Embedded Morse Theory and Relative Splitting of Cobordisms of Manifolds
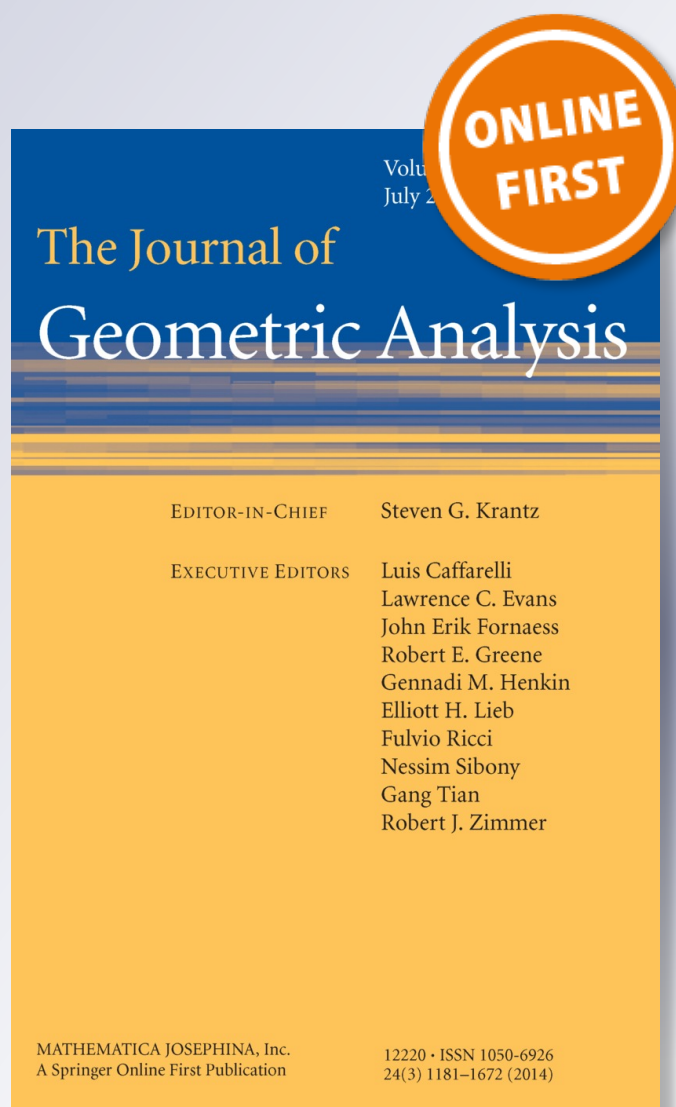
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Embedded Morse Theory and Relative Splitting of Cobordisms of Manifolds

Maciej Borodzik · Mark Powell

Abstract We prove that an embedded cobordism between manifolds with boundary can be split into a sequence of right product and left product cobordisms, if the codimension of the embedding is at least two. This is a topological counterpart of the algebraic splitting theorem for embedded cobordisms of the first author, A. Némethi and A. Ranicki. In the codimension one case, we provide a slightly weaker statement. We also give proofs of rearrangement and cancellation theorems for handles of embedded submanifolds with boundary.

1 Introduction

We investigate the Morse theory of embedded cobordisms of manifolds with boundary. An embedded cobordism (Z, Ω) is a cobordism Ω between two manifolds Σ_0 and Σ_1 in $Z \times [0, 1]$, where $\Sigma_i = \Omega \cap (Z \times \{i\})$ for $i = 0, 1$. Both Σ_0 and Σ_1 can have nonempty boundary, and $Y := \partial\Omega \setminus (\Sigma_0 \cup \Sigma_1)$ can also be nonempty. By a small perturbation it

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can be arranged that the projection map $F : Z \times [0, 1] \to [0, 1]$ restricts to a Morse function f on Ω.

An instance of an embedded cobordism is when there are embeddings of two closed $(n - 2)$-dimensional manifolds N_0 and N_1 into $Z = S^n$. That is, N_0 and N_1 are two non-spherical links in S^n, there is a cobordism between the two links and Σ_0, Σ_1 are Seifert surfaces for N_0 and N_1 respectively. A Pontrjagin–Thom construction guarantees the existence of Ω with the properties above. Morse theory can be used to study the relation between Seifert forms associated with Σ_0 and Σ_1; see [4].

Morse theory for manifolds with boundary was studied in the 1970s, by [6] and [8] independently. Recently, it was rediscovered by [10] in the context of Floer theory, and since then many articles about Morse theory for manifolds with boundary have appeared; see, for instance [3,5,11]. A recent application of it is the rapidly developing theory of bordered Heegaard Floer homology; compare [13].

Morse functions on manifolds with boundary have three types of critical point: interior, boundary stable and boundary unstable. A boundary critical point is stable or unstable according to whether the ascending or descending submanifold of the critical point lies in the boundary (see Definition 2.6 for more details).

The main result of this paper is the following theorem which shows that the critical points of an embedded cobordism of manifolds with boundary can be split, i.e., pushed to the boundary, where they become two boundary critical points, when the codimension is at least two. Thus the cobordism can be expressed as a cobordism with only boundary critical points. This theorem is the topological counterpart of [4, Main Theorem 1]. As in [5], boundary stable critical points correspond to the addition of left half-handles and boundary unstable critical points correspond to the addition of right half-handles.

Theorem 6.12 (6.11 (Global Handle Splitting Theorem)) *Let (Z, Ω) be an embedded cobordism such that $\Omega \subset Z \times [0, 1]$ has codimension 2 or more, and $\dim \Omega = n + 1$. Suppose that Ω, Σ_0 and Σ_1 have no closed connected components. Then there exists a map $F : Z \times [0, 1] \to [0, 1]$, which is homotopic through submersions (see Definition 2.14) to the projection onto the second factor, such that Ω can be expressed as a union:*

$$\Omega = \Omega_{-1/2} \cup \Omega_0 \cup \Omega_{1/2} \cup \Omega_1 \cup \Omega_{3/2} \cup \cdots \cup \Omega_{n+1/2} \cup \Omega_{n+1},$$

where $\Omega_i = \Omega \cap F^{-1}([(2i + 1)/(2n + 4), (2i + 2)/(2n + 4)])$ and

- $\Omega_{-1/2}$ *is a cobordism given by a sequence of index 0 handle attachments;*
- *if $i \in \{0, \ldots, n\}$, then Ω_i is a right product cobordism given by a sequence of elementary index i right product cobordisms;*
- *if $i + 1/2 \in \{1, \ldots, n+1\}$, then Ω_i is a left product cobordism given by a sequence of elementary index $i + 1/2$ left product cobordisms;*
- Ω_{n+1} *is a cobordism given by a sequence of index $n + 1$ handle attachments.*

We refer to Definition 2.10 for a simple explanation of elementary index i right/left product cobordisms. A more detailed description is given in [5, Sect. 2], or [3,8].

The case of codimension one is stated in Theorem 6.12. Due to problems with rearrangement of handles in codimension one, the results in codimension one are

slightly weaker, for example, we were unable to guarantee that index 1 and n handles split (see Proposition 6.10). Nevertheless, the flavor of the result is similar to the result in higher codimensions stated above.

Remark 1.1 Note that Theorem 6.11 is not just a corollary of the rearrangement theorem (Theorem 4.7 below) and the embedded analogue of the Thom–Milnor theorem that Ω can be expressed as a union $\Omega'_1 \cup \cdots \cup \Omega'_k$, where Ω'_k is a cobordism corresponding to a single critical point (interior or boundary).

The proof of Theorem 6.11 is more difficult, due to the requirement that all the interior critical points of non-extremal indices are absent. Being able to push interior critical points of non-extremal indices to the boundary is important when one wants to compare the Seifert forms as in [4]. Further explanation is given after the statement of Theorem 6.12.

We also give results detailing when critical points of Morse functions on embedded manifolds with boundary can be rearranged and canceled. These folklore results have been stated in the literature in the case of embedded manifolds with empty boundary, but we could not find detailed proofs.

The statements and proofs are new in the case of embedded manifolds with boundary. The rough idea is to adapt arguments of Milnor for the absolute case of closed manifolds, from [14,15], and to understand *ascending and descending membranes* (Definition 3.5), which appeared in the papers of Perron [16] and Sharpe [18]. Intersections involving these membranes can obstruct the rearrangement and cancellation of critical points of Morse functions on embedded manifolds, in cases when the operation could be performed in the absolute setting.

In order to state our rearrangement theorem we use the following definition.

Definition 1.2 A *configuration* Ξ of the critical points of an embedded cobordism (Z, Ω) is an assignment of a value $\Xi(z_i) \in (0, 1)$ to each critical point z_i of f.

An *admissible configuration* is a configuration satisfying the following conditions:

(A1) if z, w are critical points with indices k, l with $k < l$, then $\Xi(z) < \Xi(w)$;
(A2) if z, w have the same index k and if z is boundary stable and w is boundary unstable, then $\Xi(z) < \Xi(w)$.

Next we state our rearrangement theorem, which in the case that the codimension is at least two, essentially says that any admissible configuration can be realized.

Theorem 1.3 (4.7 (Global Rearrangement Theorem)) *Suppose* (Z, Ω) *is a cobordism. Given an admissible configuration* Ξ *of the critical points of* f, *if* $\mathrm{codim}(\Omega \subset Z \times [0, 1]) \geq 2$, *there exists a function* $G \colon Z \times [0, 1] \to [0, 1]$ *without critical points, homotopic through submersions to* F *(Definition 2.14), which restricts to a Morse function* $g \colon \Omega \to [0, 1]$, *such that* g *restricted to the critical points agrees with* Ξ *(the type and index of each critical point is preserved).*

Results on rearrangement in codimension one are given in Theorem 4.9.

The cancellation theorem is as follows; here there are no codimension restrictions.

Theorem 1.4 (5.1 (Elementary Cancellation Theorem)) *Let* (Z, Ω) *be a cobordism. Let* z *and* w *be critical points of* f *of indices* k *and* $k + 1$, *of the same type (i.e., either*

both interior, or both boundary stable, or both boundary unstable). Suppose that ξ is an embedded gradient-like vector field, which is Morse–Smale (Definition 4.3), and that there exists a single trajectory γ of ξ connecting z with w. If z, w are interior critical points, we require that $\gamma \subset \Omega$. If z, w are boundary critical points, then we require that γ is contained in the boundary of Z. Furthermore, suppose that there are no broken trajectories (see Definition 3.6) between z and w.

Then, for any neighborhood U of γ, there exists a vector field ξ' on $Z \times [0, 1]$, agreeing with ξ away from U, non-vanishing on U, and a function $F' \colon Z \times [0, 1] \to [0, 1]$ such that $f' = F'|_\Omega$ has the same critical points as f with the exception of z and w, which are regular points of f', and such that ξ' is an embedded gradient-like vector field with respect to F' (Definition 3.2).

1.1 Organization of the Paper

Section 2 reviews Morse theory for manifolds with boundary and gives the definitions and terminology that we will use throughout the paper. The Embedded Morse Lemma 2.16 is also proven.

Section 3 introduces the notion of embedded gradient-like vector fields, defines membranes, and develops embedded Morse theory. Section 3.1 shows how to recover a Morse function from an embedded gradient-like vector field. The Embedded Isotopy Lemma 3.10 is proved in Sect. 3.2.

Section 4 deals with rearrangement of critical points. Section 4.1 proves the Elementary Rearrangement Theorem 4.1, Sect. 4.2 looks at the dimensions of transverse intersections of membranes, and Sect. 4.3 gives the Global Rearrangement Theorem 4.7.

Section 5 looks at cancellation of critical points; the Elementary Cancellation Theorem 5.1 is proven in Sect. 5.1.

Section 6 proves our main result, concerning the pushing of interior critical points to the boundary, splitting a handle into two half-handles. The Elementary Handle Splitting Theorem 6.1 is proven in Sect. 6.1 and the Global Handle Splitting Theorem 6.11 is proven in Sect. 6.3.

2 Review of Morse Theory for Manifolds with Boundary

2.1 The Absolute Case

We begin by recalling various definitions from [5] for absolute cobordisms, i.e., we are not considering cobordism of embedded submanifolds yet.

Definition 2.1 An $(n + 1)$-dimensional *cobordism of manifolds with boundary* is a triple $(\Omega, \Sigma_0, \Sigma_1)$, where Ω is a manifold with boundary, $\dim \Omega = n + 1$, Σ_0 and Σ_1 are codimension 0 submanifolds of $\partial\Omega$ and

- the boundary of Ω decomposes as a union $\partial\Omega = \Sigma_0 \cup Y \cup \Sigma_1$ for some n-dimensional manifold with boundary Y;
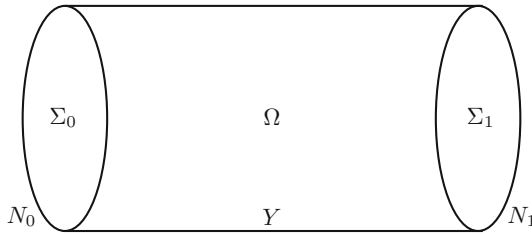
Fig. 1 An embedded cobordism

- we have $\partial \Sigma_0 = \Sigma_0 \cap Y =: N_0$, $\partial \Sigma_1 = \Sigma_1 \cap Y =: N_1$ and $\partial Y = N_0 \cup N_1$. In other words, Y is a cobordism between N_0 and N_1.

A schematic diagram is shown in Figure 1.

In many cases it is useful to assume that Ω is a manifold with corners, i.e., near $N_0 \cup N_1$ it is locally modeled on $\mathbb{R}^2_{\geq 0} \times \mathbb{R}^{n-1}$. Thus, tubular neighborhoods of Y, Σ_0 and Σ_1 are diffeomorphic to $Y \times [0, 1)$, $\Sigma_0 \times [0, 1)$ and $\Sigma_1 \times [0, 1)$ respectively.

Definition 2.2 A cobordism $(\Omega, \Sigma_0, \Sigma_1)$ is a *right product* if $\Omega \cong \Sigma_1 \times [0, 1]$. It is called a *left product* if $\Omega \cong \Sigma_0 \times [0, 1]$.

We remark that neither right nor left product cobordisms are necessarily trivial. We now recall a definition from [10].

Definition 2.3 Let $(\Omega, \Sigma_0, \Sigma_1)$ be a cobordism. A function $f : \Omega \to [0, 1]$ is called a *Morse function* if $f^{-1}(0) = \Sigma_0$, $f^{-1}(1) = \Sigma_1$, all the critical points of f which lie in the interior Int Ω are nondegenerate (the Hessian matrix is non-singular), $f_Y := f|_Y$ is a Morse function on Y and, furthermore, for each $y \in Y$ such that $df(y) \neq 0$ we have

$$\ker df(y) \not\subset T_y Y. \tag{2.1}$$

Remark 2.4 Our notation differs from that of [5] in two places. First, we use small f to denote the Morse function on Ω (the capital F is reserved for another function). Furthermore, we write N_0 and N_1 instead of M_0 and M_1, since M is reserved for the *membrane*; see Definition 3.5 below.

In [10] and [5], condition (2.1) was replaced by a condition on the gradient of f, so apparently depending on a choice of a metric. The definition that we use does not involve choosing a Riemannian metric. The next lemma shows that the two points of view are interchangeable.

Lemma 2.5 *(see [5, Lemma 1.7]) If f satisfies (2.1) at each $y \in Y$, then there exists a Riemannian metric on Ω such that ∇f is everywhere tangent to Y.*

From now on, given a Morse function M we shall assume that there is a Riemannian metric chosen so that ∇f is everywhere tangent to Y.

Let us now consider a critical point $z \in \Omega$ of f. If $z \in$ Int Ω, we shall call it an *interior* critical point. If $z \in Y$, then we shall call it a *boundary* critical point. Obviously, a boundary critical point is also a critical point of the restriction $f|_Y$. There are two types of boundary critical points. The classification depends on whether the flow of ∇f near z repels or attracts in the direction of the normal to the boundary. More precisely, we have the following definition.

Definition 2.6 A boundary critical point z is called *boundary stable* if $T_z Y$ contains the tangent space at z to the unstable manifold of z. If $T_z Y$ contains the tangent space at z to the stable manifold of z, then z is called *boundary unstable*.

Definition 2.7 The *index* of a boundary critical point z is the dimension of the stable manifold of z.

Remark 2.8 If $z \in Y$ is a boundary stable critical point of index k, then it is a critical point of $f|_Y$, but of index $k - 1$.

The change in the topology of the set $f^{-1}([0, x])$ as x crosses the critical value corresponding to a boundary critical point is described in detail in [5, Sect. 2.4]. See also [3,8].

Proposition 2.9 *Let $(\Omega, \Sigma_0, \Sigma_1)$ be a cobordism. Assume that there exists a Morse function $f : \Omega \to [0, 1]$ such that f has only boundary stable (respectively, boundary unstable) critical points. Then $(\Omega, \Sigma_0, \Sigma_1)$ is a left product cobordism (respectively, right product cobordism).*

Definition 2.10 A cobordism $(\Omega, \Sigma_0, \Sigma_1)$ is called an *elementary index i right product cobordism*, (respectively, an *elementary index i left product cobordism*) if there exists a Morse function $f : \Omega \to [0, 1]$, that has a single critical point, this critical point has index i and is boundary unstable (respectively, boundary stable).

The previous definition is the analogue in the boundary case of a single index i handle attachment corresponding to an interior critical point. The topological meaning is explained in detail in [5, Sect. 2] and in [8]. We also refer to [4] for the study of homological properties of elementary right/left product cobordisms.

2.2 The Embedded Case

We shall now set up the notation for the embedded case.

Definition 2.11 A quadruple $(Z, \Omega, \Sigma_0, \Sigma_1)$ is called an *embedded cobordism* if Z is a closed manifold, $(\Omega, \Sigma_0, \Sigma_1)$ is a cobordism and Ω is embedded in $Z \times [0, 1]$ in such a way that $\Omega \cap Z \times \{0\} = \Sigma_0$ and $\Omega \cap Z \times \{1\} = \Sigma_1$ and Ω is transverse to $Z \times \{0, 1\}$. The *codimension* of the embedding is the quantity

$$\dim Z - \dim \Sigma_0 = \dim Z - \dim \Omega + 1.$$

An embedded cobordism is *nondegenerate* if the function $F : Z \times [0, 1] \to [0, 1]$ given by projection onto the second factor restricts to a Morse function on Ω.

Every embedded cobordism can be modified by a C^2-small perturbation to a nondegenerate one. From now on we shall assume that all embedded cobordisms are nondegenerate.

Moreover, whenever we write (Z, Ω) as a cobordism, we understand the whole structure: the cobordism Ω is actually $(\Omega, \Sigma_0, \Sigma_1)$, $Y = \mathrm{cl}(\partial\Omega \setminus (\Sigma_0 \cup \Sigma_1))$, $F : Z \times [0, 1] \to [0, 1]$ is a function without critical points and $f = F|_\Omega$. We will usually denote $m = \dim Z$ and $n + 1 = \dim \Omega$.

Remark 2.12 The whole theory could be developed in a more general setting, with $Z \times [0, 1]$ replaced by a compact $(m+1)$-dimensional manifold X, and F also allowed to have Morse critical points in $X \setminus \Omega$. The special case $X = Z \times [0, 1]$ is more transparent, and making the generalization is straightforward.

Definition 2.13 Let (Z, Ω) be an embedded cobordism. The function $f : \Omega \to [0, 1]$ given by $f = F|_\Omega$ is called the *underlying Morse function*.

We shall need one more notion.

Definition 2.14 For a cobordism (Z, Ω) the family F_t of functions from $Z \times [0, 1] \to [0, 1]$ is called a *(nondegenerate) homotopy through submersions* if:

(i) for any t, $F_t(Z \times \{0\}) = 0$, $F_t(Z \times \{1\}) = 1$ and F_t does not have any critical points on $Z \times [0, 1]$;
(ii) there exists a (possibly empty) finite set $\{t_1, \ldots, t_k\} \subset (0, 1)$ such that if $t \notin \{t_1, \ldots, t_k\}$, then F_t restricts to a Morse function on Ω.

We will now choose a special metric on $Z \times [0, 1]$, which will be used later.

Lemma 2.15 *For any two open subsets $V \subset U \subset Z \times [0, 1]$, such that V contains all the critical points of f and $\overline{V} \subset U$, and for any Riemannian metric g on U there exists a Riemannian metric h on $Z \times [0, 1]$ such that we have*

$$\nabla f(x) = \nabla F(x) \quad \text{for any } x \in \Omega \setminus U.$$

Here we understand that ∇f is computed using the metric on Ω induced from the metric on $Z \times [0, 1]$ (recall that $f = F|_\Omega$). Furthermore, if $y \in Y$, we may assume that $\nabla f(y) \in T_y Y$. Finally, the new metric agrees with g on V.

Proof We split the complement of U into three cases: away from Ω, in $\Omega \setminus Y$ and finally in Y.

For each point $x \in Z \times [0, 1] \setminus (U \cup \Omega)$ we choose a ball B_x with center x, disjoint from $\Omega \cup V$ and an arbitrary metric h_x on B_x.

Now we consider $x \in \Omega \setminus (U \cup Y)$. Let $m := \dim Z + 1$. Choose a ball B_x which is disjoint from $V \cup Y$, such that there is a local coordinate system $(x_1, \ldots, x_k, x_{k+1}, \ldots, x_m)$ in B_x, centered on x, in which $\Omega \cap B_x = \{x_{k+1} = \cdots = x_m = 0\}$. As B_x does not contain any critical points of f, we may assume that $\frac{\partial F}{\partial x_1}(z) \neq 0$ for all $z \in B_x$; we potentially need to relabel the coordinate system and/or take a smaller ball for B_x.

Let us assume for now that $\frac{\partial F}{\partial x_1}(z) > 0$ for all $z \in B_x$. Choose an $m \times m$ matrix $M(z)$ such that $M_{1j}(z) = M_{j1}(z) = \frac{\partial F}{\partial x_j}(z)$ for $j = 1, \ldots, m$, with all other entries chosen so that $M(z)$ is positive definite and depends smoothly on z. This can always be achieved, since $M_{11} > 0$. The matrix M determines a metric on B_x, with respect to the standard basis $\{\partial/\partial x_i \mid i = 1, \ldots, m\}$ for $T_z Z$ induced by the coordinate system. This metric is denoted h_x, and using this metric we have $\nabla F = (1, 0, \ldots, 0) \in T_z \Omega$. (If $\frac{\partial F}{\partial z} < 0$ we get a metric such that $\nabla F = (-1, 0, \ldots, 0)$.) To see this, recall that ∇F is the unique vector field v such that $h_x(v_z, \cdot) = dF_z \in T_z^* Z$. Then observe that for any tangent vector $\begin{bmatrix} a_1 & \ldots & a_m \end{bmatrix}^T \in T_z Z$ we have:

$$\begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} M_{11} & \ldots & M_{1m} \\ \vdots & * & * \\ M_{m1} & * & * \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} = \sum_{i=1}^{m} a_i \frac{\partial F}{\partial x_i} = dF \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix},$$

where we have omitted the point $z \in Z$ from the notation. By functoriality, ∇f is the image of the orthogonal projection of ∇F onto $T_z \Omega$. In particular, for any $z \in B_x \cap \Omega$, we have $\nabla f(z) = \nabla F(z)$.

If $x \in Y \setminus U$, we use the same argument to construct a metric h_x on a ball B_x, where we assume that $B_x \cap V = \emptyset$. This time we assume additionally that the coordinate system on B_x is chosen so that $Y = \Omega \cap \{x_k = 0\}$. Then we have $\frac{\partial F}{\partial x_j}(0, \ldots, 0) \neq 0$ for some $j \in \{1, \ldots, k-1\}$, for otherwise either y is a critical point of f, or the condition (2.1) is violated. Then we choose a matrix $M(z)$ on (an again potentially smaller) B_x, which in turn induces a metric h_x on B_x, similarly to the case above, in such a way that for all $z \in B_x \cap \Omega$ we have $\nabla f(z) = \nabla F(z)$ and, if additionally $z \in Y$, then $\nabla f(z) \in T_z Y$.

The balls B_x, where x runs through all points in $Z \times [0, 1] \setminus U$, together with U constitute an open covering of $Z \times [0, 1]$, which is compact. Let ϕ_U, ϕ_x for $x \in Z \times [0, 1] \setminus U$ be a partition of unity subordinate to this covering. We define a metric $h = \phi_U \cdot g + \sum_x \phi_x \cdot h_x$. It has all the desired properties. □

We conclude the section with a standard but fundamental result.

Lemma 2.16 (Embedded Morse lemma) *Assume that $p \in \Omega$ is a critical point of f of index k. Then there exist local coordinates $x_1, \ldots, x_{n+1}, y_1, \ldots, y_{m-n}$ in $Z \times [0, 1]$ centered at p, such that in these coordinates:*

$$F(x_1, \ldots, x_{n+1}, y_1, \ldots, y_{m-n}) = -x_1^2 - x_2^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_{n+1}^2 + y_1 + F(p),$$

and such that moreover:

- *if p is an interior critical point, then the intersection of Ω with this coordinate system is given by*

$$\{y_1 = \cdots = y_{m-n} = 0\};$$

- *if p is a boundary stable critical point, then Ω is given by*

$$\{y_1 = \cdots = y_{m-n} = 0\} \cap \{x_1 \geq 0\}; \qquad (2.2)$$

- *if p is a boundary unstable critical point, then Ω is given by*

$$\{y_1 = \cdots = y_{m-n} = 0\} \cap \{x_{n+1} \geq 0\}. \qquad (2.3)$$

Proof First let us consider the case that p is an interior critical point. The Morse lemma (see, e.g., [14, Lemma 2.2]) says that there exist local coordinates $x_1, \ldots, x_{n+1}$ on an open neighborhood $V \subset \Omega$ such that $f = F|_\Omega$ is equal to $F(0) - x_1^2 - x_2^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_{n+1}^2$. Let $\tilde{y}_1, \ldots, \tilde{y}_{m-n}$ be local coordinates in the normal bundle of V in $Z \times [0, 1]$. Then $x_1, \ldots, x_{n+1}, \tilde{y}_1, \ldots, \tilde{y}_{m-n}$ form a local coordinate system in $Z \times [0, 1]$ in a neighborhood of p. By a linear change we may assume that $\frac{\partial \tilde{F}}{\partial y_j} = 0$ if $j > 1$. Let us define

$$y_1 := F(x_1, \ldots, x_{n+1}, \tilde{y}_1, \ldots, \tilde{y}_{m-n}) - (-x_1^2 - x_2^2 - \cdots + x_{k+1}^2 + \cdots + x_{n+1}^2) - F(p)$$

and $y_j := \tilde{y}_j$ for $j = 2, \ldots, m - n$. Consider now the map of open neighborhoods of $\mathbb{R}^{m+1}$ given by

$$\Phi \colon (x_1, \ldots, x_{k+1}, \tilde{y}_1, \ldots, \tilde{y}_{m-n}) \mapsto (x_1, \ldots, x_{k+1}, y_1, y_2, \ldots, y_{m-n}).$$

The derivative of Φ at 0 is diagonal and nondegenerate, hence Φ is a local diffeomorphism by the inverse function theorem. Furthermore, the set $\{y_1 = \cdots = y_{m-n} = 0\}$ is invariant under Φ. The importance of these two facts is that $(x_1, \ldots, y_{m-n})$ forms a local coordinate system near $p \in Z \times [0, 1]$. In this system Ω is given by $\{y_1 = \cdots = y_{m-n} = 0\}$ and F has the form as described in the statement.

The proof in the case of boundary critical points is analogous. Using [5, Lemma 2.6] we find local coordinates on Ω such that (2.2) or (2.3) is satisfied. Then we extend this coordinate system as in the case of an interior critical point. We leave filling in further details. $\qquad\square$

3 Embedded Gradient-Like Vector Fields

We need to develop the theory of embedded gradient-like vector fields in order to prove our subsequent results on rearrangement, cancellation and splitting.

We will soon introduce a notion of gradient-like vector fields for embedded submanifolds. First let us recall the definition of a gradient-like vector field for manifolds with boundary.

Definition 3.1 (Compare [5, Definition 1.5]) Let $(\Omega, \Sigma_0, \Sigma_1)$ be a cobordism, and let f be a Morse function. We shall say that a vector field ξ on Ω is *gradient-like with respect to f*, if the following conditions are satisfied:

(a) $\xi \cdot f := df(\xi) > 0$ away from the set of critical points of f;
(b) if p is a critical point of f of index k, then there exist local coordinates $x_1, \ldots, x_{n+1}$ in a neighborhood of p, such that

$$f(x_1, \ldots, x_{n+1}) = f(p) - (x_1^2 + \cdots + x_k^2) + (x_{k+1}^2 + \cdots + x_{n+1}^2)$$

and

$$\xi = (-x_1, \ldots, -x_k, x_{k+1}, \ldots, x_{n+1}) \text{ in these coordinates.}$$

(b′) Furthermore, if p is a boundary critical point, then the above coordinate system can be chosen so that $Y = \{x_j = 0\}$ and $\Omega = \{x_j \geq 0\}$ for some $j \in \{1, \ldots, n+1\}$.
(c) The vector field ξ is tangent to Y on all of Y.

Now let us define an analogue for embedded cobordisms. Observe that we cannot simultaneously assume that $\xi \cdot F > 0$ away from the critical points of F and that ξ is everywhere tangent to Ω, because these two conditions are mutually exclusive if $f = F|_\Omega$ has critical points. The vector field that we define below has a critical point at each critical point of f. The following definition comes from R. Sharpe's paper [18, p. 67] and turns out to provide a very useful analytic language for studying embedded cobordisms.

Definition 3.2 Let (Z, Ω) be an embedded cobordism, with $F: Z \times [0, 1] \to [0, 1]$ the projection and the underlying Morse function f. A vector field ξ on $Z \times [0, 1]$ is an *embedded gradient-like vector field* with respect to F if:

(a) for any $x \in Z \times [0, 1]$ which is not a critical point of f, we have $(\xi \cdot F)_x = dF_x(\xi_x) > 0$;
(b) for any $x \in \Omega$ we have $\xi_x \in T_x(\Omega)$, and for any $y \in Y$ we have $\xi_y \in T_y Y$;
(c) for any $p \in \Omega$ such that $df(p) = 0$, there exists an open subset $U \subset Z \times [0, 1]$ with a choice of local coordinates $(x_1, \ldots, x_{n+1}, y_1, \ldots, y_{m-n})$ centered at p such that $U \cap \Omega$ is given by $\{y_1 = \cdots = y_{m-n} = 0\}$ (if p is a boundary critical point, then $U \cap \Omega = \{x_1 \geq 0, y_1 = \cdots = y_{m-n} = 0\}$ and $Y = \{x_1 = y_1 = \cdots = y_{m-n} = 0\}$), ξ in these local coordinates has the form

$$(-x_1, -x_2, \ldots, -x_k, x_{k+1}, \ldots, x_{n+1}, (y_1^2 + \cdots + y_{m-n}^2), 0, \ldots, 0) \quad (3.1)$$

and

$$F(x_1, \ldots, y_{m-n}) = F(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_{n+1}^2 + y_1. \quad (3.2)$$

Note that

$$dF(\xi) = \sum_{i=1}^{n+1} 2x_i^2 + \sum_{\ell=1}^{m-n} y_\ell^2$$

so that (a) and (c) are consistent.

We have the following result.

Proposition 3.3 *For any cobordism (Z, Ω) there exists an embedded gradient-like vector field.*

For the convenience of the reader we present a straightforward proof.

Proof Let $p_1, \ldots, p_r$ be the critical points of $F|_{\Omega}$. For each $j \in \{1, \ldots, r\}$ we choose an open neighborhood U'_j in $Z \times [0, 1]$ such that the Embedded Morse Lemma 2.16 holds, i.e., there exist coordinates of a special form as set out in that lemma. We choose U_j to be a neighborhood of p_j such that $\overline{U_j} \subset U'_j$. Furthermore, let V be an open subset of $Z \times [0, 1]$ such that $V \cup \bigcup U'_j = Z \times [0, 1]$ and $U_j \cap V = \emptyset$ for any $j \in \{1, \ldots, r\}$. Let $\phi_V, \phi_1, \ldots, \phi_r$ be a partition of unity subordinate to the covering $V \cup \bigcup U'_j$. In particular, $\phi_j|_{U_j} \equiv 1$.

Choose a Riemannian metric on any U_j such that the local coordinates $x_1, \ldots, x_{n+1}$, $y_1, \ldots, y_{m-n}$ are orthogonal. By Lemma 2.15 there exists a Riemannian metric on $Z \times [0, 1]$ such that $\nabla F(x)$ is tangent to Ω for all $x \in \Omega \setminus V$ and $\nabla F(x)$ is tangent to Y for all $x \in Y \setminus V$.

We define a vector field on V by $\xi_V = \nabla F$ and ξ_j on U'_j by the explicit formula (3.1). Then $\xi = \phi_V \xi_V + \sum \phi_j \xi_j$ is a vector field on $Z \times [0, 1]$ which, by construction, satisfies the desired properties. $\qquad\square$

We remark that ξ has a critical point at each critical point p of f. This is not a Morse critical point, because the coordinate corresponding to y_1 vanishes up to order 2. Nevertheless, we have well-defined stable and unstable manifolds of ξ at p, which we now discuss. The following lemma is a consequence of the local description of a critical point.

Lemma 3.4 *In a neighborhood of a critical point, in local coordinates as in Definition 3.2, the stable manifold is given by*

$$\{x_{k+1} = \cdots = x_{n+1} = 0, \ y_2 = \cdots = y_{m-n} = 0, \ y_1 < 0\}.$$

The unstable manifold is given by

$$\{x_1 = \cdots = x_k = 0, \ y_2 = \cdots = y_{m-n} = 0, \ y_1 > 0\}.$$

The intersection of a stable manifold (respectively, unstable manifold) with a level set $F^{-1}(p - \varepsilon)$, for $\varepsilon > 0$ sufficiently small (respectively, with a level set $F^{-1}(p + \varepsilon)$) is a k-dimensional disc (respectively, $(n + 1 - k)$-dimensional disc). The boundary of the disc is the stable manifold of $\xi|_{\Omega}$ (respectively, the unstable manifold of $\xi|_{\Omega}$).

The following terminology is essentially due to B. Perron [16].

Definition 3.5 Let p be a critical point of f. The *ascending membrane* M^u_p is the unstable manifold of p with respect to ξ. The *descending membrane* M^s_p is the stable manifold of p with respect to ξ.

From now on, when speaking of a stable and unstable manifold of ξ, we will understand a stable and unstable manifold of $\xi|_{\Omega}$, since we will use the term membrane of Definition 3.5 for the ambient version.

3.1 Integrating the Vector Field to Recover the Morse Function

Starting with a Morse function F and a gradient-like vector field ξ, we might wish to alter the vector field ξ to ξ', and the altered vector field ξ' may then not necessarily be a gradient-like vector field for F. Under some conditions we shall be able to find a function F', such that ξ' is a gradient-like vector field with respect to F'. This is the idea of the Vector Field Integration Lemma 3.8 below.

We remark that in this paper the Integration Lemma 3.8 is only used in the proof of the Elementary Cancellation Theorem 5.1. We present the Integration Lemma separately since we think it is of interest independently from the cancellation theorem.

Before stating and proving the Vector Field Integration Lemma, first we need to introduce some more terminology. Part (a) of the next definition is standard.

Definition 3.6 Let ξ be a smooth vector field $Z \times [0, 1]$.

(a) A *trajectory* is a map $\gamma \colon A \to Z \times [0, 1]$, where A is a connected subset of $\mathbb{R}$, such that $\frac{d}{dt} \gamma(t) = \xi(\gamma(t))$. We will always assume that A is maximal, i.e., γ cannot be extended over a larger subset of $\mathbb{R}$. Note that, up to reparameterization, that is changing t to $t + a$ for some $a \in \mathbb{R}$, exactly one trajectory of ξ passes through a given point of $Z \times [0, 1]$.

(b) A *broken trajectory* is a union of trajectories $\gamma_1, \ldots, \gamma_s$ such that for any $j = 1, \ldots, s-1$ we have $\lim_{t \to \infty} \gamma_j(t) = \lim_{t \to -\infty} \gamma_{j+1}(t)$. These limits are critical points of ξ.

Note that there is no mention of the function F in the next definition, which introduces the notion of an almost gradient-like vector field. In Lemma 3.8 we will see that an almost gradient-like vector field with one extra assumption is sufficient to be able to recover a Morse function with respect to which the vector field is an embedded gradient-like vector field.

Definition 3.7 A vector field ξ on $Z \times [0, 1]$ is called *almost gradient-like* if it satisfies the following conditions:

(AG1) ξ is tangent to Ω and to Y;

(AG2) ξ has no critical points on $Z \times [0, 1] \setminus \Omega$ and no critical points on $Z \times \{0, 1\}$;

(AG3) ξ has finitely many critical points on Ω. For each critical point z of ξ, there exist local coordinates near z, denoted by $x_1, \ldots, x_{n+1}, y_1, \ldots, y_{m-n}$ such that Ω in the local coordinates is given by $y_1 = \cdots = y_{m-n} = 0$ (if z belongs to Y, there is one more equation, namely $x_1 \geq 0$) and there is an index k such that in these local coordinates ξ has the form as in (3.1):

$$(-x_1, -x_2, \ldots, -x_k, x_{k+1}, \ldots, x_{n+1}, (y_1^2 + \cdots + y_{m-n}^2), 0, \ldots, 0); \text{ and}$$

(AG4) if a trajectory $\gamma(t)$ does not hit $Z \times \{0\}$, then $\lim_{t \to -\infty} \gamma(t)$ exists (and by standard arguments observing that $\lim_{t \to -\infty} \gamma'(t) = 0$ it is a critical point of ξ). Likewise, if $\gamma(t)$ does not hit $Z \times \{1\}$, then $\lim_{t \to +\infty} \gamma(t)$ exists.

If ξ is an embedded gradient-like vector field, then it is easy to see that it is an almost gradient-like vector field. Given an almost gradient-like vector field ξ it is possible

that there does not exist any function F with respect to which ξ is an embedded gradient-like vector field. For example, the conditions (AG1)–(AG4) do not exclude the possibility that there exists a broken trajectory starting and ending at the same point. As the next result shows, this is the only obstruction.

Lemma 3.8 (Vector Field Integration Lemma) *Suppose that ξ is an almost gradient-like vector field such that there are no broken trajectories starting and ending at the same point. Then there exists a smooth function F without critical points on $Z \times [0, 1]$ such that ξ is an embedded gradient-like vector field with respect to F.*

Proof Let $z_1, \ldots, z_s$ be the critical points of ξ. We introduce a partial order relation on the critical points, namely, we say that $z_i < z_j$ if there is at least one broken trajectory starting at z_i and ending at z_j. The assumption of the lemma guarantees that this is a partial order. Let us relabel the critical points so that if $i < j$, then we cannot have $z_j < z_i$. This relabeling is, in general, not unique. The proof of Lemma 3.8 continues after the statement and proof of Lemma 3.9. □

Lemma 3.9 *There exist open neighborhoods $V_1, \ldots, V_s$ of $z_1, \ldots, z_s$ such that for any $i \leq j$, there is no trajectory which leaves V_j and then enters V_i.*

Proof of Lemma 3.9 The proof follows the ideas of [15, Assertion 1, p. 50]. Suppose that the statement is false. That is, suppose that for all open neighborhoods $V_1, \ldots, V_s$ there exists $i \leq j$ and a trajectory which leaves V_j and later enters V_i.

Let U_0^i be a coordinate neighborhood of z_i from (AG3) of Definition 3.7. For $r \geq 1$ we define $U_r^i = \{x_1^2 + \cdots + x_{n+1}^2 + y_1^2 + \cdots + y_{m-n}^2 \leq \frac{\varepsilon}{r}\}$ for some ε sufficiently small (so that $U_1^i \subset U_0^i$).

Now suppose that there are indices i, j, with $i \leq j$ and, for all r, that there is a trajectory γ_r going from U_r^j to U_r^i. Let us choose a maximal j such that this holds. Working in local coordinates we convince ourselves that γ_r must intersect ∂U_1^j and ∂U_1^i. Let w_r^1 be a point where γ_r leaves ∂U_r^j, and let w_r^2 be the point where γ_r hits ∂U_1^j after leaving ∂U_r^j. Let w_r^3 be the point where γ_r hits ∂U_r^i for the first time after w_r^2.

Since all points $w_r^2, r = 2, 3, \ldots$ belong to the sphere ∂U_1^j, up to passing to a subsequence we can assume w_r^2 converges to a point $w_0 \in \partial U_1^j$. Let γ_0 be the trajectory through w_0. We claim that $\lim_{t \to -\infty} \gamma_0(t) = z_j$. To see this, observe that for any $l > r$ the trajectory through w_l^2 hits ∂U_r^j in the past. As $w_l^2 \to w_0$ and ∂U_r^j is closed, we infer that γ_0 hits ∂U_r^j in the past as well. But r was arbitrary, so there exists a sequence t_k converging to $-\infty$, such that $\gamma_0(t_k) \to z_j$. By (AG4), γ_0 is a trajectory starting from z_j. We do not claim that γ_0 ends in z_i, because the sequence of trajectories γ_r can converge to a broken trajectory, a part of which is γ_0.

To complete the proof, let us look at $\lim_{t \to +\infty} \gamma_0(t)$. Observe that the time which the trajectory γ_r takes to go from w_r^2 to w_r^3 goes to infinity as $r \to \infty$ (because the speed of the vector field near critical points is very small). Therefore, $\gamma_0(t)$ exists for any $t > 0$; in particular, γ_0 cannot hit $Z \times \{1\}$. By (AG4), $\lim_{t \to +\infty} \gamma_0(t) = z_k$ for some $k \in \{1, \ldots, s\}$. This means that $k > j$.

Consider now neighborhoods U_r^k as defined above. As $\lim \gamma_0(t) = z_k$, it follows that γ_0 hits all the boundaries ∂U_r^k. If we fix r, it follows that for l sufficiently large γ_l hits ∂U_r^k as well. We can now relabel the trajectories so that γ_r hits ∂U_r^k and then U_r^i, i.e., we want a trajectory which hits ∂U_p^k to have index p; this may involve passing to a further subsequence in the γ_r. Then the sequence $\{\gamma_r\}_{r \geq 1}$ may be considered as a sequence of trajectories coming close to z_k first, and then to z_i. But we assumed that j is the maximal index for which this is possible, and $k > j$, so we obtain a contradiction. □

We resume the proof of Lemma 3.8. Let us now choose open neighborhoods $V_1, \ldots, V_s$ of $z_1, \ldots, z_s$ as given to us by Lemma 3.9. We define a function F to be equal to $-x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_{n+1}^2 + y_1 + j/(s+1)$ on $\overline{V}_j$, where the signs agree with the signs in (AG3). In particular, $F(z_j) = j/(s+1)$. For the sake of completeness we define $\overline{V}_0 := Z \times \{0\}$ and $\overline{V}_{s+1} := Z \times \{1\}$. We also put $F(\overline{V}_0) = 0$ and $F(\overline{V}_{s+1}) = 1$.

Now let us choose any point $z \in Z \times [0, 1] \setminus \bigcup \overline{V}_j$. Let γ be a trajectory through z. Reparameterize γ so that $\gamma(0) = z$. By (AG4) there exists $a, b \in \mathbb{R}$, $-a < 0 < b$ such that $\gamma(-a) \in \overline{V}_i$, $\gamma(b) \in \overline{V}_j$ for some i, j, and $\gamma(-a, b)$ does not intersect $V_1 \cup \cdots \cup V_s$. By Lemma 3.9 we have $i < j$. Then we define

$$F(z) = \frac{b}{a+b} F(\gamma(-a)) + \frac{a}{a+b} F(\gamma(b)).$$

Since $F(\gamma(-a)) < F(\gamma(b))$ (because $i < j$), the function F increases along γ. Thus $\xi \cdot F(z) > 0$.

In general, the function F is a continuous, piecewise smooth function with all non-smooth points lying on $\partial V_1 \cup \cdots \cup \partial V_s$. The function F satisfies $\xi \cdot F(z) > 0$ for $z \in V_j \setminus \{z_j\}$ and whenever $\xi \cdot F(z)$ is well-defined. Therefore, we can slightly perturb F to a smooth function, which still satisfies $\xi \cdot F > 0$ away from $z_1, \ldots, z_s$. □

3.2 The Embedded Isotopy Lemma

Next we are going to sketch a proof of the embedded analogue of the Isotopy Lemma of [15, Lemma 4.7].

Lemma 3.10 (Embedded Isotopy Lemma) *Let (Z, Ω) be an embedded cobordism. Suppose that there are two level sets $a, b \in [0, 1]$ with $a < b$ such that f has no critical points on $\Omega \cap f^{-1}[a, b]$. Let ξ be an embedded gradient-like vector field. Let h be a diffeomorphism of the triple $(Z \times \{b\}, \Omega \cap f^{-1}(b), Y \cap f^{-1}(b))$ to itself which is isotopic to the identity.*

Then there exists an embedded gradient-like vector field ξ', agreeing with ξ away from $F^{-1}(a, b)$, such that $\psi' = h \circ \psi$, where ψ and ψ' are the diffeomorphisms

$$\psi, \psi' : (Z, \Omega \cap f^{-1}(a), Y \cap f^{-1}(a)) \to (Z, \Omega \cap f^{-1}(b), Y \cap f^{-1}(b))$$

induced by the flows of the vector fields ξ and ξ' respectively.

Proof We follow [15, Proof of Lemma 4.7]. Let h_t, $t \in [a, b]$ be an isotopy of the triple $(Z, \Omega \cap f^{-1}(b), Y \cap f^{-1}(b))$ so that h_a is the identity and $h_b = h$. We also assume that h_t does not depend on t for t close to a and b. Let $H \colon Z \times [a, b] \to Z \times [a, b]$ be given by $(x, t) \to (h_t(x), t)$. Define a diffeomorphism $\Psi \colon Z \times [a, b] \to Z \times [a, b]$ by integrating the flow of ξ; that is, if we take a point $(x, t) \in Z \times [a, b]$, there is a trajectory of ξ passing through that point. This trajectory hits a point $(x', a) \in Z \times \{a\}$. We define $\Psi(x, t) = (x', t)$. It is easy to check that Ψ is a diffeomorphism. Now let $\Phi := \Psi^{-1} \circ H \circ \Psi \colon Z \times [a, b] \to Z \times [a, b]$ and we define $\xi' := \Phi_*(\xi)$ on $Z \times [a, b]$. Since H is the identity near $Z \times \{a, b\}$, we infer that Φ is the identity near $Z \times \{a, b\}$, and hence ξ' agrees with ξ in a neighborhood of $Z \times \{a, b\}$. We extend ξ' to $Z \times [0, 1]$ by making ξ' equal to ξ on $Z \times ([0, a] \cup [b, 1])$.

By definition, the flow of ξ' induces a diffeomorphism of the triple $(Z \times \{a\}, \Omega \cap f^{-1}(a), Y \cap f^{-1}(a))$ to $(Z, \Omega \cap f^{-1}(b), Y \cap f^{-1}(b))$, which is equal to $h \circ \psi$. Note that, by construction, ξ' is tangent to Ω at all points in $\Omega \cap f^{-1}[a, b]$ and is tangent to Y on $Y \cap f^{-1}[a, b]$. □

4 Rearrangement of Critical Points

The aim of this section is to prove the Elementary and Global Rearrangement Theorems, in Sects. 4.1 and 4.3 respectively.

4.1 The Embedded Elementary Rearrangement Theorem

The rearrangement theorem in the embedded case is stated or proved in many places, like [7,16–18]. For the convenience of the reader, and because we would also like to deal with boundary critical points, we present a proof.

Theorem 4.1 (Elementary Rearrangement Theorem) *Let* $(Z, \Omega, \Sigma_0, \Sigma_1)$ *be an embedded cobordism and let* f *be the underlying Morse function. Suppose that* f *has exactly two critical points* z_1 *and* z_2 *with* $f(z_1) < f(z_2)$. *Let* ξ *be an embedded gradient-like vector field. For* $i = 1, 2$, *let* $M_i^u \subset Z \times [0, 1]$ *(respectively,* $M_i^s \subset Z \times [0, 1]$*), be the ascending membrane of* z_i *(respectively, the descending membrane) under the flow of* ξ. *If*

$$M_1^u \cap M_2^s = \emptyset,$$

then for any two values $a, b \in [0, 1]$ *there exists a function* $G \colon Z \times [0, 1] \to [0, 1]$ *such that*

(E1) *G has no critical points;*
(E2) *$G(z_1) = a$ and $G(z_2) = b$;*
(E3) *The restriction $g := G|_\Omega$ is Morse. It has two critical points, z_1 and z_2 with the same type as f.*

Remark 4.2 We note that the new Morse function G can be chosen so that there is a nondegenerate homotopy through submersions between G and the old Morse function F.

Proof The proof goes along lines similar to [15, Sect. 4] (see also [5, Proposition 4.1]). We define $K_1 = M_1^u \cup M_1^s$ and $K_2 = M_2^u \cup M_2^s$. The emptiness of $M_1^u \cap M_2^s$ implies that $K_1 \cap K_2 = \emptyset$. Let $T_1 = K_1 \cap Z \times \{0\}$ and $T_2 = K_2 \cap Z \times \{0\}$. We see that T_1 and T_2 are not empty, because $\dim M_1^s, \dim M_2^s \geq 1$. Also $T_1 \cap T_2 = \emptyset$.

Let $W_1 \supset T_1$ and $W_2 \supset T_2$ be two disjoint open subsets of $Z \times \{0\}$. Let $\mu \colon Z \times \{0\} \to [0, 1]$ be a smooth function, such that $\mu(W_1) = 0$ and $\mu(W_2) = 1$. We extend μ to $Z \times [0, 1]$ as follows: if $x \in K_1$, we put $\mu(x) = 0$; if $x \in K_2$, we put $\mu(x) = 1$. If $x \notin (K_1 \cup K_2)$, then the trajectory of ξ through x hits $Z \times \{0\}$ in a unique point $y \in Z \times \{0\}$. Then we define $\mu(x) := \mu(y)$. This definition implies in particular that μ is constant along all the trajectories of ξ.

Following Milnor we choose a smooth function $\Psi \colon [0, 1] \times [0, 1] \to [0, 1]$ satisfying:

- $\frac{\partial \Psi}{\partial x}(x, y) > 0$ for all $(x, y) \in [0, 1] \times [0, 1]$;
- there exists $\delta > 0$, such that $\Psi(x, y) = x$ for all $x \in [0, \delta] \cup [1 - \delta, 1]$ and $y \in [0, 1]$;
- for any $s \in (-\delta, \delta)$ we have $\Psi(f(z_1) + s, 0) = a + s$ and $\Psi(f(z_2) + s, 1) = b + s$.

We now define

$$G(x) := \Psi(F(x), \mu(x)) \text{ and } g := G|_{\Omega}.$$

Observe that by the chain rule

$$\xi \cdot G = dG(\xi) = \left(\frac{\partial \Psi}{\partial x} dF + \frac{\partial \Psi}{\partial y} d\mu \right)(\xi) = \frac{\partial \Psi}{\partial x} \xi \cdot F + \frac{\partial \Psi}{\partial y} \xi \cdot \mu.$$

Since μ is constant on the trajectories of ξ, we have $\xi \cdot \mu = 0$. As $\frac{\partial \Psi}{\partial x} > 0$ and ξ is an embedded gradient-like vector field, we see that $\xi \cdot G(x) \geq 0$ with equality if and only if x is a critical point of f. On the other hand, by the definition of Ψ we have that, near a critical point of f, the function G is equal to F plus a constant. Hence G has no critical points. We compute that by construction $G(z_1) = a$ and $G(z_2) = b$:

$$G(z_1) = \Psi(F(z_1), \mu(z_1)) = \Psi(F(z_1), 1) = \Psi(f(z_1), 1) = a;$$
$$G(z_2) = \Psi(F(z_2), \mu(z_2)) = \Psi(F(z_2), 1) = \Psi(f(z_2), 1) = b.$$

Now, to check that (E3) is still satisfied, let $x \in \Omega$. We have $g(x) = \Psi(f(x), \mu(x))$. As μ is everywhere tangent to Ω, we can repeat the above argument to show that $\xi \cdot g \geq 0$, with equality only at the critical points of f. Since $g - f$ is constant near critical points, the types of the critical points are the same.

Finally, consider $x \in Y$, such that x is not a critical point of f. Then ξ is tangent to $T_x Y$ by definition and $\xi \cdot g > 0$ as g is just the restriction of G. This means that $dg(\xi) > 0$, and in particular that $T_x Y \not\subset \ker dg$. Thus, as required, g is a Morse function in the sense of Definition 2.3. $\qquad \square$

It is easy to see that the argument of Theorem 4.1 can be repeated if f has more critical points and suitable intersections of stable/unstable manifolds are empty. This is made precise in Theorem 4.7 below.

Table 1 Dimensions of various stable and unstable manifolds

Type of z	$M^s \setminus \Omega$	$M^u \setminus \Omega$	$W^s \setminus Y$	$W^u \setminus Y$	W^s_Y	W^u_Y
interior	$k+1$	$n+2-k$	k	$n+1-k$	$\emptyset$	$\emptyset$
bdy. stable	$k+1$	$n+2-k$	k	$\emptyset$	$k-1$	$n+1-k$
bdy. unstable	$k+1$	$n+2-k$	$\emptyset$	$n+1-k$	k	$n-k$

Here $\emptyset$ means that the corresponding manifold is empty, as does dimension -1

4.2 The Embedded Morse–Smale Condition

In the following we write $M^s(z)$ (respectively, $M^u(z)$), to denote the descending and ascending membranes of the critical point z. We write $W^s(z)$ and $W^u(z)$ to denote the stable and the unstable manifolds of $\xi|_\Omega$, with $W^s(z)$, $W^u(z) \subset \Omega$. If z is a boundary critical point, we denote by $W^s_Y(z)$ and $W^u_Y(z)$, respectively, the stable and unstable manifold of the vector field ξ restricted to Y.

Definition 4.3 The vector field ξ satisfies the *embedded Morse–Smale conditions* if for any two critical points z_1 and z_2 of f, the intersections of $M^s(z_1)$ with $M^u(z_2)$ are transverse in $Z \times [0, 1] \setminus \Omega$, the intersections of $W^s(z_1)$ and $W^u(z_2)$ are transverse in $\Omega \setminus Y$ and the intersections of $W^s_Y(z_1)$ with $W^u_Y(z_2)$ are transverse in Y.

Lemma 4.4 *For every embedded gradient-like vector field ξ there exists a C^2-small perturbation ξ' which satisfies the embedded Morse–Smale condition.*

Sketch of proof This is a standard result combining the fact that the transversality condition is open (see [1, Sect. 29]) together with Lemma 3.10. We leave the details to the reader.

We show, in table form, the dimensions of stable and unstable manifolds of a critical point. In Table 1, we assume that z is a critical point of Ω of index k, and we recall that dim $\Omega = n + 1$. Also recall that the index of a critical point z is the dimension of its stable manifold $W^s(z)$.

We remark that the intersection of the stable manifold of one point with the unstable manifold of another, unless empty, must have dimension at least one. Therefore, the embedded Morse–Smale condition (Definition 4.3) yields the following result.

Proposition 4.5 *Let z and w be two critical points of f of indices k and l respectively. Let $m := \dim Z$. Suppose that ξ satisfies the embedded Morse–Smale condition. Then the intersection $M^u(z) \cap M^s(w)$ is empty if at least one of the following conditions hold:*

- $k = l$, $m \geq n + 2$ *and either z is not a boundary stable critical point or w is not a boundary unstable critical point;*
- $k > l$ *and $m \geq n + 1$;*
- *z is an interior critical point, w is boundary unstable and $l - k \leq m - n - 2$;*
- *z is a boundary stable critical point, w is interior and $l - k \leq m - n - 2$.*

Proof In each case, the proof follows by checking that each of

$$\dim(M^u(z) \setminus \Omega) + \dim(M^s(w) \setminus \Omega) \leq m + 1,$$
$$\dim(W^u(z) \setminus Y) + \dim(W^s(w) \setminus Y) \leq n + 1$$

and

$$\dim W_Y^u(z) + \dim W_Y^s(w) \le n$$

are satisfied. □

4.3 The Embedded Global Rearrangement Theorem

As a corollary of Proposition 4.5, we obtain the following global rearrangement theorem. In codimension 2 or more, as in Theorem 4.7, this is the standard rearrangement theorem. In codimension 1 the situation is more complicated and will be addressed in Theorem 4.9.

The next definition was already given in the Introduction; for the convenience of the reader we recall it here.

Definition 4.6 A *configuration* Ξ of the critical points of an embedded cobordism (Z, Ω) is an assignment of a value $\Xi(z_i) \in (0, 1)$ to each critical point z_i of f.

An *admissible configuration* is a configuration satisfying the following conditions:

(A1) if z, w are critical points with indices k, l with $k < l$, then $\Xi(z) < \Xi(w)$;
(A2) if z, w have the same index k and if z is boundary stable and w is boundary unstable, then $\Xi(z) < \Xi(w)$.

The Global Rearrangement Theorem says that any admissible configuration can be realized by changing the Morse function.

Theorem 4.7 (Global Rearrangement Theorem) *Suppose (Z, Ω) is a cobordism. Given an admissible configuration Ξ of the critical points of f, if $\mathrm{codim}(\Omega \subset Z \times [0, 1]) \ge 2$, there exists a function $G\colon Z \times [0, 1] \to [0, 1]$ without critical points, homotopic through submersions to F, which restricts to a Morse function $g\colon \Omega \to [0, 1]$, such that g restricted to the critical points agrees with Ξ (the type and index of each critical point is preserved).*

Remark 4.8 If z, w have the same index k, both critical points are boundary stable, both are boundary unstable or at least one of them is interior, then we can have $g(z) < g(w)$, $g(z) = g(w)$ or $g(z) > g(w)$, as we please, provided condition (A2) is satisfied.

For example, if we have three critical points z, w and v of index k, where z is boundary stable and w is boundary unstable and v is interior, then in general we cannot arrange that $g(z) > g(v)$ and $g(v) > g(w)$ simultaneously, since this would violate (A2).

Proof of Theorem 4.7 First apply Theorem 4.1 to arrange the critical points to satisfy (A1): by the second bullet point of Proposition 4.5 this is always possible. Now critical points of the same index can be arranged into any chosen order that satisfies (A2), by the first bullet point of Proposition 4.5 and further applications of Theorem 4.1.

The conclusions of Theorem 4.7 do not hold in codimension 1, since critical points of the same index cannot in general be rearranged. Instead we have the following weaker result.

Theorem 4.9 *Suppose that Z, Ω and f are as in Theorem 4.7, but* codim$(\Omega \subset Z \times [0, 1]) = 1$. *Then there exists a function* $G: Z \times [0, 1] \rightarrow [0, 1]$ *without critical points, which restricts to a Morse function* $g: \Omega \rightarrow [0, 1]$ *having the same critical points as f, but the critical values satisfy:*

(A1′) *if z, w are critical points with indices k, l with k < l, then g(z) < g(w).*

5 Cancellation of Critical Points

In the absolute case, two critical points of indices k and $k + 1$ can be canceled if there is a single trajectory of a Morse–Smale gradient-like vector field between them. The situation is slightly more complicated in the embedded case. We present a result which is stated in [16, Lemma 2.9] and [17, Lemma 5]. For the convenience of the reader we sketch the proof. The proof also makes crucial use of the Vector Field Integration Lemma 3.8.

Note that if we have critical points z, w with indices k, $k + 1$ respectively, then we can assume that $g(z) < g(w)$, by Theorem 4.7 in the case that the codimension is 2 or more, and Theorem 4.9 in the codimension 1 case.

5.1 The Embedded Elementary Cancellation Theorem

Theorem 5.1 (Elementary Cancellation Theorem) *Let (Z, Ω) be a cobordism. Let z and w be critical points of f of indices k and k + 1, of the same type (i.e., either both interior, or both boundary stable, or both boundary unstable). Suppose that ξ is an embedded gradient-like vector field, which is Morse–Smale (Definition 4.3), and that there exists a single trajectory γ of ξ connecting z with w. If z, w are interior critical points, we require that $\gamma \subset \Omega$. If z, w are boundary critical points, then we require that $\gamma \subset Y$. Furthermore, suppose that there are no broken trajectories between z and w.*

Then, for any neighborhood U of γ, there exists a vector field ξ′ on $Z \times [0, 1]$, agreeing with ξ away from U, non-vanishing on U, and a function $F': Z \times [0, 1] \rightarrow [0, 1]$ such that $f' = F'|_\Omega$ has the same critical points as f with the exception of z and w, which are regular points of f′, and such that ξ′ is an embedded gradient-like vector field with respect to F′.

Remark 5.2 (1) In particular, note that the assumptions on a single trajectory and the lack of broken trajectories imply that the intersection of the interiors of the membranes Int $M^u(z) \cap$ Int $M^s(w)$ is empty. In codimension 3 or more, such disjointness can always be arranged by general position and Table 1: $(n + 2 - k) + (k + 1 + 1) = n + 4 \leq m + 1$ when $m \geq n + 3$. This can be viewed as the main reason why "concordance implies isotopy" in codimension 3 or more [9]. When the codimension is 1 or 2 there are obstructions from membrane intersections.

(2) Unlike Milnor [15, Theorem 5.4], we do not assume that z and w are the only critical points in $f^{-1}[f(z), f(w)]$. The assumption is replaced by the lack of broken trajectories. The statement is equivalent to that of Milnor in the absolute case, or in the embedded case with codimension 2 or more, since the critical points

can be rearranged to achieve Milnor's assumption. It can be shown that it is also equivalent in the case of codimension 1, but since rearrangement of critical points of the same index is in general not possible in codimension 1, it is nice to be able to separate rearrangement and cancellation.

(3) The new Morse function $F' : Z \times [0, 1] \to [0, 1]$ can be chosen so as to be homotopic through submersions to F.

Proof of Theorem 5.1 Milnor's approach works with a few modifications to adapt it to the embedded case and the possibility of additional critical points between z and w. The proof of Theorem 5.1 proceeds by way of Lemmas 5.3, 5.4 and 5.6, and will take the remainder of this section.

Lemma 5.3 *(cf. [15, Assertion 1, p. 50]) Let γ be as in Theorem 5.1. For any open set $U_1 \subset Z \times [0, 1]$ such that $\gamma \subset U_1$, there exists another open set U_2 with $\gamma \subset U_2 \subset U_1$ such that any trajectory of ξ which starts in U_2 and leads out of U_1 never goes back to U_2.*

Proof of Lemma 5.3 The proof resembles that of Lemma 3.9 above, hence we do not give all the details. The idea is that a trajectory which leaves and then returns will imply the existence of another trajectory between z and w, or a broken trajectory, both of which are assumed in the hypothesis of Theorem 5.1 not to exist.

Suppose the statement is false. That is, there exists an open set U_1 with $\gamma \subset U_1$ such that for all U_2 with $\gamma \subset U_2 \subset U_1$ there is a trajectory which starts in U_2, leaves U_1, and then later returns to U_2. Then there exist sequences of points $\{w_r^1\}_{r \geq 1}$, $\{w_r^2\}_{r \geq 1}$ and $\{w_r^3\}_{r \geq 1}$ and a sequence of trajectories $\{\gamma_r\}_{r \geq 1}$, such that w_r^1 and w_r^3 approach γ as $r \to \infty$, $w_r^2 \in \partial(\mathrm{cl}(U_1))$ and γ_r is a trajectory going first through w_r^1, then through w_r^2 and finally through w_r^3. By choosing successively smaller open neighborhoods U_2^r we obtain trajectories γ_r from our assumption, and then can choose $w_r^1, w_r^3 \in U_2^r \cap \gamma_r$.

Since $\partial(\mathrm{cl}\, U_1)$ is compact, we may assume that w_r^2 converges to a point w_0 (as in the proof of Lemma 3.8 we may need to pass to a subsequence). Furthermore, as w_r^1 and w_r^3 come very near γ, we can move w_r^3 along γ_r to ensure that $w_r^3 \to w$. Likewise, we may assume that $w_r^1 \to z$.

Consider γ_0, the trajectory through w_0. Since, for any neighborhood of z, there exists r_0 such that for all $r > r_0$, γ_r enters this neighborhood, we would like to claim that γ_0 starts at z. Similarly, we would like to claim that γ_0 ends at w. This is true under the assumption that there are no critical points in $F^{-1}(F(z), F(w))$.

In the general case, the sequence of trajectories $\gamma_1, \dots, \gamma_r, \dots$ either converges to a trajectory γ_0 (and then γ_0 is another trajectory between z or w by the arguments as above) or to a broken trajectory; see, e.g., [2, Corollary 6.23]. This broken trajectory is a broken trajectory between z and w, by our assumptions on the limits of w_r^1 and w_3^r. This contradicts the assumptions of the Elementary Cancellation Theorem 5.1, namely, that there is a single trajectory joining z to w, and no broken trajectories. $\square$

We continue with the proof of Theorem 5.1. Let U_z and U_w be open neighborhoods of z and w respectively, such that the Embedded Morse Lemma holds for them. Define U_{zw} to be the set of points $x \in Z \times [0, 1] \setminus (U_z \cup U_w)$ such that the trajectory through x hits U_z in the past and U_w in the future. Shrink U_z and U_w if needed, in order to

guarantee that $U_{zw}, U_z, U_w \subset U$. Indeed, if V_2 is defined to be the set U_2 given by Lemma 5.3, with $U_1 = U$, and we take $U_z, U_w \subset V_2$, then any trajectory from U_z to U_w must be contained in U. The vector field ξ flows from U_z to U_w.

The next result is an analogue of [15, Assertion 6, p. 55]. We state it in the case that z and w are interior critical points; see Remark 5.5 below for the boundary case.

Lemma 5.4 *It is possible to change the vector field ξ, inside U_{zw}, to a vector field ξ_1, which is still a gradient-like vector field for F, and flows from U_z to U_w (that is, a trajectory of ξ_1 through a point p in U_{zw} hits U_z in the past and U_w in the future) such that there is a smaller neighborhood $U_1 \subset \overline{U_z \cup U_w \cup U_{zw}}$ containing γ and a coordinate system on U_1 given by $x_1, \ldots, x_{n+1}, y_1, \ldots, y_{m-n}$ such that $z = (0, \ldots, 0), w = (1, 0, \ldots, 0), \Omega \cap U_1 = \{y_1 = \cdots = y_{m-n} = 0\}$ and ξ has the following form:*

$$(v(x_1), -x_2, \ldots, -x_{k+1}, x_{k+2}, \ldots, x_{n+1}, y_1^2 + \cdots + y_{m-n}^2, 0, \ldots, 0), \quad (5.1)$$

where v is a smooth function positive on $(0, 1)$, negative away from $[0, 1]$ and such that $v(x) = x$ (respectively, $v(x) = 1 - x$) in a neighborhood of 0 (respectively, in a neighborhood of 1). The coordinate systems on U and U_z, U_w agree (up to shifting x_1 by $+1$ on U_w).

Remark 5.5 If z and w are boundary critical points, then the statement of Lemma 5.4 should be changed as in [5, Proposition 5.2]. Namely, the local coordinate system should be such that $\Omega = \{x_1 \geq 0, y_1 = \cdots = y_{m-n} = 0\}, z = (0, \ldots, 0), w = (0, 1, 0, \ldots, 0)$ and ξ should be of the form

$$(\epsilon x_1, v(x_2), -x_3, \ldots, -x_{k+1}, -\epsilon x_{k+2}, x_{k+3}, \ldots, x_{n+1}, y_1^2 + \cdots + y_{m-n}^2, 0, \ldots, 0),$$

where $\epsilon = 1$ for boundary unstable and $\epsilon = -1$ for boundary stable critical points.

Sketch of proof of Lemma 5.4 The proof is analogous to the proof of Assertion 6 in [15]. It consists of looking at the map h between $\partial U_z \cap \partial U_{zw}$ and $\partial U_w \cap \partial U_{zw}$ induced by the flow of ξ. One writes it in coordinates of U_z and U_w. If this is the identity, we can extend the coordinates from U_z to U_{zw} using the flow of ξ, and the coordinate systems on U_z, U_w and U_{zw} match together. Otherwise, we change h by an isotopy so that it is the identity near the origin and use the Embedded Isotopy Lemma 3.10. We omit the details, which are a straightforward but tedious generalization of Milnor's approach. □

We resume the proof of Theorem 5.1. Observe that changing the vector field from ξ to ξ_1 does not create any broken trajectories between z and w, so Lemma 5.3 holds for ξ'. We now take U_1 to be the set given by Lemma 5.4, and apply Lemma 5.3 to U_1. Let U_2 be the open set which is the output of Lemma 5.3, and choose a still smaller neighborhood U_3 of γ. Let C be the supremum of $v(x_1)$ on U_3. Let ψ be a cut-off function, which is 1 on U_3 and 0 outside of U_2. We define $\xi' = \xi_1 - (C+1)\psi \partial_{x_1}$. The vector field ξ' now has no zeros on U_3, because the first coordinate of ξ' is negative. On $(U_2 \setminus U_3) \cap \Omega$, at least one of the ∂_{x_j} coordinates of ξ' is non-zero and on $(U_2 \setminus U_3) \setminus \Omega$,

the coordinate ∂_{y_1} is non-zero. So in fact, ξ' has critical points z and w removed. We want to show that ξ' is a gradient-like vector field of some function. We will need the following result.

Lemma 5.6 *Any trajectory γ' through a point $u \in U_2$ leaves U_1 in the past and in the future.*

Proof This statement is from [15, Assertion 2, p. 51] and its proof is completely analogous. □

We can now finish the proof of Theorem 5.1. By Lemma 5.6 and Lemma 5.3 a trajectory passing through U_2 remains there only for a finite time and does not go back. In particular, changing ξ to ξ' does not introduce any "circular" broken trajectories. Furthermore, the fact that any trajectory only stays in U_2 for finite time implies that ξ' has property (AG4).

By Lemma 3.8, ξ' is a gradient-like vector field of some function F' without critical points.

6 Splitting of Critical Points

The aim of this section is to prove the Elementary and Global Handle Splitting Theorems, in Sects. 6.1 and 6.3 respectively.

We begin by showing that a single interior handle can be split into two half-handles; that is, an interior critical point can be pushed to the boundary and exchanged for one boundary stable critical point and one boundary unstable critical point. Then we investigate when this theorem can be applied to split all of the interior handles simultaneously. In codimension at least two, this can be achieved; in codimension one we have a partial result.

6.1 The Embedded Elementary Handle Splitting Theorem

The following result holds for any positive codimension.

Theorem 6.1 (Elementary Handle Splitting Theorem) *Let (Z, Ω) be an embedded cobordism with $F : Z \times [0, 1] \to [0, 1]$ a projection and $f = F|_\Omega$ the underlying Morse function. Suppose that $f = F|_\Omega$ has a single interior critical point at z with index $k \in \{1, \ldots, n\}$, $f(z) = \frac{1}{2}$ and suppose that z can be connected to the set $f^{-1}(1/2) \cap Y$ by a smooth path γ contained entirely in $f^{-1}(1/2)$. Then, for any neighborhood of γ in $Z \times [0, 1]$, there exists a function $G : Z \times [0, 1] \to [0, 1]$, homotopic through submersions (Definition 2.14) to F, agreeing with F away from that neighborhood, such that G has no critical points and $g := G|_\Omega$ has two boundary critical points z^s and z^u of index k and no interior critical points, where z^s is boundary stable, z^u is boundary unstable and there is a single trajectory of ∇g on Y going from z^s to z^u.*

Proof The proof is an extension of the argument in the proof of [5, Theorem 3.1]. First, by [5, Proposition 3.5] there exists $\rho > 0$ and a "half-disc" $U \subset \Omega$ with $\gamma \subset U$

with local coordinates $(x, y, \vec{u})$, for $x \in [0, 3+\rho)$, $|y| < \rho$ and $||\vec{u}||^2 < \rho^2$, such that
the coordinates of the critical point are $z = (1, 0, \dots, 0)$ and f has the form

$$y^3 - yx^2 + y + \frac{1}{2} + \vec{u}^2.$$

Here we write $\vec{u} = (u_1, \dots, u_{n-1})$, $||\vec{u}||^2 = \sum u_j^2$ and $\vec{u}^2 = \sum \epsilon_j u_j^2$, where $\epsilon_j = \pm 1$
depending on the index of the critical point z.

We thicken U in $Z \times [0, 1]$ to a "half-disc" $W \subset Z \times [0, 1]$ of dimension $m + 1$
(i.e., codimension 0 in $Z \times [0, 1]$). Choose U and W small enough so that they lie
inside the neighborhood of γ referred to in the proof of Theorem 6.1. Now W is
diffeomorphic to a product $U \times (-\rho, \rho)^{m-n}$, which means that there exists a map
$\vec{w} : W \to (-\rho, \rho)^{m-n}$, $\vec{w} = (w_1, \dots, w_{m-n})$, so that $U = \{w_1 = \dots = w_{m-n} = 0\}$
and the collection of functions $(x, y, \vec{u}, \vec{w})$ forms a local coordinate system on W. In
this way we identify W with $[0, 3 + \rho) \times (-\rho, \rho)^m$.

Lemma 6.2 *There exists a sign $\epsilon \in \{\pm 1\}$, a choice of index $r \in \{1, \dots, m - n\}$, real
numbers $\vartheta, \tau \in (0, \rho/2)$ and a smooth function $F_\phi : Z \times [0, 1] \to [0, 1]$, such that
$F_\phi|_\Omega = F|_\Omega$, F_ϕ agrees with F away from $[0, 3+\vartheta] \times [-2\vartheta, 2\vartheta]^n \times [-\tau, \tau]^{m-n} \subset W$
and for any $v \in [0, 3 + \vartheta] \times [-\vartheta, \vartheta]^n \times \{0\}$ we have $\epsilon \frac{\partial F_\phi}{\partial w_r}(v) > 0$.*

Proof As z is not a critical point of F (only of $F|_\Omega$), there exists an integer $r \in
\{1, \dots, m - n\}$ such that $\frac{\partial F}{\partial w_r}(z) \neq 0$. We choose ϵ so that $\epsilon \frac{\partial F}{\partial w_r}(z) > 0$. By continuity
of $\frac{\partial F}{\partial w_r}$ there exists $\vartheta > 0$, $\vartheta < \rho/2$, such that $\epsilon \frac{\partial F}{\partial w_r}(v) > 0$ whenever $v \in [1 - \vartheta, 1 +
2\vartheta] \times [-\vartheta, \vartheta]^m$. Define:

$$A = ([0, 1 - \vartheta] \cup [1 + 2\vartheta, 3 + \vartheta]) \times [-\vartheta, \vartheta]^n \subset U \subset \Omega;$$
$$A' = ([0, 1 - \vartheta/2] \cup [1 + \vartheta, 3 + \vartheta]) \times [-2\vartheta, 2\vartheta]^n \subset U \subset \Omega.$$

Note that $A \subset A'$. We choose τ such that $0 < \tau < \vartheta$ and define

$$p_1 = 2 \sup \left\{ -\epsilon \frac{\partial F}{\partial w_r}(v) : v \in A \right\}$$
$$p_2 = \frac{1}{2} \inf \left\{ \left| \frac{\partial F}{\partial y}(v) \right| : v \in A' \times [-\tau, \tau]^{m-n} \subset W \right\}.$$

If $p_1 < 0$ no changes are needed and the proof is finished. If $p_1 = 0$, we redefine
p_1 to be a very small positive number. As for $v \in A'$ we have $\frac{\partial F}{\partial y}(v) \neq 0$ by direct
computation, for τ small enough we have $p_2 > 0$. We assume that

$$\tau < \frac{p_2 \vartheta}{2 p_1}.$$

If this is not true at first then choose a smaller τ; this lowers the left-hand side and cannot
lower the right-hand side, since only p_2 depends on τ and lower τ cannot lower p_2.

Now choose a cut-off function $\phi_1 \colon Z \times [0, 1] \to [0, 1]$ with support contained in $A' \times [-\tau, \tau]^{m-n}$ such that $\left|\frac{\partial \phi_1}{\partial y}\right| < \frac{2}{\vartheta}$ and $\phi_1|_A = 1$. We define now

$$F_\phi = F + \epsilon p_1 w_r \phi_1(x, y, \vec{u}, \vec{w}). \tag{6.1}$$

This function agrees with F on $Z \times [0, 1] \setminus (A' \times [-\tau, \tau]^{n-m})$ and on A' (because on U we have $w_r = 0$). Furthermore, for $v \in A$,

$$\epsilon \frac{\partial F_\phi}{\partial w_r}(v) = \epsilon \frac{\partial f}{\partial w_r}(v) + p_1 > 0,$$

so $\epsilon \frac{\partial F_\phi}{\partial w_r} > 0$ everywhere on $[0, 3 + \vartheta] \times [-\vartheta, \vartheta]^n \subset U$. To show that F_ϕ has no critical points in $A' \times [-\tau, \tau]^{m-n}$ we compute

$$\left|\frac{\partial F_\phi}{\partial y}\right| \geq \left|\frac{\partial F}{\partial y}\right| - \left|p_1 w_r \frac{\partial \phi_1}{\partial y}\right| > 2p_2 - p_1 \frac{p_2 \vartheta}{2p_1} \frac{2}{\vartheta} = p_2 > 0.$$

$\square$

Given Lemma 6.2 we write F instead of F_ϕ. Define

$$W_0(t) := [0, 3 + \rho] \times [-\rho, \rho]^n \times [-t, t]^{n-m}.$$

Furthermore, define

$$p_3(t) := \inf \left\{ \epsilon \frac{\partial f}{\partial w_r}(v) \colon v \in W_0(t) \right\}.$$

For t small enough, by continuity of $\frac{\partial f}{\partial w_r}$ we have that $p_3(t) > 0$. Choose $\varsigma > 0$ to be such a t and satisfying $\varsigma < \tau$, where τ is from Lemma 6.2. Now fix $p_3 := p_3(\varsigma) > 0$, and choose $\eta < \vartheta$, where ϑ is also from Lemma 6.2, satisfying

$$\eta < \frac{\varsigma p_3}{4}.$$

Let $\delta > 0$ be very small. Let us choose a function $b \colon U \to [0, 1]$ as in [5, equation (3.7)]. Let $\phi_2 \colon [-\varsigma, \varsigma] \to [0, 1]$ be a cut-off function, with $\phi_2 \equiv 1$ near 0 and $|\phi_2'(s)| < \frac{2}{\varsigma}$ for all $s \in [-\varsigma, \varsigma]$ (we use the notation ϕ_2 to avoid confusion with ϕ from [5, equation (3.7)]). Define

$$W_1 := [0, 3 + \eta] \times [-\eta, \eta]^n \times [-\varsigma, \varsigma]^{n-m}.$$

Note that $W_1 \subset W_0(\varsigma)$. By analogy with [5, equation (3.8)], we define

$$G(v) = \begin{cases} F(v) & \text{if } v \notin W_1 \\ F(v) - \phi_2(||\vec{w}||)(1 + \delta)b(x, y, \vec{u})y & \text{if } v = (x, y, \vec{u}, \vec{w}) \in W_1, \end{cases}$$

where $||\vec{w}||$ is the Euclidean norm of the vector $\vec{w}$.

Since $\phi_2(0) = 1$, on $g := G|_\Omega$ agrees with the function g from [5, Theorem 3.1], so the properties of the critical points of g from the statement of the present theorem are satisfied. The last condition that we need to ensure holds is that G has no critical points on W_0. But now $|(1 + \delta)b| < 2$, so

$$\left| \frac{\partial}{\partial w_r} \left(\phi_2 \left(||\vec{w}|| \right) (1 + \delta)b(x, y, \vec{u}) \right) \right| < \frac{4}{\varsigma}.$$

Now $|y| \leq \eta < \frac{p_3\varsigma}{4}$, hence $\epsilon \frac{\partial g}{\partial w_r} > 0$.

Finally, we note that the passage from F to F_ϕ and then from F_ϕ to G can be obtained by a nondegenerate homotopy through submersions; one simply writes $F_t = F + t\epsilon p_1 w_r \phi_1(x, y, \vec{u}, \vec{w})$ for $t \in [0, 1]$ and similarly in the definition of G. Thus the functions F and G are homotopic through submersions as claimed. $\qquad\square$

6.2 Moving Many Critical Points At Once

Now we pass to the problem of moving all the critical points of a cobordism to the boundary at once. Checking the condition that each critical point of f can be joined to the boundary by a curve lying entirely in one level set of f is rather complicated. We shall show that this is possible. We would mostly like to repeat the procedure from [5, Sect. 4.5]. However, in the embedded case we cannot, in general, cancel pairs of 0 and 1 handles in such a way that we preserve the isotopy class of Ω, if the codimension is 1 or 2. Therefore, the notion of a technically good function (see [5, Definition 4.8]) is not suitable for our present purpose. We need to modify the reasoning and replace the notion of a technically good function with a more convenient notion.

Definition 6.3 A function $f : \Omega \to [0, 1]$ is called *technically still acceptable* if there exist non-critical values a, b, c, d of f with $0 < a < c < d < b < 1$ such that the critical points of f are distributed in the following way. For $n > 1$:

(TSA1) The inverse image $f^{-1}[0, a]$ contains all critical points of index 0 and all boundary stable critical points of index 1. It does not contain any other critical points.

(TSA2) The inverse image $f^{-1}[a, c]$ contains all interior critical points of index 1 and no other critical points.

(TSA3) The inverse image $f^{-1}[d, b]$ contains all interior critical points of index n and no other critical points.

(TSA4) The inverse image $f^{-1}[b, 1]$ contains all critical points of index $n + 1$ and all boundary unstable critical points of index n. It does not contain any other critical points.

(TSA5) The interior critical points of index 1 lie on a single level set $f^{-1}(\rho)$ and the interior critical points of index n lie on a single level set $f^{-1}(\rho')$.

If $\dim \Omega = 2$, so $n = 1$, then for compatibility we modify the conditions (TSA1)–(TSA5). Namely, we define $d = a, b = c$, and we have $0 < a < c < 1$, so that $[c, d]$ is undefined. We assume that the interior critical points of index 1 lie in $f^{-1}(\rho)$ for some $\rho \in [a, c]$. Furthermore, critical points of index 0 are in $f^{-1}[0, a]$, as well

as boundary stable critical points of index 1; critical points of index 2 and boundary unstable critical points of index 1 are in $f^{-1}[c, 1]$.

We point out that we do not assume that there are no pairs of critical points that could be canceled (i.e., which are joined by a single trajectory, and no broken trajectories, of the gradient-like vector field, are of the same type and have indices $k, k + 1$ for some k), and also we do not assume that the critical points whose indices are between 2 and $n - 1$ are ordered. By Theorem 4.7 we can always assume that the function f is technically still acceptable in codimension two or more. We have the following analogue of [5, Proposition 4.11].

Proposition 6.4 *Suppose that neither Ω, Σ_0, nor Σ_1 have closed connected components. If $\operatorname{codim} \Omega \subset Z \times [0, 1] \geq 2$ and f is technically still acceptable, then we can rearrange F so that each interior critical point z of $f = F|_{\Omega}$ of index $1, \ldots, n$ can be joined to the boundary by a curve lying entirely in $f^{-1}(f(z))$.*

Beginning of the proof of Proposition 6.5 We shall follow the proof of [5, Proposition 4.11]. In particular, we shall use several lemmas from that paper. The majority of the proof of Proposition 6.4 will be contained in Lemmas 6.5 through 6.9.

Observe that in the present case the interval $[c, d]$ does not contain the critical value of any interior critical point with index 1 or n. First we shall prove that for any $y \in [c, d]$, the inverse image $f^{-1}(y)$ has no closed connected components, then we shall work with $f^{-1}[a, c]$ and $f^{-1}[b, d]$.

Let us recall the following lemmas from [5]. All the proofs are given in that article; we indicate how they can be modified for our embedded case as necessary.

Lemma 6.5 *(see [5, Lemma 4.13]) Let $x, y \in [0, 1]$ with $x < y$. If Ω' is a connected component of $f^{-1}[x, y]$ then either $\Omega' \cap Y = \emptyset$, or for any $u \in [x, y] \cap [c, d]$ we have $f^{-1}(u) \cap \Omega' \cap Y \neq \emptyset$.*

Lemma 6.6 *(see [5, Lemma 4.14]) For any $x \in [c, 1]$ the set $f^{-1}[0, x]$ cannot have a connected component disjoint from Y.*

Proof of Lemma 6.6 The proof in [5] relies on having canceled all possible pairs of critical points of indices 0 and 1. We present a modification of that proof which does not assume this.

Suppose for a contradiction that Ω' is a connected component of $f^{-1}[0, x]$ such that $\Omega' \cap Y = \emptyset$. Let Ω_1 be the connected component of Ω which contains Ω'. If $\Omega_1 \cap Y = \emptyset$, then either $\partial \Omega_1 = \emptyset$ or $\partial \Omega \subset \Sigma_0 \cup \Sigma_1$. In the first case Ω_1 is a closed connected component of Ω. In the second either Σ_0 or Σ_1 has a closed connected component. This contradicts the hypothesis of Proposition 6.4. The contradiction implies that $\Omega_1 \cap Y \neq \emptyset$. By Lemma 6.5, $f^{-1}(x) \cap \Omega_1 \cap Y \neq \emptyset$. In particular, $\Omega'' := (f^{-1}[0, x] \cap \Omega_1) \setminus \Omega'$ is not empty.

Now, Ω'' and Ω' are both subsets of a connected space Ω_1. Thus, there must be a critical point $z \in \Omega_1$, of index 1, which joins Ω' to Ω''. We have that $f(z) > x$. As the connected component of $f^{-1}[0, f(z))$ containing Ω' has empty intersection with Y (by Lemma 6.5), z must be an interior critical point.

Up until now we were following the proof of [5, Lemma 4.14]. Now we use a different argument. Namely, as $x > c$, $f(z) > c$ as well. But the property (TSA2) implies that there cannot be any interior critical points of index 1 in $f^{-1}[c, 1]$. This contradiction finishes the proof of the lemma.

Lemma 6.7 *(see [5, Lemma 4.16]) Let $y \in [c, d]$ be chosen so that there are no interior or boundary unstable critical points of index n with critical values in $[c, y)$. Then $f^{-1}(y)$ has no closed connected components.*

Observe that in the present situation, (TSA3) implies that the assumptions of Lemma 6.7 are automatically satisfied for $y = d$. Hence we get the following result.

Lemma 6.8 *For any $y \in [c, d]$ the set $f^{-1}(y)$ has no closed connected components.*

We are going to deal with the set $f^{-1}(y)$ for $y \in [a, c]$. The case $y \in [d, b]$ is symmetric. The following lemma holds for $n > 1$ and $n = 1$, although the proof in the two cases is different.

Lemma 6.9 *The interior critical points of f of index 1 can be rearranged, without changing f away from $f^{-1}[a, c]$, so that each critical point z of index 1 can be connected to the boundary of Y with a curve lying entirely in $f^{-1}(z)$.*

Proof The proof follows [5, proof of Proposition 4.11, case $n = 1$] with a small modification. Suppose $n > 1$.

Observe that $f^{-1}(c)$ has no closed connected components by Lemma 6.8. The interior critical points of index 1 all lie originally on the level set $f^{-1}(\rho)$ by (TSA5), for some $\rho \in [a, c]$. It follows that $f^{-1}(\rho)$ has no closed connected components; in fact, $f^{-1}(\rho)$ arises from $f^{-1}(c)$ by contracting each intersection $W^u(z) \cap f^{-1}(c)$ to a point, where z ranges through all interior critical points of index 1.

Now we proceed by induction, as in [5]. Assume that the interior critical points of index 1 are $z_1, \ldots, z_k$. Let us choose $a_1, \ldots, a_k$ such that $a < a_1 < \cdots < a_k < \rho$ (this is different from [5]). Choose a critical point $z \in f^{-1}(\rho)$ which can be connected to Y by a curve γ staying in $f^{-1}(\rho)$ and not intersecting other critical points. Assume that this is z_1. Then we rearrange the critical points, so that $f(z_1) = a_1$ and the position of other critical points is not changed. We claim that $f^{-1}(\rho)$ still has no closed connected components. This is so, because if $n > 1$ a 1-handle does not increase the number of connected components. Hence we find again a critical point in $f^{-1}(\rho)$, which we call z_2, that can be connected to Y by a curve lying entirely in $f^{-1}(\rho)$ omitting $z_3, \ldots, z_k$. We move z_2 to the level a_2.

After a finite number of moves we complete the proof.

The $n = 1$ case is slightly more difficult, because the number of connected components of the level set can increase or decrease depending on whether the ascending sphere S^0 belongs to a single connected component of $f^{-1}(a)$, or to two components. We will use the following trick. As before, let the critical points be $z_1, \ldots, z_k$. Let us choose numbers $a_1, \ldots, a_k, c_1, \ldots, c_k$ such that $a < a_1 < \cdots < a_k < \rho < c_k < \cdots < c_1 < c$. Recall that $f^{-1}(\rho)$ has no closed connected components. Let us again choose a critical point (we relabel the critical points so that this is z_1), which can be connected to Y in the level set of $f^{-1}(\rho)$ by a curve omitting all other critical

points. We have two cases: either z_1 is attached to two separate connected components of $f^{-1}(a)$, then we move it to the level a_1; or z_1 is attached to a single connected component, and then we move z_1 to the level set of c_1. In both cases, it still holds that $f^{-1}(\rho)$ has no closed connected component. We can thus find a critical point in $f^{-1}(\rho)$ (we will call it z_2) that can be connected to Y by a path in $f^{-1}(\rho)$ omitting other critical points. We move it to the level set a_2 or c_2 as above. The procedure is then repeated inductively with the remaining critical points. □

Conclusion of the proof of Proposition 6.4. For $n > 1$, by Lemma 6.8 $f^{-1}(y)$ has no closed connected components for any $y \in [c, d]$. Hence all interior critical points of index $2, \ldots, n - 1$ can be connected to the boundary by a curve lying in the corresponding level set. The case of interior critical points of index 1 is dealt with in Lemma 6.9. The same lemma also deals with the case of critical points of index $n - 1$ by applying it to $1 - f$; the proof for $n > 1$ is finished.

For $n = 1$ we use only Lemma 6.9. □

If the codimension of the embedding is one we have the following partial result.

Proposition 6.10 *Suppose that neither Ω, Σ_0, nor Σ_1 have closed connected components and* codim $\Omega \subset Z \times [0, 1] = 1$. *Then we can change F by a rearrangement so that each critical point z of $f = F|_\Omega$ of index $2, \ldots, n - 1$ can be joined to the boundary by a curve lying entirely in the level set $f^{-1})(f(z))$.*

Proof By Theorem 4.9 (Global Rearrangement Theorem in codimension one) we rearrange F so that if z and w are critical points of $f = F|_\Omega$ and the index of z is smaller than the index of w, then $F(z) < F(w)$. In particular, we can choose $c < d$ such that $F^{-1}[c, d]$ contains all critical points of f of indices between 2 and $n - 1$ inclusive, and only those critical points.

Now Lemmas 6.5, 6.6 and 6.7 hold in this case, because the codimension assumption is never used in the proofs. □

6.3 The Embedded Global Handle Splitting Theorem

Now we are going to prove one of the main results of the present paper, which is the topological counterpart of [4, Main Theorem 1].

Theorem 6.11 (Global Handle Splitting Theorem) *Let (Z, Ω) be an embedded cobordism such that $\Omega \subset Z \times [0, 1]$ has codimension 2 or more. Suppose that Ω, Σ_0 and Σ_1 have no closed connected components. Then there exists a map $F \colon Z \times [0, 1] \to [0, 1]$, which is homotopic through submersions to the projection onto the second factor, such that Ω can be expressed as a union:*

$$\Omega = \Omega_{-1/2} \cup \Omega_0 \cup \Omega_{1/2} \cup \Omega_1 \cup \Omega_{3/2} \cup \cdots \cup \Omega_{n+1/2} \cup \Omega_{n+1},$$

where $\Omega_i = \Omega \cap F^{-1}([(2i + 1)/(2n + 4), (2i + 2)/(2n + 4)])$ and

- *$\Omega_{-1/2}$ is a cobordism given by a sequence of index 0 handle attachments;*
- *if $i \in \{0, \ldots, n\}$, then Ω_i is a right product cobordism given by a sequence of elementary index i right product cobordisms;*

- if $i + 1/2 \in \{1, \ldots, n+1\}$, then Ω_i is a left product cobordism given by a sequence of elementary index $i + 1/2$ left product cobordisms;
- Ω_{n+1} is a cobordism given by a sequence of index $n + 1$ handle attachments.

Proof The proof follows the line of [5, Theorem 4.18] with the exception that we cannot in general assume that the original underlying Morse function has only boundary stable critical points.

By the Global Rearrangement Theorem (Theorem 4.7) we can rearrange the critical points of f so that the boundary stable critical points of index k have critical value $\frac{3k+1}{3n+6}$, interior critical points of index k are on the level $\frac{3k+2}{3n+6}$ and boundary unstable critical points of index k are on the level $\frac{3k+3}{3n+6}$. We point out that there are no boundary unstable critical points of index $n + 1$, nor boundary stable critical points of index 0 (see Remark 2.8).

After such rearrangements, the function f is technically still acceptable. By Proposition 6.4 we can join each interior critical point of index $1, \ldots, n$ to the boundary by a curve lying in a level set of f. Then, by Theorem 6.1 we can move these interior critical points to the boundary and split into boundary stable and unstable critical points. After this, the critical points are organized so that first come (this means that the value of f at the corresponding points is the smallest) interior index 0 critical points, then boundary unstable index 0, then boundary stable index 1, boundary unstable index 1, and so on. Finally, we have boundary unstable critical points of index n, boundary stable critical points of index $n + 1$ and interior critical points of index $n + 1$.

We can now choose noncritical values of f, $0 = f_0 < f_1 < f_2 < \cdots < f_{2n+4} = 1$ in such a way that $f^{-1}[0, f_1]$ contains only the interior critical points of index 0, $f^{-1}[f_{2i+1}, f_{2i+2}]$ contains the boundary unstable critical points of index i for $i = 0, \ldots, n$, $f^{-1}[f_{2i}, f_{2i+1}]$ contains the boundary stable critical points of index i for $i = 1, \ldots, n + 1$ and $f^{-1}[f_{2n+3}, f_{2n+4}]$ contains only the interior critical points of index $n + 1$. We define $\Omega_i := f^{-1}[f_{2i+1,2i+2}]$ (we can rescale the function F so that $f_j = \frac{j}{2n+4}$). □

In the codimension one case, the situation is much more complicated. First, the rearrangement theorem does not work in general. Since we are not allowed to switch the position of critical points of the same index, the method of proof used for Proposition 6.4 for the interval $[a, c]$ does not work. For the moment we can prove the following result.

Theorem 6.12 *Let* (Z, Ω) *be a codimension one embedded cobordism, where* Ω, Σ_0 *and* Σ_1 *have no closed connected components. Then the cobordism* (Z, Ω) *can be split into cobordisms* $\Omega_0, \Omega_1, \ldots, \Omega_M$ *for some* $M > 0$, *where* Ω_0 *contains only critical points of index* 0 *and* 1 *(of all possible types),* Ω_M *has only critical points of index* n *and* $n + 1$ *and for* $i = 1, \ldots, M - 1$, *the cobordism* Ω_i *is either a left product or a right product cobordism, and the index of critical points in* Ω_i *is less than or equal to the index of the critical points in* Ω_j *whenever* $1 < i < j < M$.

We remark that the statement is vacuous if $\dim \Omega = 2$.

Proof By Theorem 4.9 we can rearrange critical points so that the critical points of index 0 come first, followed by the critical points of index 1, then those of index 2,

and so on. Recall that we cannot, in general, rearrange critical points so that boundary unstable critical points come after other critical points of the same index. Notwithstanding, there exist c, d with $0 < c < d < 1$ such that $f^{-1}[c, d]$ contains critical points of indices between 2 and $n - 1$ and no other. We can already define $\Omega_0 = f^{-1}[0, c]$ and $\Omega_M = f^{-1}[d, 1]$. Now by Proposition 6.10 we can connect each interior critical point of index $k, k = \{2, \ldots, n - 1\}$ to the boundary Y by a curve lying entirely in a level set of f. Hence we can apply Theorem 6.1 to split the critical point. Now we define cobordisms $\Omega_1, \ldots, \Omega_{M-1}$ by the condition that each contains exactly one critical point of f. □

The codimension 1 case, although the most difficult, is also very important in applications; see [4]. We conclude the section with a rough description of how Theorem 6.12 can replace [4, Main Theorem 1] in the proof of [4, Main Theorem 2]. The former states that relative algebraic cobordisms are algebraically split. The latter states that isotopic knots have S-equivalent Seifert forms and that H-cobordant knots have H-equivalent Seifert forms (we refer to [4] for the definitions).

Suppose $Z = S^{2k+1}$ is a sphere and N_0, N_1 are $(2k-1)$-dimensional closed oriented submanifolds of Z with Seifert surfaces Σ_0 and Σ_1 respectively. The Seifert forms are defined on the torsion-free parts of $H_k(\Sigma_0, \mathbb{Z})$ and $H_k(\Sigma_1, \mathbb{Z})$. Suppose N_0 and N_1 are cobordant as submanifolds of $Z \times [0, 1]$, for example, if the knots are isotopic or concordant. We want to compare the Seifert forms related to Σ_0 and Σ_1. To this end, we find a $(2k+1)$-dimensional manifold $\Omega \subset Z \times [0, 1]$, such that $\partial\Omega = \Sigma_0 \cup Y \cup \Sigma_1$, where Y is the cobordism between N_0 and N_1. The changes between the Seifert forms related to Σ_0 and Σ_1 can be studied by splitting the cobordism Ω into simple pieces, and looking at the change corresponding to each piece. If the piece consists of a handle attachment of index $s \neq k, k + 1$, then the Seifert form is unchanged; see [12]. The action occurs in the middle dimensions k and $k + 1$. The Seifert form can change, but it is much easier to control the change if the cobordism corresponds to a half-handle attachment (that is, it corresponds to crossing a boundary critical point) and not a handle attachment (corresponding to crossing an interior critical point). So to prove [4, Main Theorem 2], one wants to move all interior critical points of index $k, k + 1$ to the boundary. By Theorem 6.12, this is possible if $k \geq 2$, so in any dimension past the classical dimension $k = 1, Z = S^3$. Of course, in the classical case, the theorem that isotopy of knots implies S-equivalence of Seifert matrices, and that Seifert matrices of concordant knots are algebraically concordant, was known long before [4].

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