$h_0 = h$. If W is a subset of U, a deformation $\Phi: P \times I \to E(U; M) \times I$ is **modulo** W if $\Phi_t(h) \mid W = h \mid W$ for all $h \in P$ and $t \in I$.

Suppose that $\Phi: P \times I \to E(U, M) \times I$ and $\psi: Q \times I \to E(U, M) \times I$ are deformations of subsets of $E(U, M)$, and suppose that $\Phi_1(P) \subset Q$. Then, the **composition of** ψ **with** Φ denoted by $\psi * \Phi: P \times I \to E(U, M) \times I$ is defined by

$$\psi * \Phi(h, t) = \begin{cases} (\Phi_{2t}(h), t) & \text{if } t \in [0, \frac{1}{2}], \\ (\psi_{2t-1}\Phi_1(h), t) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

We shall denote the cube $\{x \in E^n \mid |x_i| \leqslant r, 1 \leqslant i \leqslant n\}$ by I_r^n. We regard S^1 as the space obtained by identifying the endpoints of $[-4, 4]$ and we let $p: E^1 \to S^1$ denote the natural covering projection, that is, $p(x) = (x + 4)_{(\mathrm{mod}\ 8)} - 4$. Let T^n be the n-fold product of S^1. Then, I_r^n can be regarded as a subset of T^n for $r < 4$. Let $p^n: E^n \to T^n$ be the product covering projection and let $p^{k,n}: I_1^k \times E^n \to I_1^k \times T^n$ be the map $1_{I_1^k} \times p^n$. These maps will each be denoted by p when there is no possibility of confusion.

Let B^n be the unit n-ball in E^n and let S^{n-1} be its boundary as usual. We regard $S^{n-1} \times [-1, 1]$ as a subset of E^n by identifying (x, t) with $(1 + t/2) \cdot x$.

With the above discussions, definitions and notation out of the way, we are ready to start formulating some lemmas preliminary to the proofs of the main results of this section.

A discussion of, and a geometrical proof of, our first lemma will be postponed until the end of this section. (An **immersion** of one space into another is a continuous map which is locally an embedding.)

Immersion Lemma 5.6.1. *There is an immersion $\alpha: T^n - B^n \to E^n$ of the punctured torus into E^n.*

For a picture of α in the case $n = 2$, see Fig. **5.6.2**.

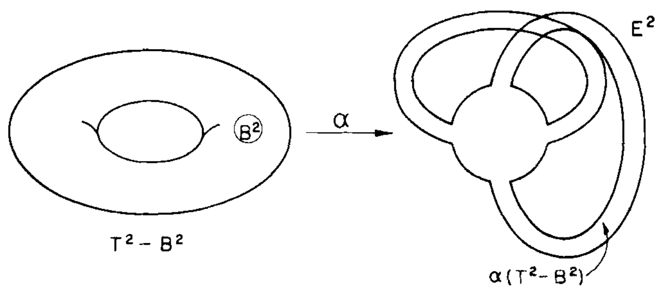

Figure 5.6.2

We are now ready to give a proof of Immersion Lemma **5.6.1**. The proof presented here was communicated to this author by R. D. Edwards. It was originated by Barden [2] and was formulated in the following picturesque form by Siebenmann. (Immersion Lemma **5.6.1** also follows from [Hirsch, 1].)

Proof of Immersion Lemma 5.6.1. We will work with the following inductive statement which is stronger than Lemma **5.6.1**.

n-DIMENSIONAL INDUCTIVE STATEMENT: *There exists an immersion f of $T^n \times I$ into $E^n \times I$ such that $f \mid T_0^n \times I$ is a product map, $f = \alpha \times 1$, where T_0^n is T^n minus an n-cell.*

We adopt the following notation for this proof: Let $I = [-1, 1] = J$, $J^n = (J)^n$, $S^1 = I \cup_\partial J$, $T^n = (S^1)^n$ and $T_0^n = T^n - \operatorname{Int} J^n$. It is easy to see that $E^n \times S^1$ can be regarded as a subset of E^{n+1} where the I-fibers of $E^n \times I$ are straight and vertical in E^{n+1} (see Fig. **5.6.9**).

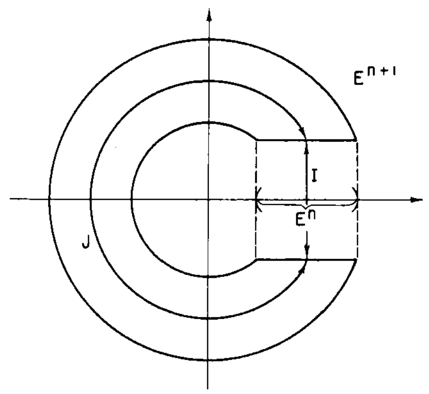

Figure 5.6.9

Assume that f and α are given by the inductive statement in dimension n. It is a simple matter to extend f to an immersion of $T^{n+1} \times I$ into $E^{n+1} \times I$, that is, just let

$$f \times 1_{S^1}: T^n \times S^1 \times I \rightarrow E^n \times S^1 \times I \subset E^{n+1} \times I$$

be the extension (see Fig. **5.6.10**). However, $f \times 1_{S^1}$ is not a product on $T_0^{n+1} \times I$, but merely on $T_0^n \times S^1 \times I$. The way to correct this is to conjugate $f \times 1_{S^1}$ with a 90° rotation (on the $I \times I$ factor) of the missing plug $(T_0^{n+1} \times I) - (T_0^n \times S^1 \times I) = \operatorname{Int} J^n \times I \times I$. The fact that $f \times 1_{S^1} \mid T_0^n \times I^2$ is a product in the I^2 factor allows one to do this.

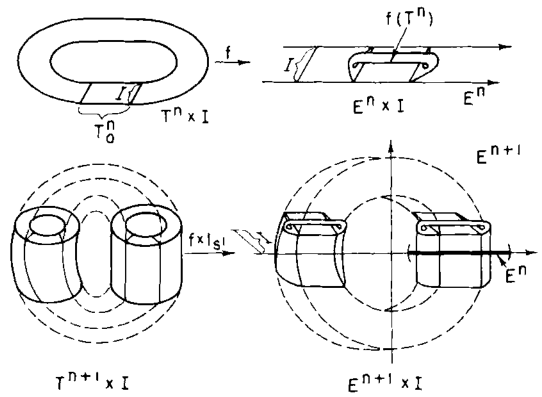

Figure 5.6.10

Assume without loss of generality that $f(T^n \times [-\frac{1}{2}, \frac{1}{2}]) \subset E^n \times [-\frac{2}{3}, \frac{2}{3}]$. Let λ be a homeomorphism of I^2 that is the identity on Bd I^2 and is a $\pi/2$-rotation on $[-\frac{2}{3}, \frac{2}{3}] \times [-\frac{2}{3}, \frac{2}{3}]$ (see Fig. **5.6.11**). Extend λ via the identity to a homeomorphism $\bar{\lambda}$: $S^1 \times I \to S^1 \times I$ (see Fig. **5.6.12**).

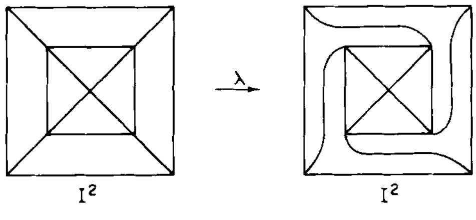

Figure 5.6.11

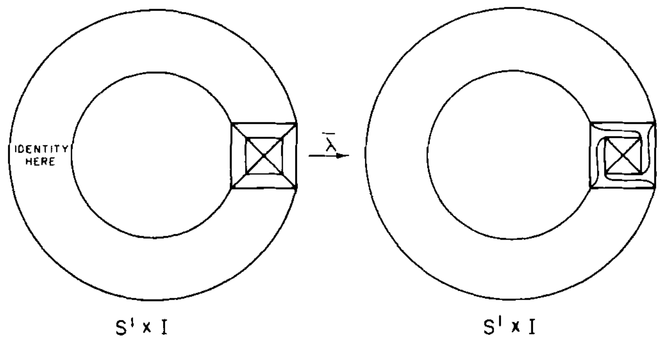

Figure 5.6.12

Consider now the following immersion h of $T^{n+1} \times I$ into $E^{n+1} \times I$,

$$h = (1_{E^n} \times \bar{\lambda}^{-1})(f \times 1_{S^1})(1_{T^n} \times \bar{\lambda}).$$

If we let $g = h \mid T^{n+1} \times [-\frac{1}{2}, \frac{1}{2}]$, then it can easily be checked that g is a product on $(T_0{}^n \times S^1) \cup (J^n \times [-\frac{1}{2}, \frac{1}{2}])$ which is a deformation retract of $T_0^{n+1} \times [-\frac{1}{2}, \frac{1}{2}]$. Thus, without loss of generality we can assume that g is a product on $T_0^{n+1} \times [\frac{1}{2}, \frac{1}{2}]$ (see Figs. **5.6.13** and **5.6.14**).

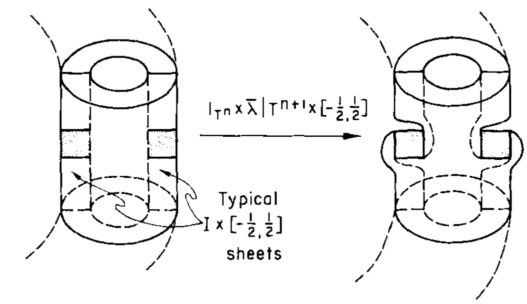

Figure 5.6.13

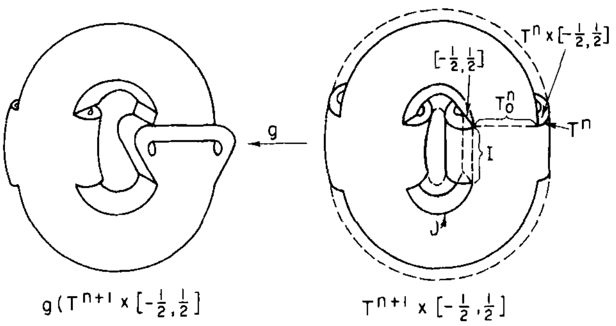

Figure 5.6.14

It is now easy to see that such a g gives rise to an immersion as desired in the theorem. This would be a trivial matter of reparametrizing the I coordinate if we knew that $g(T^{n+1} \times [-\frac{1}{2}, \frac{1}{2}]) \subset E^{n+1} \times [-\frac{1}{2}, \frac{1}{2}]$. To get such an inclusion, one can shrink T_0^{n+1} a little, with the help of an interior collar, to T_1^{n+1}, and using the fact that $g \mid T_0^{n+1} \times [-\frac{1}{2}, \frac{1}{2}]$ is a product, isotop $g(T^{n+1} \times [-\frac{1}{2}, \frac{1}{2}])$ into $E^{n+1} \times [-\frac{1}{2}, \frac{1}{2}])$ keeping $g \mid T_1^{n+1} \times [-\frac{1}{2}, \frac{1}{2}]$ fixed.

Let us conclude this section with a couple of remarks concerning how the preceding results on local contractibility relate to codimension zero taming. First note that if M^n and $\tilde{M}^n$ are PL manifolds and if $h: M \to \tilde{M}$ is a topological homeomorphism which can be approximated arbitrarily closely by PL homeomorphisms, then the results on local contractibility imply that h is ϵ-tame. For instance, it follows from Theorem **4.11.1** that stable homeomorphisms of E^n are ϵ-tame. (For an example of another use of this observation, see Theorem 1 of [Cantrell and Rushing, 1].) A strong form of the ***hauptvermutung*** for PL manifolds (Question **1.6.5**) is just the following codimension zero taming question: Can every topological homeomorphism of a PL manifold M^n onto a PL manifold $\tilde{M}^n$ be ϵ-tamed? By using some of the techniques presented in this section as well as some work of Wall, it has recently been established by Kirby and Siebenmann that this codimension-zero taming theorem holds for many manifolds and fails for others.