ALGEBRAIC TOPOLOGY IV || LECTURE NOTES

MARK POWELL

Contents

1. Michaelmas term summary

The focus of Michaelmas semester is to develop *homology.* For each nonnegative integer n, the homology $H_n(X)$ of a topological space X is an abelian group. Crucially, for homeomorphic spaces X and Y, we have that $H_n(X) \cong H_n(Y)$ for every n. So homology can be used to distinguish spaces. Homology is also highly computable, and satisfies nice properties such as functoriality: a map $f\colon X \to Y$ determines a map $f_*\colon H_n(X) \to H_n(Y)$ for every n. Our goal is to learn how to compute homology, and then apply it to prove some cool things about topological spaces, and sometimes also about algebra.

These notes are a record of what was covered in lectures, but are not exhaustive. Where proofs can be found in Hatcher, you are also referred there.

I will write PNE in the margin for Proof Non-Examinable.

2. Definition of singular homology

Let $\{A_i\}_{i\in J}$ be a family of abelian groups. Here J could be uncountable. The *direct product* is

$$\prod_{i\in J} A_i := \{\text{maps } f\colon J \to \cup_{i\in J} A_i \mid f(i) \in A_i \text{ for every } i \in J\}.$$

We write elements as tuples $(a_i)_{i\in J}$.

The *direct sum* is

$$\bigoplus_{i\in J} A_i := \{(a_i)_{i\in J} \in \prod_{i\in J} A_i \mid \text{ finitely many } a_i \neq e\}.$$

For finite families of abelian groups, the direct product and the direct sum coincide, but for infinite families the direct product contains many more elements.

Definition 2.1. A *chain complex* is a sequence of abelian groups $\{C_i\}_{i\in\mathbb{Z}}$ and homomorphisms $\partial_i\colon C_i \to C_{i-1}$ such that $0 = \partial_i \circ \partial_{i+1}\colon C_{i+1} \to C_{i-1}$. The maps ∂_i are called *boundary maps.*

Definition 2.2. The *homology* $H_*(C)$ of a chain complex (C_*, ∂_*) is

$$H_i(C) := \frac{\ker(\partial_i\colon C_i \to C_{i-1})}{\operatorname{im}(\partial_{i+1}\colon C_{i+1} \to C_i)}.$$

The standard n-simplex Δ^n is

$$\{\underline{x} \in \mathbb{R}^{n+1} \mid x_0 + x_1 + \cdots + x_n = 1\}.$$

A *singular n-simplex* of a topological space X is a continuous map $\sigma\colon \Delta^n \to X$.

For $n \in \mathbb{N}_0$, the *singular n chains* of X, $C_n(X)$, is the free abelian group generated by the singular n-simplices.

$$C_n(X) := \{n_1\sigma_1 + \cdots n_k\sigma_k \mid k \in \mathbb{N}_0, n_1, \dots, n_k \in \mathbb{Z}, \sigma_1, \dots, \sigma_k\colon \Delta^n \to X \text{ sing. simplices}\}$$

Note that $C_n(X) = 0$ for $n < 0$.

Let $v_0, \dots, v_n$ be the standard basis of $\mathbb{R}^{n+1}$. Note that each v_i corresponds to a vertex of Δ^n. There are $n+1$ inclusion maps $\iota_j\colon \Delta^{n-1} \to \Delta^n$, $j = 0, \dots, n$, defined by restricting the linear maps $\mathbb{R}^n \to \mathbb{R}^{n+1}$ that send

$$\begin{cases} v_i \mapsto v_i & i < j \\ v_i \mapsto v_{i+1} & i \geq j. \end{cases}$$

We also write this map as

$$[v_0, \dots, \widehat{v_j}, \dots, v_n].$$

Here, as is customary, the hat denotes the missing vertex / coordinate. Now we can define

$$\partial(\sigma) := \sum_{j=0}^{n} (-1)^j \sigma \circ \iota_j.$$

Then we extend the boundary map ∂ by linearity to define it on formal sums of singular simplices.

Lemma 2.3. *$\partial^2\colon C_n(X) \to C_{n-2}(X)$ is the zero map, so $(C_*(X), \partial)$ is a chain complex.*

Definition 2.4. The nth singular homology of X is

$$H_n(X) := H_n(C_*(X)) = \frac{\ker(\partial_i\colon C_i(X) \to C_{i-1}(X))}{\operatorname{im}(\partial_{i+1}\colon C_{i+1}(X) \to C_i(X))}.$$

We also write $Z_n(X) := \ker(\partial_i\colon C_i(X) \to C_{i-1}(X))$, the n-cycles, and $B_n(X) := \operatorname{im}(\partial_{i+1}\colon C_{i+1}(X) \to C_i(X))$, the n-boundaries.

Example 2.5. When X is a point, $H_0(X) \cong \mathbb{Z}$ and $H_i(X) = 0$ for $i \neq 0$.

Proposition 2.6. *Let A be the set of path components of X. Then $H_0(X) \cong \bigoplus_A \mathbb{Z}$. If X is path connected, then $H_0(X) = \mathbb{Z}$.*

Higher homology groups cannot be directly computed from the definition. We need to develop some tools and theory with which to compute. This means learning about exact sequences, which we will do in the next chapter. So that you have some intuition, here are some homologies of spaces.

Example 2.7.

(1) For $X = \mathbb{R}^n$, the homology is the same as that of a point.

$$H_i(\mathbb{R}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } i = 0 \\ 0, & \text{otherwise.} \end{cases}$$

(2) The n dimensional sphere is defined as the subspace of $\mathbb{R}^n$

$$S^n := \{\underline{x} \in \mathbb{R}^n \mid \|\underline{x}\| = 1\}.$$

The homology is

$$H_i(S^n) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, n \\ 0, & \text{otherwise.} \end{cases}$$

(3) Products of spheres, $S^n \times S^m$. First, if $n = m$ we have:
$$H_i(S^n \times S^n) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, 2n \\ \mathbb{Z} \oplus \mathbb{Z}, & \text{if } i = n \\ 0, & \text{otherwise.} \end{cases}$$
The torus $S^1 \times S^1$ is a good first case to think about. On the other hand if $n \neq m$ we have
$$H_i(S^n \times S^m) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, n, m, n+m \\ 0, & \text{otherwise.} \end{cases}$$

What are the advantages of defining homology by considering the (usually) infinite rank abelian groups generated by all possible continuous maps of an n-simplex into X? The main advantage is that it is easy to prove that homology behaves well with respect to maps between spaces i.e. that it is *functorial*.

Definition 2.8. A *chain map* $F\colon C_* \to D_*$ between chain complexes C_* and D_* is a collection of homomorphisms $F_n\colon C_n \to D_n$ such that
$$\partial^D_{n+1} \circ F_{n+1} = F_n \circ \partial^C_n \colon C_{n+1} \to D_n$$
for every $n \in \mathbb{Z}$.

In other words, the diagram
$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "\partial^C_{n+1}"] \arrow[d, "F_{n+1}"] & C_n \arrow[r, "\partial^C_n"] \arrow[d, "F_n"] & C_{n-1} \arrow[r] \arrow[d, "F_{n-1}"] & \cdots \\ \cdots \arrow[r] & D_{n+1} \arrow[r, "\partial^D_{n+1}"'] & D_n \arrow[r, "\partial^D_n"'] & D_{n-1} \arrow[r] & \cdots \end{tikzcd}$$
commutes.

Lemma 2.9. *A chain map $F\colon C_* \to D_*$ induces a map on homology*
$$\begin{array}{rcl} F_*\colon H_n(C_*) & \to & H_n(D_*) \\ [c] & \mapsto & [f(c)] \end{array}$$
for every $n \in \mathbb{N}_0$.

In other words, this map is both defined, i.e. sends cycles to cycles, and well-defined, meaning it sends boundaries to boundaries.

Proposition 2.10. *Let $f\colon X \to Y$ be a homeomorphism of topological spaces. Then the induced map $f_*\colon H_n(X) \xrightarrow{\cong} H_n(Y)$ is an isomorphism for every $n \in \mathbb{N}_0$.*

3. Exact sequences

Definition 3.1. A sequence of abelian groups and homomorphisms
$$A \xrightarrow{f} B \xrightarrow{g} C$$
is *exact at B* if $\operatorname{im}(f) = \ker g$. A sequence $\cdots \to A_{i+1} \to A_i \to A_{i-1} \to \cdots$ is exact if $A_{i+1} \to A_i \to A_{i-1}$ is exact at A_i for every i.

Note that for chain complexes, $\partial_n \circ \partial_{n+1} = 0$ means that $\operatorname{im} \partial_{n+1} \subseteq \ker \partial_n$. If a chain complex C_* is exact then $H_n(C_*) = 0$ for every n.

Definition 3.2.

(1) A *short exact sequence* is five-term exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$. So f is injective, g is surjective, and $\operatorname{im} f = \ker g$. Note that $C \cong B/\ker g \cong B/\operatorname{im} f \cong B/f(A)$.

(2) A *short exact sequence of chain complexes* is a sequence of chain maps
$$0 \to C_* \to D_* \to E_* \to 0$$
with $0 \to C_n \to D_n \to E_n \to 0$ a short exact sequence of abelian groups for every n.

Theorem 3.3. *A short exact sequence of chain complexes*
$$0 \to C_* \xrightarrow{f} D_* \xrightarrow{g} E_* \to 0$$
determines a long exact sequence in homology groups
$$\cdots \to H_{n+1}(E) \xrightarrow{\delta}\to H_n(C) \xrightarrow{f_*} H_n(D) \xrightarrow{g_*} H_n(E) \xrightarrow{\delta} \cdots$$

PNE

Here is the definition of the map δ. Let $[e] \in H_{n+1}(E)$, so $e \in E_{n+1}$ and $\partial(e) = 0$. Since g_{n+1} is surjective, there exists $d \in D_{n+1}$ with $g_{n+1}(d) = e$. Then
$$g_n \circ \partial^D(d) = \partial^E \circ g_{n+1}(d) = \partial^E(e) = 0.$$
Therefore $\partial^D(d) \in \ker(g_n) \subseteq \operatorname{im} f_n$. Since f_n is injective, there is a unique $c \in C_n$ with $f_n(c) = \partial^D(d)$. We define
$$\delta([e]) = [c].$$
We need to see that this defines a homology class, that is $\partial^C(c) = 0$, and that this is well defined: we chose a representative e and we chose d. There are also six things to show to check exactness. Details can be found in Hatcher pages 116–7.

4. Homotopies and homotopy equivalence

Definition 4.1. Let $f, g\colon X \to Y$ be continuous maps between topological spaces. A *homotopy* from f to g is a continuous map $h\colon X \times I \to Y$ with $h|_{X\times\{0\}} = f\colon X \times \{0\} = X \to Y$ and $h|_{X\times\{1\}} = g\colon X \times \{0\} = X \to Y$. We write $f \sim_h g$ or just $f \sim g$.

Lemma 4.2. *If $f, f'\colon X \to Y$ are homotopic and $g, g'\colon Y \to Z$ are homotopic then $g \circ f \sim g' \circ f'\colon X \to Z$ are also homotopic.*

Homotopy gives rise to an equivalence relation on maps $X \to Y$.

Definition 4.3. A map $f\colon X \to Y$ is a *homotopy equivalence* if there exists a map $g\colon Y \to X$ such that $f \circ g \sim \mathrm{Id}_Y$ and $g \circ f \sim \mathrm{Id}_X$. The map g is called the *homotopy inverse*. We say that the spaces X and Y are *homotopy equivalent*, and write $X \simeq Y$.

Homotopy equivalence is an equivalence relation on spaces. Please do not say or write that two *spaces* X and Y are *homotopic*. There is no meaning attached to such a phrase.

Lemma 4.4. *Any two homotopy inverses $g_1, g_2 \colon Y \to X$ for a homotopy equivalence $f \colon X \to Y$ are homotopic.*

Proof. We have a sequence of homotopies
$$g_1 = \mathrm{Id}_Y \circ g_1 \sim g_2 \circ f \circ g_1 \sim g_2 \circ \mathrm{Id}_X = g_2. \qquad \square$$

A space X is *contractible* if $X \simeq \{\mathrm{pt}\}$.

Here is the most important result for us about homotopy equivalences, which will be proven in the next section..

Theorem 4.5. *Let $f \colon X \to Y$ be a homotopy equivalence. Then $f_* \colon H_n(X) \to H_n(Y)$ is an isomorphism for every $n \in \mathbb{N}_0$.*

So spaces X and Y with different homology are not homotopy equivalent. In particular we see that $H_k(D^n) \cong H_k(\mathbb{R}^n) \cong H_k(\{\mathrm{pt}\})$, and also $H_k(\mathbb{R}^2 \setminus \{\mathrm{pt}\}) \cong H_k(S^1)$ for every k.

A similar result holds for fundamental groups, provided we are careful with base points.

Theorem 4.6. *Let $f \colon (X, x) \to (Y, y)$ be a based homotopy equivalence with $f(x) = y$. Then $f_* \colon \pi_1(X, x) \to \pi_1(Y, y)$ is an isomorphism.*

Here are some important homotopy equivalent spaces.

Definition 4.7. Let $f \colon X \to Y$ be a continuous map. The *mapping cylinder* of f, M_f, is
$$X \times I \coprod Y / (X \times \{1\} \sim f(x) \text{ for all } x \in X).$$

Lemma 4.8. *For every $f \colon X \to Y$, we have that $M_f \simeq Y$.*

Definition 4.9. The *cone* on a map f is
$$\mathrm{Cone}(f) = C_f := M_f / X \times \{0\}.$$

Lemma 4.10. *For any space X, $C_{\mathrm{Id}_X} \simeq \{\mathrm{pt}\}$. i.e. the cone on the identity map is contractible.*

An important special case of homotopy equivalences is deformation retracts. Let $i_A \colon A \subseteq X$ be a subspace.

Definition 4.11. A *deformation retract* of X onto A is a map $f \colon X \to A$ with a homotopy $H \colon X \times I \to X$ with $\mathrm{Id}_X \sim_H f$ such that $H|_{A \times \{t\}} \colon A \times \{t\} \to X$ coincides with i_A for every $t \in I$.

Definition 4.12. A *retraction* is a continuous map $r \colon X \to X$ with $r(X) = A$ and $r \circ i_A \colon A \to X$ equal to i_A.

Every nonempty space X retracts to a point for all X, but this is not true for deformation retracts.

A deformation retract determines a retract, but the converse does not hold. A deformation retract is a homotopy equivalence, but homotopy equivalence is a notion defined when neither space is a subspace of the other.

5. Chain homotopies and chain homotopy equivalences

Chain homotopies are algebraic models of homotopies, and chain homotopy equivalences model homotopy equivalences.

Definition 5.1. Two chain maps $f, g\colon C_* \to D_*$ are *chain homotopic* if there exists a homomorphism $P_n\colon C_n \to D_{n+1}$ for every n with

$$f_n - g_n = \partial \circ P_n + P_{n-1} \circ \partial \colon C_n \to D_n.$$

We write $f \sim g$.

Proposition 5.2. *If chain maps $f \sim g\colon C_* \to D_*$ then $f_* = g_*\colon H_n(C_*) \to H_n(D_*)$ for every n.*

Proof. Let $c \in C_n$ be a cycle, so $\partial c = 0$. Then $f_n(c) - g_n(c) = \partial(P(c)) + P(\partial(c)) = \partial(P(c))$, so

$$[f_n(c)] = [g_n(c) + \partial(P(c))] = [g_n(c)]. \qquad \square$$

Theorem 5.3. *If maps of spaces $f, g\colon C_*(X) \to C_*(Y)$ are homotopic, then $f_* \sim g_*\colon C_*(X) \to C_*(Y)$ are chain homotopic.*

PNE

The idea of the proof is to convert the product $\Delta^n \times I$ into a sum of simplices, and use this to take the data of a homotopy and convert it into a chain homotopy.

Definition 5.4. A chain map $f\colon C_* \to D_*$ is a *chain homotopy equivalence* if there is a chain map

$$g\colon D_* \to C_*$$

with $g \circ f \sim \mathrm{Id}_C$ and $f \circ g \sim \mathrm{Id}_D$. We write $C_* \simeq D_*$ and we say that C_* and D_* are chain homotopy equivalent. If $C_* \simeq 0$ then we say that C_* is *chain contractible.*

Lemma 5.5. *If $X \simeq Y$ then $C_*(X) \simeq C_*(Y)$.*

Proof. Let $f\colon X \to Y$ and $g\colon Y \to X$ be maps witnessing the homotopy equivalence. Then $f \circ g \sim \mathrm{Id}$ and $g \circ f \sim \mathrm{Id}$ imply that $f_* \circ g_* \sim \mathrm{Id}_*\colon C_*(Y) \to C_*(Y)$ and $g_* \circ f_* \sim \mathrm{Id}_*\colon C_*(X) \to C_*(X)$ by Theorem 5.3. It follows that $C_*(X) \simeq C_*(Y)$ as desired. $\square$

Lemma 5.6. *If $C_* \simeq D_*$ then $H_n(C_*) \cong H_n(D_*)$ for every n.*

Proof. By Proposition 5.2 we have that $g_* \circ f_* = \mathrm{Id}_*\colon H_n(C_*) \to H_n(C_*)$ and $f_* \circ g_* = \mathrm{Id}_*\colon H_n(D_*) \to H_n(D_*)$ for every n. $\square$

By combining the above facts we obtain the following conclusion.

Corollary 5.7. *If two spaces X and Y are homotopy equivalent via a homotopy equivalence $f\colon X \to Y$, then $f_*\colon H_n(X) \to H_n(Y)$ is an isomorphism for every $n \in \mathbb{N}_0$.*

6. Mayer-Vietoris sequence

The next stage in computing homology groups is to develop the Mayer-Vietoris long exact sequence. We know that a short exact sequence of chain complexes gives rise to a long exact sequence in homology. So we need an auspicious choice of short exact sequence.

Given a subset $U \subset X$ let $\mathring{U}$ denote its interior. Let $\mathcal{U} = \{U_i\}_{i \in \mathcal{I}}$ be a collection of subsets $U_i \subseteq X$ with $\{\mathring{U}_i\}$ an open cover of X. Let

$$C_n^{\mathcal{U}}(X) := \text{free abelian group on singular } n-\text{simplices} \sum_j n_j \sigma_j$$

$$\text{where for every } j,\ \sigma_j(\Delta^n) \subseteq U_{i_j} \text{ for some } i_j \in \mathcal{I}.$$

Theorem 6.1. *The inclusion $C_*^{\mathcal{U}}(X) \to C_*(X)$ is a chain homotopy equivalence.* PNE

The proof uses iterated subdivision of simplices by adding in the *barycentre* as an extra vertex, to construct a chain homotopy inverse. See Proposition 2.21 of Hatcher. We use this in the next theorem, giving the Mayer-Vietoris sequence.

Theorem 6.2 (Mayer-Vietoris long exact sequence)**.** *Let $X = \mathring{U} \cup \mathring{V}$ where U, V are subsets. Let $\mathcal{U} := \{U, V\}$. There is a short exact sequence of chain complexes*

$$0 \to C_*(U \cap V) \xrightarrow{(\iota_U, -\iota_V)} C_*(U) \oplus C_*(V) \xrightarrow{j_U + j_V} C_*^{\mathcal{U}}(X) \to 0$$

inducing a long exact sequence of homology groups

$$\cdots \to H_{n+1}(X) \xrightarrow{\delta}$$

$$\to H_n(U \cap V) \xrightarrow{((\iota_U)_*, -(\iota_V)_*)} H_n(U) \oplus H_n(V) \xrightarrow{(j_U)_* + (j_V)_*} H_n(X) \xrightarrow{\delta} \cdots$$

We use Theorem 3.3 to convert the short exact sequence of chain complexes into the long exact sequence in homology. The key step in the proof is to apply Theorem 6.1 to equate $H_n(C_*^{\mathcal{U}}(X))$ with $H_n(X)$.

Using the Mayer-Vietoris sequence we can compute the homology groups of the spheres S^m. This lets us deduce the following likely-sounding but nontrivial theorem.

Theorem 6.3. *The spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic if and only if $n = m$.*

The next theorem is rather more surprising.

Theorem 6.4. *Let $n \geq 0$ and let $f \colon D^n \to D^n$ be a continuous map. Then f has a fixed point, i.e. there is a point $x \in D^n$ with $f(x) = x$.*

7. Reduced homology

Define the *augmented chain complex* of a space X to be

$$\widetilde{C}_*(X) := \big(\cdots \to C_n(X) \xrightarrow{\partial} C_{n-1}(X) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0\big)$$

where

$$\begin{array}{rcl} C_0(X) & \to & \mathbb{Z} \\ \sum n_i \sigma_i & \mapsto & \sum n_i. \end{array}$$

Note that $\varepsilon \circ \partial = 0$. Define the *reduced homology* to be

$$\widetilde{H}_n(X) := H_n(\widetilde{C}_*(X))$$

for every $n \in \mathbb{N}_0$.

This homology makes many statements much cleaner, and will be a useful tool.

Proposition 7.1.

(1) *If $f\colon X \to Y$ is a map, then there is an induced chain map $f_*\colon \widetilde{C}_*(X) \to \widetilde{C}_*(Y)$ and thus homomorphisms*

$$f_*\colon \widetilde{H}_n(X) \to \widetilde{H}_n(Y),\ n \in \mathbb{N} \cup \{0, -1\}.$$

(2) *If $f \sim g\colon X \to Y$ then*

$$f_* \sim g_*\colon \widetilde{C}_*(X) \to \widetilde{C}_*(Y)$$

and

$$f_* = g_*\colon \widetilde{H}_*(X) \to \widetilde{H}_*(Y).$$

(3) $\widetilde{H}_k(\varnothing) = \begin{cases} \mathbb{Z} & k = -1 \\ 0 & \textit{else.} \end{cases}$

(4) *If X is nonempty, then $H_k(X) = \begin{cases} \widetilde{H}_k(X) & k > 0 \\ \widetilde{H}_0(X) \oplus \mathbb{Z} & k = 0. \end{cases}$ In particular, if n has n path components, then $\widetilde{H}_0(X) \cong \mathbb{Z}^{n-1}$.*

Proof.

(1) The diagram

$$\begin{array}{ccc} C_0(X) & \xrightarrow{f_*} & C_0(Y) \\ \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon} \\ \mathbb{Z} & \xrightarrow{=} & \mathbb{Z} \end{array}$$

commutes, so f_* gives rise to a chain map $\widetilde{C}_*(X) \to \widetilde{C}_*(Y)$ as claimed.

(2) Let $P\colon C_k(X) \to C_{k+1}(Y)$ be the prism operator defined in the proof of homotopy invariance, Theorem 5.3. Extend this by $0\colon \mathbb{Z} \to C_0(Y)$. The result is chain homotopy between the maps f and g. To see this, note that $f_* - g_* = \partial \circ P = \partial \circ P + 0 \circ \varepsilon\colon C_0(X) \to C_0(Y)$. Also $\mathrm{Id} - \mathrm{Id} = 0 = \varepsilon \circ 0\colon \mathbb{Z} \to \mathbb{Z}$, so the conditions for a chain homotopy are met.

(3) The reduced chain complex of the empty set consists of the groups $C_i(\varnothing) = 0$ for $i \geq 0$ and $C_{-1}(\varnothing) = \mathbb{Z}$. Take homology of this.

(4) The end of the chain complex is $C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z}$. This gives rise to a short exact sequence

$$0 \to \widetilde{H}_0(X) \to H_0(X) \to \mathbb{Z} \to 0.$$

We will show below in Lemma 8.2 that any such sequence *splits*, so that $H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}$.

$\square$

8. Split short exact sequences

A short exact sequence of abelian groups

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is *split* if there is a homomorphism $s\colon C \to B$ with $\mathrm{Id} = g \circ s\colon C \to C$.

Proposition 8.1. *The following are equivalent:*

1. *There is a homomorphism $p\colon B \to A$ with $\mathrm{Id} = p \circ f\colon A \to A$.*
2. *There is a homomorphism $s\colon C \to B$ with $\mathrm{Id} = g \circ s\colon C \to C$.*
3. *There is an isomorphism $\theta\colon A \oplus C \to B$ such that*

$$\begin{tikzcd}
 & & A \oplus C \arrow[dd, "\cong"', "\theta"] \arrow[dr, "p_2"] & & \\
0 \arrow[r] & A \arrow[ur, "i_1"] \arrow[dr, "f"'] & & C \arrow[r] & 0 \\
 & & B \arrow[ur, "g"'] & &
\end{tikzcd}$$

commutes, where $i_1\colon a \mapsto (a,0)$ and $p_2\colon (a,c) \mapsto c$.

Proof. First we prove that (1) implies (3). We claim that the map

$$\begin{array}{rcl} \varphi\colon B & \to & A \oplus C \\ b & \mapsto & (p(b), g(b)) \end{array}$$

is an isomorphism. Then ϕ will be the inverse of θ as in the picture. It is clearly a homomorphism. To see that it is surjective, take $(a,c) \in A \oplus C$, and choose $b \in B$ with $g(b) = c$. Then

$$\varphi(b + f(a - p(b))) = (p(b) + pf(a - p(b)), g(b) + gf(a - p(b))) = (p(b), g(b)) = (a, c).$$

To see injectivity, $(p(b), g(b)) = 0$ implies that $g(b) = 0$ so $b = f(a)$ for some $a \in A$. Then $0 = p(b) = pf(a) = a$, so $b = f(a) = 0$. Also it is easy to check that the diagram

$$\begin{tikzcd}
 & & A \oplus C \arrow[dr, "p_2"] & & \\
0 \arrow[r] & A \arrow[ur, "i_1"] \arrow[dr, "f"'] & & C \arrow[r] & 0 \\
 & & B \arrow[uu, "\varphi", "\cong"'] \arrow[ur, "g"'] & &
\end{tikzcd}$$

commutes.

To prove that (3) implies (1), let $p_1\colon A \oplus C \to A$ be the projection. Then define $p := p_1 \circ \varphi\colon B \to A$. Note that $p_1 \circ i_1 = \mathrm{Id}$. So

$$p \circ f = p_1 \circ \varphi \circ f = p_1 \circ i_1 = \mathrm{Id}\,.$$

The proof that (2) and (3) are equivalent is left as an exercise. □

Lemma 8.2. *Consider abelian groups A and B fitting into a short exact sequence $0 \to A \to B \to \mathbb{Z}^m \to 0$, for some $m > 0$. Then this short exact sequence splits, so $B \cong A \oplus \mathbb{Z}^m$.*

Proof. For each $e_i \in \mathbb{Z}^m$, choose $b_i \in B$ mapping to e_i. Then define $s\left(\sum_{i=1}^m n_i e_i\right) = \sum_{i=1}^m n_i b_i$. This is a splitting as in (2) of Proposition 8.1. □

9. Relative homology

We want to define cellular homology, since this is another excellent tool for computing homology groups. In order to do this we need relative homology groups.

Let $A \subseteq X$ be a subspace of X. Let $i\colon A \to X$ be the inclusion. This induces $i_*\colon C_n(A) \to C_n(X)$. Identify $C_n(A)$ with $i_*(C_n(A)) \subset C_n(X)$ for every $n \in \mathbb{N}_0$.

Definition 9.1. Define

$$C_n(X, A) := \frac{C_n(X)}{C_n(A)} \quad n \in \mathbb{N}_0$$

as the quotient group, factoring out by singular chains with image in $A \subset X$. The boundary map in $C_*(X)$ induces a boundary map on these chains:

$$\begin{array}{rcl} \partial\colon C_n(X, A) & \to & C_{n-1}(X, A) \\ c + C_n(A) & \mapsto & \partial c + C_{n-1}(A). \end{array}$$

This is well-defined since if $x \in C_n(A)$ then $\partial(c + x + C_n(A)) = \partial(c) + \partial(x) + C_{n-1}(A) = \partial(c) + C_{n-1}(A)$, as $\partial(x) \in C_{n-1}(A)$. Define

$$H_n(X, A) := H_n(C_*(X, A)) \quad n \in \mathbb{N}_0.$$

Remark 9.2.

(1) $H_n(X, \varnothing) \cong H_n(X)$ canonically for every $n \in \mathbb{N}_0$.
(2) Let $x_0 \in X$ be a point. Then $H_n(X, \{x_0\}) \cong \widetilde{H}_n(X)$ for every $n \in \mathbb{N}_0$.

Example 9.3. Let$a, b \in \mathbb{R}^2$ be distinct points. We have that $H_n(\mathbb{R}^2, \{a\}) = \{0\}$ for every $n \in \mathbb{N}_0$. On the other hand

$$H_n(\mathbb{R}^2, \{a, b\}) \cong \begin{cases} \mathbb{Z} & n = 1 \\ 0 & \text{else.} \end{cases}$$

Note that in both cases H_0 vanishes. We will prove this using the next theorem.

Theorem 9.4 (Long exact sequence of a pair)**.** *Let $A \subseteq X$.*

(1) There is a short exact sequence of chain complexes

$$0 \to C_*(A) \xrightarrow{i_*} C_*(X) \xrightarrow{q} C_*(X, A) \to 0$$

with associated long exact sequence in homology

$$\begin{array}{r} \cdots \xrightarrow{q_*} H_{n+1}(X, A) \xrightarrow{\delta} \\ \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} H_n(X, A) \xrightarrow{\delta} \cdots \\ \to H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{q_*} H_0(X, A) \to 0. \end{array}$$

(2) *(Naturality) If $f\colon (X,A) \to (Y,B)$ is a map of pairs, that is $f(A) \subseteq B$, then there is an induced map $f_*\colon H_n(X,A) \to H_n(Y,B)$ for all $n \in \mathbb{N}_0$ such that*

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(X,A) \arrow[r, "\delta"] \arrow[d, "f_*"] & H_n(A) \arrow[r, "i_*"] \arrow[d, "(f|)_*"] & H_n(X) \arrow[r, "q_*"] \arrow[d, "f_*"] & H_n(X,A) \arrow[r, "\delta"] \arrow[d, "f_*"] & {} \\
\cdots \arrow[r] & H_{n+1}(Y,B) \arrow[r, "\delta"] & H_n(B) \arrow[r, "i_*"] & H_n(Y) \arrow[r, "q_*"] & H_n(Y,B) \arrow[r, "\delta"] & {}
\end{tikzcd}$$

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Proof. (1) is immediate from Theorem 3.3. See p. 127 of Hatcher for details of (2). ☐

Example 9.5.

$$H_k(D^n, S^{n-1}) \cong \begin{cases} \mathbb{Z} & k = n \\ 0 & \text{else.} \end{cases}$$

10. Excision

One of the key properties of relative homology, which is also very useful trick for computations, is called *excision*. It means that if you have a pair (X,A) and $Z \subset A$, under some mild technical restrictions you can remove Z from A and X without changing the relative homology.

Theorem 10.1 (Excision)**.** *Let $Z \subseteq A \subseteq X$ be subsets such that the closure of Z is contained in the interior of A. Then the map of pairs $(X \smallsetminus Z, A \smallsetminus Z) \to (X,A)$ induces an isomorphism*

$$H_n(X \smallsetminus Z, A \smallsetminus Z) \to H_n(X,A)$$

for every $n \in \mathbb{N}_0$.

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This is equivalent to the following version of excision.

Theorem 10.2 (Excision Version II)**.** *Let $U, V \subseteq X$ be subsets of X such that* $\operatorname{Int} U \cup \operatorname{Int} V = X$*. Then the map of pairs $(V, U \cap V) \to (X,U)$ induces an isomorphism*

$$H_n(V, U \cap V) \to H_n(X,U)$$

for every $n \in \mathbb{N}_0$.

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Definition 10.3. Let $A \subseteq X$ be a closed subset with $A \neq \varnothing$. Suppose A is a deformation retract of some neighbourhood $V \subseteq X$ of A, so $A \subseteq V$. Then (X,A) is a *good pair.*

Theorem 10.4. *Let (X,A) be a good pair. Then the quotient map $q\colon (X,A) \to (X/A, A/A) = (X/A, \{\mathrm{pt}\})$ induces an isomorphism $q_*\colon H_n(X,A) \xrightarrow{\cong} H_n(X/A, \{\mathrm{pt}\}) \cong \widetilde{H}_n(X/A)$ for every $n \in \mathbb{N}_0$.*

11. Axioms for homology

Many of the theorems we have proven so far about singular homology can be made into axioms for *homology theories.*

Definition 11.1. An *ordinary homology theory* G_* is, for each $n \in \mathbb{N}_0$, an assignment (i.e. a functor) from pairs of spaces (X, A) to abelian groups, whose output we write as $G_n(X, A)$ such that, writing $G_n(X)$ for $G_n(X, \varnothing)$:

(1) (Dimension) $G_n(\{\text{pt}\}) = \{0\}$ for $n \neq 0$.
(2) (Additivity) $G_n(\coprod_{i\in\mathcal{I}} X_i) \cong \bigoplus_{i\in\mathcal{I}} G_n(X_i)$.
(3) (Functoriality and homotopy invariance) A map of pairs $(X, A) \to (Y, B)$ induces a map on homology $f_*\colon G_n(X, A) \to G_n(Y, B)$. These maps satisfy that $\text{Id}_* = \text{Id}$, $(f \circ g)_* = f_* \circ g_*$, and if $f \sim g$ then $f_* = g_*$.
(4) (Excision) Let $Z \subseteq A \subseteq X$ be subsets such that the closure of Z is contained in the interior of A. Then the map of pairs $(X \smallsetminus Z, A \smallsetminus Z) \to (X, A)$ induces an isomorphism
$$G_n(X \smallsetminus Z, A \smallsetminus Z) \to G_n(X, A)$$
for every $n \in \mathbb{N}_0$.
(5) (Long exact sequence of a pair) For a pair (X, A) there is a *natural* map $\delta\colon G_n(X, A) \to G_{n-1}(A)$, that fits into a natural long exact sequence of homology groups
$$\cdots \xrightarrow{q_*} G_{n+1}(X, A) \xrightarrow{\delta} G_n(A) \xrightarrow{i_*} G_n(X) \xrightarrow{q_*} G_n(X, A) \xrightarrow{\delta} G_{n-1}(A) \to \cdots$$

Remark 11.2.

(i) $G_0(\{\text{pt}\})$ is called the *coefficients* of the homology theory.
(ii) The Mayer-Vietoris sequence can be deduced from the axioms.
(iii) Two other famous homology theories, defined for smaller classes of spaces, are simplicial homology (for simplicial complexes) and cellular homology (for CW complexes). We will discuss CW complexes and cellular homology shortly.
(iv) A *generalised homology theory* is characterised by the same set of axioms bar one: the dimension axiom is removed. Moreover it is permitted for $n \in \mathbb{Z}$ instead of just nonnegative integers.
(v) Bordism $\Omega_n(X, A)$ and topological K-theory are two of the most famous examples of generalised homology theories.

Theorem 11.3. *An ordinary homology theory is characterised by the axioms in Definition 11.1. In particular if $G_0(\{\text{pt}\}) \cong \mathbb{Z}$ then G_n coincides with singular homology.*

12. CW complexes

Given a sequence of spaces and inclusions
$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$$
we define the colimit
$$X := \operatorname{colim}_n X_n := \bigcup_{n\in\mathbb{N}_0} X_n.$$

with the topology that $U \subseteq X$ is open if and only if $U \cap X_n$ is open for every $n \in \mathbb{N}_0$. This is called the *weak topology* on X.

Definition 12.1 (CW complex)**.** A *CW complex* is a space X together with data

(1) a filtration of spaces
$$\varnothing = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots$$
and

(2) a collection of maps $\{\varphi_i \colon S^{n-1} \to X_{n-1}\}_{i \in \mathcal{I}_n}$ called the *attaching maps*,

satisfying the following properties.

(i) For every $n \in \mathbb{N}_0$,
$$X_n = \frac{X_{n-1} \sqcup \amalg_{i \in \mathcal{I}_n} D^n_i}{\varphi_i(x) \sim x,\ i \in \mathcal{I}_n,\ x \in \partial D^n_i \cong S^{n-1}};$$

(ii) $X = \operatorname{colim}_n X_n$, with the associated weak topology.

Note that X_0 is a discrete set. The set X_n is called the n-skeleton of X.

Definition 12.2.

(1) A *CW structure* on a space Y is a CW complex X and a homeomorphism $X \cong Y$.
(2) The dimension of X is -1 if $X = \varnothing$, is n if this is the largest n with $X_{n-1} \neq X$, if such an n exists, and is $\dim X = \infty$ if no such n exists.
(3) A CW complex X is called *finite dimensional* if $\dim X < \infty$.
(4) A CW complex of dimension $\leq n$ is called an n-complex.
(5) The image of each copy of D^n is called a (closed) n-cell, and $\mathring{D}^n$ is called an open n-cell.
(6) A CW complex is *finite* if it has finitely many cells.
(7) The maps $\overline{\varphi}_i \colon D^n \to X$ extending the attaching maps are the *characteristic maps.*

Example 12.3. The nonorientable surface $\mathbb{RP}^2 = \mathbb{R}^3 \smallsetminus \{0\}/x \sim \lambda x$, where $\lambda \in \mathbb{R} \smallsetminus \{0\}$, can be thought of as $S^2/x \sim -x$ or $D^2/y \sim -y$ for $y \in S^1$. It can be decomposed into a 0-cell, a 1-cell, and a 2-cell. In homogeneous coordinates, these are:
$$\mathbb{RP}^2 = \{[1:0:0]\} \cup \{[x:1:0] \mid x \in \mathbb{R}\} \cup \{[x:y:1] \mid x, y \in \mathbb{R}\}.$$

A subcomplex of a CW complex X is a subset $Y \subseteq X$ that is the union of cells of X.

Definition 12.4.

(1) A space X is *Hausdorff* if for all $x, y \in X$, there exist open subsets $U \ni x$ and $V \ni y$ with $U \cap V = \varnothing$.
(2) A space X is *normal* if for every pair of closed sets $S, T \subseteq X$, there exist open subsets $U \subseteq S$ and $V \subseteq T$ with $U \cap V = \varnothing$.

Theorem 12.5 (Topological properties of CW complexes)**.** *Let X be a CW complex and let A be a subcomplex.*

(1) Every point of X is closed.

(2) The CW complex X is Hausdorff and normal.
(3) Every cell of X is closed and compact.
(4) A CW complex is compact if and only if it is finite.
(5) A CW complex is connected if and only if it is path connected.
(6) The subcomplex A is closed and there is a neighbourhood $V \supseteq A$ that deformation retracts to A, that is (X, A) is a good pair.

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13. Degrees of maps

In Example 2.7 we computed the homology of S^n the n-sphere:

$$H_i(S^n) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, n \\ 0, & \text{otherwise.} \end{cases}$$

Definition 13.1. Given a map $f\colon S^n \to S^n$, the induced map on n^{th} homology, $f_*\colon H_n(S^n) \to H_n(S^n)$ is a homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$, hence $f_*(a) = da$ for some integer d depending only on f. We call this integer the *degree* of f, denoted $\deg(f)$.

Remark 13.2.

(1) $\deg(\mathrm{Id}_{S^n}) = 1$ since $\mathrm{Id}_{S^n *}$ is the identity map on homology.
(2) If f is not surjective then $\deg(f) = 0$. To show this suppose f misses $x \in S^n$ then f factors as a composition $S^n \to S^n \smallsetminus \{x\} \hookrightarrow S^n$, so f_* factors as composition $H_n(S^n) \to H_n(S^n \smallsetminus \{x\}) \to H_n(S_n)$. Since $S^n \smallsetminus \{x\}$ is contractible $H_n(S^n \smallsetminus \{x\}) = 0$ so the map on homology must be 0.
(3) If $f \sim g$ then $f_* = g_*$ so $\deg(f) = \deg(g)$.
(4) $(f \circ g)_* = f_* \circ g_*$ so $\deg(f \circ g) = \deg(f)\deg(g)$.
(5) If f is a homotopy equivalence $\deg(f) = \pm 1$, since if g is its homotopy inverse then $\deg(f)\deg(g) = \deg(\mathrm{Id}) = 1$

Definition 13.3 (Suspension)**.** Given a topological space X, the suspension $\mathrm{S}\,X$ is the quotient topological space $X \times [0,1]/X \times \{0,1\}$. Given a map $f\colon X \to Y$ the suspension map $\mathrm{S}\,f$ is the map $\mathrm{S}\,f\colon \mathrm{S}\,X \to \mathrm{S}\,Y$ defined by sending $(p,t) \in X \times [0,1]$ to $(f(p),t) \in Y \times [0,1]$ then taking quotients.

Exercise 13.4. Show that $\mathrm{S}\,S^n \cong S^{n+1}$ and $\mathrm{C}\,S^n = D^{n+2}$.

Proposition 13.5. *For $i \geq 0$ there is a natural isomorphism $\widetilde{H}_{i+1}(\mathrm{S}\,X) \xrightarrow{\cong} \widetilde{H}_i(X)$, in the sense that for any map $f\colon X \to Y$ the following diagram commutes:*

(13.6)
$$\begin{tikzcd} \widetilde{H}_{i+1}(\mathrm{S}\,X) \arrow[r, "\cong"] \arrow[d, "\mathrm{S}\,f_*"] & \widetilde{H}_i(X) \arrow[d, "f_*"] \\ \widetilde{H}_{i+1}(\mathrm{S}\,Y) \arrow[r, "\cong"] & \widetilde{H}_i(X) \end{tikzcd}$$

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Proof. For a topological space X define $\mathrm{C}\,X$ by $\mathrm{C}\,X = X \times [0,1]/X \times \{0\}$, and for a map $f\colon X \to Y$ define $\mathrm{C}\,f\colon \mathrm{C}\,X \to \mathrm{C}\,Y$ by sending (x,t) to $(f(x),t)$. Taking

$q\colon \mathrm{C}\,X \to \mathrm{S}\,X$ to be the obvious quotient, the following diagram commutes:

$$\begin{tikzcd}
(\mathrm{S}\,X, \{\mathrm{pt}\}) \arrow[d, "\mathrm{S}\,f"] & (\mathrm{C}\,X, X \times \{1\}) \arrow[l, "q"'] \arrow[d, "\mathrm{C}\,f"] \\
(\mathrm{S}\,Y, \{\mathrm{pt})\}) & (\mathrm{C}\,Y, Y \times \{1\}) \arrow[l, "q"']
\end{tikzcd}$$

So does the corresponding diagram in homology.

Since $\mathrm{C}\,f$ is a map of pairs as above, we use the long exact sequence and naturality in Theorem 11.3 to obtain the following commutative diagram:

$$\begin{tikzcd}
\cdots \widetilde{H}_{i+1}(\mathrm{C}\,X) \arrow[r] \arrow[d, "\mathrm{C}\,f_*"] & \widetilde{H}_{i+1}(\mathrm{C}\,X, X \times \{1\}) \arrow[r, "\delta"] \arrow[d, "\mathrm{C}\,f_*"] & \widetilde{H}_i(X) \arrow[r, "i_*"] \arrow[d, "f_*"] & \widetilde{H}_i(\mathrm{C}\,X) \cdots \arrow[d, "\mathrm{C}\,f"] \\
\cdots \widetilde{H}_{i+1}(\mathrm{C}\,Y) \arrow[r] & \widetilde{H}_{i+1}(\mathrm{C}\,Y, Y \times \{1\}) \arrow[r, "\delta"] & \widetilde{H}_i(Y) \arrow[r, "i_*"] & \widetilde{H}_i(\mathrm{C}\,Y) \cdots
\end{tikzcd}$$

Since $\mathrm{C}\,X$ and $\mathrm{C}\,Y$ are contractible, δ is an isomorphism for $i \geq 0$.

Combining these two commutative diagrams gives:

$$\begin{tikzcd}
\widetilde{H}_{i+1}(\mathrm{S}\,X) \arrow[d, "\mathrm{S}\,f_*"] & \widetilde{H}_{i+1}(\mathrm{C}\,X, X \times \{1\}) \arrow[l, "q_*", "\cong"'] \arrow[r, "\delta", "\cong"'] \arrow[d, "\mathrm{C}\,f_*"] & \widetilde{H}_i(X) \arrow[d, "f_*"] \\
\widetilde{H}_{i+1}(\mathrm{S}\,Y) & \widetilde{H}_{i+1}(\mathrm{C}\,Y, Y \times \{1\}) \arrow[l, "q_*", "\cong"'] \arrow[r, "\delta", "\cong"'] & \widetilde{H}_i(Y)
\end{tikzcd}$$

The homomorphisms q_* are isomorphisms by Theorem 10.4, so $\delta \circ q_*^{-1}$ gives the desired isomorphism. $\square$

Corollary 13.7. *For $f\colon S^n \to S^n$ we have $\mathrm{S}\,f\colon S^{n+1} \to S^{n+1}$ and $\deg(\mathrm{S}\,f) = \deg(f)$*

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Exercise 13.8. Let $R_n\colon S^n \to S^n$ be the reflection through S^{n-1} living on the equator. Show for $n \geq 1$ that $\mathrm{S}\,R_n = R_{n+1}$.

Note also that R_1 is $\mathrm{S}\,R_0$, where $R_0\colon \{-1,1\} \to \{-1,1\}$ is the map that swaps the points. The reduced chain complex $\widetilde{C}(\{-1,1\})$ is easy to understand:

$$\cdots \to \widetilde{C}_1 \xrightarrow{d_1} \widetilde{C}_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

has $\widetilde{C}_i \cong \mathbb{Z} \oplus \mathbb{Z}$ for every $i \in \mathbb{N}_0$. The boundary map d_1 is the zero map. $\widetilde{C}_0 = \mathbb{Z}\sigma_{-1} \oplus \mathbb{Z}\sigma_1$ where σ_p is the zero simplex mapping to the point p. Hence

$$\widetilde{H}_0(\{-1,1\}) = \{(a,b) \mid a + b = 0\} = \mathbb{Z}\langle(1,-1)\rangle.$$

Since $(R_0)_*((1,-1)) = (-1,1) = -(1,-1)$ we see that $\deg(R_0) = -1$.

Remark 13.9.

(1) Reflections have degree -1 since

$$\deg(R_i) = \deg(\mathrm{S}\,R_{i-1}) = \deg(R_{i-1}) = \cdots = \deg(R_0) = -1.$$

(2) The antipodal map $a\colon S^n \to S^n$ has degree $(-1)^{n+1}$ since it is the composition of $n+1$ reflections, in each of the hyperplanes $\{x_i = 0\}$, for $i = 0, \ldots, n$.
(3) If f has no fixed points then $f \sim a$, so $\deg(f) = (-1)^{n+1}$

To see (3) note that $h_t\colon S^n \to S^n$ defined by

$$h_t(x) = \frac{(1-t)\,f(x) - tx}{\|(1-t)\,f(x) - tx\|},$$

with addition performed in $\mathbb{R}^{n+1}$, gives a homotopy from f to a. Note that the path $(1-t)\,f(x) - tx$ from $f(x)$ to $-x$ passes through 0 only if $f(x) = x$, contradicting the condition of no fixed points. Thus $h_t(x) \neq 0$ for all $x \in S^n$ and for all $t \in [0,1]$, so h_t is a well-defined homotopy. We can use this to prove the following.

Theorem 13.10 (Hairy Ball theorem)**.** *If n is even, then for every continuous vector field v on S^n, there is a point of S^n at which v vanishes.*

Proof. Suppose $x \mapsto v(x)$ is a nonvanishing vector field on S^n. By viewing $S^n \subset R^{n+1}$ we may view $v(x)$ as a vector in R^{n+1}. Let $w(x) = v(x)/\|v(x)\|$. Now $h_t\colon S^n \to S^n$ defined by $h_t(x) = \cos(t)x + \sin(t)v(x)$ for $t \in [0,\pi]$ is a homotopy from Id_{S^n} to the antipodal map a; note that $w(x)$ is a tangent to S^n at x, so is orthogonal to x, so $||h_t(x)|| = 1$. But since n is even, $\deg(a) = -1 \neq \deg(\mathrm{Id}_{S^n})$, which yields a contradiction. Hence no such v exists, and all vector fields on f have at least one point where they vanish. $\square$

14. Local degree

We require a technique for computing the degrees of maps. Suppose $n > 0$ and $f\colon S^n \to S^n$ is a map, and for some point $y \in S^n$ that $f^{-1}(y)$ consists of finitely many points $x_1, \ldots, x_m$. Let $U_1, \ldots, U_m$ be disjoint n-disc neighbourhoods of the x_i, and let V be an n-disc neighbourhood of y so that $f(U_i) \subset V$ for each i. Then we have the following diagram:

$$\begin{tikzcd}
 & H_n(U_i, U_i \smallsetminus \{x_i\}) \arrow[r, "f_*"] \arrow[dl, "\cong"'] \arrow[d, "k_i"] & H_n(V, V \smallsetminus \{y\}) \arrow[d, "\cong"] \\
H_n(S^n, S^n \smallsetminus \{x_i\}) & H_n(S^n, S^n \smallsetminus f^{-1}(y)) \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n \smallsetminus \{y\}) \\
 & H_n(S^n) \arrow[r, "f_*"] \arrow[ul, "\cong"] \arrow[u, "j"] & H_n(S^n). \arrow[u, "\cong"']
\end{tikzcd}$$

All maps are induced by inclusions, the obvious quotient or by f. The top two isomorphisms are by excision, the bottom two by the long exact sequence of the pairs.

Definition 14.1. Since the groups $H_n(U_i, U_i \smallsetminus \{x_i\})$ and $H_n(V, V \smallsetminus \{y\})$ can be canonically identified with $H_n(S^n) \cong \mathbb{Z}$ as above, we call the degree of the map $H_n(U_i, U_i \smallsetminus \{x_i\}) \xrightarrow{f_*} H_n(V, V \smallsetminus \{y\})$ the *local degree* at X_i, denoted $\deg(f|x_i)$.

Proposition 14.2. $\deg(f) = \sum_i \deg(f|x_i)$

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Proof. Identify $H_n(S^n)$ with $\mathbb{Z}$. Now identify all outer groups $H_n(S^n)$ by the isomorphisms above, hence identifying them all with $\mathbb{Z}$, making all isomorphisms in the diagram the identity. Identify $H_n(S^n, S^n \smallsetminus f^{-1}(y))$ with $\oplus_i H_n(U_i, U_i \smallsetminus x_i) = \mathbb{Z}^m$ by excision. The upper triangle commutes so $k_i(1) = (0, \dots, 0, 1, 0, \dots, 0)$ with the 1 in the i^{th} position. The lower triangle commutes so $p_i(j(1)) = 1$ for each i, so $j(1) = (1, \dots, 1) = \sum_i k_i(1)$. Commutativity of the top square says $f_*(k_i(1)) = \deg(f|x_i)$. Commutativity of the bottom square says $\deg(f) = f_*(1) = f_*(1, \dots, 1) = f_*(\sum_i k_i(1)) = \sum_i f_*(k_i(1)) = \sum_i \deg(f|x_i)$. $\square$

Example 14.3.

(1) If f is a homeomorphism, then every arrow on the diagram becomes an isomorphism, so we can diagram chase to show that $\deg(f) = \deg(f|x)$, for all $x \in X$.

(2) Considering $S^1 \subset \mathbb{C}$ we can define $f\colon S^1 \to S^1$ by $f_k(z) = z^k$ for $k > 0$. Setting $y = 0$ above, we have ℓ pre-image points $x_1, \dots, x_n$. For each i the restriction $f|_{U_i}\colon U_i \to V$ is homotopic to $r_\theta|_{U_i}\colon U_i \to V$, where r_θ is a rotation of S^1 through angle θ. Hence $\deg(f_k|x_i) = \deg(r_\theta|x_i) = 1$, since r_θ is a homeomorphism; note these maps are not globally homotopic, just their local restrictions, which means they induce the same maps on local homology. By the theorem we have $\deg(f_k) = k$. We can also prove this for $k < 0$ by noting that $f_{-k} = R_1 \circ f_k$ so $\deg(f_{-k}) = -\deg(f_k)$.

(3) Taking repeated suspensions of the above map we can construct a map of any degree from $S^m \to S^m$, for every m.

15. Cellular homology

For any CW complex X we wish to construct a chain complex for which $C_n^{CW}(X)$ is the free abelian group generated by the n-cells, and for which we can understand the boundary maps using the theorems of the previous section. Motivated by this we prove the following lemma, 2.34 of Hatcher.

Lemma 15.1. *For X a CW complex, and X^n the n-skeleton:*

(1) $H_k(X^n, X^{n-1}) = 0$ for $k \neq n$, and free abelian for $k = n$ with generators in one to one correspondence with the n-cells;

(2) $H_k(X^n) = 0$ for $k > n$;

(3) The map $H_k(X^n) \to H_k(X)$ induced by inclusion, is an isomorphism for $k < n$.

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We can now define the cellular chain group

$$C_n^{CW}(X) = H_n(X^n, X^{n-1}).$$

We define the boundary map $d_n\colon C_n^{CW}(X) \to C_{n-1}^{CW}(X)$ as the composition

$$H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$$

where the first map comes from the long exact sequence of the pair (X^n, X^{n-1}) and the second map comes from the long exact sequence of the pair (X^{n-1}, X^{n-2}).

Lemma 15.2. *$d_n \circ d_{n+1} = 0$, so $(C_*^{CW}(X), d_*)$ is a chain complex.*

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Definition 15.3. The *CW homology or cellular homology of X* is $H_n^{CW}(X) := H_n((C_*^{CW}(X), d_*)) = \ker d_n / \operatorname{im} d_{n+1}$.

Theorem 15.4. *For every CW complex X, we have an isomorphism $H_n(X) \cong H_n^{CW}(X)$. In particular, the CW homology of a space X that admits a CW structure is independent of the choice of CW structure.*

PNE

For applications, we need to be able to compute the cellular boundary maps. We can do this using degrees of maps between spheres.

Proposition 15.5. *Let $\{e_i^n\}$ be the n-cells of X, corresponding to a basis for $C_n^{CW}(X)$. Let $\{e_j^{n-1}\}$ be the n-cells of X, corresponding to a basis for $C_{n-1}^{CW}(X)$. The boundary map $d_n \colon C_n^{CW}(X) \to C_{n-1}^{CW}(X)$ is given by*

$$d_n(e_i^n) = \sum_j c_{ij} e_j^{n-1}$$

where the integer c_{ij} is the degree of the composition

$$S^{n-1} \xrightarrow{\alpha_i} X^{n-1} \to X^{n-1}/X^{n-2} = \bigvee_j S^{n-1} \xrightarrow{q_j} S^{n-1}$$

where α_i is the attaching map for e_i^n and the final map q_j identifies to the basepoint all wedge summands S^{n-1} other than the jth summand, that is the image of e_j^{n-1}.

PNE

Example 15.6. We can compute the homology of spaces such as $T^2 = S^1 \times S^1$, $\mathbb{RP}^n$, and the Klein bottle rather easily using CW homology.

Example 15.7. The product of spheres $S^n \times S^m$, has a cell decomposition $e^0 \cup e^n \cup e^m \cup e^{n+m}$. The $(\max\{n,m\})$-skeleton is $S^n \vee S^m$, so $S^n \times S^m \cong S^n \vee S^m \cup D^{n+m}$. The attaching maps are such that all the boundary maps in the cellular chain complex are zero. One can easily compute the homology of $S^n \times S^m$ using this information.

Example 15.8. Let $n \geq 1$ and let G be an abelian group. There exists a connected, simply-connected space $M(G,n)$ whose reduced homology is isomorphic to G in degree n and vanishes otherwise. If $n \geq 2$ we can assume that $M(G,n)$ is simply connected. The idea is to find a free abelian group F with a surjection $\phi \colon F \to G$, let $K := \ker \phi$, and realise the inclusion map $K \to F$ as the CW boundary map $C_{n+1}^{CW}(X) \xrightarrow{d_{n+1}} C_n^{CW}(X)$.

By taking wedge products

$$M(G_1, 1) \vee M(G_2, 2) \vee \cdots$$

we can realise any sequence $G_1, G_2, \ldots$ of abelian groups as the reduced homology of some CW complex. Note that the G_i need not be finitely generated and the resulting CW complex need be neither finite nor finite dimensional.

Definition 15.9. A map $f \colon X \to Y$ between CW complexes is said to be *cellular* if $f(X^n) \subseteq Y^n$.

Lemma 15.10. *A cellular map $f\colon X \to Y$ between CW complexes X and Y induces a chain map $C_*^{CW}(X) \to C_*^{CW}(Y)$.* PNE

Theorem 15.11. *Every map $X \to Y$ between CW complexes is homotopic to a cellular map.* PNE

This is the end of the Michaelmas term notes.

16. Epiphany term summary

In the second term we have the following aims.

(1) Study *covering spaces*, and how to classify them.
(2) Learn about *cohomology* of spaces, and its relation to homology via the *universal coefficient theorem.*
(3) Homology with different abelian groups as coefficients.
(4) Study the homology and cohomology of *manifolds. Poincaré duality* places restrictions on these groups.
(5) The *cup* product makes the cohomology groups of a space into a *ring.* The ring structure contains a lot of information about the space. We will put a fair amount of effort into being able to calculate the cup product.

17. Covering spaces

Definition 17.1. Let X be a space. A *covering space* is a space $\widetilde{X}$ and a map $p\colon \widetilde{X} \to X$ such that for every $x \in X$, there is an open neighbourhood $U \ni x$ for which the inverse image $p^{-1}(U)$ is a disjoint union of open sets $\{V_i\}$, with each $V_i \subseteq \widetilde{X}$ and $p|_{V_i}\colon V_i \to U$ a homeomorphism.

The open set U is called *evenly covered* and the sets V_i are called the *sheets of $\widetilde{X}$ over U.* The map p is called a *covering map.*

Example 17.2.

(1) Let X and Y be discrete spaces and let $f\colon Y \to X$ be a surjective map. Then f is a covering map.
(2) The map $p\colon \mathbb{R} \to S^1$ sending $t \mapsto e^{2\pi i t}$ is a covering map.
(3) The map $p_n\colon S^1 \to S^1$ sending $z \mapsto z^n$ is a covering map.
(4) The map $\sqcup^m S^1 \to S^1$ sending each component to S^1 via the identity map, is a covering map.

If X is connected and $\widetilde{X}$ is connected and simply connected, then $\widetilde{X}$ is called the *universal cover* of X. We will discuss the universal cover in more detail later.

Definition 17.3. Let $A = \{a_i\}$ be a set. The *free group* on A, F_A, is the set of *reduced* words in the symbols a_i and a_i^{-1}, plus the empty word (the identity element). The multiplication is concatenation. Here reduced means that all instances of $a_i a_i^{-1}$ have been deleted.

The fundamental group of a graph is free. In particular the fundamental group of $S^1 \vee S^1$ is free. The covering spaces of $S^1 \vee S^1$ are a fascinating source of examples worth contemplating. See class notes, Hatcher page 58, and ET problem sheet 1.

Theorem 17.4 (Homotopy lifting property)**.** *Let $p\colon \widetilde{X} \to X$ be a covering space. Let $f_t\colon Y \to X$ a homotopy, starting with $f_0\colon Y \to X$. Let $\widetilde{f_0}\colon Y \to \widetilde{X}$ be a lift of f_0, that is $p \circ \widetilde{f_0} = f_0$. Then there exists a unique homotopy $\widetilde{f_t}\colon Y \to \widetilde{X}$ of $\widetilde{f_0}$, with $p \circ \widetilde{f_t} = f_t$.*

PNE

Proposition 17.5. *Let $p\colon \widetilde{X} \to X$ be a covering space. Let $x_0 \in X$ be a basepoint, and let $\widetilde{x}_0 \in p^{-1}(x_0) \subseteq \widetilde{X}$ be a choice of lift of x_0. The map $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X)$ is injective.*

PNE

So the image $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ is abstractly isomorphic to the fundamental group of $\widetilde{X}$. This subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ is important for the classification of connected covering spaces. The image $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ is the subgroup of loops in X that lift to loops in $\widetilde{x}_0$.

Proposition 17.6. *Let $p\colon \widetilde{X} \to X$ be a covering space. Let $x_0 \in X$ be a basepoint, and let $\widetilde{x}_0 \in p^{-1}(x_0) \subseteq \widetilde{X}$ be a choice of lift of x_0. Suppose that $\widetilde{X}$ and X are path connected. The number of sheets of the covering space equals the index*

$$[\pi_1(X, x_0) : p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))].$$

PNE

In particular, the number of sheets at every point in X is the same.

Definition 17.7. A space Y is *locally path connected* if for every $y \in Y$ and for every neighbourhood U of y, there exists an open neighbourhood $V \subseteq U$ such that $y \in V$ and V is path connected.

A space Y is *semi-locally simply connected* if for every $y \in Y$, there exists a neighbourhood U of y with $\pi_1(U, y) \to \pi_1(Y, y)$ the trivial map.

Theorem 17.8 (Lifting criterion)**.** *Let $p\colon \widetilde{X} \to X$ be a covering space, with $f\colon Y \to X$ path connected and locally path connected. There exists a lift $\widetilde{f}\colon Y \to \widetilde{X}$ if and only if*

$$f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)).$$

If Y is connected and $\widetilde{f_1}, \widetilde{f_2}$ are lifts of f that agree at one point $y \in Y$, then $\widetilde{f_1} = \widetilde{f_2}$.

PNE

Theorem 17.9. *Let X be a path connected, locally path connected, and semi-locally simply connected space. Then X has a connected and simply connected covering space $\widetilde{X}$, called the universal covering space of X.*

PNE

We can now work towards the classification of connected covering spaces.

Proposition 17.10. *Let X be a path connected, locally path connected, and semi-locally simply connected space with basepoint x_0. Let $H \leq \pi_1(X, x_0)$ be a subgroup. There exists a covering space $p\colon X_H \to X$ with $p_*(\pi_1(X_H, x_0)) = H$ for some $\widetilde{x}_0 \in X_H$.*

PNE

The idea of the proof is to define X_H as a quotient of the universal cover.

Definition 17.11. Covering space $p_1\colon \widetilde{X}_1 \to X$ and $p_2\colon \widetilde{X}_2 \to X$ are *isomorphic* if there is a homeomorphism $f\colon \widetilde{X}_1 \xrightarrow{\cong} \widetilde{X}_2$ such that $p_1 = p_2 \circ f$.

Proposition 17.12. *Consider covering space $p_1\colon \widetilde{X}_1 \to X$ and $p_2\colon \widetilde{X}_2 \to X$, with base points $x_1 \in X_1$, $x_2 \in X_2$, $\widetilde{x}_1 \in p_1^{-1}(x_1) \subseteq \widetilde{X_1}$ and $\widetilde{x}_2 \in p_2^{-1}(x_2) \subseteq \widetilde{X}_2$, are isomorphic if and only if*

$$(p_1)_*(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = (p_2)_*(\pi_1(\widetilde{X}_2, \widetilde{x}_2)).$$

PNE

Note that connected and locally path connected implies path connected, so we could assume connected instead of path connected. We will do this in the statement of the main classification result.

Theorem 17.13 (Classification of connected covering spaces)**.** *Let X be a connected, locally path connected, and semi-locally simply connected space. Sending a covering space $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ to $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ induces a bijection*

$$\frac{\{\textit{Connected covering spaces } p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)\}}{\textit{basepoint preserving isomorphism}} \leftrightarrow \{\textit{Subgroups of } \pi_1(X, x_0)\}.$$

Moreover, the same correspondence induces a bijection between

$$\frac{\{\textit{Connected covering spaces } p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)\}}{\textit{isomorphism}} \leftrightarrow \frac{\{\textit{Subgroups of } \pi_1(X, x_0)\}}{\textit{conjugacy}}.$$

PNE

This gives a beautiful correspondence between algebra and topology.

18. Cohomology

We introduce groups of homomorphisms.

Let A be a group, and let G be an abelian group. Define an abelian group

$$\mathrm{Hom}(A, G) := (\{\text{group homomorphisms } A \to G\}, +, 0).$$

The group structure is defined by $(f + g)(a) = f(a) + g(a) \in G$. The identity is $0(a) = 0 \in G$ for all $a \in A$.

Let $f\colon A \to B$ be a group homomorphism. There is a homomorphism induced by f:

$$\begin{array}{rcl} f^*\colon \mathrm{Hom}(B, G) & \to & \mathrm{Hom}(A, G) \\ (\varphi\colon B \to G) & \mapsto & (\varphi \circ f\colon A \to B \to G). \end{array}$$

Let $g\colon G \to H$ be a homomorphism of abelian groups. Then g induces a homomorphism

$$\begin{array}{rcl} g_*\colon \mathrm{Hom}(A, G) & \to & \mathrm{Hom}(A, H) \\ (\varphi\colon A \to G) & \mapsto & (g \circ \varphi\colon A \to G \to H). \end{array}$$

Example 18.1.

(1) $\mathrm{Hom}(\mathbb{Z}, G) \cong G$ for all G, with the map $\varphi \mapsto \varphi(1)$ giving an isomorphism.
(2) $\mathrm{Hom}(\mathbb{Z}/n, G) \cong \ker(\cdot n\colon G \to G)$.
(3) $\mathrm{Hom}(\mathbb{Z}/n, \mathbb{Z}/m) \cong \mathbb{Z}/\gcd(n, m)$.

Lemma 18.2. *Let $\{A_i\}_{i\in I}$ be a sequence of abelian groups and let G be an abelian group. Then*

$$\mathrm{Hom}(\bigoplus_{i\in I} A_i, G) \cong \prod_{i\in I} \mathrm{Hom}(A_i, G)$$

via the map $f \mapsto \prod_{i\in I} f|_{A_i}$.

Note the change between direct sum and direct product. What is the difference? The elements of both are tuples $(a_i)_{i\in I}$, and the addition and the identity are the same, but the sets are different: in the direct product any tuples are allowed but in the direct sum only $\{(a_i)_{i\in I} \mid \text{ finitely many } a_i \neq e_{A_i}\}$.

Lemma 18.3. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups and let G be an abelian group. Suppose that C is free abelian. Then*

$$0 \to \mathrm{Hom}(C,G) \to \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G) \to 0$$

is also short exact.

Lemma 18.4. $\mathrm{Hom}(\mathbb{Z}^n,\mathbb{Z}) \cong \mathbb{Z}^n$.

Proof. For I finite, $\bigoplus_I$ and $\prod_I$ coincide. Therefore

$$\mathrm{Hom}(\mathbb{Z}^n,\mathbb{Z}) \cong \mathrm{Hom}(\bigoplus_{i=1}^n \mathbb{Z},\mathbb{Z}) \cong \prod_{i=1}^n \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \cong \prod_{i=1}^n \mathbb{Z} \cong \bigoplus_{i=1}^n \mathbb{Z} = \mathbb{Z}^n.$$

□

Definition 18.5. A *cochain complex* is a sequence of abelian groups

$$0 \to D^0 \xrightarrow{\delta_0} D^1 \xrightarrow{\delta_1} D^2 \xrightarrow{\delta_2} \cdots \to D^n \xrightarrow{\delta_n} \cdots$$

with $\delta_{i+1} \circ \delta_i = 0$ for all i.

We call the δ_i the *coboundary maps.*

The cohomology groups of a cochain complex are

$$H^n(D^*) := \frac{\ker(\delta_n \colon D^n \to D^{n+1})}{\mathrm{im}(\delta_{n-1} \colon D^{n-1} \to D^n)}.$$

Here $\ker(\delta_n \colon D^n \to D^{n+1})$ are the cocycles and $\mathrm{im}(\delta_{n-1} \colon D^{n-1} \to D^n)$ are the coboundaries.

Definition 18.6. A *cochain map* $f \colon C^* \to D^*$ consists of a homomorphism $f_i \colon C^i \to D^i$ for each i such that

$$\begin{tikzcd} C^i \arrow[r, "f_i"] \arrow[d, "\delta_i^C"] & D^i \arrow[d, "\delta_i^D"] \\ C^{i+1} \arrow[r, "f_{i+1}"] & D^{i+1} \end{tikzcd}$$

commutes for all i, meaning that both routes round the square give the same outcome for every element of C^i.

Definition 18.7. A *cochain homotopy* between cochain maps $f,g \colon C^i \to D^i$ consists of homomorphisms $h_i \colon C^i \to D^{i-1}$ for all i with $f_i - g_i = h_{i+1} \circ \delta_i + \delta_{i-1} \circ h_i$

for all i.

$$\begin{tikzcd}
\cdots \arrow[r] & C^{i-1} \arrow[r, "\delta_{i-1}"] \arrow[d, "f_{i-1}-g_{i-1}"'] & C^i \arrow[r, "\delta_i"] \arrow[d, "f_i-g_i"'] \arrow[dl, "h_i"'] & C^{i+1} \arrow[r, "\delta_{i+1}"] \arrow[d, "f_{i+1}-g_{i+1}"] \arrow[dl, "h_{i+1}"'] & \cdots \\
\cdots \arrow[r] & D^{i-1} \arrow[r, "\delta_{i-1}"] & D^i \arrow[r, "\delta_i"] & D^{i+1} \arrow[r, "\delta_{i+1}"] & \cdots
\end{tikzcd}$$

The proof of the next lemma is directly analogous to the corresponding lemma for homology groups.

Lemma 18.8.

(i) Let $f\colon C^* \to D^*$ be a cochain map. Then

$$\begin{array}{rcl} f_*\colon H^n(C) & \to & H^n(D) \\ [c] & \mapsto & [f_n(c)] \end{array}$$

is a well-defined map.

(ii) If $f \sim g\colon C^* \to D^*$ are two cochain homotopic cochain maps then $f_* = g_*\colon H^n(C^*) \to H^n(D^*)$ for all $n \in \mathbb{N}_0$.

Let (C_*, ∂_*) be a chain complex, with $\partial_n\colon C_n \to C_{n-1}$. Let G be an abelian group. For $n \in \mathbb{N}_0$, define

$$\delta_n := \partial_{n+1}^*\colon \operatorname{Hom}(C_n, G) \to \operatorname{Hom}(C_{n+1}, G).$$

We obtain a cochain complex $\operatorname{Hom}(C_*, G)$ given by

$$\operatorname{Hom}(C_0, G) \xrightarrow{\delta_0} \operatorname{Hom}(C_1, G) \xrightarrow{\delta_1} \operatorname{Hom}(C_2, G) \xrightarrow{\delta_2} \cdots$$
$$\to \operatorname{Hom}(C_n, G) \xrightarrow{\delta_n} \operatorname{Hom}(C_{n+1}, G) \xrightarrow{\delta_{n+1}} \cdots$$

This is the *cochain complex dual to* (C_*, ∂_*). The *cohomology of* C_* with coefficients in G is

$$H^n(C; G) := \ker(\delta_n)/\operatorname{im}(\delta_{n-1}).$$

Lemma 18.9.

(i) Let $f_*\colon C_* \to D_*$. Then

$$\begin{array}{rcl} f^*\colon \operatorname{Hom}(D_n, G) & \to & \operatorname{Hom}(C_n, G) \\ (\varphi\colon D_n \to G) & \mapsto & (\varphi \circ f\colon C_n \to G) \end{array}$$

is a cochain map, and therefore induces a map $f^*\colon H^n(D_*) \to H^n(C_*)$.

(ii) Two chain homotopic chains map $f, g\colon C_* \to D_*$ induce cochain homotopic cochain maps

$$f^*, g^*\colon \operatorname{Hom}(D_*, G) \to \operatorname{Hom}(C_*, G),$$

so that

$$f^* = g^*\colon H^n(D) \to H^n(C).$$

Now we define the singular cohomology of a topological space. Let (X, A) be a pair of spaces. Usually we will have $A = \emptyset$, and then we write $X = (X, \emptyset)$. Define the singular cochain group to be

$$C^n(X, A; G) := \mathrm{Hom}(C_n(X, A), G)$$

and

$$\delta_n := \partial_{n+1}^* \colon \mathrm{Hom}(C_n(X, A), G) \to \mathrm{Hom}(C_{n+1}(X, A), G).$$

We call

$$\ker(\delta_n \colon C^n(X, A; G) \to C^{n+1}(X, A; G))$$

the *singular cocycles* and

$$\mathrm{im}(\delta_{n-1} \colon C^{n-1}(X, A; G) \to C^n(X, A; G))$$

the singular coboundaries. The singular cochain complex is

$$\begin{aligned} C^0(X, A; G) &\xrightarrow{\delta_0} C^1(X, A; G) \xrightarrow{\delta_1} C^2(X, A; G) \xrightarrow{\delta_2} \cdots \\ &\to C^{n-1}(X, A; G) \xrightarrow{\delta_{n-1}} C^n(X, A; G) \xrightarrow{\delta_n} C^{n+1}(X, A; G) \to \cdots \end{aligned}$$

We then define the singular cohomology

$$H^n(X, A; G) := H^n(C^*(X, A; G)) = \ker \delta_n / \operatorname{im} \delta_{n-1}.$$

When $G = \mathbb{Z}$, we will often omit the coefficients and simply write $H^n(X, A)$.

We also have CW cohomology. Let X be a CW complex. Recall that there is the cellular chain complex

$$(C_*^{CW}(X), \partial_*^{CW}).$$

Let G be an abelian group. Define the CW cochain groups to be

$$C^i_{CW}(X; G) := \mathrm{Hom}(C_i^{CW}(X), G).$$

The CW coboundary maps are given by

$$\delta_i^{CW} := (\partial_{i+1}^{CW})^* \colon \mathrm{Hom}(C_i^{CW}(X), G) \to \mathrm{Hom}(C_{i+1}^{CW}(X), G).$$

Then the CW cohomology is

$$H^n_{CW}(X; G) := \ker(\delta_n^{CW}) / \operatorname{im}(\delta_{n-1}^{CW}).$$

Theorem 18.10. *For every CW complex X, for every abelian group G, and for every $n \in \mathbb{N}_0$, we have $H^n(X; G) \cong H^n_{CW}(X; G)$.*

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19. Ext groups

Lemma 19.1. *Let G be an abelian group and let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups. Then*

$$0 \to \mathrm{Hom}(C, G) \to \mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G)$$

is exact.

Proof. Let
$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$
be exact. Then consider the sequence of maps
$$0 \to \mathrm{Hom}(C, G) \xrightarrow{g^*} \mathrm{Hom}(B, G) \xrightarrow{f^*} \mathrm{Hom}(A, G).$$
We want to show that this is an exact sequence. Let $\varphi \in \ker g^*$. That is, $\varphi \circ g \colon B \to G$ is the zero map. Let $c \in C$. Then there is a $b \in B$ such that $g(b) = c$. Then $\varphi(c) = \varphi \circ g(b) = 0$, so $\varphi = 0$. Thus g^* is injective.

To show exactness at $\mathrm{Hom}(B, G)$, suppose that $\theta = g^*(\varphi) \in \mathrm{Hom}(B, G)$. Then $f^* \circ g^*(\varphi)(a) = \varphi \circ g \circ f(a) = \varphi(0) = 0$ for all $a \in A$. Therefore $f^* \circ g^* = 0$. Now let $\theta \in \ker f^* \subseteq \mathrm{Hom}(B, G)$. That is, $\theta \circ f \colon A \to G$ is the zero map. We want to show that $\theta = g^*(\varphi)$ for some $\varphi \colon C \to G$. Define $\varphi(c)$, for $c \in C$, by taking $b \in B$ with $g(b) = c$ and defining $\varphi(c) = \theta(b)$. Suppose that $b' \in B$ with $g(b') = c$ too. Then $\theta(b) - \theta(b') = \theta(b - b')$. Since $b - b' \in \ker g$, we have that $b - b' = f(a)$ for some $a \in A$. Then $\theta(b - b') = \theta(f(a)) = 0$, so $\theta(b) = \theta(b')$. Therefore $\varphi(c)$ is well-defined, and does not depend on the choice of b. This shows that the sequence is exact at $\mathrm{Hom}(B, G)$. □

Example 19.2. Consider the short exact sequence
$$0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to \mathbb{Z}/p \to 0$$
for $p > 1$ an integer. Then taking $\mathrm{Hom}(-, \mathbb{Z})$ yields
$$0 \to \mathrm{Hom}(\mathbb{Z}/p, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\cdot p} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \to 0$$
Now $\mathrm{Hom}(\mathbb{Z}/p, \mathbb{Z}) = 0$ and $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$, so this yields
$$0 \to 0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to 0.$$
This is *not exact*, since multiplication by p is not onto.

To measure the failure of $\mathrm{Hom}(-, G)$ to be right exact, we define the Ext groups.

Definition 19.3. Now we define the Ext groups of a pair of abelian groups H, G. An exact sequence
$$0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$
with F_i a free abelian group for $i = 0, 1$ is called a *free resolution* of H. Then
$$\mathrm{Ext}^0(H, G) := \ker(f_1^* \colon \mathrm{Hom}(F_0, G) \to \mathrm{Hom}(F_1, G)).$$
and
$$\mathrm{Ext}^1(H, G) := \mathrm{coker}(f_1^* \colon \mathrm{Hom}(F_0, G) \to \mathrm{Hom}(F_1, G)).$$
That is replace H by 0, apply $\mathrm{Hom}(-, G)$ to the resulting chain complex, and then take cohomology, to get $H^n(\mathrm{Hom}(F_*, G))$ for $n = 0, 1$.

We shall be principally concerned with Ext^1 and Ext^0, since for abelian groups these are the only ones that are nonzero, as we will soon see.

Let H be a finitely generated abelian group. Then by the classification of finitely generated abelian groups,

$$H \cong \mathbb{Z}^n \oplus \bigoplus_{i=1}^{k} \mathbb{Z}/p_i^{n_i}$$

for some n, for some k, for some primes $p_1, \ldots, p_k$, and for some integers $n_1, \ldots, n_k$. There is a resolution of length one

$$0 \to \bigoplus_{i=1}^{k} \mathbb{Z} \xrightarrow{\left(0, \bigoplus_{i=1}^{k} p_i^{n_i}\right)} \mathbb{Z}^n \oplus \bigoplus_{i=1}^{k} \mathbb{Z} \to H \to 0.$$

The next lemma is a baby version of what is often called the fundamental lemma of homological algebra. It will be sufficient to prove that the groups $\mathrm{Ext}^1(H, G)$ are well-defined.

Lemma 19.4. *Given two resolutions*

$$0 \to F_1 \to F_0 \to H \to 0$$

and

$$0 \to F_1' \to F_0' \to H \to 0$$

of H, and a homomorphism $\varphi\colon H \to H$, we have the following.

(1) There is a chain map between the resolutions inducing φ.
(2) Any two such chain maps are chain homotopic.

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Proof. We need to construct a map as shown in the next diagram.

$$\begin{tikzcd}
F_1 \arrow[r, "f_1"] & F_0 \arrow[r, "f_0"] \arrow[d, dashed, "\varphi_0"] & H \arrow[r] \arrow[d, "\varphi"] & 0 \\
F_1' \arrow[r, "f_1'"] & F_0' \arrow[r, "f_0'"] & H \arrow[r] & 0
\end{tikzcd}$$

Let $x \in F_0$ be an element of a generating set. Note that $\varphi(f_0(x)) \in H$. Choose $x' \in F_0'$ with $f_0'(x') = \varphi(f_0(x)) \in H$. Define $\varphi_0(x) = x'$. Do this for all the free generators of F_0, and then extend by linearity. This defines $\varphi_0\colon F_0 \to F_0'$. Now we want to define the map φ_1 as shown in the next diagram.

$$\begin{tikzcd}
F_1 \arrow[r, "f_1"] \arrow[d, dashed, "\varphi_1"] & F_0 \arrow[r, "f_0"] \arrow[d, "\varphi_0"] & H \arrow[r] \arrow[d, "\varphi"] & 0 \\
F_1' \arrow[r, "f_1'"] & F_0' \arrow[r, "f_0'"] & H \arrow[r] & 0
\end{tikzcd}$$

Let $y \in F_1$. Then $f_0 \circ f_1(y) = 0$, so $\varphi \circ f_0 \circ f_1(y) = 0$. By commutativity, $f_0' \circ \varphi_0 \circ f_1(y) = 0$. By exactness, there exists $y' \in F_1'$ with $f_1'(y') = \varphi_0 \circ f_1(y)$. Define

$$\varphi_1(y) = y'.$$

Do this for a generating set of F_1, and extend by linearity. This defines $\varphi_1\colon F_1 \to F_1'$. The map $\varphi_*\colon F_* \to F_*'$ is a chain map by construction.

Now we want to show that any two such maps are chain homotopic.

$$\begin{tikzcd}
F_1 \arrow[r, "f_1"] \arrow[d, shift right, "\varphi_1'"'] \arrow[d, shift left, "\varphi_1"] & F_0 \arrow[r, "f_0"] \arrow[d, shift right, "\varphi_0'"'] \arrow[d, shift left, "\varphi_0"] \arrow[dl, dashed, "g"'] & H \arrow[r] \arrow[d, "\varphi"] \arrow[dl, dashed, "0"'] & 0 \\
F_1' \arrow[r, "f_1'"] & F_0' \arrow[r, "f_0'"] & H \arrow[r] & 0
\end{tikzcd}$$

We want to define the chain homotopy g shown. Let $x \in F_0$. Then

$$f_0'(\varphi_0(x) - \varphi_0'(x)) = \varphi(f_0(x)) - \varphi(f_0(x)) = 0.$$

Therefore there exists a $y' \in F_1'$ with $f_1'(y') = \varphi_0(x) - \varphi_0'(x)$. Define

$$g(x) = y'.$$

Do this for a generating set of F_0, and extend by linearity. This defines $g\colon F_0 \to F_1'$. It has the property that $f_1' \circ g = \varphi_0 - \varphi_0'$, so satisfies the requirement for a chain homotopy at F_0. Now, let $y \in F_1$. We want to show that

$$\varphi_1(y) - \varphi_1'(y) = g \circ f_1(y).$$

Since f_1' is injective, it is enough to show that

$$f_1'(\varphi_1(y) - \varphi_1'(y)) = f_1' \circ g \circ f_1(y).$$

Now we compute:

$$\begin{aligned} f_1'(\varphi_1(y) - \varphi_1'(y)) &= f_1' \circ \varphi_1(y) - f_1' \circ \varphi_1'(y) = \varphi_0 \circ f_1(y) - \varphi_0' \circ f_1(y) \\ &= (\varphi_0 - \varphi_0')(f_1(y)) = f_1' \circ g \circ f_1(y), \end{aligned}$$

as required. Therefore g is a chain homotopy. $\square$

Now let F_*, F_*' be two resolutions of H. With $\varphi = \mathrm{Id}$, let $\varphi_*\colon F \to F'$ be a chain map inducing φ, and let $\psi_*\colon F_*' \to F_*$ be a chain map inducing $\psi = \mathrm{Id}$. Then $\varphi_* \circ \psi_*, \mathrm{Id}$ and $\psi_* \circ \varphi_*, \mathrm{Id}$ are two chain maps inducing Id on H. Then for both pairs, the two chain maps are chain homotopic. Therefore $\varphi_* \circ \psi_* \sim \mathrm{Id}$ and $\psi_* \circ \varphi_* \sim \mathrm{Id}$, so $F_* \to H \to 0$ and $F_*' \to H \to 0$ are chain homotopy equivalent. It follows, since applying $\mathrm{Hom}(-, G)$ to a chain equivalence yields a chain equivalence, that the Ext groups

$$\mathrm{Ext}^1(H, G), \mathrm{Ext}^0(H, G)$$

are well-defined, for every abelian group G. That is, for any two chain resolutions of H we obtain the same group $\mathrm{Ext}^i(H, G)$, $i = 0, 1$, up to canonical isomorphism.

Proposition 19.5. *The groups* $\mathrm{Ext}^i(H, G)$ *are independent of the choice of free resolution of* H.

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Now that we know the Ext groups Ext^0 and Ext^1 are well-defined, let us see some examples.

Example 19.6.

(1) We compute $\mathrm{Ext}^n(\mathbb{Z},\mathbb{Z})$. There is a free resolution

$$0 \to F_1 = 0 \to F_0 = \mathbb{Z} \to \mathbb{Z} \to 0.$$

So $F_* = 0 \to \mathbb{Z} \to 0$, supported in degree zero. Taking $\mathrm{Hom}(-,\mathbb{Z})$ yields

$$0 \to \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \to 0.$$

So

$$\mathrm{Ext}^0(\mathbb{Z},\mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}$$

and

$$\mathrm{Ext}^i(\mathbb{Z},\mathbb{Z}) = 0 \text{ for } i > 0.$$

(2) We compute $\mathrm{Ext}^n(\mathbb{Z}/p,\mathbb{Z})$. There is a free resolution

$$0 \to F_1 = \mathbb{Z} \xrightarrow{\cdot p} F_0 = \mathbb{Z} \to \mathbb{Z}/p \to 0.$$

So $F_* = 0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to 0$, supported in degrees zero and one. Taking $\mathrm{Hom}(-,\mathbb{Z})$ yields

$$0 \to \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \xrightarrow{\cdot p} \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \to 0.$$

So

$$\mathrm{Ext}^0(\mathbb{Z}/p,\mathbb{Z}) = 0,$$

$$\mathrm{Ext}^1(\mathbb{Z}/p,\mathbb{Z}) \cong \mathbb{Z}/p,$$

and

$$\mathrm{Ext}^i(\mathbb{Z}/p,\mathbb{Z}) = 0 \text{ for } i > 1.$$

The fact that $\mathrm{Ext}^1(\mathbb{Z}/p,\mathbb{Z}) \cong \mathbb{Z}/p$ is nonzero corresponds to the failure of $\mathrm{Hom}(-,\mathbb{Z})$ to be right exact observed in Example 19.2.

Next, we list some of the main important properties of the Ext groups.

Proposition 19.7. *Let G and H be abelian groups.*

(1) $\mathrm{Ext}^0(H,G) \cong \mathrm{Hom}(H,G)$.
(2) If H is a free abelian group, then $\mathrm{Ext}^1(H,G) = 0$.
(3) If H is a finitely generated abelian group, then $\mathrm{Ext}^1(H,\mathbb{Z})$ *is the torsion subgroup of H, that is $TH := \{h \in H \mid \exists\, n \in \mathbb{N}\smallsetminus\{0\}$ with $nh = 0\}$.*
(4) $\mathrm{Ext}^1(H,\mathbb{Q}) = 0$ *for any H.*
(5) Let $H_1,\dots,H_k$ and $G_1,\dots,G_k$ be abelian groups. Then

$$\mathrm{Ext}^1(\bigoplus_{i=1}^k H_i, G) \cong \bigoplus_{i=1}^k \mathrm{Ext}^1(H_i,G)$$

and

$$\mathrm{Ext}^1(H, \bigoplus_{i=1}^k G_i) \cong \bigoplus_{i=1}^k \mathrm{Ext}^1(H,G_i).$$

20. The universal coefficient theorem for cohomology

One of the maps in the theorem is the evaluation map, so let us define it.

Definition 20.1. Let (C_*, ∂) be a chain complex of free abelian groups, and let G be an abelian group. The evaluation map is

$$\begin{array}{rcl} \mathrm{ev}\colon H^n(C;G) & \to & \mathrm{Hom}(H_n(C),G) \\ [\varphi\colon C_n \to G] & \mapsto & \left(\begin{array}{rcl} H_n(C) & \to & G \\ [c] & \mapsto & \langle [\varphi],[c]\rangle = \varphi(c). \end{array} \right) \end{array}$$

Here the pairing $\langle [\varphi],[c]\rangle = \varphi(c)$ is often called the Kronecker pairing. It simply means evaluate a representative cocycle of a homology class on a representative cycle of a homology class.

We should show that ev is well-defined.

Lemma 20.2. *The map* ev *is a well-defined group homomorphism.*

Proof.

$$\begin{aligned} (\varphi + \delta\psi)(c + \partial d) &= \varphi(c) + \delta\psi(c) + \varphi(\partial d) + \delta\psi(\partial d) \\ &= \varphi(c) + \psi(\partial c) + \delta\varphi(d) + \psi(\partial^2 d) = \varphi(c). \end{aligned}$$

Here $\partial c = 0$ because c is a cycle and $\delta\varphi = 0$ because φ is a cocycle. Then $\partial^2 = 0$, so the last term vanishes as well. □

The universal coefficient theorem for spaces will follow directly from the following purely algebraic statement.

Theorem 20.3. *Let (C_*, ∂) be a chain complex of free abelian groups, and let G be an abelian group. For each $n \in \mathbb{N}_0$, there is a natural (in C_*) short exact sequence*

$$0 \to \mathrm{Ext}^1(H_{n-1}(C), G) \to H^n(C;G) \xrightarrow{\mathrm{ev}} \mathrm{Hom}(H_n(C),G) \to 0$$

that splits, i.e.

$$H^n(C;G) \cong \mathrm{Ext}^1(H_{n-1}(C),G) \oplus \mathrm{Hom}(H_n(C),G).$$

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Proof. Let (C_*, ∂) be a chain complex of free abelian groups, and let G be an abelian group. Write $Z_n := \ker \partial_n$ and $B_n := \operatorname{im} \partial_{n+1}$. Both are also free abelian groups. We have a short exact sequence of chain complexes. That is the rows are exact of the next diagram.

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Z_n \arrow[r] \arrow[d, "0"] & C_n \arrow[r, "\partial_n"] \arrow[d, "\partial_n"] & B_{n-1} \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d] & B_{n-2} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

Apply $\mathrm{Hom}(-, G)$ to obtain the next diagram. The rows are again exact because B_n is free abelian.

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \mathrm{Hom}(B_{n-1}, G) \arrow[r, "\partial_n^*"] \arrow[u] & \mathrm{Hom}(C_n, G) \arrow[r] \arrow[u] & \mathrm{Hom}(Z_n, G) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathrm{Hom}(B_{n-2}, G) \arrow[r] \arrow[u, "0"] & \mathrm{Hom}(C_{n-1}, G) \arrow[r] \arrow[u, "\partial_n^*"] & \mathrm{Hom}(Z_{n-1}, G) \arrow[r] \arrow[u, "0"] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}
$$

By the snake lemma, we obtain a long exact sequence in cohomology. However the left and right vertical sequences have all coboundary maps trivial, so the cohomology is equal to the cochain groups. We therefore have a long exact sequence:

$$\cdots \to \mathrm{Hom}(Z_{n-1}, G) \xrightarrow{d_{n-1}} \mathrm{Hom}(B_{n-1}, G) \to H^n(C; G) \to \mathrm{Hom}(Z_n, G) \xrightarrow{d_n} \mathrm{Hom}(B_n, G) \to \cdots$$

Here d_n, d_{n-1} denotes the connecting homomorphism. For $m = n, n-1$ we assert that $d_m = i_m^*$, where $i_m \colon : B_m \to Z_m$ is the inclusion. This is a straightforward check using the definition of the connecting homomorphism. We therefore have a short exact sequence

$$0 \to \mathrm{coker}(d_{n-1}) \to H^n(C; G) \to \ker(d_n) \to 0.$$

There is a resolution

$$0 \to B_{n-1} \xrightarrow{i_{n-1} = d_{n-1}^*} Z_{n-1} \to H_{n-1}(C) \to 0.$$

Therefore

$$\mathrm{coker}(d_{n-1}) = \mathrm{Ext}^1(H_{n-1}(C), G).$$

Similarly there is a resolution

$$0 \to B_n \xrightarrow{i_n = d_n^*} Z_n \to H_n(C) \to 0.$$

Therefore
$$\ker(d_n) = \mathrm{Ext}^0(H_n(C), G) \cong \mathrm{Hom}(H_n(C), G).$$
Putting this together yields a short exact sequence
$$0 \to \mathrm{Ext}^1(H_{n-1}(C), G) \to H^n(C; G) \to \mathrm{Hom}(H_n(C), G) \to 0$$
as desired. We omit the proof that this is a split sequence, and that it is natural, for time reasons. □

If $H_n(C)$ is finitely generated and $G = \mathbb{Z}$, then $\mathrm{Hom}(H_n(C), G)$ is a finitely generated free abelian group, so the short exact sequence must split.

21. Tensor products and Tor

Let A and B be abelian groups. Then the tensor product is a quotient of the free abelian group generated by symbols of the form $a_i \otimes b_i$, with a_i in A and b_i in B
$$A \otimes B = \{\sum_{i=1}^{n} a_i \otimes b_i\}/\sim$$
where the relations $\sim$ are generated by
$$(a + a') \otimes b = a \otimes b + a' \otimes b$$
and
$$a \otimes (b + b') = a \otimes b + a \otimes b'.$$
Note that $a \otimes 0 + a \otimes b = a \otimes (0 + b) = a \otimes b$ so $a \otimes 0 = 0$. Similarly $0 \otimes b = 0$. The identity element in either factor gives the trivial element. Here are some facts on tensor products.

(1) $A \otimes B \cong B \otimes A$.
(2) $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$.
(3) $\mathbb{Z} \otimes A \cong A$.
(4) $\mathbb{Z}/k \otimes A \cong A/kA$.
(5) $\mathbb{Z}/n \otimes \mathbb{Z}/m \cong \mathbb{Z}/n/(m\mathbb{Z}/m) \cong \mathbb{Z}/\gcd(m, n)$.
(6) If A is a finitely generated abelian group and $A \cong \mathbb{Z}^r \oplus TA$, then $\mathbb{Q} \otimes A \cong \mathbb{Q}^r$.

The Tor groups are to tensor product as Ext groups are to Hom. If $0 \to A \to B \to C \to 0$ is a short exact sequence, then
$$A \otimes G \to B \otimes G \to C \otimes G \to 0$$
is exact. The failure for $A \otimes G \to B \otimes G$ to be exact is measured by the Tor groups.

Definition 21.1. Let A and B be abelian groups. Let
$$0 \to F_1 \to F_0 \to A \to 0$$
be a free resolution of A. Then
$$\mathrm{Tor}_n(A, B) := H_n(F_* \otimes B)$$
for $n = 0, 1$. That is, replace A by 0, tensor with B, then take homology.

The proof that Tor_n is well-defined uses the fundamental lemma of homological algebra, that we showed for length one resolutions above in order to show that Ext is well defined. The next proposition is the analogue of the fact that $\mathrm{Ext}^0(G,H) \cong \mathrm{Hom}(G,H)$.

Proposition 21.2. $\mathrm{Tor}_0(A,B) \cong A \otimes B$.

22. Universal coefficient theorem for homology

Tensor product appears in two important places: in the universal coefficient theorem for homology and in the Künneth theorem.

Theorem 22.1 (Universal coefficient theorem for homology)**.** *Let X be a topological space and let G be an abelian group. For all $n \in \mathbb{N}_0$, there is a natural short exact sequence*

$$0 \to H_n(X) \otimes G \to H_n(X;G) \to \mathrm{Tor}_1(H_{n-1}(X);G) \to 0$$

that splits.

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Corollary 22.2. *For every space X and for $n \in \mathbb{N}_0$, we have that $H_n(X;\mathbb{Q}) \cong H_n(X;\mathbb{Z}) \otimes \mathbb{Q}$.*

Proof. Let $H := H_{n-1}(X)$. There is a resolution:

$$0 \to \bigoplus_{i=1}^{k} \mathbb{Z} \xrightarrow{\left(0, \bigoplus_{i=1}^{k} p_i^{n_i}\right)} \mathbb{Z}^n \oplus \bigoplus_{i=1}^{k} \mathbb{Z} \to H \to 0.$$

for some integers n, k, p_i, n_i. To compute $\mathrm{Tor}_1(H,Q)$ we tensor with $\mathbb{Q}$ to obtain

$$0 \to \bigoplus_{i=1}^{k} \mathbb{Q} \xrightarrow{\left(0, \bigoplus_{i=1}^{k} p_i^{n_i}\right)} \mathbb{Q}^n \oplus \bigoplus_{i=1}^{k} \mathbb{Q} \to 0$$

Since the p_i are nonzero, the first homology of this chain complex is trivial, so $\mathrm{Tor}_1(H,\mathbb{Q}) = 0$. The corollary then follows from the universal coefficient theorem exact sequence. □

Example 22.3. Let us consider the homology of $\mathbb{RP}^3$. With $\mathbb{Z}$ coefficients, we have $H_0(\mathbb{RP}^3;\mathbb{Z}) \cong H_3(\mathbb{RP}^2;\mathbb{Z}) \cong \mathbb{Z}$, $H_2(\mathbb{RP}^3;\mathbb{Z}) = 0$ and $H_1(\mathbb{RP}^3;\mathbb{Z}) = \mathbb{Z}/2$. We can compute the homology with $\mathbb{Z}/2$ coefficients using the cellular chain complex $C_i^{CW}(\mathbb{RP}^3) \cong \mathbb{Z}$ for $i = 0,1,2,3$ and boundary maps

$$0 \to C_3^{CW}(\mathbb{RP}^3) \cong \mathbb{Z} \xrightarrow{0} C_2^{CW}(\mathbb{RP}^3) \cong \mathbb{Z} \xrightarrow{2} C_1^{CW}(\mathbb{RP}^3) \cong \mathbb{Z} \xrightarrow{0} C_0^{CW}(\mathbb{RP}^3) \cong \mathbb{Z} \to 0.$$

Tensor with $\mathbb{Z}/2$ to obtain, using that $\mathbb{Z} \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2$, the chain complex:

$$0 \to \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \to 0.$$

Therefore $H_i(\mathbb{RP}^3;\mathbb{Z}/2) \cong H_i^{CW}(\mathbb{RP}^3;\mathbb{Z}/2) \cong \mathbb{Z}/2$ for $i = 0,1,2,3$, and is otherwise trivial.

Now let us compute with the universal coefficient theorem for homology instead. What is $\mathrm{Tor}_1(H, \mathbb{Z}/2)$? For H free abelian, $\mathrm{Tor}_1(H, \mathbb{Z}/2) = 0$. However $\mathrm{Tor}_1(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2$. To see this, note that $\mathbb{Z}/2$ has a resolution

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0.$$

Forget the right $\mathbb{Z}/2$ and tensor with $\mathbb{Z}/2$ to get

$$\mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2$$

supported in degrees 0 and 1. The H_1 of this is $\mathrm{Tor}_1(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2$. Now, by the universal coefficient theorem for homology, this means that

$$H_i(\mathbb{RP}^3; \mathbb{Z}/2) = H_i(\mathbb{RP}^3; \mathbb{Z}) \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2$$

for $i = 0, 1, 3$, since $\mathbb{Z} \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2$ for $i = 0, 3$, and since $\mathbb{Z}/2 \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2$ for $i = 1$. For $H_2(\mathbb{RP}^3; \mathbb{Z}/2)$, we have $H_2(\mathbb{RP}^3; \mathbb{Z}) = 0$ so

$$H_2(\mathbb{RP}^3; \mathbb{Z}/2) \cong \mathrm{Tor}_1(H_1(\mathbb{RP}^3; \mathbb{Z}), \mathbb{Z}/2) \cong \mathrm{Tor}_1(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2.$$

So once again we compute that $H_i(\mathbb{RP}^3; \mathbb{Z}/2) \cong \mathbb{Z}/2$ for $i = 0, 1, 2, 3$, and is otherwise trivial.

23. Understanding cohomology

Recall that

$$C^n(X; G) \cong \mathrm{Hom}(C_n(X), G) \cong \{\text{maps } \{\Delta^n \to X\} \text{ to } G\}$$

In particular,

$$C^0(X; G) \cong \mathrm{Hom}(C_0(X), G) \cong \mathrm{Map}(\{\Delta^0 \to X\}, G) \cong \{\text{functions } X \to G\}.$$

Now,

$$C^1(X; G) \cong \mathrm{Hom}(C_1(X), G) \cong \mathrm{Map}(\{\Delta^1 \to X\}, G)$$

and the coboundary map is given by

$$\begin{array}{ccc} C^0(X; G) & \to & C^1(X; G) \\ (\varphi\colon X \to G) & \mapsto & (\delta\varphi\colon \{\Delta^1 \to X\} \to G) \end{array}$$

where $\delta\varphi(\psi) := \varphi(\psi(1)) - \varphi(\psi(0))$. Therefore

$$\begin{aligned} H^0(X; G) = \ker \delta = \{\varphi\colon X \to G \mid{} & \varphi(x) = \varphi(y) \\ & \text{whenever there is } \psi\colon \Delta^1 \to X \text{ with } \psi(0) = x, \psi(1) = y\} \\ = {} & \{\varphi\colon \{\text{connected components of } X\} = \pi_0(X) \to G\} \\ = {} & \prod_{|\pi_0(X)|} G \end{aligned}$$

An analogous computation holds for the zeroth CW cohomology of CW complexes. So zeroth cohomology is not too hard to understand: functions on X that are constant on path components.

For further discussion of an intuitive idea behind cohomology, I recommend reading the beginning of Chapter 3 of Hatcher. As an alternative, we give an explicit generator for the first cohomology of the circle S^1.

Example 23.1. Consider the real line $\mathbb{R}$. Consider the function

$$\begin{aligned} \alpha_{\mathbb{Z}} \colon \mathbb{R} &\to \mathbb{Z} \\ x &\mapsto \lfloor x \rfloor \end{aligned}$$

Let $\mu\colon \Delta^1 \to \mathbb{R}$ be a singular 1-simplex. Recall that $\Delta^1 = \{(1-t,t) \in \mathbb{R}^2 \mid t \in [0,1]\}$. Denote a point in Δ^1 by the value of t. Then

$$\delta\alpha_{\mathbb{Z}}(\mu) = \alpha_{\mathbb{Z}}(\partial\mu) = \begin{cases} |\{n \in \mathbb{Z} \mid \mu(0) < n \leq \mu(1)\}| & \text{if } \mu(0) \leq \mu(1) \\ -|\{n \in \mathbb{Z} \mid \mu(1) < n \leq \mu(0)\}| & \text{if } \mu(1) \leq \mu(0). \end{cases}$$

This counts how many integers μ crosses from left to right, with sign, so that crossing an integer from right to left counts as -1.

Example 23.2. Now consider the circle S^1. Let $p\colon \mathbb{R} \to S^1$ be the function sending $t \to e^{2\pi i t} \in S^1 \subset \mathbb{C}$. Given a 1-simplex $\sigma\colon \Delta^1 \to S^1$, it lifts to a 1-simplex in $C_1(\mathbb{R};\mathbb{R})$

$$\widetilde{\sigma}\colon \Delta^1 \to \mathbb{R}$$

such that $p \circ \widetilde{\sigma} = \sigma\colon \Delta^1 \to S^1$. Such a lift is not unique; any two lifts differ by an integer $\widetilde{\sigma}(x) - \widetilde{\sigma}'(x) \in \mathbb{Z}$.

Define an element of $C^1(S^1;\mathbb{Z})$ by

$$\theta_{\mathbb{Z}}(\sigma) := \delta\alpha_{\mathbb{Z}}(\widetilde{\sigma}) = \begin{cases} |\{n \in \mathbb{Z} \mid \widetilde{\sigma}(0) < n \leq \widetilde{\sigma}(1)\}| & \text{if } \widetilde{\sigma}(0) \leq \widetilde{\sigma}(1) \\ -|\{n \in \mathbb{Z} \mid \widetilde{\sigma}(1) < n \leq \widetilde{\sigma}(0)\}| & \text{if } \widetilde{\sigma}(1) \leq \widetilde{\sigma}(0). \end{cases}$$

You should check that this is well-defined i.e. does not depend on the choice of lift $\widetilde{\sigma}$.

Now, consider the 1-simplex $\mu\colon \Delta^1 \to S^1$ defined by $\mu(t) = e^{2\pi i t}$. Lift it, to obtain for example $\widetilde{\mu}(t) = t$. Then $\widetilde{\mu}(1) - \widetilde{\mu(0)} = 1$, so

$$\theta_{\mathbb{Z}}(\mu) = 1.$$

In the next lemma, we will show that $\theta_{\mathbb{Z}}$ represents a nontrivial cohomology class in $H^1(S^1;\mathbb{Z})$. Since it evaluates to 1 on $\mu = [S^1]$, the generator of $H_1(S^1;\mathbb{Z})$, this will imply that it generates the cohomology group.

Lemma 23.3.

(1) *The singular 1-cochain $\theta_{\mathbb{Z}}$ is a cocycle, so represent an element $\theta_{\mathbb{Z}} \in H^1(S^1;\mathbb{Z})$.*
(2) *$[\theta_{\mathbb{Z}}]$ is* not *a coboundary δc for any $c \in C^0(S^1;\mathbb{Z})$.*

Proof.

(1) We will show that $\theta_{\mathbb{Z}}$ is a cocycle, that is $\delta_1\theta_{\mathbb{Z}} = 0 \in C^2(S^1;\mathbb{Z})$. Let $\sigma\colon \Delta^2 \to S^1$. We want that $(\delta_1\theta_{\mathbb{Z}})(\sigma) = \theta_{\mathbb{Z}}(\partial\sigma) = 0$. Recall the function $p\colon \mathbb{R} \to S^1$ sending $t \to e^{2\pi i t} \in S^1 \subset \mathbb{C}$. Then σ lifts to $\widetilde{\sigma}\colon \Delta^2 \to \mathbb{R}$ and,

with $\iota_j \colon \Delta^1 \to \Delta^2$ the jth face inclusion, we have

$$\begin{aligned}\theta_{\mathbb{Z}}(\partial\sigma) &= \theta_{\mathbb{Z}}\big(\sum_{j=0}^{2}(-1)^j \sigma\circ\iota_j\big)\\ &= (\delta\alpha_{\mathbb{Z}})\big(\sum_{j=0}^{2}(-1)^j\widetilde{\sigma}\circ\iota_j\big) = \alpha_{\mathbb{Z}}(\partial\circ\partial\widetilde{\sigma}) = \alpha_{\mathbb{Z}}(0) = 0.\end{aligned}$$

(2) To show that $\theta_{\mathbb{Z}}$ is not a coboundary, let $\beta\colon C^0(S^1;\mathbb{Z})$. We can think of β as a function $\beta\colon S^1 \to \mathbb{Z}$. Consider the 1-chain $\mu\colon \Delta^1 \to S^1$ sending $t \mapsto e^{2\pi i t}$. Then

$$\delta_0(\beta)(\mu) = \beta(\partial\mu) = \beta(\mu(1) - \mu(0)) = \beta(1) - \beta(1) = 0.$$

But $\theta_{\mathbb{Z}}(\mu) = 1$, so these cannot be the coboundaries of any 0-chain.

□

The cohomology class $\theta_{\mathbb{Z}} \in H^1(S^1;\mathbb{Z}) \cong \mathbb{Z}$ is a generator.

24. Manifolds and orientations

Definition 24.1. An n-dimensional smooth manifold is a second countable, Hausdorff topological space together with a collection of *charts* $\mathcal{A}$, called an *atlas*, namely open subsets $U_i \subseteq M$ and homeomorphisms $\{\varphi_i\colon U_i \to V_i \subseteq \mathbb{R}^n\}_{i\in\mathcal{I}}$, where $V_i \subseteq \mathbb{R}^n$ is an open subset, satisfying:

(i) For every $p \in M$, there exists a U_i with $p \in U_i$.
(ii) For every i, j, the function $\varphi_j \circ \varphi_i^{-1}|\colon \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is a smooth map from an open subset of $\mathbb{R}^n$ to another open subset of $\mathbb{R}^n$.
(iii) The collection of charts is maximal, meaning that if $\mathcal{A}'$ is a collection of charts $\{\varphi_i'\colon U_i' \to V_i'\}$ satisfying (i) and (ii), and $\mathcal{A}\cup\mathcal{A}'$ is also a collection of charts satisfying (i) and (ii), then $\mathcal{A}' \subseteq \mathcal{A}$.

A manifold is called closed if it is compact and has empty boundary. But since we have not defined manifolds with boundary yet, closed and compact mean the same thing.

Example 24.2.

(1) The sphere S^n is an n-dimensional manifold.
(2) A product of spheres $S^{n_1} \times \cdots \times S^{n_k}$ is a manifold of dimension $n_1 + \cdots + n_k$.
(3) The surface of genus g, Σ_g.
(4) Real projective space, $\mathbb{RP}^n$ is an n-dimensional manifold.
(5) Complex projective space

$$\mathbb{CP}^n := \mathbb{C}^{n+1}/\mathbb{C}^\times$$

is a $2n$-dimensional manifold.
(6) The orthogonal group $O(n)$ is a manifold of dimension $n(n-1)/2$.
(7) If a group G acts on S^n or $\mathbb{R}^n$ freely, smoothly, and properly, then the orbit space S^n/G or $\mathbb{R}^n/G$ is a manifold. We define these adjectives next.

(a) An action of a group G on a manifold X is *free* if $g \cdot x = x$ implies that $g = e \in G$ for every $x \in X$.
(b) An action of a group G on a manifold X is *proper* if for all $x, y \in X$, there exist open sets $U \ni x$ and $V \ni y$ such that
$$\{g \in G \mid gU \cap V \neq \emptyset\}$$
is finite.
(c) A group action is smooth if the map $x \mapsto g \cdot x$ is a smooth map $X \to X$ for all $g \in G$.

Here are some examples of manifolds arising by group actions as in the last item of the previous list. All but the last example already appears in the list above.

(i) The group $\mathbb{Z}$ acts on $\mathbb{R}$ by $n \cdot x := x + n$. The quotient is $\mathbb{R}/\mathbb{Z} \cong S^1$.
(ii) The group $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ by
$$(n, m) \cdot (x, y) = (x + n, y + m).$$
The quotient $\mathbb{R}^2/\mathbb{Z}^2 \cong T^2$.
(iii) The group $\mathbb{Z}/2$ acts on S^n by $1 \cdot x = -x$. The quotient $S^n/(\mathbb{Z}/2) \cong \mathbb{RP}^n$.
(iv) The group S^1 acts on S^{2n+1} as follows. Consider $S^{2n+1} \subset \mathbb{C}^{n+1}$ as the set of points with $|z_0|^2 + \cdots |z_n|^2 = 1$. Then define
$$e^{2\pi i\theta} \cdot (z_0, \ldots, z_n) := (e^{2\pi i\theta} \cdot z_0, \ldots, e^{2\pi i\theta} \cdot z_n).$$
The quotient is $S^{2n+1}/S^1 \cong \mathbb{CP}^n$.
(v) The group $\mathbb{Z}/p$ acts on S^3 as follows. Let p and q be coprime positive integers. Consider $S^3 \subset \mathbb{C}^2$ with $|z|^2 + |w|^2 = 1$. Write $\mathbb{Z}/p \cong C_p$ where C_p is the cyclic group generated by $\xi = e^{2\pi i/p} \in S^1$. Then
$$\xi^j \cdot (z, w) := (\xi \cdot z, \xi^q \cdot w).$$
The quotient is $S^3/C_p = L(p, q)$, the lens space.

Next we define the important notion of an orientation of a manifold.

Lemma 24.3. $H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \begin{cases} \mathbb{Z} & i = n \\ 0 & \textit{else.} \end{cases}$

Proof. Compute using the long exact sequence of a pair. □

Let $p \in M$ and let $U \ni p$ be a neighbourhood. Then
$$(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \to (U, U \setminus \{p\}) \to (M, M \setminus \{p\})$$
induces an isomorphism
$$H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong H_i(M, M \setminus \{p\})$$
for every $i \in \mathbb{N}_0$, by excision.

Definition 24.4.

(i) A *local orientation* for M at $p \in M$ is a choice of generator $\mathcal{O}_p$ for
$$H_n(M, M \setminus \{p\}) \cong \mathbb{Z}.$$

(ii) Consider a choice of local orientation at every point $p \in M$, $\{\mathcal{O}_p\}_{p\in M}$. Such a choice of local orientations is *continuous* at p if there exists an open set $U \ni p$ and a class $\alpha \in H_n(M, M\setminus U)$ such that α is sent to $\mathcal{O}_p$ under the inclusion-induced map
$$H_n(M, M\setminus U) \to H_n(M, M\setminus\{p\})$$
for every $p \in U$.

(iii) If $\{\mathcal{O}_p\}_{p\in M}$ is continuous at every p in M, then we say that $\{\mathcal{O}_p\}_{p\in M}$ is continuous.

(iv) A *homological orientation* for a manifold M (which need not exist) is a continuous choice of local orientation $\{\mathcal{O}_p\}_{p\in M}$. Sometimes we refer to it as an orientation.

(v) An n-dimensional manifold is said to be *orientable* if it admits a homological orientation. An n-manifold together with a choice of orientation is called *oriented.*

Definition 24.5. A manifold is called *closed* if it is compact and has empty boundary.

In fact, in our definition, every manifold has empty boundary, but one often expands the definition of manifold to allow boundary points. The adjective closed emphasises that there is no boundary.

There are some important restrictions on the homology of manifolds.

Theorem 24.6. *Let M be a closed, connected n-dimensional manifold.*

(i) The top dimensional homology satisfies
$$H_n(M;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & M \text{ orientable} \\ 0 & M \text{ not orientable.} \end{cases}$$
Moreover, $H_i(M;\mathbb{Z}) = 0$ for $i > n$.

(ii) In the case that M is orientable, let $\{\mathcal{O}_p\}_{p\in M}$ be an orientation, so that $(M, \{\mathcal{O}_p\}_{p\in M})$ is an oriented manifold. Then $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$ is generated by a class $[M]$ that maps to $\mathcal{O}_p$ under the map
$$H_n(M;\mathbb{Z}) \to H_n(M, M\setminus\{p\};\mathbb{Z})$$
for every $p \in M$. The homology class $[M]$ is called the fundamental class *of M.*

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We also have that $H^n(M;\mathbb{Z}) \cong \mathbb{Z}$. We call the generator $[M]^*$ with the property that $\langle [M^*], [M]\rangle = 1$ the *dual fundamental class.* For example in the previous section we constructed $\theta_{\mathbb{Z}} = [S^1]^* \in H^1(S^1;\mathbb{Z})$.

For every closed manifold M, there is an analogous result for $\mathbb{Z}/2$ coefficients. This does not require that M be orientable.

Theorem 24.7. *Let M be a closed, connected n-dimensional manifold. Then $H_n(M;\mathbb{Z}/2) \cong \mathbb{Z}/2$ and $H_i(M;\mathbb{Z}/2) = 0$ for $i > n$. Thee top-dimensional homology $H_n(M;\mathbb{Z}/2) \cong \mathbb{Z}/2$ is generated by a class $[M]_{\mathbb{Z}/2}$ that maps to a generator under the map*
$$H_n(M;\mathbb{Z}/2) \to H_n(M, M\setminus\{p\};\mathbb{Z}/2) \cong \mathbb{Z}/2$$

for every $p \in M$. The homology class $[M]_{\mathbb{Z}/2}$ is called the $\mathbb{Z}/2$-fundamental class *of M.*

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There is another very important relation between the homology and the cohomology of a manifold, called Poincaré duality.

Theorem 24.8 (Poincaré duality)**.** *Let M be an n-dimensional closed, oriented manifold. Then for any abelian group G, and for every $k \in \mathbb{Z}$,*

$$H^{n-k}(M;G) \cong H_k(M;G).$$

PNE

Again, there is a $\mathbb{Z}/2$-coefficient version where one does not need orientability.

Theorem 24.9 ($\mathbb{Z}/2$-coefficient Poincaré duality)**.** *Let M be an n-dimensional closed manifold. Then for every $k \in \mathbb{Z}$,*

$$H^{n-k}(M;\mathbb{Z}/2) \cong H_k(M;\mathbb{Z}/2).$$

PNE

We will return to Poincaré duality later after we have defined the cap product. The notion of degrees of maps between manifolds will be useful.

Definition 24.10. Let M and N be closed, oriented, connected n-dimensional manifolds with fundamental classes $[M]$ and $[N]$ and let $f\colon M \to N$. The map f induces a map on homology $f_*\colon H_n(M;\mathbb{Z}) \to H_n(N;\mathbb{Z})$. Since $H_n(M;\mathbb{Z}) \cong H_n(N;\mathbb{Z}) \cong \mathbb{Z}$, $f_*([M]) = k[N]$ for some $k \in \mathbb{Z}$. We refer to this integer k as the *degree* of f, and write $\deg(f) = k$.

Proposition 24.11. *Let M be a closed, oriented n-dimensional manifold, and let $k \in \mathbb{Z}$. There exists a degree k map $f_k\colon M \to S^n$.*

Proof. If $k = 0$ then map all of M to a single point in S^n. For $n \neq 0$, the proof constructs a degree 1 map $f_1\colon M \to S^n$. To obtain a degree k map, post-compose f_1 with the degree k map $S^n \to S^n$ constructed in Example 14.3. □

25. Cup products

As advertised previously, one of the great advantages of cohomology is that it admits a beautiful multiplicative structure. Whereas homology associates to a topological space a sequence of abelian groups, in cohomology these abelian groups are collected together to produce a single *ring*. The multiplication in this ring is called the cup product, which we shall now define. In order to make cohomology into a ring, we will need to work with coefficients that are themselves a ring. So let R be a commutative ring. For example we will typically use $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\mathbb{Z}/n$ for some n. This looks similar to before, since these are all abelian groups. Indeed any ring is an abelian group. But now we shall also use the ring structure on these abelian groups.

Here is our goal. For a space X, we want to define a map

$$\smile\colon H^i(X;R) \times H^j(X;R) \to H^{i+j}(X;R).$$

To define such a map, we will define a map on cochains $\smile\colon C^i(X;R) \times C^j(X;R) \to C^{i+j}(X;R)$, and then we will check that this induces a well-defined map on cohomology groups.

Let us recall and introduce some notation for singular simplices. A singular i-simplex of X is a continuous map $\Delta^i \to X$. We write label the vertices $0, 1, \dots, i$. We write $[v_0, \dots, v_j]$ for the face of Δ^i spanned by the vertices $v_0, \dots, v_j$, with $v_k \in \{0, \dots, i\}$ for $k = 0, \dots, j$.

We write
$$\lfloor_j \colon [0, \dots, j] \to [0, \dots, i] = \Delta^i$$
for the *front j face* of Δ^i. We write
$$\rfloor_j \colon [i-j, \dots, i] \to [0, \dots, i] = \Delta^i$$
for the *back j face* of Δ^i. Given a singular $i+j$-simplex $\sigma\colon \Delta^{i+j} \to X$, we can consider its composition with $\lfloor_i$ and $\rfloor_j$:
$$\sigma \circ \lfloor_i \colon \Delta^i \to \Delta^{i+j} \to X$$
and
$$\sigma \circ \rfloor_j \colon \Delta^j \to \Delta^{i+j} \to X.$$
Now we can *define*:
$$\begin{array}{rcl} \smile\colon C^i(X;R) \times C^j(X;R) & \to & C^{i+j}(X;R) \\ (\varphi, \psi) & \mapsto & \varphi(\sigma \circ \lfloor_i)\psi(\sigma \circ \rfloor_j) \\ & = & \varphi(\sigma|_{[0,\dots,i]})\psi(\sigma|_{[i,\dots,i+j]}). \end{array}$$
That is, we evaluate φ on the front i-face and ψ on the front j face. Note that in the last line we think of $\sigma \circ \lfloor_i$ as the restriction of σ to a face, $\sigma|_{[0,\dots,i]}$.

Theorem 25.1.

(1) Given two cocycles $\varphi \in C^i(X;R)$ and $\psi \in C^j(X;R)$ with $\delta\varphi = 0$ and $\delta\psi = 0$, then $\varphi \smile \psi \in C^{i+j}(X;R)$ is also a cocycle, and therefore represents a cohomology class $[\varphi \smile \psi] \in H^{i+j}(X;R)$.

(2) Cup product is well-defined on cohomology. That is, for any $\theta \in C^{i-1}(X;R)$ and for any $\chi \in C^{j-1}(X;R)$, we have that
$$\big((\varphi + \delta\theta) \smile (\psi + \delta\chi)\big) - \varphi \smile \psi$$
is a coboundary. Therefore
$$[(\varphi + \delta\theta) \smile (\psi + \delta\chi)] = [\varphi \smile \psi] \in H^{i+j}(X;R).$$
PNE

We need the following lemma to prove Theorem 25.1.

Lemma 25.2. *Let $\varphi \in C^i(X;R)$ and $\psi \in C^j(X;R)$. Then*
$$\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^i \varphi \smile \delta\psi \in C^{i+j+1}(X;R).$$
PNE

Proof. Let $\sigma\colon \Delta^{i+j+1} \to X$. Then we have the following computations. Here $\widehat{k}$ indicates that the number is missed out.

$$(\delta\varphi \smile \psi)(\sigma) = \sum_{k=1}^{i+1} (-1)^k \varphi(\sigma|_{[0,\dots,\widehat{k},\dots,i+1]}) \cdot \psi(\sigma_{[i+1,\dots,i+j+1]}) \tag{25.3}$$

$$(-1)^i(\varphi \smile \delta\psi)(\sigma) = \sum_{k=i}^{i+j+1} (-1)^k \varphi(\sigma_{[0,\dots,i]}) \cdot \psi(\sigma_{[i,\dots,\widehat{k},\dots,i+j+1]}) \tag{25.4}$$

The sum of the left hand sides of these two equations equals the right hand side of the equation we want to prove. In the sum of the right hand sides of these two equations, the last term of (25.3) cancels with the first term of (25.4). The rest of the terms sum to

$$\delta(\varphi \smile \psi)(\sigma) = (\varphi \smile \psi)(\partial\sigma) \tag{25.5}$$

where $\partial\sigma = \sum_{k=0}^{i+j+1} \sigma_{[0,\dots,\widehat{k},\dots,i+j+1]}$. □

Proof of Theorem 25.1. First we prove that the cup product of two cocycles is a cocycle. Suppose that $\delta\varphi = 0$ and $\delta\psi = 0$, so that both are cocycles. Then

$$\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^i \varphi \smile \delta\psi = 0 \smile \psi + (-1)^i \varphi \smile 0 = 0 + 0 = 0.$$

So $\varphi \smile \psi$ is a cocycle.

Next, by the lemma we have that

$$\varphi \smile \delta\chi = \pm\delta(\varphi \smile \chi) - (\delta\varphi \smile \chi) = \pm\delta(\varphi \smile \chi).$$

Similarly

$$\delta\theta \smile \psi = \pm\delta(\theta \smile \psi) \pm (\theta \smile \delta\psi) = \pm\delta(\theta \smile \psi).$$

So both are coboundaries. Finally

$$\delta\theta \smile \delta\chi = \pm\delta(\theta \smile \delta\chi) \pm (\theta \smile \delta^2\psi) = \pm\delta(\theta \smile \delta\chi),$$

which is also a coboundary. We make no effort to get the signs right because they are irrelevant. Therefore

$$\begin{aligned}
&\big((\varphi + \delta\theta) \smile (\psi + \delta\chi)\big) - \varphi \smile \psi \\
=&\varphi \smile \psi + \varphi \smile \delta\chi + \delta\theta \smile \psi + \delta\theta \smile \delta\chi - \varphi \smile \psi \\
=&\pm \delta(\varphi \smile \chi) \pm \delta(\theta \smile \psi) \pm \delta(\theta \smile \delta\chi) \\
=&\delta(\pm\varphi \smile \chi \pm \theta \smile \psi \pm \theta \smile \delta\chi)
\end{aligned}$$

which is a coboundary, as desired. Therefore the cup products are well-defined on cohomology, and so define maps

$$\smile : H^i(X;R) \times H^j(X;R) \to H^{i+j}(X;R),$$

as promised. □

26. Cup products on the torus

Example 26.1. One of the most instructive examples of cup product is the torus $T^2 = S^1 \times S^1$. We explicitly compute the cup product of the torus

$$\smile : H^1(S^1 \times S^1;\mathbb{Z}) \times H^1(S^1 \times S^1;\mathbb{Z}) \to H^2(S^1 \times S^1;\mathbb{Z}).$$

Recall that $H^1(S^1 \times S^1;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $H^2(S^1 \times S^1;\mathbb{Z}) \cong \mathbb{Z}$. So the cup product corresponds, after having chosen generating cohomology classes, to a pairing $\mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z}$. Such a pairing can be represented by a 2×2 matrix A, in the sense that the pairing of $\mathbf{x} = (x_1, x_2) \in \mathbb{Z}^2$ and $\mathbf{y} = (y_1, y_2) \in \mathbb{Z}^2$ is given by

$$\mathbf{x}A\mathbf{y}^T,$$

as you may recall from bilinear forms or inner product spaces in linear algebra.

We shall choose natural bases for the cohomology of the torus, and with respect to these bases the cup product is represented by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In order to compute the cup product of the torus, first we need to define some singular simplices and some singular cochains.

In Figure 1, we draw the torus, we draw the projections $p, q\colon S^1 \times S^1 \to S^1$ to the first and second factors respectively. We also show the images of two singular simplices with the property that $-\sigma_1 + \sigma_2$ represents a fundamental class, that is

$$[-\sigma_1 + \sigma_2] = [T^2] \in H_2(S^1 \times S^1; \mathbb{Z}) \cong \mathbb{Z}.$$

The figure indicates which 1-simplices correspond to the images of the 3 summands in the boundaries $\partial\sigma_i$. Then note that

$$\begin{aligned} \partial(-\sigma_1 + \sigma_2) &= -(\sigma_1|_{[0,1]} + \sigma_1|_{[1,2]} - \sigma_1|_{[0,2]}) + (\sigma_2|_{[0,1]} + \sigma_2|_{[1,2]} - \sigma_2|_{[0,2]}) \\ &= -\sigma_1|_{[0,1]} - \sigma_1|_{[1,2]} + \sigma_1|_{[0,2]}) + \sigma_2|_{[0,1]} + \sigma_2|_{[1,2]} - \sigma_2|_{[0,2]} \\ &= -\sigma_1|_{[0,1]} - \sigma_1|_{[1,2]} + \sigma_1|_{[0,2]}) + \sigma_1|_{[1,2]} + \sigma_1|_{[0,1]} - \sigma_1|_{[0,2]} \\ &= 0. \end{aligned}$$

Note that it is important here that, for example $\sigma_1|_{[0,1]}$ shows up twice here with opposite signs in the formal sum. What emphatically does *not* happen is that $\sigma_1|_{[0,1]}$ and $\sigma_1|_{[1,0]}$ show up, both with a positive sign. In the singular chain complex, recall that these two do *not* cancel. (This is a popular misapprehension, so I am trying to highlight it. In cellular homology, and in the simplicial theory, such cancellation does occur. But we are not working in those theories.)

Now, some more notation for the upcoming cup product computation. Let $\mu\colon \Delta^1 \to S^1$ be a 1-simplex given by $t \mapsto e^{2\pi i t}$, and let $\nu\colon \Delta^1 \to S^1$ be the constant 1-simplex $t \mapsto 1 \in S^1$. Let $\theta \in C^1(S^1; \mathbb{Z})$ be a cocycle with $\theta(\mu) = 1$ and $\theta(\nu) = 0$, so that $[\theta] \in H^1(S^1; \mathbb{Z}) \cong \mathbb{Z}$ is a generator. For example, we could use the cocycle $\theta_{\mathbb{Z}}$ constructed in Example 23.2.

The cohomology classes $[p^*(\theta)]$ and $[q^*(\theta)] \in H^1(S^1 \times S^1; \mathbb{Z})$ provide a generating set for $H^1(S^1 \times S^1; \mathbb{Z}) \cong \mathbb{Z}^2$, and the dual class $[T^2]^* = [-\sigma_1 + \sigma_2]^*$ is a generator for $H^2(S^1 \times S^1; \mathbb{Z}) \cong \mathbb{Z}$. Now we can compute the cup product. We focus on the computation of $p^*(\theta) \smile q^*(\theta)$, since this is the most exciting one. All other cup products $p^*(\theta) \smile p^*(\theta)$, $q^*(\theta) \smile q^*(\theta)$ and $q^*(\theta) \smile p^*(\theta)$ can be computed similarly, but also we will soon be able to deduce them from the computation below and the graded commutativity property of the cup product. We compute the cup product $p^*(\theta) \smile q^*(\theta)$ evaluated on σ_1 and evaluated on σ_2 separately.

$$\begin{aligned} \big(p^*(\theta) \smile q^*(\theta)\big)(\sigma_1) &= p^*(\theta)(\sigma_1|_{[0,1]}) \cdot q^*(\theta)(\sigma_1|_{[1,2]}) \\ &= \theta(p \circ \sigma_1|_{[0,1]}) \cdot \theta(q \circ \sigma_1|_{[1,2]}) \\ &= \theta(\nu) \cdot \theta(\nu) = 0 \cdot 0 = 0. \end{aligned}$$

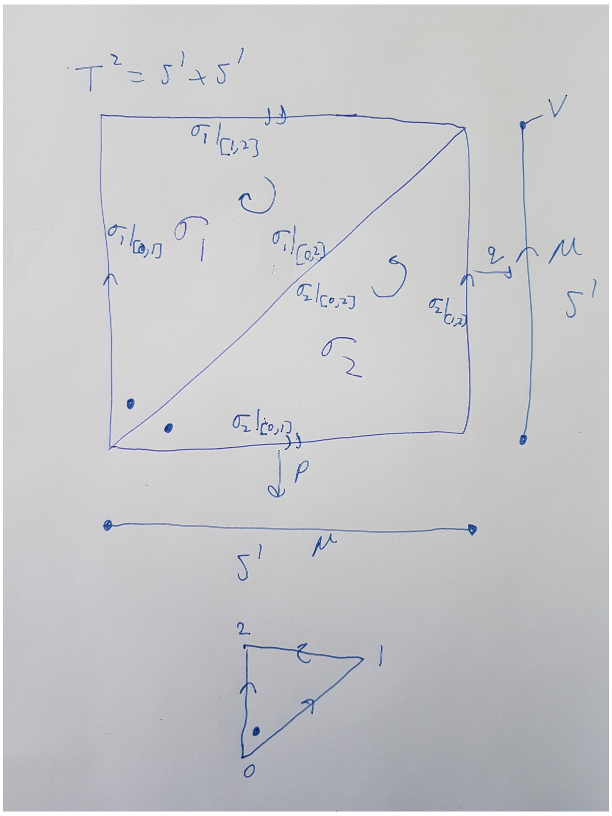

FIGURE 1. A diagram of the torus. Identify the vertical edges with each other and the horizontal edges with each other to get the torus. The projections $p, q\colon S^1 \times S^1 \to S^1$ are shown. The images of two singular 2-simplices are shown. At the bottom, a standard 2-simplex Δ^2 is drawn. The orientation is anticlockwise, and the dot indicates the vertex 0. Going anticlockwise round the edges we come to vertex 1, then vertex 2, then back to vertex 0. The dots and the orientation arrows in the main diagram of T^2 indicate how the singular 2-simplices $\sigma_i\colon \Delta^2 \to T^2$ are defined. This enables us to label the boundary 1-simplices, such as $\sigma_1|_{[0,1]}$.

Here, to identify $p \circ \sigma_1|_{[0,1]} = \nu = q \circ \sigma_1|_{[1,2]}$ we look at Figure 1, and note that these 1-simplice indeed project to points. On the other hand, we have:

$$\begin{aligned}\big(p^*(\theta) \smile q^*(\theta)\big)(\sigma_2) &= p^*(\theta)(\sigma_2|_{[0,1]}) \cdot q^*(\theta)(\sigma_2|_{[1,2]}) \\ &= \theta(p \circ \sigma_2|_{[0,1]}) \cdot \theta(q \circ \sigma_2|_{[1,2]}) \\ &= \theta(\mu) \cdot \theta(\mu) = 1 \cdot 1 = 1.\end{aligned}$$

Therefore

$$[p^*(\theta) \smile q^*(\theta)]([T^2]) = \big(p^*(\theta) \smile q^*(\theta)\big)(-\sigma_1 + \sigma_2) = 0 + 1 = 1.$$

So we have a nontrivial cup product. This corresponds to the top right entry of the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

claimed earlier. The remaining entries can be calculated with analogous computations, or as mentioned above can be deduced from the fact, to be proven shortly, that $\varphi \smile \psi = -\psi \smile \varphi \in H^2(T^2;\mathbb{Z}) \cong \mathbb{Z}$ for $\varphi, \psi \in H^1(T^2;\mathbb{Z})$.

27. The cohomology ring

Let us return to the theory of cup products.

Proposition 27.1. *Let R be a commutative ring and let X be a topological space.*

(1) The cup product is R-bilinear and associative.

(2) The conglomerate $H^*(X;R) := \bigoplus_{n\in\mathbb{N}_0} H^n(X,R)$, with the operations $+$ and $\smile$, forms a ring:

$$H^*(X;R) := \Big(\bigoplus_{n\in\mathbb{N}_0} H^n(X,R), +, \smile \Big).$$

The unit of the ring is 1_X, the cohomology class in $H^0(X;R)$ represented by the constant map $X \to R$ that sends $x \mapsto 1 \in R$ for every $x \in X$.

Proof. Let $\varphi \in C^i(X;R)$, let $\psi \in C^j(X;R)$ and let $\chi \in C^k(X;R)$. Let $(\sigma\colon \Delta^{i+j+k} \to X) \in C_{i+j+k}(X;R)$ be a singular simplex. We prove associativity:

$$\begin{aligned}(\varphi \smile \psi) \smile \chi(\sigma) &= (\varphi \smile \psi)(\sigma|_{[0,\dots,i+j]}) \cdot \chi(\sigma|_{[i+j,\dots,i+j+k]}) \\ &= \varphi(\sigma|_{[0,\dots,i]}) \cdot \psi((\sigma|_{[i,\dots,i+j]})) \cdot \chi(\sigma|_{[i+j,\dots,i+j+k]}) \\ &= \varphi(\sigma|_{[0,\dots,i]}) \cdot (\psi \smile \chi)(\sigma|_{[i,\dots,i+j+k]}) \\ &= \varphi \smile (\psi \smile \chi)(\sigma).\end{aligned}$$

The R-bilinearity is easy and is left to the reader.

Now let us check that 1_X is a unit. Let $[\varphi] \in H^n(X;R)$ and let $\sigma\colon \Delta^n \to X$ be an n-simplex in $C_n(X;R)$. Then

$$(\varphi \smile 1_X)(\sigma) = \varphi(\sigma) \cdot 1_X(\sigma|_{[n]}) = \varphi(\sigma) \cdot 1 = \varphi(\sigma).$$

Thus $[\varphi \smile 1_X] = [\varphi] \in H^n(X;R)$. Similarly $1_X \smile \varphi = \varphi$. So 1_X is an identity. Therefore $H^*(X;R)$ is a ring as asserted. □

Theorem 27.2. *Let $f\colon X \to Y$ be a continuous map of topological spaces. Then $f^*\colon H^*(Y;R) \to H^*(X;R)$ is a ring homomorphism.*

Proof. Let $\varphi \in C^k(Y;R)$ and let $\psi \in C^\ell(Y;R)$. Let $\sigma\colon \Delta^{k+\ell} \to X$ be a singular $(k+\ell)$-simplex. Then

$$\begin{aligned}(f^*(\varphi) \smile f^*(\psi))(\sigma) &= f^*(\varphi)(\sigma|_{[0,\dots,k]}) \cdot f^*(\psi)(\sigma|_{[k,\dots,k+\ell]}) \\ &= \varphi(f \circ \sigma|_{[0,\dots,k]}) \cdot \psi(f \circ \sigma|_{[k,\dots,k+\ell]}) \\ &= (\varphi \smile \psi)(f \circ \sigma) \\ &= f^*(\varphi \smile \psi)(\sigma)\end{aligned}$$

So $f^*(\varphi) \smile f^*(\psi) = f^*(\varphi \smile \psi)$. Since we already know that f^* is a homomorphism of the underlying abelian groups $H^*(Y;R) \to H^*(X;R)$, it follows that f^* is a ring homomorphism as claimed. □

Recall that if $f \sim g\colon X \to Y$ are homotopic maps, then $f^* = g^*\colon H^k(X;R) \to H^k(Y;R)$ for all $k \in \mathbb{N}_0$.

Corollary 27.3. *Suppose that $X \simeq Y$ are homotopy equivalent spaces. Then $H^*(X;R) \cong H^*(Y;R)$ are isomorphic rings.*

Proof. Let $F\colon X \to Y$ and $G\colon Y \to X$ be maps realising the homotopy equivalence. Then $f \circ g \sim \mathrm{Id}$ and $g \circ f \sim \mathrm{Id}$ implies that $g^* \circ f^* = \mathrm{Id}$ and $f^* \circ g^* = \mathrm{Id}$ as ring homomorphisms. Therefore f^* and g^* are inverse ring homomorphisms to one another. □

Now we sketch the proof that cup product is graded commutative on the level of cohomology. Note that this certainly does not hold in general on the chain level, only *after* passing to cohomology.

Theorem 27.4 (Graded commutativity)**.** *Let $[\varphi] \in H^k(X;R)$ and let $[\psi] \in H^\ell(X;R)$. Then*
$$[\varphi \smile \psi] = (-1)^{k\ell}[\psi \smile \varphi] \in H^{k+\ell}(X;R).$$
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Proof. We give a sketch of the proof. Please read [Hat] or [Fr] for the details. Define
$$\begin{array}{rcl}\rho\colon C_n(X) & \to & C_n(X) \\ \sigma & \mapsto & \varepsilon_n\overline{\sigma}\end{array}$$
where
$$\varepsilon_n := (-1)^{n(n+1)/2}.$$
Here $\overline{\sigma}(i) = \sigma(n-i)$, so $\overline{\sigma}$ is the composition
$$[0,\dots,n] \to [n,\dots,0] \xrightarrow{\sigma} X.$$
The corresponding permutation of the vertices can be written as a product of $n(n+1)/2$ transpositions, whence the definition of ε. We *claim* that ρ is a chain map and that $\rho \sim \mathrm{Id}$. We omit the proof of the claim, which is the main difficulty in the proof. See Page 211–2 of Hatcher, for example.

Now, assuming the claim, we prove the theorem. We have
$$\rho^*(\varphi) \smile \rho^*(\psi)(\sigma) = \varphi(\varepsilon_k\sigma|_{[k,\dots,0]}) \cdot \psi(\varepsilon_\ell\sigma|_{[k+\ell,\dots,k]})$$
$$\rho^*(\psi \smile \varphi) = \varepsilon_{k+\ell}\psi(\sigma|_{k+\ell,\dots,k}) \cdot \varphi(\sigma|_{[k,\dots,0]}).$$
It follows that
$$\varepsilon_k\varepsilon_\ell(\rho^*(\varphi) \smile \rho^*(\psi)) = \varepsilon_{k+\ell}\rho^*(\psi \smile \varphi).$$
It is trivial to check that
$$\varepsilon_{k+\ell} = (-1)^{k\ell}\varepsilon_k\varepsilon_\ell.$$
Therefore
$$\rho^*(\varphi) \smile \rho^*(\psi) = (-1)^{k\ell}\rho^*(\psi \smile \varphi).$$

Since $\rho \sim \text{Id}$, on passing to cohomology this yields

$$[\varphi] \smile [\psi] = (-1)^{k\ell}[\psi \smile \varphi]$$

as desired. □

28. More examples of cup products

Example 28.1. We compute the cup product of $\mathbb{RP}^2$ with $\mathbb{F}_2$ coefficients. The interesting cup product is

$$\smile H^1(\mathbb{RP}^2;\mathbb{F}_2) \times H^1(\mathbb{RP}^2;\mathbb{F}_2) \to H^2(\mathbb{RP}^2;\mathbb{F}_2)$$

which is a function $\mathbb{F}_2 \times \mathbb{F}_2 \to \mathbb{F}_2$. We have to compute the image of $(1, 1)$. The answer is 1, so the cup product is nontrivial. We have to compute this though.

See Figure 2 for a depiction of the image of some 2-simplices that sum to a fundamental class over $\mathbb{F}_2$ for $\mathbb{RP}^2$. That is,

$$[\mathbb{RP}^2]_{\mathbb{F}_2} := \sigma_1 + \sigma_2 + \tau_1 + \tau_2 \in C_2(\mathbb{RP}^2;\mathbb{F}_2)$$

generates $H_2(\mathbb{RP}^2;\mathbb{F}_2) \cong \mathbb{F}_2$.

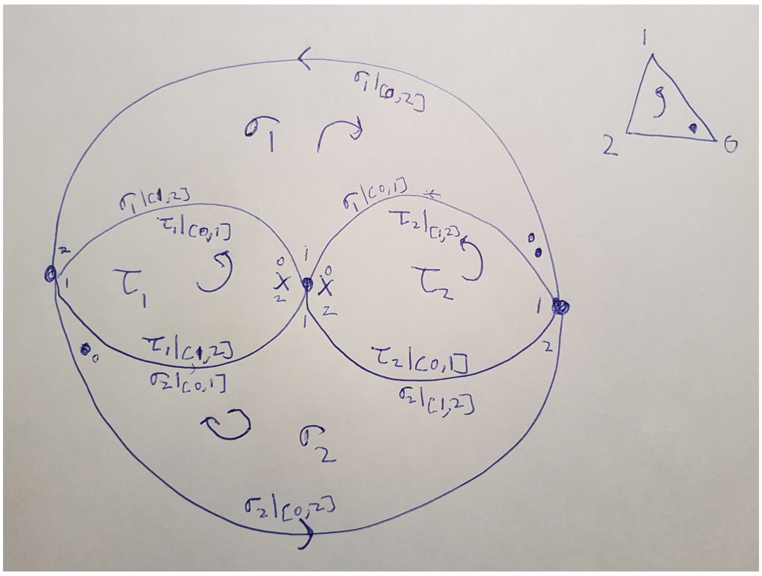

Figure 2. A diagram of $\mathbb{RP}^2$. Identify the edges with each other to get $\mathbb{RP}^2$. The images of four singular 2-simplices σ_1, σ_2, τ_1 and τ_2 are shown.

In the figure, which shows a disc, opposite points on the boundary circle are identified, as shown by the arrows. The convention for drawing σ_1 and σ_2 is the same as in our example of the torus above, and shown in Figure 1. The convention for drawing τ_1 and τ_2 is different, since these look like bigons, not triangles. Remember that we are describing the images of maps $\Delta^2 \to \mathbb{RP}^2$, not subsets homeomorphic to Δ^2. The τ_i are degenerate 2-simplices: one of the edges of each of the 2-simplices τ_i is mapped entirely to a single point, the vertex labelled with a $\times$. In both cases this edge is $\tau_i|_{[0,2]}$. Note that every singular 1-simplex appears exactly twice in $\partial(\sigma_1+\sigma_2+\tau_1+\tau_2)$, *with the same orientation.* Coming up with such a diagram is not so easy, as you will discover if you try to do it yourself for the Klein bottle.

We want to define a generator $\alpha \in H^1(\mathbb{RP}^2;\mathbb{F}_2)$ such that $(\alpha \smile \alpha)([\mathbb{RP}^2]_{\mathbb{F}_2}) = 1 \in \mathbb{F}_2$. Let us explicitly define an element α of $H^1(\mathbb{RP}^2;\mathbb{F}_2)$, then we will show it has these properties. Consider the function

$$\begin{aligned} S^2 &\to \mathbb{F}_2 \\ (x,y,z) &\mapsto \begin{cases} 1 & \text{if } (z>0) \text{ or } (z=0 \text{ and } y>0) \text{ or } (z=0=y \text{ and } x=1). \\ 0 & \text{else.} \end{cases} \end{aligned}$$

Note that for $p \in S^2$, $f(-p) = 1 - f(p)$. Let $\sigma\colon \Delta^1 \to \mathbb{RP}^2$ be a 1-simplex. Let $\pi\colon S^2 \to \mathbb{RP}^2$ be the projection that sends $p \in S^2$ to $[p] \in \mathbb{RP}^2 = S^2/\{p \sim -p\}$. Pick a lift $\widetilde{\sigma}\colon \Delta^1 \to S^2$, in the sense that $\pi \circ \widetilde{\sigma} = \sigma$. Then define a 1-cochain α by

$$\alpha(\sigma) := f(\widetilde{\sigma}(1)) - f(\widetilde{\sigma}(0)) \in \mathbb{F}_2.$$

There are choices of lifts. However the end point $\widetilde{\sigma}(1)$ changing to its antipodal point would also change the endpoint $\widetilde{\sigma}(0)$ to its antipodal point, by continuity and the fact that $\widetilde{\sigma}$ is a lift. Since $f(-p) = 1 - f(p)$, making both these changes does not affect $\alpha(\sigma)$. We note that α is a cocycle, as can be seen by lifting a 2-simplex to S^2: α applied to the boundary of a 2-simplex in $\mathbb{RP}^2$ is computed by applying f to the image of the vertices 2-simplex under the lifted map to S^2. Each vertex appears twice, since it is the vertex of two different edges, so in $\mathbb{F}_2$ all contributions vanish and so $\delta\alpha = 0$.

Let $\mu\colon \Delta^1 \to \mathbb{RP}^2$ be defined by $t \mapsto e^{\pi i t} \subset S^1 \subset S^2 \to \mathbb{RP}^2$, with S^1 sitting inside S^2 as the equator $\{z = 0\}$. Then μ is a cycle and

$$\alpha(\mu) = f(e^{\pi i}) - f(e^0) = 1 \in \mathbb{F}_2.$$

Therefore α is a generator of $H^1(\mathbb{RP}^2;\mathbb{F}_2)$ and μ is a generator of $H_1(\mathbb{RP}^2;\mathbb{F}_2)$.

Now that we have defined all our cycles and cocycles, we can proceed to compute the cup product. We have:

$$\alpha \smile \alpha(\sigma_1) = \alpha(\sigma_1|_{[0,1]}) \cdot \alpha(\sigma_1|_{[1,2]}).$$

To compute α on 1-simplices, think of the disc we have drawn in Figure 2 as the northern hemisphere of S^2, that is the points $\{z \geq 0\}$. Then $f(p) = 1$ everywhere in the interior of the disc, everywhere on the upper boundary arc, excluding the endpoints. $f(p) = 0$ on the lower boundary arc, excluding the endpoints. Finally $f(p) = 1$ for p the right hand vertex on the boundary, and $f(p) = 0$ for p the left hand vertex on the boundary. We assign to a 1-simplex the difference in the values of f at its endpoints. So any 1-simplex in Figure 2 with an endpoint at the left hand vertex has α evaluating to 1 on it, while the other 1-simplices have α evaluating to 0. This makes computing α now relatively straightforward. So we have

$$\begin{aligned}
\alpha \smile \alpha(\sigma_1) &= \alpha(\sigma_1|_{[0,1]}) \cdot \alpha(\sigma_1|_{[1,2]}) = 0 \cdot 1 = 0 \\
\alpha \smile \alpha(\sigma_2) &= \alpha(\sigma_2|_{[0,1]}) \cdot \alpha(\sigma_2|_{[1,2]}) = 1 \cdot 0 = 0 \\
\alpha \smile \alpha(\tau_1) &= \alpha(\tau_1|_{[0,1]}) \cdot \alpha(\tau_1|_{[1,2]}) = 1 \cdot 1 = 1 \\
\alpha \smile \alpha(\tau_2) &= \alpha(\tau_2|_{[0,1]}) \cdot \alpha(\tau_2|_{[1,2]}) = 0 \cdot 0 = 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
(\alpha \smile \alpha)([\mathbb{RP}^2]_{\mathbb{F}_2}) &= (\alpha \smile \alpha)(\sigma_1 + \sigma_2 + \tau_1 + \tau_2) \\
&= (\alpha \smile \alpha)(\sigma_1) + (\alpha \smile \alpha)(\sigma_2) + (\alpha \smile \alpha)(\tau_1) + (\alpha \smile \alpha)(\tau_2) \\
&= 0 + 0 + 1 + 0 = 1.
\end{aligned}$$

So we have a nontrivial cup product as asserted above.

Example 28.2. We describe the cohomology ring of a sphere S^n. We have that $H^i(S^n; R) \cong R$ if $i = 0, n$, and is 0 otherwise. The cohomology ring is

$$H^*(S^n; R) \cong R[x]/(x^2),$$

the quotient of the polynomial ring $R[x]$ by the ideal (x^2). This is known as a truncated polynomial ring. Here x is the generator of $H^n(S^n; R) \cong R$. So $H^*(S^n; R) = \{a + bx \mid a, b \in R\}$.

Next we show that cup products in wedges of spheres are pretty boring. This will provide a good source of examples of spaces with a certain set of cohomology groups, and when we show that for certain other spaces, cup products are nontrivial, this will enable us to show that these spaces are not homotopy equivalent.

Proposition 28.3. *Consider a wedge of spheres*

$$S^{n_1} \vee S^{n_2} \vee \cdots \vee S^{n_k}.$$

Here $n_i \geq 1$ and $i = 1, \ldots, k$. Then all cup products of degree at least one are zero.

The analogue actually holds for general wedges, but we will not give the proof.

Proof. Let $p_i \colon S^{n_1} \vee S^{n_2} \vee \cdots \vee S^{n_k} \to S^{n_i}$ be the projection map, sending all spheres other than the ith sphere to the basepoint. For all $n \in \mathbb{N}_0$, we have an isomorphism

$$p_1^* \oplus \cdots \oplus p_k^* \colon H^n(S^{n_1}; R) \oplus \cdots \oplus H^n(S^{n_k}; R) \xrightarrow{\cong} H^n(S^{n_1} \vee \cdots \vee S^{n_k}; R).$$

The statement means that we need to show that

$$p_i^*([S^{n_i}]^*) \smile p_j^*([S^{n_j}]^*) = 0 \in H^{n_i+n_j}(S^{n_1} \vee \cdots \vee S^{n_k}; R)$$

for all $i, j = 1, \ldots, k$. For $i \neq j$, let

$$g_{ij} \colon S^{n_1} \vee \cdots \vee S^{n_k} \to S^{n_i} \vee S^{n_j}$$

be the projection, and let

$$q_i \colon S^{n_i} \vee S^{n_j} \to S^{n_i}$$

and

$$q_j \colon S^{n_i} \vee S^{n_j} \to S^{n_j}$$

also be projections. Note that $q_i \circ g_{ij} = p_i$ and $q_j \circ g_{ij} = p_j$. We then compute:

$$\begin{aligned} p_i^*([S^{n_i}]^*) \smile p_j^*([S^{n_j}]^*) &= g_{ij}^*(q_i^*([S^{n_i}]^*)) \smile g_{ij}^*(q_j^*([S^{n_j}]^*)) \\ &= g_{ij}^*\big(q_i^*([S^{n_i}]^*) \smile q_j^*([S^{n_j}]^*)\big) \\ &= g_{ij}^*(0) = 0 \end{aligned}$$

where we use that $q_i^*([S^{n_i}]^*) \smile q_j^*([S^{n_j}]^*) \in H^{n_i+n_j}(S^{n_i} \vee S^{n_j}; R) = 0$ since $n_i \geq 1$ for all i, so certainly $q_i^*([S^{n_i}]^*) \smile q_j^*([S^{n_j}]^*) = 0$. For $i = j$, we have

$$p_i^*([S^{n_i}]^*) \smile p_i^*([S^{n_i}]^*) = p_i^*([S^{n_i}]^* \smile [S^{n_i}]^*) = p_i^*(0) = 0.$$

Since $[S^{n_i}]^* \smile [S^{n_i}]^* \in H^{2n_i}(S^{n_i}; R) = 0$. □

29. Cap products

In this section we define cap products. Let R be a commutative ring. The cap product is an R-bilinear function

$$\frown : H^k(X;R) \times H_\ell(X;R) \to H_{\ell-k}(X;R)$$

As with cup product, we define the cap product on the chain and cochain level, then we show that these maps induce well-defined maps on homology and cohomology. Then we show that these maps have the certain useful properties, in particular the cap and cup product interact in a useful way. As mentioned above, the Poincaré duality map is defined in terms of cap products.

Definition 29.1. Let $\varphi \in C^k(X;R)$ be a singular cochain. Let $\sigma\colon \Delta^\ell \to X$ with $k \leq \ell$. Then we define

$$\varphi \frown \sigma = \varphi(\sigma|_{[0,\ldots,k]}) \otimes \sigma|_{[k,\ldots,\ell]} \in R \otimes C_{k-\ell}(X) = C_{k-\ell}(X;R).$$

Extend to all of $C_\ell(X;R)$ by linearity. This gives rise to a product

$$\frown : C^k(X;R) \times C_\ell(X;R) \to C_{\ell-k}(X;R)$$

when $\ell \geq k$ and if $k > \ell$ then we take the cap product to be the zero map by definition.

To remember what is happening, you can think that the cochain is hungry, with appetite k, and the ℓ simplex is food. The cochain eats up k of the simplex, leaving an $\ell - k$ simplex left.

The next lemma will be key for showing that the cap product descends to a well-defined map on homology/cohomology.

Lemma 29.2. *Let $\varphi \in C^k(X;R)$ and let $\sigma \in C_\ell(X;R)$. Then*

$$\partial(\varphi \frown \sigma) = (-1)^k(-\delta\varphi \frown \sigma + \varphi \frown \partial\sigma) \in C_{\ell-k-1}(X;R).$$

Proof. Assume that $k \leq \ell$, or else this just says that $0 = 0$. Let $\sigma\colon \Delta^\ell \to X$ be a singular ℓ-simplex. We compute the three terms individually:

(29.3)
$$\varphi \frown \partial\sigma = \sum_{i=0}^{k}(-1)^i \varphi(\sigma|_{[0,\dots,\widehat{i},\dots,k+1]})\sigma|_{[k+1,\dots,\ell]} + \sum_{i=k+1}^{\ell}(-1)^i\varphi(\sigma|_{[0,\dots,k]})\sigma|_{[k,\dots,\widehat{i},\dots,\ell]};$$

(29.4)
$$\delta\varphi \frown \sigma = \delta\varphi(\sigma|_{[0,\dots,k+1]})\sigma|_{[k+1,\dots,\ell]} = \varphi(\partial\sigma|_{[0,\dots,k+1]})\sigma|_{[k+1,\dots,\ell]}$$

(29.5)
$$= \sum_{i=0}^{k+1}(-1)^i\varphi(\sigma|_{[0,\dots,\widehat{i},\dots,k+1]})\sigma|_{[k+1,\dots,\ell]};$$

(29.6)
$$(-1)^k\partial(\varphi \frown \sigma) = (-1)^k\partial(\varphi(\sigma|_{[0,\dots,k]})\sigma|_{[k,\dots,\ell]}) = (-1)^k\varphi(\sigma|_{[0,\dots,k]})\partial\sigma|_{[k,\dots,\ell]}$$

(29.7)
$$= \sum_{i=k}^{\ell}(-1)^i\varphi(\sigma|_{[0,\dots,k]})\sigma|_{[k,\dots,\widehat{i},\dots,\ell]}.$$

When $i = k+1$ in the (29.5) cancels with $i = k$ in (29.7). Then we are left with (29.5) + (29.7) = (29.3). That is,
$$\varphi \frown \partial\sigma = \delta\varphi \frown \sigma + (-1)^k\partial(\varphi \frown \sigma),$$
which rearranges to the statement of the lemma. □

Lemma 29.8. *The cap product induces a defined and well defined map*
$$\begin{array}{rcl} H^k(X;R) \times H_\ell(X;R) & \to & H_{\ell-k}(X;R) \\ ([\varphi],[\sigma]) & \mapsto & [\varphi \frown \sigma] \end{array}$$
for all $k, \ell \in \mathbb{N}_0$.

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Proof. Let $\varphi \in C^k(X;R)$ be a cocycle, so that $\delta\varphi = 0$, and let $\sigma \in C_\ell(X;R)$ a cycle with $\partial\sigma = 0$. Then
$$\partial(\varphi \frown \sigma) = (-1)^k(-\delta\varphi \frown \sigma + \varphi \frown \partial\sigma) = (-1)^k(-0 \frown \sigma + \varphi \frown 0) = 0.$$
So $\varphi \frown \sigma$ is a cycle. Then we want to check that cap product is well-defined.
$$\begin{aligned}
&[(\varphi + \delta\chi) \frown (\sigma + \partial\tau)] = [\varphi \frown \sigma + \delta\chi \frown \sigma + \delta\chi \frown \partial\tau + \varphi \frown \partial\tau] \\
=&[\varphi \frown \sigma \pm \chi \frown \partial\sigma \pm \partial(\chi \frown \sigma) \pm \chi \frown \partial^2\tau \pm \partial(\chi \frown \partial\tau) \pm \delta\varphi \frown \partial\tau + \partial(\varphi \frown \tau)] \\
=&[\varphi \frown \sigma \pm \chi \frown 0 \pm \partial(\chi \frown \sigma) \pm \chi \frown 0 \pm \partial(\chi \frown \partial\tau) \pm 0 \frown \partial\tau + \partial(\varphi \frown \tau)] \\
=&[\varphi \frown \sigma + \partial(\pm\chi \frown \sigma \pm \chi \frown \partial\tau \pm \varphi \frown \tau)] \\
=&[\varphi \frown \sigma].
\end{aligned}$$
So the homology class produced does not depend on the cocycle representative for $[\varphi]$, nor does it depend on the cycle representative for σ. □

Here are two basic lemmas on cap product: the identity of cohomology acts as identity on homology, and a special case where cap product is the same as evaluation.

Lemma 29.9. *Let $\sigma \in H_k(X;R)$ and let $1_X \in H^0(X;R)$ be the identity in the cohomology ring. Then $1_X \frown \sigma = \sigma$.*

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Proof.

$$1_X \frown \sigma = 1_X(\sigma_{[0]})\sigma|_{[0,\dots,k]} = 1 \cdot \sigma = \sigma.$$

□

Lemma 29.10. *Suppose that X is connected. Then*

$$\begin{array}{rcl} \frown : H^k(X;R) \times H_k(X;R) & \to & H_0(X;R) \cong R \\ ([\varphi],[\sigma]) & \mapsto & \langle \varphi, \sigma \rangle. \end{array}$$

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Proof. Let I_X denote a 0-simplex that generates $H_0(X;R)$. We compute:

$$\begin{aligned} [\varphi \frown \sigma] &= [\varphi(\sigma|_{[0,\dots,k]})\sigma_{[k]}] \\ &= \langle \varphi, \sigma \rangle I_X \mapsto \langle \varphi, \sigma \rangle \in R. \end{aligned}$$

□

30. An example of cap products

Example 30.1. We compute some cap products in $T^2 = S^1 \times S^1$. We use the same conventions as in Example 26.1. We repeat our diagram of the torus, and we use the same singular 2-simplices σ_1 and σ_2. We also show the projections $p, q : S^1 \times S^1 \to S^1$.

Let $\theta \in C^1(S^1;\mathbb{Z})$ represent a generator $[\theta] \in H^1(S^1;\mathbb{Z})$. We have singular 1-simplices $\mu, \nu : \Delta^1 \to S^1$, with $[\mu] \in H_1(S^1;\mathbb{Z})$ a generator (i.e. a fundamental class) and $[\nu] = 0 \in H_1(S^1;\mathbb{Z})$. We also have that $[-\sigma_1 + \sigma_2] = [T^2] \in H_2(T^2;\mathbb{Z})$ is a fundamental class.

We compute the cap product of a generating set of $H^1(T^2;\mathbb{Z})$ with the fundamental class $[T^2]$.

$$\begin{aligned} p^*(\theta) \frown \sigma_1 &= p^*(\theta)(\sigma_1|_{[0,1]})\sigma_1|_{[1,2]} \\ &= \theta(p \circ \sigma_1|_{[0,1]})\sigma_1|_{[1,2]} \\ &= \theta(\nu)\sigma_1|_{[1,2]} = 0 \end{aligned}$$

On the other hand

$$\begin{aligned} p^*(\theta) \frown \sigma_2 &= p^*(\theta)(\sigma_2|_{[0,1]})\sigma_2|_{[1,2]} \\ &= \theta(p \circ \sigma_2|_{[0,1]})\sigma_2|_{[1,2]} \\ &= \theta(\mu)\sigma_2|_{[1,2]} = \sigma_2|_{[1,2]} = [1 \times S^1]. \end{aligned}$$

Putting these together we obtain

$$[p^*(\theta)] \frown [T^2] = -p^*(\theta) \frown \sigma_1 + p^*(\theta) \frown \sigma_2 = [1 \times S^1].$$

Similarly, we can compute that

$$[q^*(\theta)] \frown [T^2] == -[S^1 \times 1].$$

The map

$$- \frown [T^2] : H^1(T^2;\mathbb{Z}) \to H_1(T^2;\mathbb{Z})$$

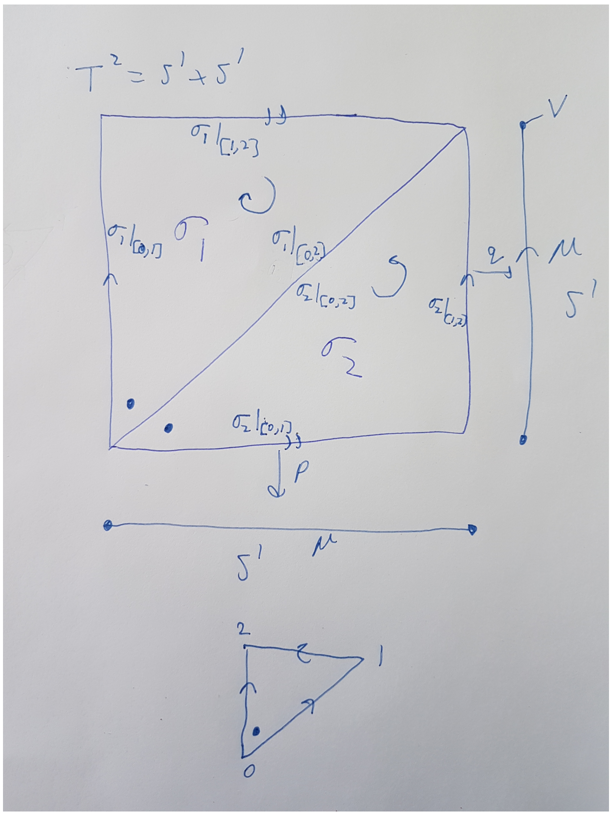

Figure 3. A diagram of the torus. Identify the vertical edges with each other and the horizontal edges with each other to get the torus. The projections $p, q \colon S^1 \times S^1 \to S^1$ are shown. The images of two singular 2-simplices are shown. At the bottom, a standard 2-simplex Δ^2 is drawn.

is an isomorphism, with

$$p^*(\theta) \mapsto [1 \times S^1]$$
$$q^*(\theta) \mapsto -[S^1 \times 1].$$

This map is in fact the Poincaré duality map, so it had to have been an isomorphism.

31. Properties of cap product

An important formula will be the following formula involving both the cup and the cap product.

Theorem 31.1 (Cup-cap formula). *Let X be a topological space, and let R be a commutative ring. Let $\varphi \in C^k(X;R)$, let $\psi \in C^\ell(X;R)$ and let $\sigma \in C_n(X;R)$. Then*

$$\varphi \frown (\psi \frown \sigma) = (\psi \smile \varphi) \frown \sigma \in C_{n-k-\ell}(X;R).$$

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Note that the order of the φ and the ψ switch. We could change them back, but then we would have to introduce a sign $(-1)^{k\ell}$.

Proof. Suppose that $k + \ell \leq n$, otherwise the statement is trivial since both sides are zero. Let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. Then:

$$\begin{aligned}\varphi \frown (\psi \frown \sigma) &= \varphi \frown \big(\psi(\sigma|_{[0,\dots,\ell]}) \cdot \sigma|_{[\ell,\dots,n]}\big)\\ &= \psi(\sigma|_{[0,\dots,\ell]}) \cdot \varphi(\sigma|_{[\ell,\dots,\ell+k]}) \cdot \sigma|_{[\ell+k,\dots,n]}\\ &= (\psi \smile \phi)(\sigma|_{[0,\dots,\ell+k]}) \cdot \sigma|_{[\ell+k,\dots,n]}\\ &= (\psi \smile \varphi) \frown \sigma.\end{aligned}$$

Extending this by linearity to general $\sigma \in C_n(X; R)$ gives the result. □

We will also need to know the behaviour of cap product with respect to maps between spaces.

Theorem 31.2. *Let $f\colon X \to Y$ be a map of spaces. Let $\varphi \in C^k(Y; \mathbb{Z})$ and let $\sigma \in C_n(X; \mathbb{Z})$. Then*

$$f_*(f^*(\varphi) \frown \sigma) = \varphi \frown f_*(\sigma).$$

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Proof. Let $\sigma\colon \Delta^n \to X$. As usual we prove the theorem for this σ and extend by linearity. We compute:

$$\begin{aligned}f_*(f^*(\varphi) \frown \sigma) &= f_*(f^*(\varphi)(\sigma|_{0,\dots,k})\sigma|_{[k,\dots,n]}\\ &= f_*\varphi(f \circ \sigma|_{0,\dots,k})\sigma|_{[k,\dots,n]}\\ &= \varphi(f \circ \sigma|_{0,\dots,k}) f \circ \sigma|_{[k,\dots,n]}\\ &= \varphi \frown f_*(\sigma).\end{aligned}$$

□

32. Applications to manifolds

One of the main reasons for introducing the cap product is its rôle in the formulation of the Poincaré duality theorem.

We can now state a more precise version of Poincaré duality.

Theorem 32.1 (Poincaré Duality)**.** *Let M be a closed, oriented n-dimensional manifold. Let $[M] \in H_n(M; \mathbb{Z})$ be a fundamental class. Then for every $r \in \mathbb{N}_0$, the map*

$$\begin{array}{rcl} PD\colon H^r(M; \mathbb{Z}) & \xrightarrow{\cong} & H_{n-r}(M; \mathbb{Z}) \\ [\varphi] & \mapsto & [\varphi \frown [M]] \end{array}$$

is an isomorphism.

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Theorem 32.2 (Poincaré Duality with $\mathbb{Z}/2$-coefficients)**.** *Let M be a closed n-dimensional manifold. Let $[M]_{\mathbb{Z}/2} \in H_n(M; \mathbb{Z}/2)$ be a $\mathbb{Z}/2$-fundamental class. Then for every $r \in \mathbb{N}_0$, the map*

$$\begin{array}{rcl} PD\colon H^r(M; \mathbb{Z}/2) & \xrightarrow{\cong} & H_{n-r}(M; \mathbb{Z}/2) \\ [\varphi] & \mapsto & [\varphi \frown [M]_{\mathbb{Z}/2}] \end{array}$$

is an isomorphism.

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Example 32.3.

(1) For S^n, we have
$$- \frown [S^n]\colon H^n(S^n;R) = R \xrightarrow{\cong} H_0(S^n;R) \cong R$$
is an isomorphism.

(2) We also computed the cap product with the fundamental class of the torus:
$$- \frown [T^2]\colon H^1(T^2;\mathbb{Z}) \cong \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} \mathbb{Z}^2 \cong H_1(T^2;\mathbb{Z})$$
which is also an isomorphism.

Now we start to discuss some applications of the whole theory. It has been quite a lot of work to develop homology, cohomology, cup and cap products, together with universal coefficients and Poincaré duality. So now we start to get some rewards for our labour.

Proposition 32.4. *Let $f\colon M \to N$ be a degree one map between n-dimensional manifolds. Then for all $k \in \mathbb{N}_0$ there is a direct summand of $H_k(M;Z)$ isomorphic to $H_k(N;\mathbb{Z})$.*

Proof. Consider the following diagram.
$$\begin{tikzcd}
H^{n-k}(M;\mathbb{Z}) \arrow[d, "-\frown[M]", "\cong"'] & H^{n-k}(N;\mathbb{Z}) \arrow[l, "f^*"'] \arrow[d, "-\frown[N]", "\cong"'] \\
H_k(M;\mathbb{Z}) \arrow[r, "f_*"] & H_k(N;\mathbb{Z}).
\end{tikzcd}$$
We claim that it commutes. To see this let $\varphi \in H^{n-k}(N;\mathbb{Z})$. Then
$$f_*(f^*(\varphi) \frown [M]) = \varphi \frown f_*([M]) = \varphi \frown [N].$$
The first equality is the functoriality for cap product (Theorem 31.2), and the second is that f is degree one so $f_*([M]) = [N]$. Let
$$f^!\colon H_k(N;\mathbb{Z}) \to H_k(M;\mathbb{Z})$$
be defined by the composition $f^! := (- \frown [M]) \circ f^* \circ (- \frown [N])^{-1}$. By commutativity of the diagram above we have that $f_* \circ f^! = \mathrm{Id}$.

Now consider the short exact sequence
$$0 \to \ker f_* \to H_k(M;\mathbb{Z}) \xrightarrow{f_*} H_k(N;\mathbb{Z}) \to 0.$$
The map $f^!\colon H_k(N;\mathbb{Z}) \to H_k(M;\mathbb{Z})$ is a homomorphism and satisfies $f_* \circ f^! = \mathrm{Id}$, so is a splitting. It follows that
$$H_k(M;\mathbb{Z}) \cong \ker f_* \oplus f^!(H_k(N;\mathbb{Z})).$$
□

In particular the existence of a degree one map $M \to N$ implies that there is a surjective homomorphism $H_k(M;\mathbb{Z}) \to H_k(N;\mathbb{Z})$ for every $k \in \mathbb{N}_0$.

Example 32.5.

(i) There is no degree one map $S^3 \to \mathbb{RP}^3$. Here $H_1(S^3) = 0$ whereas $H_1(\mathbb{RP}^3) \cong \mathbb{Z}/2$, so there can be no surjection $H_1(S^3) \to H_1(\mathbb{RP}^3)$.

(ii) There is no degree one map $S^1 \times S^2 \to \mathbb{RP}^3$. Here $H_1(S^1 \times S^2) \cong \mathbb{Z}$ and $H_1(\mathbb{RP}^3) \cong \mathbb{Z}/2$, and $\mathbb{Z}/2$ is not a direct summand of $\mathbb{Z}$.

(iii) There is no degree one map from a surface of genus one to a surface of genus two, since there can be no surjective homomorphism on first homology, which would be a surjection $\mathbb{Z}^2 \to \mathbb{Z}^4$.

Now we combine cup and cap products with Poincaré duality to compute cup products.

Recall that the complex projective plane $\mathbb{CP}^2$ is by definition

$$\mathbb{CP}^2 := \frac{\mathbb{C}^3 \smallsetminus \{(0,0,0)\}}{(z_0,z_1,z_2) \sim (\lambda z_0, \lambda z_1, \lambda z_2);\ \lambda \in \mathbb{C}\smallsetminus\{0\}}.$$

This has a CW decomposition with three cells, one cell of dimension 0,2 and 4. That is

$$\mathbb{CP}^2 = e^0 \cup e^2 \cup e^4 = S^2 \cup_\eta D^4.$$

It is a 4-dimensional, oriented manifold. Let $\varphi := [\mathbb{CP}^1]^* \in H^2(\mathbb{CP}^2;\mathbb{Z}) \cong \mathbb{Z}$ be a generator. Let $[\mathbb{CP}^2] \in H_4(\mathbb{CP}^2;\mathbb{Z})$ be a fundamental class. Let $\psi \in H^2(\mathbb{CP}^2;\mathbb{Z})$. Then

$$\langle \varphi \smile \psi, [\mathbb{CP}^2]\rangle = (\varphi \smile \psi) \frown [\mathbb{CP}^2] = \psi \frown (\varphi \frown [\mathbb{CP}^2])$$

Now, since the map $(- \frown [\mathbb{CP}^2]\colon H^2(\mathbb{CP}^2;\mathbb{Z}) \to H_2(\mathbb{CP}^2;\mathbb{Z})$ is an isomorphism by Poincaré duality, we have that $\varphi \frown [\mathbb{CP}^2]$ is a generator of $H_2(\mathbb{CP}^2;\mathbb{Z})$, so $\varphi \frown [\mathbb{CP}^2] = \pm[\mathbb{CP}^1]$.

Next, note that $\mathrm{Ext}^1(H_1(\mathbb{CP}^2;\mathbb{Z}),\mathbb{Z}) = \mathrm{Ext}^1(0,\mathbb{Z}) = 0$, so

$$\mathrm{ev}\colon H^2(\mathbb{CP}^2;\mathbb{Z}) \to \mathrm{Hom}(H_2(\mathbb{CP}^2;\mathbb{Z}),\mathbb{Z})$$

is an isomorphism by the universal coefficient theorem. We have that

$$\langle \varphi \smile \psi, [\mathbb{CP}^2]\rangle = \psi \frown (\varphi \frown [\mathbb{CP}^2]) = \mathrm{ev}(\psi)(\varphi \frown [\mathbb{CP}^2])$$

Taking $\psi = \varphi$, we have

$$\langle \varphi \smile \varphi, [\mathbb{CP}^2]\rangle = \mathrm{ev}(\varphi)(\varphi \frown [\mathbb{CP}^2]) = [\mathbb{CP}^1]^*(\pm[\mathbb{CP}^1]) = \pm 1$$

It follows that $\varphi \smile \varphi = \pm[\mathbb{CP}^2]^*$ is a dual fundamental class. In particular the cup product

$$\smile\colon H^2(\mathbb{CP}^2;\mathbb{Z}) \times H^2(\mathbb{CP}^2;\mathbb{Z}) \to H^4(\mathbb{CP}^2;\mathbb{Z})$$

is nontrivial. In general, $\psi = n\varphi$ for some $n \in \mathbb{Z}$, and so

$$\langle \varphi \smile \psi, [\mathbb{CP}^2]\rangle = \mathrm{ev}(n\varphi)(\varphi \frown [\mathbb{CP}^2]) = n \cdot [\mathbb{CP}^1]^*(\pm[\mathbb{CP}^1]) = \pm n.$$

Proposition 32.6. *The spaces $S^2 \vee S^4$ and $\mathbb{CP}^2$ are not homotopy equivalent.*

Note that the homology and cohomology of these spaces coincide, namely a $\mathbb{Z}$ in degrees 0,2, and 4. So one really needs cup products to distinguish the homotopy types of these spaces.

Proof. Suppose that there is a homotopy equivalence $f\colon S^2 \vee S^4 \to \mathbb{CP}^2$. Then we have a commutative diagram as follows:

$$\begin{tikzcd} H^2(\mathbb{CP}^2) \times H^2(\mathbb{CP}^2) \arrow[r, "\smile"] \arrow[d, "f^* \times f^*"] & H^4(\mathbb{CP}^2) \arrow[d, "f^*"] \\ H^2(S^2 \vee S^4) \times H^2(S^2 \vee S^4) \arrow[r, "\smile"] & H^4(S^2 \vee S^4) \end{tikzcd}$$

This commutes by functoriality of the cup product. Since f is a homotopy equivalence, the vertical maps are isomorphisms. Start with $(1,1) \in \mathbb{Z} \times \mathbb{Z} \cong H^2(\mathbb{CP}^2) \times H^2(\mathbb{CP}^2)$. This maps to $\pm 1 \in H^4(\mathbb{CP}^2)$ and therefore to ± 1 in $H^4(S^2 \vee S^4)$. On the other hand $(1,1)$ maps to $\pm(1,1) \in H^2(S^2 \vee S^4) \times H^2(S^2 \vee S^4)$. Since the cup product of wedges of spheres vanishes, this maps to $0 \in H^4(S^2 \vee S^4)$. This violates $f^*(\varphi) \smile f^*(\varphi) = f^*(\varphi \smile \varphi)$, i.e. the commutativity of the diagram. So no such homotopy equivalence f can exist. □

Recall that $\mathbb{CP}^2 = S^2 \cup_\eta D^4$. The 4-cell D^4 is attached by a map $\eta\colon S^3 \to S^2$. This is a famous map called the Hopf map. If this map were null homotopic, then we would have that $S^2 \cup_\eta D^4 \simeq S^2 \vee S^4$. Since we just showed this is not the case, we must have that the map $\eta\colon S^3 \to S^2$ is nontrivial. The set of (based) homotopy classes of maps from S^3 to S^2 form a group, a higher dimensional analogue of the fundamental group, called $\pi_3(S^2)$. We have just shown that $\pi_3(S^2)$ is nontrivial. In fact $\pi_3(S^2) \cong \mathbb{Z}$, and η is a generator, but we shall not prove that here.

Example 32.7. We compute the cup products of $S^2 \times S^2$.

$$\smile\colon H^2(S^2 \times S^2) \cong \mathbb{Z}^2 \times H^2(S^2 \times S^2) \cong \mathbb{Z}^2 \to H^4(S^2 \times S^2) \cong \mathbb{Z}$$

This can be represented by a 2×2 matrix. Let $p_i\colon S^2 \times S^2 \to S^2$ be the projection to the ith factor and let $\theta \in H^2(S^2)$ be a generator. Then $p_1^*(\theta)$, $p_2^*(\theta)$ are generators of $H^2(S^2 \times S^2) \cong \mathbb{Z}^2$. We have that

$$p_i^*(\theta) \smile p_i^*(\theta) = p_i^*(\theta \smile \theta) = p_i^*(0) = 0.$$

So the matrix looks like

$$\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}.$$

We need to find x. We know that the off-diagonal entries are equal by the symmetry of the cup product. Let $\varphi = (1,0)$ and let $\psi = (0,1)$ in $H^2(S^2 \times S^2) \cong \mathbb{Z}^2$. Then

$$(\varphi \smile \psi) \frown [S^2 \times S^2] = \psi \frown (\varphi \frown [S^2 \times S^2]) = \mathrm{ev}(\psi)(\varphi \frown [S^2 \times S^2]) = m$$

for some $m \in \mathbb{Z}$, that we shall determine. This m is such that

$$\varphi \frown [S^2 \times S^2] = n \cdot [S^2 \times \mathrm{pt}] + m \cdot [\mathrm{pt} \times S^2].$$

Note that

$$0 = (\varphi \smile \varphi) \frown [S^2 \times S^2] == \mathrm{ev}(\varphi)(\varphi \frown [S^2 \times S^2] = n.$$

So $\varphi \frown [S^2 \times S^2] = m \cdot [\mathrm{pt} \times S^2]$. Since φ is a generator of $\mathbb{Z}^2$, and $- \frown [S^2 \times S^2]\colon H^2(S^2 \times S^2) \to H_2(S^2 \times S^2)$ is an isomorphism, then $m \cdot [\mathrm{pt} \times S^2]$ must also be a generator. For the vector $(m, 0)$ to be part of a generating set of $\mathbb{Z}^2$, there must be a $(y, z) \in \mathbb{Z}^2$ with

$$\begin{pmatrix} m & y \\ 0 & z \end{pmatrix}$$

having determinant ± 1. But the determinant is my, and for this to be ± 1 we must have $m = \pm 1$. So

$$\varphi \smile \psi = \pm [S^2 \times S^2]^*$$

and the matrix for the cup product, with respect to the chosen basis $p_i^*(\theta)$, is

$$\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}.$$

We could change our choice of fundamental class to its negative, if we want to remove minus signs that might appear here.

Proposition 32.8. *The spaces $S^2 \times S^2$ and $\mathbb{CP}^2 \# \mathbb{CP}^2$ are not homotopy equivalent.*

Proof. Both spaces are closed, orientable 4-manifolds with homology $\mathbb{Z}, 0, \mathbb{Z}^2, 0\mathbb{Z}$. With respect to the usual bases, the cup product on $S^2 \times S^2$ is represented by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ while the cup product on $\mathbb{CP}^2$ is represented by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Suppose that there is a homotopy equivalence $S^2 \times S^2 \to \mathbb{CP}^2 \# \mathbb{CP}^2$. Let $\beta = (1, 0) \in H^2(\mathbb{CP}^2 \# \mathbb{CP}^2)$ be a generator: then $\beta \smile \beta = \pm [\mathbb{CP}^2 \# \mathbb{CP}^2]^*$. Therefore

$$f^*(\beta) \smile f^*(\beta) = f^*(\beta \smile \beta) = f^*(\pm [\mathbb{CP}^2 \# \mathbb{CP}^2]^*) = \pm [S^2 \times S^2]^*.$$

This is an odd multiple ± 1 of the dual fundamental class. Suppose that $f^*(\beta) = (a, b)$ for some (a, b) in $H^2(S^2 \times S^2) \cong \mathbb{Z}^2$. Then $f^*(\beta) \smile f^*(\beta)$ is computed by

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b & a \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 2ab.$$

This is even for every (a, b), so can never equal an odd multiple of the dual fundamental class. This contradiction yields that $S^2 \times S^2$ and $\mathbb{CP}^2 \# \mathbb{CP}^2$ are not homotopy equivalent. □

Example 32.9. Here is an application of cup products on $\mathbb{CP}^n$. We want to show that there is no retract $r \colon \mathbb{CP}^m \to \mathbb{CP}^n$, when $m > n \geq 1$. For a subspace $X \subseteq Y$, with $i \colon X \to Y$ the inclusion map, a retract is a continuous map $r \colon Y \to X$ with $r \circ i = \mathrm{Id} \colon X \to X$.

Assume for a contradiction that there is a retract $r \colon \mathbb{CP}^m \to \mathbb{CP}^n$. Then $r \circ i = \mathrm{Id}$ implies that

$$i^* \circ r^* = \mathrm{Id}^* \colon H^2(\mathbb{CP}^n) \xrightarrow{r^*} H^2(\mathbb{CP}^m) \xrightarrow{i^*} H^2(\mathbb{CP}^n).$$

Since i^* and Id are isomorphisms on second cohomology, we deduce that so is r^*. Let $x := [\mathbb{CP}^1]^*_n \in H^2(\mathbb{CP}^n) \cong \mathbb{Z}$ be a generator. Then $r^*(x) = \pm [\mathbb{CP}^1]^*_m$ is a generator of $H^2(\mathbb{CP}^m)$. Therefore, since $x^m \in H^{2m}(\mathbb{CP}^n) = 0$, we have

$$0 = r^*(0) = r^*(x^m) = r^*(x)^m = (\pm [\mathbb{CP}^1]^*)^m = \pm [\mathbb{CP}^m]^* \neq 0 \in H^{2m}(\mathbb{CP}^m).$$

This is a contradiction, so no retraction r exists, as desired.

33. The Borsuk-Ulam Theorem

The Borsuk-Ulam theorem is a beautiful theorem that combines a lot our expertise. See Hatcher p. 174-6 for the full exposition, that I explained in the lecture. We start with a proposition.

Proposition 33.1. *Let $f\colon S^n \to S^n$ be such that $f(-x) = -f(x)$. Then* $\deg f \in \mathbb{Z}$ *is odd.*

PNE

We will need the short exact sequence associated to a 2-sheeted covering space $p\colon \widetilde{X} \to X$:

$$0 \to C_*(X;\mathbb{Z}/2) \xrightarrow{\tau} C_*(\widetilde{X};\mathbb{Z}/2) \xrightarrow{p_*} C_*(X;\mathbb{Z}/2) \to 0.$$

Here the map τ is a *transfer map*, sending a simplex $\sigma\colon \Delta^n \to X$ to the sum $\widetilde{\sigma}_1 + \widetilde{\sigma}_2$ of the lifts $\widetilde{\sigma}_i$ of σ to $\widetilde{X}$.

Sketch of proof. To prove the proposition, apply the resulting long exact sequence in homology with $\mathbb{Z}/2$-coefficients to $p\colon S^n \to \mathbb{RP}^n$ and the map between the long exact sequences induced by f. One proves by induction that f induces an isomorphism $\bar{f}_*\colon H_i(\mathbb{RP}^n;\mathbb{Z}/2) \to H_i(\mathbb{RP}^n;\mathbb{Z}/2)$ for every $0 \leq i \leq n$, and then deduces that $f_*\colon H_n(S^n;\mathbb{Z}/2) \to H_n(S^n;\mathbb{Z}/2)$ is an isomorphism. It follows that the degree of f is odd. □

Theorem 33.2 (Borsuk-Ulam Theorem)**.** *Let $g\colon S^n \to \mathbb{R}^n$ be a map. There exists a point $x \in S^n$ with $g(x) = g(-x)$.*

PNE

The classical illustration of this theorem is that there is a pair of antipodal points on the earth's surface that have exactly the same temperature and pressure (assuming these are both constant functions $S^2 \to \mathbb{R}$.

Proof. Define

$$\begin{aligned} f\colon S^n &\to \mathbb{R}^n \\ x &\mapsto g(x) - g(-x) \end{aligned}$$

Note that $f(-x) = -f(x)$. If there is an $x \in S^n$ with $f(x) = 0$, then we are done. Suppose not, and define

$$\begin{aligned} \bar{f}\colon S^n &\to S^{n-1} \\ x &\mapsto f(x)/\|f(x)\|. \end{aligned}$$

Restrict this to S^{n-1}, to obtain $f'\colon S^{n-1} \to S^{n-1}$. Note that f' also satisfies $f'(-x) = -f'(x)$. Therefore f' has odd degree by Proposition 33.1. On the other hand, f' was obtained by restriction, so it extends over a disc D^n, for example the northern or southern hemispheres of S^n. It follows that f' is null homotopic, that is homotopic to a constant map, and therefore has degree zero. Since 0 is even, we have a contradiction, which proves the Borsuk-Ulam theorem. □

34. Proof of $\mathbb{F}_2$ coefficient Poincaré duality

This section did not appear in lectures and is included for interest only. We shall outline a proof of the Poincaré duality theorem with $\mathbb{F}_2$ coefficients, using simplicial structures. This gives a nice intuition of the idea behind Poincaré duality, without having to go into cohomology with compact supports, as in e.g. Hatcher. For the full proof we refer to [Hat] or [Br].

Theorem 34.1. *Let M be a closed n-dimensional manifold. For every $k \in \mathbb{N}_0$, there is an isomorphism*

$$PD\colon H^{n-k}(M;\mathbb{F}_2) \xrightarrow{\cong} H_k(M;\mathbb{F}_2).$$

PNE

Before we give a detailed sketch of the proof we introduce some notation, as well as the notion of barycentric subdivision. Recall that the m-simplex is

$$\Delta^m := \{(t_0,\dots,t_m) \in \mathbb{R}^{m+1} \mid \sum_i t_i = 1\}.$$

Definition 34.2.

(1) The *barycentre* of Δ^m is
$$B_m := (\frac{1}{m+1},\dots,\frac{1}{m+1}) \in \Delta^m.$$
(2) Let $P_0,\dots,P_k \in \Delta^m$ be a collection of points. Then we define a map
$$\begin{array}{rcl}[P_0,\dots,P_k]\colon \Delta^k & \to & \Delta^m \\ (t_0,\dots,t_k) & \mapsto & \sum_{i=0}^k t_i P_i\end{array}$$
(3) Let $\langle P_0,\dots,P_k\rangle \subseteq \Delta^m$ denote the image of the map $[P_0,\dots,P_k]$.
(4) Let $0 \leq n_1 < \cdots < n_r \leq m$. Then $\langle v_{n_1},\dots,v_{n_r}\rangle$ an r-dimensional face of Δ^m.
(5) The image of B_r under $[v_{n_1},\dots,v_{n_r}]\colon \Delta^r \to \Delta^m$ is the *barycentre of the face.* Given a face τ we write $\underline{\tau}$ for the barycentre of τ.
(6) The set of singular simplices
$$\big\{[\underline{\sigma}_0,\dots,\underline{\sigma}_k]\colon \Delta^k \to \Delta^m \mid \sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_k \subset \sigma\big\}$$
is the *barycentric subdivision* of Δ^m.

Note that the barycentric subdivision of a face is equal to the barycentric subdivision of Δ^m restricted to a face.

Let M be a closed, oriented n-manifold. Then M admits a simplicial structure $\{\varphi_i\colon \Delta^{n_i} \to M\}_{i\in I}$. By definition, this is a collection of maps with:

(1) $\varphi_i\colon \Delta^{n_i} \to M$ injective.
(2) For all $x \in M$, there exists a unique i with x in the interior of the image of φ_i.
(3) Every face of a simplex is again a simplex.
(4) If φ_i and φ_j intersect, they do so in a face.

Given $P_0,\dots,P_k \in \varphi_i(\Delta^{n_i})$, define

$$[P_0,\dots,P_k] := \varphi_i \circ [\varphi_i^{-1}(P_0),\dots,\varphi_i^{-1}(P_k)]\colon \Delta^k \to \Delta^{n_i} \xrightarrow{\varphi_i} M.$$

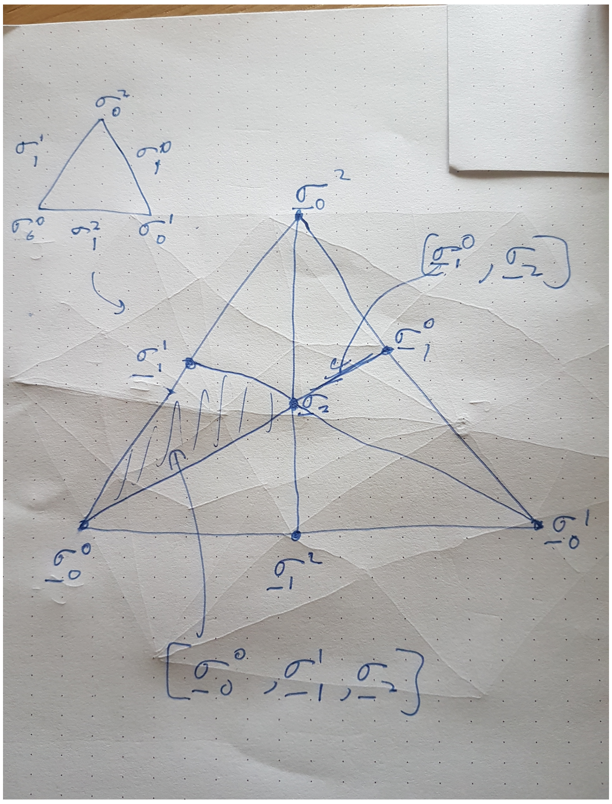

FIGURE 4. Barycentric subdivision of a 2-simplex.

Proof of Theorem 34.1. Here is a detailed sketch of a proof. Let M be a closed, oriented n-manifold. Choose a simplicial structure $\{\varphi_i \colon \Delta^{n_i} \to M\}_{i\in I}$ for M. We have a barycentric subdivision of this simplicial structure, obtained by applying the barycentric subdivision defined above to every simplex in M. This makes sense by the observation made above that the barycentric subdivision of a face is equal to the barycentric subdivision of Δ^m restricted to a face.

Now let σ be a k simplex of M. Then define the *dual cell* to σ as the union

$$\sigma^d := \bigcup_{\sigma=\sigma_0\subset\sigma_1\subset\cdots\subset\sigma_s} \langle \underline{\sigma}_0, \underline{\sigma}_1, \cdots, \underline{\sigma}_s\rangle.$$

The *boundary* of the dual cell is

$$\partial\sigma^d := \bigcup_{\sigma\subset\sigma_1\subset\cdots\subset\sigma_s} \langle \underline{\sigma}_1, \cdots, \underline{\sigma}_s\rangle.$$

There are two claims in this proof, whose proofs we shall omit. These omissions are why this is just a sketch of the proof.

Claim.

$$(\sigma^d, \partial\sigma^d) \cong (D^{n-k}, S^{n-k})$$

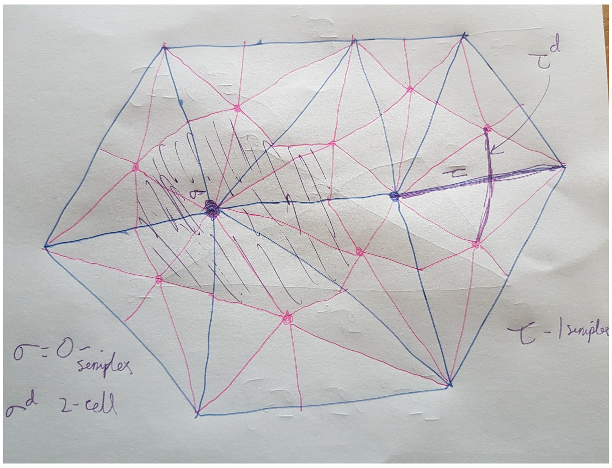

FIGURE 5. A triangulation of a hexagon and its barycentric subdivision are shown. A 0-simplex σ and a 1-simplex τ are chosen, and their dual cells are depicted.

The proof of this claim uses crucially that M is a manifold. It is not too hard to construct topological spaces with a simplicial structure where this claim would fail.

Claim. *Suppose that σ and τ are simplices. Then τ is a face of σ if and only if $\sigma^d \subset \tau^d$.*

We have two CW complex structures on M. Let $S_k(M)$ be the set of k-simplices in $\{\varphi_i\}$. Then the CW complex X is the space M with CW structure having as k-cells the elements of $S_k(M)$. On the other hand the CW complex Y is the space M with k-cells

$$\{\sigma^d \mid \sigma \text{ an } (n-k)\text{-simplex in } S_{n-k}(M)\}.$$

This is a CW structure by the two claims above.

Given a k-cell e in Y, its *dual* e^d is the simplex $\sigma = e^d$ with $\sigma^d = e$.

Now let $\sigma \in S_k(M)$ and define

$$PD(\sigma)\colon C^{CW}_{n-k}(Y) \to \mathbb{F}_2$$

to be the map determined by sending σ^d to 1 and all other $(n-k)$-cells to 0.

Claim. *The map*

$$\begin{aligned} PD\colon C^{CW}_k(X) \otimes \mathbb{F}_2 &\to \operatorname{Hom}(C^{CW}_{n-k}(Y), \mathbb{F}_2) \\ \textstyle\sum_{i=1}^m \sigma_i \otimes a_i &\mapsto \textstyle\sum_{i=1}^m a_i PD(\sigma_i) \end{aligned}$$

induces isomorphisms

$$H^{CW}_k(X;\mathbb{F}_2) \xrightarrow{\cong} H^{n-k}_{CW}(Y;\mathbb{F}_2)$$

for every $k \in \mathbb{N}_0$.

By the claim,

$$H_k(M;\mathbb{F}_2) \cong H_k^{CW}(X;\mathbb{F}_2) \xrightarrow{\cong} H^{n-k}_{CW}(Y;\mathbb{F}_2) \cong H^k(M;\mathbb{F}_2)$$

since singular and CW homology are isomorphic. This completes the proof of Poincaré duality modulo the claims.

We indicate the proof of this last claim. The inverse to the map is

$$\begin{array}{rcl} \mathrm{Hom}(C^{CW}_{n-k}(Y),\mathbb{F}_2) & \to & C^{CW}_k(X)\otimes\mathbb{F}_2 \\ f & \mapsto & \sum_{\sigma\in S_k(M)} \sigma\otimes f(\sigma^d). \end{array}$$

You should check that this indeed an inverse. Therefore the maps are isomorphisms between the chain groups. To see that they induce a chain isomorphism, we need to see that the following diagram commutes.

$$\begin{tikzcd} C_k^{CW}(X)\otimes\mathbb{F}_2 \arrow[r, "PD"] \arrow[d, "\partial"] & \mathrm{Hom}(C^{CW}_{n-k}(Y),\mathbb{F}_2) \arrow[d, "\partial^*"] \\ C_{k-1}^{CW}(X)\otimes\mathbb{F}_2 \arrow[r, "PD"] & \mathrm{Hom}(C^{CW}_{n-(k-1)}(Y),\mathbb{F}_2) \end{tikzcd}$$

Then the map PD is chain map and an isomorphism on chain groups, so is a chain isomorphism and so induces an isomorphism between homology and cohomology.

To see that the diagram commutes, let $\sigma\in S_k(M)$. We want to show that

$$PD(\partial\sigma) = \partial^*(PD(\sigma)) \in \mathrm{Hom}(C^{CW}_{n-(k-1)}(Y),\mathbb{F}_2).$$

That is, for every $\tau\in S_{k-1}(M)$, we have an $(n-(k-1))$-cell τ^d, and we want that

$$PD(\partial\sigma)(\tau^d) = PD(\sigma)(\partial\tau^d)\in\mathbb{F}_2. \tag{34.3}$$

Let $y_1,\dots,y_m$ be the $(n-k)$-cells in the boundary of τ^d, which is an $(n-k+1)$-cell. The only possible $(n-k)$-cells in τ^d are in its boundary. Then we have:

$$\begin{aligned} PD(\partial\sigma)(\tau^d)=1 &\Leftrightarrow \tau \text{ is a face of } \sigma \\ &\Leftrightarrow \text{there is an } i \text{ with } y_i=\sigma^d \\ &\Leftrightarrow \text{there is an } i \text{ with } y_i^d=\sigma \\ &\Leftrightarrow PD(\sigma)(\partial\tau^d)=1. \end{aligned}$$

This proves (34.3), which completes the proof of the claim, and therefore completes our sketch proof of $\mathbb{F}_2$ coefficient Poincaré duality. □

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Department of Mathematical Sciences, Durham University, United Kingdom
Email address: `mark.a.powell@durham.ac.uk`