CONCORDANCE AND ISOTOPY INVARIANTS OF SURFACES

MARK POWELL

Abstract. I give a survey of some link homotopy, concordance, and isotopy invariants for compact orientable surfaces embedded in 4-manifolds, as well as a discussion of some theorems about them.

1. Introduction

This is a survey of link homotopy, concordance, and isotopy invariants of surfaces knots and links, as well as some of the exciting results that have been proven about them.

1.1. **Definitions and conventions.** I consider compact, orientable surfaces Σ properly immersed or embedded in a compact, orientable 4-manifold N under a map $f\colon \Sigma \to N$. I do not assume that Σ is connected. Images $f(\Sigma) \subseteq N$ will typically be denoted R, R', or similar. The assumption that embeddings or immersion are proper means that either the surface Σ is closed, or if $\partial\Sigma \neq \emptyset$, then the boundary is embedded in ∂N.

Definition 1.1.

(1) If R is embedded, then it is called a *surface link.*
(2) If moreover all components are spherical then R is called a *2-link.*
(3) If the surface is connected and embedded (but not necessarily spherical) then R is called a *surface knot.*
(4) If moreover the surface is S^2, then R is called a *2-knot.*
(5) If the connected components of Σ have disjoint image in N (but are not necessarily embedded) then f is called a *surface link map.*
(6) If moreover all components of R are spherical then f is called a *link map.*

I will work in either the smooth or topological categories. When a statement works in both categories I will use CAT to denote either. I could also consider the piecewise linear category PL, but this is essentially interchangeable with the smooth category in this setting, provided we consider locally flat embeddings and immersions, so I will not consider it. In CAT = Top all embeddings and immersions are assumed to be locally flat without further comment, while in CAT = Diff all embeddings and immersions are assumed to be smooth.

Definition 1.2 (Link homotopy)**.** Surface link maps f_0 and f_1 are *link homotopic* if they are homotopic via a homotopy $f_t\colon \Sigma \to N$ such that f_t is a surface link map for all $t \in [0,1]$.

Definition 1.3 (Concordance)**.** CAT surfaces links R and R' in N, with the same underlying surface Σ, and such that $\partial R = \partial R'$, are *concordant rel. boundary* if there is a proper CAT embedding $C\colon \Sigma \times [0,1] \to N \times [0,1]$ such that

$$C(\Sigma \times \{0\}) = R \subseteq N \times \{0\} \text{ and } C(\Sigma \times \{1\}) = R \subseteq N \times \{1\},$$

and such that

$$C(\partial\Sigma \times [0,1]) = \partial R \times [0,1] = \partial R' \times [0,1] \subseteq \partial N \times [0,1].$$

Definition 1.4 (Isotopy)**.** CAT surfaces links R and R' in N, with the same underlying surface Σ, and such that $\partial R = \partial R'$, are CAT *isotopic rel. boundary* if there is a CAT isotopy $F_t\colon N \to N$, $t \in [0,1]$, such that $F_0 = \mathrm{Id}$, $F_t|_{\partial N} = \mathrm{Id}$ for all $t \in [0,1]$, and such that $F_1(R_i) = R_i'$ for $i = 1, \ldots, n$.

Note that in all three relations, the ordering of the components is preserved.

2020 *Mathematics Subject Classification.* 57K40. 57K10, 57N35. 57N70,
Key words and phrases. 4-manifolds, embedded surfaces, concordance, isotopy.

1.2. **Aims and limitations.** I will try to answer the following questions.

(a) What are some link homotopy, concordance, and isotopy obstructions for surface link maps and surface links in 4-manifolds?
(b) Under what circumstances have the invariants been defined?
(c) When certain invariants coincide for two surfaces, or the case of a relative invariant when it vanishes, what can we deduce about the two surfaces?

I will try to sketch the definitions of the invariants that I consider. Frequently they are defined, after auxiliary choices, in some value set which is relatively easy to understand. Then in order to make them into genuine invariants, one has to take an appropriate quotient of the value set. I will usually not describe the quotient carefully, other than to point out the choices that have to be accounted for. I will not prove that any of the invariants are well-defined, and I will not say anything about the proof of the theorems stated.

I will not consider all possible invariants, especially of isotopy. Anthony Conway has produced a survey of isotopy invariants of 2-knots, which I recommend reading.

Notation. I will refer to the components of a surface Σ as $\Sigma_1 \sqcup \cdots \sqcup \Sigma_n$, the map into N as $f\colon \Sigma \to N$ with components

$$f_1 \sqcup \cdots \sqcup f_n \colon \Sigma_1 \sqcup \cdots \sqcup \Sigma_n \to N,$$

and the images as $R_i := f_i(\Sigma_i)$ for $i = 1, \ldots, n$, and $R := f(\Sigma)$.

Acknowledgements. I am grateful to the organisers of the BIRS conference on 'Topology in dimension 4.5' for providing the impetus to write this survey, and to Anthony Conway and Jason Joseph for some helpful suggestions. I was partially supported by EPSRC New Investigator grant EP/T028335/1 and EPSRC New Horizons grant EP/V04821X/1.

2. Link homotopy obstructions

2.1. **Free homotopy classes of the components.** The free homotopy classes of the components $[f_i] \in [\Sigma_i, N]$, or for the nonempty boundary case in homotopy classes of maps of pairs: $[(\Sigma_i, \partial\Sigma_i), (N, \partial N)]$, are invariants of surface link maps up to link homotopy.

For the one component case this is a complete invariant, otherwise it is not.

2.2. **4-dimensional Milnor's invariants.** In [AMY21], Audoux-Meilhan-Yasuhara, building on earlier work by [AMW17], and extending classical Milnor's invariants for links in S^3, defined 4-dimensional Milnor's invariants.

Let R be a surface link in S^4, that is an embedded closed surface in S^4 with components $R_1 \sqcup \cdots \sqcup R_n$. For a multi-index I containing elements of $\{1, \ldots, n\}$, and $i \in \{1, \ldots, n\}$, such that the concatenation Ii contains no repeated index, the *4-dimensional Milnor invariant* is a function

$$M^{Ii}_{R_i} \colon H_1(R_i) \to \mathbb{Z}/\Delta(Ii)\mathbb{Z}$$

where $\Delta(Ii) \in \mathbb{Z}$ is an integer which encodes the indeterminacy of the invariants. It is obtained, for $|I| = q - 1$, by considering elements of $H_1(R_i)$ in the quotient $\pi_1(S^4 \setminus R)/\pi_1(S^4 \setminus R)_q$ by the qth term of the lower central series.

Example 2.1. As an example, consider an unknotted S^2, take a meridian, thicken it to a solid torus $S^1 \times D^2$, and take the boundary. We have a link with $\Sigma = S^2 \sqcup T^2$, and

$$M^{12}_{R_2} \colon \mathbb{Z}^2 \to \mathbb{Z}; \ \ (x, y) \mapsto y.$$

Advancing one step, consider an unlink $S^2 \sqcup S^2 \subseteq S^4$, with meridians μ_1 and μ_2. Let γ be a curve representing $[\mu_1, \mu_2]$, thicken it to $S^1 \times D^2$, and take the boundary. We obtain a link with $\Sigma = S^2 \sqcup S^2 \sqcup T^2$, and

$$M^{123}_{R_3} \colon \mathbb{Z}^2 \to \mathbb{Z}; \ \ (x, y) \mapsto y.$$

2.3. **The Kirk and Stirling invariants.** Let $f = f_1 \sqcup f_2 \colon S^2 \sqcup S^2 \to S^4$ be a 2-component link map. For each component, f_i represents a class in $\pi_2(S^4 \setminus f_j(S^2))$, with $i \neq j$. Taking the equivariant self intersection number for each i

$$\sigma_i(f) := \lambda(f_i, f_i) \in \mathbb{Z}[\mathbb{Z}],$$

with values in $\mathbb{Z}[H_1(S^4 \setminus f_j(S^2))] = \mathbb{Z}[\mathbb{Z}]$, we obtain

$$\sigma(f) := (\sigma_1(f), \sigma_2(f)) \in \mathbb{Z}[\mathbb{Z}] \oplus \mathbb{Z}[\mathbb{Z}].$$

This is called the *Kirk invariant* of f.

Theorem 2.2 (Schneiderman-Teichner)**.** *The map σ is an injection from the group of link homotopy classes $LM^4_{2,2}$ of 2-component link maps in S^4 to $\mathbb{Z}[\mathbb{Z}] \oplus \mathbb{Z}[\mathbb{Z}]$.*

Scott Stirling generalised the Kirk invariant to an invariant of n component link maps in S^4, in his PhD thesis. I will describe the 3-component case, for link maps $f_1 \sqcup f_2 \sqcup f_3 \colon S^2 \sqcup S^2 \sqcup S^2 \to S^4$. For any 3-component link map, together with a basing of its components, for each 2 element subset $\{i,j\} \subseteq \{1,2,3\}$, there is a surjection $\pi_1(S^4 \setminus (f_i(S^2) \cup f_j(S^2))) \to F/F_3$, where F is the free group on two generators and F_3 is the third lower central subgroup of F. We can then, for $k \notin \{i,j\}$, consider

$$\sigma_k(f) := \lambda(f_k, f_k) \in \mathbb{Z}[F/F_3].$$

Then

$$\sigma^3(f) := (\sigma_1(f), \sigma_2(f), \sigma_3(f)) \in \mathbb{Z}[F/F_3]^3$$

is an invariant of based link maps up to based link homotopy. Stirling defined an action of $\mathbb{Z}^3$ on $\mathbb{Z}[F/F_3]^3$, taking account of the choice of basing paths, and the orbit $[\sigma^3(f)] \in \mathbb{Z}[F/F_3]^3/\mathbb{Z}^3$ is the *Stirling invariant* of f. If two 3-component link maps are link homotopic then they have equal Stirling invariant.

Theorem 2.3 (Stirling)**.** *There are 3-component link maps f, f' in S^4 that have the same image, but are not link homotopic as link maps.*

3. Concordance obstructions

Concordant embedded surfaces are link homotopic by Bartels-Teichner [BT99], so the obstructions in the previous section are also obstructions to concordance.

3.1. **Free homotopy classes of the components.** The free homotopy class of the components is in particular also a concordance obstruction. For (connected) surface knots in a simply-connected 4-manifold, the free homotopy class determines the concordance class, by [Sun15].

Theorem 3.1 (Sunukjian)**.** *Let R and R' be closed* CAT *surface knots in a closed simply-connected 4-manifold N, with the same underlying surface Σ, such that $[R] = [R'] \in H_2(N;\mathbb{Z})$. Then R and R' are* CAT *concordant.*

3.2. **The Sato-Levine invariant.** This invariant was defined in [Sat84], and also studied in [Coc84b]. It is also named after Levine; apparently this is based on simultaneous and independent work of Levine on the same invariant, but Levine's version was never published and I have not seen it.

Consider a 2-component surface link $R = R_1 \sqcup R_2$ in S^4. Suppose that for each $i = 1,2$ there is a Seifert manifold V_i for R_i in S^4, that is disjoint from the other component. The intersection $F := V_1 \cap V_2$ is a closed surface. A section of the normal bundle to V_i, for each i, determines a framing of the normal bundle of F. Therefore is a framed surface, and so represents an element of $\Omega_2^{\text{fr}} \cong \pi_4(S^2) \cong \mathbb{Z}/2$. This element is independent of the choice of Seifert manifolds, the concordance class of R, and all orientation choices.

An example of a surface link of $T^2 \sqcup T^2$ realising this invariant is as follows. Start with a Whitehead link. Twist spin it around a disjoint axis, doing an odd number of full rotations. There are Seifert manifolds that intersect in a torus with a nontrivial framing of its normal bundle, i.e. representing the nontrivial element of Ω_2^{fr}.

Cochran [Coc85] defined an infinite sequence of invariants that extend the Sato-Levine invariant, called $\beta^i(L)$. Orr defined a sequence of homotopy invariants [Orr87]. Cochran [Coc87] showed that Orr's homotopy invariants vanish for 2-links. Then Orr [Orr91] got his revenge, showing that the Sato-Levine invariant, as well as all of Cochran's β^i invariants, vanish also for 2-links.

3.3. **4-dimensional Milnor's invariants.** The 4-dimensional Milnor's invariants discussed in Section 2.2 from [AMY21] are also CAT concordance invariants, but in addition this holds without the restriction that the index Ii is non-repeating. For every multi-index I and integer i in the labels $\{1,\dots,n\}$, the orbit of each map $M^I_{R_i}\colon H_1(R_i)\to \mathbb{Z}/\Delta(Ii)$ under the action of the automorphism group of the intersection pairing $\lambda(\Sigma_i)$ by precomposition, is a concordance invariant.

There is also an earlier construction of concordance invariants with similar properties to these by Orr [Orr87]. The two approaches are rather different, but both deserve to be called 4-dimensional Milnor's invariants.

3.4. **The Freedman-Quinn invariant.** Let Σ be a disjoint union of copies of S^2 or D^2 in a compact, orientable 4-manifold N, and let R and R' be CAT embeddings of Σ. Suppose that R and R' are homotopic rel. boundary. By picking basing paths carefully we may and will assume that R and R' are based homotopic rel. boundary. I will define the Freedman-Quinn invariant $\mathrm{fq}(R,R')$ for Σ connected; for Σ disconnected we have such an obstruction for each component.

Choose a based homotopy between R and R' and use it to form an immersion $H\colon \Sigma\times I\looparrowright N\times I$. Extend this to a map

$$\overline{H}\colon \Sigma\times I\to N\times I\hookrightarrow N\times I\times\mathbb{R},$$

and perturb to be in general position. This is a generic CAT immersion of a 3-manifold in a 6-manifold, and so the singularities are isolated double points. Then by computing the Wall self-intersection number we obtain the Freedman-Quinn invariant

$$\mathrm{fq}(R,R') := \mu(\overline{H}) \in \frac{\mathbb{Z}[\pi_1(N\times I\times\mathbb{R})]}{\langle g+g^{-1}, g\in\pi_1(N),1\rangle, \mu(\pi_3(N\times I\times\mathbb{R}))}.$$

The first indeterminacy gives the usual value group of μ, made into a homotopy invariant rather than just a regular homotopy invariant by ignoring the coefficient of 1. The second indeterminacy renders the choice of homotopy H immaterial. It turns out that $\mathrm{fq}(R,R') = \overline{\mathrm{fq}(R,R')}$, which combined with the relations $g+g^{-1}=0$ imply that $\mathrm{fq}(R,R')$ takes values in a subset $(\mathbb{Z}/2)[T]/\mu(\pi_3(N))$, where $T\subseteq\pi_1(N)$ is the subset of elements of order exactly two. If R and R' are concordant, then $\mathrm{fq}(R,R')=0$.

A CAT embedded surface $R\subseteq N$ is *π_1-negligible* if the inclusion induced map $\pi_1(N\setminus R)\to\pi_1(N)$ is an isomorphism. This holds if and only if R admits a collection of immersed geometric dual spheres, one for each R_i. In the presence of dual immersed framed spheres, fq is the complete obstruction to concordance.

Theorem 3.2 (Freedman-Quinn, Stong)**.** *Let R and R' be* CAT *embedded π_1-negligible surfaces with the same underlying surface Σ, which is a union of spheres and discs. Suppose that R and R' are homotopic rel. boundary, and that there exists a collection of dual immersed spheres for R, denoted $\{G_j\}_{j=1}^n$, with trivial normal bundles and $|R_i \pitchfork G_j| = \delta_{ij}$. Suppose that* $\mathrm{fq}(R,R')=0$*. Then R and R' are* CAT *concordant rel. boundary.*

Problem 3.3. Define the Freedman-Quinn invariant for all orientable surfaces.

Remark 3.4. Klug-Miller [KM21] generalised the Freedman-Quinn invariant to π_1-trivial surfaces, i.e. such that $f_*\colon\pi_1(\Sigma)\to\pi_1(N)$ is the trivial homomorphism for all basepoints. They called such surfaces π_1-negligible, but this term already had a concrete meaning from [FQ90], so I will retain the usage from [FQ90] and instead call these surfaces π_1-trivial. The proof in Klug-Miller that fq is well-defined in this setting starts by modifying a homotopy rel. boundary so that a curve α in Σ is fixed for all $t\in[0,1]$. It is not clear to me that this is always possible.

3.5. **The Stong km invariant.** Theorem 3.2 had an assumption on the dual spheres that they have trivial normal bundle. Such a dual sphere exists if and only if there is a dual sphere on which w_2 vanishes. Stong [Sto93] studied the case that there is an immersed dual sphere that is not framed, i.e. he studied uniqueness of π_1-negligible embeddings in general.

Suppose R and R' are CAT embeddings of Σ, a disjoint union of spheres or discs, in a compact oriented 4-manifold N. Assume R and R' are homotopic rel. boundary, and that $\mathrm{fq}(R,R')=0$. Let

$$H\colon\Sigma\times I\looparrowright N\times I$$

be a rel. boundary based homotopy from R to R' realising $\mathrm{fq}(R, R') = 0$, i.e. $\mu(\overline{H}) = 0 \in \mathbb{Z}[\pi_1(N)]/\langle g + g^{-1}, 1\rangle$. Then there is defined a secondary concordance obstruction

$$\mathrm{Skm}(H) \in H_1(N; \mathbb{Z}/2).$$

Stong [Sto93] called this a 'Kervaire-Milnor invariant', by analogy with the fact that Freedman-Quinn had defined a secondary embedding obstruction for immersed surfaces, and named that after Kervaire-Milnor. This embedding obstruction was related to the work of Kervaire and Milnor, but went further. Stong's invariant is not closely related to Kervaire-Milnor's paper, since Stong's invariant is about uniqueness of embeddings up to isotopy. Moreover it is confusing to have two different invariants in the same field with the same name. Therefore I will call this instead *Stong's km invariant* and denote it Skm.

Theorem 3.5 (Stong)**.** *Let H be a homotopy between R and R' as above, with $\mu(\overline{H}) = 0$.*

(1) We have $\mathrm{Skm}(H) = 0$ if H is homologous in $H_3(N \times I, (N \times \{0, 1\}) \cup (\partial N \times I); \mathbb{Z}[\pi_1(N)])$ to a concordance between R and R' in $N \times I$.
(2) If R and R' are π_1-negligible, and $\mathrm{Skm}(H) = 0 \in H_1(N; \mathbb{Z}/2)$, then H is homologous to a π_1-negligible concordance from R to R'.
(3) If every component of R (equivalently R') admits a framed immersed dual sphere then $\mathrm{Skm}(H) = 0$ for every H.

Problem 3.6. Define Skm for general surfaces.

Problem 3.7. What is the indeterminacy relating to different choices of H that makes Skm into an invariant just of R and R'?

The Stong km invariant also extends to nonorientable manifolds, in which case there is an extra indeterminacy in the value group; one has to quotient by a subgroup of $H_1(N; \mathbb{Z}/2)$ called Γ by Stong, which is a subgroup of the subgroup generated by the elements $g \in \pi_1(N)$ of order exactly two and on which the orientation character is nonvanishing. See [Sto93, Section 3] for details.

Stong and Wang [SW00] used Skm in their computation of the topological pseudo-mapping class group for closed oriented 4-manifolds with fundamental group $\mathbb{Z}$.

3.6. **Blowing down and covering links.** The following observation can be useful to obstruct concordances.

Proposition 3.8 (Melvin)**.** *Suppose surfaces R and R' in a 4-manifold N both have a connected component which is an embedded sphere and whose normal bundle has Euler number ± 1, R_1 and R_1' say. If R and R' are concordant, then the remaining components of the surface $R \setminus R_1$ and $R' \setminus R_1'$ determine surfaces in the results N_1 and N_1' of blowing down N along R_1 and R_1' respectively. These surfaces are concordant in some h-cobordism between N_1 and N_1'.*

This reminded me of the covering link method, which is well-known in classical link concordance, and was used spectacularly in high dimensions by Cochran and Orr [CO93] to show that there are links not concordant to boundary links (with vanishing Milnor's invariants in the classical dimension). So I have written a version of this method for concordance of surface links.

Proposition 3.9. *Suppose surfaces R and R' in a 4-manifold N both have a connected component which is an embedded sphere and that lies in an embedded D^4, R_1 and R_1' say. If R and R' are concordant, then the 4-manifolds N_1 and N_1' obtained by taking the p-fold branched cover of N along R_1 and R_1' respectively are $\mathbb{Z}_{(p)}$-homology cobordant, and if we take surface links in N_1 and N_1' formed by taking preimages of the remaining components of R and R', then these surface links are concordant in some $\mathbb{Z}_{(p)}$-homology cobordism between N_1 and N_1'.*

To form the branched cover we use a map

$$\pi_1(N \setminus \nu R_1) \to H_1(N \setminus \nu R_1) \to H_1(N \setminus \nu R_1)/H_1(N) \cong \mathbb{Z} \to \mathbb{Z}/p.$$

Here the inclusion map $H_1(N \setminus \nu R_1) \to H_1(N)$ splits because R_1 is contained in an embedded D^4, and so every loop can be isotoped to miss the D^4, canonically up to isotopy. Therefore we can consider $H_1(N)$ as a subgroup of $H_1(N \setminus \nu R_1)$, and take the quotient as above.

4. Isotopy obstructions

Isotopic surfaces are concordant, so all the concordance obstructions discussed above are also isotopy obstructions. The following obstructions, on the other hand, do not obstruct concordance, but do obstruct isotopies between surfaces. This is not at all an exhaustive list of possible isotopy invariants. It focusses mostly on situations where I can describe some positive results, in which the invariants under consideration have been shown to precisely detect isotopy. Anthony Conway's notes, also written for this conference, give a thorough survey of 2-knot invariants.

4.1. **Homotopy type of the complement.** The most commonly used obstructions to surface links being isotopic is the homotopy type of the exterior $E_R := N \setminus \nu R$. This means that one can use $\pi_1(E_R)$, $\pi_2(E_R)$ as a $\mathbb{Z}[\pi_1(E_R)]$-module, or the k-invariant in $H^3(\pi_1(E_R);\pi_2(E_R))$. One can also use the homotopy type of the pair $(E_R, \partial E_R)$, in which case a common invariant is the intersection pairing of the exterior $\lambda\colon \pi_2(N \setminus \nu R) \times \pi_2(N \setminus \nu R) \to \mathbb{Z}[\pi_1(N \setminus \nu R)]$. Sometimes these invariants suffice to detect the isotopy class, for example as in the following two theorems of [Boy93, Theorem F] and [CP20].

Theorem 4.1 (Boyer)**.** *Let N be a closed, oriented, simply connected 4-manifold and let R and R' be* Top *embedded closed, connected surfaces, with the same underlying surface Σ, of any genus, in N. Suppose that $[R] = [R'] \in H_2(N;\mathbb{Z})$, both are primitive classes, and that $\pi_1(N \setminus R) \cong \{1\} \cong \pi_1(N \setminus R')$. Then R and R' are* Top *ambiently isotopic.*

Theorem 4.2 (Conway-Powell)**.** *Let N be a closed, oriented, simply connected 4-manifold and let R and R' be* Top *embedded closed surfaces in N with $\pi_1(N \setminus R) \cong \mathbb{Z} \cong \pi_1(N \setminus R')$. Suppose that the equivariant intersection forms $\lambda_{N\setminus\nu R}$ and $\lambda_{N\setminus\nu R'}$ are isometric. Then R and R' are* Top *ambiently isotopic.*

4.2. **The Freedman-Quinn invariant.** For spheres with a common embedded framed dual sphere, the Freedman-Quinn invariant is also the complete obstruction to isotopy. Gabai [Gab20] proved that the isotopy classes of spheres in a fixed homotopy class can be put into one of a collection of normal forms, one for each element of $(\mathbb{Z}/2)[T]$. Schwartz [Sch19] gave specific examples where distinct spheres are not isotopic.

Schneiderman and Teichner [ST22] explained the Freedman-Quinn invariant, reformulated it in terms of the self-intersection number, observed that Gabai's normal forms coincide with the values that can be taken by fq, and showed that the different normal forms are distinguished by this invariant. Therefore Gabai's different normal forms are not concordant, and concordance implies isotopy for spheres with a common framed embedded dual sphere.

Theorem 4.3 (Gabai, Schneiderman-Teichner)**.** *Let R and R' be* CAT *embedded surfaces with the same underlying surface Σ, which is a sphere. Suppose that R and R' are homotopic, and that there exists a framed embedded sphere G that is geometrically dual to both R and R'. Suppose that* $\mathrm{fq}(R,R') = 0$. *Then R and R' are* CAT *isotopic.*

Schwartz [Sch22] constructed a pair of homotopic spheres with a common embedded dual whose normal bundle is nontrivial, with vanishing of the Freedman-Quinn and Stong Skm invariants, but such that the spheres are not isotopic. So the assumption on the normal bundle is necessary.

4.3. **The Dax invariant.** The Dax invariant is based on work of Dax, as the name suggests, but it was interpreted in the format I will describe by Gabai [Gab21].

Let R and R' be the images of proper smooth embeddings $D^2 \hookrightarrow N$. Suppose that R and R' are homotopic rel. boundary. Then the Dax invariant is a relative invariant

$$\mathrm{Dax}(R,R') \in \frac{\mathbb{Z}[\pi_1(N)]}{\langle \mathrm{dax}(\pi_3(N)), 1\rangle}.$$

The quotient by the subgroup $\mathrm{dax}(\pi_3(N))$ makes it so that the invariant does not depend on the choice of homotopy made in order to define the invariant.

Theorem 4.4 (Dax, Gabai)**.** *If R and R' are isotopic rel. boundary then* $\mathrm{Dax}(R,R') = 0$.

Here is an outline of the definition. Choose a homotopy rel. boundary $H\colon D^2 \times I \looparrowright N \times I$ with $H(x,y) \in N \times \{t\}$. Form

$$H'\colon D^2 \times I \to N \times I \times I,$$

and perturb this to a smooth generic immersion. Foliate D^2 by arcs going from -1 to 1, to obtain

$$H'' \in \pi_2(\mathrm{Imm}(D^1, N \times I^2), \mathrm{Emb}(D^1, N \times I^2)).$$

Represent this by a map $H'' \colon I^2 \to \mathrm{Imm}(D^1, N \times I^2)$. There exist finitely many $t_i \in I^2$ such that $H''(t_i)$ is not embedded; for every other $t \in I^2$ its image under H'' is embedded. At each t_i, $H''(t_i)$ is an immersed arc with one double point. We can associate to it $g_i \in \pi_1(N)$ and $\varepsilon_i \in \{\pm 1\}$. See [Gab21, p. 14] for the details, which I omit. The relation $g + g^{-1}$ does not apply, so this count refines the fq invariant. Define

$$\mathrm{Dax}(R, R') = \Big[\sum_i \varepsilon_i g_i\Big].$$

Problem 4.5. Adapt the definition for the topological category.

Schwartz [Sch21] gave a condition under which the Dax invariant detects isotopy of discs with dual spheres.

Theorem 4.6 (Schwartz)**.** *Suppose that* $\mathrm{dax}(\pi_3(N)) = 0$*. If* $\partial R = \partial R' \subseteq \partial N$ *has a geometrically dual framed embedded sphere* G *in the interior of* N*,* G *is* π_1*-negligible, and* $\mathrm{Dax}(R, R') = 0$*, then* R *and* R' *are smoothly isotopic rel. boundary.*

The next result is from [KT22], which gives another such condition.

Theorem 4.7 (Kosanović-Teichner)**.** *If* $\partial R = \partial R' \subseteq \partial N$ *has a geometrically dual embedded sphere in* ∂N*, and* $\mathrm{Dax}(R, R') = 0$*, then* R *and* R' *are smoothly isotopic rel. boundary.*

4.4. **Seiberg-Witten, Heegaard Floer, and Khovanov homology.** Seiberg-Witten invariants of the complement of a surface knot have been used to find exotic surfaces in 4-manifolds, for example using the rim surgery method introduced by Fintushel and Stern.

Heegaard-Floer and Khovanov cobordism maps have been used to great effect, for example recently to find exotic discs and exotic surfaces in D^4 with boundary the same knot in S^3.

Theorem 4.8 (Hayden)**.** *There is a pair of smooth slice discs in* D^4 *for a fixed knot* K *in* S^3 *that are topologically but not smoothly isotopic rel. boundary.*

This theorem now has three independent proofs: the original one by Hayden, one using Khovanov cobordism maps by Hayden-Sundberg, and one by Dai-Mallick-Stoffregen using involutive Heegaard-Floer homology.

Theorem 4.9 (Juhasz-Miller-Zemke)**.** *For each* $g \geq 1$*, there is an infinite family of smooth embeddings of* $\Sigma_{g,1}$ *in* D^4*, all with boundary the same knot* K*, that are all topologically isotopic rel. boundary, but such that no pair from the family are smoothly isotopic.*

4.5. **Blowing down and covering links.** The blow down and covering link methods discussed in Section 3.6 also provide isotopy obstructions, except they are more straightforward in this setting. If two surfaces are isotopic, then blowing down some components results in isotopic surfaces. Similarly passing to a covering link results in isotopic surfaces.

5. Open problems

Problem 5.1. Is every closed, orientable surface R in S^4 with $\pi_1(S^4 \setminus R) \cong \mathbb{Z}$ smoothly unknotted.

Problem 5.2. Is every 2-link in S^4 slice?

See [Coc84a], [Coc84b], [Coc85], [Orr87], [Coc87], and [Orr91] for some past attempts and context on the problem.

Problem 5.3. Where possible, extend the definitions of the obstructions or invariants above to more general surfaces, or in more general ambient 4-manifolds.

In [KPRT22] we defined intersection numbers and the Kervaire-Milnor invariant for general surfaces in any 4-manifold. Can one do the same thing for fq and Skm?

Problem 5.4. Characterise concordance of all surface links with dual spheres in all 4-manifolds using computable invariants.

Problem 5.5. Classify n-component link maps in S^4 up to link homotopy using computable invariants, for $n \geq 3$.

References

[AMW17] B. Audoux, J.-B. Meilhan, and E. Wagner, *On codimension two embeddings up to link-homotopy*, Journal of Topology **10** (2017), no. 4, 1107–1123.

[AMY21] B. Audoux, J.-B. Meilhan, and A. Yasuhara, *Milnor concordance invariant for knotted surfaces and beyond*, 2021. Preprint, available at arxiv:2109.14578.

[Boy93] S. Boyer, *Realization of simply-connected* 4*-manifolds with a given boundary*, Comment. Math. Helv. **68** (1993), no. 1, 20–47.

[BT99] A. Bartels and P. Teichner, *All two dimensions links are null homotopic*, Geom. Topol. **3** (1999), 235–252.

[CO93] T. D. Cochran and K. E. Orr, *Not all links are concordant to boundary links*, Ann. Math. (2) **138** (1993), no. 3, 519–554.

[Coc84a] T. Cochran, *Slice links in* S^4, Trans. Am. Math. Soc. **285** (1984), 389–401.

[Coc84b] T. D. Cochran, *On an invariant of link cobordism in dimension four*, Topology Appl. **18** (1984), 97–108.

[Coc85] T. D. Cochran, *Geometric invariants of link cobordism*, Comment. Math. Helv. **60** (1985), 291–311.

[Coc87] T. D. Cochran, *Link concordance invariants and homotopy theory*, Invent. Math. **90** (1987), 635–645.

[CP20] A. Conway and M. Powell, *Embedded surfaces with infinite cyclic knot group*, 2020. Preprint, available at arXiv:2009.13461.

[FQ90] M. Freedman and F. Quinn, *Topology of* 4*-manifolds*, Princeton Mathematical Series, vol. 39, Princeton University Press, 1990.

[Gab20] D. Gabai, *The 4-dimensional light bulb theorem*, J. Am. Math. Soc. **33** (2020), no. 3, 609–652 (English).

[Gab21] D. Gabai, *Self-referential discs and the light bulb lemma*, Comment. Math. Helv. **96** (2021), no. 3, 483–513.

[KM21] M. R. Klug and M. Miller, *Concordance of surfaces in 4-manifolds and the Freedman-Quinn invariant*, J. Topol. **14** (2021), no. 2, 560–586.

[KPRT22] D. Kasprowski, M. Powell, A. Ray, and P. Teichner, *Embedding surfaces in 4-manifolds*, 2022. Preprint, available at arxiv:2201.03961.

[KT22] D. Kosanović and P. Teichner, *A new approach to light bulb tricks: disks in 4-manifolds*, 2022. Preprint, available at arxiv:2209.12015.

[Orr87] K. E. Orr, *New link invariants and applications*, Comment. Math. Helv. **62** (1987), 542–560.

[Orr91] K. E. Orr, *Link concordance invariants and Massey products*, Topology **30** (1991), no. 4, 699–710.

[Sat84] N. Sato, *Cobordisms of semi-boundary links*, Topology Appl. **18** (1984), 225–234.

[Sch19] H. R. Schwartz, *Equivalent non-isotopic spheres in 4-manifolds*, J. Topol. **12** (2019), no. 4, 1396–1412.

[Sch21] H. R. Schwartz, *A 4-dimensional light bulb theorem for disks*, 2021. Preprint, available at arxiv:2109.13397.

[Sch22] H. R. Schwartz, *Duals of non-zero square*, Math. Res. Lett. **29** (2022), no. 1, 275–284 (English).

[ST22] R. Schneiderman and P. Teichner, *Homotopy versus isotopy: spheres with duals in 4-manifolds*, Duke Math. J. **171** (2022), no. 2, 273–325.

[Sto93] R. Stong, *Uniqueness of* π_1*-negligible embeddings in 4-manifolds: A correction to theorem 10.5 of Freedman and Quinn*, Topology **32** (1993), no. 4, 677–699.

[Sun15] N. S. Sunukjian, *Surfaces in 4-manifolds: concordance, isotopy, and surgery*, Int. Math. Res. Not. **2015** (2015), no. 17, 7950–7978.

[SW00] R. Stong and Z. Wang, *Self-homeomorphisms of 4-manifolds with fundamental group* $\mathbb{Z}$, Topology Appl. **106** (2000), no. 1, 49–56.

School of Mathematics and Statistics, University of Glasgow, United Kingdom
Email address: `mark.powell@glasgow.ac.uk`