id}$;
(ii) $\mathrm{Sq}^n(x) = x^2 = x \cup x$ if $n = \deg x$;
(iii) $\mathrm{Sq}^n(x) = 0$ if $n > \deg x$; and
(iv) $\mathrm{Sq}^m(xy) = \sum_{i+j=n} \mathrm{Sq}^i(x)\,\mathrm{Sq}^j(x)$. This is called the Cartan formula.

These properties characterise the Steenrod operations Sq^n. They correspond to the cohomology ring

$$H^*(K(\mathbb{Z}/2,p);\mathbb{Z}/2),$$

which is a polynomial algebra generated by certain iterates of the Steedrod operations. These operations are important operations in algebraic topology. For example, they are used to define the Stiefel-Whitney classes of vector bundles.

Here is an example of a computation of Steedrod squares. Let $X = \mathbb{RP}^\infty$ and let $y \in H^2(\mathbb{RP}^\infty;\mathbb{Z}/2)$ be equal to x^2, where $x \in H^1(\mathbb{RP}^\infty;\mathbb{Z}/2)$ is a generator of the degree one cohomology. Then

$$\mathrm{Sq}^1(y) = \mathrm{Sq}^1(x^2) = \mathrm{Sq}^1(x)\,\mathrm{Sq}^0(x) + \mathrm{Sq}^0(x)\,\mathrm{Sq}^1(x) = x\cup x\cup x + x\cup x\cup x = 2x\cup x\cup x = 0.$$

This example shows that not all of the Steenrod operations of $\mathbb{RP}^\infty$ are nontrivial, even though many of them are.

We finish this section be using the Steedrod operations to give a proof that the first stable homotopy group is nontrivial.

Theorem 17.6. *Let* $f\colon S^3 \to S^2$ *represent a generator of* $\pi_3(S^2) \cong \mathbb{Z}$. *Then* $\Sigma^k f\colon S^{3+k} \to S^{2+k}$ *is nontrivial for all* k. *That is,* $\pi_1^S \neq 0$.

Proof. The mapping cone $C_f = D^4 \cup_f S^2$. The mapping cone of Σf $C_{\Sigma f} \cong \Sigma C_f = D^5 \cup_{\Sigma f} S^3$. Iterating this, we obtain that

$$\Sigma^k C_f = D^{4+k} \cup_{\Sigma^k f} S^{2+k}.$$

Note that $0 \neq \mathrm{Sq}^2\colon H^2(C_f;\mathbb{Z}/2) \to H^4(C_f;\mathbb{Z}/2)$, because the cup product of $\mathbb{CP}^2 \simeq S^2 \cup_f D^4$ is nontrivial. Since the Steenrod operations are stable, we have

$$0 \neq \mathrm{Sq}^2\colon H^{2+k}(\Sigma^k C_f;\mathbb{Z}/2) \to H^{4+k}(\Sigma^k C_f;\mathbb{Z}/2).$$

Now if there is a k such that $\Sigma^k f \sim *$, then there exists a retraction map $r\colon \Sigma^k C_f \to S^{k+2}$. This is because attaching D^{4+k} is homotopy equivalent to wedging with a

S^{4+k}, and then the retraction map can identify this S^{4+k} to the basepoint. By naturality we have a commutative square

$$\begin{tikzcd} H^{2+k}(S^{2+k};\mathbb{Z}/2) \arrow[r, "r^*", "\cong"'] \arrow[d] & H^{2+k}(\Sigma^k C_f;\mathbb{Z}/2) \arrow[d, "\mathrm{Sq}^2"] \\ H^{4+k}(S^{4+k};\mathbb{Z}/2) \arrow[r, "r^*"] & H^{4+k}(\Sigma^k C_f;\mathbb{Z}/2) \end{tikzcd}$$

The bottom left group is zero, the others are all $\mathbb{Z}/2$. The right then down composition is nontrivial, but it also factors through zero. This is a contradiction, so we deduce that no retraction r can exist. □

18. Obstruction theory

Let (X, A) be a CW pair. In this section we try to solve the following problems. The exposition follows [DK].

(1) The Extension problem. We are given a map $f\colon A \to Y$, for some space Y, and we want to know whether f can be extended to a map $X \to Y$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & Y \\ X \arrow[ur, dashed] & \end{tikzcd}$$

(2) Given two maps $f_0, f_1\colon X \to Y$ and a homotopy of $f_0|_A$ to $f_1|_A$, can we extend this to a homotopy between f_0 and f_1?

$$\begin{tikzcd} X \times \{0,1\} \cup A \times I \arrow[r] \arrow[d] & Y \\ X \times I \arrow[ur, dashed] & \end{tikzcd}$$

Note that this is not the same as the HEP, since here we specify f_1, whereas in the HEP no f_1 is specified.

(3) A lifting problem. Let $p\colon E \to B$ be a fibration. We have a map $X \to B$, and we want to know whether it can be lifted to a map $X \to E$.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[r, "f"] \arrow[ur, dashed] & B \end{tikzcd}$$

(4) A relative lifting problem. This is the same as the previous case, but the lift is already fixed for us on A.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \arrow[r, "f"] \arrow[ur, dashed] & B \end{tikzcd}$$

(5) The section problem. This is the lifting problem with $X = B$ and $f = \mathrm{Id}$. It asks whether a fibration has a section.

For the relative lifting problem, if $X = A \times I$, then the HLP for $p\colon E \to B$ says that this problem is soluble.

Note that we assume $A \to X$ is a cofibration and $E \to B$ is a fibration. The problems above are soluble for generic maps of CW complexes if and only if they are soluble for $A \to Mf$ and $P_p \to B$ respectively, that is replacing maps by their homotopy cofibre and fibre.

Here is the first main observation of obstruction theory. The strategy is to try to extend maps cell by cell using the CW structure of X. At some point we will run into an obstruction to extending the map, and this will be the primary obstruction to solving the given problem. If the primary obstruction does not vanish, the map cannot be extended, at least not without going back and altering the map as already defined on earlier cells. If the primary obstruction vanishes, there are potentially further obstructions, but we will not study them. In favourable cases, the primary obstruction is the only obstruction, and we are able to determine whether the map extends, or whether the map lifts.

Lemma 18.1. *Let X be an n-dimensional CW complex and let Y be an n-connected space. Then any map $f\colon X \to Y$ is null homotopic.*

Proof. We aim to deform the map f on the k-skeleton to be null homotopic, inductively. Suppose that f has been deformed so that the $(k-1)$-skeleton maps to the basepoint of Y. Then for $k \leq n$, the composition $D^k \to X \xrightarrow{f} Y$, where $D^k \to X$ is the characteristic map of a k-cell, factors through $D^k \to D^k/\partial D^k = S^k \to Y$. Moreover this map is null-homotopic, since Y is n-connected, so the map $S^k \to Y$ extends to a map $H\colon D^{k+1} \to Y$. H can be thought of as a homotopy $H\colon f \sim *$. Define a homotopy on the k-skeleton using this map. We then have to extend this to a homotopy of f. To achieve this, use that $X^{(k)} \to X$ is a cofibration, so the HEP gives

$$\begin{tikzcd}
X^{(k)} \arrow[rr, "i"] \arrow[dd, "i_0"'] & & X \arrow[dl, "f"'] \arrow[dd, "i_0"] \\
& Y & \\
X^{(k)} \times I \arrow[ur, "H"] \arrow[rr] & & X \times I \arrow[ul, dotted, "\widetilde{H}"']
\end{tikzcd}$$

The map $\widetilde{H}$ extends the homotopy to a homotopy of X, so f is homotopic to a map that sends the k skeleton $X^{(k)}$ to the basepoint $*$. This completes the proof of the inductive step. $\square$

Now we define an obstruction to extending a map. Suppose that the problem

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & Y \\
X \arrow[ur, dashed] &
\end{tikzcd}$$

has been solved over the n-skeleton $X^{(n)}$ of (X, A). So there is a map $X^{(n)} \to Y$ that agrees with the given map $A \to Y$ on A. For each n-cell e^{n+1}, consider the composition

$$S^n \to \partial e^{n+1} \to X^{(n)} \xrightarrow{f^{(n)}} Y \in \pi_n(Y).$$

This determines a cochain

$$\theta(f^{(n)}) \in C^{n+1}(X; \pi_n(Y)).$$

We will show that $\partial^*(\theta(f^n)) = 0$, so that $[\theta(f^n)] \in H^{n+1}(X; \pi_n(Y)) = [X, K(\pi_n(Y), n+1)]$. We will show that if $[\theta(f^n)] = 0$ then one can modify f on $X^{(n)}$ and then extend it to a map $\widetilde{f}\colon X^{(n+1)} \to Y$.

The homotopy problem is a special case of the extension problem. Let

$$(X', A') := (X \times I, X \times \partial I \cup A \times I).$$

This leads to the obstruction

$$H^{n+1}(X \times I, X \times \{0,1\}; \pi_n(Y)) \cong H^n(X; \pi_n(Y)) = [X, K(\pi_n(Y), n)].$$

To see the isomorphism we use the long exact sequence of a pair

$$H^n(X \times I) \xrightarrow{\Delta} H^n(X \times \{0,1\}) \to H^{n+1}(X \times I, X \times \{0,1\}) \to H^n(X \times I) \xrightarrow{\Delta} H^{n+1}(X \times \{0,1\})$$

In general $\operatorname{coker}(B \xrightarrow{(f,f)} A \oplus A) \cong A \oplus A/f(B)$. In this case f is the identity map $\mathrm{Id}\colon H^n(X) \to H^n(X)$. The diagonal map is also injective, so that the map $H^{n+1}(X \times I, X \times \{0,1\}) \to H^n(X \times I)$ is the zero map. It follows that $H^n(X \times \{0,1\}) \to H^{n+1}(X \times I, X \times \{0,1\})$ is an isomorphism.

If $Y = K(\pi, n)$ then these are precisely the obstructions. In general these are the primary obstructions, but the complete obstruction theory is much less clean, when the target is not an Eilenberg-Maclane space. Maps $f, g \in [X, Y] = [X, K(\pi, n)]$ are homotopic if and only if they have equivalent classes in $H^n(X; \pi_n(K(\pi, n))) = H^n(X; \pi)$. This explains how it was arrived at to define cohomology using spectra.

Recall that if $\{T_n\}$ is a spectrum, such that $\Sigma T_n \to T_{n+1}$ is an Ω-spectrum i.e. the adjoint $T_n \to \Omega T_{n+1}$ is a weak homotopy equivalence.

Theorem 18.2. *Let T_n be an Ω-spectrum. Then*

$$\widetilde{E}^n(X) \cong [X, T_n]$$

is a generalised homology theory.

Suppose that Y is n-simple, that is $[S^n, Y] = \pi_n(Y)$, since $\pi_1(Y)$ acts trivially on $\pi_n(Y)$. Let (X, A) be a relative CW complex and let $n \geq 1$. Let $g\colon X_n \to Y$ be a map. let

$$\phi_i\colon (D^{n+1}, S^n) \to (X^{(n+1)}, X^n)$$

be the characteristic map of the ith $(n+1)$-cell e_i^{n+1}. Then the composition

$$g \circ \phi_i|_S^n\colon S^n \to Y$$

gives an element of $\pi_n(Y)$.

Definition 18.3. Define the obstruction cochain $\theta^{n+1}(g) \in C^{n+1}(X, A; \pi_n(Y))$ by $\theta^{n+1}(g)(e_i^{n+1}) = [g \circ \phi_i|_{S^n}]$, and extend linearly.

Here is the main theorem of obstruction theory.

Theorem 18.4.

(1) The obstruction cocycle $\theta^{n+1}(g) = 0$ if and only if g extends to a map $X^{(n+1)} \to Y$.
(2) The obstruction cohomology class $[\theta(g)] = 0 \in H^{n+1}(X, A; \pi_n(Y))$ if and only if the restriction $g|_{X^{(n-1)}} \colon X^{(n-1)} \to Y$ extends to a map $X^{(n+1)} \to Y$.

The obstruction cohomology class vanishing says that we can alter the map on the n-skeleton in such a way that it extends over the $(n+1)$-skeleton. Before we start the proof, we give another slightly more formal definition of $\theta(g)$.

The nth cellular chain group is

$$C_n(X, A) = H_n(X^{(n)}, X^{(n-1)})$$

with boundary map

$$\partial \colon H_n(X^{(n)}, X^{(n-1)}) \xrightarrow{\partial} H_{n-1}(X^{(n-1)}) \to H_{n-1}(X^{(n-1)}, X^{(n-2)}).$$

The Hurewicz map gives a surjective map

$$\pi_{n+1}(X^{(n+1)}, X^{(n)}) \to H_{n+1}(X^{(n+1)}, X^{(n)})$$

The kernel is

$$K = \{x(\alpha(x))^{-1} \mid x \in \pi_{n+1}(X^{(n+1)}, X^{(n)}), \alpha \in \pi_1(X^{(n)})\}$$

Define

$$\pi^+_{n+1}(X^{(n+1)}, X^{(n)}) := \pi_{n+1}(X^{(n+1)}, X^{(n)}) \cong H_n(X^{(n+1)}, X^{(n)}).$$

There is a factorisation since Y is n-simple.

$$\begin{tikzcd}
\pi_{n+1}(X^{(n+1)}, X^{(n)}) \arrow[r, "\partial"] \arrow[dr] & \pi_n(X^{(n)}) \arrow[r, "g_*"] & \pi_n(Y) \\
 & \pi^+_{n+1}(X^{(n+1)}, X^{(n)}) \arrow[ur, "\overline{g \circ \partial}"'] &
\end{tikzcd}$$

We then have a map

$$C_{n+1}(X, A) = H_{n+1}(X^{(n+1)}, X^{(n)}) \xrightarrow{\simeq} \pi^+_{n+1}(X^{(n+1)}, X^{(n)}) \xrightarrow{\overline{g \circ \partial}} \pi_n(Y).$$

This defines $\theta^{n+1}(g)$ algebraically.

Proposition 18.5. *$\theta^{n+1}(g)$ is a cocycle.*

Proof. Write h for the Hurewicz homomorphism. We have a commutative diagram

$$\begin{tikzcd}
\pi_{n+2}(X^{(n+2)}, X^{(n+1)}) \arrow[r, two heads, "h"] \arrow[d, "\partial"'] & H_{n+1}(X^{(n+2)}, X^{(n+1)}) \arrow[d, "\partial"] \\
\pi_{n+1}(X^{(n+1)}) \arrow[r, "h"] \arrow[d] & H_{n+1}(X^{(n+1)}) \arrow[d] \\
\pi_{n+1}(X^{(n+1)}, X^{(n)}) \arrow[r, "h"] \arrow[d, "\partial"'] & H_{n+1}(X^{(n+1)}, X^{(n)}) \arrow[d, "\theta^{n+1}(g)"] \\
\pi_n(X^{(n)}) \arrow[r, "g_*"] & \pi_n(Y)
\end{tikzcd}$$

The last two maps on of the left hand column are from the long exact sequence of a pair, and so their composition vanishes. The entire composition of the right hand column is the map $\partial^*\theta^{n+1}(g)$. Also the top Hurewicz map is surjective. It follows that the composition of the right hand column vanishes, and therefore $\partial^*\theta^{n+1}(g) = 0$, so $\theta^{n+1}(g)$ is a cocycle. □

The first part of the main theorem, that if the obstruction cocycle vanishes, then the map extends over $X^{(n+1)}$, follows from the argument of Lemma 18.1. Since the boundaries of the $(n+1)$-cells map trivially into $\pi_n(Y)$, there are extensions of these boundaries to maps of discs to Y. We need to show that if the cohomology class vanishes, then we can change the map on X^n, fixing $X^{(n+1)}$, so that the outcome extends over $X^{(n+1)}$, i.e. so that the outcome is has vanishing obstruction cocycle.

Lemma 18.6. *Let $f_0, f_1\colon X^{(n)} \to Y$ be two maps such that $f_0|_{X^{(n-1)}} \sim f_1|_{X^{(n-1)}}$. Then a homotopy determines a difference cochain*

$$d \in C^n(X, A; \pi_n(Y))$$

satisfying $\partial^*(d) = \theta^{n+1}(f_0) - \theta^{n+1}(f_1)$.

Proof. Let $\widehat{X} := X \times I$ and let $\widehat{A} := A \times I$. Then $(\widehat{X}, \widehat{A})$ is a relative CW complex, with

$$\widehat{X}^{(k)} = X^{(k)} \times \partial I \cup X^{(k-1)} \times I.$$

A map $\widehat{X}^{(n)} \to Y$ is a pair of maps $f_0, f_1\colon X^{(n)} \to Y$, together with a homotopy $G\colon X^{(n-1)} \to Y$ between $f_0|_{X^{(n-1)}}$ and $f_1|_{X^{(n-1)}}$, the restrictions to $X^{(n-1)}$. This gives rise to an obstruction cocycle

$$\theta(f_0, G, f_1) \in C^{n+1}(\widehat{X}, \widehat{A}; \pi_n(Y))$$

that obstructs extending $f_0 \cup G \cup f_1$ to $\widehat{X}^{(n+1)}$. Take the restriction of this cocycle to cells of the form $e^n \times I$, to define the *difference* cochain

$$d(f_0, G, f_1) \in C^n(X, A; \pi_n(Y))$$

That is,

$$d(f_0, G, f_1)(e^n_i) = (-1)^{n+1}\theta(f_0, G, f_1)(e^n_i \times I).$$

We have

$$
\begin{aligned}
0 &= \partial^*\theta(f_0, G, f_1)(e_i^{n+1} \times I) \\
&= \theta(f_0, G, f_1)(\partial(e_i^{n+1} \times I)) \\
&= \theta(f_0, G, f_1)(\partial(e_i^{n+1}) \times I + (-1)^{n+1}(\theta(f_0, G, f_1)(e_i^{n+1} \times \{1\}) - \theta(f_0, G, f_1)(\varepsilon_i^{n+1} \times \{0\}))) \\
&= (-1)^{n+1}\big(\partial^* d(f_0, G, f_1)(e_i^{n+1}) + \theta^{n+1}(f_1)(e_i^{n+1}) - \theta^{n+1}(f_0)(\varepsilon_i^{n+1})\big)
\end{aligned}
$$

Therefore

$$\partial^* d(f_0, G, f_1)(e_i^{n+1}) = \theta^{n+1}(f_0)(e_i^{n+1}) - \theta^{n+1}(f_1)(\varepsilon_i^{n+1})$$

□

Corollary 18.7. *If f_0 is homotopic to f_1, and f_1 extends to $X^{(n+1)}$, then $\theta^{n+1}(f_1) = 0$, and so $\theta^{n+1}(f_0)$ is null homologous.*

We want a converse to this corollary. That is, if $\theta^{n+1}(f_0)$ is null homologous, then there exists an extension to the $n+1$-skeleton up to homotopy on the n-skeleton.

Proposition 18.8. *Let $f_0 \colon X \to Y$ be a map and let $G \colon X^{(n-1)} \times I \to Y$ be a homotopy with $G_0 = f_0|_{X^{(n-1)}}$. Let $d \in C^n(X, A; \pi_n(Y))$ be a cochain. Then there exists a map $f \colon X^{(n)} \to Y$ such that $G_1 = f_1|_{X^{(n-1)}}$ and $d = d(f_0, G, f_1)$.*

We start the proof with a lemma.

Lemma 18.9. *For any map $f \colon D^n \times \{0\} \cup S^{n-1} \times I \to Y$ and for any $\alpha \in [\partial(D^n \times I), Y]$, there exists a map $F \colon \partial(D^n \times I) \to Y$ such that F represents the homotopy class α and restricts to f.*

Proof. Let $D := D^n \times \{0\} \cup S^{n-1} \times I$. Let $K \colon \partial(D^n \times I)$ be any map representing α. We are given a map $f \colon D \to Y$. Since D is contractible, f and $K|_D$ are homotopic maps. Let $h \colon D \times I \to Y$ be such a homotopy. Apply the HEP to the following diagram:

$$
\begin{tikzcd}
D \arrow[rr, "i"] \arrow[dd, "i_0"'] & & \partial(D^n \times I) \arrow[dl, "K"'] \arrow[dd, "i_0"] \\
& Y & \\
D \times I \arrow[ur, "h"] \arrow[rr, "i\times\mathrm{Id}"'] & & \partial(D^n \times I) \times I. \arrow[ul, dotted, "\widetilde{h}"']
\end{tikzcd}
$$

We obtain a map $\widetilde{h} \colon \partial(D^n \times I) \times I \to Y$, such that $F := \widetilde{h}_1$ restricts to f and is homotopic to K, i.e. represents α. □

Proof of Proposition 18.8. Recall that we are given $f_0 \colon X^{(n)} \to Y$ and $G \colon X^{(n-1)} \times I \to Y$ such that $G_0 = f_0|_{X^{n-1}}$. We are also given a chain $d \in C^n(X, A; \pi_n(Y)$. Our task is to show that there exists a map $f_1 \colon X^{(n)} \to Y$ such that $G_1 = f_1|_{X^{(n-1)}}$ and such that $d = d(f_0, G, f_1)$.

Let e_i^n be an n-cell of X, and let

$$\varphi_i \colon (D^n, S^{n-1}) \to (X^{(n)}, X^{(n-1)})$$

be its characteristic map. Let

$$f = f_0 \circ \varphi_i \cup G \circ (\varphi_i|_{S^{n-1}} \times \mathrm{Id}_I)$$

where

$$f_0 \circ \varphi_i \colon D^n \times \{0\} \to X \to Y$$

and

$$G \circ (\varphi_i|_{S^{n-1}} \times \mathrm{Id}_I) \colon S^{n-1} \times I \to X^{(n-1)} \times I \to Y$$

Let $\alpha = d(e_i^n) \in \pi_n(Y)$. Note that α is represented by a map $S^n = \partial(D^n \times I) \to Y$. Apply the lemma to get a map

$$F_i \colon \partial(D^n \times I) \to Y$$

representing α whose restriction to $D^n \times \{0\} \cup S^{n-1} \times I$ is equal to f. Define

$$f_1 \colon X^{(n)} \to Y$$

by

$$f_1(\varphi_i(x)) = F_i(x, 1)$$

for $x \in e_i^n \cong D^n$. Now

$$d(f_0, G, f_1)(e_i^n) = d(e_i^n)$$

and $G_1 = f_1|_{X^{(n-1)}}$ by construction. □

Proof of Theorem 18.4. Let $g \colon X^{(n)} \to Y$ and suppose that $\theta(g) = \partial^*(d)$. Define

$$\begin{array}{rcl} G \colon X^{(n-1)} \times I & \to & Y \\ (x, t) & \mapsto & Y \end{array}$$

for all $t \in I$. By the preceding proposition, there exists a map $g' \colon X^{(n)} \to Y$ such that $d = d(g, G, g')$ and

$$G(-, 1) = g'|_{X^{(n-1)}} = g|_{X^{(n-1)}}.$$

Then $\theta(g) - \theta(g') = \partial^*(d)$. Therefore $\theta(g') = 0$ so g' extends to $X^{(n+1)}$. This completes the proof of the main theorem of obstruction theory. □

Next we examine some of the consequences. This obstruction is very useful when there is exactly one potentially non-vanishing obstruction. But if there are more, then the obstruction theory quickly becomes more complicated.

Let us consider the case of extending homotopies.

$$\begin{tikzcd} X \times \{0,1\} \cup A \times I \arrow[r, "f"] \arrow[d] & Y \\ X \times I \arrow[ur, dashed] & \end{tikzcd}$$

Recall that we define the pair

$$(X^*, A^*) := (X, A) \times (I, \partial I) = (X \times I, X \times \partial I \cup A \times I).$$

A map $F \colon (X^*)^{(n)} \to Y$ is two maps $f_0, f_1 \colon X^{(n)} \to Y$ and a homotopy $f_0|_{X^{(n-1)}} \sim f_1|_{X^{(n-1)}}$ Recall that we have an obstruction

$$d(f_0, f_1) := \theta^{n+1}(F) \in H^{n+1}(X^*, A^*; \pi_n(Y)) \cong H^n(X, A; \pi_n(Y)).$$

Theorem 18.10. *Let (X, A) be a relative CW complex, and let Y be n-simple. Let $f_0, f_1 \colon X \to Y$ be maps with $f_0|_A = f_1|_A$, and let $F \colon X^{(n-1)} \times I \to Y$ be a homotopy relative to A between $f_0|_{X^{(n-1)}}$ and $f_1|_{X^{(n-1)}}$. Then $\theta^{n+1}(F) = 0 \in H^n(X, A; \pi_n(Y))$ if and only if $F|_{X^{(n-2)}}$ extends to a homotopy $f_0|_{X^{(n)}} \sim f_1|_{X^{(n)}}$.*

This theorem follows directly from the previous theorem, using the translation with (X^*, A^*) given above.

Now we return to discussing the extension problem.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & Y \\ X \arrow[ur, dashed] & \end{tikzcd}$$

Note that if $H^{(n+1)}(X, A; \pi_n(Y)) = 0$ for all n, then we can always extend maps. If Y is $(n-1)$-connected and $f \colon A \to Y$ is a map, then the *primary obstruction* to extending f to $X^{(n+1)}$ is the obstruction $\gamma^{(n+1)}(f) = \theta^{(n+1)}(f) \in H^{n+1}(X, A; \pi_n(Y))$.

If $H^{(n+1)}(X, A; \pi_n(Y)) \neq 0$ for exactly one n, then there is a single obstruction, and obstruction theory works rather well.

Suppose that $Y = K(\pi, n)$ is an Eilenberg-MacLane space. Then there is an obstruction $\gamma^{(n+1)}(f) \in H^{n+1}(X, A; \pi_n(Y)) = H^{n+1}(X, A; \pi)$ to extending f to the $(n+1)$-skeleton of X. Suppose that $\gamma^{(n+1)}(f) = 0$. Then we can change f relative to $X^{(n-1)}$ so that it extends to a map $g \colon X^{(n+1)} \to Y$. Then g extends to a map $X \to Y$ since $\pi_k(Y) = 0$ for all $k > n$.

We can consider the indexing of possible choices of extension up to homotopy. Let $g, g' \colon X \to Y$ be maps. They are homotopic if and only if

$$d(g, g') = 0 \in H^n(X, A; \pi_n(Y)) = H^n(X, A; \pi).$$

In fact, homotopy classes are in one to one correspondence with $H^n(X, A; \pi)$, therefore we see the identification of cohomology with $[X, K(\pi, n)]$.

Example 18.11. Let $X = S^3 \smallsetminus \nu K$ be the exterior of a knot $K \subset S^3$, where νK is a regular neighbourhood of K. The reader should compute the homology groups of X as an exercise. Let

$$f \colon \partial(\nu K) = \partial X = S^1 \times S^1 \to S^1$$

be given by the projection $(x, y) \mapsto x$. We have a primary obstruction to extending the map over all of X in

$$H^2(X, \partial X; \pi_1(S^1)) = H^2(X, \partial X; \mathbb{Z}) \cong H_1(X; \mathbb{Z})$$

where the last isomorphism is by Poincaré-Lefschetz duality. This obstruction depends on the precise identification of ∂X with $S^1 \times S^1$ used. To see whether it vanishes, check that the boundary of 2-cells of X map to zero in $\pi_1(S^1) \cong \mathbb{Z}$. The homotopy classes of maps extending the given map correspond to $H^1(X, \partial X; \mathbb{Z}) \cong H_2(X; \mathbb{Z}) = 0$, so if an extension exists, if is unique.

On the other hand, if we have no requirement on the map on the boundary, then there is no obstruction, since the primary obstruction lives in $H^2(X; \mathbb{Z}) \cong H_1(X, \partial X; \mathbb{Z}) = 0$. Make a choice of map on the 1-skeleton. Then it automatically extends to a map $X \to S^1$. The homotopy classes of maps are in one to one correspondence with $H^1(X;\mathbb{Z}) = [X, S^1] \cong \mathbb{Z}$.

Given a map $X \to S^1$, the inverse image of a regular value is a Seifert surface for the knot K.

Now we consider the case of fibrations. Recall that this is following lifting problem:

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[r, "f"] \arrow[ur, dashed] & B \end{tikzcd}$$

where $p\colon E \to B$ is a fibration. Suppose that F is n-simple, and suppose that g is defined on the n-skeleton of X. Let e^{n+1} be an $(n+1)$-cell of X. Then the boundary gives a map $S^n \to X^{(n)} \xrightarrow{g} E \to B$ that is null homotopic, since f gives an extension over the e^{n+1} of the composite. Therefore $S^n \to X^{(n)} \to E$ is homotopic to map $S^n \to F$, by the HLP. Recall that path lifting gave us a map $\pi_1(B) \to \mathrm{hAut}(F)$. We therefore have an induced map

$$(h_\alpha)_*\colon [S^n, F] \to [S^n, F].$$

Note that $[S^n, F] \cong \pi_n(F)$ since F is n-simple. We therefore have a representation $\rho\colon \pi_1(B) \to \mathrm{Aut}(\pi_n(F))$. The composition

$$\pi_1(X) \to \pi_1(B) \to \mathrm{Aut}(\pi_n(F))$$

allows us to define the obstruction cochain with twisted coefficients in

$$\theta^{n+1}(g) \in C^{n+1}(X; \pi_n(F)_\rho).$$

Theorem 18.12. *Let X be a CW complex and let $g\colon X^{(n)} \to E$ be a lift of $f\colon X \to B$ on the n-skeleton. Suppose that F is n-simple. An obstruction class $\theta^{n+1}(g) \in H^{n+1}(X; \pi_n(F)_\rho)$ is defined, and if $\theta^{n+1}(g) = 0$, then g can be changed on the n-skeleton, relative to the $(n-1)$-skeleton, and then extended over the $(n+1)$-skeleton.*

This is essentially the same ideas as the previous theorems in this section. Finally we apply the fibrations obstruction to the problem of finding sections for vector bundles. Let $E \to B$ be an oriented n-dimensional vector bundle and let $E_0 := E \smallsetminus \{0 - \text{section}\}$. Finding a section of a vector bundle is the same as finding a lift of in the following diagram.

$$\begin{tikzcd} & E_0 \arrow[d] \\ B \arrow[r, "\mathrm{Id}"] \arrow[ur, "s"] & B \end{tikzcd}$$

The associated primary obstruction is

$$e(p) \in H^n(B; \pi_{n-1}(\mathbb{R}^n \smallsetminus \{0\})) = H^n(B; \mathbb{Z}).$$

Note that $\mathbb{R}^n \smallsetminus \{0\}$ is n simple since $\pi_1(S^{n-1}) = 0$ for $n > 2$. Also $\pi_1(B)$ acts trivially on $\pi_{n-1}(S^n - 1)$ since the transition functions in an oriented vector bundle lie in $\mathrm{GL}(n, \mathbb{R}^n)_+$. Therefore there is no twisting. The first obstruction to finding a section of a vector bundle is $e(p) \in H^n(B;\mathbb{Z})$; this is called the *Euler class* of the vector bundle.

Theorem 18.13. *Let $p\colon TB \to B$ be the tangent bundle of an oriented manifold B. Then*
$$\langle e(p), [B]\rangle = \chi(B).$$

For example, TS^2 has no nonzero section, because $\chi(S^2) = 2$. This is the hairy balls theorem.

19. Spectral sequences

19.1. **Introduction.** We finish the course with a brief introduction to spectral sequences. Our goal is to give a formalism to compute the homology of a total space of a fibration $F \to E \to B$ in terms of the homology of B and F. This will be via the Leray-Serre spectral sequence. This is one fantastic example of a spectral sequence. There are many others. We will start with homology spectral sequences, but then we will also later the cohomology version. As usual, the main difference between the homology and cohomology versions is that the maps go in the opposite direction.

A spectral sequence can be thought of as a book of modules. There are differentials on each page, and we can take homology to turn the page of the book. At the end of the book, all is revealed. The first or second page of the book might be something that we can compute. Sometimes page k is equal to page $k+1$, for all $k \geq N$, for some N. In this case we say that the sequence collapses at the Nth page. After that nothing more interesting happens in the book. Like when the one ring is destroyed 100 pages before the end of LOTR, and you think what is going to happen now, and the answer is nothing. In such cases we have a chance of being able to compute something.

19.2. **Algebraic formalism of a spectral sequence arising from a filtration.** Let R be a commutative PID. A bigraded module E is a collection $E_{s.t}$ of R modules $s,t \in \mathbb{Z}$. A differential d of bidegree $(-r, r-1)$ is a homomorphism
$$d\colon E_{s,t} \to E_{s-r,t+r-1}$$
for all $s,t \in \mathbb{Z}$, such that $d^2 = 0$. We can take homology via:
$$H_{s,t}(E) := \frac{\ker(d\colon E_{s,t} \to E_{s-r,t+r-1})}{\operatorname{im}(E_{s+r,t-r+1})}.$$

Note that if $E_q := \bigoplus_{s+t=q} E_{s,t}$, then the differential defines a map $\partial\colon E_q \to E_{q-1}$ such that (E_q, ∂) is a chain complex with homology $\bigoplus_{s+t=q} H_{s,t}(E)$.

Definition 19.1. An E^k-spectral sequence (often we drop E^k-from the notation, it just records the first page) is a sequence (E^r, d^r), with $r \geq k$, such that E^r is a bigraded module, d^r a differential of bidegree $(-r, r-1)$, and for $r \geq k$ we have $H(E^r) \cong E^{r+1}$.

We can draw diagrams of pages, with a grid, and we show the differentials with diagonal arrows that show the bidegree. The following simple observation is nevertheless rather powerful.

Lemma 19.2. *If $E^r_{p,q} = 0$ for some r, then $E^s_{p,q} = 0$ for all $s > r$.*

Define Z^k to be the bigraded module with $Z^k_{s,t} = \ker(d^k \colon E^k_{s,t} \to E^k_{s-k,t+k-1})$, and define B^k to be the bigraded module with $B^k_{s,t} = d^k(E^k_{s+k,t-k+1})$. Then $B^k \subseteq Z^k$ and $E^{k+1} = Z^k/B^k$.

Next, let $Z(E^{k+1})$ be the bigraded module with $Z(E^{k+1})_{s,t} = \ker(d^{k+1} \colon E^{k+1}_{s,t} \to E^{k+1}_{s-k-1,t+k})$. Let $B(E^{k+1})$ be the bigraded module with $B(E^{k+1}_{s,t}) = d^{k+1}(E^{k+1}_{s+k+1,t-k})$. There exist bigraded submodules Z^{k+1}, B^{k+1} satisfying $B^k \subseteq B^{k+1} \subset Z^{k+1} \subset Z^k$ and such that $Z^{k+1}_{s,t}/B^k_{s,t} = Z(E^{k+1})_{s,t}$ and $B(E^{k+1})_{s,t} = B^{k+1}_{s,t}/B^k_{s,t}$. Iterating, this yields a sequence of submodules

$$B^k \subseteq B^{k+1} \subseteq \cdots \subseteq B^{r-1} \subseteq B^r \subseteq \cdots \subseteq Z^r \subseteq Z^{r+1} \subseteq \cdots Z^{k+1} \subseteq Z^k.$$

Define $E^{r+1} = Z^r/B^r$, $Z^\infty = \bigcap Z^r$ and $B^\infty = \bigcup B^r$. Then $E^\infty = Z^\infty/B^\infty$ is the *limit* of the spectral sequence.

An E^k-spectral sequence is said to converge if for all s, t there is an $R(s,t) \geq k$ such that for $r \geq R$, $d^r \colon E^r_{s,t} \to E^r_{s-r,t+r-1}$ is trivial. Then $E^{r+1}_{s,t}$ is a quotient of $E^r_{s,t}$, and we can write $E^\infty_{s,t} = \operatorname{colim}_j E^{R(s,t)+j}_{s,t}$. The sequence converges in the strong sense if there is an N with $R(s,t) \leq N$ for all s, t.

An example to have in mind is the following. Let $F \to E \to B$ be a fibration with $\pi_1(B) = 0$. Then $E^2_{p,q} = H_p(B;\mathbb{Q}) \otimes H_q(F;\mathbb{Q})$ and $\bigoplus_{p+q=n} E^\infty_{p,q} = H_n(E;\mathbb{Q})$. So there is a spectral sequence that computes the homology of the total space in terms of the homology of the base and the fibre. This is a special case of the Leray-Serre spectral sequence, that we will discuss in greater detail below.

Now we show how a spectral sequence can arise from filtrations.

Definition 19.3. A *filtration* of an R-module A is a sequence of submodules F_sA, with $s \in \mathbb{Z}$, such that

$$\cdots \subseteq F_{s-1}A \subseteq F_sA \subseteq F_{s+1}A \subseteq F_{s+2}A \subseteq \cdots$$

If A is graded, $A = \{A_t\}$, then F_sA is graded, with $F_sA = \{F_sA_t\}$. The *associated graded* module is

$$G(A)_s = F_s(A)/F_{s-1}(A).$$

If A is graded then the associated graded becomes bigraded, with $G(A)_{s,t} = F_sA_t/F_{s-1}A_t$. The filtration F_sA is said to be *convergent* if $\bigcap_s F_sA = 0$ and $\bigcup_s F_sA = A$.

Note that $G(A)$ does not determine A, only up to extension problems. If R is a field, then $G(A)$ determines A up to isomorphism. Also if ever $G(A) = 0$ then $A = 0$.

A filtration is bounded below if for all t, there is an $s(t)$ such that $F_{s(t)}A_t = 0$.

A filtration on a chain complex C is a filtration compatible with the differentials, i.e. so that each term F_sC is a chain complex F_sC_t. This gives a filtration of the homology $F_sH_*(C) := \operatorname{im}(H_*(F_sC) \to H_*(C))$. Then $\bigcup F_sH_*(C) = H_*(C)$.

Theorem 19.4. *Let F_sC be a convergent filtration on a bounded below chain complex C. There is a convergent E^1-spectral sequence with*

$$E^1_{s,t} \cong H_{s+t}(F_sC/F_{s-1}C)$$

such that d^1 corresponds to the boundary operator of the triple $(F_sC, F_{s-1}C, F_{s-2}C)$, that is

$$H_{s+t}(F_sC/F_{s-1}C) \to H_{s+t-1}(F_{s-1}C) \to H_{s+t-1}(F_{s-1}C/F_{s-2}C),$$

and E^∞ is isomorphic to the bigraded module $GH_*(C)$ associated to the filtration

$$F_sH_*(C) = \mathrm{im}(H_*(F_sC) \to H_*(C)).$$

That is, we compute the homology $H_*(C)$ (well, a graded module that describes the iterated quotients of some filtration of $H_*(C)$), in terms of the homology $H_*(F_sC/F_{s-1}C)$ of quotients of a filtration of the chain complex. We just have to take homology (turn the page) enough times until it stabilises. Then hope we can solve the extension problem.

Often spectral sequences are first quadrant, meaning $E^r_{s,t} = 0$ for $s, t < 0$. Then such sequences automatically converge, since the differentials get longer and so eventually land or originate outside the first quadrant.

We will not give the proof of this theorem due to time constraints. See [Sp, p. 469] for the proof. We want instead to give some examples of computations using this technology. The rough idea is that later differentials approximate the actual differentials on C, and later modules E^r better approximate $\ker \partial$.

19.3. **The spectral sequence of a fibration.** Let $p\colon E \to B$ be a fibration, and let π be a coefficient module. Suppose also that B is a CW complex. Define

$$E^{(s)} = p^{-1}(B^{(s)})$$

for $s \geq 0$, and $E^{(s)} = \emptyset$ for $s < 0$. We have $E^{(s)} \subset E^{(s+1)}$, so $E^{(s)}$ is a filtration on E with $\bigcup E^{(s)} = E$. Let $C_* = C_*(E; \pi)$. This induces a filtration on C_* by

$$F_s(C) = C_*(E^{(s)}; \pi).$$

The filtration $F_s(C)$ is bounded below and convergent. We have

$$F_s(C)/F_{s-1}(C) = C_*(E^{(s)}, E^{(s-1)}).$$

Then if we take homology, we get the E^1 page of a spectral sequence. We have a convergent spectral sequence, for any coefficient module π, with

$$E^1_{s,t} \cong H_{s+t}(E^{(s)}, E^{(s-1)}; \pi)$$

with d^1 the boundary operator of $(E^{(s)}, E^{(s-1)}, E^{(s-2)})$. Then the E^∞ page gives a bigraded module associated to some filtration of $H_*(E; \pi)$.

Theorem 19.5. *For all $s \geq 0$, there are isomorphisms that fit into a commutative diagram:*

$$\begin{tikzcd} H_s(B^{(s)}, B^{(s-1)}; H_n(F;\pi)) \arrow[r, "\cong"] \arrow[d, "\partial"] & H_{s+n}(E^{(s)}, E^{(s-1)}; \pi) \arrow[d, "\partial"] \\ H_{s-1}(B^{(s-1)}, B^{(s-2)}; H_n(F;\pi)) \arrow[r, "\cong"] & H_{s+n-1}(E^{(s-1)}, E^{(s-2)}; \pi) \end{tikzcd}$$

Thus $E^1_{s,t}$ is isomorphic to the cellular chain complex of B, with coefficients in $H_n(F;\pi)$. We take homology to get the E^2-page. This yields the following theorem, which is the main result of this section.

What we have done actually works for any generalised homology theory, so we state the theorem in this generality.

Theorem 19.6 (Leray-Serre spectral sequence)**.** *Let h_* be a generalised homology theory, let $p\colon E \to B$ be a fibration with B a path connected CW complex. Let $F := p^{-1}(B)$. There is a convergent E^2-spectral sequence with*

$$E^2_{s,t} = H_s(B; h_t(F))$$

converging to $h_*(E)$. That is, E^∞ is a bigraded module associated to the filtration of $h_*(E)$ defined by $F_s h_*(E) = \operatorname{im}(h_*(E^{(s)}) \to H_*(E))$.

Recall that this means the following:

(1) There is a filtration
$$0 \leq F_{0,n} \subseteq F_{1,n-1} \subseteq \cdots \subseteq F_{n,n-p} \subseteq \cdots \subseteq h_n(E)$$
with $\bigcup_p F_{p,n-p} = h_n(E)$.

(2) There is a spectral sequence $E^r_{p,q}$ with differentials $d^r\colon E^r_{p,q} \to E^r_{p-r,q+r-1}$ and $\ker d^r_{p,q}/\operatorname{im} d^r \cong E^{r+1}_{p,q}$, with isomorphisms $E^r_{p,q} = H_p(B; h_q(F))$.

(3) For all $p, q \geq 0$, there is an $r_{p,q}$ such that for all $r \geq r_{p,q}$ we have that $d^r\colon E^r_{p,q} \to E^r_{p-r,q+r-1}$ is the zero map. Then $E^{r+1}_{p,q} \cong E^r_{p,q}/d^r(E^r_{p+r,q-r+1})$, so that $E^\infty_{p,q} = \operatorname{colim}_r E^r_{p,q} \cong F_{p,q}/F_{p-1,q+1} = G(h_n(E))_p$. That is, the E^∞ terms give the steps in a filtration of $h_n(E)$.

19.4. **Examples.** We warn that these notes to not contain diagrams, and the reader has to supply them for him or her self (or refer to class notes). Such diagrams are almost essential for keeping track of the bidegrees when following or making a spectral sequence computation.

Example 19.7. Consider the path space fibration $\Omega S^k \to PS^k \to S^k$ of the k-sphere, with $k \geq 2$. Since $\pi_1(S^k) = 0$, the coefficients are untwisted in the E^2 page

$$E^2_{p,q} = H_p(S^k; H_q(\Omega S^k)) = \begin{cases} H_q(\Omega S^k) & p = 0, k \\ 0 & \text{else.} \end{cases}$$

Since $H_n(PS^n) = 0$ for all $n \neq 0$, we since the Leray Serre spectral sequence is first quadrant, we have that $E^\infty_{p,q} = 0$ unless $(p,q) = (0,0)$.

Since the differentials are of degree $(-r, r+1)$, the differentials either start or end at 0, or we have $r = k$, and $d^k \colon E^k_{k,q} \to E^k_{0,q+k-1}$. Thus $E^2_{p,q} = \cdots = E^k_{p,q}$ and $E^{k+1}_{p,q} = \cdots = E^\infty_{p,q} = 0$ for $(p,q) \neq (0,0)$. But also

$$E^{k+1}_{p,q} = \begin{cases} \ker(d^k \colon E^k_{k,q} \to E^k_{0,q+k-1}) & (p,q) = (k,q) \\ \operatorname{coker}(d^k \colon E^k_{k,q} \to E^k_0, q+k-1) & (p,q) = (0, q+k-1) \end{cases}.$$

Thus each map in d^k is an isomorphism. Therefore we obtain

$$H_q(\Omega S^k) \cong H_{q+k-1}(\Omega S^k).$$

It follows that $H_q(\Omega S^k) = \mathbb{Z}$ when $q = a(k-1)$ for some $a \geq 0$ and vanishes otherwise.

Example 19.8 (The Atiyah-Hirzebruch spectral sequence)**.** Consider the fibration pt $\to X \to X$ and let h_* be a generalised homology theory. Then we obtain a spectral sequence with

$$E^2_{p,q} = H_p(X; h_*(\mathrm{pt})) \Rightarrow h_{p+q}(X).$$

This has untwisted coefficients. This enables us to compute generalised homology theories in terms of ordinary homology, and knowledge of the theory of a point. For example, with $h_* = \Omega$, the oriented bordism theory, we have $\Omega_1 = \Omega_2 = \Omega_3 = 0$ and $\Omega_4 = \mathbb{Z} = \Omega_0$. The AHSS yields $\Omega_i(X) = H_i(X)$ for $i = 1, 2, 3$. Note that the map $X \to \mathrm{pt}$ splits via $\mathrm{pt} \to X$, so we get a splitting $\Omega_i(X) \cong \Omega_i(\mathrm{pt}) \oplus \widetilde{\Omega}_i(X)$. It follows that any differential with image $E^r_{0,n}$ vanishes. We can therefore compute

$$\Omega_4(X) = H_4(X) \oplus \Omega_4 = H_4(X) \oplus \mathbb{Z}.$$

Here $(M, f) \mapsto (f([M]), \sigma(M))$, the image of the fundamental class of M in the fourth homology of X, and the signature of the intersection form on the second real coefficient homology of X.

19.5. **Gysin sequence.**

Theorem 19.9. *Let R be a commutative ring and let $F \to E \to B$ be a fibration, with F an R-homology sphere. Suppose that $\pi_1(B)$ acts trivially on $H_n(F; R) = R$ if $i = 0, n$ and 0 otherwise. There exists an exact sequence*

$$H_r(E) \xrightarrow{f_*} H_r(B) \to H_{r-n-1}(B) \to H_{r-1}(E) \xrightarrow{f_*} H_{r-1}(B) \to \dots$$

Proof. Homology is with R coefficients if not mentioned.

$$E^2_{p,q} = \begin{cases} H_p(B; R) & q = 0, n \\ 0 & \text{else.} \end{cases}$$

The nontrivial differential is $d^{n+1} \colon E^{n+1}_{p,0} \to E^{n+1}_{p-n-1,n}$. Therefore

$$E^{n+1}_{p,q} \cong E^2_{p,q} \cong H_p(B; H_q(F)) \cong \begin{cases} H_p(B; R) & q = 0, n \\ 0 & \text{else.} \end{cases}$$

and

$$E^{\infty}_{p,q} = \begin{cases} 0 & q \neq 0, n \\ \ker(d^{n+1}) & q = 0 \\ \operatorname{coker} d^{n+1} & q = n. \end{cases}$$

Thus $H_r(E)$ is filtered by

$$0 \subseteq E^{\infty}_{r-n,n} \cong F_{r-n,n} \subseteq F_{r,0} = H_r(E).$$

Here $F_{r-n,n} = \operatorname{coker} d^{n+1}$ and $F_{r,0}/F_{r-n,n} = \ker d^{n+1}$. Therefore $0 \to E^{\infty}_{r-n,n} \to H_r(E) \to E^{\infty}_{r,0} \to 0$ is exact. The fact that we know the kernel and cokernel of d^{n+1} also tells us that

$$0 \to E^{\infty}_{p,0} \to E^{n+1}_{p,0} \xrightarrow{d^{n+1}} E^{n+1}_{p-n-1,n} \to E^{\infty}{}_{p-n-1,n} \to 0.$$

We get a map $H_p(E) \to H_p(B)$ by combining

$$H_p(E) \to E^{\infty}_{p,0} \to E^{n+1}_{p,0} = H_p(B),$$

and we get a map $H_p(B) = E^{\infty}_{p,0} \to E^{n+1}_{p-n-1,n} = H_{p-n-1}(B)$. Finally we get a map $H_{p-n-1}(B) \to H_{p-1}(E)$ by

$$H_{p-n-1}(B) = E^{n+1}_{p-n-1,n} \to E^{\infty}_{p-n-1,n} \to H_{p-1}(E).$$

□

19.6. **Cohomology spectral sequences.** There are also cohomology spectral sequences. In particular, we have the Leray Serre cohomology spectral sequence of a fibration, with h^q a generalised cohomology theory, and

$$E_2^{p,q} = H^p(B; h^q(F)) \Rightarrow h^{p+q}(E).$$

The differentials have bidegree $(r, 1-r$, that is they go $E_r^{p,q} \to E_r^{p+r,q+1-r}$. If the generalised cohomology theory has products, then the spectral sequence also has products:

$$E_\ell^{p,q} \times E_\ell^{r,s} \to E_\ell^{p+r,q+s}.$$

If $d_t \colon E_t \to E_t$ is a differential, and $a \in E_t^{p,q}$, $b \in E_t^{r,s}$, we have

$$d_t(a \cdot b) = d_t(a) \cdot b + (-1)^{p+q} a \cdot d_t(b).$$

The induced product on E^∞ coincides with the cup product on $h^*(E)$.

Example 19.10. To finish, we start a computation of the cohomology ring of $K(\mathbb{Z}, 2)$, which can be modelled with the complex projective space $\mathbb{CP}^\infty$. We want to show that the cohomology ring is isomorphic to the polynomial ring $\mathbb{Z}[c]$, where c has degree 2. Use $K(\mathbb{Z}, 1) \to * \to K(\mathbb{Z}, 2)$, the path space fibration, and the Leray Serre cohomology spectral sequence of it. The path space is contractible, so the E^∞ page vanishes away from $(0,0)$. The E_2-page is $H^p(K(\mathbb{Z}, 2); \mathbb{Z})$ for $q = 0, 1$ and is zero otherwise. On the E_2-page the differential is of degree $(2, -1)$, and all the other differential vanish. So the d_2 differentials map by an isomorphism from $H^p(K(\mathbb{Z}, 2); \mathbb{Z}) \to H^{p+2}(K(\mathbb{Z}, 2); \mathbb{Z})$. The homology of $\mathbb{CP}^\infty$ is therefore $\mathbb{Z}$ in even nonnegative dimensions and 0 in other dimensions. Next, we use the derivation rule. If $1 \in E_2^{0,1}$ and $c \in E_2^{2,1}$ are generators, then $c \cdot 1 \in E^{2,2} = 0$. Therefore

$== d(0) = d(c{\cdot}1) = d(c){\cdot}1+(-1)^3 c{\cdot}d(1) = (\text{gen of } E_2^{4,0}){\cdot}1-c{\cdot}c = (\text{gen of } E_2^{4,1})-c{\cdot}c$. Thus the generator of $E_2^{4,1}$ is equal to $c \cdot c$, with one $c \in E^{2,1}$ and one in $E^{2,0}$. Therefore the cup product is nontrivial.

We can continue this type of argument.

$$d(c_{2,1} \cdot c_{2,0}) = d(c) \cdot c + (-1)^3 c \cdot d(c) = \text{gen of } E_2^{4,0} \cdot c - 0.$$

But we saw that $c_{2,1} \cdot c_{2,0} \in E_2^{4,1}$ is a generator, and the differential $d(c_{2,1} \cdot c_{2,0})$ is the generator of $E_2^{6,0}$. Therefore c^3 is also nontrivial. This type of argument can be made into an induction to complete the proof that the cohomology ring is $\mathbb{Z}[c]$ with $\deg c = 2$.

The exercise sheet outlines extended examples to use spectral sequences to compute the stable homotopy groups $\pi_1^S \cong \mathbb{Z}/2$ and $\pi_2^S = \mathbb{Z}/2$. While it is also possible to compute these ones using framed bordism theory and the Pontryagin-Thom construction, in general spectral sequences have been a huge tool in computations of homotopy groups and related objects. The computations given in the example sheet are but preliminary examples. Serre was able to use the spectral sequence of a fibration to show that all the stable homotopy groups of spheres in positive degree are finite groups.

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Département de Mathématiques, Université du Québec à Montréal, Canada
E-mail address: mark@cirget.ca