THE ENGULFING THEOREM AND UNIQUENESS OF PL STRUCTURES ON $\mathbb{R}^{\mathbf{n}}$ FOR $\mathbf{n} \geq 5$

DIEGO SANTORO

1 Introduction

In these notes we prove that for $n \geq 5$ there exists a unique PL structure on $\mathbb{R}^n$ up to PL isomorphism. The proof will be mostly based on the so called Engulfing Theorem, that is presented in the second section. In the first section we recall some basic notions and results of PL theory.

Conventions. We will usually omit the prefix PL: so for example manifold stands for PL manifold, isomorphism for PL isomorphism and so on. We will explicitly state the category in which we are working when it is necessary. When not explicitly stated, manifolds are supposed to be without boundary, and they can be compact or not.

2 Basic notions and useful theorems in PL theory

In this section we recall some definitions and results regarding PL theory that will be needed later. This section will contain no proof. We refer to [RS72], [Zee63] and [Buo] for details and proofs.

Definition 2.1. Let $n \geq 0, m \geq 0$ be two natural numbers. An *n-simplex* A in some Euclidean space E^m is the convex hull of n linearly independent points, called *vertices*. A simplex B spanned by a subset of vertices of A is called a *face* of A, and we write $B < A$. The number n is called the *dimension* of A.

Definition 2.2. A (locally finite) *simplicial complex* K is a collection of simplices in some Euclidean space E, such that:

- if $A \in K$ and B is a face of A then $B \in K$.
- If $A, B \in K$ then $A \cap B$ is a common face, possibly empty, of both A and B.
- Each simplex of K has a neighbourhood in E which intersects only a finite number of simplices of K.

We define the *dimension* of K to be the maximal dimension of a simplex in K.

Given a simplicial complex K we denote

$$|K| = \bigcup_{A \in K} A$$

its underlying topological space, and we call it a *Euclidean polyhedron*.

We say that K' is a *subdivision* of K if $|K'| = |K|$ and each simplex of K' is contained in some simplex of K.

Definition 2.3. Let K, L be two simplicial complexes. We say that a map $f\colon K \to L$ is *simplicial* if for each simplex $A \in K$ its image $f(A)$ is a simplex in L and the restriction of f on A is linear. We say that f is *piecewise linear*, abbreviated PL, if there exists a subdivision K' of K such that f maps each simplex of K' linearly into some simplex of L.

Remark 2.4. Notice that the map f in the previous definition is defined on the Euclidean polyhedron $|K|$ and $|L|$, but we write $f\colon K \to L$ as an abuse of notation to stress the dependence on the simplicial complexes.

Remark 2.5. Notice also that in the definition of PL map we do not ask for the map $f\colon |K'| \to |L|$ to be simplicial. However it is true that if K and L are finite complexes and $f\colon |K| \to |L|$ is a PL map, then there exists subdivision K' of K and L' of L such that $f\colon K' \to L'$ is simplicial.

Definition 2.6. A *triangulation* of a topological space X is a simplicial complex K and a homeomorphism $t\colon |K| \to X$. A *polyhedron* is a pair $(P, \mathcal{F})$, where P is a topological space and $\mathcal{F}$ is a maximal collection of PL compatible triangulations; that is to say, given $t_1\colon |K_1| \to P$ and $t_2\colon |K_2| \to P$ in $\mathcal{F}$ we have that $t_1 \circ t_2^{-1}\colon K_2 \to K_1$ is a PL isomorphism.

Given X_1, X_2 two polyhedra, we say that $f\colon X_1 \to X_2$ is a PL map if there exists triangulations of X_1 and X_2 such that f is PL with respect to these triangulations.

Fact: A theorem of Runge ensures that an open set U of a simplicial complex K, or more precisely of $|K|$, can be triangulated, i.e. underlies a locally finite simplicial complex, in such a way that the inclusion map is PL. Furthermore such a triangulation is unique up to a PL isomorphism. For a proof see [AH35].

By virtue of the previous fact, it makes sense to give the following definition.

Definition 2.7. A (PL) manifold M of dimension n is a polyhedron such that every point $x \in M$ has a neighbourhood (PL) isomorphic to an open set in $\mathbb{R}^n_{\geq 0} = \{x \in \mathbb{R}^n | x_n \geq 0\}$.

Remark 2.8. In the previous definition the open sets in M and the open sets in $\mathbb{R}^n_{\geq 0}$ are endowed with the PL structure induced by the given PL structure on M and the standard PL structure on $\mathbb{R}^n_{\geq 0}$ respectively.

We denote with ∂M the set of points that are mapped to the boundary of $\mathbb{R}^n_{\geq 0}$ by some (and hence all) such local isomorphisms, and call it the *boundary* of M. We denote its complement with $\mathrm{Int}(M)$ and call it the *interior* of M.

Recall that, unless explicitly stated, in these notes we will suppose that our manifolds are without boundary.

2.1 Regular neighbourhoods

Definition 2.9. Let P be a polyhedron. A subset $P_0 \subseteq P$ is a *subpolyhedron* if there exists a triangulation of P which restricts to a triangulation of P_0.

Definition 2.10. Let K be a simplicial complex, and let $K_0 \subseteq K$ a subcomplex. Suppose that there exists a simplex $A = v * B \in K$ (i.e. A is the cone with vertex v and base the face B) where $v \in A$ is a vertex such that $K = K_0 \cup A$ and $K_0 \cap A = v * \partial B$. In this case we say that there is a *elementary simplicial collapse* from K to K_0, and we denote it by $K \overset{e.s.}{\searrow} K_0$. A *simplicial collapse* is a finite number of elementary simplicial collapses, and if K has a simplicial collapse to K_0 we denote this by $K \overset{s}{\searrow} K_0$.

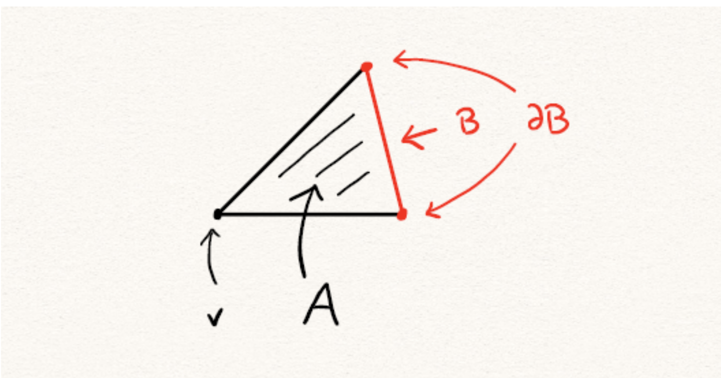

Figure 1

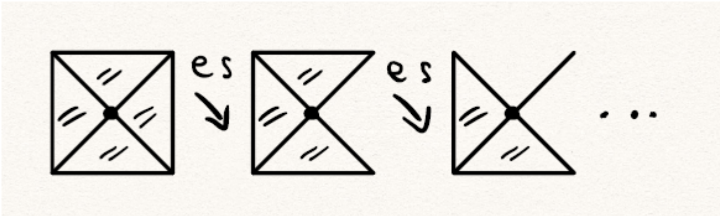

FIGURE 2. A sequence of elementary simplicial collapses.

Definition 2.11. Let P be a polyhedron, and let $P_0 \subseteq P$ a subpolyhedron. Suppose that there exists, for some natural number m, a (PL) m-ball $B \subseteq P$ such that $P = P_0 \cup B$ and $K_0 \cap B$ is a (PL) $(m-1)$-ball in ∂B. In this case we say that there is an *elementary collapse* from K to K_0, and we denote it by $K \overset{e}{\searrow} K_0$. A collapse is a finite number of elementary collapses, and if K collapses to K_0 we denote this by $K \searrow K_0$.

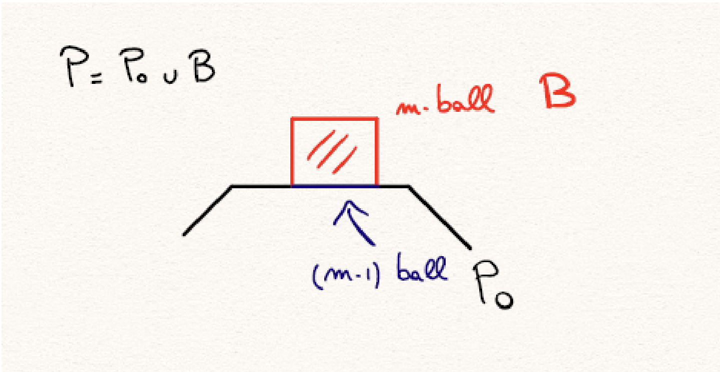

FIGURE 3

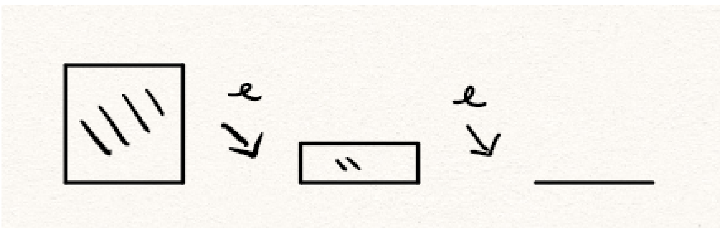

FIGURE 4. A sequence of elementary collapses.

Remark 2.12. The difference between the definition of collapse and simplicial collapse lies in the fact that a polyhedron does not have a "canonical" triangulation. It is obvious that a simplicial collapse is a collapse, but it is not true that if $P \searrow P_0$ then for any triangulation (K, K_0) of the pair (P, P_0) we have a simplicial collapse $K \overset{s}{\searrow} K_0$. It is however true that it is possible to find a subdivision (K', K_0') of (K, K_0) such that $K' \overset{s}{\searrow} K_0'$.

Definition 2.13. Let M be a closed (PL) n-manifold and let X be a subpolyhedron in M. A regular neighbourhood of X in M is any subpolyhedron N in M such that:

- N is an n-manifold with boundary,
- N is a topological neighbourhood of X in M,
- $N \searrow X$

For proofs of the following results we refer to [RS72].

Theorem 2.14. *Any second derived neighbourhood of X in M is a regular neighbourhood of X in M. Moreover any two regular neighbourhoods N_1, N_2 of X in M are ambiently isotopic in M, keeping fixed any arbitrary regular neighbourhood $N \subseteq N_1 \cap N_2$ and the complement of any arbitrary open set $U \supseteq N_1 \cup N_2$.*

Lemma 2.15. *Suppose X, Y are two subpolyhedra in a manifold M, and suppose that $X \searrow Y$. Then any regular neighbourhood of X is a regular neighbourhood of Y.*

As a corollary of the previous theorem and lemma we have the following.

Corollary 2.16. *Suppose X, Y are subpolyhedra in a manifold M, and suppose that $X \searrow Y$. Then any two regular neighbourhoods N_X and N_Y of X and Y are ambiently isotopic in M, via an isotopy keeping fixed any arbitrary regular neighbourhood of Y in $N_X \cap N_Y$ and the complement of any arbitrary open set $U \supseteq N_X \cup N_Y$.*

One way to construct regular neighbourhood is the following. Suppose that X is a subpolyhedron of a closed manifold M and consider a triangulation (T, T_0) of the pair (M, X).

Define $f_X \colon T \to [0,1]$ to be the unique simplicial map defined by mapping each vertex of T_0 to 0 and the other vertices to 1. We say that T_0 is *full* in T if $f_X^{-1}(0) = T_0$. If T_0 is full in M then for any $t \in (0,1)$ the preimage $f_X^{-1}([0,t])$ is a regular neighbourhood of X in M.

Remark 2.17. It can happen that T_0 is not full in T, but it is always possible to find a subdivision (T', T_0') of (T, T_0) such that T_0' is full in T'. Also notice that $f_X^{-1}(1)$ is always full in T.

Since we will work also with non compact manifolds and non compact polyhedra it is important to mention that regular neighbourhoods can be defined also in this setting and analogous results hold. The main difference is that also infinite sequences of elementary collapses are allowed. We refer to the paper [Sco67] for details.

Of course, in case of infinite regular neighbourhoods it is not possible in general to have uniqueness up to ambient isotopy with compact support. In any case the following lemma will be enough for our purposes.

Lemma 2.18. *Suppose that X, Y are subpolyhedra of M and suppose that $X \searrow Y$ (finite collapse). If $Y \subseteq U$, where U is an open set in M, then there exists an ambient isotopy of M with compact support that maps X in U.*

We end this subsection stating a lemma that we will play a key role in the following section.

Lemma 2.19. *Suppose that $P_0 \subseteq P$ are compact polyhedra and that $P \searrow P_0$. Also suppose that $S \subseteq P$ is a subpolyhedron. Then there exists a subpolyhedron $S^+ \supseteq S$ such that $P \searrow P_0 \cup S^+$ and $\dim S^+ \leq \dim S + 1$.*

2.2 General position

We need the following theorem, which roughly states that it is possible, with a slight perturbation, to promote a continuous map to a PL map that is "generic", in the sense that the image of this map has transverse self-intersections. Moreover it is possible to keep the map unchanged on a subpolyhedron on which it is already PL and generic.

To quantify the amount of perturbation, we will fix any metric compatible with the topology of our polyhedron. If (Z, d) is a metric space, we say that a map $f \colon Y \times I \to Z$ is an ε-homotopy if $d(f(y,0), f(y,t)) < \varepsilon$ for all $y \in Y$ and $t \in I$.

Theorem 2.20. *Let $P_0 \subseteq P^p$ be a subpolyhedron with $cl(P \setminus P_0)$ compact. Let $f \colon P \to M^m$ be a closed and continuous map with $p \leq m$ such that f is a PL embedding when restricted to P_0, and let $\varepsilon > 0$ be given. Then there is an ε-homotopy rel P_0 from f to a map g and a triangulation T of P such that:*

– *for every simplex $A \in T$ the restriction $g_{|A}$ is a PL embedding;*
– *for every $A, B \in T$, we have that $g^{-1}(g(B)) \cap A = (A \cap B) \cup S(A, B)$, where $S(A, B)$ is a subpolyhedron of A of dimension*

$$\dim(S(A, B)) \leq \dim A + \dim B - m.$$

Here are some comments to clarify the second condition in Theorem 2.20. The set $g^{-1}g(B) \cap A$ is by definition the set of points in A that share their image with some point in B. Since g is an embedding when restricted to any simplex, this set parametrises the intersection between $g(A)$ and $g(B)$ in M. The second condition then asks that this intersection (apart from the obvious set $g(A \cap B)$) is a polyhedron and is generic. The following figures should help the comprehension of this request.

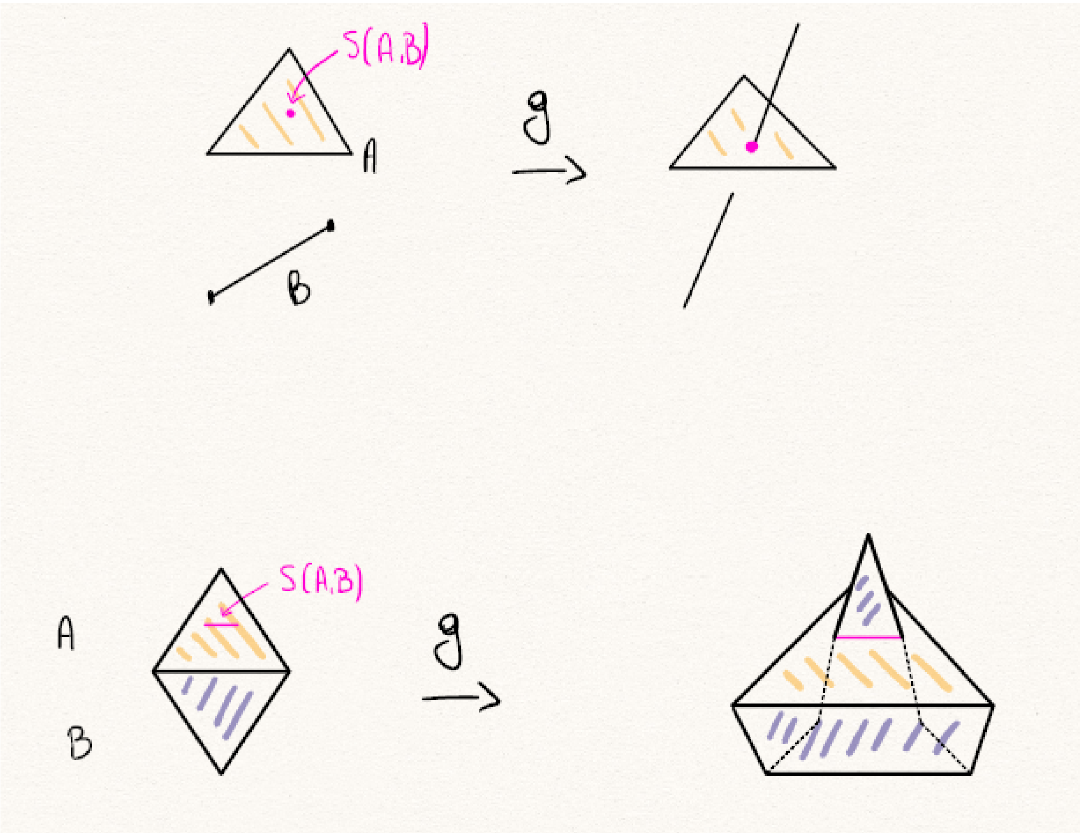

FIGURE 5

Remark 2.21. In general it is false that $A \cap B$ and $S(A, B)$ are disjoint, since we require $S(A, B)$ to be a subpolyhedron. For instance, in the case depicted in Figure 6 the set $S(A, B)$ contains also two points in $A \cap B$.

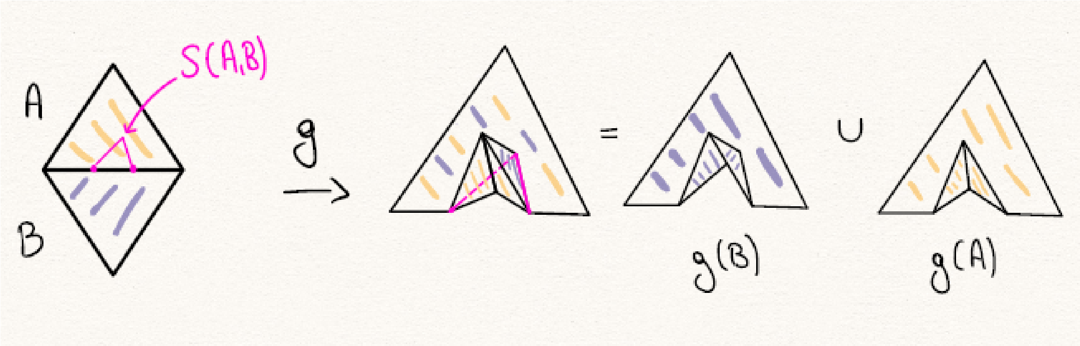

FIGURE 6

If we let A, B vary we can define the *singular set* of g:

$$S(g) = \bigcup_{A,B \in T} S(A, B).$$

It is not difficult to prove that $S(g)$ is a subpolyhedron of dimension at most $2p - m$ and that $S(g) = cl\{p \in P | g^{-1}g(p) \neq p\}$. In particular g is injective on $P \setminus S(g)$.

Details about general position arguments can be found in [Zee63] and [RS72].

In the following section we will need to use some collapses in the domain of a PL map to induce collapses on the image. If we have a PL embedding of course it is possible to mirror such a collapse on the image of the map. In general we are able to do so if the collapse takes place away from the singular set of the map.

Lemma 2.22. *Let P, Q be two polyhedra. Let $g\colon P \to Q$ be a PL map and suppose that $S \supseteq S(g)$ is a subpolyhedron of P that contains the singular set of g. Then if $P \searrow S$ also $g(P) \searrow g(S)$.*

For a proof of the previous lemma we refer to [Zee63].

3 The Engulfing Theorem

In this section we will state and prove the main theorem of these notes, the Engulfing Theorem. The sense of this theorem is to promote an homotopical, and hence algebraic, statement into a geometric one. As an example, consider the following question.

Question 3.1. Suppose that C is a compact set in a manifold M^n such that the inclusion $C \hookrightarrow M$ is nullhomotopic. Is C contained in an n-ball?

Of course if a set is contained in a ball then its inclusion is nullhomotopic, but at a first sight it is very difficult to give an answer to Question 3.1. As a consequence of the Engulfing Theorem we will improve our understanding of this problem and have a satisfying partial answer to this question.

There are several versions of the Engulfing Theorem, which is a technique more than a theorem in itself. We will present here the Stallings' version of the theorem [Sta62], which is the one that suits our needs.

Theorem 3.2 (Stallings' Engulfing Theorem)**.** *Let M^n be a PL manifold, U an open subset of M, P a subpolyhedron of M of dimension p. Suppose that:*

- *(M, U) is p-connected;*
- *$P \cap (M \setminus U)$ is compact;*
- *$p \leq n - 3$.*

Then there is a compact $E \subseteq M$, and there is an isomorphism $h\colon M \to M$, such that

$$P \subseteq h(U) \quad \textit{and} \quad h_{|M \setminus E} = \mathrm{Id}_{|M \setminus E}.$$

Recall that (M, U) is said to be p-connected if the relative homotopy groups $\pi_i(M, U)$ all vanish for $i \leq p$. Notice that, since the polyhedron P has dimension p, the hypothesis of p-connectedness is the sufficient algebraic condition to deduce that it is possible to homotope the remaining part of P inside the open subset U, as the following lemma proves.

Lemma 3.3. *Suppose that P is a subpolyhedron of dimension p of a manifold M. Suppose that $P_0 \subseteq P$ is a subpolyhedron of P and that U is an open set in M such that $P_0 \subseteq U$ and (M, U) is p-connected. Then there exists a homotopy $f\colon P \times I \to M$ rel P_0 such that $f(P \times \{1\}) \subseteq U$.*

Proof. The hypothesis that $\pi_k(M, U) = 0$ means that each map of pairs $(D^k, \partial D^k) \to (M, U)$ can be homotoped, relative to the boundary, to a map $D^k \to U$.

Assume inductively that the $(k-1)$-skeleton of P is already contained in U and consider the k-skeleton $P^{(k)}$ of P. Each simplex A in the k-skeleton can be homotoped into U rel ∂A, since $k \leq p$ and (M, U) is p-connected. In this way we can define an homotopy on $P_0 \cup P^{(k)}$ that is constant on P_0 and that takes $P_0 \cup P^{(k)}$ into U. Since the pair $(P, P_0 \cup P^{(k)})$ satisfies the homotopy extension property, we are able to extend this homotopy on the whole P, completing the inductive step. $\square$

The Engulfing Theorem improves the previous lemma in the much stronger result that the open set U can be enlarged to "engulf" the whole polyhedron P.

We will not start by proving the complete statement of the Engulfing Theorem, but we will first give some proofs of it when the codimension of P is big enough and when P is compact for the following reasons:

- the basic ideas of the final proof are already present in these simpler cases;
- the problems that one encounters when trying to generalise these simpler proofs to the general case give enough motivation to endure some technicalities of the final proof.

Step 1: P compact, $\mathbf{2(p+1) - n < 0}$.

Denote with P_0 the biggest subcomplex of P contained in U. It follows from the hypotheses and Lemma 3.3 that there exists a continuous homotopy $f\colon P \times I \to M$ relative to P_0 such that $f(P \times \{1\}) \subseteq U$. We can apply Theorem 2.20 to obtain a new map $g\colon P \times I \to M$ that is a PL map, that coincides with f on $P \times \{0\}$ and such that $g(P \times 1) \subseteq U$.

Moreover the singular set $S(g)$ has dimension at most $2(p+1) - n < 0$ and therefore g is an embedding. Since $P \times I \searrow P \times 1$ and g is a PL embedding, we have that $g(P \times I) \searrow g(P \times 1)$ and therefore by Corollary 2.16 (or also Lemma 2.18) there exists an ambient isotopy of M with compact support mapping P inside U. If we call h^{-1} the isomorphism at the end of this isotopy, we have that $P \subseteq h(U)$.

Step 2: P compact, $\mathbf{p \leq n - 4}$.

We try now to improve the hypothesis on the codimension. In this case it is not true a priori that g is an embedding, because the dimension of its singular set can be positive. This is a problem, because the collapse $P \times I \searrow P \times \{1\}$ does not induce a collapse on the image. We want to get rid of the singular set.

First good idea: We can suppose that by induction we are able to engulf subpolyhedra of dimension $p' < p$. If we are able to show that the dimension of $S(g)$ is strictly smaller than p we can engulf its image by induction. We have

$$dim(S(g)) = 2p + 2 - n < p \iff p < n - 2$$

Therefore, if $p \leq n - 3$ we can engulf the image of the singular set by inductive hypothesis.

Problem: It is not true in general that $P \times I \searrow P \times \{1\} \cup S(g)$.

Second good idea: Using Lemma 2.19 we can find $S^+(g)$, such that $S(g) \subseteq S^+(g)$ and $P \times I \searrow P \times \{1\} \cup S^+(g)$. Since dim $S^+(g) \leq$ dim $S(g) + 1$ we need

$$2p + 3 - n < p \iff p < n - 3.$$

Since by hypothesis we have $p \leq n - 4$ we can suppose that also the image of $S^+(g)$ is contained in U.

At this point, since the collapse $P \times I \searrow P \times \{1\} \cup S^+(g)$ takes place away from singular set, we can use Lemma 2.22 to mirror this collapse on the image. Since the image of $P \times \{1\} \cup S^+(g)$ has been engulfed from U, we can conclude as in Step 1.

As a result of the previous discussion, we have that the Engulfing Theorem holds for compact polyhedron P of codimension $\leq n - 4$. We will now present the proof of the more general result, that allows P to be non compact and of codimension $\leq n - 3$.

We assert that it is not difficult to drop the compactness hypothesis, due to the hypothesis of compactness of the set $P \cap (M \setminus U)$. What needs a more clever idea is to allow for codimension $n - 3$. The key observation is that, in order to engulf P we only need to engulf $g(P \times \{0\})$ and not the whole image of $P \times I$; if we pay attention to this and manage U to carefully select what portion of $g(P \times I)$ to engulf, we will be able to prove the case $p = n - 3$.

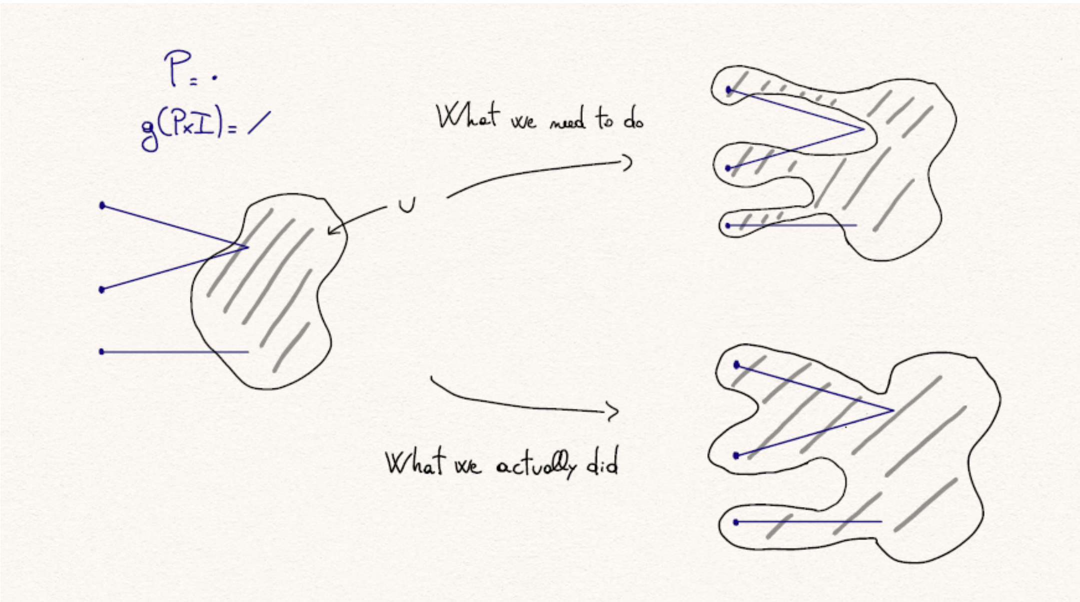

FIGURE 7

Proof of Theorem 3.2. It is clear that we can suppose that $P \setminus P_0$ has only one simplex Δ, by using an induction argument on the number of simplices in $P \setminus P_0$. We denote with q the dimension of Δ, and we notice that by hypothesis $q \leq p \leq n - 3$. The hypotheses of the theorem yield a continuous map $F\colon \Delta \times I \to M$ such that $F_{|\Delta \times \{0\}}$ is the inclusion of Δ in M and $F(\Delta \times \{1\}) \subseteq U$. Now we consider the polyhedron $K = \Delta \times I \cup_{\Delta \times \{0\}} P$ and we can glue the inclusion of P with the map F to obtain a map $f\colon K \to M$. We can apply Theorem 2.20 to obtain a map g that is PL and a triangulation T of K such that

- g is an embedding restricted to any simplex of T;
- given simplices A and B, $A \cap g^{-1}(g(B)) = (A \cap B) \cup S(A, B)$ where $S(A, B)$ is a compact subpolyhedron of A of dimension $\leq \dim A + \dim B - n$.

Moreover, up to passing to a subdivision, we can also suppose that T simplicially collapses to $K_0 = P_0 \cup (\partial\Delta \times I) \cup (\Delta \times \{1\})$. This follows from the fact that $\Delta \times I$ collapses to $(\partial\Delta \times I) \cup (\Delta \times \{1\})$.

In other words we have a finite number of simplices $A_1, \ldots, A_s$ such that:

- $K = K_0 \cup A_1 \cup \cdots \cup A_s$,
- each A_i has a vertex v_i and a face B_i such that $A_i = v_i * B_i$ and

$$(K_0 \cup A_1 \cup \cdots \cup A_{i-1}) \cap A_i = v_i * \partial B_i.$$

We denote with K_i the union $(K_0 \cup A_1 \cup \cdots \cup A_i)$ and with D_i its p-skeleton. Our aim is to engulf $g(D_i)$. Notice that $D_i = K_i$ except when the simplex Δ has dimension $q = p$, and that in any case $P \subseteq D_s$, the p-skeleton of K.

We can suppose by induction that the statement of the Engulfing Theorem holds for $q' < q$, i.e. that the statement of the theorem holds for subpolyhedra of M of dimension $q' < q$. Also suppose by induction that $g(D_{i-1})$ has already been engulfed. We now prove that it is possible to engulf $g(D_i)$.

Exactly as in Step 2 we have the problem that the collapse of $D_{i-1} \cup A_i$ to D_{i-1} does not induce a collapse of the images, since g is a priori not injective on $D_{i-1} \cup A_i$. But exactly as before we can consider the set

$$\Sigma_i = \cup \{S(A_i, B) | B \text{ is a simplex in } D_{i-1}\}.$$

The set Σ_i is the singular set of the map g restricted to $D_{i-1} \cup A_i$ and Σ_i is a compact subpolyhedron of A_i of dimension

$$\dim \Sigma_i \leq \dim A_i + p - n \leq q + 1 + (n-3) - n \leq q - 2.$$

Since $\dim \Sigma_i \leq q-2$, when we consider the set Σ_i^+ from Lemma 2.19 we have that $\dim \Sigma_i^+ \leq q-1$ and so we can apply the inductive hypothesis and obtain a compactly supported isomorphism $h \colon M \to M$ such that U engulfs $g(D_i \cup \Sigma_i^+)$. Since now $A_i \searrow (v_i * \partial B_i) \cup \Sigma_i^+$ we can use Lemma 2.22 to mirror this collapse on the image and deduce that there exists an isomorphism $h' \colon M \to M$ with compact support such that $h(U)$ contains $g(A_i \cup D_{i-1})$. The composition $h' \circ h$ gives the engulfing of $g(D_i)$ from U.

Since $P \subseteq D_s$, we have proved the Engulfing Theorem. $\square$

Remark 3.4. Notice that in the proof of the inductive step we actually managed to engulf the whole image of the simplex A_i, so it could seem that at the end the open set U engulfed the whole image of K. The important point is that when trying to engulf the image of the next simplex A_{i+1} we cannot impose that the open set keeps containing $g(A_i)$ during the isotopy, but only its p-skeleton.

So what happens is that at each step the open set U loses some pieces of what it has already engulfed. This is not a problem as long as none of these pieces belongs to the p-skeleton of the image of K and this is something we can control. The next figure is a schematic picture of what can happen.

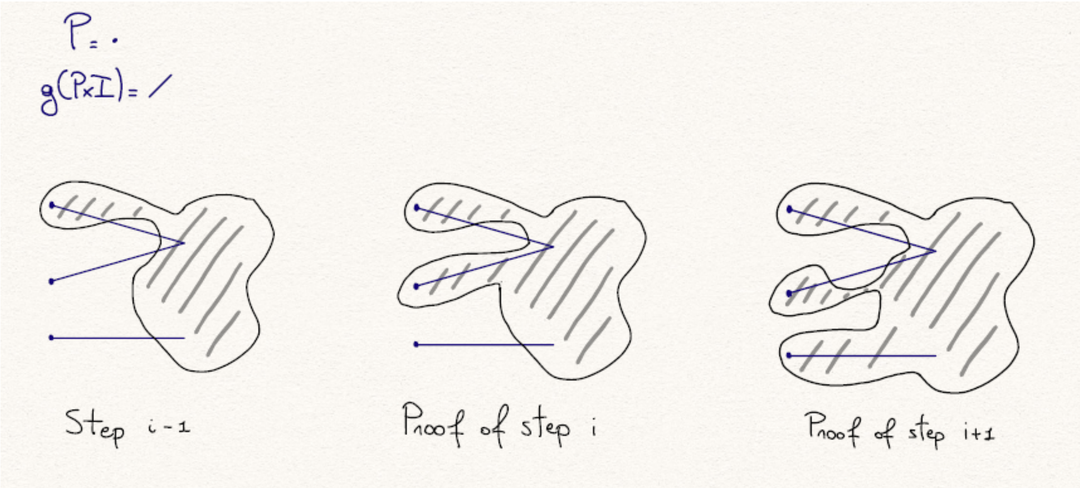

FIGURE 8

As a corollary of the Engulfing Theorem we have

Corollary 3.5. *Suppose that M^n is a contractible PL manifold and that $C \subseteq M$ is a compact subpolyhedron in M of dimension $\leq n-3$. Then C is contained in an n-ball.*

Proof. Take U to be any n-ball in M. Since both M and U are contractible, as a consequence of the long exact sequence of homotopy groups of the pair we have that (M, U) is p-connected for all p. Since C is compact, the set $C \cap (M \setminus U)$ is compact. Moreover the dimension of C is $\leq n-3$ by hypothesis and therefore we can apply the Engulfing Theorem to find an isomorphism $h\colon M \to M$ such that C is contained in the n-ball $h(U)$. □

4 Uniqueness of PL structures on $\mathbb{R}^n$

In this last section we will use the Engulfing Theorem to prove

Theorem 4.1 (Uniqueness of PL structure)**.** *Let $n \geq 5$. Then there exists a unique PL structure on $\mathbb{R}^n$ up to isomorphism.*

Remark 4.2. It is proved with other techniques that $\mathbb{R}^n$ has a unique PL structure if $n \leq 3$ [Moi52]. On the other hand, it can be showed that $\mathbb{R}^4$ has uncountably many different PL structures [Tau87].

We recall the following definition.

Definition 4.3. A topological space X is said to be *simply connected at infinity* if for any compact set $C \subseteq X$ there exists a compact D such that $C \subseteq D \subseteq X$ and $X \setminus D$ is simply connected.

Theorem 4.4. *Let M^n be a connected and oriented manifold with possibly empty boundary. Then any two cooriented embeddings of n-balls in* $\mathrm{Int}(M)$ *are ambiently isotopic.*

Theorem 4.1 will be a corollary of the following proposition.

Proposition 4.5. *Suppose that M^n, $n \geq 5$, is a contractible manifold that is simply connected at infinity. Then any compact subset of M is contained in an n-ball.*

Proof of Theorem 4.1. Let M^n be contractible and simply connected at infinity. Then the existence of a countable compact exhaustion of M and Proposition 4.5 imply that M is the union of $\{F_i\}_{i\in\mathbb{N}}$, where each F_i is a n-ball, and $F_i \subseteq \mathrm{Int}\, F_{i+1}$. We now prove that all the manifolds obtained in this way are isomorphic, and since $\mathbb{R}^n$ can be obtained in this way the theorem follows.

Suppose that we have $F_i \subseteq \operatorname{Int} F_{i+1}$ and $G_i \subseteq \operatorname{Int} G_{i+1}$ two pair of nested n-balls and fix any isomorphism $f_i \colon F_i \to G_i$. If we are able to show that there exists an isomorphism $f_{i+1} \colon F_{i+1} \to G_{i+1}$ that extends f_i we have finished, since we can iterate this process countably many times, starting with a fixed isomorphism $f_1 \colon F_1 \to G_1$, to obtain an isomorphism $\bigcup_{i \in \mathbb{N}}\{F_i\} \to \bigcup_{i \in \mathbb{N}}\{G_i\}$.

Suppose that we have fixed f_i and consider any isomorphism $f'_{i+1} \colon F_{i+1} \to G_{i+1}$ with the property that its restriction to F_i is cooriented with f_i. Then by Theorem 4.4 we know that there exists an isomorphism $H \colon G_{i+1} \to G_{i+1}$ such that $(H \circ f'_{i+1})_{|F_i} = f_i$. Simply define $f_{i+1} = H \circ f'_{i+1}$. □

Our aim now is to prove Proposition 4.5. We start with some simple lemmas.

Lemma 4.6. *Suppose M^n is a manifold that is contractible and simply connected at infinity. Then for any compact set $C \subseteq M$ there exists a compact set D such that $C \subseteq D \subseteq M$ and $(M, M \setminus D)$ is 2-connected.*

Proof. Consider D such that $M \setminus D$ is simply connected. Consider the long exact sequence of homotopy groups of the pair

$$\cdots \to \pi_2(M) \to \pi_2(M, M \setminus D) \to \pi_1(D) \to \pi_1(M) \to \pi_1(M, M \setminus D) \to \dots$$

Since M is contractible and $M \setminus D$ is simply connected, we deduce that $(M, M \setminus D)$ is 2-connected. □

Lemma 4.7. *Suppose M^n, $n \geq 5$, is a manifold that is contractible and simply connected at infinity. Let $T^{(2)}$ denote the 2-skeleton of a triangulation T of M and let $C \subseteq M$ be a compact subset. Then there exists an isomorphism $h \colon M \to M$ whose support is compact and contains C and such that $M \setminus C$ engulfs $T^{(2)}$, i.e. $T^{(2)} \subseteq h(M \setminus C)$.*

Proof. Consider a compact set D such that $C \subseteq D \subseteq M$ and $(M, M \setminus D)$ is 2-connected. Since $T^{(2)}$ is a 2-dimensional polyhedron, $n \geq 5$, and $T^{(2)} \cap D$ is compact, being D compact, we can apply the Engulfing Theorem and find a compactly supported isomorphism $h \colon M \to M$ such that $T^{(2)} \subseteq h(M \setminus D) \subseteq h(M \setminus C)$. If the support of h does not contain C we can simply consider its union with C, that is still compact. □

Proof of Proposition 4.5. Consider $C \subseteq M$ a compact subset and consider $T^{(2)}$ the 2-skeleton of some triangulation of M. We know that, up to isomorphism of M, we can suppose that $C \cap T^{(2)} = \emptyset$. Define K as the polyhedron obtained by adding to $T^{(2)}$ all the closed simplices of T that are contained in $M \setminus C$. Consider $C(K)$, the complement of K. $C(K)$ is defined in the following way:

- consider the first barycentric subdivision of T and denote it by $\widetilde{T}$;
- consider the unique simplicial map $f_K \colon \widetilde{T} \to [0,1]$ defined by mapping the vertices of $\widetilde{T}$ that are in K to 0 and the other vertices in 1;
- define $C(K) = f_K^{-1}(1)$.

Claim. *The subpolyhedron $C(K)$ is compact and has dimension $\leq n-3$.*

We postpone the proof of the claim to the end of the proof.

Since $C(K)$ has dimension $\leq n-3$ and is compact, by virtue of Corollary 3.5 we can suppose that $C(K)$ is contained in an n-ball A.

To conclude the proof of the proposition it is sufficient to observe that since $C(K)$ is compact there exists $t_1 \in (0,1)$ such that $C(K) \subseteq f_K^{-1}([t_1, 1]) \subseteq A$. Moreover since C is compact and contained in $M \setminus K$ there exists $t_2 \in (0,1)$ such that $C \subseteq f_K^{-1}([t_2, 1])$.

Since both $f_K^{-1}([t_1, 1])$ and $f_K^{-1}([t_2, 1])$ are regular neighbourhoods of $C(K)$ in M, by virtue of Theorem 2.14, it is possibile to find an isomorphism $h \colon M \to M$ such that $C \subseteq h(A)$, which is an n-ball. □

Proof of the claim. It is easy to prove that $C(K)$ contains only a finite number of vertices. In fact its vertices are contained in the simplices of T that intersect the compact C, and therefore are contained in a finite number of simplices. This implies the compactness of T.

The bound on the dimension of $C(K)$ follows from the fact that any k-simplex of $\widetilde{T}$ intersects a $(n-k)$-simplex of T. In fact the operation of first barycentric subdivision can be described in the following way:

- **Step 0:** Do nothing. Rename the 0-skeleton of T by $T_0^{(0)}$
- **Step 1:** Add to each edge of T its barycenter and subdivide $T^{(1)}$ by taking the cones with vertices these barycenters and base $T_0^{(0)}$. In this way we obtain a new triangulation of the 1-skeleton of T. Denote the new 0-skeleton with $T_1^{(0)}$ and the new 1-skeleton with $T_1^{(1)}$.
- **Step 2:** Add the barycenters of the 2-simplices of T and take the cones with vertices these barycenters and base $T_1^{(1)}$. In this way we obtain a new triangulation of the 2-skeleton of T. Denote the new 0-skeleton, 1-skeleton and 2-skeleton with $T_2^{(0)}$, $T_2^{(1)}$ and $T_2^{(2)}$.
- Iterate this process up to the n-skeleton. By construction $T_n^{(n)}$ is the barycentric subdivision $\widetilde{T}$.

Using this description it is easy to prove that:

- each simplex in $T_1^{(1)}$ contains a vertex of $T_0^{(0)} = T^{(0)}$;
- each simplex of $T_2^{(2)}$ contains an edge of $T_1^{(1)}$;
- each 3-simplex of $T_3^{(3)}$ contains a 2-simplex of $T_2^{(2)}$. Analogously each 2-simplex of $T_3^{(2)}$ contains a 1-simplex of $T_2^{(1)}$ and each 1-simplex of $T_3^{(1)}$ contains a vertex of $T_2^{(0)}$;
- by induction, each k-simplex of $T_m^{(k)}$, with $k \leq m$, contains a $(k-h)$-simplex of $T_{m-h}^{(k-h)}$, with $h \leq k$.

The schematic picture of Figure 9 should help to visualise this "cascade" situation.

In particular, each simplex of dimension $\geq n-2$ in $\widetilde{T} = T_n^{(n)}$ must contain a simplex in $T_2^{(2)}$. Since $|T_2^{(2)}| = |T^{(2)}|$ and K contains the 2-skeleton of T, it follows that any such simplex must intersect K. This implies that any simplex of $C(K)$ has dimension at most $n-3$. $\square$

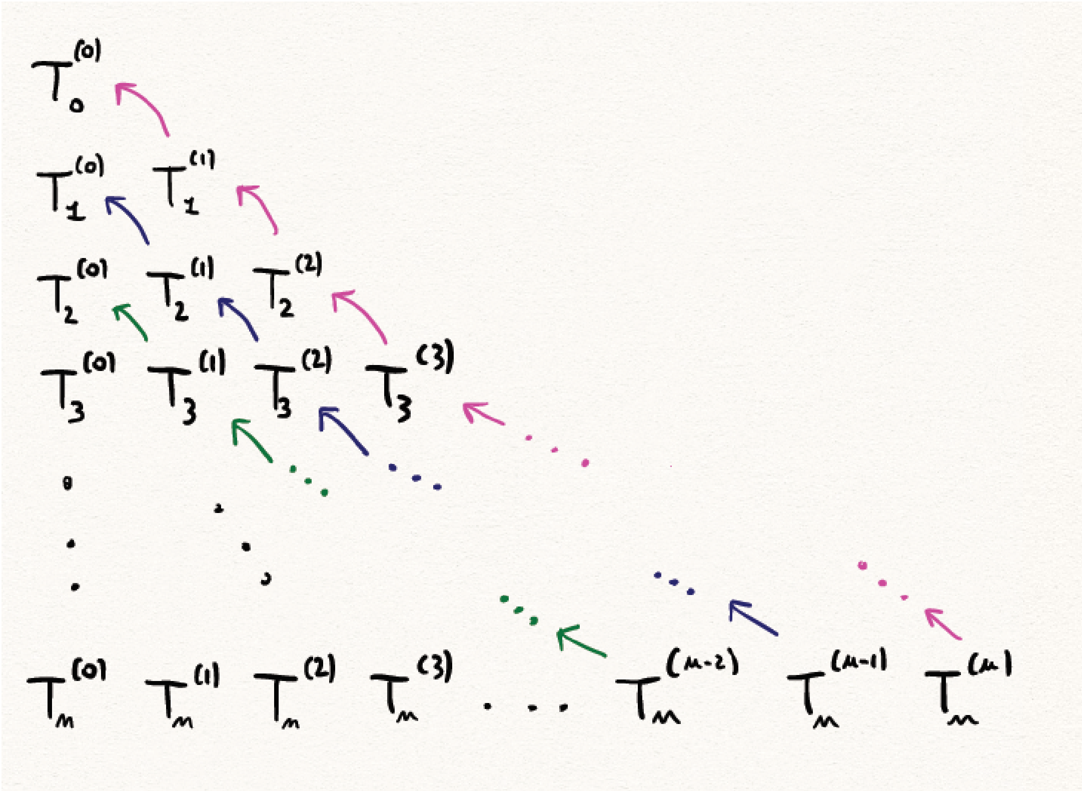

FIGURE 9

Remark 4.8. Notice that even if to prove the uniqueness of PL structures on $\mathbb{R}^n$ when $n \geq 5$ it was crucial to engulf compact sets, we actually needed the full strength of Stallings' version of the Engulfing Theorem, since in Lemma 4.7 we needed to engulf $T^{(2)}$ that is a non compact polyhedron.

Remark 4.9. Consider an exotic $\mathbb{R}^4$, i.e. a PL structure on $\mathbb{R}^4$ that is non isomorphic to the standard one. In such a PL manifold there must exist a compact set C that is not contained in any PL 4-ball, otherwise from the proof of Theorem 4.1 we would find an isomorphism with the standard $\mathbb{R}^4$.

As a further corollary of what we have proven so far we have a proof of the so called *weak Poincaré conjecture* in dimension ≥ 5.

Corollary 4.10 (High dimensional weak Poincaré conjecture)**.** *Suppose that M^n is a closed PL manifold homotopy equivalent to S^n, with $n \geq 5$. Then $M \cong_{Top} S^n$.*

Proof. Consider a point $p \in M$. We can use Proposition 4.11 (which is proved later) to deduce that $M \setminus \{p\}$ is contractible and simply connected at infinity. By virtue of Theorem 4.1 $M \setminus \{p\} \cong_{PL} \mathbb{R}^n$. Therefore M is the one-point compactification of $\mathbb{R}^n$, and hence a topological sphere. □

Proposition 4.11. *Let $n \geq 3$. Suppose that M^n is a closed topological manifold homotopy equivalent to S^n and consider a point $p \in M$. Then $M \setminus \{p\}$ is simply connected at infinity and contractible.*

Proof. We divide the proof in two parts.

Part 1: By definition there exist continuous $f\colon M \to S^n$ and $g\colon S^n \to M$ such that $gf \sim \mathrm{Id}_M$ and $fg \sim \mathrm{Id}_{S^n}$. Consider the north pole $N = (0, \dots, 0, 1) \in S^n$ and without loss of generality we can suppose that $p = g(N)$. Since any rotation of S^n is isotopic to the identity, we can compose f with a rotation of S^n and suppose that $g(p) = N$. We now prove that it is possible to find $g' \colon S^n \to M$ and homotopies $g'f \sim \mathrm{Id}_M$ and $fg' \sim \mathrm{Id}_{S^n}$ that fix respectively p and N.

First consider an arbitrary homotopy $\Phi \colon M \times I \to M$ between gf and Id_M. The image of $\{p\} \times I$ via this homotopy is a continuous loop $\gamma \colon I \to M$. We want to compose this homotopy with an isotopy of M that at each time brings back the point $\gamma(t)$ to p. To do this we consider an ambient isotopy of M extending the curve γ, i.e. an isotopy $\chi \colon M \times I \to M$ such that $\chi_t(p) = \gamma(t)$ and $\chi_1 = \mathrm{Id}_M$. The homotopy $\chi^{-1}\Phi \colon M \times I \to M$ defined by

$$(x, t) \quad \mapsto \quad \chi_t^{-1}(\Phi_t(x))$$

satisfies:

– $(\chi^{-1}\Phi)_t(p) = \chi_t^{-1}(\gamma(t)) = p$.
– $(\chi^{-1}\Phi)_1(x) = \chi_1^{-1}(\Phi_1(x)) = x$ for all $x \in M$.
– $(\chi^{-1}\Phi)_0(x) = \chi_0^{-1}(\Phi_0(x)) = \chi_0^{-1}(g(f(x))$.

Since $\chi_0^{-1}(p) = p$ we can replace g with $g' = \chi_0^{-1}g$. Now $g' \colon S^n \to M$ is such that $g'f \sim \mathrm{Id}_M$ fixing p.

We now want to proceed analogously with f.

First of all, notice that since χ_0^{-1} is isotopic to the identity of M, it is still true that $fg' \sim \mathrm{Id}_{S^n}$. Consider an arbitrary homotopy $\Psi \colon S^n \times I \to S^n$ between fg' and Id_{S^n}. Also in this case the image of $\{N\} \times I$ is a continuous loop δ in S^n. In the same way as before, we want to find an isotopy of S^n that at each time brings back the point $\delta(t)$ to N, but we can do this in a smarter way. In fact there is a fibration $\pi \colon \mathrm{SO}(n+1) \to S^n$ defined by

$$A \mapsto A(N).$$

Since the fibrations have the path lifting property, we can lift the path $\delta \colon I \to S^n$ to a path $\tilde{\delta} \colon I \to \mathrm{SO}(n+1)$ such that $\tilde{\delta}(1) = \mathrm{Id}$.

We can now define a homotopy $\tilde{\delta}^{-1}\Psi \colon S^n \times I \to S^n$ by

$$(y, t) \mapsto \tilde{\delta}_t^{-1}(\Psi_t(y)).$$

This homotopy satisfies:

– $(\tilde{\delta}^{-1}\Psi)_t(N) = \tilde{\delta}_t^{-1}(\delta(t)) = N$
– $(\tilde{\delta}^{-1}\Psi)_1(y) = \tilde{\delta}_1^{-1}(\Psi_1(y)) = y$ for all $y \in S^n$.
– $(\tilde{\delta}^{-1}\Psi)_0(y) = \tilde{\delta}_0^{-1}(\Psi_0(y)) = \tilde{\delta}_0^{-1}(f(g'(x)))$ for all $y \in S^n$.

So we have proven that there is a homotopy between Id_{S^n} and $\tilde{\delta}_0^{-1}fg'$ keeping N fixed. Since $\tilde{\delta}_0$ is a rotation of S^n that fixes N (it is a lifting via the fibration π of $\delta(0) = 0$), of course we deduce that there is also such a homotopy between Id_{S^n} and fg', that is what we wanted to prove.

Part 2: It follows from **Part 1** that $M \setminus \{p\}$ is homotopy equivalent to $\mathbb{R}^n$. This of course implies that $M \setminus \{p\}$ is contractible.

Notice that simply connectedness at infinity is not a homotopical invariant, since for example $\mathbb{R}^2$ is not simply connected at infinity but is homotopy equivalent to $\mathbb{R}^3$, which is. However our situation is way simpler. In fact consider a compact $C \subseteq M \setminus \{p\}$ and consider a small open n-ball in M containing p and not intersecting C. The complement of this ball in $M \setminus \{p\}$ is a compact D that contains C. By construction the complement of D is homeomorphic to a punctured n-ball, which is simply connected if $n \geq 3$. $\square$

References

[AH35] P. Alexandroff and H. Hopf. Topologie. 1935.

[Buo] Sandro Buoncristiano. Fragments of geometric topology from the sixties. `https://maths.dur.ac.uk/users/mark.a.powell/sandro.pdf`.

[Moi52] Edwin E. Moise. Affine structures in 3-manifolds: V. the triangulation theorem and hauptvermutung. *Annals of Mathematics*, 56(1):96–114, 1952.

[RS72] Colin Patrick Rourke and Brian Joseph. Sanderson. *Introduction to piecewise-linear topology*. Springer-Verlag Berlin, New York, 1972.

[Sco67] Anne Scott. Infinite regular neighborhoods. *J. London Math. Soc.*, 42:245–253, 1967.

[Sta62] John Stallings. The piecewise-linear structure of Euclidean space. *Proc. Cambridge Philos. Soc.*, 58:481–488, 1962.

[Tau87] Clifford Henry Taubes. Gauge theory on asymptotically periodic 4-manifolds. *J. Differential Geom.*, 25(3):363–430, 1987.

[Zee63] Eric Christopher Zeeman. *Seminar on Combinatorial Topology*. Number v. 2,No. 8 in Seminar on Combinatorial Topology. Institut des hautes etudes scientifique, 1963.

Scuola Normale Superiore, Pisa
Email address: diego.santoro@sns.it