ALGEBRAIC TOPOLOGY IV ‖ EPIPHANY TERM LECTURE NOTES

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This course is about cohomology of spaces, products on cohomology, and manifolds. Cohomology is useful primarily because it can be made into a ring using the cup product. We will study the cohomology ring and apply it to manifolds, especially of dimension 2, 3 and 4. For manifolds homology and cohomology can be related to each other in two different ways, using Poincaré duality and using the universal coefficient theorem. The interplay between these two relations can also be extremely powerful.

1. CW complexes and manifolds

1.1. **CW complexes.** Here is a better definition of CW complexes. It is important that the decomposition into cells is remembered as part of the data of a CW complex. Merely defining it as a space that "was built in a certain way" loses too much data that is needed when working with these spaces.

Definition 1.1 (Finite dimensional CW complex)**.**

(a) A CW complex X of dimension -1 is $\emptyset$. Its *topological realisation* $|X|$ is also the empty set.

(b) Inductively define higher CW complexes. A CW complex of dimension $\leq n$, X^n, consists of

   (i) A CW complex X^{n-1} of dimension $\leq n-1$, with topological realisation $|X^{n-1}|$.

   (ii) A set of maps $\{\varphi_i \colon S^{n-1} \to |X^{n-1}|\}_{i \in I_n}$, the attaching maps.

   The topological realisation $|X^n|$ of X is
$$|X^n| := \big(|X^{n-1} \sqcup \bigsqcup_{i \in I_n} D^n\big)/\sim$$
   where for all $i \in I_n$, $x \in S^{n-1} = \partial D^n_i$, we identify $x \sim \varphi_i(x) \in |X^{n-1}|$.

(c) A (finite dimensional) CW structure on a topological space Y is a CW complex X and a homeomorphism $f \colon |X| \to Y$.

(d) The map $\overline{\varphi_i} \colon D^n \to |X^n|$ such that $S^{n-1} \to D^n \xrightarrow{\overline{\varphi_i}} |X^n|$ equals φ_i is called a *characteristic map*.

1.2. **Manifolds.** An n-dimensional topological manifold M is a topological space that is Hausdorff, second countable (countable basis of open sets), and locally homeomorphic to $\mathbb{R}^n$. That is, for all $x \in M$, there is an open set U_x containing x and a homeomorphism $\varphi_x \colon U_x \to V \subset \mathbb{R}^n$ where V is an open subset of $\mathbb{R}^n$.

Example 1.2.

(i) The sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$.
(ii) Products of spheres $S^n \times S^m$, of dimension $n+m$.
(iii) The n-torus $T^n = S^1 \times \cdots \times S^1$, the product of n copies of the circle S^1.
(iv) Real projective space $\mathbb{RP}^n = \mathbb{R}^{n+1} \smallsetminus \{0\}/\sim$, where $x \sim \lambda x$ for all $\lambda \in \mathbb{R} \smallsetminus \{0\}$.
(v) Similarly complex projective space $\mathbb{CP}^n = \mathbb{C}^{n+1} \smallsetminus \{0\}/\sim$, where $z \sim \lambda z$ for all $\lambda \in \mathbb{C} \smallsetminus \{0\}$.
(vi) The surface Σ_g of genus g is a 2-dimensional manifold.
(vii) The Klein bottle $\mathbb{K}$.
(viii) Quotients of S^n or $\mathbb{R}^n$ by a free, proper, continuous action of a group are manifolds.
(ix) Products of manifolds are manifolds.

For this last item, we will provide several examples of manifolds obtained as orbits spaces of free actions. First, the definitions of these adjectives associated with group actions.

(a) An action of a group G on a manifold X is *free* if $g \cdot x = x$ implies that $g = e \in G$ for every $x \in X$.
(b) An action of a group G on a manifold X is *proper* if for all $x, y \in X$, there exist open sets $U \ni x$ and $V \ni y$ such that
$$\{g \in G \mid gU \cap V \neq \emptyset\}$$
is finite.
(c) A group action is continuous if the map $x \mapsto g \cdot x$ is a continuous map $X \to X$ for all $g \in G$.

Now the promised examples of manifolds defined by quotients by group actions.

Example 1.3.

(1) The group $\mathbb{Z}$ acts on $\mathbb{R}$ by $n \cdot x := x + n$. The quotient is $\mathbb{R}/\mathbb{Z} \cong S^1$.
(2) The group $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ by $(n, m) \cdot (x, y) = (x + n, y + m)$. The quotient $\mathbb{R}^2/\mathbb{Z}^2 \cong T^2$.
(3) The group $\mathbb{Z}/2$ acts on S^n by $1 \cdot x = -x$. The quotient $S^n/(\mathbb{Z}/2) \cong \mathbb{RP}^n$.
(4) The group S^1 acts on S^{2n+1} as follows. Consider $S^{2n+1} \subset \mathbb{C}^{n+1}$ as the set of points with $|z_0|^2 + \cdots |z_n|^2 = 1$. Then $e^{2\pi i\theta} \cdot (z_0, \ldots, z_n) := (e^{2\pi i\theta} \cdot z_0, \ldots, e^{2\pi i\theta} \cdot z_n)$. The quotient is $S^{2n+1}/S^1 \cong \mathbb{CP}^n$.
(5) The group $\mathbb{Z}/p$ acts on S^3 as follows. Let p and q be coprime positive integers. Consider $S^3 \subset \mathbb{C}^2$ with $|z|^2 + |w|^2 = 1$. Write $\mathbb{Z}/p \cong C_p$ where C_p is the cyclic group generated by $\xi = e^{2\pi i/p} \in S^1$. Then
$$\xi^j \cdot (z, w) := (\xi \cdot z, \xi^q \cdot w).$$
The quotient is $S^3/C_p = L(p, q)$, the lens space.

1.3. **Smooth manifolds.** All the manifolds we talk about will be *smooth*.

Definition 1.4. Let $U \subseteq \mathbb{R}^n$ be open. A function
$$f = (f_1, \ldots, f_n) \colon U \to \mathbb{R}^m$$
is *smooth* is all partial derivatives of f_i exist for all i.

Definition 1.5.

(i) An n-dimensional *chart* for a manifold M at $x \in M$ is a homeomorphism $\Phi \colon U \to V$ where $U \ni x$ is an open neighbourhood of x, and $V \subset \mathbb{R}^n$ is open.
(ii) An n-dimensional *atlas* for M is a collection of n-dimensional charts $\{\Phi_i \colon U_i \to V_i\}_{i \in I}$ with $\bigcup_i U_i = M$, where I is some indexing set.
(iii) An atlas $\{\Phi_i \colon U_i \to V_i\}_{i \in I}$ is *smooth* if the transition map

$$\Phi_j \circ \Phi_i^{-1} \colon \Phi_i(U_i \cap U_j) \to \Phi_j(U_i \cap U_j)$$

is smooth as a map between open subsets of $\mathbb{R}^n$.

Definition 1.6. A *smooth manifold* is a topological manifold M with a choice of smooth atlas.

When we say "manifold" we will usually implicitly mean smooth manifold.

1.4. **Manifolds with boundary.** An n-dimensional topological manifold with boundary M is a topological space that is Hausdorff, second countable, and locally homeomorphic to either $\mathbb{R}^n$ or to the half space

$$H^n := \{\underline{x} \in \mathbb{R}^n \mid x_1 \geq 0\}.$$

We write

$$\partial M := \{x \in M \text{ with } U_x = H^n\}.$$

For example, D^n, or a surface $\Sigma_{g,1}$.

We will mostly consider *closed manifolds.* By definition a closed manifold is a compact manifold with empty boundary $\partial M = \emptyset$.

2. Homology groups of manifolds

We start with a recollection of homology groups, then we discuss homology of manifolds.

2.1. **Recap of singular and cellular homology.** Homology theories are covariant functors

$$H_i \colon \{\text{Top spaces}\} \to \{\text{Abelian groups}\}$$

for each i, satisfying some axioms: homotopy invariance, long exact sequence of a pair, excision, and taking a fixed value on a point: $H_0(\mathrm{pt}) = \mathbb{Z}$.

2.1.1. *Singular homology.* Singular homology sends

$$X \mapsto H_i(X; \mathbb{Z})$$

or $H_i(X)$ for short, defined as follows. The singular chain complex

$$C_i(X) := \mathbb{Z}\Big\{\text{singular simplices } \Delta^i \to X\Big\}$$

is the free abelian group generated by the singular simplices. Recall that Δ^i is the i-simplex

$$\{(t_0, \dots, t_i) \in \mathbb{R}^{i+1} \mid 0 \leq t_i \leq 1, \sum_{i=0}^{i} t_i = 1\}.$$

The vertices $t_j = 1$ are naturally ordered by We denote the jth face inclusion respecting the ordering of vertices by $\iota_j \colon \Delta^{i-1} \to \Delta^i$. Then the boundary map

$$\partial_i \colon C_i(X) \to C_{i-1}(X)$$

is defined by

$$\partial_i(\sigma \colon \Delta^i \to X) = \sum_{j=0}^{i} (-1)^{j+1} (\sigma \circ \iota_j \colon \Delta^{i-1} \to X).$$

The cycles are

$$Z_i(X) := \ker(\partial_i \colon C_i(X) \to C_{i-1}(X))$$

and the boundaries are

$$B_i(X) := \ker(\partial_{i+1} \colon C_{i+1}(X) \to C_i(X)).$$

The ith singular homology of X is

$$H_i(X) := \frac{Z_i(X)}{B_i(X)}.$$

2.1.2. *Cellular homology.* Sometimes CW homology is easier for computing homology. This is a functor from spaces to abelian groups

$$X \mapsto H_i^{CW}(X; \mathbb{Z})$$

or $H_i^{CW}(X)$ for short. Define the CW chains by

$$C_i^{CW}(X) := \mathbb{Z}\Big\{ i - \text{cells of } X \Big\}$$

and

$$\partial_i^{CW} \colon C_i^{CW}(X) \to C_{i-1}^{CW}(X)$$

is defined by

$$\partial_i(D^i, \overline{\varphi_j}) = \sum_{k \in I_{i-1}} n_k \, (D^{i-1}, \overline{\varphi_k})$$

where n_k is the degree of the composition

$$S^i \xrightarrow{\varphi_j} X^{i-1}/X^{i-2} \to D^{i-1}/\partial D^{i-1} \cong S^{i-1}$$

where the second map is given by collapsing onto the kth $(i-1)$-cell. Here the degree of a map $f \colon S^{i-1} \to S^{i-1}$ is $d \in Z$ such that $\mathbb{Z} \cong H_{i-1}(S^{i-1}) \to H_{i-1}(S^{i-1}) \cong \mathbb{Z}$ is given by multiplication by d. There is a sign issue here that is important to get right. We will discuss it again later.

Theorem 2.1. *For every finite dimensional CW complex X, we have that*

$$H_i(X) \cong H_i^{CW}(X).$$

2.2. **Top dimensional homology of a manifold.** We will discuss the following theorem.

Theorem 2.2. *Let M be a connected nonempty n-dimensional manifold. Then*
$$H_n(M;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & M \text{ closed and orientable} \\ 0 & \text{otherwise.} \end{cases}$$
Also, $H_m(M;\mathbb{Z}) = 0$ for $m > n$.

Recall that we also know for connected manifolds that $H_0(M;\mathbb{Z}) = \mathbb{Z}$.

In order to make sense of this theorem, we need to discuss what it means for a manifold to be orientable. For example, S^2 and T^2 are orientable surfaces, whereas $\mathbb{RP}^2$ and $\mathbb{K}$ are non-orientable; two sided surfaces are orientable and one-sided surfaces are non-orientable.

First, note that this theorem can be applied to show that a given closed manifold is orientable or nonorientable.

Corollary 2.3. *For $n \geq 1$, the $2n$-manifold $\mathbb{RP}^{2n}$ is nonorientable, whereas $\mathbb{RP}^{2n-1}$ is orientable.*

Proof. There is a cell structure on $\mathbb{RP}^m$ with $C_i^{CW}(\mathbb{RP}^m;\mathbb{Z}) \cong \mathbb{Z}$ for $0 \leq i \leq m$ and $\partial_i^{CW} = 1+(-1)^i$. The top dimensional homology is therefore $\ker(1+(-1)^i \colon \mathbb{Z} \to \mathbb{Z})$ which vanishes for i even and is $\mathbb{Z}$ for i odd. Then Theorem 2.2 implies the result. $\square$

To given an algebraic topology definition of orientability, we need the next lemma.

Lemma 2.4. *Let $n \geq 1$ and let M be a closed n-manifold. Then $H_n(M, M\setminus\{\mathrm{pt}\}) \cong \mathbb{Z}$ for every point in M.*

Proof. We give the proof for $n \geq 2$. The proof for $n = 1$ is similar but slightly different since removing a point from a neighbourhood of that point changes the number of connected components. By excision, $H_n(M, M\setminus\{\mathrm{pt}\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{\mathrm{pt}\})$. Then we may compute using the long exact sequence of a pair:
$$H_n(\mathbb{R}^n) \to H_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{\mathrm{pt}\}) \to H_{n-1}(\mathbb{R}^n\setminus\{\mathrm{pt}\}) \to H_{n-1}(\mathbb{R}^n).$$
Here $H_n(\mathbb{R}^n) = H_{n-1}(\mathbb{R}^n) = 0$, so
$$H_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{\mathrm{pt}\}) \cong H_{n-1}(\mathbb{R}^n\setminus\{\mathrm{pt}\}) \cong H_{n-1}(S^{n-1}) \cong \mathbb{Z},$$
using the homotopy equivalence $\mathbb{R}^n\setminus\{\mathrm{pt}\} \simeq S^{n-1}$. $\square$

Definition 2.5. An *orientation* on a n-dimensional connected manifold M is an assignment of a generator $\mathcal{O}_p \in H_n(M, M\setminus\{p\};\mathbb{Z})$ for every $p \in M$, such that: for every $p, q \in M$ and for every $\mathbb{R}^n \subset M$ with $p, q \in \mathbb{R}^n$, there is an element $\mathcal{O}_{p,q} \in H_n(M, M\setminus\mathbb{R}^n;\mathbb{Z})$ such that for $r = p, q$, under the maps $H_n(M, M\setminus\mathbb{R}^n;\mathbb{Z}) \to H_n(M, M\setminus\{r\};\mathbb{Z})$ induced by the inclusion $M\setminus\mathbb{R}^n \subseteq M\setminus\{r\}$, we have $\mathcal{O}_{p,q} \mapsto \mathcal{O}_r$.

Definition 2.6. An n-manifold is *orientable* if an orientation $\{\mathcal{O}_p\}_{p\in M}$ exists.

All of S^n, $\mathbb{CP}^n$, T^n, $L(p,q)$, Σ_g and $\mathbb{RP}^{2n+1}$ are orientable.

Sketch proof of Theorem 2.2. First, every n-manifold admits a *simplicial structure*: a family of injective maps

$$\{\varphi_i\colon \Delta^{n_i} \to X\}_{i\in I}$$

with $(X, \{\varphi_i\})$ a simplicial complex, where simplices intersect in faces and the union of the images of the φ_i is X. There are some other conditions, we will not go into all the details.

We say that a simplex c has *order* r if there are precisely r simplices of dimension $k+1$ with c as a face. We will use the following lemma.

Lemma 2.7. *For every simplicial structure on a closed n-manifolds M:*

(i) The simplicial structure has finitely many simplices if and only if there are finitely many n simplices.
(ii) If c is an $(n-1)$ dimensional simplex of M, then the order of c is two.

Since this is just a sketch, we will not prove the lemma in detail.

Now,

$$H_n(M) \cong H_n^{CW}(M) \cong \ker(C_n^{CW}(M) \to C_{n-1}^{CW}(M)),$$

thinking of the simplicial structure as a cellular structure. A generator of the cycles in $C_n^{CW}(M)$ is a sum of n-dimensional simplices with appropriate orientations. Start with one n-simplex σ, and choose an $(n-1)$-dimensional face of it τ. The order of this face is two, so there is exactly one other n-simplex σ' that also has τ as a face. Orient (choose the sign $\pm$ of) σ' in an n-chain so that in $\partial^{CW}(\sigma+\sigma')$, τ is cancelled. Since M is compact, there are finitely many n-simplices, so we can continue by induction. The appropriate choice of signs exists to produce a cycle if and only if M is orientable. This procedure in fact creates a generator of $H_n(M) \cong \mathbb{Z}$, for M closed and orientable, but we will not show this precisely.

If M has nonempty boundary, then the $(n-1)$-simplices of the boundary have order one, so the induction stops there without creating an n-cycle. As mentioned above, if M were nonorientable, then there is no appropriate choice of signs to form a cycle. So in these two cases, $H_n(m) = 0$. This completes our sketch proof of Theorem 2.2. □

2.3. Fundamental classes and degrees of maps.

Definition 2.8. If a connected nonempty n-manifold M is closed and orientable, then a choice of generator $[M] \in H_n(M;\mathbb{Z})$ is called a *fundamental class*.

There are two such choices corresponding to two possible orientations: an orientation determines a fundamental class and vice versa.

Definition 2.9. Let $f\colon M \to N$ be a map between closed, oriented, connected n-manifolds with fundamental classes $[M]$ and $[N]$. The degree of f is the integer d such that

$$f_*([M]) = d[N] \in H_n(N;\mathbb{Z}).$$

3. Cohomology

3.1. Hom groups. We have to introduce groups of homomorphisms.

Let A be a group, and let G be an abelian group. Define an abelian group

$$\mathrm{Hom}(A,G) := (\{\text{group homomorphisms } A \to G\}, +, 0).$$

The group structure is defined by $(f+g)(a) = f(a) + g(a) \in G$. The identity is $0(a) = 0 \in G$ for all $a \in A$.

Let $f\colon A \to B$ be a group homomorphism. There is a homomorphism induced by f:

$$\begin{array}{rcl} f^*\colon \mathrm{Hom}(B,G) & \to & \mathrm{Hom}(A,G) \\ (\varphi\colon B \to G) & \mapsto & (\varphi \circ f\colon A \to B \to G). \end{array}$$

Let $g\colon G \to H$ be a homomorphism of abelian groups. Then g induces a homomorphism

$$\begin{array}{rcl} g_*\colon \mathrm{Hom}(A,G) & \to & \mathrm{Hom}(A,H) \\ (\varphi\colon A \to G) & \mapsto & (g \circ \varphi\colon A \to G \to H). \end{array}$$

Example 3.1.

(1) $\mathrm{Hom}(\mathbb{Z},G) \cong G$ for all G, with the map $\varphi \mapsto \varphi(1)$ giving an isomorphism.
(2) $\mathrm{Hom}(\mathbb{Z}/n,G) \cong \ker(\cdot n\colon G \to G)$.
(3) $\mathrm{Hom}(\mathbb{Z}/n,\mathbb{Z}/m) \cong \mathbb{Z}/\gcd(n,m)$.

Lemma 3.2. *Let $\{A_i\}_{i\in I}$ be a sequence of abelian groups and let G be an abelian group. Then*

$$\mathrm{Hom}(\bigoplus_{i\in I} A_i, G) \cong \prod_{i\in I} \mathrm{Hom}(A_i, G)$$

via the map $f \mapsto \prod_{i\in I} f|_{A_i}$.

Note the change between direct sum and direct product. What is the difference? The elements of both are tuples $(a_i)_{i\in I}$, and the addition and the identity are the same, but the sets are different: in the direct product any tuples are allowed but in the direct sum only $\{(a_i)_{i\in I} \mid \text{ finitely many } a_i \neq e_{A_i}\}$.

Lemma 3.3. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups and let G be an abelian group. Suppose that C is free abelian. Then*

$$0 \to \mathrm{Hom}(C,G) \to \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G) \to 0$$

is also short exact.

The proof is an exercise on the problem sheet.

Lemma 3.4. $\mathrm{Hom}(\mathbb{Z}^n,\mathbb{Z}) \cong \mathbb{Z}^n$.

Proof. For I finite, $\bigoplus_I$ and $\prod_I$ coincide. Therefore

$$\mathrm{Hom}(\mathbb{Z}^n,\mathbb{Z}) \cong \mathrm{Hom}(\bigoplus_{i=1}^n \mathbb{Z},\mathbb{Z}) \cong \prod_{i=1}^n \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \cong \prod_{i=1}^n \mathbb{Z} \cong \bigoplus_{i=1}^n \mathbb{Z} = \mathbb{Z}^n.$$

□

3.2. **Algebra of cochain complexes.** Later, we will apply the algebra of cochain complexes in this section to cochain complexes arising from spaces, in order to define and work with cohomology.

Definition 3.5. A *cochain complex* is a sequence of abelian groups

$$0 \to D^0 \xrightarrow{\delta_0} D^1 \xrightarrow{\delta_1} D^2 \xrightarrow{\delta_2} \cdots \to D^n \xrightarrow{\delta_n} \cdots$$

with $\delta_{i+1} \circ \delta_i = 0$ for all i.

We call the δ_i the *coboundary maps.*

The cohomology groups of a cochain complex are

$$H^n(D^*) := \frac{\ker(\delta_n \colon D^n \to D^{n+1})}{\operatorname{im}(\delta_{n-1} \colon D^{n-1} \to D^n)}.$$

Here $\ker(\delta_n \colon D^n \to D^{n+1})$ are the cocycles and $\operatorname{im}(\delta_{n-1} \colon D^{n-1} \to D^n)$ are the coboundaries.

Definition 3.6. A *cochain map* $f \colon C^* \to D^*$ consists of a homomorphism $f_i \colon C^i \to D^i$ for each i such that

$$\begin{tikzcd} C^i \arrow[r, "f_i"] \arrow[d, "\delta_i^C"] & D^i \arrow[d, "\delta_i^D"] \\ C^{i+1} \arrow[r, "f_{i+1}"] & D^{i+1} \end{tikzcd}$$

commutes for all i, meaning that both routes round the square give the same outcome for every element of C^i.

Definition 3.7. A *cochain homotopy* between cochain maps $f, g \colon C^i \to D^i$ consists of homomorphisms $h_i \colon C^i \to D^{i-1}$ for all i with $f_i - g_i = h_{i+1} \circ \delta_i + \delta_{i-1} \circ h_i$ for all i.

$$\begin{tikzcd} \cdots \arrow[r] & C^{i-1} \arrow[r, "\delta_{i-1}"] \arrow[d, "f_{i-1}-g_{i-1}"'] & C^i \arrow[r, "\delta_i"] \arrow[d, "f_i-g_i"] \arrow[dl, "h_i"'] & C^{i+1} \arrow[r, "\delta_{i+1}"] \arrow[d, "f_{i+1}-g_{i+1}"] \arrow[dl, "h_{i+1}"'] & \cdots \\ \cdots \arrow[r] & D^{i-1} \arrow[r, "\delta_{i-1}"] & D^i \arrow[r, "\delta_i"] & D^{i+1} \arrow[r, "\delta_{i+1}"] & \cdots \end{tikzcd}$$

Lemma 3.8.

(i) Let $f \colon C^* \to D^*$ be a cochain map. Then

$$\begin{array}{rcl} f_* \colon H^n(C) & \to & H^n(D) \\ [c] & \mapsto & [f_n(c)] \end{array}$$

is a well-defined map.

(ii) If $f \sim g \colon C^* \to D^*$ are two cochain homotopic cochain maps then $f_* = g_* \colon H^n(C^*) \to H^n(D^*)$ for all $n \in \mathbb{N}_0$.

The proof is essentially the same as the analogous fact for homology.

Now we have a long exact sequence in cohomology from a short exact sequence of cochain complexes $0 \to C^* \to B^* \to A^* \to 0$. That is, we have a commuting diagram

$$\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \arrow[d] & \\
0 \arrow[r] & C^n \arrow[r, "f^n"] \arrow[d, "\delta_n"] & B^n \arrow[r, "g^n"] \arrow[d, "\delta_n"] & A^n \arrow[r] \arrow[d, "\delta_n"] & 0 \\
0 \arrow[r] & C^{n+1} \arrow[r, "f^{n+1}"] \arrow[d] & B^{n+1} \arrow[r, "g^{n+1}"] \arrow[d] & A^{n+1} \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

We define a *connecting homomorphism*, that is a homomorphism

$$d\colon H^n(A^*) \to H^{n+1}(C^*).$$

Let $a \in A^n$ be a cocycle, that is $\delta_n(a) = 0$. Then $a = g^n(b)$ for some $b \in B^n$. Note that $g^{n+1}(\delta_n(b)) = \delta_n(g^n(b)) = \delta_n(a) = 0$. Therefore there exists $c \in C^{n+1}$ such that $f^{n+1}(c) = \delta_n(b)$. Define

$$d([a]) = [c] \in H^{n+1}(C^*).$$

It is a routine but slightly long check that this is well-defined, omitted here. This connecting homomorphism enables us to fit the cohomology groups together into a long exact sequence

$$\cdots \to H^n(C) \xrightarrow{f^*} H^n(B) \xrightarrow{g^*} H^n(A) \xrightarrow{d} H^{n+1}(C) \xrightarrow{g^*} H^{n+1}(B) \to \cdots$$

We will apply this to obtain certain long exact sequences, in particular the long exact sequence of a pair and the Mayer-Vietoris sequence.

Cochain complexes will, for us, most often arise from taking Hom of a chain complex.

Let (C_*, ∂_*) be a chain complex, with $\partial_n\colon C_n \to C_{n-1}$. Let G be an abelian group. For $n \in \mathbb{N}_0$, define

$$\delta_n := \partial_{n+1}^*\colon \operatorname{Hom}(C_n, G) \to \operatorname{Hom}(C_{n+1}, G).$$

We obtain a cochain complex $\operatorname{Hom}(C_*, G)$ given by

$$\begin{aligned}\operatorname{Hom}(C_0, G) \xrightarrow{\delta_0} \operatorname{Hom}(C_1, G) \xrightarrow{\delta_1} \operatorname{Hom}(C_2, G) \xrightarrow{\delta_2} \cdots \\ \to \operatorname{Hom}(C_n, G) \xrightarrow{\delta_n} \operatorname{Hom}(C_{n+1}, G) \xrightarrow{\delta_{n+1}} \cdots\end{aligned}$$

This is the *cochain complex dual to* (C_*, ∂_*). The *cohomology of* C_* with coefficients in G is

$$H^n(C; G) := \ker(\delta_n)/\operatorname{im}(\delta_{n-1}).$$

Lemma 3.9.

(i) Let $f_*\colon C_* \to D_*$. Then

$$\begin{array}{rcl} f^*\colon \operatorname{Hom}(D_n, G) & \to & \operatorname{Hom}(C_n, G) \\ (\varphi\colon D_n \to G) & \mapsto & (\varphi \circ f\colon C_n \to G) \end{array}$$

is a cochain map, and therefore induces a map $f^*\colon H^n(D_*) \to H^n(C_*)$.

(ii) Two chain homotopic chains map $f, g\colon C_* \to D_*$ *induce cochain homotopic cochain maps*

$$f^*, g^*\colon \operatorname{Hom}(D_*, G) \to \operatorname{Hom}(C_*, G),$$

so that

$$f^* = g^*\colon H^n(D) \to H^n(C).$$

Proof. For the proof of (i), just note that $f_{n-1} \circ \partial_n = \partial_n \circ f_n$ implies that

$$\delta_{n-1} \circ f^*_{n-1} = \partial^*_n \circ f^*_{n-1} = f^*_n \circ \partial^*_n = f^*_n \circ \delta_{n-1}.$$

Then apply Lemma 3.8 (i) to see that cochain maps induce maps on cohomology. To prove (ii), start with $\partial \circ h + h \circ \partial = f - g$, and take the dual, to obtain

$$h^* \circ \delta + \delta \circ h^* = f^* - g^*.$$

Thus h^* gives a cochain homotopy, so by Lemma 3.8 (ii), f^* and g^* induce the same map on cohomology. □

3.3. **Cohomology of a topological space.** Now we define the singular cohomology of a topological space. Let (X, A) be a pair of spaces. Usually we will have $A = \emptyset$, and then we write $X = (X, \emptyset)$. Define the singular cochain group to be

$$C^n(X, A; G) := \operatorname{Hom}(C_n(X, A), G)$$

and

$$\delta_n := \partial^*_{n+1}\colon \operatorname{Hom}(C_n(X, A), G) \to \operatorname{Hom}(C_{n+1}(X, A), G).$$

We call

$$\ker(\delta_n\colon C^n(X, A; G) \to C^{n+1}(X, A; G))$$

the *singular cocycles* and

$$\operatorname{im}(\delta_{n-1}\colon C^{n-1}(X, A; G) \to C^n(X, A; G))$$

the singular coboundaries. The singular cochain complex is

$$C^0(X, A; G) \xrightarrow{\delta_0} C^1(X, A; G) \xrightarrow{\delta_1} C^2(X, A; G) \xrightarrow{\delta_2} \cdots$$
$$\to C^{n-1}(X, A; G) \xrightarrow{\delta_{n-1}} C^n(X, A; G) \xrightarrow{\delta_n} C^{n+1}(X, A; G) \to \cdots$$

We then define the singular cohomology

$$H^n(X, A; G) := H^n(C^*(X, A; G)) = \ker \delta_n / \operatorname{im} \delta_{n-1}.$$

When $G = \mathbb{Z}$, we will often omit the coefficients and simply write $H^n(X, A)$.

3.4. **CW cohomology.** Let X be a CW complex. Recall that there is the cellular chain complex

$$(C_*^{CW}(X), \partial_*^{CW}).$$

Let G be an abelian group. Define the CW cochain groups to be

$$C^i_{CW}(X; G) := \mathrm{Hom}(C_i^{CW}(X), G).$$

The CW coboundary maps are given by

$$\delta_i^{CW} := (\partial_{i+1}^{CW})^* \colon \mathrm{Hom}(C_i^{CW}(X), G) \to \mathrm{Hom}(C_{i+1}^{CW}(X), G).$$

Then the CW cohomology is

$$H^n_{CW}(X; G) := \ker(\delta_n^{CW}) / \mathrm{im}(\delta_{n-1}^{CW}).$$

Theorem 3.10. *For every CW complex X, for every abelian group G, and for every $n \in \mathbb{N}_0$, we have $H^n(X; G) \cong H^n_{CW}(X; G)$.*

Example 3.11. Let us consider the cohomology of the sphere S^n. It has a CW structure with two cells: a 0-cell and an n-cell. The resulting CW chain complex is

$$C_*^{CW}(S^n) = \cdots \to \mathbb{Z} \to 0 \to \cdots \to 0 \to \mathbb{Z}$$

with nonzero chain groups in degrees 0 and n. Since $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$ and $\mathrm{Hom}(0, \mathbb{Z}) = 0$, the cochain complex is

$$C_*^{CW}(S^n) = \mathbb{Z} \to 0 \to \cdots \to 0 \to \mathbb{Z} \to \cdots$$

where now the $\mathbb{Z}$ on the left is in degree 0 and the $\mathbb{Z}$ on the right is in degree n. Therefore the CW cohomology of S^n is

$$H^i_{CW}(S^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.12. Now we use general G coefficients, with G an abelian group. Let Σ_g be the surface of genus g. There is a CW structure with one 0-cell, $2g$ 1-cells, and one 2-cell. The associated CW chain complex is

$$\mathbb{Z} \xrightarrow{0} \bigoplus_{2g} \mathbb{Z} \xrightarrow{0} \mathbb{Z}.$$

Since $\mathrm{Hom}(\mathbb{Z}, G) \cong G$, this gives rise to a cochain complex that also takes the form

$$G \xrightarrow{0} \bigoplus_{2g} G \xrightarrow{0} G.$$

Then the cellular cohomology is

$$H^i_{CW}(\Sigma_g; G) \cong \begin{cases} G & i = 0, 2 \\ \bigoplus^{2g} G & i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

3.5. **Understanding cohomology.** Recall that

$$C^n(X;G) \cong \mathrm{Hom}(C_n(X),G) \cong \{\text{maps } \{\Delta^n \to X\} \text{ to } G\}$$

In particular,

$$C^0(X;G) \cong \mathrm{Hom}(C_0(X),G) \cong \mathrm{Map}(\{\Delta^0 \to X\},G) \cong \{\text{functions } X \to G\}.$$

Now,

$$C^1(X;G) \cong \mathrm{Hom}(C_1(X),G) \cong \mathrm{Map}(\{\Delta^1 \to X\},G)$$

and the coboundary map is given by

$$\begin{array}{ccc} C^0(X;G) & \to & C^1(X;G) \\ (\varphi\colon X \to G) & \mapsto & (\delta\varphi\colon \{\Delta^1 \to X\} \to G) \end{array}$$

where $\delta\varphi(\psi) := \varphi(\psi(1)) = \varphi(\psi(0))$. Therefore

$$\begin{aligned} H^0(X;G) = \ker\delta = \{\varphi\colon X \to G \mid{} & \varphi(x) = \varphi(y) \\ & \text{whenever there is } \psi\colon \Delta^1 \to X \text{ with } \psi(0)=x, \psi(1)=y\} \\ & = \{\varphi\colon \{\text{connected components of } X\} = \pi_0(X) \to G\} \\ & = \prod_{|\pi_0(X)|} G \end{aligned}$$

An analogous computation holds for the zeroth CW cohomology of CW complexes. So zeroth cohomology is not too hard to understand: functions on X that are constant on path components.

For further discussion of an intuitive idea behind cohomology, I recommend reading the beginning of Chapter 3 of Hatcher. As an alternative, we give an explicit generator for the first cohomology of the circle S^1.

Example 3.13. Consider the real line $\mathbb{R}$. Consider the functions

$$\begin{aligned} \alpha_{\mathbb{R}}\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x \end{aligned}$$

in $C^0(\mathbb{R};\mathbb{R})$ (here the first $\mathbb{R}$ is a topological space and the second $\mathbb{R}$ is an abelian group) and

$$\begin{aligned} \alpha_{\mathbb{Z}}\colon \mathbb{R} &\to \mathbb{Z} \\ x &\mapsto \lfloor x \rfloor \end{aligned}$$

Let $\mu\colon \Delta^1 \to \mathbb{R}$ be a singular 1-simplex. Recall that $\Delta^1 = \{(1-t,t) \in \mathbb{R}^2 \mid t \in [0,1]\}$. Denote a point in Δ^1 by the value of t. Then

$$\delta\alpha_{\mathbb{R}}(\mu) = \alpha_{\mathbb{R}}(\partial\mu) = \alpha_{\mathbb{R}}(\mu(1)) - \alpha_{\mathbb{R}}(\mu(0)) = \mu(1) - \mu(0).$$

Also

$$\delta\alpha_{\mathbb{Z}}(\mu) = \alpha_{\mathbb{Z}}(\partial\mu) = \begin{cases} |\{n \in \mathbb{Z} \mid \mu(0) < n \le \mu(1)\}| & \text{if } \mu(0) \le \mu(1) \\ -|\{n \in \mathbb{Z} \mid \mu(1) < n \le \mu(0)\}| & \text{if } \mu(1) \le \mu(0). \end{cases}$$

This counts how many integers μ crosses from left to right, with sign, so that crossing an integer from right to left counts as -1.

Example 3.14. Now consider the circle S^1. Let $p\colon \mathbb{R} \to S^1$ be the function sending $t \to e^{2\pi i t} \in S^1 \subset \mathbb{C}$. Given a 1-simplex $\sigma\colon \Delta^1 \to S^1$, it lifts to a 1-simplex in $C_1(\mathbb{R};\mathbb{R})$

$$\widetilde{\sigma}\colon \Delta^1 \to \mathbb{R}$$

such that $p \circ \widetilde{\sigma} = \sigma\colon \Delta^1 \to S^1$. Such a lift is not unique; any two lifts differ by an integer $\widetilde{\sigma}(x) - \widetilde{\sigma}'(x) \in \mathbb{Z}$.

Define an element of $C^1(S^1;\mathbb{R})$ by

$$\theta_{\mathbb{R}}(\sigma) := \delta\alpha_{\mathbb{R}}(\widetilde{\sigma}) = \widetilde{\sigma}(1) - \widetilde{\sigma}(0).$$

Define an element of $C^1(S^1;\mathbb{Z})$ by

$$\theta_{\mathbb{Z}}(\sigma) := \delta\alpha_{\mathbb{Z}}(\widetilde{\sigma}) = \begin{cases} |\{n \in \mathbb{Z} \mid \widetilde{\sigma}(0) < n \leq \widetilde{\sigma}(1)\}| & \text{if } \widetilde{\sigma}(0) \leq \widetilde{\sigma}(1) \\ -|\{n \in \mathbb{Z} \mid \widetilde{\sigma}(1) < n \leq \widetilde{\sigma}(0)\}| & \text{if } \widetilde{\sigma}(1) \leq \widetilde{\sigma}(0). \end{cases}$$

You should check that this is well-defined i.e. does not depend on the choice of lift $\widetilde{\sigma}$. We have that $\theta_{\mathbb{R}} \in C^1(S^1;\mathbb{R})$ and $\theta_{\mathbb{Z}} \in C^1(S^1;\mathbb{Z})$

Now, consider the 1-simplex $\mu\colon \Delta^1 \to S^1$ defined by $\mu(t) = e^{2\pi i t}$. Lift it, to obtain for example $\widetilde{\mu}(t) = t$. Then $\widetilde{\mu}(1) - \widetilde{\mu(0)} = 1$, so

$$\theta_{\mathbb{R}}(\mu) = 1$$

and

$$\theta_{\mathbb{Z}}(\mu) = 1.$$

In the next lemma, we will show that $\theta_{\mathbb{R}}$ and $\theta_{\mathbb{Z}}$ represent nontrivial cohomology classes in $H^1(S^1;\mathbb{R})$ and $H^1(S^1;\mathbb{Z})$ respectively. Since they evaluate to 1 on $\mu = [S^1]$, this will imply that they generate these cohomology groups.

Lemma 3.15.

(1) *The singular 1-cochains $\theta_{\mathbb{R}}$ and $\theta_{\mathbb{Z}}$ are cocycles, so represent elements $\theta_{\mathbb{R}} \in H^1(S^1;\mathbb{R})$ and $\theta_{\mathbb{Z}} \in H^1(S^1;\mathbb{Z})$.*

(2) *$[\theta_{\mathbb{R}}]$ and $[\theta_{\mathbb{Z}}]$ are* not *coboundaries δc for any $c \in C^0(S^1;\mathbb{R})$, respectively $C^0(S^1;\mathbb{Z})$.*

Proof.

(1) We will show that $\theta_{\mathbb{R}}$ is a cocycle, that is $\delta_1\theta_{\mathbb{R}} = 0 \in C^2(S^1;\mathbb{R})$. A similar argument works for $\theta_{\mathbb{Z}}$. Let $\sigma\colon \Delta^2 \to S^1$. We want that $(\delta_1\theta_{\mathbb{R}})(\sigma) = \theta_{\mathbb{R}}(\partial\sigma) = 0$. Recall the function $p\colon \mathbb{R} \to S^1$ sending $t \to e^{2\pi i t} \in S^1 \subset \mathbb{C}$. Then σ lifts to $\widetilde{\sigma}\colon \Delta^2 \to \mathbb{R}$ and, with $\iota_j\colon \Delta^1 \to \Delta^2$ the jth face inclusion, we have

$$\begin{aligned} \theta_{\mathbb{R}}(\partial\sigma) &= \theta_{\mathbb{R}}\big(\sum_{j=0}^{2}(-1)^j \sigma \circ \iota_j\big) \\ &= (\delta\alpha_{\mathbb{R}})\big(\sum_{j=0}^{2}(-1)^j \widetilde{\sigma} \circ \iota_j\big) = \alpha_{\mathbb{R}}(\partial \circ \partial\widetilde{\sigma}) = \alpha_{\mathbb{R}}(0) = 0. \end{aligned}$$

(2) To show that $\theta_{\mathbb{R}}$ and $\theta_{\mathbb{Z}}$ are not coboundaries, let $\beta\colon C^0(S^1;\mathbb{R})$, and similarly $\beta \in C^0(S^1;\mathbb{Z})$. We can think of β as a function $\beta\colon S^1 \to \mathbb{R}$, respectively $\beta\colon S^1 \to \mathbb{Z}$. Consider the 1-chain $\mu\colon \Delta^1 \to S^1$ sending $t \mapsto e^{2\pi i t}$. Then
$$\delta_0(\beta)(\mu) = \beta(\partial\mu) = \beta(\mu(1) - \mu(0)) = \beta(1) - \beta(1) = 0.$$
But $\theta_{\mathbb{R}}(\mu) = 1$, and also $\theta_{\mathbb{Z}}(\mu) = 1$, so these cannot be the coboundaries of any 0-chain.

□

The cohomology class $\theta_{\mathbb{Z}} \in H^1(S^1;\mathbb{Z}) \cong \mathbb{Z}$ is a generator. It evaluates to 1 on the fundamental class. For this reason we refer to it as the dual fundamental class and write $\theta_{\mathbb{Z}} = [S^1]^* \in H^1(S^1;\mathbb{Z})$.

Definition 3.16. Let M be a compact, oriented n-dimensional manifold. Call the canonical generator
$$[M]^* \in H^n(M, \partial M;\mathbb{Z})$$
the *dual fundamental class* of M, with the property that $\langle [M]^*, [M]\rangle = 1$.

3.6. **Properties of cohomology.** The following basic properties of cohomology are closely related to the homology versions. Let G be an abelian group.

(1) Let $f\colon (X,A) \to (Y,B)$ be maps of pairs of spaces. Then for all $n \in \mathbb{N}_0$, $f_*\colon C_n(X,A) \to C_n(Y,B)$ is a chain map, which induces a map on cohomology
$$f^*\colon H^n(Y,B;G) \to H^n(X,A;G).$$
(2) The assigment $(X,A) \to H^n(X,A;G)$ defines a contravariant functor.
(3) Let $f,g\colon (X,A) \to (Y,B)$ be homotopic maps of pairs. They induce chain homotopic chain maps, and therefore cochain homotopic cochain maps, whence equal maps on cohomology:
$$f^* = g^*\colon H^n(Y,B;G) \to H^n(X,A;G).$$
(4) Let $f\colon X \to Y$ be a homotopy equivalence. Then it follows from the previous item that
$$f^*\colon H^n(Y,B;G) \to H^n(X,A;G)$$
is an isomorphism of groups.
(5) Let (X,A) be a pair of spaces. Then there is a long exact sequence:
$$\begin{aligned} &0 \to H^0(X,A;G) \to H^0(X;G) \to H^0(A;G) \\ &\xrightarrow{\delta^*} H^1(X,A) \to H^1(X;G) \to H^1(A;G) \\ &\to \cdots \\ &\xrightarrow{\delta^*} H^n(X,A) \xrightarrow{q^*} H^n(X;G) \xrightarrow{i^*} H^n(A;G) \\ &\xrightarrow{\delta^*} H^{n+1}(X,A) \to H^{n+1}(X;G) \to H^{n+1}(A;G) \\ &\to \cdots \end{aligned}$$

with $i\colon A \to X$ the inclusion map, $q_*\colon C_*(X) \to C_*(X)/C_*(A) = C_*(X,A)$ the quotient map, and δ^* the connecting homomorphism.

We also have excision for cohomology.

Theorem 3.17 (Excision for cohomology)**.** *Let X be a space, and let G be an abelian group. Let $Z \subset A \subset X$ be subsets such that the closure of Z is contained in the interior of A. Then the map of pairs*

$$i\colon (X \smallsetminus Z, A \smallsetminus Z) \to (X, A)$$

induces an isomorphism

$$i^*\colon H^n(X, A; G) \xrightarrow{\cong} H^n(X \smallsetminus Z, A \smallsetminus Z; G)$$

for every $n \in \mathbb{N}_0$.

For example, let M be a manifold and let $p \in M$ be an interior point. Let U be an open subset containing p that is homeomorphic to $\mathbb{R}^n$. Then with G coefficients

$$\begin{aligned} H^n(M, M \smallsetminus \{p\}) &\cong H^n(M \smallsetminus (M \smallsetminus U), (M \smallsetminus \{p\}) \smallsetminus (M \smallsetminus U)) \cong H^n(U, U \smallsetminus \{p\}) \\ &\cong H^n(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \{0\}) \cong G. \end{aligned}$$

So as with homology, the local homology of a manifold does not depend on the point p, and is isomorphic to G in degree equal to the dimension of the manifold.

We also have a Mayer-Vietoris sequence for cohomology. It is similar to the Mayer-Vietoris sequence for homology, but the arrows are reversed. We will define excisive triads after stating the theorem.

Theorem 3.18 (Mayer-Vietoris sequence)**.** *Let (X, A, B) be an* excisive triad. *Let G be an abelian group, and suppose that $X = A \cup B$. Then there is a natural long exact sequence*

$$\begin{aligned} \cdots \to H^n(X; G) \xrightarrow{i^* \oplus i^*} H^n(A; G) \oplus H^n(B; G) \xrightarrow{i^* \oplus -i^*} H^n(A \cap B; G) \\ \xrightarrow{d} H^{n+1}(X; G) \to \cdots \end{aligned}$$

where d is the connecting homomorphism.

What is an excisive triad? Let (X, A_1, A_2) be a triad, i.e. $A_1, A_2 \subseteq X$ and $A_1 \cup A_2 = X$. Define

$$C_n^{\{A_1, A_2\}}(X) := \Big\{ \sum_{j=1}^m a_j \sigma_j \in C_n(X) \;\Big|\; \text{for each } j, \text{ there is } i \in \{1,2\} \text{ with } \operatorname{im}(\sigma_j) \subseteq A_i \Big\}.$$

Definition 3.19. The triad (X, A_1, A_2) is *excisive* if inclusion induces a chain homotopy equivalence

$$C_*^{\{A_1, A_2\}}(X) \to C_*(X).$$

Proposition 3.20. *Let (X, A_1, A_2) be a triad. Suppose one of the following holds.*

(i) $A_1 \subseteq A_2$.
(ii) $A_2 \subseteq A_1$.
(iii) $A_1 = \emptyset$.
(iv) $A_2 = \emptyset$.

(v) $A_1 = A_2$.
(vi) A_1 *and* A_2 *are open in* X.
(vii) X *is a CW complex and* A_1 *and* A_2 *are subcomplexes.*
(viii) X *is a manifold and* A_1 *and* A_2 *are closed submanifolds, with* $A_1 \cap A_2$ *a boundary component of* A_1 *and* A_2.

Then (X, A_1, A_2) *is excisive.*

The idea is roughly that one can define a homotopy inverse to the chain map $C_*^{\{A_1,A_2\}}(X) \to C_*(X)$ by subdividing simplices in X so that the image of every simplex lies in A_1, A_2, or both. Now for excisive triads we can prove the Mayer-Vietoris theorem.

Proof of Theorem 3.18. Suppose that (X, A, B) is an excisive triad with $X = A \cup B$. Then

$$0 \to C_*(A \cap B) \xrightarrow{i_* \oplus -i_*} C_*(A) \oplus C_*(B) \to C_*^{\{A,B\}}(X) \to 0$$

is a short exact sequence. Therefore

$$0 \to \operatorname{Hom}(C_*^{\{A,B\}}(X), G) \to \operatorname{Hom}(C_*(A), G) \oplus \operatorname{Hom}(C_*(B), G) \to \operatorname{Hom}(C_*(X), G) \to 0$$

is exact, since $C_*^{\{A,B\}}(X)$ is free abelian. Now this short exact sequence of cochain complexes gives rise to a long exact sequence in cohomology. Note that

$$H^n(\operatorname{Hom}(C_*^{\{A,B\}}(X), G)) \cong H^n(\operatorname{Hom}(C_*(X), G)) \cong H^n(X; G)$$

since $C_*^{\{A,B\}}(X)$ and $C_*(X)$ are chain homotopy equivalent. Therefore we obtain the long exact sequence

$$\begin{aligned} \cdots \to H^n(X; G) &\xrightarrow{i^* \oplus i^*} H^n(A; G) \oplus H^n(B; G) \xrightarrow{i^* \oplus -i^*} H^n(A \cap B; G) \\ &\xrightarrow{d} H^{n+1}(X; G) \to \cdots \end{aligned}$$

required. □

4. Relationships between homology and cohomology

There are two relationships between homology and cohomology. We state them here, to give you the idea, and then we explain them. We will study the first one, the universal coefficient theorem (UCT), in more detail shortly. The UCT is for any topological space. The second relationship, Poincaré duality, is only for manifolds. We will return to studying this in more detail later, because we need the cap product to do so properly. But for context, and to have the two main relationships between homology and cohomology in one place, we state both now.

Roughly, in the case $G = \mathbb{Z}$ the Ext group $\operatorname{Ext}^1(H_{n-1}(X); \mathbb{Z})$ appearing in the next theorem is the *torsion* in $H_{n-1}(X)$. For an abelian group P, the torsion subgroup TP is the subgroup of elements $p \in P$ such that $np = 0$ for some $n \in \mathbb{Z}$. We will study Ext groups in detail soon, so this rough description is to give you enough of an idea to parse the next theorem.

Theorem 4.1 (Universal coefficient theorem). *Let X be a topological space and let G be an abelian group. There is a split, natural, short exact sequence for each $n \in \mathbb{N}_0$*

$$0 \to \operatorname{Ext}^1(H_{n-1}(X), G) \to H^n(X; G) \to \operatorname{Hom}(H_n(X), G) \to 0.$$

In the statement, *split* means that there is a map $\operatorname{Hom}(H_n(X), G) \to H^n(X; G)$ which when post-composed with the map back to $\operatorname{Hom}(H_n(X), G)$ yields the identity on $\operatorname{Hom}(H_n(X), G)$. This implies that

$$H^n(X; G) \cong \operatorname{Hom}(H_n(X), G) \oplus \operatorname{Ext}^1(H_{n-1}(X), G).$$

The *naturality* means that if there is a map $X \to Y$, there is an associated map of short exact sequences induced by that map. But this map of short exact sequences does not respect the splitting, in general. Naturality will be rather unimportant in this course.

As mentioned above, the idea for $\mathbb{Z}$ coefficients is that $\operatorname{Hom}(H_n(X), \mathbb{Z})$ sees the "free part" of $H_n(X)$, that is $H_n(X)/TH_n(X)$, whereas $\operatorname{Ext}^1(H_{n-1}(X), \mathbb{Z})$ sees the torsion subgroup $TH_{n-1}(X)$.

For example in $\mathbb{Z}^6 \oplus \mathbb{Z}/7 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/9$, $\mathbb{Z}^6$ might be called the "free part" (but beware this is not canonically defined) and the torsion subgroup is $\mathbb{Z}/7 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/9$. This is a canonically defined subgroup.

Here is the second important relationship between homology and cohomology.

Theorem 4.2 (Poincaré duality). *Let M be a compact, orientable, n-dimensional manifold, possibly with $\partial M \neq \emptyset$. There are isomorphisms*

$$H^{n-r}(M, \partial M; \mathbb{Z}) \cong H_r(M; \mathbb{Z})$$

and

$$H^{n-r}(M; \mathbb{Z}) \cong H_r(M, \partial M; \mathbb{Z})$$

for $0 \leq r \leq n$.

If $\partial M = \emptyset$, so that M is closed, then we have $H^{n-r}(M; \mathbb{Z}) \cong H_r(M; \mathbb{Z})$. This also holds for other coefficient groups. In particular, using $\mathbb{Z}/2$ coefficients, we can drop the assumption that M be orientable.

Theorem 4.3 ($\mathbb{Z}/2$ Poincaré duality). *Let M be a compact, n-dimensional manifold, possibly with $\partial M \neq \emptyset$. There are isomorphisms*

$$H^{n-r}(M, \partial M; \mathbb{Z}/2) \cong H_r(M; \mathbb{Z}/2)$$

and

$$H^{n-r}(M; \mathbb{Z}/2) \cong H_r(M, \partial M; \mathbb{Z}/2)$$

for $0 \leq r \leq n$.

4.1. **An application.** Let M be a closed, connected, oriented 4-dimensional manifold with $\pi_1(M) = \{1\}$, so that $H_1(M; \mathbb{Z}) = 0$. We try to describe the homology and cohomology of M. Since M is connected and is a closed, oriented 4-manifold, we know that

$$H_0(M; \mathbb{Z}) \cong H^0(M; \mathbb{Z}) \cong H_4(M; \mathbb{Z}) \cong H^4(M; \mathbb{Z}).$$

By Poincaré duality, we have
$$H^3(M;\mathbb{Z}) \cong H_1(M;\mathbb{Z}) = 0.$$
We have
$$H^1(M;\mathbb{Z}) \cong \mathrm{Hom}(H_1(M;\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Ext}^1(H_0(M;\mathbb{Z}),\mathbb{Z}) = \mathrm{Hom}(0,\mathbb{Z}) \oplus \mathrm{Ext}^1(\mathbb{Z},Z).$$
Since $\mathbb{Z}$ is torsion-free, $\mathrm{Ext}^1(\mathbb{Z},\mathbb{Z}) = 0$, so $H^1(M;\mathbb{Z}) = 0$. This implies by Poincaré duality that
$$H_3(M;\mathbb{Z}) \cong H^1(M;\mathbb{Z}) = 0.$$
Finally,
$$\begin{aligned} H_2(M;\mathbb{Z}) \cong H^2(M;\mathbb{Z}) &\cong \mathrm{Hom}(H_2(M;\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Ext}^1(H_1(M;\mathbb{Z}),\mathbb{Z}) \\ &\cong \mathrm{Hom}(H_2(M;\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Ext}^1(0,\mathbb{Z}) \cong \mathrm{Hom}(H_2(M;\mathbb{Z}),\mathbb{Z}). \end{aligned}$$
Since $\mathrm{Hom}(P,\mathbb{Z})$ is torsion free for every finitely generated abelian group P, this means that
$$H_2(M;\mathbb{Z}) \cong H^2(M;\mathbb{Z}) \cong \mathbb{Z}^n$$
for some n. So just the knowledge that a 4-manifold has $H_1(M;\mathbb{Z}) = 0$ enabled us to deduce quite a lot.

5. Universal coefficient theorem

As promised, we proceed to explain and prove the universal coefficient theorem.

5.1. Ext groups.

Let $F\colon \mathsf{Ab} \to \mathsf{Ab}$ be a contravariant functor between abelian groups.

Definition 5.1.

(i) We say that the functor F is *left exact* if whenever
$$A \to B \to C \to 0$$
is exact, then
$$0 \to F(C) \to F(B) \to F(A)$$
is exact.

(ii) We say that F is *right exact* if whenever
$$0 \to A \to B \to C$$
is exact, then
$$F(C) \to F(B) \to F(A) \to 0$$
is exact.

Lemma 5.2. *Let G be an abelian group. Then* $\mathrm{Hom}(-,G)$ *is left exact.*

This was already part of what was asked in Lemma 3.3, but with the extra assumption that C is free abelian. In the case that C is free abelian, the sequence $0 \to A \to B \to C \to 0$ splits, so $B \cong A \oplus C$, and then $\mathrm{Hom}(B,G) \cong \mathrm{Hom}(A,G) \oplus \mathrm{Hom}(C,G)$. A bit more work shows this version. Let us now prove the more general Lemma 5.2, without restrictions on C.

Proof. Let

$$A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be exact. Then consider the sequence of maps

$$0 \to \mathrm{Hom}(C, G) \xrightarrow{g^*} \mathrm{Hom}(B, G) \xrightarrow{f^*} \mathrm{Hom}(A, G).$$

We want to show that this is an exact sequence. Let $\varphi \in \ker g^*$. That is, $\varphi \circ g \colon B \to G$ is the zero map. Let $c \in C$. Then there is a $b \in B$ such that $g(b) = c$. Then $\varphi(c) = \varphi \circ g(b) = 0$, so $\varphi = 0$. Thus g^* is injective.

To show exactness at $\mathrm{Hom}(B, G)$, suppose that $\theta = g^*(\varphi) \in \mathrm{Hom}(B, G)$. Then $f^* \circ g^*(\varphi)(a) = \varphi \circ g \circ f(a) = \varphi(0) = 0$ for all $a \in A$. Therefore $f^* \circ g^* = 0$. Now let $\theta \in \ker f^* \subseteq \mathrm{Hom}(B, G)$. That is, $\theta \circ f \colon A \to G$ is the zero map. We want to show that $\theta = g^*(\varphi)$ for some $\varphi \colon C \to G$. Define $\varphi(c)$, for $c \in C$, by taking $b \in B$ with $g(b) = c$ and defining $\varphi(c) = \theta(b)$. Suppose that $b' \in B$ with $g(b') = c$ too. Then $\theta(b) - \theta(b') = \theta(b - b')$. Since $b - b' \in \ker g$, we have that $b - b' = f(a)$ for some $a \in A$. Then $\theta(b - b') = \theta(f(a)) = 0$, so $\theta(b) = \theta(b')$. Therefore $\varphi(c)$ is well-defined, and does not depend on the choice of b. This shows that the sequence is exact at $\mathrm{Hom}(B, G)$. □

Example 5.3. Consider the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to \mathbb{Z}/p \to 0$$

for $p > 1$ an integer. Then taking $\mathrm{Hom}(-, \mathbb{Z})$ yields

$$0 \to \mathrm{Hom}(\mathbb{Z}/p, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\cdot p} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \to 0$$

Now $\mathrm{Hom}(\mathbb{Z}/p, \mathbb{Z}) = 0$ and $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$, so this yields

$$0 \to 0 \to \mathbb{Z} \xrightarrow{\cdot p}\to \mathbb{Z} \to 0.$$

This is *not exact*, since multiplication by p is not onto. So we see that $\mathrm{Hom}(-\mathbb{Z})$ is not right exact.

To measure the failure of $\mathrm{Hom}(-, G)$ to be right exact, we define the Ext groups.

Definition 5.4. Now we define the Ext groups of a pair of abelian groups H, G. A sequence of groups

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

with F_i a free abelian group, and $\ker f_i = \mathrm{im}\, f_{i+1}$ for all i, is called a *free resolution* of H. Then

$$\mathrm{Ext}^n(H, G) := H^n(F_*; G).$$

That is replace H by 0, apply $\mathrm{Hom}(-, G)$ to the resulting chain complex, and then take cohomology, to get $H^n(\mathrm{Hom}(F_*, G))$.

We shall be principally concerned with Ext^1 and Ext^0, since for abelian groups these are the only ones that are nonzero, as we will soon see.

Let H be a finitely generated abelian group. Then by the classification of finitely generated abelian groups,

$$H \cong \mathbb{Z}^n \oplus \bigoplus_{i=1}^{k} \mathbb{Z}/p_i^{n_i}$$

for some n, for some k, for some primes $p_1, \ldots, p_k$, and for some integers $n_1, \ldots, n_k$. There is a resolution of length one

$$0 \to \bigoplus_{i=1}^{k} \mathbb{Z} \xrightarrow{\left(0, \bigoplus_{i=1}^{k} p_i^{n_i}\right)} \mathbb{Z}^n \oplus \bigoplus_{i=1}^{k} \mathbb{Z} \to H \to 0.$$

The next lemma is a baby version of what is often called the fundamental lemma of homological algebra. It will be sufficient to prove that the groups $\mathrm{Ext}^1(H, G)$ are well-defined for finitely generated abelian groups H.

Lemma 5.5. *Given two resolutions*

$$0 \to F_1 \to F_0 \to H \to 0$$

and

$$0 \to F_1' \to F_0' \to H \to 0$$

of H, and a homomorphism $\varphi\colon H \to H$, we have the following.

(1) There is a chain map between the resolutions inducing φ.
(2) Any two such chain maps are chain homotopic.

Proof. We need to construct a map as shown in the next diagram.

$$\begin{tikzcd} F_1 \arrow[r, "f_1"] & F_0 \arrow[r, "f_0"] \arrow[d, dashed, "\varphi_0"] & H \arrow[r] \arrow[d, "\varphi"] & 0 \\ F_1' \arrow[r, "f_1'"] & F_0' \arrow[r, "f_0'"] & H \arrow[r] & 0 \end{tikzcd}$$

Let $x \in F_0$ be an element of a generating set. Note that $\varphi(f_0(x)) \in H$. Choose $x' \in F_0'$ with $f_0'(x') = \varphi(f_0(x)) \in H$. Define $\varphi_0(x) = x'$. Do this for all the free generators of F_0, and then extend by linearity. This defines $\varphi_0\colon F_0 \to F_0'$. Now we want to define the map φ_1 as shown in the next diagram.

$$\begin{tikzcd} F_1 \arrow[r, "f_1"] \arrow[d, dashed, "\varphi_1"] & F_0 \arrow[r, "f_0"] \arrow[d, "\varphi_0"] & H \arrow[r] \arrow[d, "\varphi"] & 0 \\ F_1' \arrow[r, "f_1'"] & F_0' \arrow[r, "f_0'"] & H \arrow[r] & 0 \end{tikzcd}$$

Let $y \in F_1$. Then $f_0 \circ f_1(y) = 0$, so $\varphi \circ f_0 \circ f_1(y) = 0$. By commutativity, $f_0' \circ \varphi_0 \circ f_1(y) = 0$. By exactness, there exists $y' \in F_1'$ with $f_1'(y') = \varphi_0 \circ f_1(y)$. Define

$$\varphi_1(y) = y'.$$

Do this for a generating set of F_1, and extend by linearity. This defines $\varphi_1\colon F_1 \to F_1'$. The map $\varphi_*\colon F_* \to F_*'$ is a chain map by construction.

Now we want to show that any two such maps are chain homotopic.

$$
\begin{tikzcd}
F_1 \arrow[r, "f_1"] \arrow[d, shift right, "\varphi_1'"'] \arrow[d, shift left, "\varphi_1"] & F_0 \arrow[r, "f_0"] \arrow[d, shift right, "\varphi_0'"'] \arrow[d, shift left, "\varphi_0"] \arrow[dl, dashed, "g"'] & H \arrow[r] \arrow[d, "\varphi"] \arrow[dl, dashed, "0"'] & 0 \\
F_1' \arrow[r, "f_1'"] & F_0' \arrow[r, "f_0'"] & H \arrow[r] & 0
\end{tikzcd}
$$

We want to define the chain homotopy g shown. Let $x \in F_0$. Then

$$f_0'(\varphi_0(x) - \varphi_0'(x)) = \varphi(f_0(x)) - \varphi(f_0(x)) = 0.$$

Therefore there exists a $y' \in F_1'$ with $f_1'(y') = \varphi_0(x) - \varphi_0'(x)$. Define

$$g(x) = y'.$$

Do this for a generating set of F_0, and extend by linearity. This defines $g\colon F_0 \to F_1'$. It has the property that $f_1' \circ g = \varphi_0 - \varphi_0'$, so satisfies the requirement for a chain homotopy at F_0. Now, let $y \in F_1$. We want to show that

$$\varphi_1(y) - \varphi_1'(y) = g \circ f_1(y).$$

Since f_1' is injective, it is enough to show that

$$f_1'(\varphi_1(y) - \varphi_1'(y)) = f_1' \circ g \circ f_1(y).$$

Now we compute:

$$
\begin{aligned}
f_1'(\varphi_1(y) - \varphi_1'(y)) &= f_1' \circ \varphi_1(y) - f_1' \circ \varphi_1'(y) = \varphi_0 \circ f_1(y) - \varphi_0' \circ f_1(y) \\
&= (\varphi_0 - \varphi_0')(f_1(y)) = f_1' \circ g \circ f_1(y),
\end{aligned}
$$

as required. Therefore g is a chain homotopy. □

Now let F_*, F_*' be two resolutions of H. With $\varphi = \mathrm{Id}$, let $\varphi_*\colon F \to F'$ be a chain map inducing φ, and let $\psi_*\colon F_*' \to F_*$ be a chain map inducing $\psi = \mathrm{Id}$. Then $\varphi_* \circ \psi_*, \mathrm{Id}$ and $\psi_* \circ \varphi_*, \mathrm{Id}$ are two chain maps inducing Id on H. Then for both pairs, the two chain maps are chain homotopic. Therefore $\varphi_* \circ \psi_* \sim \mathrm{Id}$ and $\psi_* \circ \varphi_* \sim \mathrm{Id}$, so $F_* \to H \to 0$ and $F_*' \to H \to 0$ are chain homotopy equivalent. It follows, since applying $\mathrm{Hom}(-, G)$ to a chain equivalence yields a chain equivalence, that the Ext groups

$$\mathrm{Ext}^1(H, G), \mathrm{Ext}^0(H, G)$$

are well-defined, for every abelian group G. That is, for any two chain resolutions of H we obtain the same group $\mathrm{Ext}^i(H, G)$, $i = 0, 1$, up to canonical isomorphism.

Now that we know the Ext groups Ext^0 and Ext^1 are well-defined, let us see some examples.

Example 5.6.

(1) We compute $\mathrm{Ext}^n(\mathbb{Z}, \mathbb{Z})$. There is a free resolution

$$0 \to F_1 = 0 \to F_0 = \mathbb{Z} \to \mathbb{Z} \to 0.$$

So $F_* = 0 \to \mathbb{Z} \to 0$, supported in degree zero. Taking $\mathrm{Hom}(-, \mathbb{Z})$ yields

$$0 \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \to 0.$$

So

$$\mathrm{Ext}^0(\mathbb{Z}, \mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$$

and

$$\mathrm{Ext}^i(\mathbb{Z}, \mathbb{Z}) = 0 \text{ for } i > 0.$$

(2) We compute $\mathrm{Ext}^n(\mathbb{Z}/p, \mathbb{Z})$. There is a free resolution

$$0 \to F_1 = \mathbb{Z} \xrightarrow{\cdot p} F_0 = \mathbb{Z} \to \mathbb{Z}/p \to 0.$$

So $F_* = 0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to 0$, supported in degrees zero and one. Taking $\mathrm{Hom}(-, \mathbb{Z})$ yields

$$0 \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\cdot p} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \to 0.$$

So

$$\mathrm{Ext}^0(\mathbb{Z}/p, \mathbb{Z}) = 0,$$
$$\mathrm{Ext}^1(\mathbb{Z}/p, \mathbb{Z}) \cong \mathbb{Z}/p,$$

and

$$\mathrm{Ext}^i(\mathbb{Z}/p, \mathbb{Z}) = 0 \text{ for } i > 1.$$

The fact that $\mathrm{Ext}^1(\mathbb{Z}/p, \mathbb{Z}) \cong \mathbb{Z}/p$ is nonzero corresponds to the failure of $\mathrm{Hom}(-, \mathbb{Z})$ to be right exact observed in Example 5.3.

Next, we list some of the main important properties of the Ext groups.

Proposition 5.7. *Let G and H be abelian groups.*

(1) $\mathrm{Ext}^0(H, G) \cong \mathrm{Hom}(H, G)$.
(2) $\mathrm{Ext}^i(H, G) = 0$ *for* $i \geq 2$.
(3) $\mathrm{Ext}^1(H, G) = \mathrm{coker}\big(\mathrm{Hom}(F_0, G) \to \mathrm{Hom}(F_1, G)\big)$, *where* $0 \to F_1 \to F_0 \to H \to 0$ *is a free resolution of length one.*
(4) If H is a free abelian group, then $\mathrm{Ext}^1(H, G) = 0$.
(5) $\mathrm{Ext}^1(\mathbb{Z}/n, G) \cong G/nG$.
(6) If H is a finitely generated abelian group, then $\mathrm{Ext}^1(H, \mathbb{Z})$ *is the torsion subgroup of H, that is* $TH := \{h \in H \mid \exists\, n \in \mathbb{N} \setminus \{0\} \text{ with } nh = 0\}$.
(7) $\mathrm{Ext}^1(H, \mathbb{Q}) = 0$ *for any H.*
(8) Let $H_1, \ldots, H_k$ and $G_1, \ldots, G_k$ be abelian groups. Then

$$\mathrm{Ext}^1\Big(\bigoplus_{i=1}^k H_i, G\Big) \cong \bigoplus_{i=1}^k \mathrm{Ext}^1(H_i, G)$$

and

$$\mathrm{Ext}^1\Big(H, \bigoplus_{i=1}^k G_i\Big) \cong \bigoplus_{i=1}^k \mathrm{Ext}^1(H, G_i).$$

We will not prove all of these in detail.

Proof. (1) Consider a resolution $F_1 \to F_0 \to H \to 0$. This is exact. Therefore

$$0 \to \mathrm{Hom}(H, G) \to \mathrm{Hom}(F_0, G) \to \mathrm{Hom}(F_1, G)$$

is exact since $\mathrm{Hom}(-, G)$ is left exact. Therefore

$$\mathrm{Hom}(H, G) = \ker(\mathrm{Hom}(F_0, G) \to \mathrm{Hom}(F_1, G)) = \mathrm{Ext}^0(H, G)$$

where the last equality is by definition.

(2) Since H is abelian, there is a resolution $0 \to F_1 \to F_0 \to H \to 0$. Then

$$\mathrm{Ext}^i(H, G) = H^i(\mathrm{Hom}(F_0, G) \to \mathrm{Hom}(F_1, G))$$

which, since the nonzero groups are in degrees zero and one, equals 0 for $i \geq 2$.

(3) We have

$$\mathrm{Ext}^1(H, G) = H^1(\mathrm{Hom}(F_0, G) \to \mathrm{Hom}(F_1, G)) = \mathrm{coker}(\mathrm{Hom}(F_0, G) \to \mathrm{Hom}(F_1, G)).$$

(4) Since H is free abelian, there is a free resolution with $F_0 = H$ and $F_i = 0$ for $i \geq 1$.

(5) We have

$$\mathrm{Ext}^1(\mathbb{Z}/n, G) = H^1(\mathrm{Hom}(\mathbb{Z}, G) \xrightarrow{\cdot n} \mathrm{Hom}(\mathbb{Z}, G)) = \mathrm{coker}(G \xrightarrow{\cdot n} G) \cong G/nG.$$

(6) Let H be a finitely generated abelian group. As noted above, by the classification of finitely generated abelian groups,

$$H \cong \mathbb{Z}^n \oplus \bigoplus_{i=1}^{k} \mathbb{Z}/p_i^{n_i}$$

for some n, for some k, and for some primes $p_1, \dots, p_k$ and some integers $n_1, \dots, n_k$. The torsion subgroup is $\bigoplus_{i=1}^{k} \mathbb{Z}/p_i^{n_i}$ There is a resolution of length one

$$0 \to \bigoplus_{i=1}^{k} \mathbb{Z} \xrightarrow{\left(0, \bigoplus_{i=1}^{k} p_i^{n_i}\right)} \mathbb{Z}^n \oplus \bigoplus_{i=1}^{k} \mathbb{Z} \to H \to 0.$$

Applying $\mathrm{Hom}(-, \mathbb{Z})$ to this yields

$$0 \to \mathbb{Z}^n \oplus \bigoplus_{i=1}^{k} \mathbb{Z} \xrightarrow{\left(0, \bigoplus_{i=1}^{k} p_i^{n_i}\right)} \bigoplus_{i=1}^{k} \mathbb{Z} \to 0.$$

Taking H^1 yields

$$\mathrm{Ext}^1(H, \mathbb{Z}) \cong \bigoplus_{i=1}^{k} \mathbb{Z}/p_i^{n_i} = TH,$$

the torsion subgroup of H.

(7) I am omitting the proof of this. Note that it uses that $\mathbb{Q}$ is *divisible*, and $\mathbb{Q}$ can be replaced with any divisible abelian group here.

(8) I am also omitting a detailed proof of this. It is not too hard if one follows through the definitions.

□

Now we move on to the universal coefficient theorem, which was the reason for introducing the Ext groups. One of the maps in the theorem is the evaluation map, so let us define it.

Definition 5.8. Let (C_*, ∂) be a chain complex of free abelian groups, and let G be an abelian group. The evaluation map is

$$\begin{array}{rcl} \mathrm{ev}\colon H^n(C;G) & \to & \mathrm{Hom}(H_n(C), G) \\ [\varphi\colon C_n \to G] & \mapsto & \left(\begin{array}{rcl} H_n(C) & \to & G \\ [c] & \mapsto & \langle [\varphi], [c] \rangle = \varphi(c). \end{array} \right) \end{array}$$

Here the pairing $\langle [\varphi], [c] \rangle = \varphi(c)$ is often called the Kronecker pairing. It simply means evaluate a representative cocycle of a homology class on a representative cycle of a homology class.

In order to make the last "definition" really a definition, we should show that ev is well-defined.

Lemma 5.9. *The map* ev *is a well-defined group homomorphism.*

Proof. We will just show that it is well-defined, considering the effect of changes in representative cocycles and cycles.

$$\begin{aligned} (\varphi + \delta\psi)(c + \partial d) &= \varphi(c) + \delta\psi(c) + \varphi(\partial d) + \delta\psi(\partial d) \\ &= \varphi(c) + \psi(\partial c) + \delta\varphi(d) + \psi(\partial^2 d) = \varphi(c). \end{aligned}$$

Here $\partial c = 0$ because c is a cycle and $\delta\varphi = 0$ because φ is a cocycle. Then $\partial^2 = 0$, so the last term vanishes as well. □

The universal coefficient theorem for spaces will follow directly from the following purely algebraic statement.

Theorem 5.10. *Let (C_*, ∂) be a chain complex of free abelian groups, and let G be an abelian group. For each $n \in \mathbb{N}_0$, there is a natural short exact sequence*

$$0 \to \mathrm{Ext}^1(H_{n-1}(C), G) \to H^n(C;G) \xrightarrow{\mathrm{ev}} \mathrm{Hom}(H_n(C), G) \to 0$$

that splits, i.e.

$$H^n(C;\Gamma) \cong \mathrm{Ext}^1(H_{n-1}(C), G) \oplus \mathrm{Hom}(H_n(C), G).$$

Proof. Let (C_*, ∂) be a chain complex of free abelian groups, and let G be an abelian group. Write $Z_n := \ker \partial_n$ and $B_n := \operatorname{im} \partial_{n+1}$. Both are also free abelian groups. We have a short exact sequence of chain complexes. That is the rows are exact of the next diagram.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Z_n \arrow[r] \arrow[d, "0"] & C_n \arrow[r, "\partial_n"] \arrow[d, "\partial_n"] & B_{n-1} \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d] & B_{n-2} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Apply $\mathrm{Hom}(-, G)$ to obtain the next diagram. The rows are again exact because B_n is free abelian.

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \mathrm{Hom}(B_{n-1}, G) \arrow[r, "\partial_n^*"] \arrow[u] & \mathrm{Hom}(C_n, G) \arrow[r] \arrow[u] & \mathrm{Hom}(Z_n, G) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathrm{Hom}(B_{n-2}, G) \arrow[r] \arrow[u, "0"] & \mathrm{Hom}(C_{n-1}, G) \arrow[r] \arrow[u, "\partial_n^*"] & \mathrm{Hom}(Z_{n-1}, G) \arrow[r] \arrow[u, "0"] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

By the snake lemma, we obtain a long exact sequence in cohomology. However the left and right vertical sequences have all coboundary maps trivial, so the cohomology is equal to the cochain groups. We therefore have a long exact sequence:

$$\cdots \to \mathrm{Hom}(Z_{n-1}, G) \xrightarrow{d_{n-1}} \mathrm{Hom}(B_{n-1}, G) \to H^n(C; G) \to \mathrm{Hom}(Z_n, G) \xrightarrow{d_n} \mathrm{Hom}(B_n, G) \to \cdots$$

Here d_n, d_{n-1} denotes the connecting homomorphism. For $m = n, n-1$ we assert that $d_m = i_m^*$, where $i_m \colon : B_m \to Z_m$ is the inclusion. This is a straightforward check using the definition of the connecting homomorphism. We therefore have a short exact sequence

$$0 \to \mathrm{coker}(d_{n-1}) \to H^n(C; G) \to \ker(d_n) \to 0.$$

There is a resolution

$$0 \to B_{n-1} \xrightarrow{i_{n-1} = d_{n-1}^*} Z_{n-1} \to H_{n-1}(C) \to 0.$$

Therefore

$$\mathrm{coker}(d_{n-1}) = \mathrm{Ext}^1(H_{n-1}(C), G).$$

Similarly there is a resolution

$$0 \to B_n \xrightarrow{i_n = d_n^*} Z_n \to H_n(C) \to 0.$$

Therefore
$$\ker(d_n) = \mathrm{Ext}^0(H_n(C), G) \cong \mathrm{Hom}(H_n(C), G).$$
Putting this together yields a short exact sequence
$$0 \to \mathrm{Ext}^1(H_{n-1}(C), G) \to H^n(C; G) \to \mathrm{Hom}(H_n(C), G) \to 0$$
as desired. I will omit the proof that this is also a split sequence, and that it is natural, for time reasons. □

We end the discussion of the universal coefficient theorem by noting that with field coefficients, the relationship between homology and cohomology is much simpler.

Theorem 5.11. *Let (X, A) be a pair of topological spaces. Let $\mathbb{F}$ be a field. Then*
$$\begin{array}{rcl} \mathrm{ev}\colon H^n(X, A; \mathbb{F}) & \to & \mathrm{Hom}_{\mathbb{F}}(H_n(X, A; \mathbb{F}), \mathbb{F}) \\ \varphi & \mapsto & \left(\begin{array}{rcl} H_n(X, A; \mathbb{F}) & \to & \mathbb{F} \\ \sigma & \mapsto & \langle \varphi, \sigma \rangle \end{array} \right) \end{array}$$
is an isomorphism for every $n \in \mathbb{N}_0$.

6. Tensor products, Tor and the Universal coefficient theorem for homology

6.1. Tensor products.

Let A and B be abelain groups. Then the tensor product is a quotient of the free abelian group generated by symbols of the form $a_i \otimes b_i$, with a_i in A and b_i in B
$$A \otimes B = \{\sum_{i=1}^{n} a_i \otimes b_i\}/\sim$$
where the relations $\sim$ are generated by
$$(a + a') \otimes b = a \otimes b + a' \otimes b$$
and
$$a \otimes (b + b') = a \otimes b + a \otimes b'.$$
Here are some facts on tensor products. The proofs are omitted, and you should think about them to help you understand tensor products.

(1) $A \otimes B \cong B \otimes A$.
(2) $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$.
(3) $\mathbb{Z} \otimes A \cong A$.
(4) $\mathbb{Z}/k \times A \cong A/kA$.
(5) $\mathbb{Z}/n \otimes \mathbb{Z}/m \cong \mathbb{Z}/n/(m\mathbb{Z}/m) \cong \mathbb{Z}/\gcd(m, n)$.
(6) If A is a finitely generated abelian group and $A \cong \mathbb{Z}^r \oplus TA$, then $\mathbb{Q} \otimes A \cong \mathbb{Q}^r$.

6.2. **Tor groups.** The Tor groups are to tensor product as Ext groups are to Hom. If $0 \to A \to B \to C \to 0$ is a short exact sequence, then

$$A \otimes G \to B \otimes G \to C \otimes G \to 0$$

is exact. So $- \otimes G$ is a covariant right exact functor from abelian groups to abelian groups. The failure to be left exact is measured by the Tor groups.

Definition 6.1. Let A and B be abelian groups. Let

$$\cdots \to F_2 \to F_1 \to F_0 \to A \to 0$$

be a free resolution of A. Then

$$\mathrm{Tor}_n(A, B) := H_n(F_* \otimes B).$$

That is, replace A by 0, tensor with B, then take homology.

The proof that Tor_n is well-defined uses the fundamental lemma of homological algebra, that we showed for length one resolutions above in order to show that Ext is well defined. The next proposition is the analogue of the fact that $\mathrm{Ext}^0(G, H) \cong \mathrm{Hom}(G, H)$.

Proposition 6.2. $\mathrm{Tor}_0(A, B) \cong A \otimes B$.

We will not provide any proofs of these statements, since they are rather similar in character to the Ext group proofs we have already covered, and I think it best to move onto cup products as soon as possible.

6.3. **Universal coefficient theorem for homology.** Tensor product appears in two important places: in the universal coefficient theorem for homology and in the Künneth theorem for cohomology.

Theorem 6.3 (Universal coefficient theorem for homology)**.** *Let X be a topological space and let G be an abelian group. For all $n \in \mathbb{N}_0$, there is a natural short exact sequence*

$$0 \to H_n(X) \otimes G \to H_n(X; G) \to \mathrm{Tor}_1(H_{n-1}(X); G) \to 0$$

that splits.

Corollary 6.4. *For every space X and for $n \in \mathbb{N}_0$, we have that $H_n(X; \mathbb{Q}) \cong H_n(X; \mathbb{Z}) \otimes \mathbb{Q}$.*

Proof. Let $H := H_{n-1}(X)$. There is a resolution of length one:

$$0 \to \bigoplus_{i=1}^{k} \mathbb{Z} \xrightarrow{\left(0, \bigoplus_{i=1}^{k} p_i^{n_i}\right)} \mathbb{Z}^n \oplus \bigoplus_{i=1}^{k} \mathbb{Z} \to H \to 0.$$

for some integers n, k, p_i, n_i. To compute $\mathrm{Tor}_1(H, Q)$ we tensor with $\mathbb{Q}$ to obtain

$$0 \to \bigoplus_{i=1}^{k} \mathbb{Q} \xrightarrow{\left(0, \bigoplus_{i=1}^{k} p_i^{n_i}\right)} \mathbb{Q}^n \oplus \bigoplus_{i=1}^{k} \mathbb{Q} \to 0$$

Since the p_i are nonzero, the first homology of this chain complex is trivial, so $\mathrm{Tor}_1(H, \mathbb{Q}) = 0$. The corollary then follows from the universal coefficient theorem exact sequence. $\square$

Example 6.5. Let us consider the homology of $\mathbb{RP}^3$. With $\mathbb{Z}$ coefficients, we have $H_0(\mathbb{RP}^3;\mathbb{Z}) \cong H_3(\mathbb{RP}^2;\mathbb{Z}) \cong \mathbb{Z}$, $H_2(\mathbb{RP}^3;\mathbb{Z}) = 0$ and $H_1(\mathbb{RP}^3;\mathbb{Z}) = \mathbb{Z}/2$. We can compute the homology with $\mathbb{Z}/2$ coefficients using the cellular chain complex $C_i^{CW}(\mathbb{RP}^3) \cong \mathbb{Z}$ for $i = 0,1,2,3$ and boundary maps

$$0 \to C_3^{CW}(\mathbb{RP}^3) \cong \mathbb{Z} \xrightarrow{0} C_2^{CW}(\mathbb{RP}^3) \cong \mathbb{Z} \xrightarrow{2} C_1^{CW}(\mathbb{RP}^3) \cong \mathbb{Z} \xrightarrow{0} C_0^{CW}(\mathbb{RP}^3) \cong \mathbb{Z} \to 0.$$

Tensor with $\mathbb{Z}/2$ to obtain, using that $\mathbb{Z} \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2$, the chain complex:

$$0 \to \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \to 0.$$

Therefore $H_i(\mathbb{RP}^3;\mathbb{Z}/2) \cong H_i^{CW}(\mathbb{RP}^3;\mathbb{Z}/2) \cong \mathbb{Z}/2$ for $i = 0,1,2,3$, and is otherwise trivial.

Now let us compute with the universal coefficient theorem for homology instead. What is $\mathrm{Tor}_1(H,\mathbb{Z}/2)$? For H free abelian, $\mathrm{Tor}_1(H,\mathbb{Z}/2) = 0$. However $\mathrm{Tor}_1(\mathbb{Z}/2,\mathbb{Z}/2) \cong \mathbb{Z}/2$. To see this, note that $\mathbb{Z}/2$ has a resolution

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0.$$

Tensor with $\mathbb{Z}/2$ to get

$$\mathbb{Z}/2 \xrightarrow{0} \xrightarrow{\mathbb{Z}/2}$$

supported in degrees 0 and 1. The H_1 of this is $\mathrm{Tor}_1(\mathbb{Z}/2,\mathbb{Z}/2) = \mathbb{Z}/2$. Now, by the universal coefficient theorem for homology, this means that

$$H_i(\mathbb{RP}^3;\mathbb{Z}/2) = H_i(\mathbb{RP}^3;\mathbb{Z}) \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2$$

for $i = 0,1,3$, since $\mathbb{Z} \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2$ for $i = 0,3$, and since $\mathbb{Z}/2 \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2$ for $i = 1$. For $H_2(\mathbb{RP}^3;\mathbb{Z}/2)$, we have $H_2(\mathbb{RP}^3;\mathbb{Z}) = 0$ so

$$H_2(\mathbb{RP}^3;\mathbb{Z}/2) \cong \mathrm{Tor}_1(H_1(\mathbb{RP}^3;\mathbb{Z}),\mathbb{Z}/2) \cong \mathrm{Tor}_1(\mathbb{Z}/2,\mathbb{Z}/2) = \mathbb{Z}/2.$$

So once again we compute that $H_i(\mathbb{RP}^3;\mathbb{Z}/2) \cong \mathbb{Z}/2$ for $i = 0,1,2,3$, and is otherwise trivial.

6.4. **Künneth theorem for cohomology.** Now for another very important use of tensor product: the Künneth theorem that allows us to compute the cohomology of a product of space $X \times Y$, given knowledge of the cohomology of X and of Y.

Theorem 6.6. *Let X and Y be topological spaces. Suppose that the homology groups of X and Y are finitely generated. Then for all $n \in \mathbb{N}_0$, there is a short exact sequence*

$$0 \to \bigoplus_{k+\ell=n} H^k(X;\mathbb{Z}) \otimes H^\ell(Y;\mathbb{Z}) \to H^n(X \times Y;\mathbb{Z}) \to \bigoplus_{k+\ell=n+1} \mathrm{Tor}_1(H^k(X;\mathbb{Z}), H^\ell(Y;\mathbb{Z})) \to 0$$

which splits.

For examples, see the problem sheet which focussed on applications of this theorem.

7. Cup products

As advertised previously, one of the great advantages of cohomology is that it admits a beautiful multiplicative structure. Whereas homology associates to a topological space a sequence of abelian groups, in cohomology these abelian groups are collected together to produce a single *ring*. The multiplication in this ring is called the cup product, which we shall now define.

In order to make cohomology into a ring, we will need to work with coefficients that are themselves a ring. So let R be a commutative ring. For example we will typically use $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\mathbb{Z}/n$ for some n.

This looks similar to before, since these are all abelian groups. Indeed any ring is an abelian group. But now we shall also use the ring structure on these abelian groups.

7.1. Definition of cup products.

Here is our goal. For a space X, we want to define a map

$$\smile\colon H^i(X;R) \times H^j(X;R) \to H^{i+j}(X;R).$$

To define such a map, we will define a map on cochains $\smile\colon C^i(X;R) \times C^j(X;R) \to C^{i+j}(X;R)$, and then we will check that this induces a well-defined map on cohomology groups.

Let us recall and introduce some notation for singular simplices. A singular i-simplex of X is a continuous map $\Delta^i \to X$. We write label the vertices $0, 1, \dots, i$. We write $[v_0, \dots, v_j]$ for the face of Δ^i spanned by the vertices $v_0, \dots, v_j$, with $v_k \in \{0, \dots, i\}$ for $k = 0, \dots, j$.

We write

$$\lfloor_j\colon [0, \dots, j] \to [0, \dots, i] = \Delta^i$$

for the *front j face* of Δ^i. We write

$$\rfloor_j\colon [i-j, \dots, i] \to [0, \dots, i] = \Delta^i$$

for the *back j face* of Δ^i. Given a singular $i+j$-simplex $\sigma\colon \Delta^{i+j} \to X$, we can consider its composition with $\lfloor_i$ and $\rfloor_j$:

$$\sigma \circ \lfloor_i\colon \Delta^i \to \Delta^{i+j} \to X$$

and

$$\sigma \circ \rfloor_j\colon \Delta^j \to \Delta^{i+j} \to X.$$

Now we can *define*:

$$\begin{array}{rcl} \smile\colon C^i(X;R) \times C^j(X;R) & \to & C^{i+j}(X;R) \\ (\varphi, \psi) & \mapsto & \varphi(\sigma \circ \lfloor_i)\psi(\sigma \circ \rfloor_j) \\ & = & \varphi(\sigma|_{[0,\dots,i]})\psi(\sigma|_{[i,\dots,i+j]}). \end{array}$$

That is, we evaluate φ on the front i-face and ψ on the front j face. Note that in the last line we think of $\sigma \circ \lfloor_i$ as the restriction of σ to a face, $\sigma|_{[0,\dots,i]}$.

Theorem 7.1.

(1) Given two cocycles $\varphi \in C^i(X;R)$ and $\psi \in C^j(X;R)$ with $\delta\varphi = 0$ and $\delta\psi = 0$, then $\varphi \smile \psi \in C^{i+j}(X;R)$ is also a cocycle, and therefore represents a cohomology class $[\varphi \smile \psi] \in H^{i+j}(X;R)$.

(2) *Cup product is well-defined on cohomology. That is, for any $\theta \in C^{i-1}(X;R)$ and for any $\chi \in C^{j-1}(X;R)$, we have that*

$$\big((\varphi + \delta\theta) \smile (\psi + \delta\chi)\big) - \varphi \smile \psi$$

is a coboundary. Therefore

$$[(\varphi + \delta\theta) \smile (\psi + \delta\chi)] = [\varphi \smile \psi] \in H^{i+j}(X;R).$$

We need the following lemma to prove Theorem 7.1.

Lemma 7.2. *Let $\varphi \in C^i(X;R)$ and $\psi \in C^j(X;R)$. Then*

$$\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^i \varphi \smile \delta\psi \in C^{i+j+1}(X;R).$$

Proof. Let $\sigma \colon \Delta^{i+j+1} \to X$. Then we have the following computations. Here $\widehat{k}$ indicates that the number is missed out.

$$(\delta\varphi \smile \psi)(\sigma) = \sum_{k=1}^{i+1} (-1)^k \varphi(\sigma|_{[0,\dots,\widehat{k},\dots,i+1]}) \cdot \psi(\sigma_{[i+1,\dots,i+j+1]}) \tag{7.3}$$

$$(-1)^i(\varphi \smile \delta\psi)(\sigma) = \sum_{k=i}^{i+j+1} (-1)^k \varphi(\sigma_{[0,\dots,i]}) \cdot \psi(\sigma_{[i,\dots,\widehat{k},\dots,i+j+1]}) \tag{7.4}$$

The sum of the left hand sides of these two equations equals the right hand side of the equation we want to prove. In the sum of the right hand sides of these two equations, the last term of (7.3) cancels with the first term of (7.4). The rest of the terms sum to

$$\delta(\varphi \smile \psi)(\sigma) = (\varphi \smile \psi)(\partial\sigma) \tag{7.5}$$

where $\partial\sigma = \sum_{k=0}^{i+j+1} \sigma_{[0,\dots,\widehat{k},\dots,i+j+1]}$. □

Proof of Theorem 7.1. First we prove that the cup product of two cocycles is a cocycle. Suppose that $\delta\varphi = 0$ and $\delta\psi = 0$, so that both are cocycles. Then

$$\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^i \varphi \smile \delta\psi = 0 \smile \psi + (-1)^i \varphi \smile 0 = 0 + 0 = 0.$$

So $\varphi \smile \psi$ is a cocycle.

Next, by the lemma we have that

$$\varphi \smile \delta\chi = \pm\delta(\varphi \smile \chi) - (\delta\varphi \smile \chi) = \pm\delta(\varphi \smile \chi).$$

Similarly

$$\delta\theta \smile \psi = \pm\delta(\theta \smile \psi) \pm (\theta \smile \delta\psi) = \pm\delta(\theta \smile \psi).$$

So both are coboundaries. Finally

$$\delta\theta \smile \delta\chi = \pm\delta(\theta \smile \delta\chi) \pm (\theta \smile \delta^2\psi) = \pm\delta(\theta \smile \delta\chi),$$

which is also a coboundary. We make no effort to get the signs right because they are irrelevant. Therefore

$$\begin{aligned}&\big((\varphi+\delta\theta)\smile(\psi+\delta\chi)\big)-\varphi\smile\psi\\ =&\varphi\smile\psi+\varphi\smile\delta\chi+\delta\theta\smile\psi+\delta\theta\smile\delta\chi-\varphi\smile\psi\\ =&\pm\delta(\varphi\smile\chi)\pm\delta(\theta\smile\psi)\pm\delta(\theta\smile\delta\chi)\\ =&\delta(\pm\varphi\smile\chi\pm\theta\smile\psi\pm\theta\smile\delta\chi)\end{aligned}$$

which is a coboundary, as desired. Therefore the cup products are well-defined on cohomology, and so define maps

$$\smile\colon H^i(X;R)\times H^j(X;R)\to H^{i+j}(X;R),$$

as promised. □

Example 7.6. One of the most instructive examples of cup product is the torus $S^1\times S^1$. We explicitly compute the cup product of the torus

$$\smile\colon H^1(S^1\times S^1;\mathbb{Z})\times H^1(S^1\times S^1;\mathbb{Z})\to H^2(S^1\times S^1;\mathbb{Z}).$$

Recall that $H^1(S^1\times S^1;\mathbb{Z})\cong\mathbb{Z}\oplus\mathbb{Z}$ and $H^2(S^1\times S^1;\mathbb{Z})\cong\mathbb{Z}$. So the cup product corresponds, after having chosen generating cohomology classes, to a pairing $\mathbb{Z}^2\times\mathbb{Z}^2\to\mathbb{Z}$. Such a pairing can be represented by a 2×2 matrix A, in the sense that the pairing of $\mathbf{x}=(x_1,x_2)\in\mathbb{Z}^2$ and $\mathbf{y}=(y_1,y_2)\in\mathbb{Z}^2$ is given by

$$\mathbf{x}A\mathbf{y}^T,$$

as you may recall from bilinear forms or inner product spaces in linear algebra.

We shall choose natural bases for the cohomology of the torus, and with respect to these bases the cup product is represented by

$$\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

In order to compute the cup product of the torus, first we need to define some singular simplices and some singular cochains.

8. Cap products

9. Calculations of cup products using Poincaré duality

10. Proof of $\mathbb{F}_2$ coefficient Poincaré duality

11. Linking forms and lens spaces

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