HOMOTOPY INVARIANCE FOR MICROBUNDLES

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1 Homotopy Invariance for Microbundles

The whole content of this section very closely follows [Mil64], chapter 6. Our goal is to prove the following.

Theorem 1.1. *Let $\mathfrak{X}$ be a microbundle with base space B, and let B' be a paracompact space. Let $f \simeq g\colon B' \to B$ be homotopic maps. Then the induced microbundles are isomorphic, i.e.*

$$f^*(\mathfrak{X}) \cong g^*(\mathfrak{X})$$

This has a well-known equivalent for fiber bundles. We'll start by introducing a few notions about map germs. Those will be put to use on microbundles to define bundle map germs. It turns out that the proofs (e.g. Lemma 1.14) look a lot like their fiber bundle analogues once we use those definitions.

1.1 Bundle germs

Definition 1.2. A *map-germ* from (X, A) to (Y, B) is an equivalence class of the elements of the following set

$$\{(f, U) \mid X \supset U \supset A \text{ a neighborhood and a map of pairs } f\colon (U, A) \to (Y, B)\}$$

with the equivalence relation $(f, U) \sim (g, V)$ if and only if there is a neighborhood $N \supset A$ with $f|_V$.

We denote map germs as capital letters $F\colon (X, A) \Rightarrow (Y, B)$.

Remember that we introduced microbundles with the goal to construct tangent bundles of topological manifolds. One can think of those map-germs as workaround for derivatives.

First, observe that we can compose two map germs $(X, A) \overset{F}{\Rightarrow} (Y, B) \overset{G}{\Rightarrow} (Z, C)$, by taking representatives (f, U) and (g, V) and defining a map $g \circ f|_{f^{-1}(V)}\colon f^{-1}(V) \to V \to g(V)$. Since $f^{-1}(V)$ is a neighborhood of A we can set $F \circ G = [(f \circ g|_{f^{-1}(V)}, f^{-1}(V))]$.

Secondly, we observe that there is a standard identity map germ $\mathrm{Id}\colon (X, A) \Rightarrow (X, A)$. This enables us to make another definition:

Definition 1.3. A *homeomorphism-germ* (or homeo-germ) is a map germ with a two-sided inverse, i.e. $F\colon (X, A) \Rightarrow (Y, B)$ is a homeo-germ if there is $G\colon (Y, B) \Rightarrow (X, A)$ such that $F \circ G = \mathrm{Id}_{(Y,B)}$ and $G \circ F = \mathrm{Id}_{(X,A)}$.

The following is a helpful observation to keep control of the definitions introduced so far.

Proposition 1.4. *A map-germ F is a homeo-germ if and only if there is a representative $(f, U) \in F$ that maps U homeomorphically onto its image, and $f(U)$ is a neighborhood of B.*

Proof. " $\Longleftarrow$ ": We take the inverse f^{-1} defined on $f(U)$ as a representative for the map-germ inverse G.

" $\Longrightarrow$ ": Let G be the map-germ inverse and take representatives (f, U), (g, V) with U open, such that $f(U) \subset V$ and $g \circ f = \mathrm{Id}_U$. We know there is an open subset $V' \subset V$ such that $g(V') \subset U$ and $f \circ g|_{V'} = \mathrm{Id}_{V'}$. In particular $g|_{V'}$ and $f|_U$ are injective. Take $U' \coloneqq (f|_U)^{-1}(V')$, which is open with $U' \subset U$. From injectivity follows $f(U') = V'$, which is open and has the continuous inverse $g|_{V'}$. □

Now let's bring those map germs into the context of microbundles. Consider a microbundle $\mathfrak{X}$ consisting of

$$B \xrightarrow{i} E \xrightarrow{j} B.$$

Definition 1.5. The map germ $J\colon (E, i(B)) \Rightarrow (B, B)$ induced by j is called the *projection germ.*

To make our notation a little easier, we will write B instead of (B, B). Furthermore, we identify $i(B)$ with B, for example we just denote the projection germ as $J\colon (E, B) \Rightarrow B$.

Now let's introduce another microbundle $\mathfrak{X}'\colon B' \xrightarrow{i'} E' \xrightarrow{j'} B'$. After all, we are interested in maps between microbundles. This $\mathfrak{X}'$ has the projection germ $J'\colon (E', B') \Rightarrow B'$.

Definition 1.6. Suppose $B = B'$. An *isomorphism germ* (or iso-germ) from $\mathfrak{X}$ to $\mathfrak{X}'$ is a homeo-germ $F\colon (E, B) \Rightarrow (E', B)$ that is *fibre preserving*, i.e. $J' \circ F = J$.

Indeed, this definition translates isomorphisms of microbundles into germ-language:

Proposition 1.7. *An iso-germ exists from $\mathfrak{X}$ to $\mathfrak{X}'$ exists if and only if $\mathfrak{X} \cong \mathfrak{X}'$ as microbundles.*

Proof. " $\Longrightarrow$ ": Take a representative (f, V), which we choose with $f\colon V \xrightarrow{\cong} f(V)$ using Proposition 1.4 This means $f(V)$ is an open neighborhood of B in E'. Fiber preservation implies $j' \circ f|_V = j|_V$. We get the diagram for microbundle isomorphisms:

$$\begin{tikzcd} & V \arrow[dd, "f"', "\cong"] \arrow[dr, "j"] & \\ B \arrow[ur, "i"] \arrow[dr, "i'"'] & & B \\ & f(V) \arrow[ur, "j'"'] & \end{tikzcd}$$

.

" $\Longleftarrow$ ": Given such a diagram, we take (f, V) to represent the iso-germ. □

More generally, we want to consider maps between microbundles on different base spaces $B \neq B'$ but with the same fiber dimension.

Definition 1.8. Let $F\colon (E, B) \Rightarrow (E', B')$ be a map germ, with some representative $f\colon U \to E'$. We say F is a *bundle map germ* from $\mathfrak{X}$ to $\mathfrak{X}'$ if there is a neighborhood $V \supset B$ with $V \subset U$ such that for every $b \in B$ exists $b' \in B'$ so that f maps $V \cap j^{-1}(b)$ injectively to $j'^{-1}(b')$.

$$f|_{V \cap j^{-1}(b)}\colon V \cap j^{-1}(b) \rightarrowtail j'^{-1}(b')$$

We denote such a bundle map germ by $F\colon \mathfrak{X} \Rightarrow \mathfrak{X}'$.

Let's look at this definition for a moment. We should ensure that the existence of such a V does not depend on the choice of representative (f, U). Well, any other representative (f', U') can be restricted to some $W \supset B$ so that $f'|_W = f|_W$. Now $V \cap W$ fulfills the definition for (f', U').

Given a bundle map germ $F\colon \mathfrak{X} \Rightarrow \mathfrak{X}'$, the definition above ensures that the following diagram commutes:

$$\begin{tikzcd} (E, B) \arrow[r, Rightarrow, "F"] \arrow[d, Rightarrow, "J"] & (E', B') \arrow[d, Rightarrow, "J'"] \\ B \arrow[r, "F|_B"] & B' \end{tikzcd}$$

We say that $F|_B$ is *covered by a bundle map germ F*. But be aware that the condition $f\colon V \cap j^{-1}(b) \rightarrowtail j'^{-1}(b')$ is stronger than $J' \circ F = F|_B \circ J$.

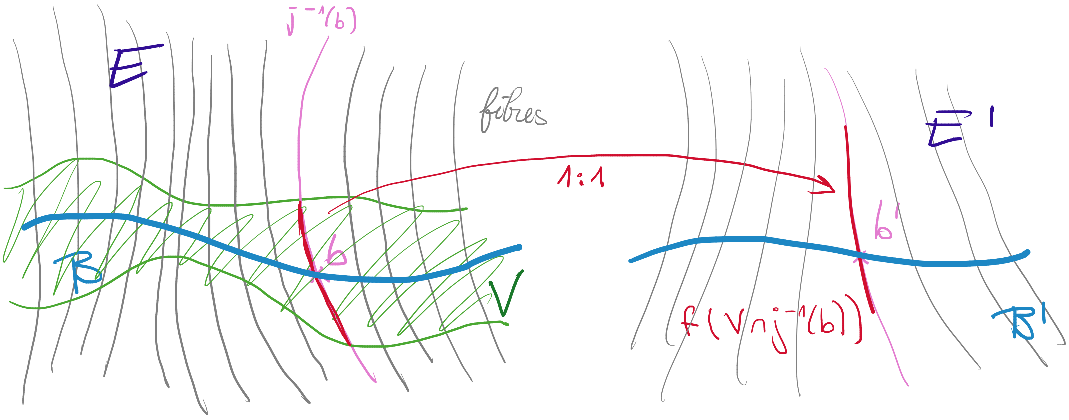

FIGURE 1. Visualization of a bundle map germ

1.2 Proof of Homotopy Invariance

Remember that we are trying to prove Homotopy Invariance for microbundles.

Theorem 1.1. Let $\mathfrak{X}$ be a microbundle with base space B, and let B' be a paracompact space. Let $f \simeq g \colon B' \to B$ be homotopic maps. Then the induced microbundles are isomorphic, i.e.

$$f^*(\mathfrak{X}) \cong g^*(\mathfrak{X})$$

With the definitions above we have developed sufficient language to give a proof. We will need two more ingredients.

Lemma 1.9. Suppose $\mathfrak{X}$ and $\mathfrak{X}'$ are microbundles over the same base space $B = B'$, and suppose $F \colon \mathfrak{X} \Rightarrow \mathfrak{X}'$ is a bundle map germ covering Id_B. Then F is an iso-germ.

Lemma 1.14. Let $\mathfrak{X}$ be a microbundle over $B \times [0,1]$, where B is paracompact. Then the standard retraction

$$r \colon B \times [0,1] \to B \times [1]$$

is covered by a bundle map germ $R \colon \mathfrak{X} \to \mathfrak{X}|_{B \times [1]}$.

For now we assume those two lemmas hold and prove them later.

Proof of Theorem 1.1. Let $\mathfrak{X}$ be a microbundle with base space B, and let B' be a paracompact space. Let $H \colon B' \times [0,1] \to B$ be a homotopy from $H_0 = f$ to $H_1 = g$. Let $R \colon H^*\mathfrak{X} \Rightarrow H^*\mathfrak{X}|_{B' \times [1]}$ be the bundle map germ covering the standard retraction from Lemma 1.14. Look at the following diagram:

$$\begin{tikzcd}
f^*\mathfrak{X} \arrow[r, Rightarrow] \arrow[d, Rightarrow, "J"] & H^*\mathfrak{X} \arrow[r, Rightarrow, "R"] \arrow[d, Rightarrow] & H^*\mathfrak{X}|_{B \times [1]} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & g^*\mathfrak{X} \arrow[d, Rightarrow, "J"] \\
B' \arrow[r, "\mathrm{Id}_B \times (0)"] \arrow[rrr, bend right, "\mathrm{Id}_B"'] & B' \times [0,1] \arrow[r, "r"] & B' \times [1] \arrow[r] & B'
\end{tikzcd}$$

Here the left and right bundle map germs are the obvious ones. Observe that the composition of the bottom maps is $\mathrm{Id}_{B'}$. Taking the composition of the bundle map germs on top therefore leads to a bundle map germ $f^*\mathfrak{X} \Rightarrow g^*\mathfrak{X}$ that covers the identity. Lemma 1.9 finishes the proof. □

1.3 Proof of Ingredients

Lemma 1.9. *Suppose $\mathfrak{X}$ and $\mathfrak{X}'$ are microbundles over the same base space $B = B'$, and suppose $F\colon \mathfrak{X} \Rightarrow \mathfrak{X}'$ is a bundle map germ covering Id_B. Then F is an iso-germ.*

Proof. It is clear from the definition (see diagram 1.1), that a bundle map germ covering the identity is fiber preserving. We have to concern ourselves with showing that F is a homeomorphism germ.

We start by proving a special case before we move on to the general case. Assume $\mathfrak{X}$ and $\mathfrak{X}'$ are trivial, i.e. $E = E' = B \times \mathbb{R}^n$. For $x \in \mathbb{R}^n$ and $\varepsilon > 0$ we will denote the open ball of radius ε at x as $D_\varepsilon(x)$. We want to show any bundle map germ $F\colon E \to E'$ covering the identity is a homeo germ. Take a representative $g\colon U \to E'$ with $B \subset U \subset B \times \mathbb{R}^n$ open. The definition of bundle map germ combined with the information that F covers Id_B tells us that g maps $U \cap j^{-1}(b)$ injectively to $j'^{-1}(b)$. (W.l.o.g. we have chosen U small enough.) Hence g is injective and fiber preserving.

Claim: Every map $g\colon U \to B \times \mathbb{R}^n$ that is injective and fiber preserving is an open mapping.

Observe first, that g can be expressed as $g(b, x) = (b, g_b(x))$ for $b \in B$, with $g_b\colon U_b \to \mathbb{R}^n$ defined on the open set $U_b := j^{-1}(b) \cap U$. By definition of the bundle map germ, g_b is injective, thus Invariance of domain implies that every g_b is an open mapping. Now given some point $p_0 := (x_0, b_0) \in B \times \mathbb{R}^n$, we write $g_{b_0}(x_0) =: x_1$ and $g(p_0) = g(b_0, x_0) = (b_0, x_1) =: p_1$. In order to show that g is open, we have to show that for any open neighborhood U_0 of p_0 there is an open neighborhood U_1 of p_1 such that $g(U_0) \supset U_1$.

Given U_0 we start by choosing $\varepsilon > 0$ so that $D_\varepsilon(x_0) \subset \mathrm{proj}_{\mathbb{R}^n}(U_0)$. Since g_{b_0} is an open map, there is $\delta > 0$ so that $D_{2\delta}(x_1) \subset g_{b_0}(D_\varepsilon(x_0))$. There exists a neighborhood V of b_0 in B such that

$$|g_b(x) - g_b(x_0)| < \delta \quad \forall b \in V,\ x \in D_\varepsilon(x_0)$$

Hence for each $b \in V$ holds $D_\delta(x_1) \subset g_b(D_\varepsilon(x_0))$, thus $g(V \times D_\varepsilon(x_0)) \supset V \times D_\delta(x_1) =: U_1$.

A consequence of the Claim is that g is an embedding. It maps U homeomorphically onto $g(U)$ (which is open in E') and we can apply Proposition 1.4 to see that F is a homeo-germ.

Now the general case. Let $\mathfrak{X}$ and $\mathfrak{X}'$ be microbundles over B and let $F\colon \mathfrak{X} \Rightarrow \mathfrak{X}'$ be a bundle map germ covering Id_B. Take a representative $f\colon U \to E'$ of F, where we choose U small enough to assume f is injective and fiber preserving.

For any $b \in B$ exists a neighborhood W_b of $i(b)$ in U such that $\mathfrak{X}|_{W_b}$ is trivial. Set $C_b := j(W_b)$. Clearly the restriction $F|_{\mathfrak{X}|_{C_b}}$ covers the identity on C_b. We can choose W_b small enough that $\mathfrak{X}'|_{C_b}$ is trivial as well, and then we can apply the first case. This means $f|_{W_b}$ is a homeomorphism onto its image, with $f(W_b) \subset E'$ open.

Now we define $W := \bigcup_{b \in B} W_b$ and obtain $f\colon W \xrightarrow{\cong} f(W)$, where $f(W)$ is open in E'. Proposition 1.4 asserts that we get a homeo-germ. □

Corollary 1.10. *If a map $g\colon B \to B'$ is covered by a bundle map germ $G\colon \mathfrak{X} \Rightarrow \mathfrak{X}'$, then $\mathfrak{X} \cong g^*\mathfrak{X}'$.*

Proof. The gist is that G induces a bundle map germ $F\colon \mathfrak{X} \Rightarrow g^*\mathfrak{X}'$ that covers the identity Id_B. Then we can apply Lemma 1.9 above.

Let's spell out how we get this F. Start with a representative $g\colon V \to E'$ of G, such that for any $b \in B$ exists $b' \in B'$ with $g\colon j^{-1}(b) \cap V \rightarrowtail j'^{-1}(b')$ injective. Remember that the induced bundle $g^*\mathfrak{X}'$ is the pullback in the following diagram:

$$\begin{tikzcd}
V \arrow[dr, dotted, "f"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "j|_V"'] & & \\
& g^*E' \arrow[r] \arrow[d, "g^*j'"] & E' \arrow[d, "j'"] \\
& B \arrow[r, "g"'] & B'
\end{tikzcd}$$

The universal property of pullbacks induces the dotted map f. Since $j|_V = g^*j' \circ f$, we immediately get that $f|_B$ is the identity. Also f represents a bundle map germ, because the diagram reduces to the following when we start with $\{b'\} \subset B'$:

$$\begin{tikzcd}
V \cap j^{-1}(b) \arrow[dr, dotted, "f"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "j"'] & & \\
& g^*j'^{-1}(b) \arrow[r] \arrow[d, "g^*j'"] & j'^{-1}(b') \arrow[d, "j'"] \\
& \{b\} \arrow[r, "g"'] & \{b'\}
\end{tikzcd}$$

In the diagram restrictions are left out for improved readability. Since the restriction of g is injective, so is f. In conclusion is (f, V) a representative for our bundle map germ F. $\square$

For the second ingredient we have to make some observations about how to piece bundle maps together before we can build one that covers the standard retraction.

Lemma 1.11. *Let $\mathfrak{X}$ be a microbundle over B, and let $\{B_\alpha\}_{\alpha\in A}$ be a locally finite collection of closed sets that cover B. Suppose for all $\alpha \in A$ we have bundle map germs to some microbundle $\mathfrak{N}$:*

$$F_\alpha \colon \mathfrak{X}|_{B_\alpha} \Rightarrow \mathfrak{N}$$

such that for any $\alpha, \beta \in A$ the restrictions of F_α and F_β agree, i.e.:

$$F_\alpha|_{\mathfrak{X}|_{B_\alpha\cap B_\beta}} = F_\beta|_{\mathfrak{X}|_{B_\alpha\cap B_\beta}}$$

Then there is a bundle map germ $F\colon \mathfrak{X} \Rightarrow \mathfrak{N}$ extending the F_α, i.e. $F|_{\mathfrak{X}|_{B_\alpha}} = F_\alpha$ for all $\alpha \in A$.

Proof. Take $f_\alpha\colon U_\alpha \to E'$ some representative for F_α. By definition there are open neighborhoods $U_{\alpha\beta}$ of $B_\alpha \cap B_\beta$ inside $U_\alpha \cap U_\beta$ such that $f_\alpha|_{U_{\alpha\beta}} = f_\beta|_{U_{\alpha\beta}}$. Define the following set:

$$U = \left\{ \quad e \in E \quad \middle| \quad \begin{matrix} j(e) \in B_\alpha \implies e \in U_\alpha \\ j(e) \in B_\alpha \cap B_\beta \implies e \in U_{\alpha\beta} \end{matrix} \quad \right\}.$$

Claim: This set U is open.

Take some $e_0 \in U$. As $\{B_\alpha\}$ is a locally finite cover of B, we have some neighborhood V_0 of $j(e_0)$ that intersects only finitely many, let's say $B_{\alpha_1}, \dots, B_{\alpha_k}$. Look at $W := \bigcap_{1\le i<j\le k} U_{\alpha_i\alpha_j}$. Since we only intersect finitely many sets W is open. Define $V_1 := j^{-1}(V_0) \cap W$. This fulfills $e_0 \in V_1 \subset U$ and V_1 is open.

We can define $f\colon U \to E'$ that extends the f_α. This is the representative for F. $\square$

Proposition 1.12. *Let $\mathfrak{X}$ be a microbundle over $B \times [0,1]$ such that both $\mathfrak{X}|_{B\times[0,\frac{1}{2}]}$ and $\mathfrak{X}|_{B\times[\frac{1}{2},1]}$ are trivial. Then $\mathfrak{X}$ is trivial.*

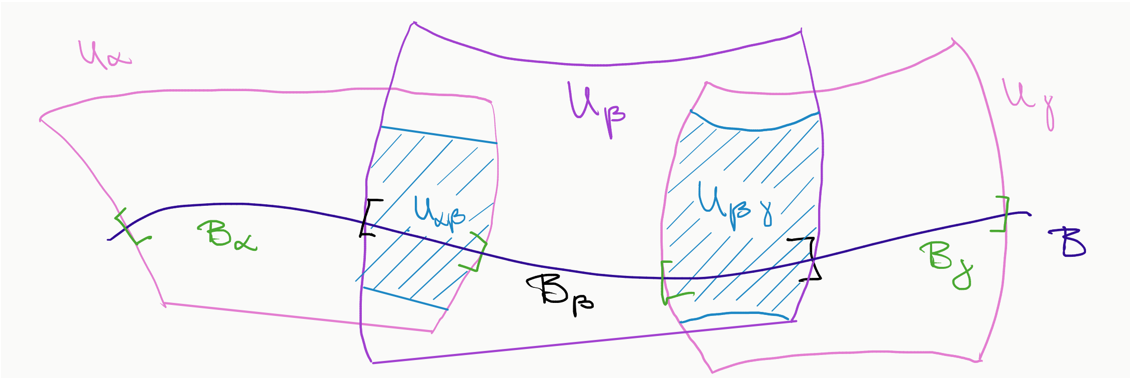

FIGURE 2. Visualization of the $U_{\alpha\beta}$

Proof. Look at the "obvious" restriction map $f\colon B \times [0,1] \to B \times [\frac{1}{2}]$. Since $\mathfrak{X}|_{B\times[0,\frac{1}{2}]}$ and $\mathfrak{X}|_{B\times[\frac{1}{2},1]}$ are trivial, we can cover the maps $f_1\colon B \times [0, \frac{1}{2}]$ and $f_2\colon B \times [\frac{1}{2}, 1]$ with bundle map germs:

$$F_1\colon \mathfrak{X}|_{B\times[0,\frac{1}{2}]} \Rightarrow \mathfrak{X}|_{B\times[\frac{1}{2}]}$$
$$F_2\colon \mathfrak{X}|_{B\times[\frac{1}{2},1]} \Rightarrow \mathfrak{X}|_{B\times[\frac{1}{2}]}$$

Now we can apply Lemma 1.11 to the locally finite covering $B \times [0, \frac{1}{2}]$, $B \times [\frac{1}{2}, 1]$ to obtain a bundle map germ $F\colon \mathfrak{X} \Rightarrow \mathfrak{X}|_{B\times[\frac{1}{2}]}$ which covers the restriction $f\colon B\times[0,1] \to B\times[\frac{1}{2}]$. Corollary 1.10 tells us $\mathfrak{X} \cong f^*\mathfrak{X}|_{B\times[\frac{1}{2}]}$. Since $\mathfrak{X}|_{B\times[\frac{1}{2}]}$ is trivial, we see that $f^*\mathfrak{X}|_{B\times[\frac{1}{2}]}$ is trivial, and finally deduce that $\mathfrak{X}$ is trivial. □

The next lemma is important for finding the neighborhoods on which we can start building.

Lemma 1.13. *Let $\mathfrak{X}$ be a microbundle over $B\times[0,1]$. Then for every $b \in B$ exists a neighborhood V of b such that $\mathfrak{X}|_{V\times[0,1]}$ is trivial.*

Proof. Fix $b \in B$. For any $t \in [0,1]$ we choose an open neighborhood $V_t \times (t - \varepsilon_t, t + \varepsilon_t)$ of (b,t) so that $\mathfrak{X}$ is trivial on there. The compact set $b \times [0,1]$ can now be covered with finitely many of the sets $(t - \varepsilon_t, t + \varepsilon_t)$. Let those sets be centered at $0 = t_0 < t_1 < \cdots < t_n = 1$, and define $V = \bigcap_{i=0}^n V_i$. $V \subset B$ is open because all V_{t_i} are open. Now we make a refinement $0 = t'_0 < t'_1 < \cdots < t'_m = 1$ so that $|t'_{j-1} - t'_j| < \min_{i=0,\dots,n} \varepsilon_{t_i}$ for all $1 \le j \le m$. This ensures that $\mathfrak{X}|_{V\times[t'_{j-1},t'_j]}$ is trivial for all j. Now we (repeatedly) apply Proposition 1.12 to see that $\mathfrak{X}|_{V\times[0,1]}$ is trivial. □

Finally we can prove the last ingredient.

Lemma 1.14. *Let $\mathfrak{X}$ be a microbundle over $B \times [0,1]$, where B is paracompact. Then the standard retraction*

$$r\colon B \times [0,1] \to B \times [1]$$

is covered by a bundle map germ $R\colon \mathfrak{X} \to \mathfrak{X}|_{B\times[1]}$.

Proof. Lemma 1.13 gives us a covering $\{V_b\}_{b\in B}$ of B with every $\mathfrak{X}|_{V_b\times[0,1]}$ trivial. Paracompactness of B gives us a locally finite refinement $\{V_\alpha\}_{\alpha\in A}$. Now we choose functions $\lambda_\alpha\colon B \to [0,1]$ so that $\operatorname{supp}\lambda_\alpha \subset V_\alpha$ for all $\alpha \in A$ and $\max_{\alpha\in A}\lambda_\alpha(b) = 1$ for all $b \in B$.

Define the retractions $r_\alpha\colon B\times[0,1] \to B\times[0,1]$ by

$$r_\alpha(b,t) = (b, \max\{t, \lambda_\alpha(b)\}).$$

If we assign some ordering to A and were to define r as the composition of all r_α in that order, it is well defined because locally we only have finitely many $\lambda_\alpha(b)$. In particular r is the standard retraction:

$$r(b,t) = (b, \max_{\alpha\in A}\{t, \lambda_\alpha(b)\}) = (b,1).$$

This gives us an idea of what our next steps for the construction of R are:

(1) Cover each r_α with a bundle map germ $R_\alpha \colon \mathfrak{X} \Rightarrow \mathfrak{X}$.
(2) Choose an ordering of A and let the desired bundle map germ $R\colon \mathfrak{X} \Rightarrow \mathfrak{X}|_{B\times[1]}$ be the composition of the R_α in that order.

Step (1): We can write $B\times[0,1]$ as the union of the following closed sets:

$$C_\alpha := (\operatorname{supp}\lambda_\alpha)\times[0,1]$$
$$D_\alpha := \{(b,t) \mid t \geq \lambda_\alpha(b)\}$$

Since $C_\alpha \subset V_\alpha \times [0,1]$ we have that $\mathfrak{X}|_{C_\alpha}$ is trivial. Hence the identity map germ of $\mathfrak{X}|_{C_\alpha\cap D_\alpha}$ extends to a bundle map germ $\mathfrak{X}|_{C_\alpha} \Rightarrow \mathfrak{X}|_{C_\alpha\cap D_\alpha}$ that covers $r_\alpha|_{C_\alpha}$. Piece this germ together with the identity map germ on $\mathfrak{X}|_{D_\alpha}$ by Lemma 1.11 to obtain R_α.

Step (2): We have to argue that taking an "infinite" composition makes sense. We use that locally all but finitely many R_α are the identity.

More precisely, we define R on $\{B_\beta\}$, some locally finite covering of B by closed sets, and then glue. Each B_β intersects only finitely many V_α, let's say $V_{\alpha_1}, V_{\alpha_2}, \dots, V_{\alpha_k}$ with $\alpha_1 < \alpha_2 < \cdots < \alpha_k$ in our order. The bundle map germ $R_{\alpha_k}\cdots R_{\alpha_2}R_{\alpha_1}$ restricts to

$$R(\beta) := \mathfrak{X}|_{B_\beta\times[0,1]} \Rightarrow \mathfrak{X}|_{B_\beta\times[1]}$$

Lastly, we piece together these $R(\beta)$ with the help of Lemma 1.11. □

1.4 Corollaries to Homotopy Invariance

The most important corollary is the most obvious:

Corollary 1.15. *Every microbundle over a paracompact, contractible base space is trivial.*

Another interesting result is the following:

Corollary 1.16. *Assume we have a map $f\colon A\to B$ with A paracompact. Denote the mapping cone as $Cf = B\cup_f CA$. Then a microbundle $\mathfrak{X}$ over B can be extended to a microbundle over Cf if and only if the induced microbundle $f^*\mathfrak{X}$ is trivial.*

Proof. " $\Longrightarrow$ ": The composition $A \xrightarrow{f} B \xhookrightarrow{\text{incl}} Cf$ is always nullhomotopic since the image lies in $CA \simeq \{*\}$. If $\mathfrak{X}$ extends to a microbundle $\mathfrak{X}'$ over Cf, then clearly $\mathfrak{X}'|_B \cong \mathfrak{X}$. Thus $f^*\mathfrak{X} \cong (\text{incl}\circ f)^*\mathfrak{X}'$, which must be trivial by Theorem 1.1.

" $\Longleftarrow$ ": Consider the mapping cylinder $Zf = B\cup_f (A\times[0,1])$, where we glue $(a,1)\cong f(a)$ for all $a\in A$. Because B is a retract of Zf we can extend $\mathfrak{X}$ to a microbundle $\mathfrak{X}''$ over Zf. Now suppose that $f^*\mathfrak{X}$ is trivial. This implies that $\mathfrak{X}''|A\times[0]$ is trivial and thus $\mathfrak{X}''|_{A\times[0,\frac{1}{2}]}$ is trivial as well. This means we have some open set $U \subset E(\mathfrak{X}''|_{A\times[0,\frac{1}{2}]})$ such that $U \cong A\times[0,\frac{1}{2}]\times\mathbb{R}^n$. Hence we can remove a closed subset from $E(\mathfrak{X}'')$ and then assume $E(\mathfrak{X}''|_{A\times[0,\frac{1}{2}]}) \xrightarrow{h} \cong A\times[0,\frac{1}{2}]\times\mathbb{R}^n$. This homeomorphism h is compatible with the projections and inclusions.

Collapsing $A\times[0]$ in Zf to a single point yields Cf. We can create $E(\mathfrak{X}')$ by collapsing $h^{-1}(A\times[0]\times\{x\})$ for each $x\in\mathbb{R}^n$ in $E(\mathfrak{X}'')$. The microbundle structure of $\mathfrak{X}''$ now induces a microbundle structure on $\mathfrak{X}'$ over the basespace Cf. □

The application of this that comes to mind is taking $A = \mathbb{S}^n$ the sphere to extend microbundles along a CW-structure. This corollary is essential for proving that stable isomorphism classes form a group over finite CW-complexes (with the Whitney sum as operation).

1.5 Proof using Kister's Theorem

While one can use Kister's Theorem to prove the Homotopy Invariance for the cases that interest us most, it is unwise to do so. More specifically Corollary 1.15 is used in the proof of Kister's Theorem when we work over simplicial complexes, because it implies that a microbundle over a single simplex is trivial. Still, it is a fun exercise.

Corollary 1.17. *Assume Kister's Theorem holds, and that Homotopy Invariance holds for fiber bundles. Let $\mathfrak{X}$ be a microbundle with base space B. Assume B is a topological manifold or a finite simplicial complex and that B' is paracompact. Let $f \simeq g\colon B' \to B$ be homotopic maps. Then the induced microbundles are isomorphic, i.e.*

$$f^*(\mathfrak{X}) \cong g^*(\mathfrak{X})$$

Proof. Let $E_1 \subset E$ be so that $E_1 \to B$ is a fiber bundle ξ (with $\mathrm{Homeo}_0(\mathbb{R}^n)$ as the structure group). Clearly $|\xi| \cong \mathfrak{X}$. Homotopy Invariance for fiber bundles tells us $f^*\xi \cong g^*\xi$. That means the underlying microbundles are isomorphic as well: $|f^*\xi| \cong |g^*\xi|$. The rest is showing that the underlying microbundles are isomorphic to the original induced microbundles, i.e. $|f^*\xi| \cong f^*|\xi| \cong f^*\mathfrak{X}$. □

I can not claim with absolute certainty that triviality over simplices is the only instance of Homotopy Invariance used in the proof of Kister's Theorem, but this simpler statement can be proven faster than our general statement.

Proposition 1.18. *Let $\mathfrak{X}$ be a microbundle over the standard n-simplex σ. Then $\mathfrak{X}$ is trivial.*

Proof. By definition, there are local trivialisations, i.e. we have open sets $\{B_\alpha\}$ covering σ such that all $\mathfrak{X}|_{B_\alpha}$ are trivial. Since σ is compact we can take a finite subcover $B_1, \ldots, B_m$. Now take a barycentric refinement of σ so that any subsimplex σ_α is contained in some B_{r_α}. In particular, we have bundle map germs $\mathfrak{X}|_{\sigma_\alpha} \Rightarrow \mathfrak{e}^n_{\sigma_\alpha}$ that cover the identity. Now Lemma 1.11 tells us that we get a bundle map germ $\mathfrak{X} \Rightarrow \mathfrak{e}^n_\sigma$ covering the identity. Lemma 1.9 and Proposition 1.7 complete the proof. □

References

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