Casson towers and slice links

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Abstract. We prove that a Casson tower of height 4 contains a flat embedded disc bounded by the attaching circle, and we prove disc embedding results for height 2 and 3 Casson towers which are embedded into a 4-manifold, with some additional fundamental group assumptions. In the proofs we create a capped grope from a Casson tower and use a refined height raising argument to establish the existence of a symmetric grope which has two layers of caps, data which is sufficient for a topological disc to exist, with the desired boundary. As applications, we present new slice knots and links by giving direct applications of the disc embedding theorem to produce slice discs, without first constructing a complementary 4-manifold. In particular we construct a family of slice knots which are potential counterexamples to the homotopy ribbon slice conjecture.

1. Introduction

This paper presents results on *Casson towers* of height 2, 3 and 4 in dimension four, and applications to the problem of slicing knots and links.

The disc embedding problem is one of the most important questions in 4-manifold topology. Roughly speaking, when the disc embedding problem can be solved, the surgery and s-cobordism programme for the classification of 4-manifolds can be carried out as in the high dimensional case. In fact disc embedding in these contexts is essentially equivalent to the Whitney trick, which is a key ingredient for geometrically realising the algebraic cancellation of intersection data.

M. Freedman solved the disc embedding problem in simply connected topological 4-manifolds, and as a consequence he was able to classify such manifolds [FQ90, Chapter 10] using surgery theory. Freedman's solution built upon the work of A. Casson, who introduced the influential idea of a Casson tower. A Casson tower arises as the trace of repeated attempts to eliminate intersections of an immersed disc, the goal being to find a flat embedded disc [Cas86]. Briefly speaking, the *height* of a Casson tower is the number of stages of iterated attempts. A Casson tower T, itself a 4-manifold, is endowed with a framed circle $C = C(T)$ embedded in its boundary. We ask whether there exists a flat embedded disc with framed boundary C. See Definition 2.1 for details.

Casson considered a tower of infinite height, which is now called a *Casson handle* [Cas86]. He showed that a Casson handle is *proper homotopy equivalent* to an open 2-handle. In the original proof of the celebrated disc embedding theorem in dimension 4 [Fre82b], Freedman showed that a Casson handle is *homeomorphic* to an open 2-handle, and consequently contains a flat embedded disc with framed boundary $C(T)$. A key ingredient of the proof was Freedman's reimbedding theorem [Fre82b, Theorem 4.4], which says that a height 6 Casson tower contains within it a height 7 tower (see [Biž94] for a detailed exposition). Iterating this, it follows that a given height 6 tower T contains a Casson handle, and consequently contains a flat embedded disc with framed boundary $C(T)$. Gompf and Singh improved this disc embedding result by showing that height 5 Casson towers are sufficient for reimbedding [GS84].

From this a natural question arises: what is the minimal height of a Casson tower required to obtain an embedded disc?

In Theorems A, B and C below we give disc embedding results for Casson towers of height 4, 3 and 2 respectively, under increasingly strong assumptions on fundamental groups. The height 2 result is particularly useful for the study of knot and link concordance, since it is often feasible to construct such a tower in D^4 bounded by a knot or link.

Work of Freedman in the 1980s and 90s [Fre82a, Fre85, Fre88, Fre93], and also later work such as that of Freedman-Teichner [FT95b], Friedl-Teichner [FT05] and Cochran-Friedl-Teichner [CFT09], produced slice knots and links of great interest. Particular focus was placed on the question of which Whitehead doubles are slice (see Conjecture 1.1 below), since topological surgery problems in dimension four can be reduced to atomic problems [CF84] which have solutions precisely when such links are slice.

Using our height 2 Casson tower embedding theorem (Theorem C), we extend the class of known slice knots to include the new family of slice knots described in Theorem D. We apply the disc embedding theorem to construct slice discs directly, rather than using the topological surgery machine employed by many of the papers mentioned above. Our slice knots relate closely to the Topological Whitehead Double Conjecture 1.1, give potential counterexamples to the Homotopy Ribbon Slice Conjecture 1.2, and suggest a possible connection between the Homotopy Ribbon Slice Conjecture 1.2 and the 4-dimensional surgery conjecture.

1.1. Casson towers of height four, three, and two

We proceed to introduce our disc embedding results for Casson towers of height four, three, and two. Let W be a 4-manifold with boundary. A framed link $L \subset \partial W$ is *slice* in W if L bounds a collection of disjointly embedded flat discs in W, as framed manifolds.

Height four. Our first main result implies that a *height 4* Casson tower is in fact sufficient to obtain a flat embedded disc. In fact we give a stronger result. Briefly, define a *distorted Casson tower* by introducing plumbings of the top stage discs into discs of stage two or higher in a Casson tower (see Definition 4.4).

Theorem A. *A distorted Casson tower T of height 4 contains a topologically embedded flat disc bounded by $C(T)$ as a framed manifold.*

In other words, $C(T)$ is slice in T. Since a Casson tower is vacuously a distorted Casson tower, Theorem A holds for an ordinary (non-distorted) Casson tower of height 4. This assertion seems to have been expected to be true by the experts, but to the knowledge of the authors, no proof has appeared in the literature; compare [Ray13, Footnote 1].

Height three. It is not known in general whether a height 3 Casson tower T contains an embedded disc with boundary $C(T)$. Progress has been made by looking at special cases, as instigated in [CF84]. Freedman proved that the simplest Casson tower of height 3, namely the tower with a single double point at each stage, contains a disc [Fre88][1]. We remark that completing the analogous argument to our proof of Theorem A for a height 3 Casson towers would seem to require the surgery conjecture for non-abelian free groups. The corresponding statement to Theorem A for general height 3 Casson towers would therefore be rather interesting. The main difficulty, as so often in this subject, is to achieve π_1-nullity.

Instead of looking for null homotopies internally in Casson towers, we can consider embedded Casson towers in a 4-manifold, and then try to find null homotopies inside the 4-manifold. For our height 3 result, we use the notion of a *good group*. In this paper we use

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[1] In fact, Freedman showed that a two component link called "Whitehead$_3$" bounds slicing discs in the 4-ball whose complement has free fundamental group. This link is associated to the simplest Casson tower T of height 3, as explained in our Section 6.1. It turns out that each of the two slicing discs is ambiently isotopic to the standard disc in the 4-ball by [FQ90, 11.7A]. It follows that the exterior of one slicing disc is T and the other slicing disc is bounded by $C(T)$.

the definition of [FT95a], which defines a group to be good if it satisfies the π_1-null disc lemma. Note that this differs from the definition of a good group in [FQ90]. For a precise description and related discussion, see Definition 3.1 and the paragraph following it. A result of Freedman-Teichner and Krushkal-Quinn [FT95a, KQ00] tells us that a group of subexponential growth is good, as is any group obtained from good groups by extensions and direct limits.

A. Ray considered a framed grope bounded by $C(T)$ in a Casson tower T [Ray13]. Denote the first stage surface of this grope by $\Sigma(T)$. This is an oriented surface embedded in T with $\partial\Sigma(T) = C(T)$. See Figure 5 and Proposition 4.1 for more details. For a disjoint union of Casson towers $T = \bigsqcup T_i$, denote $C(T) := \bigsqcup C(T_i)$ and $\Sigma(T) := \bigsqcup \Sigma(T_i)$. Denote a tubular neighbourhood of $C(T)$ in ∂T by $\partial_-(T)$.

Theorem B. *Let W be a 4-manifold with boundary and suppose that $T = \bigsqcup T_i$ is a collection of disjoint Casson towers T_i of height 3 in W such that $\partial_-(T) \subset \partial W$ and the image of $\pi_1(T_i \smallsetminus \Sigma(T_i)) \to \pi_1(W \smallsetminus \Sigma(T))$ is a good group for each i. Then the framed link $C(T) \subset \partial W$ is slice in W.*

We remark that this result concerns links and not just knots.

Height two. For the height 2 case, we obtain a slicing result under a stronger simple connectivity hypothesis. In the statement T_{p-q} denotes the union of the stages p through q inclusive for a Casson tower T; see Definition 2.2 for a more precise description.

Theorem C. *Let W be a 4-manifold with boundary and suppose T is a Casson tower of height 2 embedded in W such that the second stage T_{2-2} of T lies in a codimension zero simply connected submanifold $V \subseteq \overline{W \smallsetminus T_{1-1}}$. Then the knot $C(T) \subset \partial W$ is slice in W.*

In Theorems B and C, the slice discs are contained in a neighbourhood of a union of the tower itself and a collection of null homotopies for double point loops constructed during the proofs.

Our proofs and gropes. After Freedman's original proof of the disc embedding theorem using Casson towers, the grope technology (see Definitions 2.3–2.6) has been developed in subsequent work by Quinn [Qui82], Edwards [Edw84], Freedman-Quinn [FQ90], Freedman-Teichner [FT95a], Krushkal-Quinn [KQ00] and others. It turned out that gropes are effective for proving the disc embedding theorem in the non-simply connected case.

Gropes are in fact a key ingredient of our proofs. For height four and three, our arguments hinge on Ray's construction of a framed grope inside a Casson tower [Ray13]. It enables us to connect the grope and Casson tower techniques.

In the grope setting the minimal data required for the existence of a topological disc has been quite well optimised in the decades since the original reference [FQ90] was written. (The optimisation has not been enough for the surgery conjecture to be known, of course). Up to date grope combinatorics were partially written up in [FF, Chapter 8] by the second author and W. Politarczyk as part of the lecture notes for Freedman's lectures for the Max Planck Institute for Mathematics semester on 4-manifolds in 2013. In the hope that they represent a useful addition to the literature, details relevant to the current paper which cannot be found in the earlier literature (e.g. [FQ90]) are given below: see Grope Height Raising Lemma 3.7 and Cap Separation Lemma 3.8. In fact the proof of the latter lemma has not appeared anywhere before to the best of our knowledge. The inclusion of these details is further justified by the following corollary, which is proven by combining Ray's grope construction with the Grope Height Raising Lemma 3.7.

Corollary 7.1. *A Casson tower T of height 3 contains an embedded grope of height n, with the same attaching circle $C(T)$ as the Casson tower, for all n.*

This improves a result of Ray [Ray13, Theorem A (i)]. Further discussion of this corollary can be found in Section 7.

The proof of the height 2 result, Theorem C, requires an entirely new construction, given in Proposition 4.5, of *capped* gropes from Casson towers embedded in a 4-manifold, under a certain fundamental group condition. This depends on new geometric arguments and some quite delicate combinatorics. The application to slice knots discussed next utilises this construction.

1.2. Applications to slicing knots and links

We apply our results on Casson towers to present new slice knots and slice links in S^3. As usual, we say a knot or link in S^3 is *slice* if it is slice in D^4.

New slice knots. To state our results on knots, we recall that Milnor called a link L in S^3 *homotopically trivial* if its components admit disjoint null-homotopies [Mil54]. That is, if there are maps $h_i \colon D^2 \to S^3$ such that $L = \bigsqcup_i h_i(S^1)$ and $h_i(D^2) \cap h_j(D^2) = \emptyset$ for $i \neq j$. We also recall that a band sum operation on a link L is performed along an embedded band $D^1 \times D^1$ which joins two components of L and such that $\mathrm{Int}(D^1) \times D^1$ is disjoint from L. Denote the untwisted Whitehead double of a link L by $\mathrm{Wh}(L)$. The link $\mathrm{Wh}(L)$ is only defined up to a sign choice for the clasp when doubling each component of L. Our theorems hold for any choices of signs.

Theorem D. *Suppose that L is an m-component homotopically trivial link, and K is a knot obtained from $\mathrm{Wh}(L)$ by applying $m-1$ band sum operations. Then K is slice.*

Our proof of Theorem D uses Theorem C on Casson towers of height 2. The details are discussed in Section 5.

Theorem D specialises to several interesting cases. First, taking L to be a knot we see that Theorem D has, as a special case, the result of Freedman that the Whitehead double of any knot is slice. When L is a knot there are no band sums. More generally, when the bands miss standard genus one Seifert surfaces for the components of $\mathrm{Wh}(L)$, we obtain a knot of Alexander polynomial one, to which Freedman's slicing result applies. The novel aspect of Theorem D is that arbitrary bands are allowed. By applying Theorem D for L a link and suitably complicated bands, we obtain a large family of new slice knots. For instance, the following corollary gives a way to construct intriguing examples.

Corollary E. *Suppose L is an m-component homotopically trivial link, and R is a ribbon knot. Consider a split union $\mathrm{Wh}(L) \sqcup R$ in S^3, and choose m disjoint bands which join each component of $\mathrm{Wh}(L)$ to R, such that in addition the bands are disjoint from an immersed ribbon disc for R in S^3 and are disjoint from Seifert surfaces for $\mathrm{Wh}(L)$. Then the knot K obtained from $\mathrm{Wh}(L) \sqcup R$ by these band sum operations along the arcs is slice.*

The additional assumptions on the bands is not necessary to conclude that K is slice, but we include it so that we can discuss the ribbon knot R meaningfully. For example, a slice knot K from Corollary E has the same Alexander polynomial as R; see Proposition 5.3. An explicit example is given in Figure 1. To construct this knot apply band sum operations to $\mathrm{Wh}(L) \sqcup R$, where L is the 3-component link obtained from the Whitehead link by adjoining an untwisted parallel of one of the components, and R is the ribbon knot 8_8. By Corollary E, the knot K in Figure 1 is slice. This K is a hyperbolic knot (verified by SnapPea), and consequently is prime and non-satellite.

To the knowledge of the authors, previously known methods and results are not able to show that all of our knots are slice, except for in some special cases. Section 5.2 contains more details on the failure of the topological surgery method to slice these knots. Another possible approach to slice a knot K given by Corollary E would be to show that the *link* $\mathrm{Wh}(L) \sqcup R$ is slice. This is an important conjecture in the theory of topological 4-manifolds.

Conjecture 1.1 (Topological Whitehead Double Conjecture)**.** *The Whitehead double $\mathrm{Wh}(L)$ of a link L is freely slice if and only if L is homotopically trivial.*
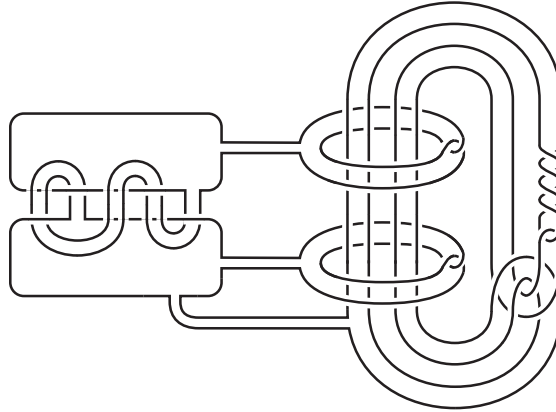
FIGURE 1. An example of a new slice knot

Here a link is called *freely slice* if there are slice discs whose complement has free fundamental group. Conjecture 1.1 was stated explicitly in [CFT09, Conjecture 1.1], but was implicit in several earlier works such as [Fre88, FL89, FT95b].

The only if direction of the conjecture implies that topological surgery does not work for the rank two free group [CF84, FL89]. Freedman confirmed the conjecture for knots and 2-component links [Fre88]. See [Kru99, Kru08, Kru15] for recent progress towards the only if direction in the 3-component case.

The best known result toward the if direction of Conjecture 1.1 is a theorem of Freedman and Teichner that if a link L is homotopically trivial$^+$, then $\mathrm{Wh}(L)$ is (freely) slice [FT95b], where L is said to be *homotopically trivial*$^+$ if any link obtained from L by adjoining a zero-linking parallel copy of one of the components is homotopically trivial.

Since Conjecture 1.1 remains open, this approach is not sufficient to slice the knots of Corollary E when one uses a link L which is homotopically trivial but not homotopically trivial$^+$. For instance, this is the case for the knot of Figure 1.

Homotopy ribbon-slice conjecture. Recall that the ribbon slice conjecture claims that every slice knot is a ribbon knot. More precisely, the statement depends on the category: in the smooth case, one asks whether a knot bounds a smooth slicing disc if and only if it is a ribbon knot. In the topological case, first note that not all slice knots are ribbon, since there are slice knots which are not smoothly slice. Following [CG83], we say that a knot K in S^3 is *homotopy ribbon* if there is a slicing disc Δ in D^4 for which the inclusion induces an epimorphism $\pi_1(S^3 \smallsetminus K) \twoheadrightarrow \pi_1(D^4 \smallsetminus \Delta)$. As stated in, for instance, [LO00, Section 3.5] (also cf. [CG83, Question 6.2], [Kir95, Problem 4.22]), one asks the following in the topological category.

Conjecture 1.2 (Homotopy Ribbon Slice Conjecture)**.** *A knot is slice if and only if it is homotopy ribbon.*

We remark that the homotopy ribbon property is essential in the study of ribbon obstructions in the literature. See for example [CG83, Bon83, CG86, Miy94, Fri04, DN06].

Our geometric method, which applies the disc embedding theorem directly to construct a slicing disc, gives potential counterexamples to the homotopy ribbon slice conjecture. In particular, we ask an explicit question:

Question 1.3. Is the slice knot in Figure 1 homotopy ribbon?

We remark that no potential counterexample to the homotopy ribbon slice conjecture was known—more precisely, every previously known slice knot is either smoothly slice or homotopy ribbon, or both. For instance, many slice knots in the literature which are not

smoothly slice are knots with Alexander polynomial one. These are homotopy ribbon by Freedman's result. Slice knots obtained by the results in [FT05] are homotopy ribbon, too. Several papers in the literature (e.g. [HLR12, CHH13, CH15]) also consider slice knots produced by satellite constructions using companion knots of Alexander polynomial one. They are all homotopy ribbon. The essential reason for this is that all use (building blocks obtained by) a topological surgery construction of a slice disc exterior, with the fundamental group of a ribbon disc exterior, as in [FQ90] and [FT05]. The smoothly slice knots presented in [GST10, AT13], which are not known to be ribbon, can be seen to be homotopy ribbon by inspecting their constructions.

We remark that the if direction of Conjecture 1.1 would imply that the slice knots given by Corollary E are homotopy ribbon, and in particular that the answer to Question 1.3 is yes. It is also interesting to note that according to [CF84, Proposition 2], in order to solve 4-dimensional surgery problems, one needs the Whitehead doubled links in Conjecture 1.1 to be homotopy ribbon. We discuss more related questions at the end of Section 5.

Smooth status. The smooth status of our knots is also interesting. We think our knots are unlikely to be smoothly slice (particularly when all the clasps have the same sign); compare [Lev12]. For some special cases of Corollary E, we computed the Rasmussen s-invariant to be nonzero, aided by a computer. Thus at least some of our examples are not smoothly slice. Recall that the Alexander polynomial satisfies $\Delta_K(t) = \Delta_R(t)$, in the notation of Corollary E. The following natural question arises.

Question 1.4. Is the slice knot in Figure 1 smoothly concordant to an Alexander polynomial one knot?

In particular for the knot K of Figure 1 we have

$$\Delta_K(t) = \Delta_{8_8}(t) = 2t^{-2} - 6t^{-1} + 9 - 6t + 2t^2 = (2t^2 - 2t + 1)(2t^{-2} - 2t^{-1} + 1).$$

We think the answer to Question 1.4 is likely to be no, but we do not know at present how to perform the computation of d-invariants which we think will be necessary to prove this. The existence of topologically but not smoothly slice knots with this property was shown in [HLR12]. We remark that their examples were constructed using satellite operations which tied in Alexander polynomial one knots. We conjecture that there are slice knots produced by Corollary E, which are linearly independent from the examples in [HLR12] in the smooth knot concordance group modulo Alexander polynomial one knots.

New slice links. Using (distorted) Casson towers of height 4 and Theorem A, we prove the following two results on links. To state the first, we consider the following operation, which is called *ramified Whitehead doubling*: for a given knot, take some number of un-twisted parallel copies, and then replace each parallel copy by its untwisted Whitehead double. Either sign may be used for the clasp. We may iterate, by applying this operation again to each component produced in the previous step. If we repeat this n times, where the number of parallel copies used in each iteration may change, then we say that the result is obtained by an *n-fold ramified Whitehead doubling*. Define a *ramified* Wh_n *link* to be a link obtained from the Hopf link by replacing one component with its n-fold ramified Whitehead double.

Theorem F. *Any ramified* Wh_n *link is slice for* $n \geq 4$.

Freedman showed that the unramified Wh_3, and consequently unramified Wh_n for $n \geq 3$, are slice [Fre88]. Although Theorem F is for $n \geq 4$ only, the ramified $n = 4$ case gives new slice links. The case $n \geq 5$ was shown by Gompf-Singh [GS84].

As a second application of Theorem A, we prove the following:

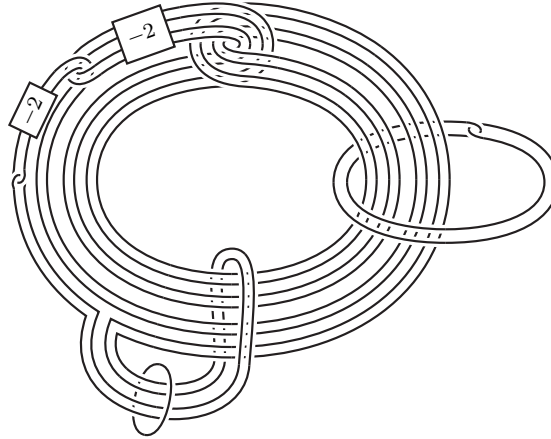**Theorem G.** *The link in Figure 2 is slice.*

FIGURE 2. A slice link obtained from a distorted Casson tower of height 4. Each box labeled -2 designates two left-handed full twists.

In fact, more generally, we specify a class of slice links, related to distorted Casson towers of height 4, which contains the link in Figure 2. See Section 6.2, and particularly Theorem 6.1.

We finish the introduction with a couple of additional remarks. Our proofs of slicing results are more geometric and direct than many previous applications of the disc embedding theorem to slicing knots and links. Our method is similar in character to Kervaire and Levine's programme [Ker65, Lev69] for high dimensional knots, in that we construct the *slice discs* in D^4, while most previous slicing results (e.g. [Fre88, FT95b, FT05, CFT09, Dav06]) proceed by first constructing a slice disc *exterior* using 4-dimensional topological surgery, following the homology surgery slicing strategy of Cappell-Shaneson [CS74]. The only exception known to the authors is an alternative proof of Garoufalidis-Teichner that Alexander polynomial one knots are slice [GT04]. Of course there is a reason for this: the surgery method is often remarkably effective.

Recall that while one can visualise a ribbon disc, (for example by drawing "movie pictures" of cross sections), it is nigh on impossible to visualise a slice disc for a knot or link which is topologically but not smoothly slice. Using the slicing theorems of this paper, one can at least understand a neighbourhood, in the 4-ball, used in the construction of the disc. The reader may also perhaps see it as virtuous, when constructing slice discs, to minimise the number of times Freedman's disc embedding theorem is used; the slicing constructions of this paper only use it once per slice disc. This can be contrasted with the number of Freedman discs required to employ topological surgery and h-cobordism when using the homology surgery method, or even in Garoufalidis-Teichner [GT04].

Organisation of the paper. In Section 2 we give preliminary definitions of Casson towers, gropes, and related objects. In Section 3, we prove that the existence of a height 1.5 grope with a certain fundamental group condition gives rise to a flat embedded disc with the desired boundary. In Section 4 we prove our main disc embedding results for Casson towers: Theorems A, B, and C. Sections 5 and 6 present new slice knots and links respectively. Section 7 discusses the grope filtration of knots and Casson towers.

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2. Preliminary definitions

All 4-manifolds in this paper are compact and oriented. First we give the definitions of a
Casson tower and a grope, which are the two main objects we will be working with.

Definition 2.1 ([Cas86])**.** A *Casson tower* is a 4-manifold T with a framed embedded
circle $C = C(T)$ in its boundary, defined as below. We write ∂T as a union of two
codimension zero submanifolds ∂_+ and ∂_-, where ∂_- is a (closed) tubular neighbourhood
of C in ∂T and $\partial_+ = \overline{\partial T \smallsetminus \partial_-}$. We call C and ∂_- the *attaching circle* and the *attaching
region* respectively.

A Casson tower has a height $n \in \mathbb{N}$. A *Casson tower T_1 of height 1*, which we will also
call a *plumbed handle*, is a thickened disc $D^2 \times D^2$, with $C := \partial D^2 \times 0$, $\partial_- := \partial D^2 \times D^2$, and
with some number of self-plumbings performed in the interior of $D^2 \times 0$. A self-plumbing
is performed by taking two discs $D_1, D_2 \subset \operatorname{Int} D^2 \times 0$ and then identifying $D_1 \times D^2$ and
$D_2 \times D^2$, viewing each 2-disc as the unit disc in $\mathbb{C}$, via $(z, w) \sim (w, z)$ to produce a positive
self-intersection of $D^2 \times 0$, or $(z, w) \sim (\bar{w}, \bar{z})$ to produce a negative self-intersection. The
core disc is defined to be the image of $D^2 \times 0$ in T_1, which is now an immersed disc.
The attaching circle C is framed by the restriction of the unique framing of $D^2 \times 0$ *before*
plumbing. Equivalently, if we perform k_+ positive and k_- negative plumbings, then the
framing on C is obtained by twisting the restriction to C of the unique framing of the
core disc in T_1 by $k_- - k_+$. To each double point of the core disc, there is an associated
double point loop on the core disc, which departs the double point into a sheet and comes
back through the other sheet, avoiding all other double points. Isotope this loop to obtain
an embedded circle in ∂_+, which we call a *double point loop in ∂_+*. We assume that
double point loops are disjoint. Double point loops are framed in such a way there is a
diffeomorphism of the plumbed handle with 2-handles attached along this framing to the
4-ball which takes the attaching circle to the zero framed unknot in the 3-sphere. There is
a unique such framing.

The framings can be explicitly described using a standard Kirby diagram of a plumbed
handle, shown in Figure 3: the circles a_i are double point loops on ∂_+, and the framings
on C and a_i described above are the zero-framing in Figure 3 (see [Cas86, Lemma 2 of
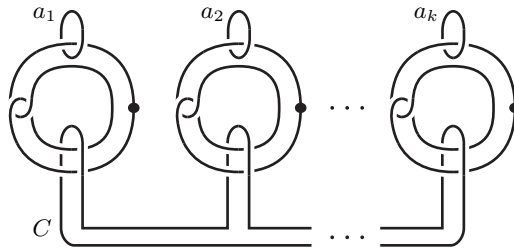Lecture I]).

FIGURE 3. A standard Kirby diagram of a plumbed handle, together with
the attaching circle C and double point loops a_i on the boundary of the
plumbed handle. Each plumbing corresponds to a dotted circle, where the
sign of the plumbing determines the sign of the clasp.

For $n \geq 2$, a Casson tower of height n, denoted T_n, is constructed inductively by taking
a height one Casson tower and, for each double point loop, identifying a neighbourhood of

the double point loop on ∂_+ with the attaching region of some Casson tower T_{n-1} of height $n-1$, along the preferred framings of the double point loop and $C(T_{n-1})$. A different height $n-1$ Casson tower may be used for each double point loop. The attaching circle $C(T_n)$ and the attaching region ∂_- of the new Casson tower T_n are just those of the original height one tower.

Definition 2.2. The *kth stage* of a Casson tower T_n is the material that was introduced by the $(n-k+1)$th inductive step in the construction, where taking a height one tower counts as the first step. Following [Fre82b], we denote the union of the stages of a Casson tower T from p through q inclusive by T_{p-q}.

Definition 2.3 ([FQ90, FT95a]). A *grope* of height n ($n \in \mathbb{N}$) is a pair (2-complex, base circles) of a certain type described inductively below. A *grope of height* 1 is a disjoint union of oriented connected surfaces each of which has connected nonempty boundary. The boundary circles are the base circles. Take a grope G_1 of height 1, and let $\{\alpha_1, \ldots, \alpha_{2g}\}$ be a standard symplectic basis of circles for the first homology of G_1. Then a *grope of height* $n+1$ is formed by, for each i, attaching some grope of height n with a single base circle to G_1, identifying the base circle of this height n grope with α_i. A different height n grope may be used for each α_i. The base circles of the height $n+1$ grope are defined to be the base circles of G_1. We often call the base circles the *boundary* of the grope. The *kth stage* of the grope is the union of the surfaces that were introduced by the $(n-k+1)$th inductive step in the construction, where taking a height one grope counts as the first step. We denote a grope of height n by G_n, and denote the stages from p through q inclusive by G_{p-q}.

A *capped grope* is constructed by attaching a disc to each of a symplectic basis of curves for the top stage surfaces. These discs are referred to as the *caps*. The surface stages are called the *body* of the grope. A capped grope of height n, sometimes also known as a *capped grope with n surface stages*, will be denoted G_n^c.

We note that, in this paper, a capped grope has a multi-component body in general. It is called a *union-of-discs-like* capped grope in [FQ90]. As a special case, if the body is connected, it is called a *disc-like* capped grope. A 2-complex obtained by attaching 2-discs to capped gropes along each boundary component is called a *union-of-spheres-like* capped grope. These only differ from union-of-discs-like capped gropes in that the bottom stage surfaces are closed. In this paper a capped grope refers by default to a union-of-discs-like capped grope. On the other hand, in the sequel, we will construct "transverse" capped gropes by glueing together union-of-discs-like capped gropes. Transverse capped gropes are always union-of-spheres-like.

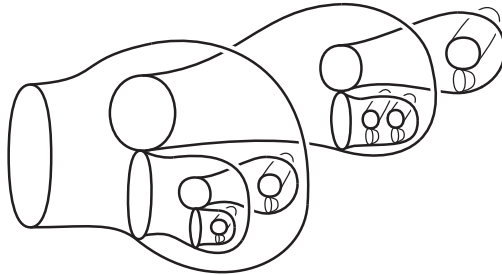As shown in Figure 4, a capped grope has a standard model embedded in $\mathbb{R}^3$.

FIGURE 4. A standard model of a capped grope of height 3 in $\mathbb{R}^3$.

Definition 2.4. Start with a model (capped) grope embedded in $\mathbb{R}^3$, and embed this in $\mathbb{R}^4$ via $\mathbb{R}^3 \hookrightarrow \mathbb{R}^3 \times \mathbb{R} \cong \mathbb{R}^4$. Take a thickening of the model in $\mathbb{R}^4$. We refer to this

thickening as a *framed (capped) grope*, which is a compact 4-manifold with boundary. The *attaching region* ∂_- of a framed (capped) grope is the compact 3-manifold with boundary given by thickening the base circles of the bottom surface stage. The attaching region is a submanifold of the boundary of the framed (capped) grope.

Note that each surface and disc component of a model (capped) grope in $\mathbb{R}^4$ has a canonical framing of its normal bundle given by taking a 1-dimensional framing in $\mathbb{R}^3$, and extending via the trivial line bundle when we take the product with $\mathbb{R}$.

We always regard the base circles of a framed (capped) grope as framed circles endowed with the induced framing. Similarly, the symplectic basis curves of the top stage surfaces of a framed grope have an induced framing. In the case of a framed capped grope, the framing for the symplectic basis curves is equal to the restriction of the unique framing of the caps.

The next definition is of proper immersions. Briefly, a properly immersed grope has embedded body, and immersed caps which are disjoint from the body.

Definition 2.5 ([FQ90])**.** Take a model framed capped grope as in Definition 2.4, and introduce plumbings into the model, by plumbing together caps and introducing self-plumbings in caps. A *proper immersion* of a capped grope into a 4-manifold M with boundary is an embedding of this plumbed model into M such that the attaching region ∂_- maps to ∂M.

We remark that for a proper immersion it is allowed to plumb two caps attached to any, possibly different, body components.

We will also denote a framed grope of height n by G_n, and a framed capped grope of height n by G_n^c. From now on when we refer to a capped grope, we will mean a framed capped grope, often a properly immersed framed capped grope. However we will also refer to geometric operations, such as tubing and taking parallel copies, on surfaces which are part of the underlying 2-complex, hopefully without causing confusion. When there is concern about ambiguity, we will denote a (further) thickening of a (framed) capped grope by νG_n^c.

We will be interested in improving a capped grope to a *one storey capped tower*. Briefly, a proper immersion of a one storey capped tower is a capped grope with caps for the caps, that is, discs bounded by the double point loops of the caps. The first layer of caps should have self-intersections only. The second layer of caps, called the *tower caps*, must be disjoint from the body *and* the caps of the capped grope. The official definition is next.

Definition 2.6 ([FQ90])**.** A *one storey capped tower* is obtained from a model framed capped grope by performing finger moves that introduce self-plumbings into the caps, and then by adjoining disjoint Whitney discs and accessory discs. We call the Whitney discs and accessory discs the *tower caps*. A *proper immersion* of a one storey capped tower into a 4-manifold M is obtained by introducing plumbings (not necessarily just self-plumbings) into the tower caps and then embedding the plumbed model into M. Note that tower caps still miss the entire capped grope.

A reference for finger moves, Whitney discs, and accessory discs for readers not familiar with them is [FQ90, Sections 1.5 and 3.1]. In this paper, we do not need their definitions nor do we need any properties of tower caps, since we are not going to work with capped towers. Rather, once we have one we will observe (Lemma 3.5 and Theorem 3.6) that this is sufficient to activate the Freedman-Quinn machine and produce an embedded disc.

3. Obtaining a disc from a framed capped grope

This section proves Theorem 3.4 below, which is our main technical result. It sharpens the minimal grope data required to produce a disc, and will be used to deduce all of our

various statements on Casson towers. In order to state the theorem we will introduce some terminology and background.

Definition 3.1 ([FT95a])**.** We say that a group π is *good* if for any properly immersed disc-like capped grope G^c of height 2 (obtained from a model framed capped grope by plumbing) and for any homomorphism $\pi_1(G^c) \to \pi$, there is an immersed disc in G^c whose framed boundary is equal to the base circle of G^c and whose double point loops have trivial image in π.

In the book of Freedman and Quinn [FQ90], a different definition is used. They call a group good if the disc embedding theorem [FQ90, p. 5] holds. Definition 3.1, which is from [FT95a], describes a (potentially) smaller class of groups than the definition of [FQ90]. A good group in the sense of Definition 3.1 is sometimes called a *Null Disc Lemma (NDL) group*; see for example [KT01].

Definition 3.2. Following [KQ00, Section 3], a group π has *subexponential growth* if given any finite subset $S \subseteq \pi$, there is an integer n such that the set of all products of elements of S with length n determine fewer than 2^n elements of the group π.

Theorem 3.3 ([FT95a, KQ00])**.** *Any group of subexponential growth is a good group.*

In addition, the class of good groups is closed under extensions and direct limits [FT95a, Lemma 1.2].

For $n \in \mathbb{N}$, a *capped grope of height $n.5$* is constructed by attaching a height n capped grope to one curve from each dual pair of curves in a symplectic basis for the first homology of a height one grope, and then attaching height $n-1$ capped gropes to the remaining curves. (By convention, a height 0 capped grope is a union of discs.) A *proper immersion* is defined by allowing plumbings of the caps, similarly to the height n case.

Now we state the main theorem of this section. The theorem is probably known to the experts, but a detailed proof has not appeared.

Theorem 3.4 (Disc Embedding for Capped Gropes of Height ≥ 1.5)**.** *For some $n \in \frac{1}{2}\mathbb{N}$ which is at least 1.5, let $(G_n^c, \partial_-) \to (M, \partial M)$ be a properly immersed capped grope of height n in a 4-manifold M, and let νG_n^c be a further thickening of G_n^c. Suppose that the image of the inclusion induced map*

$$\pi_1(\nu G_n^c \smallsetminus G_{1-1}, *) \longrightarrow \pi_1(M \smallsetminus G_{1-1}, *)$$

is a good group, for all choices of basepoint $*$ in $\nu G_n^c \smallsetminus G_{1-1}$. Then there are disjoint flat embedded discs in M with the same framed boundary as G_n^c.

Recall that the body of a capped grope need not be connected. We also remark that in order to check that the hypothesis is true for all basepoints, it is enough to check it for some choice of basepoint in each connected component of $\nu G_n^c \smallsetminus G_{1-1}$.

Next we state some results from the literature which will be used during the proof of Theorem 3.4. In the following lemma, the phrase "G_n^c has transverse spheres" means that each connected component of the bottom stage of G_n^c has a transverse sphere intersecting G_n^c in precisely one point, which lies in that connected component of the bottom stage. Also, we say that a properly immersed capped grope G_n^c in M is *π_1-null* if any loop in the image of G_n^c is null-homotopic in M.

Lemma 3.5 ([FQ90, Section 3.3])**.** *Suppose $G_n^c \to M$ is a proper immersion of a capped grope of height n at least 2, into a 4-manifold M, which is π_1-null and has transverse spheres. Then the embedding of the body of G_n^c extends to a proper immersion of a union-of-discs-like one storey capped tower with arbitrarily many surface stages.*

The first step in the proof of Lemma 3.5 given in [FQ90, Section 3.3] uses grope height raising (see Lemma 3.7 below) to find a capped grope with arbitrary height in a neighbourhood of the given capped grope. This capped grope is then improved to a one storey capped

tower. Freedman and Quinn's statement begins with a proper immersion of a capped grope of height at least 3. In our statement we have replaced height 3 with height 2, in light of Lemma 3.7 below.

The strategy of the proof of Theorem 3.4 will be to arrange a situation where Lemma 3.5 can be applied. We will then be able to apply the following theorem of [FQ90].

Theorem 3.6 (Freedman-Quinn). *A neighbourhood of a properly immersed one storey capped tower with at least four surface stages contains an embedded flat topological disc with the same framed boundary.*

Proof. Follow the arguments of the second and third sentences of [FQ90, Proof of Theorem 5.1A]. The given data is sufficient to perform *tower height raising with control*. Begin with the tower height raising proposition in [FQ90, Section 3.5], and introduce control, to produce an infinite convergent tower as in [FQ90, Sections 3.6–8]. See in particular [FQ90, Proposition 3.8]. A convergent infinite tower is then shown to be homeomorphic, relative to its attaching region ∂_-, to an open 2-handle via "the design" and Bing shrinking [FQ90, Chapter 4]. See [FQ90, Theorem 4.1].   □

A key step in the proof of Theorem 3.4 is to perform *grope height raising*.

Lemma 3.7 (Grope Height Raising Lemma). *Let $G_{1.5}^c$ be a height 1.5 capped grope which is properly immersed in a 4-manifold. For any $n \in \mathbb{N}$ there exists, inside a neighbourhood of $G_{1.5}^c$, a properly immersed height n capped grope G_n^c with the same framed boundary.*

The statement that a height 1.5 capped grope can have its height raised arbitrarily is not contained in [FQ90] (the best statement given is in an exercise in their Section 2.7, and involves height 2.5). The outline of the proof of the Grope Height Raising Lemma was explained to the second author by F. Quinn and P. Teichner, in the discussion sessions associated to the MPIM lecture series of M. Freedman [FF]. For the convenience of the reader we include the details below, after stating and proving Lemma 3.8.

Consider a framed capped grope G_n^c. Divide the surfaces (including the caps) above the first stage into two sides, labelled as the + and − sides, as follows. For a dual pair of curves in a symplectic basis for the first stage surface, the surface attached to one curve is labelled + and the surface attached to the dual curve is labelled −. A surface of stage 3 or higher has the same label as that of the surface to which it is attached. We therefore have + and − side height $n-1$ capped gropes. When beginning with a height $n.5$ capped grope, choose labels so that we have + side height n capped gropes and − side height $n-1$ capped gropes. In particular, for height 1.5, we just have caps on the − side.

The following lemma is an important preliminary construction in the height raising process. When starting with a grope of height at least 3, this step can be avoided by using the argument of [FQ90]. When starting with a grope of height 1.5 or 2 however, the Cap Separation Lemma 3.8 below seems to be necessary.

Lemma 3.8 (Cap Separation Lemma). *For any $n \in \frac{1}{2}\mathbb{N}$ with $n \geq 1.5$, within a neighbourhood of a height n capped grope, there is a height n capped grope with the same framed boundary and with the + side caps disjoint from the − side caps.*

We believe some of the details of the proof we give to be new.

In the arguments below we will use the *symmetric contraction* described in [FQ90]. For readers not familiar with this notion, we state the key properties. Given a properly immersed capped surface Σ^c (i.e. a height 1 capped grope) in a 4-manifold, [FQ90, Section 2.3] associates an immersed genus zero surface Σ' in a given neighbourhood of Σ^c, with $\partial\Sigma' = \partial\Sigma^c$, such that any immersed surface S disjoint from the body of Σ^c is regularly homotopic to a surface S' disjoint from Σ', via a homotopy supported in a given neighbourhood of the union of the caps of Σ^c. We say that Σ' is obtained from Σ^c by *symmetric contraction*, and S' is obtained by *pushing S off the contraction*. We remark

that S' may have additional self-intersections; more generally, for two such surfaces S_1 and S_2, the pushed-off surfaces S_1' and S_2' may intersect in more points than their antecedents S_1 and S_2.

In the remainder of this paper, unless specified otherwise, 'contraction' and the verb 'to contract' refer by default to the symmetric contraction.

Proof of Cap Separation Lemma 3.8. We need to remove intersections between the $+$ and $-$ side caps. Let F_- denote the transverse capped grope for the $-$ side, which is constructed from two parallel copies of the $+$ side capped grope and an annulus which joins them together in a neighbourhood of the attaching circle for the $+$ side; see [FQ90, Section 2.6]. Note that the number of components of F_- is equal to the genus of the bottom stage surface.

Contract the top stage of F_- and push the $-$ side caps off the contraction. Note that when we push off the contraction we may obtain new intersections of the $-$ side caps, but that is acceptable. Contract further if necessary, to obtain a collection of transverse spheres F_-', each of which is dual to a $-$ side stage 2 surface or cap. We have that F_-' is disjoint from the $-$ side apart from the transverse point, but F_-' may intersect the $+$ caps, since we did not push those off the contraction. For a given $-$ cap we want one of these transverse spheres in order to eliminate intersections with the $+$ caps; for each such intersection, push the intersection down to a $-$ side surface of stage 2 if necessary, and tube the $+$ side cap into a parallel copy of the F_-' transverse sphere. We may obtain new intersections between $+$ side caps, since F_-' may intersect $+$ side caps, and from the fact that F_-' will probably not be embedded. But this is acceptable too. □

We are now ready to give the proof of Lemma 3.7. The proof uses arguments known to the experts; a variant appeared in [FQ90, Section 2.7]. Ours is based on that typed up by the second author and W. Politarczyk in [FF, Section 8.3], which itself derived from the Freedman lectures and discussion sessions.

Proof of Grope Height Raising Lemma 3.7. By the Cap Separation Lemma 3.8, we may suppose that we have a properly immersed grope of height 1.5 where the $+$ caps are disjoint from the $-$ caps.

Let F_- denote the transverse capped surface for the $-$ side, which is constructed from two parallel copies of the $+$ side capped surface. Tube each intersection of the $-$ side caps into a parallel copy of F_-. This turns the $-$ side caps into capped surfaces. We have now raised the height by one on the $-$ side. We started with $(+, -)$ heights being $(1, 0)$ and now we have $(1, 1)$.

Next we raise height on the $+$ side. To achieve this we repeat the above process, with $+$ and $-$ reversed. That is, first we apply Lemma 3.8 to once again separate the caps on the $+$ and $-$ sides. Then we let F_+ denote the transverse capped surface for the new $+$ side. Tube intersections of $+$ side caps into parallel copies of F_+. This creates a grope with $(+, -)$ height $(2, 1)$.

Now repeat as many times as desired, alternating $+$ and $-$ sides, but with $F_\pm$ as transverse capped gropes instead of capped surfaces. From $(2, 1)$ we go to $(2, 3)$, then $(5, 3)$, then $(5, 8)$ and so on. The $(+, -)$ heights grow according to the Fibonacci numbers. We remark that we could instead apply the argument in [FQ90, Section 2.7] once we get both sides of height at least 2; it raises height a little slower but creates fewer intersections along the way.

To finish, once both sides have height at least n, contract until both sides have height exactly n, so that we have a height n symmetric grope. □

Remark 3.9. The proof of Lemma 3.7 can be applied to raise the height of a grope of any height $n \geq 2$. In this case one obtains a Fibonacci sequence beginning with $(n - 1, n - 1)$.

With the preliminaries at last complete, there now follows the proof of Theorem 3.4.

Proof of Theorem 3.4. Our goal is to extend our capped grope to a properly immersed one storey capped tower, so that we can apply Theorem 3.6.

We may assume that $n = 1.5$, since the first step is to perform *grope height raising*. (One can either first contract the grope until it is of height 1.5, or use Remark 3.9.) In each step below, first we state what to do, and then discuss how.

Step 1. Extend the body of $G_{1.5}^c$ to a properly immersed framed capped grope G_6^c of height 6 in $\nu G_{1.5}^c$.

This is done by applying Grope Height Raising Lemma 3.7, to raise the height of $G_{1.5}^c$ by 4.5.

Step 2. Extend $G_{1-5} \subset G_6^c$ to a height 5 properly immersed framed capped grope G_5^c in a neighbourhood of G_6^c, such that all of the cap intersections of G_5^c are self-intersections.

The argument for Step 2 is as follows. Contract the top stage surfaces of G_6^c one at a time, inductively, pushing caps of all the other remaining top stage surfaces off the contraction before contracting the next top stage surface. The result is a height 5 capped grope G_5^c with the desired property.

Step 3. Modify G_5^c in such a way that each stage 2 surface Σ of G_5^c has a transverse sphere which meets G_5^c at exactly one point, and that point belongs to Σ; also, caps attached to distinct top level surfaces are disjoint.

This is achieved using what is by now a standard argument ([FQ90, Section 2.6]), as follows. We use the following notation: for a surface Σ in the body of a capped grope G^c, let G_Σ^c be the capped grope consisting of the surfaces and caps on the top of Σ, including Σ itself. Let Σ be a stage 2 surface of G_5^c, and let Σ' be the stage 2 surface dual to Σ i.e. the attaching curves of Σ and Σ' are dual curves on the first stage surface G_{1-1}. Construct a transverse capped grope of height 4 for Σ by taking two parallel copies of $G_{\Sigma'}^c$, and attaching an annulus cobounded by the boundary of the parallel copies, which lies in a regular neighbourhood of $\partial \Sigma'$, and which meets Σ at exactly one point. Note that the caps of the transverse capped grope may meet the caps of $G_{\Sigma'}^c$, but no other caps. Contract the top stage of the transverse capped grope, push intersections with caps of $G_{\Sigma'}^c$ off the contraction, and then totally contract the remaining stages of the transverse capped grope. This gives us a new capped grope, which we still call G_5^c, together with transverse spheres for the stage 2 surfaces. Each transverse sphere meets the new G_5^c at a single point. The transverse spheres are not mutually disjoint, but that is permitted. In particular, the two transverse spheres associated to Σ and Σ', which are attached to a dual pair of curves on a stage 1 surface, intersect each other in two points. These two points lie in a neighbourhood of the intersection point between the pair of curves on the stage 1 surface. Note that while the push off operation introduces intersections of caps which are not self-intersections, caps of the new G_5^c which are attached to distinct top stages are still disjoint, since the top stage contraction-push-off can only introduce intersections between caps attached to the same top stage surface. This uses Step 2.

Step 4. Extend $G_{1-3} \subset G_5^c$ to a properly immersed framed height 3 capped grope G_3^c whose caps lie in a regular neighbourhood of G_5^c and whose double point loops are null homotopic in $M \smallsetminus G_{1-1}$.

Consider a stage 4 surface Σ of G_5^c. Choose a regular neighbourhood W_Σ of G_Σ^c in the exterior of G_{1-3}. Note that G_Σ^c has height 2. The capped gropes G_Σ^c are mutually disjoint by the penultimate sentence of Step 3. Thus we may assume that the neighbourhoods W_Σ are disjoint. Since $W_\Sigma \subset \nu G_n^c \smallsetminus G_{1-1}$, the image of $\pi_1(W_\Sigma) \to \pi_1(M \smallsetminus G_{1-1})$ lies in the image of the fundamental group of the component of $\nu G_n^c \smallsetminus G_{1-1}$ containing Σ, which is a good group, by the hypothesis. By Definition 3.1, it follows that there is an immersed disc in W_Σ, whose boundary is equal to the base circle of G_Σ^c, and whose double point loops

are trivial in $\pi_1(M \smallsetminus G_{1-1})$. Replacing each G_Σ^c with such an immersed disc, we obtain the desired properly immersed framed capped grope of height 3.

Step 5. Construct a one storey capped tower and then a flat embedded disc.

Now consider the union of G_F^c taken over all stage 2 surfaces F of G_3^c. This is a capped grope of height 2, which is properly immersed in $M \smallsetminus G_{1-1}$, is π_1-null, and has transverse spheres, by Steps 3 and 4. It follows from Lemma 3.5 that the body of $\bigcup_F G_F^c$ extends to a properly immersed one storey capped tower with 3 surface stages in $M \smallsetminus G_{1-1}$. Attach this one storey capped tower to G_{1-1}, to obtain a one storey capped tower with 4 surface stages. Theorem 3.6 then yields a flat embedded disc as claimed. $\qquad\square$

4. Casson towers of height four, three and two

In this section we apply Theorem 3.4 on gropes to obtain results on Casson towers. We consider Casson towers of height four, three and two, in that order. As the height of the Casson tower decreases, we need stronger assumptions on fundamental groups in order to deduce the existence of embedded discs.

The following construction of Ray allows us to pass from Casson towers to gropes. Recall that the symplectic basis curves of top stage surfaces of a framed grope is framed by the induced framing, and double point loops of a plumbed handle are framed as in Definition 2.1.

Proposition 4.1 ([Ray13, Proposition 3.1]). *A Casson tower T of height n contains an embedded framed grope G_n of height n with base circle equal to the attaching curve $C(T)$ as framed circles. Moreover the union of the standard symplectic basis curves on the top stage surfaces of G_n is, as a framed 1-submanifold, isotopic to the union of 2^n parallel copies of the double point loops of the top stage of T, via disjointly embedded framed annuli whose interior is disjoint from G_n.*

We note that the first stage surface G_{1-1} of the grope $G \subset T$ is denoted by $\Sigma(T)$ in the introduction.

To employ the grope technology, we need capped gropes. The following innocent observation is useful in producing capped gropes in a Casson tower. We will also present a more involved construction of a capped grope in Lemma 4.5.

Lemma 4.2 (Capped grope in a Casson tower). *A Casson tower T of height $n+1$ contains a properly immersed capped grope G_n^c of height n, with base circle equal to $C(T)$ as framed circles. The body of G_n^c is the grope G_n for the subtower T_{1-n} from Proposition 4.1.*

Proof. Let G_n be the framed embedded grope in T_{1-n} obtained by applying Proposition 4.1 to T_{1-n}. We will attach caps to G_n, constructed from parallel copies of the core discs of the top stage plumbed handles of T (together with parallels of the annuli given in Proposition 4.1). The only issue is that the caps should be framed. For this purpose, we arrange that the top stage core discs of T induce the preferred framing on the double point loops of the nth stage plumbed handles to which they are attached. That is, each core disc of a stage $n+1$ plumbed handle should have the signed count of its self-plumbings equal to zero. We achieve this by locally introducing the requisite number of self-plumbings of appropriate sign. Now, from this and from the framing property in Proposition 4.1, it follows that we obtain framed caps. Thus we have a properly immersed framed capped grope G_n^c extending G_n, inside T. $\qquad\square$

We also need the following lemma on a fundamental group arising from the construction in Proposition 4.1.

Lemma 4.3. *Let T be a Casson tower and let $\Sigma = \Sigma(T)$. Then*

$$\pi_1(T_{1-1} \setminus \Sigma) \cong \langle \mu, a_1, \ldots, a_k \mid [\mu, a_i], \ i = 1, \ldots, k \rangle$$

where μ is a meridian to Σ and the a_i are the double point loops of T_{1-1}.

Proof. For convenience, we assume that the first stage of T has one (negative) double point. We will indicate along the way how to adapt the proof for the general case.

We recall Ray's construction from [Ray13, Proof of Proposition 3.1]. The first stage surface Σ is shown in Figure 5, where the plumbed handle T_{1-1} is described as a Kirby diagram. More precisely, Σ is obtained by pushing the interior of the surface in Figure 5 slightly into the interior of T_{1-1}. The curve a_1 is the double point loop, which is the attaching circle for the next stage of the Casson tower T.
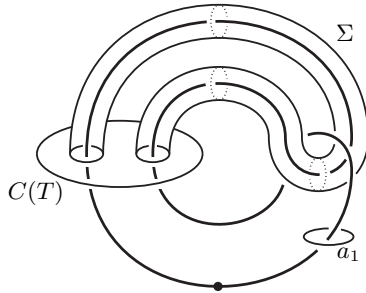

FIGURE 5. Kirby diagram of a plumbed handle together with Ray's genus one surface.

We remark that the commutator relation $[\mu, a_1]$ can be seen as follows: the normal circle bundle of Σ restricted on one of the dotted circles in Figure 5 is a torus disjoint from Σ, whose symplectic basis curves are (isotopic to) a meridian of Σ and the curve a_1. In what follows we will prove that this commutator relation suffices to present $\pi_1(T_{1-1} \setminus \Sigma)$.

Consider a collar neighbourhood $\partial T_{1-1} \times I$ of ∂T_{1-1}. We may assume that the height function $\partial T_{1-1} \times I \to I$ restricts to a Morse function for Σ with 3 critical points, corresponding to two 1-handles and one 2-handle of Σ; the 1-handles are shown as dashed lines in Figure 6.
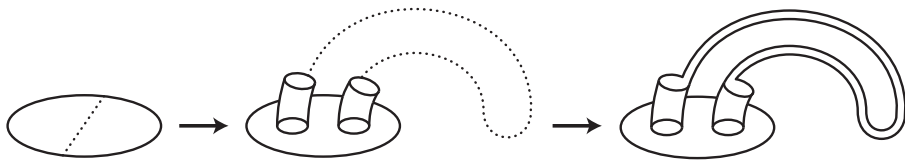

FIGURE 6. A handle decomposition of the surface Σ.

The handle decomposition of Σ gives rise to a handle decomposition of the exterior of Σ in the collar neighbourhood $\partial T_{1-1} \times I$ of ∂T_{1-1}:

$$(\partial T_{1-1} \times I) \setminus \Sigma = (\partial T_{1-1} \setminus \partial \Sigma) \times I \cup (\text{two 2-handles}) \cup (\text{one 3-handle}).$$

An i-handle in a handle decomposition of a surface embedded in a 4-manifold corresponds to an $(i+1)$-handle in a handle decomposition of the exterior of the surface (for a proof, see for example [GS99, Proposition 6.2.1]). The handle attachments are shown in Figure 7, where the attaching circles and spheres are drawn with dashed lines. The dashed lines showing the 1-handle attachments in the left and middle diagrams of Figure 6 correspond to the 2-handle attachments in the construction of the exterior of the surface shown in the top left and top centre diagrams of Figure 7.
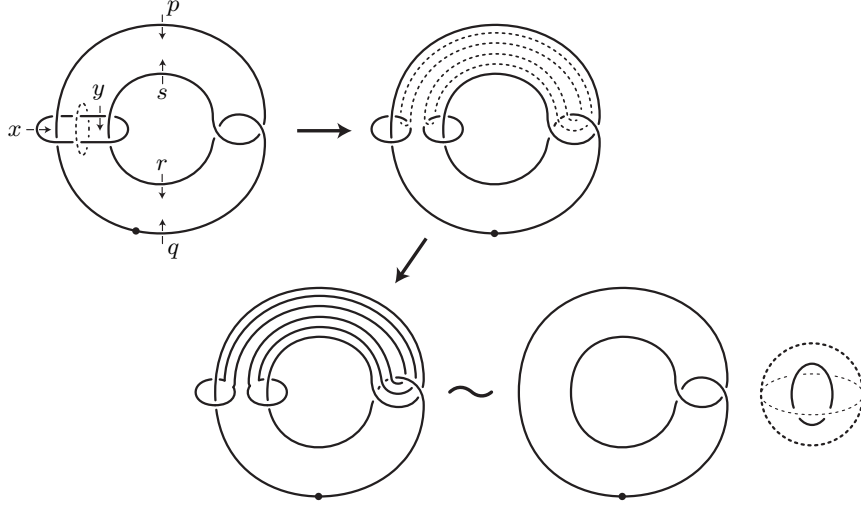
FIGURE 7. A handle decomposition of $(\partial T_{1-1} \times I) \smallsetminus \Sigma$. An arrow indicates the attaching of a handle, and the $\sim$ symbol indicates an isotopy. It is much easier to draw the attaching 2-sphere for the 3-handle after the isotopy.

As shown in Figure 5, $\partial T_{1-1} \smallsetminus \partial \Sigma$ is the exterior of the Whitehead link, with a boundary component (corresponding to the dotted circle) filled in with a solid torus along the zero framing. Therefore, starting with the Wirtinger presentation

$$\langle x, y, p, q, r, s \mid y = p^{-1}xp, \; q = xpx^{-1}, \; r = sqs^{-1}, \; s = q^{-1}pq, \; r = xsx^{-1} \rangle,$$

of the Whitehead link, where the generators are those shown in the first diagram in Figure 7, and then adding three more relators

$$x^{-1}s^{-1}xq^{-1}p^2, \; xy^{-1}, \; xqx^{-1}q^{-1}$$

which are from the Dehn filling and the 2-handle attachments respectively, we obtain a presentation of $\pi_1((\partial T_{1-1} \times I) \smallsetminus \Sigma)$. Simplifying the presentation, we obtain:

$$\pi_1((\partial T_{1-1} \times I) \smallsetminus \Sigma) \cong \langle x, q \mid [x, q] \rangle.$$

Observe that $x = \mu$ and $q = a_1$. In the case that the clasp is a positive clasp, the relators $r = sqs^{-1}$ and $s = p^{-1}qp$ are replaced by $s = rpr^{-1}$ and $r = p^{-1}qp$. It is not too hard to check that the computation above has the same outcome with these alterations. The relator corresponding to Dehn filling also changes to $x^{-1}pxrp^{-2}$, but this relator is superfluous to simplifying the presentation in both cases.

Turn the handle decomposition of T_{1-1}, into a 0-handle and a 1-handle, given by the Kirby diagram in Figure 5 upside down. We see that $T_{1-1} \smallsetminus \Sigma$ is obtained by attaching a 3-handle and a 4-handle to $(\partial T_{1-1} \times I) \smallsetminus \Sigma$. In general, turning a handle decomposition

$$\bigcup h_0 \cup \cdots \cup \bigcup h_3$$

of a connected 4-manifold with nonempty boundary $(M, \partial M)$ upside down gives us a decomposition rel. boundary

$$\partial M \times I \cup \bigcup h_3^* \cup \cdots \cup \bigcup h_0^*,$$

where h_i^* is the $(4-i)$-handle dual to the i-handle h_i. Since neither a 3-handle nor a 4-handle affect the fundamental group,

$$\pi_1(T_{1-1} \smallsetminus \Sigma) \cong \pi_1((\partial T_{1-1} \times I) \smallsetminus \Sigma) \cong \langle \mu, a_1 \mid [\mu, a_1] \rangle.$$

For $k > 1$ double points, take k copies of Figure 5, with the crossings in the clasp switched where appropriate, and connect sum the $C(T)$ curves together. This performs a boundary connect sum operation on the surfaces. Similarly, take multiple copies of the first three diagrams of Figure 7, and connect sum the $C(T)$ curves together i.e. the copies of the curve with meridians x and y in the first diagram of Figure 7. This composite curve represents $\partial\Sigma$. There is still only one 2-handle of Σ, therefore only one 3-handle of $(\partial T_{1-1} \times I) \smallsetminus \Sigma$. So in the ramified case the analogue of the final diagram of Figure 7 will still have just a single dashed 2-sphere. The Seifert-Van Kampen theorem applies to show that the effect of the connect sum operations is to identify the meridians labelled x in all the copies of the diagrams from Figure 7; all of these become the meridian μ. Indeed this is the only effect. By the computation above, μ commutes with all the double point loops. We therefore have the presentation

$$\pi_1(T_{1-1} \smallsetminus \Sigma) \cong \langle \mu, a_1, \ldots, a_k \mid [\mu, a_i], \, i = 1, \ldots, k \rangle. \qquad \square$$

4.1. Casson towers of height four and three

In this subsection we will prove Theorems A and B from the introduction.

Definition 4.4. Take a Casson tower of height n and introduce any number of plumbings between any top stage handle and any other handle in stage two or higher. A 4-manifold with boundary, together with a framed embedded circle C in its boundary, that is obtained in this way, is called a *distorted Casson tower of height n*.

Note that a Casson tower of height n is a distorted Casson tower of height n. As another example, a distorted tower of height n may arise if we have a height $n-1$ tower T embedded in a 4-manifold $(M, \partial M)$ with $\partial_- \subset \partial M$, and the double point loops of the top stage are null-homotopic in the complement in M of the first stage T_{1-1} of T. Then a neighbourhood of the union of the height $n-1$ tower and the null-homotopies of the double point loops gives rise to a distorted height n Casson tower. Some care is needed to frame the null-homotopies. Null-homotopies of different double point loops may intersect each other, and stages two or higher of the height $n-1$ Casson tower.

Theorem A says that a distorted Casson tower T of height 4 contains a disc bounded by $C(T)$. We note that we do not make any assumptions about embedding a distorted Casson tower of height 4; the distorted Casson tower itself is considered as the ambient manifold.

Proof of Theorem A. Let T be a distorted Casson tower of height 4. First, we apply Lemma 4.2 to T_{1-3} to obtain a properly immersed capped grope G_2^c in T_{1-3}, which is bounded by the framed circle $C(T)$.

Recall that a plumbed handle is diffeomorphic to a 4-ball with 1-handles attached, and that the fundamental group is generated by the double point loops. By induction, the fundamental group of a Casson tower is generated by the top stage double point loops. Applying this to our case, we see that the inclusion induced map $\pi_1(T_{2-3}) \to \pi_1(T)$ is trivial, since the 4th stage discs give null-homotopies for the double point loops of the 3rd stage plumbed handles. By Lemma 4.3 and a straightforward Seifert-Van Kampen theorem computation for $T_{1-3} \smallsetminus G_{1-1} = (T_{1-1} \smallsetminus G_{1-1}) \cup T_{2-3}$, it follows that the image of $\pi_1(T_{1-3} \smallsetminus G_{1-1})$ in $\pi_1(T \smallsetminus G_{1-1})$ under the inclusion induced map is isomorphic to $\mathbb{Z}$, generated by a meridian of G_{1-1}. From this it also follows that the image of $\pi_1(\nu G_2^c \smallsetminus G_{1-1})$ in $\pi_1(T \smallsetminus G_{1-1})$ is the same $\mathbb{Z}$. An infinite cyclic group has subexponential growth and is therefore good by Theorem 3.3, so the hypothesis of Theorem 3.4 is satisfied (in our case, G_2^c is connected and therefore so is $\nu G_2^c \smallsetminus G_{1-1}$). Applying Theorem 3.4 we can find a flat embedded disc inside T as claimed. $\qquad \square$

In order for a Casson tower of height 3 to suffice for the existence of an embedded disc, we will need to embed the tower into a 4-manifold, with a fairly strong assumption on

fundamental groups. Recall that for a Casson tower T, there is a surface $\Sigma(T)$ contained in the first stage T_{1-1}, by Proposition 4.1 (see also Figure 5). Theorem B says the following: *Let W be a 4-manifold with boundary and suppose that $T = \bigsqcup T_i$ is a collection of disjoint Casson towers T_i of height 3 in W such that $\partial_-(T) \subset \partial W$ and the image of $\pi_1(T_i \smallsetminus \Sigma(T_i)) \to \pi_1(W \smallsetminus \Sigma(T))$ is a good group for each i. Then the framed link $C(T) \subset \partial W$ is slice in W.*

We note that even if W is simply connected, it is quite possible that the image of this fundamental group in $\pi_1(W \smallsetminus G_{1-1})$ will not satisfy the good property hypothesis. For example if T has more than one component, $\pi_1(W \smallsetminus G_{1-1})$ might contain a non-abelian free group generated by meridians to the connected components of G_{1-1}.

Proof of Theorem B. The proof begins the same as the proof of Theorem A. Let G_2^c be the height 2 capped grope in T, constructed in the same way as in that proof. Note that now G_2^c may not be connected. A component of νG_2^c, say V, lies in some component T_i of T. The inclusion induced homomorphisms on fundamental groups factor as

$$\pi_1(V \smallsetminus G_{1-1}, *) \longrightarrow \pi_1(T_i \smallsetminus G_{1-1}, *) \longrightarrow \pi_1(W \smallsetminus G_{1-1}, *)$$

for any choice of basepoint $*$ in V. The hypothesis of the theorem, that the image of the second map is good, implies that the image of this composite homomorphism is also good. This holds because the good property is closed under taking subgroups. Thus we can apply Theorem 3.4 to obtain the flat embedded discs that we seek. $\qquad\square$

We remark that, as seen from the above proofs, Theorem A is indeed a consequence of Theorem B. We point out that so far we only used the Disc Embedding Theorem 3.4 for a height 2 capped group, although it holds for any height ≥ 1.5.

4.2. Casson towers of height two

This section contains our strongest conclusions in terms of the height of Casson towers, using the strongest assumption, namely triviality, on fundamental groups. Height 2 Casson towers seem to be the most useful for slicing knots and links in D^4, since in practice, contrivances notwithstanding, it is often difficult to construct tall Casson towers. Applications will be given in Section 5.

Our arguments for height 2 Casson towers also involve capped gropes. For this case, we need the full power, in terms of height, of Theorem 3.4. That is, we apply the theorem to a height 1.5 capped grope. Also, we need a new construction of a capped grope from an *embedded* Casson tower, which is given below. The construction relies upon the properties of the embedding of the tower, and will not work without an ambient manifold.

Proposition 4.5 (Capped grope from an embedded Casson tower)**.** *Suppose $n > 0$, (T, ∂_-) is a Casson tower of height $n + 1$ embedded in a 4-manifold $(M, \partial M)$, and the double point loops of the top stage of T are null-homotopic in $M \smallsetminus T_{1-n}$. Then there is a properly immersed capped grope $G_{n.5}^c$ of height $n.5$ in M, which extends the grope G_n for the subtower T_{1-n} from Proposition 4.1. In particular, the first stage surface of $G_{n.5}^c$ is $\Sigma(T)$, and the attaching circle of $G_{n.5}^c$ and $C(T)$ are equal as framed circles.*

In fact, we only need the $n = 1$ case of Proposition 4.5 in this paper, but we state and prove it for general $n > 0$ for possible later use, since this does not require any additional complication.

For the proof of Proposition 4.5, we begin with a couple of lemmata. The following lemma cleans up a certain type of naturally occurring capped grope to a properly immersed height 2 framed capped grope. We need to use the notion of a twisted cap, defined as follows. Suppose G is a framed grope embedded in a 4-manifold W. An immersed disc D in W bounded by a symplectic basis curve of a top stage surface Σ of G is called a ± 1 *twisted cap* if the interior of a collar neighbourhood of $\partial D \subset D$ is disjoint from G, and a

push-off of ∂D along Σ induces a section of the normal bundle of D with relative Euler number ± 1. That is, the push-off gives rise to the framing on ∂D obtained by twisting the restriction of the unique framing on D once (either positively or negatively).

Lemma 4.6. *Suppose we are given a height 2 capped grope G^c which is immersed (not properly) in a 4-manifold M, satisfying the following. The surface stages are disjointly embedded and framed. Each dual pair of curves on a second stage surface has two caps, one ± 1 twisted cap which is disjoint from the body of the grope, and one framed cap which potentially intersects other caps and second stage surfaces. Then there is a properly immersed height 1.5 framed capped grope in a neighbourhood of G^c with the same first stage surface.*

Proof. Divide the second stage surfaces and caps into two sides, the $+$ and $-$ sides, as described just prior to Lemma 3.8. There are two problems to be dealt with, namely the twisted caps need to be framed and their dual caps need to be made disjoint from the body of the grope. We have the freedom to reduce height by one on the $-$ side. We will modify the $-$ side first and then improve the $+$ side.

Call the ± 1 twisted caps the *small* caps, and the other caps, which can intersect second stage grope surfaces as well as each other, the *big* caps.

Forget the $-$ side big caps and apply the boundary twisting operation [FQ90, Section 1.3] to the $-$ side small caps, so that they are framed with respect to the $-$ side surfaces. This introduces an intersection of each $-$ side small cap with the $-$ side surface to which it is attached. Then use the $-$ side small caps to perform *asymmetric surgery* on the $-$ side surfaces, changing them to immersed discs, which are the new $-$ caps. The intersection of the small caps with the second stage surface introduced by the boundary twisting gives rise to self-intersections of the new $-$ caps. These new $-$ caps may also intersect other caps, but that is permitted.

Now create transverse spheres to the $+$ side surface stages using two parallel copies of the new $-$ side caps, and the annulus in the normal circle bundle to the attaching circle of the $-$ cap, as we have done in several other instances in this paper (see the first paragraph of the proof of the Cap Separation Lemma 3.8 and Step 3 of the proof of Theorem 3.4). The transverse spheres we construct are immersed and may intersect $+$ and $-$ side caps.

Boundary twist the $+$ side small caps to frame them with respect to the $+$ side surfaces. This creates intersections of the $+$ side small caps with the $+$ side second stage surfaces. Now we just have to remove intersections of the $+$ side caps, both big and small, with the $+$ side surfaces. To achieve this, tube anything that intersects a $+$ side surface into a transverse sphere. We obtain a properly immersed height 1.5 framed capped grope as required.   □

Lemma 4.7. *In a plumbed handle T with k plumbings, there is a genus k framed surface F with $\partial F = C(T)$ as framed circles, which has symplectic basis curves $\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k$ satisfying the following. Each α_i is dual to β_i, each α_i bounds a ± 1 twisted cap whose interior is disjoint from F. The union $\bigsqcup_i \beta_i$ is parallel to the union of the double point loops of T on $\partial_+ T$, via k framed annuli disjointly embedded in T, whose interior is disjoint from F and from the ± 1 twisted caps.*

Proof. The surface F is obtained from the core disc of T by a standard construction that resolves singularities by increasing the genus: replace the standard local model $(D^2 \times D^2, D^2 \times 0 \cup 0 \times D^2)$ of an intersection point by a twisted annulus in $S^3 = \partial(D^2 \times D^2) \subset D^2 \times D^2$ bounded by the Hopf link $S^1 \times 0 \cup 0 \times S^1$. To verify the framing assertions, we use a Kirby diagram argument as follows.

The Kirby diagram of a plumbed handle with k plumbings in Figure 3 is isotopic to the diagram in Figure 8. Observe that $C(T)$ bounds a surface F which is a band sum of k untwisted annuli with k twisted 1-handles attached; see Figure 8. Let α_i be the core circle
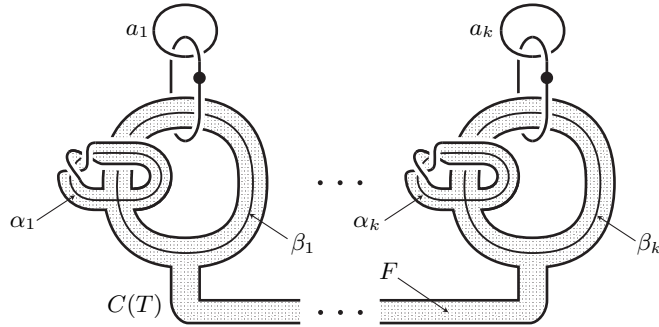
FIGURE 8. A Kirby diagram of a plumbed handle T with k plumbings, and a genus k surface F bounded by $C(T)$.

of the ith twisted 1-handle, and β_i be the core of the ith untwisted annulus; see Figure 8 again. The curves α_i and β_i form a symplectic basis.

From Figure 8, we see that F induces the zero framing on the β_i, and β_i is parallel to the double point loop a_i via a framed annulus inducing the 0-framing on both, as desired. Also, α_i bounds a ± 1 twisted cap whose interior lies in the interior of D^4 and so is disjoint from everything else. Since F is a Seifert surface for $C(T)$, F induces the zero framing on $C(T)$, which is the preferred framing by Definition 2.1. $\square$

Proof of Proposition 4.5. First apply Proposition 4.1 to find a framed grope G_n in T_{1-n} with framed boundary $C(T)$. Consider the top stage of T in $V := \overline{M - T_{1-n}}$. By Lemma 4.7, in each stage $n+1$ plumbed handle of T, we have a framed surface bounded by its attaching circle, with ± 1 twisted caps which we call a *small cap*, and annuli cobounded by the dual basis curve and the double point loop. By hypothesis, the double point loop is null-homotopic in V. Attach a null-homotopy to each annulus to obtain a cap dual to the ± 1 twisted cap; we call it a *big cap*. This terminology was already used in the proof of Lemma 4.6. We may assume that each big cap is framed, by applying boundary twist if necessary. This gives a capped grope of height 1, which is immersed in V but not properly immersed in general; the big caps may intersect other surfaces and caps.

Take 2^n push-offs of each of these height 1 capped grope and attach them to G_n to obtain a height $n+1$ capped grope G^c_{n+1}. The body of G^c_{n+1} and the big caps are compatibly framed, while the small caps are twisted. Now the big caps may intersect stage $n+1$ surfaces and other caps of G^c_{n+1}, but are disjoint from G_{1-n}, since the big caps lie in V. Note that now there are intersections between the small caps, which were introduced when we took push-offs, since the small caps are twisted. However the small caps are disjoint from the body of G^c_{n+1}.

Consider $G^c_{n-(n+1)}$, the top two surface stages of G^c_{n+1} together with all the small and big caps. Since $G^c_{n-(n+1)}$ is a height 2 capped grope satisfying the hypotheses of Lemma 4.6, there is a properly immersed height 1.5 capped grope with the same base surfaces, which lies in a neighbourhood of $G^c_{n-(n+1)}$. Replace $G^c_{n-(n+1)}$ in G^c_{n+1} with this height 1.5 capped grope to obtain a properly immersed capped grope $G^c_{n.5}$ of height $n.5$ as desired. $\square$

We are now ready to give the proof of Theorem C from the introduction, which says: *let W be a 4-manifold with boundary and suppose T is a Casson tower of height 2 embedded in W such that the second stage T_{2-2} of T lies in a codimension zero simply connected submanifold $V \subseteq \overline{W \smallsetminus T_{1-1}}$. Then the knot $C(T) \subset \partial W$ is slice in W.*

Proof of Theorem C. Apply Proposition 4.5 to the given Casson tower T of height 2, with $M := T \cup V$, to obtain a properly immersed capped grope $G^c_{1.5}$ in $T \cup V$.

Observe that $(T \cup V) \smallsetminus G_{1-1} = V \cup (T_{1-1} \smallsetminus G_{1-1})$, where V and $(T_{1-1} \smallsetminus G_{1-1})$ are glued along neighbourhoods of the attaching curves for T_{2-2}. By Lemma 4.3 and a straightforward application of the Seifert-Van Kampen theorem, it follows that $\pi_1(V \cup (T_{1-1} \smallsetminus G_{1-1})) \cong \mathbb{Z}$, which is good by Theorem 3.3. Apply Theorem 3.4 to $G_{1.5}^c$ in $M := T \cup V$, to yield a flat embedded disc in $T \cup V$ bounded by $C(T)$. $\square$

Remark 4.8. In the above proof of Theorem C, we have shown that $C(T)$ is slice in the submanifold $T \cup V \subset W$.

5. Slice knots

In this section we apply the results on Casson towers of Section 4 to produce a new family of slice knots in S^3.

5.1. Band sums of Whitehead doubles

In this subsection, as promised in the introduction, we use Theorem C to give a proof of Theorem D, which we state here again for the reader's convenience: *suppose L is an m-component homotopically trivial link, and K is a knot obtained from $\mathrm{Wh}(L)$ by applying $m-1$ band sum operations. Then K is slice.*

We begin, in Lemma 5.1, with a well-known observation on Whitehead doubles and plumbed handles. To state it we recall the definition of the (untwisted) Whitehead double of a framed link in a general 3-manifold. Let $\mathrm{Wh} \subset S^1 \times D^2$ be the standard untwisted Whitehead knot, that is, it is obtained by taking the exterior of a component of a Whitehead link and then identifying it with $S^1 \times D^2$ under the zero framing. (There are two possibilities, Wh_+ and Wh_-, depending on the sign of the clasp.) The zero framing on the Whitehead link induces a framing on $\mathrm{Wh} \subset S^1 \times D^2$ which we call the zero framing. For a framed link L in a general 3-manifold M, form an untwisted Whitehead double $\mathrm{Wh}(L)$ of L, which is a framed link, by replacing a tubular neighbourhood of each component of L with $(S^1 \times D^2, \mathrm{Wh})$ under the framing of L. We also recall that the attaching circle and double point loops of a plumbed handle are framed as in Definition 2.1.

Lemma 5.1 (Plumbed handles for Whitehead doubles)**.** *Suppose L is an m-component framed link in a 3-manifold M. Then there exist plumbed handles T_i $(i = 1, \ldots, m)$ disjointly embedded in $M \times [0,1]$ such that each T_i has exactly one self-plumbing with double point loop α_i, $\bigsqcup_i C(T_i) = \mathrm{Wh}(L) \times 0$, $\bigsqcup_i \alpha_i = L \times 1$ as framed links, $T_i \cap (M \times 0) = \partial_-(T_i)$, and $T_i \cap (M \times 1)$ is a tubular neighbourhood of $\alpha_i \subset \partial_+(T_i)$.*

We call the plumbed handles in Lemma 5.1 the *standard plumbed handles* between $\mathrm{Wh}(L)$ and L.

Proof. The best geometric way to understand our plumbed handles T_i is to construct the core discs directly: undo the clasp of the Whitehead doubling operations on L via a regular homotopy, and cap off the resulting trivial link with disjoint discs. We obtain an immersion of m discs in $M \times [0,1]$ bounded by $\mathrm{Wh}(L) \times 0$, and then by thickening this, we obtain the plumbed handles T_i. Furthermore, in this construction, by regarding the regular isotopy as a movie picture of the core discs, it can be seen that the double point loops on the core discs can be pushed to $L \times 1$ along embedded annuli. It follows that we may thicken the core discs in such a way that $T_i \cap (M \times 1)$ is a tubular neighbourhood of the double point loop $\alpha_i \subset \partial_+(T_i)$. The framing condition can also be verified by investigating the movie picture carefully.

The above assertions can be verified rigorously by the following alternative description. Recall that a plumbed handle T with one self-plumbing, together with the attaching circle C and the double point loop a_1, is described by the standard Kirby diagram in Figure 3 ($k = 1$ for now), where C and a_1 are zero framed by Definition 2.1. In particular $T \cong$

$S^1 \times D^3$. By straightening the dotted circle in the Kirby diagram, it follows that if we write $T = S^1 \times D^2 \times I$, then we may assume $C = \text{Wh} \times 0 \subset S^1 \times D^2 \times 0$ as framed circles, and $a_1 = S^1 \times 0 \times 1 \subset S^1 \times D^2 \times 1$, framed by the product structure. Now, the framing of L gives us an identification of a tubular neighbourhood of $L \times [0,1] \subset M \times [0,1]$ with $L \times D^2 \times I = \bigsqcup^m (S^1 \times D^2 \times I) = \bigsqcup^m T$, m disjoint plumbed handles. By the above and by the definition of $\text{Wh}(L)$, it follows that the attaching circles of these plumbed handles form the framed link $\text{Wh}(L) \times 0$, and the double point loops form the framed link $L \times 1$. $\qquad\square$

Proof of Theorem D. Attach the bands used in the band sum operations for $\text{Wh}(L) = \text{Wh}(L) \times 0$ to the annuli $\text{Wh}(L) \times [0, \frac{1}{2}] \subset S^3 \times [0, \frac{1}{2}]$ and push them slightly, to obtain a planar surface with $m + 1$ boundary components in $S^3 \times [0, \frac{1}{2}]$ cobounded by $K \times 0$ and $\text{Wh}(L) \times \frac{1}{2}$. The zero framings on $\text{Wh}(L)$ and K extend to a framing of the planar surface. Thicken the planar surface in $S^3 \times [0, \frac{1}{2}]$ and attach the standard plumbed handles in $S^3 \times [\frac{1}{2}, 1]$ between $\text{Wh}(L)$ and L given by Lemma 5.1. This constructs a single plumbed handle T_{1-1} embedded in $S^3 \times [0, 1]$. It has $K \times 0$ as the attaching circle and $L \times 1$ as the double points, by Lemma 5.1. We will use T_{1-1} as the first stage of a Casson tower.

Next, view $S^3 \times [0, 1]$ as a collar neighbourhood of the boundary of $D^4 = S^3 \times [0, 1] \cup_{S^3 \times 1}$ (smaller D^4). Since L is homotopically trivial, there are disjoint immersed discs in the smaller D^4, which can be thickened to plumbed handles whose attaching circles form $L \times 1$. We want to use these plumbed handles for the second stage. Observe the following general fact, which follows from Definition 2.1: the preferred framing of the attaching circle of a plumbed handle embedded in D^4 with $\partial_- \subset S^3$ is the zero framing in S^3. Apply this to our case: since the double point loops $L \times 1$ of T_{1-1} are zero framed by Lemma 5.1, it follows that we can attach these plumbed handles to T_{1-1}, in the smaller D^4, to yield a height 2 Casson tower, say T.

By construction, $C(T) = K$, the second stage T_{2-2} of T lies in the smaller D^4, and the first stage T_{1-1} lies in the collar $S^3 \times [0, 1]$ of the boundary of the bigger 4-ball D^4. Since the smaller D^4 is simply connected, we can apply Theorem C to obtain a flat embedded disc bounded by K as claimed. $\qquad\square$

The following is an immediate corollary. Recall that a link L in S^3 is *weakly slice* if it bounds a flat embedding of a planar surface in D^4.

Corollary 5.2. *The Whitehead double of any homotopically trivial link is weakly slice.*

Proof. If L is an m-component homotopically trivial link, then a knot K obtained by $m - 1$ bands sum operations on $\text{Wh}(L)$ is slice by Theorem D. Attaching the bands to a slicing disc and pushing them slightly, we obtain a punctured disc bounded by $\text{Wh}(L)$. $\qquad\square$

As another consequence of Theorem D, we prove Corollary E, which says the following: *suppose L is an m-component homotopically trivial link, and R is a ribbon knot. Consider a split union $\text{Wh}(L) \sqcup R$ in S^3, and choose m disjoint bands which join each component of $\text{Wh}(L)$ to R, such that in addition the bands are disjoint from an immersed ribbon disc for R in S^3 and are disjoint from Seifert surfaces for $\text{Wh}(L)$. Then the knot K obtained from $\text{Wh}(L) \sqcup R$ by these band sum operations along the arcs is slice.*

Proof of Corollary E. We first observe that a ribbon knot can be viewed as the result of band sum operations performed on a trivial link. Given a ribbon immersion $D^2 \looparrowright S^3$, by removing an ϵ-neighbourhood of the singularities meeting the boundary of D^2, we obtain disjoint embedded discs, which are bounded by a trivial link. This is indeed undoing band sum operations, since each removed ϵ-neighborhood can be replaced as a band.

Now, choose a ribbon embedding bounded by the given ribbon knot R. We may assume that the feet of the bands used to produce K from $\text{Wh}(L) \sqcup R$ are disjoint from the ribbon singularities. Undo the band sum operations, to obtain $\text{Wh}(L) \sqcup R$ from K, and then undo the band sum operations for the ribbon knot R as in the previous paragraph, to transform

$\mathrm{Wh}(L) \sqcup R$ into a split union of $\mathrm{Wh}(L)$ and a trivial link. A trivial link is the Whitehead double of a trivial link, say L_0, from which it follows that our knot K is obtained by band sum operations from the Whitehead double of the split union $L \sqcup L_0$. Since both L and L_0 are homotopically trivial, $L \sqcup L_0$ is homotopically trivial. By Theorem D, K is a slice knot. $\qquad\square$

5.2. Attempts to apply a surgery method

A standard surgery theoretic slicing process for a given knot K with zero-surgery manifold M_K starts with an epimorphism $\pi_1(M_K) \to G$ onto an appropriate ribbon group G; here a ribbon group is the fundamental group of the complement of a slicing disc in D^4 obtained by resolving singularities of a ribbon immersion $D^2 \looparrowright S^3$. Then one applies topological surgery over the group G, to obtain a slice disc exterior whose fundamental group is G. (For implementations of this strategy for knots, see for example [FQ90, FT05], while for links see [CFT09].) With current knowledge the surgery strategy can only be completed for certain special cases, since it is unknown whether surgery works for all ribbon groups. There are only two ribbon groups for which surgery is known to work: $\mathbb{Z}$ and $G_{6_1} := \langle a, t \mid ta^2t^{-1} = a \rangle$, a ribbon group for the Stevedore knot 6_1. They were used in [FQ90, FT05] respectively. We will show why this surgery approach fails to find slice disc exteriors for our knots from Corollary E in many cases.

Proposition 5.3. *Suppose K is a slice knot obtained by band sum operations on $\mathrm{Wh}(L) \sqcup R$ as in Corollary E.*

(1) *The knots K and R have S-equivalent Seifert matrices. Consequently, $\Delta_K(t) = \Delta_R(t)$.*

(2) *There is an epimorphism of $\pi_1(M_K)$ onto $\pi_1(M_R)$ which takes a meridian to a meridian.*

Before proving Proposition 5.3, we discuss some of its consequences. First, from Proposition 5.3 (2), it follows that for any ribbon group G for R, there is an epimorphism of $\pi_1(S^3 \smallsetminus K)$ onto G, as mentioned in the introduction. The second consequence is that if $\Delta_R(t)$ is not one and is not divisible by $t - \frac{1}{2}$ (over $\mathbb{Q}$), then the topological surgery method to construct a slice disc exterior, which we discussed in the introduction, does not work for K. In fact, to apply topological surgery, one needs to start with an epimorphism of $\pi_1(S^3 \smallsetminus K)$ onto a ribbon group G for which surgery is known to work; the only such ribbon groups are $\mathbb{Z}$ and $G_{6_1} := \langle a, t \mid ta^2t^{-1} = a \rangle$, a ribbon group for the Stevedore knot 6_1. For the $G = \mathbb{Z}$ case, it is known that the surgery programme slices a knot if and only if $\Delta_K(t) = 1$, essentially because defining a surgery problem requires a degree one normal map with target $S^1 \times D^3$ which restricts to a $\mathbb{Z}[\mathbb{Z}]$ homology equivalence on the boundary. By Proposition 5.3 (1), it follows that surgery cannot be carried out if $\Delta_R(t) \neq 1$. Also, for $G = G_{6_1}$, if there were an epimorphism $\pi_1(S^3 \smallsetminus K) \to G_{6_1}$, then it would imply that $\Delta_K(t)$ is divisible by the "Alexander polynomial" $\Delta_{G_{6_1}}(t)$ of G_{6_1}, which is defined to be the order of the module $(G'_{6_1}/G''_{6_1}) \otimes \mathbb{Q} \cong H_1(G'_{6_1}; \mathbb{Q})$ over the PID $\mathbb{Q}[t^{\pm 1}]$ as usual. Indeed, G_{6_1} is isomorphic to the Baumslag-Solitar group $\left(\mathbb{Z}[t^{\pm 1}]/(t - \frac{1}{2})\right) \rtimes \langle t \rangle$, and we have $\Delta_{G_{6_1}}(t) = t - \frac{1}{2}$. From this it follows that if $\Delta_R(t)$ is not divisible by $t - \frac{1}{2}$, then there is no epimorphism of $\pi_1(S^3 \smallsetminus K)$ onto G_{6_1}. It is conceivable that K is smoothly concordant to a knot J with $\Delta_J = \Delta_{6_1}$, such that J can be sliced using [FT05]. In this eventuality the resulting slice disc would *not* be homotopy ribbon.

Next we prove Proposition 5.3.

Proof of Proposition 5.3. First we prove part (1). The standard genus 1 Seifert surface of each component of $\mathrm{Wh}(L)$ has Seifert matrix $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. Since L is homotopically trivial, each pairwise linking number of L is zero. It follows that the diagonal block sum of m copies

of $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and a Seifert matrix of R is a Seifert matrix of K. Consequently, K has a Seifert matrix S-equivalent to that of R.

The proof of part (2) follows immediately from Lemma 5.4 below.   □

Lemma 5.4. *Suppose L is an m-component boundary link with Seifert surface V, and J is a knot. Suppose K is a knot obtained from the split union $L \sqcup J$ by m band sum operations along bands which join a component of L to J and whose interior is disjoint from V. Then there is an epimorphism $\pi_1(M_K) \to \pi_1(M_J)$ which takes a meridian to a meridian.*

Proof. Let γ_i be the core arc of the band joining J and the ith component L_i of L, for $i = 1, \ldots, m$. Using Kirby calculus it is not too hard to see (see e.g. [COT04, Proof of Theorem 4.1]) that the 3-manifold M_K is obtained from $M_{L \sqcup J}$ by zero-framed surgery along m curves, say α_i, each of which bounds an embedded 2-disc that meets $L \sqcup J$ at two transverse intersection points, contains γ_i, and induces the framing of γ_i. See Figure 9.
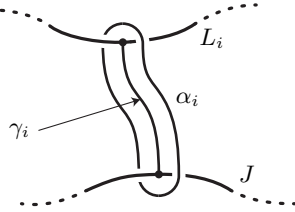

FIGURE 9. Band sum and surgery.

The standard Pontryagin-Thom construction applied to the m-component Seifert surface V gives a map

$$S^3 \smallsetminus \nu(L) \longrightarrow \bigvee^m S^1$$

which takes $S^3 \smallsetminus \nu(V)$ to the wedge point; this induces an epimorphism $\phi \colon \pi_1(M_L) \to F :=$ free group of rank m, which takes a meridian of L_i to the ith generator $x_i \in F$.

Note that since $M_{L \sqcup J} \cong M_L \# M_J$, we have that $\pi_1(M_{L \sqcup J}) = \pi_1(M_L) * \pi_1(M_J)$. Furthermore, from the hypothesis that the interior of the arc γ_i is disjoint from the Seifert surface V, it follows that $(\phi * \mathrm{Id})(\alpha_i) \in F * \pi_1(M_J)$ is of the form $x_i \zeta_i$ where $\zeta_i \in \pi_1(M_J)$. Here $\phi * \mathrm{Id} \colon \pi_1(M_L) * \pi_1(M_J) \to F * \pi_1(M_J)$ is the map defined by sending elements of $\pi_1(M_J)$ in the free product to their image under ϕ.

Let W be the cobordism between M_K and $M_{L \sqcup J}$ obtained by attaching m 2-handles along the curves α_i to the product $M_{L \sqcup J} \times [0, 1]$. We have an epimorphism

$$(5.1) \quad \pi_1(W) \cong \frac{\pi_1(M_{L \sqcup J})}{\langle \alpha_1, \ldots, \alpha_m \rangle} = \frac{\pi_1(M_L) * \pi_1(M_J)}{\langle \alpha_1, \ldots, \alpha_m \rangle} \xrightarrow{\phi * \mathrm{Id}} \frac{F * \pi_1(M_J)}{\langle x_1 \cdot \zeta_1, \ldots, x_m \cdot \zeta_m \rangle} \cong \pi_1(M_J).$$

Also, turning W upside down, W is obtained by attaching 2-handles to $M_K \times [0, 1]$. It follows that the inclusion induces an epimorphism $\pi_1(M_K) \to \pi_1(W)$. Composing this with (5.1), we obtain the desired epimorphism $\pi_1(M_K) \to \pi_1(M_J)$. By construction, this takes a meridian to a meridian.   □

Friedl and Teichner proposed conjectures related to necessary and sufficient conditions for being homotopy ribbon, in [FT05, Conjectures 1.6 and 1.8]. It is an interesting question whether the slice knots produced by Corollary E satisfy their proposed conditions, that is, the Ext condition in [FT05, Conjecture 1.6] and the Poincaré duality condition in [FT05, Conjecture 1.8].

6. Slice links

In this section we present new slice links in S^3, using Theorem A on distorted Casson towers. We focus on a constructions of links involving iterated Whitehead doubling, which is naturally related to Casson towers.

6.1. Kirby diagrams for distorted Casson towers

In this subsection we discuss Kirby diagrams for arbitrary distorted Casson towers, prior to their application in our construction of another family of new slice links, presented in subsection 6.2.

First we recall that a plumbing operation between two 2-handles in a Kirby diagram gives us a new 1-handle and a clasp (whose sign is equal to the sign of the plumbing) between the attaching circles of the 2-handles, as shown in Figure 10. As a reference, see for instance [GS99, Example 6.1.3].
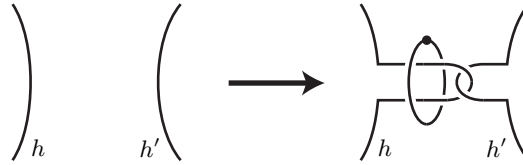

FIGURE 10. Plumbing of 2-handles in a Kirby diagram.

Start with a standard Kirby diagram of a Casson tower. For example, see Figure 11, which is a Casson tower of height 4. (It is a good exercise, for those not familiar with this diagram, to build it using the above plumbing operation; or see [Cas86], [Fre82b, Section 2], [FQ90, Chapter 12], [GS99, Example 6.1.3].) So far all the plumbing operations are self-plumbings. As our temporary convention, a circle without a dot or a label designates a zero-framed attaching circle of a 2-handle.
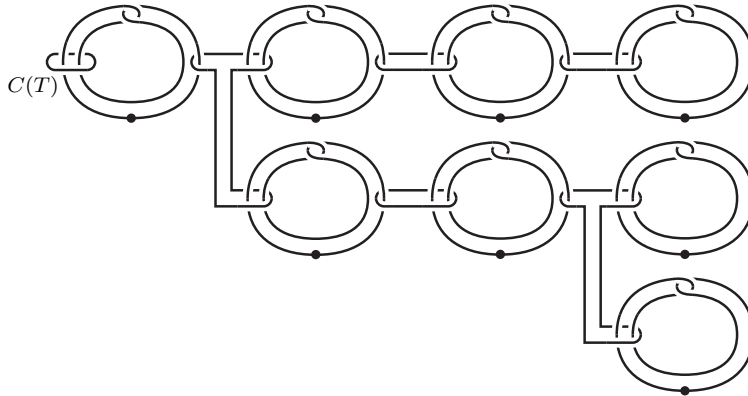

FIGURE 11. A Casson tower of height 4. Unlabelled circles without a dot are zero-framed.

By applying the above plumbing operation for 2-handles, we can plumb a stage 4 disc to another disc of stage two or higher. For instance, by plumbing a stage 4 disc in Figure 11 to a stage 2 disc and by plumbing another stage 4 disc to a stage 3 disc, we obtain a distorted Casson tower described by the Kirby diagram in Figure 12.
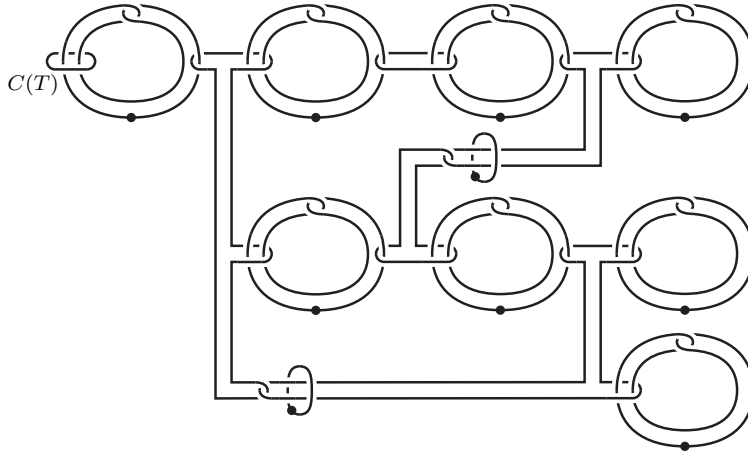
FIGURE 12. A Kirby diagram of a distorted Casson tower of height 4.

Elimination of all 2-handles. In case of a (non-distorted) Casson tower, it is well known that one can eliminate 1- and 2-handles in pairs to obtain a Kirby diagram without 2-handles. A standard procedure, which is a "top-to-bottom" elimination, is as follows. Start with a diagram such as Figure 11 (for which this procedure will be "right-to-left"). Slide each rightmost 1-handle, which is associated to a self-intersection of the top stage disc, under the adjacent 1-handle on its left. Then eliminate the 1-handle under which it slid, together with the linking 2-handle. Iterate this, to obtain a Kirby diagram with k 1-handles, where k is the number of self-intersections of the final stage discs. In fact $C(T)$ remains as an unknotted circle, and the 1-handles form a ramified iterated Whitehead double of a meridian of $C(T)$.

For a distorted Casson tower diagram, the plumbings performed between 2-handles as in Figure 10 may prevent the elimination of a 1-handle and a linking 2-handle in the above procedure. Instead, in order to obtain a Kirby diagram of a distorted Casson tower without 2-handles, we will perform the "bottom-to-top" elimination discussed below, which is "left-to-right" in case of Figure 12.

For this elimination the (well known) modification shown in Figure 13 is useful; R denotes an arbitrary tangle diagram, and the hatched band designates parallel strands. The box with label -2 designates two negative full twists. (If we had a negative clasp in the first picture, we would have $+2$ instead.) The first step in Figure 13 is an isotopy which straightens the 1-handle, and the second step is handle sliding and cancellation.
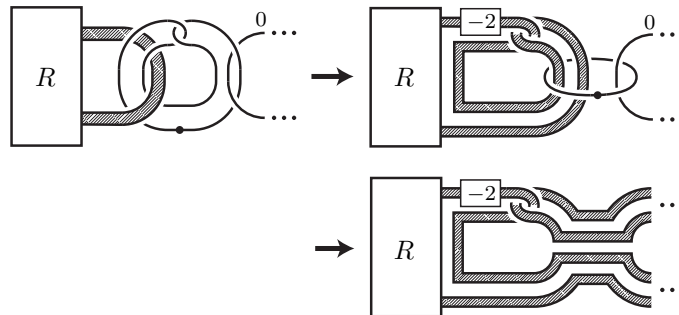

FIGURE 13. A modification of a Kirby diagram.

Now start with a Kirby diagram drawn as in Figure 12. First apply the move in Figure 13 to the leftmost part to eliminate the leftmost 1-handle, and a 2-handle linking it. By this

$C(T)$ becomes the Whitehead double of the attaching circle of the eliminated 2-handle; see Figure 14. Repeatedly apply the move in Figure 13 to eliminate the next leftmost 1-handles and 2-handles in pairs. Eventually we obtain a Kirby diagram with k 1-handles and no 2-handles, where k is the number of intersections of the top stage discs and stage ≥ 2 discs. For instance, if we start with Figure 12, we have $k = 5$ since there are 3 self-intersections of stage 4 discs, and 2 "distorting" intersections between stage 4 and lower stage discs. See Figures 15 and 16.

A consequence of this is that a distorted Casson handle is diffeomorphic to the boundary connected sum of k copies of $S^1 \times D^3$.
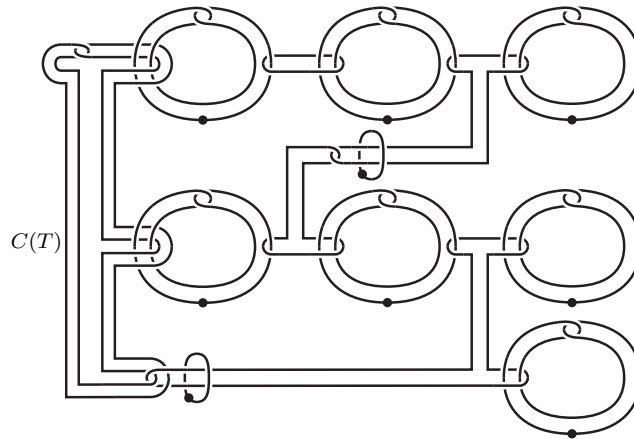

FIGURE 14. A distorted Casson tower diagram with a 1-handle and a 2-handle eliminated.
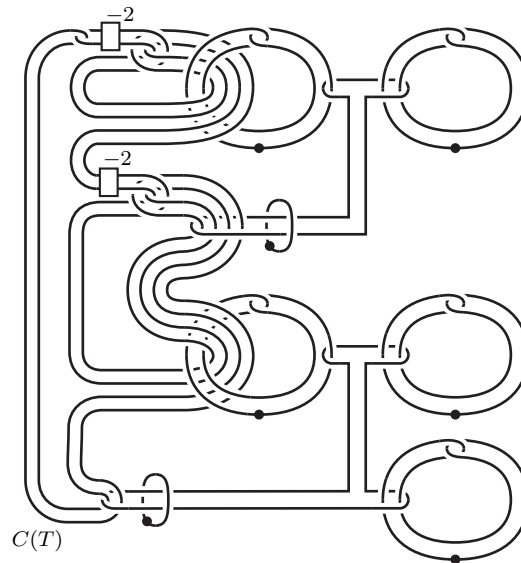

FIGURE 15. Further elimination of handles in a distorted Casson tower diagram.
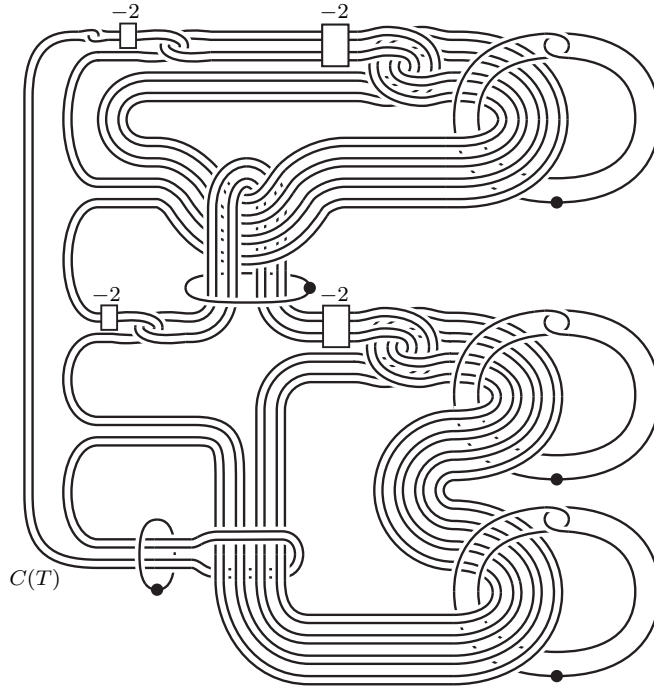
FIGURE 16. A distorted Casson tower diagram with all 2-handles eliminated.

6.2. Distorted 4-fold iterated ramified Whitehead doubles

In this subsection we give a new family of slice links. The main ingredients are Theorem A on distorted Casson towers of height 4 and the Kirby diagrams we obtained in Section 6.1. First we begin with a general construction of links, without requiring a distorted Casson tower. Then we will relate such a link to a distorted Casson tower, by connecting combinatorial choices involved in the construction of the link to intersection data of the corresponding distorted Casson tower.

Construction of links. We start with the split union of arbitrary number of Hopf links. Choose one component from each Hopf link, and denote the union of the chosen components by L_1. Denote the union of the other components by L_2. In what follows, Whitehead doubles and parallels are always untwisted, and taken in a tubular neighbourhood which is thin enough to be disjoint from anything we have considered previously. Also, a band sum is always assumed to be between components of split sublinks along a "straight" band; more precisely, whenever two components J and J' of a link are joined by a band, there is a separating 2-sphere S in S^3 disjoint from the link, and the band passes through S exactly once and is disjoint from anything we have considered previously. This determines the result of the band sum uniquely up to isotopy. Now the construction is described below.

(1) Replace each component of L_2 with $\mathrm{Wh}(L_2)$. Perform some band sum operations to combine distinct components of L_1 and call the result L_1'. Remember a meridian of each component of L_1' for later use, without adding it to the link.
(2) Replace L_1' with $\mathrm{Wh}(L_1')$, perform some band sum operations to combine distinct components of $\mathrm{Wh}(L_1')$, and call the result L_1''. The sublink $\mathrm{Wh}(L_2)$ is left unchanged. Remember a meridian of L_1''.
(3) Perform (2) once again for L_1'' in place of L_1' and call the result L_1'''. Remember a meridian of each component of L_1''' for later use.

(4) Perform (2) once again for L_1''' in place of L_1'. This time we perform band sum operations on $\mathrm{Wh}(L_1''')$ until we obtain a knot, say J.

(5) Perform the following operation some number of times: choose a remembered meridian of a component of L_1' and a remembered meridian of a component of either L_1', L_1'' or L_1'''. Band sum them, add a meridional circle of the band to our link, and modify J by performing ± 1 surgery on the banded together meridians, then ∓ 1 surgery on each of the meridians individually. This introduces a clasp between strands enclosed by the meridians. If the same meridian is chosen more than once during the iteration, use a parallel copy.

The final outcome is the union of $\mathrm{Wh}(L_2)$ and J modified in Step (5). Remembered meridians are not included.

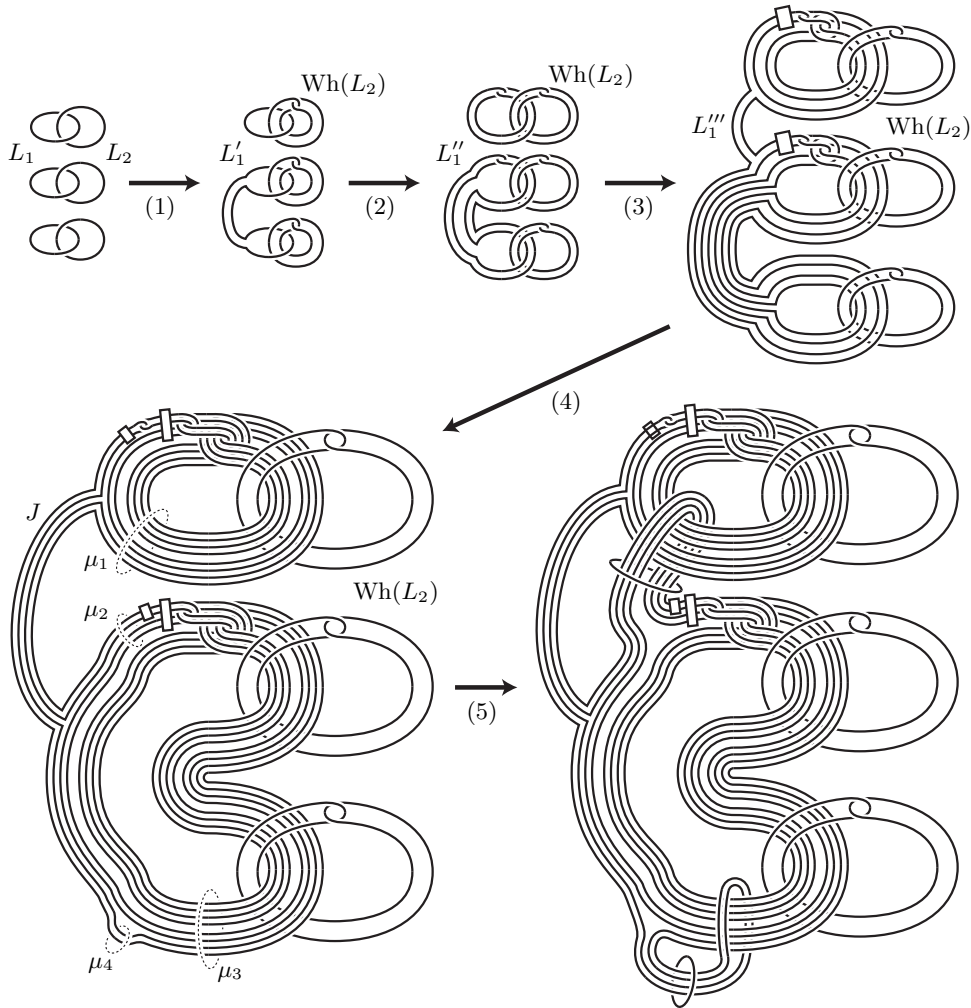Maybe our construction is best understood by an example: see Figure 17.

FIGURE 17. A construction of a slice link. Each box designates -2 full twists. The meridians μ_1, μ_2, μ_3, and μ_4 are those of L_1', L_1'', L_1', and L_1''', respectively, and Step (5) is performed for the pairs (μ_1, μ_2) and (μ_3, μ_4) to obtain the last link as the final outcome.

Theorem 6.1. *Any link constructed as above is slice.*

By Theorem 6.1, the last link in Figure 17 is slice. As another example which is simpler, the link in Figure 2 is slice. Thus Theorem G is a consequence of Theorem 6.1. Indeed, the link in Figure 2 is obtained by applying the above construction to a distorted Casson tower of height 4 which has one plumbed handle with one self-plumbing at each stage and has one distorting intersection between the stage 4 and stage 2 discs.

Proof of Theorem 6.1. We claim that a link L obtained by the above construction is the union of the curve $C(T)$ and the dotted circles representing 1-handles in a Kirby diagram of a height 4 distorted Casson tower T without 2-handles. For instance, observe that Figure 16 and the final picture in Figure 17 are identical. In fact, our construction of L corresponds to a top-to-bottom construction of a distorted Casson tower, as follows. For each component of $\mathrm{Wh}(L_2)$ in Step (1), take a disc with a single local self-plumbing, which we call a pre-stage-4 disc. Whenever we perform band sum of components of L_1 in Step (1), take a boundary connected sum of the associated pre-stage-4 discs. The resulting discs with (multi-)self-plumbings are the stage 4 discs of our tower T. In Step (2), whenever we take a Whitehead double of a component, take a disc with a single local self-plumbing, which we call a pre-stage-3 disc, and attach the associated stage 4 disc to the pre-stage-3 disc along the double point loop. Again, whenever we perform a band sum, take the boundary connected sum of the pre-stage-3 discs. The result is stage 3 discs with the stage 4 discs attached. Continue in the same way for steps (3) and (4) to produce stages 2 and 1. We arrive at a non-distorted Casson tower of height 4. Finally, for each triple of ± 1 surgeries occurring in step (5), plumb a stage 4 disc to a stage 4, 3 or 2 disc, where the choice of the meridians determines which discs to plumb. The 2-handle elimination procedure described in Section 6.1 applies to the standard Kirby diagram of the resulting distorted Casson tower, from which the claim follows. The lemma stated below now completes the proof. $\square$

Lemma 6.2. *Let L be the union of the curve $C(T)$ and the dotted circles representing 1-handles in a Kirby diagram of a height 4 distorted Casson tower T without 2-handles. Then, as a link in S^3, L is slice.*

Proof. The Kirby diagram without 2-handles determines an embedding of T into the 4-ball, to wit, T is the exterior of the standard slicing discs Δ_i bounded by the dotted circles (which form a trivial link). By Theorem A, the curve $C(T)$ bounds a flat disc Δ in T. As Δ is disjoint from the discs Δ_i, L is slice. $\square$

The above lemma also applies to give another family of slice links: recall that Theorem F in the introduction states that *any ramified* Wh_n *link is slice for $n \geq 4$.*

Proof of Theorem F. It suffices to show that any ramified Wh_4 link is slice. Recall that a ramified Wh_4 link L is the union of $C(T)$, and the dotted circles in a Kirby diagram obtained by a top-to-bottom elimination of 2-handles applied to the standard Kirby diagram of a (non-distorted) Casson tower of height 4; here $C(T)$ remains as an unknotted circle and the other components form a 4th iterated ramified Whitehead double of a meridian of $C(T)$. By Lemma 6.2, L is slice. $\square$

7. The grope filtration of knots and Casson towers of height 3

In this section we make the observation that we can use the improved initial hypothesis in the Grope Height Raising Lemma to slightly extend results from [Ray13] on the grope filtration of the knot concordance group. The grope filtration first appeared in the literature in [CT07], although it was already implicit in [COT03]. By definition a knot in S^3 lies in the nth term $\mathcal{G}^{(n)}$ of the filtration if it bounds an embedded framed grope G_n in D^4. Ray shows that a knot in S^3 which bounds a Casson tower of height 3 is (n)-solvable for

all n. This follows from the corollary below, by [COT03, Theorem 8.11]. She also shows that a knot which bounds a Casson tower of height n bounds a grope of height n [Ray13, Theorem A (i)]. In fact, a height 3 Casson tower is enough to obtain this conclusion for all n.

Corollary 7.1. *A Casson tower T_3 of height 3 contains an embedded framed grope G_n of height n, with the same attaching circle as the Casson tower, for all n.*

Proof. Apply Proposition 4.1 to construct a properly immersed framed capped grope of height 2 inside T_3, as described in the beginning of the proof of Theorem A. Apply grope height raising, as in Lemma 3.7, to obtain a properly immersed framed capped grope of height n, and then ignore the caps. □

It is interesting to contrast Corollary 7.1 with Theorem B.

As mentioned above, Ray showed in [Ray13] that a link in S^3 which bounds a Casson tower of height 3 in D^4 is (n)-solvable, in the sense of [COT03], for all n. Corollary 7.1 shows that the link also lies in the intersection of the grope filtration. A link which bounds a grope of height $n+2$ is (n)-solvable [COT03, Theorem 8.11], but by [Ott14, Corollary 6.8] the converse does not hold for links of at least 2^{n+2} components. So for finite terms of the filtrations, having a height $n+2$ embedded grope is stronger than having an (n)-solution. As observed in [Ray13], we can deduce the existence of an (n)-solution from the existence of an immersed grope of height $n+2$ with the bottom two stages embedded. It is not known whether the infinite intersections of the filtrations coincide.

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