SIMPLE HOMOTOPY TYPES OF EVEN DIMENSIONAL MANIFOLDS

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Abstract. Given a closed n-manifold, we consider the set of simple homotopy types of n-manifolds within its homotopy type, called its simple homotopy manifold set. We characterise it in terms of algebraic K-theory, the surgery obstruction map, and the homotopy automorphisms of the manifold. We use this to construct the first examples, for all $n \geq 4$ even, of closed n-manifolds that are homotopy equivalent but not simple homotopy equivalent. In fact, we construct infinite families with pairwise the same properties, and our examples can be taken to be smooth for $n \geq 6$.

Our key examples are homotopy equivalent to the product of a circle and a lens space. We analyse the simple homotopy manifold sets of these manifolds, determining exactly when they are trivial, finite, and infinite, and investigating their asymptotic behaviour. The proofs involve integral representation theory and class numbers of cyclotomic fields. We also compare with the relation of h-cobordism, and produce similar detailed quantitative descriptions of the manifold sets that arise.

1. Introduction

1.1. **Background and main results.** One of the earliest triumphs in manifold topology was the classification of 3-dimensional lens spaces up to homotopy equivalence, and up to homeomorphism, due to Seifert-Threlfall, Reidemeister, Whitehead, and Moise [TS33, Rei35, Whi41, Moi52]; see [Coh73] for a self-contained treatment and [Vol13] for a detailed history. The two classifications do not coincide e.g. $L(7,1)$ and $L(7,2)$ are famously homotopy equivalent but not homeomorphic.

The homeomorphism classification made use of Reidemeister torsion to distinguish homotopy equivalent lens spaces. While trying to understand this more deeply, J.H.C. Whitehead defined the notion of *simple homotopy equivalence* [Whi39, Whi41, Whi49, Whi50]. A homotopy equivalence $f\colon X \to Y$ between CW complexes is said to be *simple* if it is homotopic to the composition of a sequence of elementary expansions and collapses; see Section 2 for the precise definition. Chapman [Cha74] showed that every homeomorphism $f\colon X \to Y$ between compact CW complexes is a simple homotopy equivalence, and so we have:

$$\text{homeomorphism} \Rightarrow \text{simple homotopy equivalence} \Rightarrow \text{homotopy equivalence}.$$

Whitehead showed that the homeomorphism classification of 3-dimensional lens spaces coincides with the classification up to simple homotopy equivalence, so there are many examples of homotopy but not simple homotopy equivalent lens spaces, e.g. $L(7,1)$ and $L(7,2)$. Higher dimensional lens spaces give rise to similar examples in all odd dimensions ≥ 5 [Coh73], and infinite such families of odd-dimensional manifolds were produced by Jahren-Kwasik [JK15].

This article constructs the first examples of closed manifolds, in all even dimensions $2k \geq 4$, that are homotopy but not simple homotopy equivalent. In fact, we produce infinite families.

From now on, $n \geq 4$ will be an integer and an *n-manifold* will be a compact, connected, CAT n-manifold where CAT $\in \{\text{Diff}, \text{PL}, \text{TOP}\}$ is the category of either smooth, piecewise linear, or topological manifolds. Manifolds will also be assumed closed unless otherwise specified. We will also frequently make use of the following hypothesis.

Hypothesis 1.1. If $n = 4$, we assume CAT = TOP and restrict to manifolds M such that $\pi_1(M)$ is good in the sense of Freedman (see [FQ90, KOPR21]).

Theorem A. *Let $n \geq 4$ be even and let* CAT *be as in Hypothesis 1.1. Then there exists an infinite collection of orientable* CAT *n-manifolds that are all homotopy equivalent but are pairwise not simple homotopy equivalent.*

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It is currently open whether all topological 4-manifolds are homeomorphic to CW complexes. Thus, the definition of simple homotopy equivalence given previously does not apply when $n = 4$ and CAT = TOP. For a more general definition that does apply in this case, see Definition 2.3.

We note that if $M_1 \simeq N_1$ and $M_2 \simeq N_2$ are two pairs of homotopy equivalent (but not necessarily simple homotopy equivalent) odd dimensional manifolds, then $M_1 \times M_2 \simeq_s N_1 \times N_2$ (Corollary 2.33), so even dimensional examples cannot be constructed in such a straightforward way from odd dimensional examples.

1.2. **Simple homotopy manifold sets.** Beyond finding the first examples of even-dimensional manifolds that are homotopy equivalent but not simple homotopy equivalent, in order to quantify this phenomenon we study the following manifold sets. Writing $\simeq$ for homotopy equivalence and $\simeq_s$ for simple homotopy equivalence, for a CAT n-manifold M we introduce the *simple homotopy manifold set*

$$\mathcal{M}^h_s(M) := \{\text{CAT } n\text{-manifolds } N \mid N \simeq M\}/\simeq_s$$

This is the set of manifolds homotopy equivalent to M up to simple homotopy equivalence. Our main examples will be manifolds homotopy equivalent to $S^1 \times L$, where L is a lens space.

We prove that for manifolds of the form $M = S^1 \times L$, the size of $\mathcal{M}^h_s(M)$ can be trivial, finite and arbitrarily large, and infinite, and we determine precisely when each eventuality occurs. In particular the existence of the latter case implies Theorem A. For an integer $m \geq 2$, let C_m denote the cyclic group of order m, and let C_∞ denote the infinite cyclic group.

Theorem B. *Let $n \geq 4$ be even and let* CAT *be as in Hypothesis 1.1. Let $M^n_m = S^1 \times L$, where L is an $(n-1)$-dimensional lens space with $\pi_1(L) \cong C_m$, $m \geq 2$. Then*
(a) $|\mathcal{M}^h_s(M^n_m)|$ *only depends on n and m, and is independent of the choice of L or* CAT*;*
(b) $|\mathcal{M}^h_s(M^n_m)| = 1$ *if and only if $m \in \{2, 3, 5, 6, 7, 10, 11, 13, 14, 17, 19\}$;*
(c) $|\mathcal{M}^h_s(M^n_m)| = \infty$ *if and only if m is not square-free;*
(d) $|\mathcal{M}^h_s(M^n_m)| \to \infty$ *as $m \to \infty$, uniformly in n.*

The ingredients of the proof will be discussed in Sections 1.4 to 1.6. They rely on an analysis of the *Whitehead group* [Whi50]. For a group G, Whitehead defined this group, $\mathrm{Wh}(G)$, and associated to a homotopy equivalence $f : X \to Y$ its Whitehead torsion $\tau(f) \in \mathrm{Wh}(\pi_1(Y))$. He proved the fundamental result that $\tau(f) = 0$ if and only if f is simple.

First we will prove Theorem E (see Section 1.4), which characterises simple homotopy manifold sets in a general setting. It identifies $\mathcal{M}^h_s(M)$ with the orbit set of an action of $\mathrm{hAut}(M)$ on a subgroup of $\mathrm{Wh}(\pi_1(M))$, which depends on the surgery obstruction map in the surgery exact sequence of M. This is analogous to how the set of manifolds homotopy equivalent to a given Poincaré complex can be computed as a quotient of its structure set. However, we prove (see Corollary 1.3) that for some fundamental groups, including $C_\infty \times C_m$, the relevant subgroup of $\mathrm{Wh}(\pi_1(M))$ depends only on $\pi_1(M)$ and the orientation character of M. To study this subgroup for $S^1 \times L$, we will use representation theory and number theory, as explained in Section 1.6.

The action of a homotopy automorphism $g \in \mathrm{hAut}(M)$ on $\mathrm{Wh}(\pi_1(M))$ is determined by its Whitehead torsion $\tau(g) \in \mathrm{Wh}(\pi_1(M))$ and the induced homomorphism $\pi_1(g) \in \mathrm{Aut}(\pi_1(M))$ (see Definition 4.3). To understand this action for the manifolds $S^1 \times L$, the key result is the following (see Theorem 6.6):

Theorem 1.2. *Every homotopy automorphism $f : S^1 \times L \to S^1 \times L$ is simple.*

This result explains why we are able to so accurately compute simple homotopy manifold sets for this class of manifolds. Note though that further work is still required, since the action of $\mathrm{hAut}(S^1 \times L)$ on $\pi_1(S^1 \times L)$ is nontrivial. See Section 1.5 for further details.

1.3. **Comparison with h-cobordism.** Recall that a cobordism $(W; M, N)$ of closed manifolds is an *h-cobordism* if the inclusion maps $M \to W$ and $N \to W$ are homotopy equivalences. The notion of an h-cobordism is central to manifold topology. Smale's h-cobordism theorem [Sma61, Sma62, Mil65], together with its extensions to other categories and dimension 4 in [Sta67, KS77, FQ90], states that under Hypothesis 1.1, every simply-connected h-cobordism is CAT-equivalent to the product $M \times I$. To generalise the h-cobordism theorem to non-simply connected manifolds, one also considers s-cobordisms. An *s-cobordism* $(W; M, N)$ is a cobordism for which the inclusions $M \to W$ and $N \to W$ are simple homotopy equivalences. The CAT s-cobordism theorem [Bar63, Maz63, Sta67, KS77, FQ90] states that under Hypothesis 1.1, CAT equivalence classes of h-cobordisms based on M are in bijection with $\mathrm{Wh}(\pi_1(M))$, with the bijection given by taking the Whitehead torsion of the inclusion $M \to W$ (Theorem 3.1). In particular every s-cobordism is a product. This theorem underpins manifold classification in dimension at least 4.

The s-cobordism theorem is one of the tools that we can use to construct homotopy equivalent but not simple homotopy equivalent manifolds (see Sections 3.1 and 4.2), and we note that the examples constructed this way are always h-cobordant. This leads us to consider two refined versions of the problem of finding homotopy but not simple homotopy equivalent manifolds, where the manifolds are also required to be h-cobordant, or required not to be h-cobordant. We introduce the corresponding variations of $\mathcal{M}^h_s(M)$:

$$\mathcal{M}^{\mathrm{hCob}}_s(M) \coloneqq \{\mathrm{CAT}\ \ n\text{-manifolds } N \mid N \text{ is } h\text{-cobordant to } M\} \,/ \simeq_s$$
$$\mathcal{M}^{h}_{s,\mathrm{hCob}}(M) \coloneqq \{\mathrm{CAT}\ \ n\text{-manifolds } N \mid N \simeq M\} \,/ \langle \simeq_s, \mathrm{hCob}\rangle$$

where $\langle \simeq_s, \mathrm{hCob}\rangle$ denotes the equivalence relation generated by simple homotopy equivalence and h-cobordism. We also note that the sets $\mathcal{M}^{\mathrm{hCob}}_s(M)$ and $\mathcal{M}^{h}_{s,\mathrm{hCob}}(M)$ will arise naturally in the computation of $\mathcal{M}^h_s(M)$ in Theorem E below.

The next theorem shows that we obtain many examples where $\mathcal{M}^{\mathrm{hCob}}_s(M)$ is nontrivial. Thus, even if we restrict to h-cobordant manifolds, many of the simple homotopy sets we consider remain large. As with $\mathcal{M}^h_s(M)$, the sets $\mathcal{M}^{\mathrm{hCob}}_s(M)$ can be trivial, finite, or infinite, all for manifolds of the form $M = S^1 \times L$, where L is a lens space. Again using our number theoretic analysis of the Whitehead group, we determine precisely when each eventuality occurs, and we analyse the asymptotic behaviour.

Theorem C. *Let $n \geq 4$ be even and let* CAT *be as in Hypothesis 1.1. Let $M^n_m = S^1 \times L$, where L is an $(n-1)$-dimensional lens space with $\pi_1(L) \cong C_m$, $m \geq 2$. Then*

(a) $|\mathcal{M}^{\mathrm{hCob}}_s(M^n_m)|$ *only depends on n and m, but it is independent of the choice of L or* CAT*;*
(b) $|\mathcal{M}^{\mathrm{hCob}}_s(M^n_m)| = 1$ *if and only if $m \in \{2,3,5,6,7,10,11,13,14,15,17,19,29\}$;*
(c) $|\mathcal{M}^{\mathrm{hCob}}_s(M^n_m)| = \infty$ *if and only if m is not square-free;*
(d) $|\mathcal{M}^{\mathrm{hCob}}_s(M^n_m)| \to \infty$ *as $m \to \infty$ uniformly in n.*

On the other hand, we also discover a wealth of interesting behaviour when we also factor out by h-cobordism.

Theorem D. *Let $n \geq 4$ be even and let* CAT *be as in Hypothesis 1.1. Let $M^n_m = S^1 \times L$, where L is an $(n-1)$-dimensional lens space with $\pi_1(L) \cong C_m$, $m \geq 2$. Then*

(a) $|\mathcal{M}^{h}_{s,\mathrm{hCob}}(M^n_m)|$ *only depends on n and m, but it is independent of the choice of L or* CAT*;*
(b) $\liminf_{m\to\infty}\Big(\sup_n |\mathcal{M}^{h}_{s,\mathrm{hCob}}(M^n_m)|\Big) = 1$*;*
(c) $|\mathcal{M}^{h}_{s,\mathrm{hCob}}(M^n_m)| < \infty$ *for all n and m;*
(d) $\limsup_{m\to\infty}\Big(\inf_n |\mathcal{M}^{h}_{s,\mathrm{hCob}}(M^n_m)|\Big) = \infty$.

Note that part (b) says that there are infinitely many m with $|\mathcal{M}^{h}_{s,\mathrm{hCob}}(M^n_m)| = 1$ for every n, while part (d) says that $\inf_n |\mathcal{M}^{h}_{s,\mathrm{hCob}}(M^n_m)|$ is unbounded in m.

1.4. **Characterisation of simple homotopy manifold sets.** In Theorem E below, we characterise the simple homotopy manifold sets $\mathcal{M}^h_s(M)$, $\mathcal{M}^{\mathrm{hCob}}_s(M)$, and $\mathcal{M}^{h}_{s,\mathrm{hCob}}(M)$, for a CAT n-manifold M, in terms of the Whitehead group, the homotopy automorphisms of M, and the surgery obstruction map. All our main theorems above arise by applying these characterisations to the manifolds $S^1 \times L$ and computing the objects that appear in Theorem E.

An orientation character $w\colon G \to C_2$ determines an involution $x \mapsto \overline{x}$ on the Whitehead group $\mathrm{Wh}(G)$ (see Section 2.2), and we will write $\mathrm{Wh}(G,w)$ to specify that $\mathrm{Wh}(G)$ is equipped with this involution. Define:

$$\mathcal{J}_n(G,w) = \{y \in \mathrm{Wh}(G,w) \mid y = -(-1)^n \overline{y}\} \leq \mathrm{Wh}(G,w),$$
$$\mathcal{I}_n(G,w) = \{x - (-1)^n \overline{x} \mid x \in \mathrm{Wh}(G,w)\} \leq \mathcal{J}_n(G,w).$$

The Tate cohomology group $\widehat{H}^{n+1}(C_2; \mathrm{Wh}(G,w))$ is canonically identified with $\mathcal{J}_n(G,w)/\mathcal{I}_n(G,w)$ (see Proposition 9.5), and we will denote the quotient map by

$$\pi\colon \mathcal{J}_n(G,w) \to \widehat{H}^{n+1}(C_2; \mathrm{Wh}(G,w)).$$

If $w \equiv 1$ is the trivial character, then we omit it from the notation.

There is an action of $\mathrm{hAut}(M)$ on $\mathrm{Wh}(G,w)$ (as a set) such that if $f\colon N \to M$ is a homotopy equivalence and $g \in \mathrm{hAut}(M)$, then $\tau(f)^g = \tau(g \circ f)$. Let $q\colon \mathrm{Wh}(G,w) \to \mathrm{Wh}(G,w)/\mathrm{hAut}(M)$ denote the quotient map, let $\varrho\colon \widehat{H}^{n+1}(C_2; \mathrm{Wh}(G,w)) \to L^s_n(\mathbb{Z}G,w)$ be the homomorphism from the Ranicki-Rothenberg exact sequence (3.1) (see [Sha69], [Ran80, §9]), and let $\sigma_s\colon \mathcal{N}(M) \to L^s_n(\mathbb{Z}G,w)$ be the surgery obstruction map (see Section 3.2). The following theorem is the basis of our main results (see Theorems 4.11 and 4.16).

Theorem E. *Let M be a* CAT *n-manifold with fundamental group G and orientation character $w\colon G \to \{\pm 1\}$, satisfying Hypothesis 1.1. There is a commutative diagram*

$$\begin{array}{ccccc}
\mathcal{M}^{\mathrm{hCob}}_s(M) & \longrightarrow & \mathcal{M}^h_s(M) & \longrightarrow & \mathcal{M}^h_{s,\mathrm{hCob}}(M) \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
q(\mathcal{I}_n(G,w)) & \longrightarrow & (\varrho\circ\pi)^{-1}(\operatorname{Im}\sigma_s)/\mathrm{hAut}(M) & \longrightarrow & \varrho^{-1}(\operatorname{Im}\sigma_s)/\mathrm{hAut}(M)
\end{array}$$

where each row is a short exact sequence of pointed sets, and each vertical arrow is a bijection.

The subset $\mathcal{J}_n(G,w)$ is invariant under the action of $\mathrm{hAut}(M)$ on $\mathrm{Wh}(G,w)$, and there is an induced action on $\widehat{H}^{n+1}(C_2; \mathrm{Wh}(G,w))$. In the statement above, the subsets

$$(\varrho\circ\pi)^{-1}(\operatorname{Im}\sigma_s) \subseteq \mathcal{J}_n(G,w) \quad \text{and} \quad \varrho^{-1}(\operatorname{Im}\sigma_s) \subseteq \widehat{H}^{n+1}(C_2; \mathrm{Wh}(G,w))$$

are both invariant under the induced actions of $\mathrm{hAut}(M)$. Note that if the subset $\mathcal{I}(G,w)$ is invariant under the action of $\mathrm{hAut}(M)$ on $\mathrm{Wh}(G,w)$, then $q(\mathcal{I}_n(G,w)) = \mathcal{I}_n(G,w)/\mathrm{hAut}(M)$. It follows from the exact sequence (3.1) that the image of the homomorphism $\psi\colon L^h_{n+1}(\mathbb{Z}G,w) \to \widehat{H}^{n+1}(C_2; \mathrm{Wh}(G,w))$ is contained in $\varrho^{-1}(\operatorname{Im}\sigma_s)$ and so we obtain the following corollary.

Corollary 1.3. *Let M be a* CAT *n-manifold with fundamental group G and orientation character $w\colon G \to \{\pm 1\}$, satisfying Hypothesis 1.1. If $\psi\colon L^h_{n+1}(\mathbb{Z}G,w) \to \widehat{H}^{n+1}(C_2; \mathrm{Wh}(G,w))$ is surjective, then we have*

$$\mathcal{M}^h_s(M) \cong \mathcal{J}_n(G,w)/\mathrm{hAut}(M), \quad \mathcal{M}^h_{s,\mathrm{hCob}}(M) \cong \widehat{H}^{n+1}(C_2; \mathrm{Wh}(G,w))/\mathrm{hAut}(M).$$

The advantage of Corollary 1.3 is that $\mathcal{J}_n(G,w)$ and $\widehat{H}^{n+1}(C_2; \mathrm{Wh}(G,w))$ (and, more generally, $\operatorname{Im}\psi$) only depend on G and w, while $\operatorname{Im}\sigma_s$ depends on M a priori. This will allow us to apply Corollary 1.3 by separately analysing the involution on $\mathrm{Wh}(G,w)$ and the homotopy automorphisms of M. Note also that, in cases where M can be regarded as a manifold in multiple categories, the sets $q(\mathcal{I}_n(G,w))$, $\mathcal{J}_n(G,w)/\mathrm{hAut}(M)$ and $\widehat{H}^{n+1}(C_2; \mathrm{Wh}(G,w))/\mathrm{hAut}(M)$ do not depend on the choice of CAT.

To prove Theorem E, first we observe that if N is a manifold homotopy equivalent to M, then the set of Whitehead torsions of all possible homotopy equivalences $f\colon N \to M$ forms an orbit of the action of $\mathrm{hAut}(M)$ on $\mathrm{Wh}(G,w)$. By considering the different restrictions and equivalence relations on these manifolds N, we obtain maps from the various simple homotopy manifold sets of M to subsets/subquotients of $\mathrm{Wh}(G,w)/\mathrm{hAut}(M)$. We then verify that these maps are injective, and the remaining task is to determine their images.

For $\mathcal{M}^{\mathrm{hCob}}_s(M)$, we use the s-cobordism theorem. For every $x \in \mathrm{Wh}(G,w)$ there is an h-cobordism $(W;M,N)$ with Whitehead torsion x, and then $\tau(f) = -x + (-1)^n\overline{x}$ for the induced homotopy equivalence $f\colon N \to M$ (see Proposition 2.38). This shows that every element of $\mathcal{I}_n(G,w)$ can be realised as the torsion of a homotopy equivalence $N \to M$ for some N that is h-cobordant to M. For $\mathcal{M}^h_{s,\mathrm{hCob}}(M)$, we use the surgery exact sequence combined with the exact sequence (3.1) to describe the set of values of $\pi(\tau(f))$ for all homotopy equivalences $f\colon N \to M$ (see Proposition 3.6). Finally, the characterisation of $\mathcal{M}^h_s(M)$ is obtained by combining the results on $\mathcal{M}^{\mathrm{hCob}}_s(M)$ and $\mathcal{M}^h_{s,\mathrm{hCob}}(M)$.

1.5. **Outline of the proof of Theorems B, C and D.** Recall that a lens space $L := S^{2k-1}/C_m$ of dimension $2k-1 \geq 3$ is a quotient of S^{2k-1} by a free action of a finite cyclic group C_m for some $m \geq 2$. The action is determined by a k-tuple of integers $(q_1, \dots, q_k)$ with q_j coprime to m for all j. The fundamental group is $\pi_1(L) \cong C_m$.

First we consider $\mathcal{M}^{\mathrm{hCob}}_s(S^1 \times L)$ and the proof of Theorem B (and hence Theorem A). The proof is based on applying Corollary 1.3 to $S^1 \times L$. This is possible, because the map $\psi\colon L^h_{n+1}(\mathbb{Z}[C_\infty \times C_m]) \to \widehat{H}^{n+1}(C_2; \mathrm{Wh}(C_\infty \times C_m))$ is surjective for $n = 2k$ (see Proposition 3.12), and we get that $\mathcal{M}^h_s(S^1 \times L) \cong \mathcal{J}_n(C_\infty \times C_m)/\operatorname{hAut}(S^1 \times L)$. For the homotopy automorphisms of $S^1 \times L$, we will use the following, expanding on Theorem 1.2 (see Theorems 6.5 and 6.6).

Theorem 1.4.

(a) *Every homotopy automorphism $f\colon S^1 \times L \to S^1 \times L$ is simple.*

(b) *If $\pi_1\colon \operatorname{hAut}(S^1 \times L) \to \operatorname{Aut}(C_\infty \times C_m)$ is the map given by taking the induced automorphism on the fundamental group, then $\operatorname{Im}(\pi_1) = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \in \operatorname{Aut}(C_\infty \times C_m) \mid c^k \equiv \pm 1 \text{ mod } m \right\}$.*

Part (a) implies that the action of $\operatorname{hAut}(S^1 \times L)$ on $\mathrm{Wh}(C_\infty \times C_m)$ factors through the action of $\operatorname{Aut}(C_\infty \times C_m)$, which acts on $\mathrm{Wh}(C_\infty \times C_m)$ via automorphisms, using the functoriality of Wh (see Definition 4.3 and Remark 4.9). Therefore every orbit has cardinality at most $|\operatorname{Aut}(C_\infty \times C_m)| < 2m^2$, and $|\mathcal{J}_n(C_\infty \times C_m)/\operatorname{hAut}(S^1 \times L)| = 1$ if and only if $\mathcal{J}_n(C_\infty \times C_m) = 0$. Moreover, since it follows from part (b) that $\operatorname{Im}(\pi_1\colon \operatorname{hAut}(S^1 \times L) \to \operatorname{Aut}(C_\infty \times C_m))$ is independent of the choice of the integers $(q_1, \dots, q_k)$, the same is true for $\mathcal{J}_n(C_\infty \times C_m)/\operatorname{hAut}(S^1 \times L)$, proving part (a) of Theorem B.

So it remains to study the involution on $\mathrm{Wh}(C_\infty \times C_m)$ and prove the corresponding statements about $|\mathcal{J}_n(C_\infty \times C_m)|$. First, the fundamental theorem for $K_1(\mathbb{Z}C_m[t, t^{-1}])$ gives rise to a direct sum decomposition (see Theorem 5.9):

$$\mathrm{Wh}(C_m \times C_\infty) \cong \mathrm{Wh}(C_m) \oplus \widetilde{K}_0(\mathbb{Z}C_m) \oplus NK_1(\mathbb{Z}C_m)^2$$

where $\widetilde{K}_0$ is the reduced projective class group, and NK_1 is the so-called Nil group. All summands have natural involutions, which are compatible with this isomorphism. Using that $\mathcal{J}_n(C_m) = 0$ (see Proposition 5.13), we obtain the following decomposition for $\mathcal{J}_n(C_\infty \times C_m)$ (see Proposition 5.10):

$$\mathcal{J}_n(C_\infty \times C_m) \cong \{x \in \widetilde{K}_0(\mathbb{Z}C_m) \mid x = -\overline{x}\} \oplus NK_1(\mathbb{Z}C_m).$$

In this decomposition the first component is always finite (see Lemma 5.12), so Theorem B (c) is proved by the following (see Theorems 5.7 and 5.8).

Theorem 1.5 (Bass-Murthy, Martin, Weibel, Farrell)**.** *If m is square-free then $NK_1(\mathbb{Z}C_m) = 0$. Otherwise $NK_1(\mathbb{Z}C_m)$ is infinite.*

For the remaining parts (b) and (d) of Theorem B, we need to study the involution on $\widetilde{K}_0(\mathbb{Z}C_m)$ and prove the analogous statements about $\{x \in \widetilde{K}_0(\mathbb{Z}C_m) \mid x = -\overline{x}\}$ in the case where m is square-free. This will be achieved using methods from algebraic number theory, and discussed further in Section 1.6 (see Theorem 1.6 below).

Now we consider $\mathcal{M}^{\mathrm{hCob}}_s(S^1 \times L)$ and the proof of Theorem C. According to Theorem E we need to analyse $\mathcal{I}_n(C_\infty \times C_m)$. We can do this similarly to $\mathcal{J}_n(C_\infty \times C_m)$. The former has the following decomposition (see Proposition 5.10):

$$\mathcal{I}_n(C_\infty \times C_m) \cong \{x - \overline{x} \mid x \in \widetilde{K}_0(\mathbb{Z}C_m)\} \oplus NK_1(\mathbb{Z}C_m).$$

It follows from Theorem 1.4 that $\mathcal{I}_n(C_\infty \times C_m)$ is invariant under the action of $\operatorname{hAut}(S^1 \times L)$, so $q(\mathcal{I}_n(C_\infty \times C_m))$ can be expressed as $\mathcal{I}_n(C_\infty \times C_m)/\operatorname{hAut}(S^1 \times L)$. Thus by Theorem E we have $\mathcal{M}^{\mathrm{hCob}}_s(S^1 \times L) \cong \mathcal{I}_n(C_\infty \times C_m)/\operatorname{hAut}(S^1 \times L)$. By Theorems 1.4 and 1.5, together with Theorem 1.7 below, this leads to Theorem C.

Finally, we consider $\mathcal{M}^h_{s,\mathrm{hCob}}(S^1 \times L)$ and the proof of Theorem D. By Corollary 1.3, this is isomorphic to $\widehat{H}^{n+1}(C_2; \mathrm{Wh}(C_\infty \times C_m))/\operatorname{hAut}(S^1 \times L)$. The decomposition of $\mathrm{Wh}(C_\infty \times C_m)$ induces an isomorphism $\widehat{H}^{n+1}(C_2; \mathrm{Wh}(C_\infty \times C_m)) \cong \widehat{H}^{n+1}(C_2; \widetilde{K}_0(\mathbb{Z}C_m))$. Hence, as above, it remains to consider the involution on $\widetilde{K}_0(\mathbb{Z}C_m)$, but the behaviour of $\widehat{H}^{n+1}(C_2; \mathrm{Wh}(C_\infty \times C_m))$ is different from $\mathcal{J}_n(C_\infty \times C_m)$ and $\mathcal{I}_n(C_\infty \times C_m)$, in particular it is always finite. The key algebraic input comes from Theorem 1.8 below.

1.6. **The involution on $\widetilde{K}_0(\mathbb{Z}C_m)$.** In light of the discussion in Section 1.5, detailed information regarding the involution on $\widetilde{K}_0(\mathbb{Z}C_m)$ is required in order to complete the proofs of parts (b) and (d) in each of Theorems B, C, and D. The purpose of Part 3 is to explore this involution in detail and to establish the following three theorems which are the key algebraic ingredients behind the proofs of Theorems B, C, and D respectively.

Theorem 1.6. *Let $m \geq 2$ be a square-free integer. Then*
- (i) $|\{x \in \widetilde{K}_0(\mathbb{Z}C_m) \mid \overline{x} = -x\}| = 1$ *if and only if* $m \in \{2, 3, 5, 6, 7, 10, 11, 13, 14, 17, 19\}$*; and*
- (ii) $|\{x \in \widetilde{K}_0(\mathbb{Z}C_m) \mid \overline{x} = -x\}| \to \infty$ *super-exponentially in m.*

Theorem 1.7. *Let $m \geq 2$ be a square-free integer. Then*
- (i) $|\{x - \overline{x} \mid x \in \widetilde{K}_0(\mathbb{Z}C_m)\}| = 1$ *if and only if* $m \in \{2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 29\}$*; and*
- (ii) $|\{x - \overline{x} \mid x \in \widetilde{K}_0(\mathbb{Z}C_m)\}| \to \infty$ *super-exponentially in m.*

Theorem 1.8. *Let $m \geq 2$ be an integer. Then*
- (i) $|\{x \in \widetilde{K}_0(\mathbb{Z}C_m) \mid \overline{x} = -x\}/\{x - \overline{x} \mid x \in \widetilde{K}_0(\mathbb{Z}C_m)\}| = 1$ *for infinitely many m; and*
- (ii) $\sup_{n \leq m} |\{x \in \widetilde{K}_0(\mathbb{Z}C_m) \mid \overline{x} = -x\}/\{x - \overline{x} \mid x \in \widetilde{K}_0(\mathbb{Z}C_m)\}| \to \infty$ *exponentially in m.*

We will now explain the strategy of proof of these three theorems, as well as some of the key ingredients. The first thing to note is that, whilst $\widetilde{K}_0(R)$ is difficult to compute for an arbitrary ring, it is often computable in the case where $R = \mathbb{Z}G$ for a finite group G since finitely generated projective $\mathbb{Z}G$-modules P are all locally free, i.e. $\mathbb{Z}_p \otimes_{\mathbb{Z}} P$ is a free $\mathbb{Z}_pG$ module for all primes p, where $\mathbb{Z}_p$ denotes the p-adic integers. In particular, $\widetilde{K}_0(\mathbb{Z}G) \cong C(\mathbb{Z}G)$ where $C(\cdot)$ denotes the locally free class group (see Section 8.1). The general strategy for determining the involution on $\widetilde{K}_0(\mathbb{Z}C_m)$ is to note that, since $\widetilde{K}_0(\mathbb{Z}C_m) \cong C(\mathbb{Z}C_m)$, we can fit $\widetilde{K}_0(\mathbb{Z}C_m)$ into a short exact sequence of abelian groups:

$$0 \to D(\mathbb{Z}C_m) \to \widetilde{K}_0(\mathbb{Z}C_m) \to \bigoplus_{d|m} C(\mathbb{Z}[\zeta_d]) \to 0$$

where $D(\mathbb{Z}C_m)$ denotes the kernel group of $\mathbb{Z}C_m$ (see Section 8.2) and $C(\mathbb{Z}[\zeta_d])$ denotes the ideal class group of $\mathbb{Z}[\zeta_d]$, where ζ_d denotes a primitive dth root of unity. The standard involution on $\widetilde{K}_0(\mathbb{Z}C_m)$ restricts to $D(\mathbb{Z}C_m)$ and induces the involution given by conjugation on each $C(\mathbb{Z}[\zeta_d])$ (see Section 10.1).

The proofs of Theorems 1.6 and 1.7 are intertwined and can be found in Section 11.1. The approach we will take is to use Lemma 9.3, and its consequence Lemma 11.4, which allow us to obtain information about the orders $|\{x \in \widetilde{K}_0(\mathbb{Z}C_m) \mid \overline{x} = -x\}|$ and $|\{x - \overline{x} \mid x \in \widetilde{K}_0(\mathbb{Z}C_m)\}|$ from the orders of the corresponding groups $|\{x \in A \mid \overline{x} = -x\}|$ and $|\{x - \overline{x} \mid x \in A\}|$ in the cases where $A = D(\mathbb{Z}C_m)$ or $C(\mathbb{Z}[\zeta_d]))$ for some $d \mid m$.

The key lemma which makes this approach possible is the following, which is a part of Proposition 10.4. Let $h_m = |C(\mathbb{Z}[\zeta_m])|$ denotes the class number of the mth cyclotomic field, and recall that it splits as a product $h_m = h_m^+ h_m^-$ for integers $h_m^+ = |C(\mathbb{Z}[\zeta_m + \zeta_m^{-1}])|$ and h_m^- known as the plus and minus parts of the class number. For an integer m, we let $\mathrm{odd}(m)$ denote the odd part of m, i.e. the unique odd integer r such that $m = 2^k r$ for some k.

Lemma 1.9. *We have that* $\mathrm{odd}(h_m^-) \leq |\{x - \overline{x} \mid x \in C(\mathbb{Z}[\zeta_m])\}|$.

It was shown by Horie [Hor89], using results from Iwasawa theory [Fri81], that there exists finitely many m for which $\mathrm{odd}(h_m^-) = 1$ and $\mathrm{odd}(h_m^-) \to \infty$ (see Proposition 10.7). Since Lemma 1.9 also gives a lower bound on $|\{x \in C(\mathbb{Z}[\zeta_m]) \mid \overline{x} = -x\}|$, this is enough to prove part (ii) of Theorems 1.6 and 1.7 and to reduce the proof of part (i) to checking finitely many cases. These cases are dealt with via a variety of methods such as analysing the group structure on $C(\mathbb{Z}[\zeta_m])$ (see the proof of Proposition 11.6) and the studying the involution on $D(\mathbb{Z}C_m)$ by relating it to maps between units groups (see Sections 10.4 and 11.1).

The proof of Theorem 1.8 can be found in Section 11.2. Our approach will be based on the isomorphism

$$\frac{\{x \in \widetilde{K}_0(\mathbb{Z}C_m) \mid \overline{x} = -x\}}{\{x - \overline{x} \mid x \in \widetilde{K}_0(\mathbb{Z}C_m)\}} \cong \widehat{H}^1(C_2; \widetilde{K}_0(\mathbb{Z}C_m))$$

where $\widetilde{K}_0(\mathbb{Z}C_m)$ is viewed as a $\mathbb{Z}C_2$-module with the C_2-action given by the involution (see Proposition 9.5). The short exact sequence above induces a 6-periodic exact sequence on Tate cohomology (Proposition 9.7). This, combined with the fact that $\widehat{H}^n(C_2; A)$ for A finite only depends on its 2-Sylow subgroup $A_{(2)}$ (Proposition 9.9) gives the following, which is a part of Lemma 11.13.

Lemma 1.10. *If h_m is odd, then $\widehat{H}^1(C_2; \widetilde{K}_0(\mathbb{Z}C_m)) \cong \widehat{H}^1(C_2; D(\mathbb{Z}C_m))$. In particular, if h_m and $|D(\mathbb{Z}C_m)|$ are both odd, then $\widehat{H}^1(C_2; \widetilde{K}_0(\mathbb{Z}C_m)) = 0$.*

Using Iwasawa theory, Washington [Was75] showed that h_{3^n} is odd for all $n \geq 1$ (see also work of Ichimura–Nakajima [IN12] which extends this to more primes $p \leq 509$). Since C_{3^n} is a 3-group, $|D(\mathbb{Z}C_{3^n})|$ is an abelian 3-group and so is odd [CR87, Theorem 50.18]. By Lemma 1.10, this implies that $|\widehat{H}^1(C_2; \widetilde{K}_0(\mathbb{Z}C_{3^n}))| = 1$ for all $n \geq 1$ which gives Theorem 1.8 (i).

On the other hand, by Weber's theorem, h_{2^n} is odd for all $n \geq 1$ (see Lemma 10.10). Using a result of Kervaire–Murthy [KM77] which relates $D(\mathbb{Z}C_{2^n})$ with $D(\mathbb{Z}C_{2^{n+1}})$ (Proposition 10.19), and the 6-periodic exact sequence on Tate cohomology (Lemma 11.13), we show that $|\widehat{H}^1(C_2; D(\mathbb{Z}C_{2^n}))|$ is unbounded as $n \to \infty$. By Lemma 1.10, this implies Theorem 1.8 (ii).

Organisation of the paper. The paper will be structured into three parts. In Part 1 we develop the necessary background on simple homotopy equivalence, Whitehead torsion and h-cobordisms. We will then prove Theorem E, which is our main general result. In Part 2 we study the manifolds $L \times S^1$, leading to the proofs of Theorems B, C, and D subject to results about $\widetilde{K}_0(\mathbb{Z}C_m)$. Part 3 follows on from Section 1.6: we develop the necessary background on integral representation theory and algebraic number theory. We then study the involution of $\widetilde{K}_0(\mathbb{Z}C_m)$, leading to proofs of Theorems 1.6-1.8.

Conventions. The following will be in place throughout this article, unless otherwise specified. As above, $n \geq 4$ will be an integer and an n-manifold will be a compact connected CAT n-manifold where CAT $\in \{\text{Diff}, \text{PL}, \text{TOP}\}$. They will be assumed closed except where it is clear from the context that they are not, e.g. thickenings and cobordisms. We will frequently assume Hypothesis 1.1 but will state this as needed. Groups will be assumed to be finitely presented. Rings R will be assumed to have a multiplicative identity, and R-modules will be assumed to be finitely generated.

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Part 1. General results

In this part we will establish general results regarding simple homotopy equivalence. This will be the basis for our applications in Part 2. In Section 2, we will recall the basic theory of simple homotopy equivalence, culminating in constraints on the Whitehead torsion of homotopy equivalences between manifolds. Section 3 concerns the two methods which we will use for constructing manifolds: via h-cobordisms (Section 3.1) and via the surgery exact sequence (Section 3.2). In Section 4 we study the simple homotopy manifolds sets and prove Theorem E.

2. Preliminaries

In this section we recall the definition of simple homotopy equivalence, the Whitehead group and the Whitehead torsion, as well as some of their basic properties. Our main sources are Milnor [Mil66], Cohen [Coh73], and Davis-Kirk [DK01].

2.1. **Simple homotopy equivalence.** Let X be a CW complex and let $\phi\colon D^n \to X$ be a cellular map. Divide the boundary of the closed $(n+1)$-cell e^{n+1} into two n-discs, $\partial e^{n+1} \cong D^n \cup_{S^{n-1}} D^n$, gluing along the first copy of $D^n \subseteq \partial e^{n+1}$. Then the inclusion $X \to X \cup_\phi e^{n+1}$ is called an *elementary expansion*. There is a deformation retract $X \cup_\phi e^{n+1} \to X$ in the other direction, and this is called an *elementary collapse*.

Definition 2.1. A homotopy equivalence $f\colon X \to Y$ between finite CW complexes is *simple* if f is homotopic to a map that is a composition of finitely many elementary expansions and collapses

$$X = X_0 \to X_1 \to X_2 \to \cdots \to X_k = Y.$$

For a homotopy equivalence $f\colon X \to Y$, Whitehead introduced an invariant $\tau(f)$, the *Whitehead torsion* of f, which lies in the *Whitehead group* $\mathrm{Wh}(\pi_1(Y))$ of $\pi_1(Y)$. We shall define both the Whitehead group and the Whitehead torsion shortly. The motivation for the Whitehead torsion is the following beautiful result, which completely characterises whether or not a homotopy equivalence is simple.

Theorem 2.2 (Whitehead [Whi50]). *A homotopy equivalence $f\colon X \to Y$ between CW complexes X, Y is simple if and only if $\tau(f) = 0 \in \mathrm{Wh}(\pi_1(Y))$.*

In dimensions $n \neq 4$, every closed n-manifold admits an n-dimensional CW structure; see Kirby-Siebenmann [KS77, III.2.2] for $n \geq 5$, [Rad26] for $n = 2$ and [Moi52] for $n = 3$. In dimension 4, it is an open question whether this holds. Certainly every smooth or PL n-manifold admits an n-dimensional triangulation, and hence a CW structure.

We explain how the notion of simple homotopy equivalence makes sense for 4-manifolds, even those for which we do not know whether they admit a CW structure. The procedure, which is due to Kirby-Siebenmann [KS77, III, §4] works for any dimension, so we work in this generality.

Let M be a topological n-manifold. Embed M in high-dimensional Euclidean space. By [KS77, III, §4], there is a normal disc bundle $D(M) \to M$ that admits a triangulation and hence a CW structure. The inclusion map $z_M\colon M \to D(M)$ of the 0-section is a homotopy equivalence. Let z_M^{-1} denote the homotopy inverse of z_M.

Definition 2.3. We say that a homotopy equivalence $f\colon M \to N$ between topological manifolds is *simple* if the composition $z_N \circ f \circ z_M^{-1}\colon D(M) \to D(N)$ is simple. Kirby-Siebenmann [KS77, III, §4] showed that whether or not this composition is simple does not depend on the choice of normal disc bundle nor on the choice of triangulation.

Remark 2.4. Simple homotopy theory in fact extends beyond topological manifolds. By West's resolution of the Borsuk conjecture [Wes77], a compact ANR has a canonical simple type. Hanner [Han51] showed that compact topological manifolds are compact ANRs.

If M, N are smooth or PL, we can ask whether f is simple using the canonical class of triangulations of M and N, or by forgetting the smooth/PL structures and using the Kirby-Siebenmann method from Definition 2.3.

Proposition 2.5 (Kirby-Siebenmann [KS77, III.5.1]). *For* CAT $\in \{\mathrm{Diff}, \mathrm{PL}\}$, *a homotopy equivalence $f\colon M \to N$ between* CAT *manifolds M and N is simple with respect to their canonical class of triangulations if and only if it is simple with respect to Definition 2.3.*

Hence we have a coherent notion of simple homotopy equivalence in all three manifold categories. We remark that Proposition 2.5 can also be proven using the following theorem of Chapman.

Theorem 2.6 (Chapman [Cha74]). *Let $f\colon X \to Y$ be a homeomorphism between compact, connected CW complexes. Then f is a simple homotopy equivalence.*

Thus for closed manifolds M and N that admit a CW structure, one can deduce from Chapman's Theorem 2.6 that the question of whether $f\colon M \xrightarrow{\simeq} N$ is simple does not depend on the CW structures. Thus the extra work using disc bundles in Definition 2.3 is only required for non-smoothable topological 4-manifolds.

2.2. The Whitehead group.

Definition 2.7. For a ring R, let $\mathrm{GL}(R) = \mathrm{colim}_n \mathrm{GL}_n(R)$, where we take the colimit with respect to the inclusions $\mathrm{GL}_n(R) \hookrightarrow \mathrm{GL}_{n+1}(R)$, $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Define

$$K_1(R) \coloneqq \mathrm{GL}(R)^{\mathrm{ab}}.$$

The Whitehead lemma [Mil66, Lemma 1.1] states that the commutator subgroup of $\mathrm{GL}(R)$ is equal to the subgroup $E(R)$ generated by elementary matrices (i.e. the matrices E such that $A \mapsto EA$ is an elementary row operation). It follows that we can also write

$$K_1(R) = \mathrm{GL}(R)/E(R).$$

Definition 2.8. Define the *Whitehead group* of a group G to be $\mathrm{Wh}(G) = K_1(\mathbb{Z}G)/\pm G$, where the map $\pm G \to K_1(\mathbb{Z}G)$ is the composition $\pm G \subseteq GL_1(\mathbb{Z}G) \subseteq \mathrm{GL}(\mathbb{Z}G) \to K_1(\mathbb{Z}G)$.

Note that Wh is a functor from the category of groups to the category of abelian groups. The map induced by a homomorphism $\theta\colon G \to H$ will be denoted by $\theta_*\colon \mathrm{Wh}(G) \to \mathrm{Wh}(H)$. Similarly, for a continuous map $f\colon X \to Y$ between topological spaces, the map induced by $\pi_1(f)\colon \pi_1(X) \to \pi_1(Y)$ will be denoted by $f_*\colon \mathrm{Wh}(\pi_1(X)) \to \mathrm{Wh}(\pi_1(Y))$.

The isomorphism type of $\mathrm{Wh}(G)$ is known in some cases. For us the following examples will be relevant.

Proposition 2.9 (Stallings [Sta65])**.** *If G is a finitely generated free group, then* $\mathrm{Wh}(G) = 0$.

Proposition 2.10 ([Coh73, (11.5)])**.** *If C_m is the finite cyclic group of order m, then*

$$\mathrm{Wh}(C_m) \cong \mathbb{Z}^{\lfloor m/2 \rfloor + 1 - \delta(m)}$$

where $\delta(m)$ is the number of positive integers dividing m.

The Whitehead group $\mathrm{Wh}(G)$ of a group G is equipped with a natural involution, i.e. an automorphism $x \mapsto \overline{x}$ such that $\overline{\overline{x}} = x$. Equivalently, it has a $\mathbb{Z}C_2$-module structure, where the generator of C_2 acts by the involution. We will describe this involution below.

Let R be a ring with involution, i.e. a ring equipped with a map $x \mapsto \overline{x}$ that is an involution on R as an abelian group, and satisfies $\overline{xy} = \overline{y} \cdot \overline{x}$. This induces an involution on $\mathrm{GL}(R)$, sending $A = (A_{ij}) \in \mathrm{GL}_n(R)$ to $\overline{A} = (\overline{A_{ji}}) \in \mathrm{GL}_n(R)$, the conjugate transpose of A. Note that this is perhaps non-standard notation for the conjugate transpose. We use this convention so that the notation for involutions is consistent. This involution preserves the subgroup $E(R) \subseteq \mathrm{GL}(R)$ and so induces an involution $K_1(R)$.

If G is a group and $w\colon G \to \{\pm 1\}$ is an orientation character, i.e. a homomorphism from G to $\{\pm 1\}$, then the integral group ring $\mathbb{Z}G$ has an involution given by $\sum_{i=1}^k n_i g_i \mapsto \sum_{i=1}^k w(g_i) n_i g_i^{-1}$ for $n_i \in \mathbb{Z}$ and $g_i \in G$. The resulting involution on $K_1(\mathbb{Z}G)$ preserves $\pm G$ and so induces an involution on $\mathrm{Wh}(G)$.

Definition 2.11. We will write $\mathrm{Wh}(G, w)$ for the abelian group $\mathrm{Wh}(G)$ equipped with the involution determined by the orientation character $w\colon G \to \{\pm 1\}$. This can equivalently be regarded as a $\mathbb{Z}C_2$-module. When $w \equiv 1$ is the trivial homomorphism, we will omit it from the notation and write $\mathrm{Wh}(G)$ for $\mathrm{Wh}(G, 1)$.

Note that the choice of w only affects the involution, and $\mathrm{Wh}(G, w)$ is equal to $\mathrm{Wh}(G)$ as an abelian group. Next we define two subgroups of $\mathrm{Wh}(G, w)$ which will play an important rôle in the rest of this article.

Definition 2.12. For a group G and a homomorphism $w\colon G \to \{\pm 1\}$, define:

$$\mathcal{J}_n(G, w) \coloneqq \{y \in \mathrm{Wh}(G, w) \mid y = -(-1)^n \overline{y}\} \leq \mathrm{Wh}(G, w)$$
$$\mathcal{I}_n(G, w) \coloneqq \{x - (-1)^n \overline{x} \mid x \in \mathrm{Wh}(G, w)\} \leq \mathrm{Wh}(G, w)$$

We have $\mathcal{I}_n(G, w) \leq \mathcal{J}_n(G, w)$, and it follows from the definition of the Tate cohomology groups (see Proposition 9.5) that there is a canonical isomorphism

$$\mathcal{J}_n(G, w)/\mathcal{I}_n(G, w) \cong \widehat{H}^{n+1}(C_2; \mathrm{Wh}(G, w)).$$

We will denote the quotient map by

$$\pi\colon \mathcal{J}_n(G, w) \to \widehat{H}^{n+1}(C_2; \mathrm{Wh}(G, w)).$$

2.3. **The Whitehead torsion of a chain homotopy equivalence.** Now we define the Whitehead torsion in the algebraic setting. Let G be a group and let $f\colon C_* \to D_*$ be a chain homotopy equivalence between finitely generated, free, based (left) $\mathbb{Z}G$-module chain complexes. Consider the algebraic mapping cone $(\mathscr{C}(f), \partial_*)$, which is also a finitely generated, free, based $\mathbb{Z}G$-module chain complex. Since f is a chain homotopy equivalence, $\mathscr{C}(f)$ is chain contractible; see e.g. [Ran02, Proposition 3.14]. Choose a chain contraction $s\colon \mathscr{C}(f)_* \to \mathscr{C}(f)_{*+1}$. Now consider the modules

$$\mathscr{C}(f)_{\mathrm{odd}} := \bigoplus_{i=0}^{\infty} \mathscr{C}(f)_{2i+1} \quad \text{and} \quad \mathscr{C}(f)_{\mathrm{even}} := \bigoplus_{i=0}^{\infty} \mathscr{C}(f)_{2i}.$$

These are finitely generated, free, based $\mathbb{Z}G$-modules. Since $\mathscr{C}(f)$ is contractible, its Euler characteristic vanishes, and so these modules are of equal, finite rank. The collection of boundary maps $\oplus_{i=0}^{\infty}\partial_{2i+1}$ with odd degree domain, and the collection of the maps in the chain contraction $\oplus_{i=0}^{\infty}s_{2i+1}$ with odd degree domain, both give rise to homomorphisms from $\mathscr{C}(f)_{\mathrm{odd}}$ to $\mathscr{C}(f)_{\mathrm{even}}$. Using the given bases, their sum is an element of $\mathrm{GL}(\mathbb{Z}G)$ [Coh73, (15.1)], and hence represents an element of the Whitehead group.

Definition 2.13. The *Whitehead torsion* of f is the equivalence class

$$\tau(f) := \Big[\bigoplus_{i=0}^{\infty}(\partial_{2i+1} + s_{2i+1})\colon \mathscr{C}(f)_{\mathrm{odd}} \to \mathscr{C}(f)_{\mathrm{even}}\Big] \in \mathrm{Wh}(G).$$

This equivalence class is independent of the choice of the chain contraction s [Coh73, (15.3)].

Remark 2.14. Since $\mathscr{C}(f)_{\mathrm{odd}}$ and $\mathscr{C}(f)_{\mathrm{even}}$ are finitely generated, the sum in the definition of $\tau(f)$ is a finite sum.

The equivalence class $\tau(f)$ remains invariant under permutations of the bases of C_* and D_*, or if a basis element is multiplied by (-1) or an element of G [Coh73, (15.2) and (10.3)].

Now we list some useful facts about τ. We fix the group G, and all chain complexes will be assumed to be finitely generated, free, based, left $\mathbb{Z}G$-module chain complexes.

Proposition 2.15 ([DK01, Theorem 11.27])**.** *Let $f, g\colon C_* \to D_*$ be homotopic chain homotopy equivalences. Then $\tau(f) = \tau(g)$.*

Lemma 2.16 ([DK01, Theorem 11.28])**.** *Let $f\colon C_* \to D_*$ and $g\colon D_* \to E_*$ be chain homotopy equivalences. Then $\tau(g \circ f) = \tau(f) + \tau(g)$. In particular $\tau(\mathrm{Id}) = 0$.*

We say that a short exact sequence $0 \to C_*' \to C_* \to C_*'' \to 0$ of chain complexes is based if the basis of C_* consists of the image of the basis of C_*' and an element from the preimage of each basis element of C_*''.

Lemma 2.17. *Let $0 \to C_*' \to C_* \to C_*'' \to 0$ and $0 \to D_*' \to D_* \to D_*'' \to 0$ be based short exact sequences of chain complexes, and let (f', f, f'') be a morphism between them, where $f'\colon C_*' \to D_*'$, $f\colon C_* \to D_*$, and $f''\colon C_*'' \to D_*''$ are chain homotopy equivalences. Then $\tau(f) = \tau(f') + \tau(f'')$.*

Proof. The given data determines a based short exact sequence $0 \to \mathscr{C}(f') \to \mathscr{C}(f) \to \mathscr{C}''(f) \to 0$ of mapping cones, so the statement follows from [DK01, Theorem 11.23]. □

The following lemma gives a useful way to compute Whitehead torsion in favourable special cases. We will apply it in Section 6.

Lemma 2.18. *Let $f\colon C_* \to D_*$ be a chain map such that $f_i\colon C_i \to D_i$ is an isomorphism for each i. Then $[f_i] \in \mathrm{Wh}(G)$ for each i and $\tau(f) = \sum_{i=0}^{\infty}(-1)^i[f_i]$.*

Proof. Let $n = \max\{i \mid C_i \neq 0\}$, we will prove the statement by induction on n. If $n = 0$, then $\mathscr{C}(f)_{\mathrm{odd}} = \mathscr{C}(f)_1 \cong C_0$, $\mathscr{C}(f)_{\mathrm{even}} = \mathscr{C}(f)_0 \cong D_0$, and $\partial_1 = f_0$, so $\tau(f) = [f_0]$ by Definition 2.13.

For the induction step define C_*' by $C_i' = C_i$ if $i < n$ and $C_i = 0$ otherwise, and define C_*'' by $C_n'' = C_n$ and $C_i = 0$ if $i \neq n$. Define D_*', D_*'', $f'\colon C_*' \to D_*'$, and $f''\colon C_*'' \to D_*''$ analogously, and apply Lemma 2.17. □

For a chain complex C_* and an integer k we will denote by C_{k+*} the same chain complex with shifted grading $i \mapsto C_{k+i}$. Similarly, the cochain complex C_{k-*} is defined by changing the grading to $i \mapsto C_{k-i}$. The notation C^{k+i} and C^{k-*} is defined analogously for a cochain complex C^*.

Lemma 2.19. *Let $f\colon C_* \to D_*$ be a chain homotopy equivalence. For every $k \in \mathbb{Z}$, it can also be regarded as a chain homotopy equivalence $f\colon C_{k+*} \to D_{k+*}$, and we have $\tau(f\colon C_{k+*} \to D_{k+*}) = (-1)^k \tau(f\colon C_* \to D_*)$.*

Proof. If we shift the grading by an even number, then $\mathscr{C}(f)_{\mathrm{odd}}$ and $\mathscr{C}(f)_{\mathrm{even}}$ remain unchanged. If we shift the grading by an odd number, then $\mathscr{C}(f)_{\mathrm{odd}}$ and $\mathscr{C}(f)_{\mathrm{even}}$ are swapped, and by [Coh73, (15.1)], $\tau(f)$ changes its sign. □

Definition 2.20. Let $f\colon C^* \to D^*$ be a homotopy equivalence of cochain complexes of finitely generated, free, based, left $\mathbb{Z}G$-modules. It can be regarded as a homotopy equivalence of chain complexes $f\colon C^{-*} \to D^{-*}$, and we define $\tau(f\colon C^* \to D^*) \coloneqq \tau(f\colon C^{-*} \to D^{-*})$.

Next we describe how we can define the dual of a left R-module as another left R-module, if R is a ring with involution.

Definition 2.21. Suppose that R is a ring with involution. Let X and Y be a left and a right R-module respectively. Then we define the abelian groups

$$Y \otimes_R X = Y \otimes_{\mathbb{Z}} X / (yr \otimes x = y \otimes rx) \quad \forall r \in R, x \in X, y \in Y;$$

$$\mathrm{Hom}_R^{lr}(X,Y) = \{\varphi \in \mathrm{Hom}_{\mathbb{Z}}(X,Y) \mid \forall r \in R, x \in X \mid \varphi(rx) = \varphi(x)\overline{r}\}\,.$$

If Y has a left S-module structure for some ring S (which is compatible with its right R-module structure), then $Y \otimes_R X$ is a left S-module with $s(y \otimes x) = (sy) \otimes x$, and $\mathrm{Hom}_R^{lr}(X,Y)$ is a left S-module with $(s\varphi)(x) = s\varphi(x)$.

Remark 2.22. $\mathrm{Hom}_R^{lr}(X,Y)$ is equal to $\mathrm{Hom}_R^{r}(\overline{X},Y)$, the group of right R-module homomorphisms $\overline{X} \to Y$, as a subgroup of $\mathrm{Hom}_{\mathbb{Z}}(X,Y)$ (or as a left S-module). Here $\overline{X}$ denotes the right R-module that is equal to X as an abelian group, with its multiplication given by $xr = \overline{r} \cdot x$, where $\cdot$ denotes multiplication in X.

Definition 2.23. Suppose that R is a ring with involution. Let X be a left R-module. Its dual is the left R-module $X^* = \mathrm{Hom}_R^{lr}(X,R)$.

Now suppose that G is equipped with a group homomorphism $w\colon G \to \{\pm 1\}$, i.e. an orientation character. This determines an involution on the group ring $\mathbb{Z}G$, and also on the Whitehead group $\mathrm{Wh}(G,w)$ (see Section 2.2). So if C_* is a finitely generated, free, based, left $\mathbb{Z}G$-module chain complex then, using Definition 2.23, we can define the dual cochain complex C^*, which also consists of finitely generated, free, based, left $\mathbb{Z}G$-modules.

Lemma 2.24. *Let $f\colon C_* \to D_*$ be a chain homotopy equivalence and let $f^*\colon D^* \to C^*$ be its dual. Then $\tau(f^*) = \overline{\tau(f)} \in \mathrm{Wh}(G,w)$.*

Proof. First note that if $g\colon X \to Y$ is an isomorphism between finitely generated, free, based, left $\mathbb{Z}G$-modules, and $g^*\colon Y^* \to X^*$ is its dual, then the matrix of g^* is the conjugate transpose of the matrix of g, hence $\tau(g^*) = \overline{\tau(g)} \in \mathrm{Wh}(G,w)$.

Now let $(\mathscr{C}(f), \partial^f_*)$ denote the mapping cone of f, so that $\mathscr{C}(f)_i = C_{i-1} \oplus D_i$. We regard $f^*\colon D^* \to C^*$ as a homotopy equivalence $f^*\colon D^{-*} \to C^{-*}$ of chain complexes, and define its mapping cone $(\mathscr{C}(f^*), \partial^{f^*}_*)$. Then $\mathscr{C}(f^*)_i = (D_{-i+1})^* \oplus (C_{-i})^* \cong (\mathscr{C}(f)_{-i+1})^*$, and this implies that $\mathscr{C}(f^*)_{\mathrm{odd}} \cong (\mathscr{C}(f)_{\mathrm{even}})^*$ and $\mathscr{C}(f^*)_{\mathrm{even}} \cong (\mathscr{C}(f)_{\mathrm{odd}})^*$. Moreover, $\partial^{f^*}_i\colon \mathscr{C}(f^*)_i \to \mathscr{C}(f^*)_{i-1}$ is the dual of $\partial^f_{-i+2}\colon \mathscr{C}(f)_{-i+2} \to \mathscr{C}(f)_{-i+1}$.

Let $s_*\colon \mathscr{C}(f)_* \to \mathscr{C}(f)_{*+1}$ be a chain contraction. We define $s'_*\colon \mathscr{C}(f^*)_* \to \mathscr{C}(f^*)_{*+1}$ by $s'_i = (s_{-i})^*\colon \mathscr{C}(f^*)_i \cong (\mathscr{C}(f)_{-i+1})^* \to \mathscr{C}(f^*)_{i+1} \cong (\mathscr{C}(f)_{-i})^*$. Then it follows by dualising the formula $s_*\partial^f + \partial^f s_* = \mathrm{Id}$ that s'_* is a chain contraction of $\mathscr{C}(f^*)$.

Thus the map $\bigoplus_i \big(\partial^{f^*}_{2i+1} + s'_{2i+1}\big)\colon \mathscr{C}(f^*)_{\mathrm{odd}} \to \mathscr{C}(f^*)_{\mathrm{even}}$ is the dual of the map $\bigoplus_i \big(\partial^f_{2i+1} + s_{2i+1}\big)\colon \mathscr{C}(f)_{\mathrm{odd}} \to \mathscr{C}(f)_{\mathrm{even}}$. By our earlier observations in the first paragraph of the proof, this implies that $\tau(f^*) = \overline{\tau(f)}$. □

Finally we consider the effect of changing the underlying (group) ring.

Definition 2.25. Let A and B be groups, X a left $\mathbb{Z}B$-module and $\theta \in \mathrm{Hom}(A,B)$. The left $\mathbb{Z}A$-module X_θ is defined as follows. The underlying abelian group of X_θ is the same as that of X. For every $a \in A$ and $x \in X_\theta$ let $ax = \theta(a) \cdot x$, where $\cdot$ denotes multiplication in X.

Similarly, if Y is a right $\mathbb{Z}B$-module, then the right $\mathbb{Z}A$-module Y^θ is equal to Y as an abelian group, and $ya = y \cdot \theta(a)$ for every $a \in A$ and $y \in Y^\theta$, where $\cdot$ denotes multiplication in Y.

Lemma 2.26. *Let A and B be groups equipped with orientation characters $w_A \colon A \to \{\pm 1\}$ and $w_B \colon B \to \{\pm 1\}$ respectively. Let X be a left $\mathbb{Z}A$-module and let $\theta \colon A \to B$ be an isomorphism such that $w_B \circ \theta = w_A$. Then we have the following isomorphisms of left $\mathbb{Z}B$-modules.*

(a) $\mathbb{Z}B^\theta \otimes_{\mathbb{Z}A} X \cong X_{\theta^{-1}}$.

(b) $\mathrm{Hom}^{lr}_{\mathbb{Z}B}(X_{\theta^{-1}}, \mathbb{Z}B) = \mathrm{Hom}^{lr}_{\mathbb{Z}A}(X, \mathbb{Z}B^\theta) \cong \mathrm{Hom}^{lr}_{\mathbb{Z}A}(X, \mathbb{Z}A)_{\theta^{-1}}$, *in particular* $(X_{\theta^{-1}})^* \cong (X^*)_{\theta^{-1}}$.

Proof. (a) If Y is a right $\mathbb{Z}A$-module and a left R-module for some ring R, then it follows from the definition of the tensor product (see Definition 2.21) that $Y \otimes_{\mathbb{Z}A} X = Y^{\theta^{-1}} \otimes_{\mathbb{Z}B} X_{\theta^{-1}}$ (as left R-modules). By applying this for $Y = \mathbb{Z}B^\theta$ and $R = \mathbb{Z}B$, we get that $\mathbb{Z}B^\theta \otimes_{\mathbb{Z}A} X = \mathbb{Z}B \otimes_{\mathbb{Z}B} X_{\theta^{-1}}$. Of course $X_{\theta^{-1}} \cong \mathbb{Z}B \otimes_{\mathbb{Z}B} X_{\theta^{-1}}$ via the map $x \mapsto 1 \otimes x$.

(b) (cf. [Nic23, Proposition 3.9]) For Y as above, it also follows from the definitions that $\mathrm{Hom}^{lr}_{\mathbb{Z}A}(X, Y) = \mathrm{Hom}^{lr}_{\mathbb{Z}B}(X_{\theta^{-1}}, Y^{\theta^{-1}})$, so $\mathrm{Hom}^{lr}_{\mathbb{Z}A}(X, \mathbb{Z}B^\theta) = \mathrm{Hom}^{lr}_{\mathbb{Z}B}(X_{\theta^{-1}}, \mathbb{Z}B) = (X_{\theta^{-1}})^*$.

The isomorphism θ induces an isomorphism $\mathbb{Z}\theta \colon \mathbb{Z}A \to \mathbb{Z}B$ of abelian groups (or rings). This map is also an isomorphism $\mathbb{Z}\theta \colon \mathbb{Z}A \to \mathbb{Z}B^\theta$ of right $\mathbb{Z}A$-modules. Hence it induces an isomorphism $(\mathbb{Z}\theta)_* \colon \mathrm{Hom}^{lr}_{\mathbb{Z}A}(X, \mathbb{Z}A) \to \mathrm{Hom}^{lr}_{\mathbb{Z}A}(X, \mathbb{Z}B^\theta)$ of abelian groups. We can check that $(\mathbb{Z}\theta)_*$ is also an isomorphism $\mathrm{Hom}^{lr}_{\mathbb{Z}A}(X, \mathbb{Z}A)_{\theta^{-1}} \to \mathrm{Hom}^{lr}_{\mathbb{Z}A}(X, \mathbb{Z}B^\theta)$ of left $\mathbb{Z}B$-modules. So we get that $(X^*)_{\theta^{-1}} = \mathrm{Hom}^{lr}_{\mathbb{Z}A}(X, \mathbb{Z}A)_{\theta^{-1}} \cong \mathrm{Hom}^{lr}_{\mathbb{Z}A}(X, \mathbb{Z}B^\theta)$. □

Lemma 2.27. *Let A and B be groups and let $\theta \colon A \to B$ be an isomorphism. Let $f \colon C_* \to D_*$ be a chain homotopy equivalence of finitely generated, free, based, left $\mathbb{Z}B$-module chain complexes, which can also be regarded as a chain homotopy equivalence $f \colon (C_*)_\theta \to (D_*)_\theta$ of $\mathbb{Z}A$-module chain complexes. We have*

$$\tau(f \colon (C_*)_\theta \to (D_*)_\theta) = \theta_*^{-1}(\tau(f \colon C_* \to D_*)) \in \mathrm{Wh}(A).$$

Proof. First note that if $g \colon X \to Y$ is an isomorphism between finitely generated, free, based, left $\mathbb{Z}B$-modules, and we regard it as an isomorphism $g \colon X_\theta \to Y_\theta$ of $\mathbb{Z}A$-modules, then its matrix changes by applying θ^{-1} to each entry, hence $\tau(g \colon X_\theta \to Y_\theta) = \theta_*^{-1}(\tau(g \colon X \to Y)) \in \mathrm{Wh}(A)$.

Now let $(\mathscr{C}(f), \partial_*)$ denote the mapping cone of f and let $s_* \colon \mathscr{C}(f)_* \to \mathscr{C}(f)_{*+1}$ be the chain contraction used to define $\tau(f)$ over $\mathbb{Z}B$. Then s_* is also a chain contraction for $(\mathscr{C}(f)_\theta, \partial_*)$, the mapping cone over $\mathbb{Z}A$.

So we can compute $\tau(f \colon (C_*)_\theta \to (D_*)_\theta)$ from

$$\bigoplus_i \big(\partial_{2i+1} + s_{2i+1}\big) \colon (\mathscr{C}(f)_\theta)_{\mathrm{odd}} = (\mathscr{C}(f)_{\mathrm{odd}})_\theta \to (\mathscr{C}(f)_\theta)_{\mathrm{even}} = (\mathscr{C}(f)_{\mathrm{even}})_\theta,$$

which is $\bigoplus_i \big(\partial_{2i+1} + s_{2i+1}\big) \colon \mathscr{C}(f)_{\mathrm{odd}} \to \mathscr{C}(f)_{\mathrm{even}}$ regarded as an isomorphism of $\mathbb{Z}A$-modules, hence $\tau(f \colon (C_*)_\theta \to (D_*)_\theta) = \theta_*^{-1}(\tau(f \colon C_* \to D_*))$. □

2.4. **The Whitehead torsion of a homotopy equivalence.** Now let X and Y be finite CW complexes with universal covers $\widetilde{X}$ and $\widetilde{Y}$, and let $F := \pi_1(X)$ and $G := \pi_1(Y)$. The cellular chain complex of Y with $\mathbb{Z}G$ coefficients is $C_*(Y; \mathbb{Z}G) = \mathbb{Z}G \otimes_{\mathbb{Z}G} C_*(\widetilde{Y}) \cong C_*(\widetilde{Y})$, which is a finitely generated, free, left $\mathbb{Z}G$-module chain complex. Choose a lift of each cell of Y in $\widetilde{Y}$ to obtain a basis of $C_*(\widetilde{Y})$, which is well-defined up to ordering, signs and multiplication by elements of G. Similarly, the cellular chain complex of X with $\mathbb{Z}F$ coefficients, $C_*(X; \mathbb{Z}F) \cong C_*(\widetilde{X})$, is a finitely generated, free, left $\mathbb{Z}F$-module chain complex with a basis well-defined up to ordering and multiplication by elements of $\pm F$.

Let $f \colon X \to Y$ be a cellular homotopy equivalence, and let $\theta = \pi_1(f) \colon F \to G$. The right $\mathbb{Z}G$-module $\mathbb{Z}G$ corresponds to a local coefficient system on Y, which is pulled back to the local coefficient system on X corresponding to the right $\mathbb{Z}F$-module $\mathbb{Z}G^\theta$. Therefore f induces a chain homotopy equivalence $f_* \colon C_*(X; \mathbb{Z}G^\theta) \to C_*(Y; \mathbb{Z}G)$ of left $\mathbb{Z}G$-module chain complexes. Note that by Lemma 2.26 (a) we have

$$C_*(X; \mathbb{Z}G^\theta) = \mathbb{Z}G^\theta \otimes_{\mathbb{Z}F} C_*(\widetilde{X}) \cong C_*(\widetilde{X})_{\theta^{-1}} \cong C_*(X; \mathbb{Z}F)_{\theta^{-1}},$$

so this is also a finitely generated, free chain complex with a basis that is well-defined up to ordering and multiplication by elements of $\pm G$.

Definition 2.28. The *Whitehead torsion* of the cellular homotopy equivalence $f\colon X \to Y$ is $\tau(f) := \tau(f_*)$, where $f_*\colon C_*(X;\mathbb{Z}G^\theta) \to C_*(Y;\mathbb{Z}G)$ is the induced chain homotopy equivalence.

By Remark 2.14, it follows that $\tau(f_*)$ is well-defined, even though the bases of $C_*(X;\mathbb{Z}G^\theta)$ and $C_*(Y;\mathbb{Z}G)$ are well-defined only up to ordering and multiplication by elements of $\pm G$.

Proposition 2.29 ([Coh73, Statement 22.1])**.** *Let $f, g\colon X \to Y$ be homotopic cellular homotopy equivalences between finite CW complexes. Then $\tau(f) = \tau(g) \in \mathrm{Wh}(G)$.*

Proof. This follows immediately from Proposition 2.15, because homotopic homotopy equivalences induce homotopic chain homotopy equivalences. □

Now we can extend the definition of Whitehead torsion to arbitrary homotopy equivalences. If $f\colon X \to Y$ is a homotopy equivalence between finite CW complexes, then it is homotopic to a cellular homotopy equivalence $f'\colon X \to Y$, and we define $\tau(f) := \tau(f')$. By Proposition 2.29 this is independent of the choice of f'. Moreover, it follows that if $f, g\colon X \to Y$ are arbitrary homotopy equivalences and $f \simeq g$, then $\tau(f) = \tau(g)$.

Now that we have defined Whitehead torsion, it is worth recalling its key rôle: Theorem 2.2 states that a homotopy equivalence $f\colon X \to Y$ between CW complexes X and Y is simple if and only if its Whitehead torsion $\tau(f) = 0 \in \mathrm{Wh}(\pi_1(Y))$.

We collect a few key properties of Whitehead torsion.

Proposition 2.30 ([Coh73, Corollary 5.1A])**.** *Let X and Y be finite CW complexes and let $f\colon X \to Y$ be a cellular map. Let Cyl_f denote the mapping cylinder of f. Then the inclusion $Y \to \mathrm{Cyl}_f$ is a simple homotopy equivalence.*

Proposition 2.31 ([Coh73, Statement 22.4])**.** *Let X, Y, and Z be finite CW complexes and let $f\colon X \to Y$ and $g\colon Y \to Z$ be homotopy equivalences. Then*

$$\tau(g \circ f) = \tau(g) + g_*(\tau(f)).$$

Proposition 2.32 ([Coh73, Statement 23.2])**.** *For $i = 1, 2$, let X_i, Y_i be finite CW complexes and let $f_i\colon X_i \to Y_i$ be a homotopy equivalence. Let $i_1\colon Y_1 \hookrightarrow Y_1 \times Y_2$ and $i_2\colon Y_2 \hookrightarrow Y_1 \times Y_2$ be natural inclusion maps defined by fixing a point in Y_2 and Y_1 respectively. For the product map $f_1 \times f_2\colon X_1 \times X_2 \to Y_1 \times Y_2$, we have*

$$\tau(f_1 \times f_2) = \chi(Y_2) \cdot i_{1*}(\tau(f_1)) + \chi(Y_1) \cdot i_{2*}(\tau(f_2))$$

Corollary 2.33. *Let $f_1\colon M_1 \to N_1$ and $f_2\colon M_2 \to N_2$ be homotopy equivalences between odd dimensional manifolds. Then $f_1 \times f_2\colon M_1 \times M_2 \to N_1 \times N_2$ is a simple homotopy equivalence.*

Proof. We have $\tau(f_1 \times f_2) = 0$, by Proposition 2.32 and the fact that $\chi(N_1) = \chi(N_2) = 0$. □

Proposition 2.34. *Let $f\colon X \to Y$ be a homeomorphism between finite CW complexes. Then $\tau(f) = 0$.*

Proof. By Theorem 2.6, f is a simple homotopy equivalence. Hence $\tau(f) = 0$ by Theorem 2.2. □

The following generalises an observation of Wall. He pointed this out in the case where M, N are simple Poincaré complexes [Wal74, p. 612], i.e. finite Poincaré complexes X for which the chain duality isomorphism $C^*(\widetilde{X}) \to C_*(\widetilde{X})$ is a simple chain homotopy equivalence. At the time, it was known that smooth manifolds are simple Poincaré complexes [Wal99, Theorem 2.1] but the case of topological manifolds was not established until the work of Kirby-Siebenmann [KS77, III.5.13].

Proposition 2.35. *Let M and N be closed n-dimensional topological manifolds. Let $G = \pi_1(N)$ with orientation character $w\colon G \to \{\pm 1\}$, and let $f\colon M \to N$ be a homotopy equivalence. Then $\tau(f) \in \mathcal{J}_n(G, w)$.*

Proof. Let $F = \pi_1(M)$ and $\theta = \pi_1(f)\colon F \to G$. Since f is a homotopy equivalence, $f_*([M]) = \pm[N]$, where $[M]$ and $[N]$ denote the (twisted) fundamental classes of M and N respectively. Since Poincaré duality is given by taking cap product with the fundamental class, and the cap product is natural, we get a diagram of left $\mathbb{Z}G$-module chain complexes that commutes up to chain homotopy and sign:

$$\begin{array}{ccc} C^{n-*}(M;\mathbb{Z}G^\theta) & \xleftarrow{C^{n-*}(f)} & C^{n-*}(N;\mathbb{Z}G) \\ \downarrow{\scriptstyle \mathrm{PD}} & & \downarrow{\scriptstyle \mathrm{PD}} \\ C_*(M;\mathbb{Z}G^\theta) & \xrightarrow{C_*(f)} & C_*(N;\mathbb{Z}G) \end{array}$$

Poincaré duality is a simple chain homotopy equivalence by [KS77, III.5.13], so we have that $\tau(\mathrm{PD}\colon C^{n-*}(N;\mathbb{Z}G) \to C_*(N;\mathbb{Z}G)) = 0$. By Lemmas 2.26 and 2.27, the Poincaré duality map

$$\mathrm{PD}\colon C^{n-*}(M;\mathbb{Z}G^\theta) \cong C^{n-*}(M;\mathbb{Z}F)_{\theta^{-1}} \to C_*(M;\mathbb{Z}G^\theta) \cong C_*(M;\mathbb{Z}F)_{\theta^{-1}}$$

is simple too. It follows from Proposition 2.15 and Lemma 2.16 that $\tau(C^{n-*}(f)) + \tau(C_*(f)) = 0$. Therefore it is enough to prove $\tau(C^{n-*}(f)) = (-1)^n\overline{\tau(C_*(f))}$.

By Lemma 2.19, we know that $\tau(C^{n-*}(f)) = (-1)^n\tau(C^{-*}(f))$, and it follows from Definition 2.20 that $\tau(C^{-*}(f)) = \overline{\tau(C^*(f))}$. Finally Lemma 2.24 shows that $\tau(C^*(f)) = \overline{\tau(C_*(f))}$. We deduce $\tau(C^{n-*}(f)) = (-1)^n\overline{\tau(C_*(f))}$, completing the proof. □

We recall the definitions of h- and s-cobordisms.

Definition 2.36. A cobordism $(W; M, N)$ of closed manifolds is an *h-cobordism* if the inclusion maps $i_M\colon M \to W$ and $i_N\colon N \to W$ are homotopy equivalences. If in addition i_M and i_N are simple homotopy equivalences then W is an *s-cobordism*.

Definition 2.37. Suppose that W is an h-cobordism between closed n-dimensional manifolds M and N. Let $G = \pi_1(W)$ and let $w\colon G \to \{\pm 1\}$ be the orientation character of W. We will write $\tau(W, M)$ and $\tau(W, N)$ to denote the Whitehead torsion of the inclusions $M \to W$ and $N \to W$ respectively. We will refer to the composition $N \to M$ of the inclusion $N \to W$ and the homotopy inverse of the inclusion $M \to W$ as *the homotopy equivalence induced by W*.

Proposition 2.38. *Let W be an h-cobordism between closed, n-dimensional manifolds M and N. Let $G = \pi_1(W)$ and let $w\colon G \to \{\pm 1\}$ be the orientation character of W.*

(a) *We have that $\tau(W, N) = (-1)^n\overline{\tau(W, M)} \in \mathrm{Wh}(G, w)$.*
(b) *If $f\colon N \to M$ denotes the homotopy equivalence induced by W, then we have*
$$\tau(f) = -\tau(W, M) + (-1)^n\overline{\tau(W, M)} \in \mathrm{Wh}(G, w)$$
where $\pi_1(M)$ is identified with G via inclusion. In particular $\tau(f) \in \mathcal{I}_n(G, w)$.

Proof. For (a) this is the "duality theorem" [Mil66, p. 394], translated into our conventions; see also the Remark on [Mil66, p. 398]. Then (b) follows from part (a) and Proposition 2.31. □

3. Realising elements of Wh(G) by maps between manifolds

Throughout this section, fix $n \geq 4$, a finitely presented group G, and CAT $\in \{\mathrm{Diff}, \mathrm{PL}, \mathrm{TOP}\}$, satisfying Hypothesis 1.1.

3.1. **Realising via h-cobordisms.** If $(W; M, N)$ is an h-cobordism, then we will use the isomorphism $\pi_1(i_M)$ to identify $\pi_1(M)$ with $\pi_1(W)$, and regard $\tau(W, M) \in \mathrm{Wh}(\pi_1(W))$ as an element of $\mathrm{Wh}(\pi_1(M))$. Here is the complete statement of the s-cobordism theorem, for closed manifolds. It is due to Smale, Barden, Mazur, Stallings, Kirby-Siebenmann, and Freedman-Quinn [Sma62, Bar63, Maz63, Sta67, KS77, FQ90].

Theorem 3.1 (s-cobordism theorem)**.** *Let M be a closed,* CAT *n-manifold with $\pi_1(M) \cong G$, satisfying Hypothesis 1.1.*

(a) *Let $(W; M, M')$ be an h-cobordism over M. Then W is trivial over M, i.e. $W \cong M \times [0, 1]$, via a* CAT*-isomorphism restricting to the identity on M, if and only if its Whitehead torsion $\tau(W, M) \in \mathrm{Wh}(G)$ vanishes.*
(b) *For every $x \in \mathrm{Wh}(G)$ there exists an h-cobordism $(W; M, M')$ with $\tau(W, M) = x$.*
(c) *The function assigning to an h-cobordism $(W; M, M')$ its Whitehead torsion $\tau(W, M)$ yields a bijection from the* CAT*-isomorphism classes relative to M of h-cobordisms over M to the Whitehead group* $\mathrm{Wh}(G)$.

See [KPR22, Theorem 3.5] for details of the proof of part (b) in the case $n = 4$. In this case there is an extra subtlety that does not occur for $n \geq 5$, because one has to check that M' has the same fundamental group as M.

Recall that $\mathcal{I}_n(G, w) \coloneqq \{y + (-1)^{n+1}\overline{y} \mid y \in \mathrm{Wh}(G, w)\} \leq \mathrm{Wh}(G, w)$.

Corollary 3.2. *Let M be a closed,* CAT *n-manifold with $\pi_1(M) \cong G$ and orientation character $w\colon G \to \{\pm 1\}$, satisfying Hypothesis 1.1. For every $x \in \mathcal{I}_n(G, w)$ there exists a closed,* CAT *n-manifold N and a homotopy equivalence $f\colon N \to M$ induced by an h-cobordism between M and N such that $\tau(f) = x$.*

Proof. Let $x \in \mathcal{I}_n(G, w)$ and write $x = -y + (-1)^n\overline{y}$ for some $y \in \mathrm{Wh}(G, w)$. Apply Theorem 3.1 to obtain an h-cobordism $(W; M, N)$ from M to some n-manifold N with $\tau(W, M) = y$. If f is the homotopy equivalence induced by W, then $\tau(f) = -y + (-1)^n\overline{y} = x$ by Proposition 2.38 (b). □

Corollary 3.3. *Let M and N be closed,* CAT *n-manifolds satisfying Hypothesis 1.1. Suppose M has $\pi_1(M) \cong G$ and orientation character $w\colon G \to \{\pm 1\}$. If there is a homotopy equivalence $f\colon N \to M$ such that $\tau(f) \in \mathcal{I}_n(G, w)$, then there exists a closed,* CAT *n-manifold P that is simple homotopy equivalent to N and h-cobordant to M.*

Proof. By Corollary 3.2 there is an h-cobordism between M and some n-manifold P such that $\tau(g) = \tau(f)$ for the induced homotopy equivalence $g\colon P \to M$. Moreover, if g^{-1} denotes the homotopy inverse of g, then $g^{-1} \circ f\colon N \to P$ is a simple homotopy equivalence, because $\tau(g^{-1} \circ f) = \tau(g^{-1}) + g_*^{-1}(\tau(f)) = \tau(g^{-1}) + g_*^{-1}(\tau(g)) = \tau(g^{-1} \circ g) = \tau(\mathrm{Id}) = 0$ by Proposition 2.31. □

3.2. **Realising via the surgery exact sequence.** We recall the surgery exact sequence. Let M be a closed CAT n-dimensional manifold with $\pi_1(M) = G$ and orientation character $w\colon G \to \{\pm 1\}$, for some $n \geq 4$, satisfying Hypothesis 1.1.

Definition 3.4. The *homotopy structure set* of M, denoted $\mathcal{S}^h(M)$, is by definition the set of pairs (N, f), where N is a closed CAT n-manifold and $f\colon N \to M$ is a homotopy equivalence, considered up to h-cobordism over M. That is, $[N, f] = [N', f'] \in \mathcal{S}^h(M)$ if and only if there is an h-cobordism $(W; N, N')$, with inclusion maps $i\colon N \to W$ and $i'\colon N' \to W$, together with a map $F\colon W \to M$ such that $F \circ i = f$ and $F \circ i' = f'$.

We can similarly define the *simple homotopy structure set* $\mathcal{S}^s(M)$ to be the set of pairs (N, f), where N is a closed CAT n-manifold and $f\colon N \to M$ is a simple homotopy equivalence, considered up to s-cobordism over M.

The Browder-Novikov-Sullivan-Wall surgery exact sequence for $x \in \{h, s\}$ is as follows [Wal99, FQ90]; see also [Lüc02, Theorem 5.12], [OPR21].

$$\mathcal{N}(M \times [0,1], M \times \{0,1\}) \xrightarrow{\sigma_x} L^x_{n+1}(\mathbb{Z}G, w) \xrightarrow{W_x} \mathcal{S}^x(M) \xrightarrow{\eta_x} \mathcal{N}(M) \xrightarrow{\sigma_x} L^x_n(\mathbb{Z}G, w).$$

Here, $\mathcal{N}(M \times [0,1], M \times \{0,1\})$ and $\mathcal{N}(M)$ denote the sets of the normal bordism classes of degree one normal maps over $M \times [0,1]$ and M respectively. These sets do not depend on the decoration x.

The groups $L^x_n(\mathbb{Z}G, w)$ are the surgery obstruction groups. Elements of $L^h_n(\mathbb{Z}G, w)$ are represented by nonsingular Hermitian forms over finitely generated free $\mathbb{Z}G$-modules for n even, and by nonsingular formations over finitely generated free $\mathbb{Z}G$-modules for n odd, with the involution on $\mathbb{Z}G$ determined by w. See e.g. [Ran80]. In the case of $L^s_n(\mathbb{Z}G, w)$ the forms/formations are also required to be based and simple.

The maps labelled σ_x are the surgery obstruction maps. For the definition of σ_s, we take a degree one normal map $(f, b)\colon N \to M$, perform surgery below the middle dimension to make the map $[n/2]$-connected, and then produce the based, simple form or formation (for n even or odd respectively) of the surgery kernel in the middle dimension(s), to obtain an element of $L^s_n(\mathbb{Z}G, w)$. To define the map σ_h we perform the same procedure, and then forget the data of the bases, to obtain an element of $L^h_n(\mathbb{Z}G, w)$. One of the main theorems of surgery [Wal99],[Lüc02] is that the maps σ_x, for $x \in \{h, s\}$, are well-defined.

The map W_x is the Wall realisation map. Given a $z \in L^x_{n+1}(\mathbb{Z}G, w)$ and $[M_0, f_0] \in \mathcal{S}^x(M)$, Wall realisation produces a new element $[M_1, f_1] \in \mathcal{S}^x(M)$ together with a degree one normal bordism between (M_0, f_0) and (M_1, f_1) whose surgery obstruction equals z. This determines an action of $L^x_{n+1}(\mathbb{Z}G, w)$ on $\mathcal{S}^x(M)$, and the map W_x is defined by acting on the equivalence class of the identity map, $[M, \mathrm{Id}_M] \in \mathcal{S}^x(M)$.

The s and h decorated L groups are related by the exact sequence:

$$\cdots \to L^s_{n+1}(\mathbb{Z}G,w) \to L^h_{n+1}(\mathbb{Z}G,w) \xrightarrow{\psi} \widehat{H}^{n+1}(C_2;\mathrm{Wh}(G,w)) \to L^s_n(\mathbb{Z}G,w) \to L^h_n(\mathbb{Z}G,w) \to \cdots \tag{3.1}$$

Remark 3.5. The only proof we could find in the literature for this sequence is due to Shaneson [Sha69, Section 4]. There, Shaneson attributed its derivation to Rothenberg, by a different (unpublished) proof.

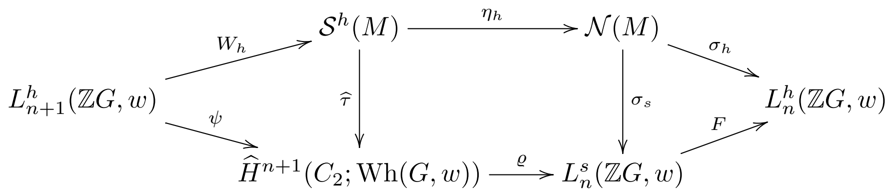
Here for the definition of the Tate group, C_2 acts on $\mathrm{Wh}(G,w)$ via the involution. Recall that $\widehat{H}^{n+1}(C_2;\mathrm{Wh}(G,w)) \cong \mathcal{J}_n(G,w)/\mathcal{I}_n(G,w)$ and $\pi\colon \mathcal{J}_n(G,w) \to \mathcal{J}_n(G,w)/\mathcal{I}_n(G,w)$ is the quotient map. Define a map

$$\begin{aligned} \widehat{\tau}\colon \mathcal{S}^h(M) &\to \widehat{H}^{n+1}(C_2;\mathrm{Wh}(G,w)) \\ [N,f] &\mapsto \pi(\tau(f)). \end{aligned}$$

We check that this map is well-defined. By Proposition 2.35, the Whitehead torsion of a homotopy equivalence between manifolds lies in $\mathcal{J}_n(G,w)$. If we change the representative of $[N,f]$ we obtain an h-cobordism between N and some N' over M, and by Proposition 2.38 this changes the torsion by an element of $\mathcal{I}_n(G,w)$. The map ψ from (3.1) determines an action of $L^h_{n+1}(\mathbb{Z}G,w)$ on $\widehat{H}^{n+1}(C_2;\mathrm{Wh}(G,w))$, given by $x^z = x + \psi(z)$ for $x \in \widehat{H}^{n+1}(C_2;\mathrm{Wh}(G,w))$ and $z \in L^h_{n+1}(\mathbb{Z}G,w)$. We will now establish two properties of $\widehat{\tau}$.

Proposition 3.6. *The map $\widehat{\tau}$ is $L^h_{n+1}(\mathbb{Z}G,w)$-equivariant. That is, for all $z \in L^h_{n+1}(\mathbb{Z}G,w)$ and $[M_0,f_0] \in \mathcal{S}^h(M)$, we have $\widehat{\tau}([M_0,f_0]^z) = \widehat{\tau}([M_0,f_0])^z$.*

Proposition 3.7. *There is a commutative diagram*



between the surgery exact sequence and the exact sequence (3.1).

Remark 3.8. This commutative diagram in Proposition 3.7 is part of a braid that appeared in Jahren and Kwasik's preprint [JK17, Diagram (6)], but was removed for the published version [JK18]. In the case CAT = TOP, see also Ranicki [Ran86, p.359–360]. Neither reference contains a proof.

In the proofs of these propositions we will need to work with Whitehead torsion in a more general setting than (chain) homotopy equivalences, so we will use Milnor's definition [Mil66, Section 3–4]. Given a chain complex C_* of free $\mathbb{Z}G$-modules with free homology $H_* := H_*(C_*)$, if bases are chosen for both C_* and H_*, then the Whitehead torsion $\tau(C_*) \in \mathrm{Wh}(G)$ is defined. In particular, if H_* is trivial, then it has a unique basis, so it is enough to choose a basis for C_*.

Given a chain homotopy equivalence $f\colon C_* \to D_*$ between based chain complexes, Cohen [Coh73, §16] showed that the definition of $\tau(f)$ from Section 2 coincides with Milnor's definition of the Whitehead torsion of the mapping cone $\mathscr{C}(f)$ of f, with basis determined by the bases of C_* and D_* (which is the same as the Whitehead torsion of the pair (Cyl_f, C_*)).

More generally, $\tau(C_*)$ is also defined when H_* is only stably free, equipped with a stable basis. A module X is called *stably free* if $X \oplus \mathbb{Z}G^k$ is free for some $k \geq 0$, and a *stable basis* of X is by definition the data of an integer k and a basis of $X \oplus \mathbb{Z}G^k$ for some k. Note that in this more general setting, $\tau(C_*)$ is no longer unambiguously defined when H_* is trivial and C_* is based, because the stable basis of the trivial module is not unique. But it has a canonical stable basis (given by its unique basis), so when H_* is trivial, we will assume that it is equipped with the canonical stable basis, unless we specify a different one.

In the proofs of Propositions 3.6 and 3.7, every space X will be equipped with a fixed map to M. This allows us to take chain complexes and homology groups with $\mathbb{Z}G$ coefficients. That is, we will write $C_*(X)$ for $C_*(X;\mathbb{Z}G^\theta)$ if $f\colon X \to M$ is the fixed map and $\theta = \pi_1(f)\colon \pi_1(X) \to \pi_1(M) = G$.

Proof of Proposition 3.6. The case of odd n was proved in Shaneson [Sha69, Lemma 4.2], so we will assume that $n = 2q$ is even.

An element $z \in L^h_{n+1}(\mathbb{Z}G, w)$ is represented by a formation $(H; L_0, L_1)$, where H is a $(-1)^q$-symmetric hyperbolic form over $\mathbb{Z}G$ (of rank $2k$) and L_0 and L_1 are lagrangians in H. Let $[M_0, f_0] \in \mathcal{S}^h(M)$. First we recall the definition of the action of z on $[M_0, f_0]$.

Consider $M_0 \times I$ and add k trivial q-handles, then we obtain a cobordism W_0 between M_0 and $N := M_0 \# k(S^q \times S^q)$. There is an isometry $H \cong H_q(\#^k S^q \times S^q)$ such that L_0 corresponds to the subgroup generated by the $* \times S^q$ (recall that all homology is with $\mathbb{Z}G$ coefficients by default in this proof). Then W_0 can also be constructed from $N \times I$ by adding k $(q+1)$-handles along a basis of L_0.

Next add k $(q+1)$-handles to $N \times I$ along a basis of L_1. This yields a cobordism W_1 between N and some M_1, and we define $W = W_0 \cup_N W_1$. It is a cobordism between M_0 and M_1 over M with the map $F \colon W \to M$, which restricts to the composition of the projection $M_0 \times I \to M_0$ and f_0 on $M_0 \times I$, and sends the extra handles to a point. Let

$$f_1 := F\big|_{M_1} \colon M_1 \to M, \ F_i := F\big|_{W_i} \colon W_i \to M \ (i = 0, 1), \ g := F\big|_{N} \colon N \to M.$$

Then $[M_0, f_0]^z = [M_1, f_1]$.

Fix a basis of H. On L_i we consider the basis which corresponds to the gluing maps of the handles that are added to $N \times I$ to construct W_i, this determines a dual basis on L_i^*. The adjoint of the intersection form on H is an isomorphism $H \to H^*$, and we get a split short exact sequence

$$0 \to L_i \to H \to L_i^* \to 0 \tag{3.2}$$

A splitting determines an isomorphism $H \cong L_i \oplus L_i^*$, and its Whitehead torsion, denoted $x_i \in \mathrm{Wh}(G, w)$, is independent of the choice of the splitting (see [Mil66, Section 2]).

We note that changing the basis of H has the same effect on x_0 and x_1, so $x_1 - x_0$ is independent of the choice of basis. In the case when the basis is given by the homology classes corresponding to the spheres in $H_q(\#^k S^q \times S^q)$, we have $x_0 = 0$ and $\psi(z) = \pi(x_1)$ by definition; see [Sha69, p. 312] and the correspondence between the different descriptions of the odd-dimensional L-groups given at the end of [Wal99, Chapter 6]. Therefore we have, independently of the choice of basis of H, that

$$\psi(z) = \pi(x_1 - x_0) \in \widehat{H}^{n+1}(C_2; \mathrm{Wh}(G, w)). \tag{3.3}$$

Now consider the triple $(\mathrm{Cyl}_{F_i}, W_i, N)$, where Cyl_{F_i} denotes the mapping cylinder of F_i. We compute the relative homology groups as follows:

(1) $H_*(W_i, N) \cong H_{q+1}(W_i, N) \cong L_i$;
(2) $H_*(\mathrm{Cyl}_{F_i}, N) \cong H_{q+1}(\mathrm{Cyl}_{F_i}, N) \cong \ker(H_q(g)) \cong H$, because $\mathrm{Cyl}_{F_i} \simeq M \simeq M_0$, and using the construction of N as $N = M_0 \# k(S^q \times S^q)$;
(3) $H_*(\mathrm{Cyl}_{F_i}, W_i) \cong H_{q+1}(\mathrm{Cyl}_{F_i}, W_i) \cong \ker(H_q(F_i)) \cong \ker(H_q(g))/\ker(H_q(N \to W_i)) \cong H/L_i \cong L_i^*$, because W_i is constructed from $M_0 \times I \simeq M \simeq \mathrm{Cyl}_{F_i}$ by adding $(q+1)$-handles.

So all of these homology groups are free (over $\mathbb{Z}G$), and we equip them with the previously chosen bases. The long exact sequence

$$\begin{aligned} \cdots \to H_{q+2}(\mathrm{Cyl}_{F_i}, W_i) \to H_{q+1}(W_i, N) \to \\ H_{q+1}(\mathrm{Cyl}_{F_i}, N) \to H_{q+1}(\mathrm{Cyl}_{F_i}, W_i) \to H_q(W_i, N) \to \cdots \end{aligned} \tag{3.4}$$

is therefore isomorphic to the short exact sequence (3.2). If we regard it as a based free chain complex with vanishing homology, then by the sign conventions in the definition (see [Mil66, p. 365]) its Whitehead torsion is $(-1)^q x_i$. The cobordism W_i is simple homotopy equivalent to N with a $(q+1)$-cell attached for each basis element of L_i, so

$$C_*(W_i, N) \cong C_{q+1}(W_i, N) \cong L_i \cong H_{q+1}(W_i, N) \cong H_*(W_i, N)$$

with the same choice of basis, therefore, again by the definition, $\tau(W_i, N) = 0$. Finally, $\mathrm{Cyl}_{F_i} \simeq_s M \simeq_s \mathrm{Cyl}_g$ by Proposition 2.30, so $\tau(\mathrm{Cyl}_{F_i}, N) = \tau(\mathrm{Cyl}_g, N)$. Hence by [Mil66, Theorem 3.2] we have

$$\tau(\mathrm{Cyl}_g, N) = \tau(\mathrm{Cyl}_{F_i}, W_i) + (-1)^q x_i,$$

which implies that

$$x_1 - x_0 = (-1)^{q+1}(\tau(\mathrm{Cyl}_{F_1}, W_1) - \tau(\mathrm{Cyl}_{F_0}, W_0)).$$

Next consider the triple $(\mathrm{Cyl}_{F_i}, W_i, M_i)$. Since W_i can be obtained from $M_i \times I$ by adding k trivial q-handles, we have $W_i \simeq_s M_i \vee (\bigvee^k S^q)$. We choose the basis corresponding to the spheres in both $C_*(W_i, M_i) \cong C_q(W_i, M_i) \cong C_q(\bigvee^k S^q)$ and $H_*(W_i, M_i) \cong H_q(W_i, M_i) \cong H_q(\bigvee^k S^q)$, so that $\tau(W_i, M_i) = 0$. Since $f_i \colon M_i \to M$ is a homotopy equivalence and $\mathrm{Cyl}_{F_i} \simeq_s M \simeq_s \mathrm{Cyl}_{F_i}$, we have $H_*(\mathrm{Cyl}_{F_i}, M_i) \cong H_*(\mathrm{Cyl}_{F_i}, M_i) = 0$ and $\tau(\mathrm{Cyl}_{F_i}, M_i) = \tau(\mathrm{Cyl}_{F_i}, M_i) = \tau(f_i)$. In the homological long exact sequence of the triple $(\mathrm{Cyl}_{F_i}, W_i, M_i)$ all terms are trivial except for the isomorphism

$$0 \to H_{q+1}(\mathrm{Cyl}_{F_i}, W_i) \xrightarrow{\cong} H_q(W_i, M_i) \to 0 \tag{3.5}$$

We equipped $H_{q+1}(\mathrm{Cyl}_{F_i}, W_i) \cong L_i^*$ with the dual of the basis of L_i. Since the handles added to $M_i \times I$ to obtain W_i (determining the basis of $H_q(W_i, M_i)$) are the duals of the handles added to $N \times I$ to obtain W_i (corresponding to the basis if L_i), the isomorphism (3.5) preserves the basis. Therefore the Whitehead torsion of the long exact sequence (3.4) vanishes. So by [Mil66, Theorem 3.2] we have $\tau(f_i) = \tau(\mathrm{Cyl}_{F_i}, W_i)$. Therefore $x_1 - x_0 = (-1)^{q+1}(\tau(f_1) - \tau(f_0))$, which implies that $\pi(x_1 - x_0) = \pi(\tau(f_1) - \tau(f_0))$. Combining with (3.3), we have $\pi(\tau(f_1) - \tau(f_0)) = \pi(x_1 - x_0) = \psi(z)$. Hence, as required, we have:

$$\begin{aligned}\widehat{\tau}([M_0, f_0]^z) &= \widehat{\tau}([M_1, f_1]) = \pi(\tau(f_1)) \\ &= \pi(\tau(f_0)) + \pi(\tau(f_1) - \tau(f_0)) = \pi(\tau(f_0)) + \pi(x_1 - x_0) \\ &= \widehat{\tau}([M_0, f_0]) + \psi(z) = \widehat{\tau}([M_0, f_0])^z.\end{aligned}$$

□

Proof of Proposition 3.7. The first triangle commutes by Proposition 3.6 combining with the fact that $\widehat{\tau}([M, \mathrm{Id}]) = 0$. The last triangle commutes by the definitions of σ_s and σ_h, see e.g. [Ran86, p.359–360]. The map σ_h is by definition the map σ_s followed by the map $F \colon L_n^s(\mathbb{Z}G, w) \to L_n^h(\mathbb{Z}G, w)$ that forgets bases. So we need to prove that the square commutes. Let $[N, f] \in \mathcal{S}^h(M)$, then $\eta_h([N, f]) \in \mathcal{N}(M)$ is the normal bordism class of $f \colon N \to M$ (with an appropriate bundle map), and $\widehat{\tau}([N, f]) = \pi(\tau(f))$. We need to determine the image of these elements in $L_n^s(\mathbb{Z}G, w)$.

First assume that $n = 2q$ is even. Since f is q-connected, $\sigma_s(\eta_h([N, f]))$ is defined as a form on $\ker(H_q(f)) \cong H_{q+1}(\mathrm{Cyl}_f, N)$, which is trivial (because f is a homotopy equivalence), and it is equipped with a certain stable basis. The stable basis of $H_{q+1}(\mathrm{Cyl}_f, N) = 0$ is chosen such that with this choice $\tau(\mathrm{Cyl}_f, N) = 0$ (where $H_i(\mathrm{Cyl}_f, N) = 0$ is equipped with the canonical stable basis if $i \neq q + 1$). If $H_{q+1}(\mathrm{Cyl}_f, N)$ were also equipped with the canonical stable basis, then the Whitehead torsion of (Cyl_f, N) would be equal to $\tau(f)$. Therefore the transition matrix between the chosen and the standard stable basis (when both are regarded as bases of $\mathbb{Z}G^k$ for some $k \geq 0$) has Whitehead torsion $(-1)^q\tau(f)$. In $L_n^s(\mathbb{Z}G, w)$ the same element $\sigma_s(\eta_h([N, f]))$ is also represented by a standard hyperbolic form with a basis such that the transition matrix between the chosen and the standard basis has Whitehead torsion $(-1)^q\tau(f)$.

The image of $\pi(\tau(f))$ in $L_n^s(\mathbb{Z}G, w)$ is represented by a standard hyperbolic form with a basis with the property that if x denotes the Whitehead torsion of the transition matrix between the chosen and the standard basis, then $x \in \mathcal{J}_n(G, w)$ and $\pi(x) = \pi(\tau(f))$ (see [Sha69, p. 312]). In particular, the representative of $\sigma_s(\eta_h([N, f]))$ constructed above also represents $\varrho(\widehat{\tau}([N, f]))$, noting that $\pi((-1)^q\tau(f)) = \pi(\tau(f))$, because $\widehat{H}^{n+1}(C_2; \mathrm{Wh}(G, w))$ is 2-torsion. Hence the square commutes if n is even.

Next consider the case when $n = 2q + 1$ is odd. Let U denote a tubular neighbourhood of a disjoint union of embeddings $S^q \to N$ representing a generating set of $\ker(\pi_q(f))$ and let $N_0 = N \setminus \mathrm{Int}\, U$. We identify $H_q(\partial U)$ with the standard hyperbolic form, then $H_{q+1}(U, \partial U)$ (more precisely, its image under the boundary map) corresponds to the standard lagrangian. We can assume that $f\big|_U$ is constant, so $f\big|_{N_0}$ is a map of pairs $(N_0, \partial U) \to (M, *)$. Then $\ker(H_{q+1}(N_0, \partial U) \to H_{q+1}(M, *)) \cong H_{q+2}(\mathrm{Cyl}_{f|_{N_0}}, N_0 \cup \mathrm{Cyl}_{f|_{\partial U}})$ determines another lagrangian in $H_q(\partial U)$. We equip $H_q(\partial U)$ and $H_{q+1}(U, \partial U)$ with their standard bases, and $H_{q+2}(\mathrm{Cyl}_{f|_{N_0}}, N_0 \cup \mathrm{Cyl}_{f|_{\partial U}})$ with a stable basis such that $\tau(\mathrm{Cyl}_{f|_{N_0}}, N_0 \cup \mathrm{Cyl}_{f|_{\partial U}}) = 0$. Then $\sigma_s(\eta_h([N, f]))$ is represented by the formation $(H_q(\partial U); H_{q+1}(U, \partial U), H_{q+2}(\mathrm{Cyl}_{f|_{N_0}}, N_0 \cup \mathrm{Cyl}_{f|_{\partial U}}))$. Since f is a homotopy equivalence, we can take the empty generating set for $\ker(\pi_q(f))$. Then $H_q(\partial U) = 0$, so we have the trivial formation, with the standard basis on the ambient form and on the first lagrangian. The stable basis on the second lagrangian, $H_{q+2}(\mathrm{Cyl}_f, N)$, is chosen such that $\tau(\mathrm{Cyl}_f, N) = 0$. Therefore the transition matrix between this stable basis and the standard one has Whitehead torsion $(-1)^{q+1}\tau(f)$. In

$L^s_n(\mathbb{Z}G,w)$ the same element $\sigma_s(\eta_h([N,f]))$ is also represented by a formation on the standard hyperbolic form given by the standard lagrangians, such that the ambient form and the first lagrangian are equipped with their standard bases, and for the second lagrangian the transition matrix between the chosen and the standard basis has Whitehead torsion $(-1)^{q+1}\tau(f)$.

The image of $\pi(\tau(f))$ in $L^s_n(\mathbb{Z}G,w)$ is represented by a formation on the standard hyperbolic form given by the standard lagrangians, such that the ambient form and the first lagrangian are equipped with their standard bases, and if x denotes the Whitehead torsion of the transition matrix between the chosen and the standard basis of the second lagrangian, then $x\in\mathcal{J}_n(G,w)$ and $\pi(x)=\pi(\tau(f))$. Again we use that we can ignore signs because $\widehat{H}^{n+1}(C_2;\mathrm{Wh}(G,w))$ is 2-torsion. So the representative of $\sigma_s(\eta_h([N,f]))$ constructed above also represents $\varrho(\widehat{\tau}([N,f]))$, showing that the square also commutes if n is odd. □

Corollary 3.9.

(a) $\operatorname{Im}\widehat{\tau}=\varrho^{-1}(\operatorname{Im}\sigma_s)$.
(b) $\operatorname{Im}\psi\subseteq\varrho^{-1}(\operatorname{Im}\sigma_s)$.

Proof. First we prove (a). The commutativity of the square in Proposition 3.7 implies that $\operatorname{Im}\widehat{\tau}\subseteq\varrho^{-1}(\operatorname{Im}\sigma_s)$. For the other direction assume that $x\in\widehat{H}^{n+1}(C_2;\mathrm{Wh}(G,w))$ and $\varrho(x)=\sigma_s(y)$ for some $y\in\mathcal{N}(M)$. Then $\sigma_h(y)=F\circ\sigma_s(y)=F\circ\varrho(x)=0$, so $y=\eta_h(v)$ for some $v\in\mathcal{S}^h(M)$. Then $\varrho(x-\widehat{\tau}(v))=\varrho(x)-\sigma_s(\eta_h(v))=\varrho(x)-\sigma_s(y)=0$, so $x-\widehat{\tau}(v)=\psi(z)$ for some $z\in L^h_{n+1}(\mathbb{Z}G,w)$. So by Proposition 3.6 $x=\widehat{\tau}(v)^z=\widehat{\tau}(v^z)\in\operatorname{Im}\widehat{\tau}$.

For (b), since Id_M has vanishing surgery obstruction, we have $\operatorname{Im}\psi=\ker\varrho\subseteq\varrho^{-1}(\operatorname{Im}\sigma_s)$. □

Corollary 3.10. *Let $x\in\widehat{H}^{n+1}(C_2;\mathrm{Wh}(G,w))$. If $x\in\operatorname{Im}\psi$, then there is a* CAT *$n$-manifold N and a homotopy equivalence $f\colon N\to M$ such that $\pi(\tau(f))=x$.*

Proof. By Corollary 3.9, $x=\widehat{\tau}([N,f])$ for some $[N,f]\in\mathcal{S}^h(M)$. □

Remark 3.11. There is a common generalisation of the two realisation techniques used in Section 3. Let $L^{s,\tau}_n(\mathbb{Z}G,w)$ denote the surgery obstruction groups defined in [Kre85, Section 4]. Similarly to $L^s_n(\mathbb{Z}G,w)$, the elements of $L^{s,\tau}_n(\mathbb{Z}G,w)$ are represented by based nonsingular forms/formations, but, unlike the elements of $L^s_n(\mathbb{Z}G,w)$, they are not assumed to be simple. Thus there is a natural forgetful map $L^{s,\tau}_n(\mathbb{Z}G,w)\to L^h_n(\mathbb{Z}G,w)$ and $L^s_n(\mathbb{Z}G,w)$ is the kernel of a map $L^{s,\tau}_n(\mathbb{Z}G,w)\to\mathrm{Wh}(G,w)$ (see [Kre85]). There is also an action of $\mathrm{Wh}(G,w)$ on $L^{s,\tau}_n(\mathbb{Z}G,w)$ (which determines a map $\mathrm{Wh}(G,w)\to L^{s,\tau}_n(\mathbb{Z}G,w)$ by acting on 0) given by changing the basis; this action is transitive on the fibers of the $L^{s,\tau}_n(\mathbb{Z}G,w)\to L^h_n(\mathbb{Z}G,w)$.

Wall's construction for realising elements of $L^s_n(\mathbb{Z}G,w)$ [Wal99] can also be applied to $L^{s,\tau}_n$, but it will only produce homotopy equivalences, not simple homotopy equivalences. The Whitehead torsion of the resulting homotopy equivalence is given by the previously mentioned map $L^{s,\tau}_n(\mathbb{Z}G,w)\to\mathrm{Wh}(G,w)$, and in fact the image of this map is in $\mathcal{J}_n(G,w)$.

The realisation of $L^h_n(\mathbb{Z}G,w)$, and the map W_h, can be regarded as a special case of this construction, by first choosing a basis for the form/formation representing an element of $L^h_n(\mathbb{Z}G,w)$, equivalently, choosing a lift in $L^{s,\tau}_n(\mathbb{Z}G,w)$. As the choice of basis is not unique, the homotopy equivalence we obtain is not unique, and its Whitehead torsion (the value of $\widehat{\tau}$) is only well-defined in the quotient $\widehat{H}^{n+1}(C_2;\mathrm{Wh}(G,w))$. The realisation part of the s-cobordism theorem is also a special case via the map $\mathrm{Wh}(G,w)\to L^{s,\tau}_n(\mathbb{Z}G,w)$. The Whitehead torsion of the homotopy equivalence induced by the h-cobordism we get is of course in $\mathcal{I}_n(G,w)$.

In the rest of this section we will consider the orientable ($w\equiv 1$) case, and omit w from the notation.

Proposition 3.12. *Let $G_m:=C_\infty\times C_m$, with $m\geq 2$. For every even integer $n=2k\geq 0$, the map*

$$\psi\colon L^h_{n+1}(\mathbb{Z}[G_m])\to\widehat{H}^{n+1}(C_2;\mathrm{Wh}(G_m))$$

is surjective.

Proof. We verify that the forgetful map $F\colon L^s_{2k}(\mathbb{Z}G_m)\to L^h_{2k}(\mathbb{Z}G_m)$ is injective. The conclusion then follows from the exact sequence (3.1). First we apply Shaneson splitting to the domain and the codomain, to obtain a commutative diagram whose rows are split short exact sequences and whose vertical maps are the forgetful maps [Ran86].

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L^s_{2k}(\mathbb{Z}C_m) & \longrightarrow & L^s_{2k}(\mathbb{Z}G_m) & \longrightarrow & L^h_{2k-1}(\mathbb{Z}C_m) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow F & & \downarrow & & \\ 0 & \longrightarrow & L^h_{2k}(\mathbb{Z}C_m) & \longrightarrow & L^h_{2k}(\mathbb{Z}G_m) & \longrightarrow & L^p_{2k-1}(\mathbb{Z}C_m) & \longrightarrow & 0. \end{array}$$

By the five lemma it suffices to show that the left and right vertical maps are injective. In [Bak76], Bak gave computations of $L^X_i(\mathbb{Z}G)$ for $X \in \{s,h,p\}$ and G a finite group whose 2-hyperelementary subgroups are abelian, which certainly holds for finite cyclic groups. See also [Bak75, Bak78] First we note that, by [Bak76, Theorem 8], the forgetful map $L^h_{2k-1}(\mathbb{Z}C_m) \to L^p_{2k-1}(\mathbb{Z}C_m)$ is injective. So the right vertical map is injective.

To prove that the left vertical map is injective we also use [Bak76]. For a finite group G whose 2-Sylow subgroup G_2 is normal and abelian, Bak defined $r_2 := \operatorname{rk} H^1(C_2; \mathrm{Wh}(G))$. On [Bak76, p. 386], he noted that if G_2 is cyclic, as in our case $G = C_m$, then $r_2 = 0$. Therefore by the sequence (3.1), as shown on [Bak76, p. 390], it follows that $L^s_{2k}(\mathbb{Z}C_m) \to L^h_{2k}(\mathbb{Z}C_m)$ is injective. Hence $F\colon L^s_{2k}(\mathbb{Z}G_m) \to L^h_{2k}(\mathbb{Z}G_m)$ is injective as desired. □

4. Simple homotopy manifold sets

In this section we prove Theorem E about the characterisation of simple homotopy manifold sets. First we look at the analogous problem in the setting of CW complexes, which has a simpler answer. Then we consider the case of manifolds, which will rely on the results of Section 3.

4.1. **CW complexes.** Fix a CW complex X and let $G = \pi_1(X)$. Our goal is to understand the set $\mathcal{C}^h_s(X)$ defined below.

Definition 4.1. Let $\mathcal{C}^h_s(X) := \{\text{CW complexes } Y \mid Y \simeq X\} / \simeq_s$.

We will need some auxiliary definitions.

Definition 4.2. If Y is a CW complex homotopy equivalent to X, then let

$$t_X(Y) = \{\tau(f) \mid f\colon Y \to X \text{ is a homotopy equivalence}\} \subseteq \mathrm{Wh}(G).$$

Definition 4.3. For $x \in \mathrm{Wh}(G)$ and $g \in \mathrm{hAut}(X)$ let $x^g = g_*(x) + \tau(g)$. Clearly $x^{\mathrm{Id}} = x$ and $x^{g \circ g'} = (x^{g'})^g$ (see Proposition 2.31), so this defines an action of $\mathrm{hAut}(X)$ on the set $\mathrm{Wh}(G)$.

With this notation we have the following theorem.

Theorem 4.4. *The map t_X induces a well-defined bijection*

$$\widetilde{t}_X\colon \mathcal{C}^h_s(X) \to \mathrm{Wh}(G)/\operatorname{hAut}(X).$$

The proof will consist of the following sequence of lemmas.

Lemma 4.5. *If Y is a CW complex homotopy equivalent to X, then $t_X(Y)$ is an orbit of the action of $\mathrm{hAut}(X)$ on $\mathrm{Wh}(G)$.*

Proof. By Proposition 2.31 we have $\tau(g \circ f) = \tau(f)^g$ for every homotopy equivalence $f\colon Y \to X$ and $g \in \mathrm{hAut}(X)$. If $f\colon Y \to X$ is a homotopy equivalence and $g \in \mathrm{hAut}(X)$, then $g \circ f$ is also a homotopy equivalence, so if $x \in t_X(Y)$, then $x^g \in t_X(Y)$ for every g. On the other hand, if $f, f'\colon Y \to X$ are homotopy equivalences, then there is a $g \in \mathrm{hAut}(X)$ such that $f' \simeq g \circ f$, showing that if $x, x' \in t_X(Y)$, then $x' = x^g$ for some g. □

Lemma 4.6. *If Y is a CW complex homotopy equivalent to X and $Y \simeq_s Z$, then $t_X(Y) = t_X(Z)$.*

Proof. Let $h\colon Z \to Y$ be a simple homotopy equivalence. If $x \in t_X(Y)$, i.e. $x = \tau(f)$ for some homotopy equivalence $f\colon Y \to X$, then $f \circ h\colon Z \to X$ is a homotopy equivalence with $\tau(f \circ h) = \tau(f) = x$ by Proposition 2.31. This shows that $t_X(Y) \subseteq t_X(Z)$. We get similarly that $t_X(Z) \subseteq t_X(Y)$, therefore $t_X(Y) = t_X(Z)$. □

Lemma 4.7. *If Y and Z are CW complexes homotopy equivalent to X and $t_X(Y) = t_X(Z)$, then $Y \simeq_s Z$.*

Proof. Let $x \in t_X(Y) = t_X(Z)$ be an arbitrary element, then there are homotopy equivalences $f\colon Y \to X$ and $f'\colon Z \to X$ with $\tau(f) = \tau(f') = x$. Let $f^{-1}\colon X \to Y$ denote the homotopy inverse of f, then by Proposition 2.31 $0 = \tau(\mathrm{Id}_Y) = \tau(f^{-1}) + f_*^{-1}(\tau(f)) = \tau(f^{-1}) + f_*^{-1}(x)$. Hence we have $\tau(f^{-1} \circ f') = \tau(f^{-1}) + f_*^{-1}(\tau(f')) = \tau(f^{-1}) + f_*^{-1}(x) = 0$, showing that $f^{-1} \circ f'\colon Z \to Y$ is a simple homotopy equivalence. □

Lemma 4.8 ([Coh73, (24.1)])**.** *For every $x \in \mathrm{Wh}(G)$ there is a CW complex Y and a homotopy equivalence $f\colon Y \to X$ such that $\tau(f) = x$.* □

Proof of Theorem 4.4. By Lemma 4.5, t_X takes values in $\mathrm{Wh}(G)/\operatorname{hAut}(X)$ and by Lemma 4.6 it induces a well-defined map $\widetilde{t}_X$ on $\mathcal{C}_s^h(X)$. Lemmas 4.7 and 4.8 imply that $\widetilde{t}_X$ is injective and surjective, respectively. □

Remark 4.9. There are two special cases when the action of $\operatorname{hAut}(X)$ on $\mathrm{Wh}(G)$ has a simpler description.

First, assume that $\pi_1(g) = \mathrm{Id}_G$ for every $g \in \operatorname{hAut}(X)$. Then $\tau(g \circ g') = \tau(g) + \tau(g')$ for every $g, g' \in \operatorname{hAut}(X)$ and $x^g = x + \tau(g)$ for every $x \in \mathrm{Wh}(G)$ and $g \in \operatorname{hAut}(X)$. This implies that $\{\tau(g) \mid g \in \operatorname{hAut}(X)\}$ is a subgroup of $\mathrm{Wh}(G)$ and $\mathrm{Wh}(G)/\operatorname{hAut}(X)$ is the corresponding quotient group.

Second, assume that $\tau(g) = 0$ for every $g \in \operatorname{hAut}(X)$. Then $x^g = g_*(x)$ for every $x \in \mathrm{Wh}(G)$ and $g \in \operatorname{hAut}(X)$. This means that the action of $\operatorname{hAut}(X)$ factors through the map $\pi_1\colon \operatorname{hAut}(X) \to \operatorname{Aut}(G)$, in particular $\operatorname{hAut}(X)$ acts via automorphisms of the group $\mathrm{Wh}(G)$.

4.2. **Manifolds.** Now we consider the problem in the manifold setting. Fix a closed connected CAT n-manifold M and let $G = \pi_1(M)$ with orientation character $w\colon G \to \{\pm 1\}$. Then we can consider either all n-manifolds that are homotopy equivalent to M, or those that are h-cobordant to M, up to simple homotopy equivalence, or manifolds homotopy equivalent to M up to the equivalence relation generated by simple homotopy equivalence and h-cobordism.

Definition 4.10. Let

$$\begin{aligned}\mathcal{M}_s^h(M) &:= \{\text{closed CAT } n\text{-manifolds } N \mid N \simeq M\} / \simeq_s \\ \mathcal{M}_s^{\mathrm{hCob}}(M) &:= \{\text{closed CAT } n\text{-manifolds } N \mid N \text{ is } h\text{-cobordant to } M\} / \simeq_s \\ \mathcal{M}_{s,\mathrm{hCob}}^h(M) &:= \{\text{closed CAT } n\text{-manifolds } N \mid N \simeq M\} / \langle \simeq_s, \mathrm{hCob}\rangle\end{aligned}$$

where $\langle \simeq_s, \mathrm{hCob}\rangle$ denotes the equivalence relation generated by simple homotopy equivalence and h-cobordism.

As before, $\operatorname{hAut}(M)$ acts on the set $\mathrm{Wh}(G, w)$, and a subset $t_M(N) \subseteq \mathrm{Wh}(G, w)$ is defined for every manifold N that is homotopy equivalent to M. By Proposition 2.35 if $f\colon N \to M$ is a homotopy equivalence between manifolds, then $\tau(f) \in \mathcal{J}_n(G, w)$, so we can also define

$$u_M(N) = \pi(t_M(N)) = \{\pi(\tau(f)) \mid f\colon N \to M \text{ is a homotopy equivalence}\}$$

which is a subset of $\widehat{H}^{n+1}(C_2; \mathrm{Wh}(G, w))$.

Theorem 4.11.

(a) *The subset $\mathcal{J}_n(G, w) \subseteq \mathrm{Wh}(G, w)$ is invariant under the action of $\operatorname{hAut}(M)$. The action of $\operatorname{hAut}(M)$ on $\mathcal{J}_n(G, w)$ induces an action on the set $\widehat{H}^{n+1}(C_2; \mathrm{Wh}(G, w))$.*

(b) *The map t_M induces well-defined injective maps*

$$\widetilde{t}_M\colon \mathcal{M}_s^h(M) \to \mathcal{J}_n(G, w)/\operatorname{hAut}(M) \text{ and } \widetilde{t}'_M\colon \mathcal{M}_s^{\mathrm{hCob}}(M) \to q(\mathcal{I}_n(G, w)),$$

where $q\colon \mathcal{J}_n(G, w) \to \mathcal{J}_n(G, w)/\operatorname{hAut}(M)$ denotes the quotient map, so that

$$q(\mathcal{I}_n(G, w)) \subseteq \mathcal{J}_n(G, w)/\operatorname{hAut}(M).$$

Similarly, the map u_M induces a well-defined map

$$\widetilde{u}_M\colon \mathcal{M}_{s,\mathrm{hCob}}^h(M) \to \widehat{H}^{n+1}(C_2; \mathrm{Wh}(G, w))/\operatorname{hAut}(M).$$

(c) *There is a commutative diagram*

$$\begin{array}{ccccc}
\mathcal{M}_s^{\mathrm{hCob}}(M) & \hookrightarrow & \mathcal{M}_s^h(M) & \twoheadrightarrow & \mathcal{M}_{s,\mathrm{hCob}}^h(M) \\
\downarrow{\scriptstyle \widetilde{t}'_M} & & \downarrow{\scriptstyle \widetilde{t}_M} & & \downarrow{\scriptstyle \widetilde{u}_M} \\
q(\mathcal{I}_n(G,w)) & \hookrightarrow & \mathcal{J}_n(G,w)/\operatorname{hAut}(M) & \twoheadrightarrow & \widehat{H}^{n+1}(C_2;\mathrm{Wh}(G,w))/\operatorname{hAut}(M)
\end{array}$$

In each row the first map is injective and the second map is surjective. In the top row the composition of the maps is trivial, while the bottom row is an exact sequence of pointed sets.

(d) *If Hypothesis 1.1 are satisfied, then $\widetilde{t}'_M$ is surjective, $\widetilde{u}_M$ is injective, and the top row is an exact sequence of pointed sets.*

(e) *If Hypothesis 1.1 are satisfied, then, using the notation from Section 3.2, the subsets*

$$\varrho^{-1}(\operatorname{Im}\sigma_s) \subseteq \mathcal{J}_n(G,w) \text{ and } (\varrho\circ\pi)^{-1}(\operatorname{Im}\sigma_s) \subseteq \widehat{H}^{n+1}(C_2;\mathrm{Wh}(G,w))$$

are invariant under the action of $\operatorname{hAut}(M)$*, and we have*

$$\operatorname{Im}\widetilde{t}_M = (\varrho\circ\pi)^{-1}(\operatorname{Im}\sigma_s)/\operatorname{hAut}(M) \text{ and } \operatorname{Im}\widetilde{u}_M = \varrho^{-1}(\operatorname{Im}\sigma_s)/\operatorname{hAut}(M).$$

This implies Theorem E (see also Theorem 4.16 below). Note that parts (a), (b) and (c) do not require Hypothesis 1.1 and so apply to smooth/PL 4-manifolds as well as topological 4-manifolds with arbitrary fundamental group.

We will use the following lemmas in the proof of Theorem 4.11.

Lemma 4.12. *Let N and P be manifolds homotopy equivalent to M. If $N \simeq_s P$ or N is h-cobordant to P, then $u_M(N) = u_M(P)$.*

Proof. If $N \simeq_s P$, then $t_M(N) = t_M(P)$ by Lemma 4.6, so $u_M(N) = \pi(t_M(N)) = \pi(t_M(P)) = u_M(P)$.

If N is h-cobordant to P and $h\colon P \to N$ is a homotopy equivalence induced by an h-cobordism, then $\pi(\tau(f\circ h)) = \pi(\tau(f)) + \pi(f_*(\tau(h))) = \pi(\tau(f))$ for every homotopy equivalence $f\colon N \to M$ (because $f_*(\tau(h)) \in \mathcal{I}_n(G,w)$ by Proposition 2.38). Hence $u_M(N) \subseteq u_M(P)$. We get similarly that $u_M(P) \subseteq u_M(N)$, therefore $u_M(N) = u_M(P)$. □

Lemma 4.13. *Suppose that Hypothesis 1.1 are satisfied, and N and P are manifolds homotopy equivalent to M. If $u_M(N) = u_M(P)$, then there is a manifold Q that is simple homotopy equivalent to N and h-cobordant to P.*

Proof. Since $u_M(N) = u_M(P)$, there are homotopy equivalences $f\colon N \to M$ and $g\colon P \to M$ such that $\pi(\tau(f)) = \pi(\tau(g))$, equivalently, $\tau(f) - \tau(g) \in \mathcal{I}_n(G,w)$. If g^{-1} denotes the homotopy inverse of g, then $\tau(g^{-1}\circ f) = \tau(g^{-1}) + g_*^{-1}(\tau(f)) = \tau(g^{-1}) + g_*^{-1}(\tau(g)) + g_*^{-1}(\tau(f)-\tau(g)) = \tau(g^{-1}\circ g) + g_*^{-1}(\tau(f)-\tau(g)) = g_*^{-1}(\tau(f)-\tau(g))$ by Proposition 2.31. So we can apply Corollary 3.3 to the homotopy equivalence $g^{-1}\circ f\colon N \to P$. □

Lemma 4.14. *Suppose that N and P are manifolds satisfying Hypothesis 1.1. If there is a manifold Q that is simple homotopy equivalent to N and h-cobordant to P, then there is a manifold R that is h-cobordant to N and simple homotopy equivalent to P.*

Proof. Apply Corollary 3.3 to the composition of a homotopy equivalence $P \to Q$ induced by an h-cobordism and a simple homotopy $Q \to N$. □

Proposition 4.15. *Suppose that N and P are manifolds satisfying Hypothesis 1.1. Then the following are equivalent.*

(a) *The manifolds N and P are equivalent under the equivalence relation generated by simple homotopy equivalence and h-cobordism.*

(a) *There is a manifold Q that is simple homotopy equivalent to N and h-cobordant to P.*

Proof. A chain between N and P of alternating simple homotopy equivalences and h-cobordisms can be reduced to a chain of length two using Lemma 4.14. □

Now we are ready to begin the proof of Theorem 4.11.

Proof of Theorem 4.11. (a) If $g \in \mathrm{hAut}(M)$, then $\tau(g) \in \mathcal{J}_n(G,w)$ by Proposition 2.35. Moreover, $\pi_1(g) \circ w = w$, hence $g_*\colon \mathrm{Wh}(G,w) \to \mathrm{Wh}(G,w)$ is compatible with the involution, so if $x = -(-1)^n \overline{x}$, then $g_*(x) = -(-1)^n \overline{g_*(x)}$. Hence, if $x \in \mathcal{J}_n(G,w)$, then $x^g \in \mathcal{J}_n(G,w)$, i.e. $\mathrm{hAut}(M)$ acts on $\mathcal{J}_n(G,w)$.

If $x, y \in \mathcal{J}_n(G,w)$ and $x - y \in \mathcal{I}_n(G,w)$, then $x^g - y^g = (g_*(x) + \tau(g)) - (g_*(y) + \tau(g)) = g_*(x-y) \in \mathcal{I}_n(G,w)$ (because g_* is compatible with the involution). Therefore there is an induced action on $\widehat{H}^{n+1}(C_2; \mathrm{Wh}(G,w))$.

(b) If $f\colon N \to M$ is a homotopy equivalence for some n-manifold N, then $\tau(f) \in \mathcal{J}_n(G,w)$ by Proposition 2.35. Hence $t_M(N) \subseteq \mathcal{J}_n(G,w)$ for every manifold N that is homotopy equivalent to M. Lemmas 4.5, 4.6, and 4.7 can be used again to show that t_M induces well-defined injective maps $\widetilde{t}_M\colon \mathcal{M}^h_s(M) \to \mathcal{J}_n(G,w)/\,\mathrm{hAut}(M)$ and $\widetilde{t}'_M\colon \mathcal{M}^{\mathrm{hCob}}_s(M) \to \mathcal{J}_n(G,w)/\,\mathrm{hAut}(M)$. If N is h-cobordant to M and $f\colon N \to M$ is a homotopy equivalence induced by an h-cobordism, then $\tau(f) \in \mathcal{I}_n(G,w)$ by Proposition 2.38, showing that $t_M(N) \in q(\mathcal{I}_n(G,w))$.

The analogue of Lemma 4.5 shows that $u_M(N)$ is an orbit of the action of $\mathrm{hAut}(M)$ on $\widehat{H}^{n+1}(C_2; \mathrm{Wh}(G,w))$ (using that $\pi(\tau(g \circ f)) = \pi(\tau(f)^g) = \pi(\tau(f))^g$ for every homotopy equivalence $f\colon N \to M$ and $g \in \mathrm{hAut}(M)$). Lemma 4.12 shows that u_M induces a well-defined map on $\mathcal{M}^h_{s,\mathrm{hCob}}(M)$.

(c) This follows immediately from the definitions.

(d) The surjectivity of $\widetilde{t}'_M$ and injectivity of $\widetilde{u}_M$ follow from Corollary 3.2 and Lemma 4.13, respectively.

For the exactness of the top row, assume that N is equivalent to M under the equivalence relation generated by simple homotopy equivalence and h-cobordism. By Proposition 4.15 there is a manifold P that is simple homotopy equivalent to N and h-cobordant to M. This represents an element of $\mathcal{M}^{\mathrm{hCob}}_s(M)$, which is mapped to the equivalence class of N in $\mathcal{M}^h_s(M)$.

(e) First consider $\varrho^{-1}(\operatorname{Im}\sigma_s)$, which is equal to $\operatorname{Im}\widehat{\tau}$ by Corollary 3.9. This is $\mathrm{hAut}(M)$-invariant, because for any $[N,f] \in \mathcal{S}^h(M)$ and $g \in \mathrm{hAut}(M)$ we have $\pi(\tau(f))^g = \pi(\tau(g \circ f)) = \widehat{\tau}([N, g\circ f])$. Since the action of $\mathrm{hAut}(M)$ on $\widehat{H}^{n+1}(C_2; \mathrm{Wh}(G,w))$ is induced by its action on $\mathcal{J}_n(G,w)$, and $\varrho^{-1}(\operatorname{Im}\sigma_s)$ is $\mathrm{hAut}(M)$-invariant, $\pi^{-1}(\varrho^{-1}(\operatorname{Im}\sigma_s))$ is $\mathrm{hAut}(M)$-invariant too.

Now suppose that $[N] \in \mathcal{M}^h_{s,\mathrm{hCob}}(M)$. Then there is a homotopy equivalence $f\colon N \to M$ and $\widetilde{u}_M([N])$ is the orbit of $\pi(\tau(f)) = \widehat{\tau}([N,f]) \in \operatorname{Im}\widehat{\tau} = \varrho^{-1}(\operatorname{Im}\sigma_s)$. This shows that $\operatorname{Im}\widetilde{u}_M \subseteq \varrho^{-1}(\operatorname{Im}\sigma_s)/\,\mathrm{hAut}(M)$. On the other hand, if $x \in \varrho^{-1}(\operatorname{Im}\sigma_s) = \operatorname{Im}\widehat{\tau}$, then $x = \pi(\tau(f))$ for some homotopy equivalence $f\colon N \to M$, and then $[N] \in \mathcal{M}^h_{s,\mathrm{hCob}}(M)$ and $\widetilde{u}_M([N])$ is the orbit of x. Therefore $\operatorname{Im}\widetilde{u}_M \supseteq \varrho^{-1}(\operatorname{Im}\sigma_s)/\,\mathrm{hAut}(M)$.

By the commutativity of the diagram in part (c), $\operatorname{Im}\widetilde{t}_M$ is contained in the inverse image of $\operatorname{Im}\widetilde{u}_M = \varrho^{-1}(\operatorname{Im}\sigma_s)/\,\mathrm{hAut}(M)$, which is $\pi^{-1}(\varrho^{-1}(\operatorname{Im}\sigma_s))/\,\mathrm{hAut}(M)$. In the other direction, if $x \in \mathcal{J}_n(G,w)$ and $\pi(x) \in \varrho^{-1}(\operatorname{Im}\sigma_s) = \operatorname{Im}\widehat{\tau}$, then there exists a manifold N_1 and a homotopy equivalence $f_1\colon N_1 \to M$ with $\pi(\tau(f_1)) = \pi(x)$. It follows that $x = \tau(f_1) + y$ for some $y \in \mathcal{I}_n(G,w)$. Use $(f_1)_*$ to identify $\pi_1(N_1)$ with $G = \pi_1(M)$ (since f_1 is a homotopy equivalence, the orientation character of N is also w). By Corollary 3.2 there exists a manifold N_2 and a homotopy equivalence $f_2\colon N_2 \to N_1$ with $\tau(f_2) = y$. By Proposition 2.31 we have $\tau(f_1 \circ f_2) = \tau(f_1) + (f_1)_*(\tau(f_2)) = \tau(f_1) + \tau(f_2) = \tau(f_1) + y = x$ (where $(f_1)_* = \mathrm{Id}$ because we used f_1 to identify $\pi_1(N_1)$ with G). So for $[N_2] \in \mathcal{M}^h_s(M)$ we get that $\widetilde{t}_M([N_2])$ is the orbit of x, showing that $\operatorname{Im}\widetilde{t}_M \supseteq (\varrho \circ \pi)^{-1}(\operatorname{Im}\sigma_s)/\,\mathrm{hAut}(M)$. □

In particular we have the following, which is immediate from Theorem 4.11.

Theorem 4.16. *If Hypothesis 1.1 are satisfied, then there are bijections*

$$\widetilde{t}'_M\colon \mathcal{M}^{\mathrm{hCob}}_s(M) \xrightarrow{\cong} q(\mathcal{I}_n(G,w))$$
$$\widetilde{t}_M\colon \mathcal{M}^h_s(M) \xrightarrow{\cong} (\varrho \circ \pi)^{-1}(\operatorname{Im}\sigma_s)/\,\mathrm{hAut}(M)$$
$$\widetilde{u}_M\colon \mathcal{M}^h_{s,\mathrm{hCob}}(M) \xrightarrow{\cong} \varrho^{-1}(\operatorname{Im}\sigma_s)/\,\mathrm{hAut}(M).$$

With extra input from the map ψ, the latter two statements become cleaner.

Corollary 4.17. *If Hypothesis 1.1 are satisfied and ψ is surjective, then there are bijections $\mathcal{M}^h_s(M) \cong \mathcal{J}_n(G,w)/\,\mathrm{hAut}(M)$ and $\mathcal{M}^h_{s,\mathrm{hCob}}(M) \cong \widehat{H}^{n+1}(C_2; \mathrm{Wh}(G,w))/\,\mathrm{hAut}(M)$.*

Proof. By Corollary 3.9 (b), $\widehat{H}^{n+1}(C_2; \mathrm{Wh}(G,w)) = \operatorname{Im}\psi \subseteq \varrho^{-1}(\operatorname{Im}\sigma_s)$. Hence $\varrho^{-1}(\operatorname{Im}\sigma_s) = \widehat{H}^{n+1}(C_2; \mathrm{Wh}(G,w))$, $(\varrho \circ \pi)^{-1}(\operatorname{Im}\sigma_s) = \mathcal{J}_n(G,w)$. □

Remark 4.18. If $\tau(g) \in \mathcal{I}_n(G,w)$ for every $g \in \mathrm{hAut}(M)$, then the proof of Theorem 4.11 (a) shows that $\mathcal{I}_n(G,w)$ is also an invariant subset under the action of $\mathrm{hAut}(M)$, and $q(\mathcal{I}_n(G,w)) = \mathcal{I}_n(G,w)/\operatorname{hAut}(M)$. It also means that $\pi(\tau(g)) = 0$ for every g, so $\mathrm{hAut}(M)$ acts on the group $\widehat{H}^{n+1}(C_2; \mathrm{Wh}(G,w))$ via automorphisms.

As in Remark 4.9, if $\tau(g) = 0$ for every g, then $\mathrm{hAut}(M)$ acts on the groups $\mathcal{I}_n(G,w)$ and $\mathcal{J}_n(G,w)$ via automorphisms. And if $\pi_1(g) = \mathrm{Id}_G$ for every g, then $\mathcal{J}_n(G,w)/\operatorname{hAut}(M)$ and $\widehat{H}^{n+1}(C_2; \mathrm{Wh}(G,w))/\operatorname{hAut}(M)$ are quotient groups of $\mathcal{J}_n(G,w)$ and $\widehat{H}^{n+1}(C_2; \mathrm{Wh}(G,w))$.

Now we consider some corollaries of Theorems 4.11 and 4.16.

Definition 4.19. Let M be a closed, CAT n-manifold with $\pi_1(M) \cong G$ and orientation character $w\colon G \to \{\pm 1\}$. We define the subsets

$$T(M) = \{\tau(g) \mid g \in \mathrm{hAut}(M)\} \subseteq \mathcal{J}_n(G,w)$$
$$U(M) = \{\pi(\tau(g)) \mid g \in \mathrm{hAut}(M)\} \subseteq \widehat{H}^{n+1}(C_2; \mathrm{Wh}(G,w))$$

Proposition 4.20. *Let M be a closed,* CAT *n-manifold satisfying Hypothesis 1.1. Let $G = \pi_1(M)$ with orientation character $w\colon G \to \{\pm 1\}$. Then $|\mathcal{M}^{\mathrm{hCob}}_s(M)| > 1$ if and only if $\mathcal{I}_n(G,w) \setminus T(M)$ is nonempty.*

Proof. Note that $T(M) = t_M(M) \subseteq \mathcal{J}_n(G,w)$ is the orbit of 0 under the action of $\mathrm{hAut}(M)$. So $q(\mathcal{I}_n(G,w))$ contains more than one element if and only if $\mathcal{I}_n(G,w) \setminus T(M)$ is nonempty. By Theorem 4.16 this is equivalent to $\mathcal{M}^{\mathrm{hCob}}_s(M)$ containing more than one element. □

Proposition 4.21. *Let M be a closed,* CAT *n-manifold satisfying Hypothesis 1.1. Let $G = \pi_1(M)$ with orientation character $w\colon G \to \{\pm 1\}$. Then $|\mathcal{M}^h_{s,\mathrm{hCob}}(M)| > 1$ if and only if $\varrho^{-1}(\operatorname{Im}\sigma_s) \setminus U(M)$ is nonempty. In particular, it is a sufficient condition that $\operatorname{Im}(\psi) \setminus U(M)$ is nonempty.*

Proof. Again $U(M) = u_M(M) \subseteq \widehat{H}^{n+1}(C_2; \mathrm{Wh}(G,w))$ is the orbit of 0 under the action of $\mathrm{hAut}(M)$. So $\varrho^{-1}(\operatorname{Im}\sigma_s)/\operatorname{hAut}(M)$ contains more than one element if and only if $\varrho^{-1}(\operatorname{Im}\sigma_s) \setminus U(M)$ is nonempty. By Theorem 4.16 this is equivalent to $\mathcal{M}^h_{s,\mathrm{hCob}}(M)$ containing more than one element. Finally, by Corollary 3.9(b) we have $\varrho^{-1}(\operatorname{Im}\sigma_s) \setminus U(M) \supseteq \operatorname{Im}(\psi) \setminus U(M)$. □

Proposition 4.22. *Let M be a closed,* CAT *n-manifold satisfying Hypothesis 1.1. Then the following are equivalent:*

(i) $|\mathcal{M}^h_s(M)| > 1$.
(ii) *Either* $|\mathcal{M}^{\mathrm{hCob}}_s(M)| > 1$ *or* $|\mathcal{M}^h_{s,\mathrm{hCob}}(M)| > 1$.
(iii) $(\varrho \circ \pi)^{-1}(\operatorname{Im}\sigma_s) \setminus T(M)$ *is nonempty.*

Proof. (i) ⇔ (ii). By Theorem 4.11 (c) and (d) we have $|\mathcal{M}^h_s(M)| = 1$ if and only if $|\mathcal{M}^{\mathrm{hCob}}_s(M)| = 1$ and $|\mathcal{M}^h_{s,\mathrm{hCob}}(M)| = 1$.

(i) ⇔ (iii). Follows from Theorem 4.16 since $T(M)$ is the orbit of 0. □

Part 2. The simple homotopy manifold set of $S^1 \times L$

In this part we consider $S^1 \times L$, the product of the circle with a lens space L. We will prove Theorems B, C, and D about the simple homotopy manifold sets of $S^1 \times L$. As described in Sections 1.4 and 1.5, we need to study the involution on $\mathrm{Wh}(C_\infty \times C_m)$ and the group $\mathrm{hAut}(S^1 \times L)$. Section 5 contains our results about the groups $\mathcal{J}_n(C_\infty \times C_m)$, $\mathcal{I}_n(C_\infty \times C_m)$ and $\widehat{H}^{n+1}(C_2; \mathrm{Wh}(C_\infty \times C_m))$ (for n even), which rely on the results of Part 3. In Section 6, we prove that all homotopy automorphisms of $S^1 \times L$ are simple, and determine the automorphisms of $C_\infty \times C_m$ induced by them. In Section 7, we combine these results to prove Theorems B, C, and D.

5. The involution on $\mathrm{Wh}(C_\infty \times C_m)$

In order to understand its involution, we first consider a direct sum decomposition of the Whitehead group $\mathrm{Wh}(C_\infty \times G)$. This decomposition is derived from the fundamental theorem for $K_1(\mathbb{Z}G[t,t^{-1}])$ [Bas68]; see also [Ran86], [Wei13, III.3.6]. Whilst this theorem appeared for the first time in Bass' book, the paper of Bass-Heller-Swan [BHS64, Theorem 2'] is often mentioned in conjunction with it, as an early version which contained several key ideas, and the theorem for left regular rings, appeared there. See [Bas68, pXV] for further discussion.

We will use the version given by Ranicki [Ran86]. We start by defining the terms appearing in the decomposition in Section 5.1. Section 5.2 contains the decomposition of $\mathrm{Wh}(C_\infty \times C_m)$ and the induced decompositions of $\mathcal{J}_n(C_\infty \times C_m)$, $\mathcal{I}_n(C_\infty \times C_m)$ and $\widehat{H}^{n+1}(C_2; \mathrm{Wh}(C_\infty \times C_m))$. These will be combined with the results of Part 3 in Section 5.3, to prove Theorems 5.14, 5.15, and 5.17, which will be the key algebraic ingredients in the proofs of Theorems B, C, and D respectively.

5.1. **The $\widetilde{K}_0$ and NK_1 groups.** First we consider the K_0 groups. For a ring R, we will define the abelian group $K_0(R)$, and, for a group G, the reduced group $\widetilde{K}_0(\mathbb{Z}G)$. A convenient reference for this material is [Wei13]. In Part 3, we will study the involution on $\widetilde{K}_0(\mathbb{Z}G)$ in more detail when G is a finite cyclic group.

Definition 5.1. For a ring R let $P(R)$ denote the set of isomorphism classes of (finitely generated left) projective R-modules, which is a monoid under direct sum. Define $K_0(R)$ to be the Grothendieck group of this monoid, i.e. the abelian group generated by symbols $[P]$, for every $P \in P(R)$, subject to the relations $[P_1 \oplus P_2] = [P_1] + [P_2]$ for $P_1, P_2 \in P(R)$.

The assignment $R \mapsto K_0(R)$ defines a functor from the category of rings to the category of abelian groups. If $f\colon R \to S$ is a ring homomorphism, then $K_0(f)\colon K_0(R) \to K_0(S)$ is the map induced by extension of scalars $P \mapsto f_\#(P) := S \otimes_R P$ (see Definition 2.21) for $P \in P(R)$.

Definition 5.2. For a group G, let

$$\widetilde{K}_0(\mathbb{Z}G) := K_0(\mathbb{Z}G)/K_0(\mathbb{Z}),$$

where the map $K_0(\mathbb{Z}) \to K_0(\mathbb{Z}G)$ is induced by the inclusion $i\colon \mathbb{Z} \to \mathbb{Z}G$.

Definition 5.3. For a group G and a (finitely generated) projective $\mathbb{Z}G$-module P, the *rank* of P is defined to be

$$\mathrm{rk}(P) := \mathrm{rk}_{\mathbb{Z}}(\varepsilon_\#(P))$$

where $\varepsilon\colon \mathbb{Z}G \to \mathbb{Z}$ denotes the augmentation map and $\mathrm{rk}_{\mathbb{Z}}$ denotes the rank of a free $\mathbb{Z}$-module.

We have that $K_0(\mathbb{Z}) \cong \mathbb{Z}$ and, for $P \in P(\mathbb{Z}G)$, the rank $\mathrm{rk}(P)$ coincides the the image of $[P]$ under the composition $K_0(\mathbb{Z}G) \to K_0(\mathbb{Z}) \cong \mathbb{Z}$ induced by ε. Since $\varepsilon \circ i = \mathrm{Id}_{\mathbb{Z}}$, we have a splitting of abelian groups $K_0(\mathbb{Z}G) \cong \mathbb{Z} \oplus \widetilde{K}_0(\mathbb{Z}G)$ given by $[P] \mapsto (\mathrm{rk}(P), [P])$. In particular, $\widetilde{K}_0(\mathbb{Z}G)$ is the set of equivalence classes of projective $\mathbb{Z}G$-modules P where $P \sim Q$ if $P \oplus \mathbb{Z}G^i \cong Q \oplus \mathbb{Z}G^j$ for some $i, j \geq 0$.

For R a ring with involution, we can define a natural involution on $K_0(R)$. If $P \in P(R)$, then $P^* \in P(R)$ (see Definition 2.23) since $P \oplus Q \cong R^n$ implies that $P^* \oplus Q^* \cong R^n$.

Definition 5.4. The *standard involution* on $K_0(R)$ is given by $[P] \mapsto -[P^*]$. This map preserves the rank of projective $\mathbb{Z}G$-modules and so induces an involution on $\widetilde{K}_0(\mathbb{Z}G)$.

Next we define the Nil groups $NK_1(R)$ and recall some of their properties.

Definition 5.5. For a ring R, let

$$NK_1(R) := \mathrm{coker}(K_1(R) \to K_1(R[t])),$$

where the map $K_1(R) \to K_1(R[t])$ is induced by the inclusion $R \to R[t]$.

Note that NK_1 is a functor from the category of rings to the category of abelian groups. If R (and hence $R[t]$) is a ring with involution, then $K_1(R[t])$ also has a natural involution (see Section 2.2). This involution preserves the image of $K_1(R) \to K_1(R[t])$, and hence induces an involution on $NK_1(R)$, which we will denote by $x \mapsto \overline{x}$.

Definition 5.6. We equip $NK_1(R)^2 = NK_1(R) \oplus NK_1(R)$ with the involution $(x, y) \mapsto (\overline{y}, \overline{x})$.

We will need the following result of Farrell [Far77] (note that Nil, which is also written as Nil_0, coincides with NK_1 by [Bas68, Chapter XII; 7.4(a)]).

Theorem 5.7 (Farrell). *For any ring R, if $NK_1(R) \neq 0$, then $NK_1(R)$ is not finitely generated.*

The following can be proven by combining results of Bass-Murthy [BM67], Martin [Mar75], and Weibel [Wei09].

Theorem 5.8. *Let $m \geq 1$. Then $NK_1(\mathbb{Z}C_m) = 0$ if and only if m is square-free.*

Proof. If m is square-free, then $NK_1(\mathbb{Z}C_m) = 0$ by [BM67, Theorem 10.8 (d)]. If $n \mid m$ and $NK_1(\mathbb{Z}C_n) \neq 0$, then $NK_1(\mathbb{Z}C_m) \neq 0$ (by, for example, [Mar75, Theorem 3.6]). It remains to show that $NK_1(\mathbb{Z}C_{p^2}) \neq 0$ for all primes p. This was achieved in the case where p is odd in [Mar75, Theorem B] and in the case $p = 2$ in [Wei09, Theorem 1.4]. □

5.2. **Bass-Heller-Swan decomposition of $\mathrm{Wh}(C_\infty \times G)$.** By [Ran86, p.329, p.357] we have the following theorem.

Theorem 5.9 (Bass-Heller-Swan decomposition). *Let G be a group. Then there is an isomorphism of $\mathbb{Z}C_2$-modules, which is natural in G:*

$$\mathrm{Wh}(C_\infty \times G) \cong \mathrm{Wh}(G) \oplus \widetilde{K}_0(\mathbb{Z}G) \oplus NK_1(\mathbb{Z}G)^2$$

where the $\mathbb{Z}C_2$-module structure of each component is determined by the involution defined in Section 2.2, Definition 5.4 and Definition 5.6 respectively.

Proposition 5.10. *Let G be a group. The decomposition of Theorem 5.9 restricts to the following isomorphisms:*

$$\mathcal{J}_n(C_\infty \times G) \cong \mathcal{J}_n(G) \oplus \{x \in \widetilde{K}_0(\mathbb{Z}G) \mid x = -(-1)^n \overline{x}\} \oplus NK_1(\mathbb{Z}G)$$
$$\mathcal{I}_n(C_\infty \times G) \cong \mathcal{I}_n(G) \oplus \{x - (-1)^n \overline{x} \mid x \in \widetilde{K}_0(\mathbb{Z}G)\} \oplus NK_1(\mathbb{Z}G)$$

where $NK_1(\mathbb{Z}G)$ is embedded into $NK_1(\mathbb{Z}G)^2$ by the map $x \mapsto (x, -(-1)^n \overline{x})$.

Proof. Since Theorem 5.9 gives a decomposition of $\mathbb{Z}C_2$-modules, $\mathcal{I}_n(C_\infty \times G)$ and $\mathcal{J}_n(C_\infty \times G)$ are decomposed into the corresponding subgroups of the components on the right-hand side. If $(x, y) \in NK_1(\mathbb{Z}G)^2$, then $\overline{(x,y)} = (\overline{y}, \overline{x})$, so $(x, y) = -(-1)^n \overline{(x,y)}$ if and only if $y = -(-1)^n \overline{x}$, so

$$\{(x, y) \in NK_1(\mathbb{Z}G)^2 \mid (x, y) = -(-1)^n \overline{(x,y)}\} = \{(x, -(-1)^n \overline{x}) \mid x \in NK_1(\mathbb{Z}G)\}$$

which is isomorphic to $NK_1(\mathbb{Z}G)$. Similarly, we have

$$(x, y) - (-1)^n \overline{(x,y)} = (x - (-1)^n \overline{y}, -(-1)^n \overline{x - (-1)^n \overline{y}}).$$

Therefore we have that

$$\{(x, y) - (-1)^n \overline{(x,y)} \mid (x, y) \in NK_1(\mathbb{Z}G)^2\} = \{(x, -(-1)^n \overline{x}) \mid x \in NK_1(\mathbb{Z}G)\}$$

which is isomorphic to $NK_1(\mathbb{Z}G)$. □

We immediately get the following corollary; see also [Ran86, p.358].

Corollary 5.11. *Let G be a group. The decomposition of Theorem 5.9 induces an isomorphism*

$$\widehat{H}^{n+1}(C_2; \mathrm{Wh}(C_\infty \times G)) \cong \widehat{H}^{n+1}(C_2; \mathrm{Wh}(G)) \oplus \widehat{H}^{n+1}(C_2; \widetilde{K}_0(\mathbb{Z}G)).$$

5.3. **The groups $\mathcal{J}_n(C_\infty \times C_m)$, $\mathcal{I}_n(C_\infty \times C_m)$, and $\widehat{H}^{n+1}(C_2; \mathrm{Wh}(C_\infty \times C_m))$.** In this section we prove, using results from Part 3, our main results about the groups $\mathcal{J}_n(C_\infty \times C_m)$, $\mathcal{I}_n(C_\infty \times C_m)$, and $\widehat{H}^{n+1}(C_2; \mathrm{Wh}(C_\infty \times C_m))$ for n even.

First note that by Proposition 5.10 and Theorem 5.7, for any group G and any n, if $NK_1(\mathbb{Z}G) \neq 0$, then $\mathcal{J}_n(C_\infty \times G)$ and $\mathcal{I}_n(C_\infty \times G)$ are not finitely generated. If G is finite and n is even, then we also have the following.

Lemma 5.12. *Suppose that n is even and G is a finite group. Then the following are equivalent:*

(i) $|\mathcal{J}_n(C_\infty \times G)| < \infty$
(ii) $|\mathcal{I}_n(C_\infty \times G)| < \infty$
(iii) $NK_1(\mathbb{Z}G) = 0$.

Proof. Let $SK_1(\mathbb{Z}G) = \ker(K_1(\mathbb{Z}G) \to K_1(\mathbb{Q}G))$ where the map is induced by inclusion $\mathbb{Z}G \subseteq \mathbb{Q}G$. It was shown by Wall [Wal74, Proposition 6.5] (see also [Oli88, Theorem 7.4]) that $SK_1(\mathbb{Z}G)$ is isomorphic to the torsion subgroup of $\mathrm{Wh}(G)$. Let $\mathrm{Wh}'(G) = \mathrm{Wh}(G)/SK_1(\mathbb{Z}G)$ be the free part. The standard involution on $\mathrm{Wh}(G)$ induces an involution on $\mathrm{Wh}'(G)$, and Wall showed this induced involution is the identity (see [Oli88, Corollary 7.5]).

It follows that if $x = -\overline{x} \in \mathrm{Wh}(G)$, then x maps to $0 \in \mathrm{Wh}'(G)$, so $x \in SK_1(\mathbb{Z}G)$. Hence $\mathcal{I}_n(G) \leq \mathcal{J}_n(G) \leq SK_1(\mathbb{Z}G)$. If G is finite, then $SK_1(\mathbb{Z}G)$ is finite [Wal74] and $\widetilde{K}_0(\mathbb{Z}G)$ is finite [Swa60] (see also Proposition 8.4 (ii)). Therefore the lemma follows from Proposition 5.10 and Theorem 5.7. □

Proposition 5.13. *Suppose that n is even. Then for every $m \geq 2$ we have $\mathcal{J}_n(C_m) = 0$, and hence $\mathcal{I}_n(C_m) \cong \widehat{H}^{n+1}(C_2; \mathrm{Wh}(C_m)) = 0$.*

Proof. By Proposition 2.10, $\mathrm{Wh}(C_m)$ is a finitely generated free abelian group. By [Bas74, Proposition 4.2], the involution acts trivially on $\mathrm{Wh}(G)$ for all G finite abelian. Hence $\mathcal{J}_n(C_m) = \{x \in \mathrm{Wh}(C_m) \mid x = -x\} = 0$. □

The next three theorems are the main results of Section 5. They will be established as a consequence of Theorems 11.1, 11.2 and 11.3, which will be proven in Section 11 (which is in Part 3). These results were stated in the introduction as Theorems 1.6, 1.7 and 1.8 respectively.

Theorem 5.14. *Suppose that n is even and $m \geq 2$ is an integer.*
- (i) $|\mathcal{J}_n(C_\infty \times C_m)| = 1$ *if and only if* $m \in \{2, 3, 5, 6, 7, 10, 11, 13, 14, 17, 19\}$;
- (ii) $|\mathcal{J}_n(C_\infty \times C_m)| = \infty$ *if and only if m is not square-free;*
- (iii) $|\mathcal{J}_n(C_\infty \times C_m)| \to \infty$ *super-exponentially in m.*

Theorem 5.15. *Suppose that n is even and $m \geq 2$ is an integer.*
- (i) $|\mathcal{I}_n(C_\infty \times C_m)| = 1$ *if and only if* $m \in \{2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 29\}$;
- (ii) $|\mathcal{I}_n(C_\infty \times C_m)| = \infty$ *if and only if m is not square-free;*
- (iii) $|\mathcal{I}_n(C_\infty \times C_m)| \to \infty$ *super-exponentially in m.*

Remark 5.16. To write Theorem 5.14 (i) another way, the $m \geq 2$ for which $|\mathcal{J}_n(C_\infty \times C_m)| = 1$ are as follows:
$$m = \begin{cases} p, & \text{where } p \leq 19 \text{ is prime} \\ 2p, & \text{where } p \leq 7 \text{ is an odd prime.} \end{cases}$$
Furthermore, Theorem 5.15 (i) states that $|\mathcal{I}_n(C_\infty \times C_m)| = 1$ if and only if $|\mathcal{J}_n(C_\infty \times C_m)| = 1$ or $m \in \{15, 29\}$.

Proof of Theorems 5.14 and 5.15. First note that Theorem 5.14 (ii) and Theorem 5.15 (ii) immediately follow from Theorem 5.8 and Lemma 5.12. For the proofs of (i) and (iii), we can therefore restrict to the case where m is square-free, when $NK_1(\mathbb{Z}C_m) = 0$. By Propositions 5.10 and 5.13 we then have:
$$\mathcal{J}_n(C_\infty \times C_m) \cong \{x \in \widetilde{K}_0(\mathbb{Z}C_m) \mid \overline{x} = -x\}$$
$$\mathcal{I}_n(C_\infty \times C_m) \cong \{x - \overline{x} \mid x \in \widetilde{K}_0(\mathbb{Z}C_m)\}.$$
The results follow from Theorems 11.1 and 11.2 (see Section 11). □

Theorem 5.17. *Suppose that n is even and $m \geq 2$ is an integer.*
- (i) $|\widehat{H}^{n+1}(C_2; \mathrm{Wh}(C_\infty \times C_m))| = 1$ *for infinitely many m.*
- (ii) $|\widehat{H}^{n+1}(C_2; \mathrm{Wh}(C_\infty \times C_m))| < \infty$ *for every m.*
- (iii) $\sup_{l \leq m} |\widehat{H}^{n+1}(C_2; \mathrm{Wh}(C_\infty \times C_l))| \to \infty$ *exponentially in m.*

Note that (i) and (iii) imply that
$$\liminf_{m\to\infty} |\widehat{H}^{n+1}(C_2; \mathrm{Wh}(C_\infty \times C_m))| = 1, \limsup_{m\to\infty} |\widehat{H}^{n+1}(C_2; \mathrm{Wh}(C_\infty \times C_m))| = \infty.$$

Proof. It follows from Corollary 5.11 and Proposition 5.13 that
$$\widehat{H}^{n+1}(C_2; \mathrm{Wh}(C_\infty \times C_m)) \cong \widehat{H}^1(C_2; \widetilde{K}_0(\mathbb{Z}C_m)) \cong \frac{\{x \in \widetilde{K}_0(\mathbb{Z}C_m) \mid x = -\overline{x}\}}{\{x - \overline{x} \mid x \in \widetilde{K}_0(\mathbb{Z}C_m)\}}.$$

Hence parts (i) and (iii) follows from Theorem 11.3, which will be proved in Section 11. Part (ii) follows from the fact that $\widetilde{K}_0(\mathbb{Z}C_m)$ is finite [Swa60] (see also Proposition 8.4 (ii)). □

6. Homotopy automorphisms of $S^1 \times L$

Fix $k, m \geq 2$ and $q_1, \dots, q_k$ such that $\gcd(m, q_j) = 1$. Let $L = L_{2k-1}(m; q_1, \dots, q_k)$ be a $(2k-1)$-dimensional lens space with $\pi_1(L) \cong C_m$. We will be interested in the product $S^1 \times L$ and its homotopy automorphisms. Fix basepoints of S^1 and L and let $i_1 \colon S^1 \to S^1 \times L$ and $i_2 \colon L \to S^1 \times L$ denote the standard embeddings, we will identify S^1 and L with their images in $S^1 \times L$. Let $G = C_\infty \times C_m \cong \pi_1(S^1 \times L)$.

6.1. **The automorphism induced on the fundamental group.** Our first goal is to determine which automorphisms of G can be realised by homotopy automorphisms of $S^1 \times L$, i.e. to describe the image of the map $\pi_1 \colon \operatorname{hAut}(S^1 \times L) \to \operatorname{Aut}(G)$. Note that the automorphisms of $G = C_\infty \times C_m$ can be expressed as matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$, where $a \in \{\pm 1\}$, $b \in C_m$ and $c \in C_m^\times \cong \operatorname{Aut}(C_m)$.

Lemma 6.1 ([Coh73, Statement 29.5])**.** *Let $c \in C_m^\times$. There is a homotopy automorphism $f \colon L \to L$ such that $\pi_1(f) = c$ if and only if $c^k \equiv \pm 1 \bmod m$.*

Lemma 6.2.

(a) *There is a diffeomorphism $f \in \operatorname{Diff}(S^1 \times L)$ such that $\pi_1(f) = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.*
(b) *There is a diffeomorphism $f \in \operatorname{Diff}(S^1 \times L)$ such that $\pi_1(f) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.*

Proof. (a) Take $f = r \times \operatorname{Id}_L$ for an orientation-reversing diffeomorphism $r \colon S^1 \to S^1$.

(b) Let $S^{2k-1} = \{(z_1, \dots, z_k) \mid \sum |z_j|^2 = 1\} \subseteq \mathbb{C}^k$. Recall that $L = L_{2k-1}(m; q_1, \dots, q_k)$ is the quotient of S^{2k-1} by the C_m-action generated by $(z_1, \dots, z_k) \mapsto (\zeta^{q_1} z_1, \dots, \zeta^{q_k} z_k)$, where $\zeta = e^{2\pi i/m}$. Define $\phi \colon S^1 \to \operatorname{Diff}(L)$ by $e^{2\pi i t} \mapsto ([z_1, \dots, z_k] \mapsto [\zeta^{q_1 t} z_1, \dots, \zeta^{q_k t} z_k])$. Then we can define a suitable diffeomorphism f by $f(x, y) = (x, \phi(x)(y))$. □

Lemma 6.3. *For every homotopy automorphism $f \colon S^1 \times L \to S^1 \times L$ there is a homotopy automorphism $\bar{f}_2 \colon L \to L$ such that $f \circ i_2 \simeq i_2 \circ \bar{f}_2 \colon L \to S^1 \times L$.*

Proof. Since $\pi_1(L) \cong C_m$ is the torsion subgroup of $\pi_1(S^1 \times L) \cong C_\infty \times C_m$, the restriction of $\pi_1(f)$ is an isomorphism $\pi_1(L) \to \pi_1(L)$, and hence $\operatorname{Im} \pi_1(f \circ i_2) = \operatorname{Im} \pi_1(\rho)$ where the covering $\rho \colon \mathbb{R} \times L \to S^1 \times L$ is the product of the universal covering of S^1 and Id_L. By the lifting criterion [Hat02, Proposition 1.33], $f \circ i_2$ has a lift $\widetilde{f}_2 \colon L \to \mathbb{R} \times L$ such that $f \circ i_2 = \rho \circ \widetilde{f}_2$. Let $\pi \colon \mathbb{R} \times L \to L$ denote the projection. Since $\mathbb{R}$ is contractible, the universal covering map $\mathbb{R} \to S^1$ is null-homotopic. It follows that $\rho \simeq i_2 \circ \pi$. Thus

$$f \circ i_2 = \rho \circ \widetilde{f}_2 \simeq i_2 \circ \pi \circ \widetilde{f}_2 = i_2 \circ \bar{f}_2,$$

where by definition $\bar{f}_2 = \pi \circ \widetilde{f}_2 \colon L \to L$.

This implies that $\pi_1(\bar{f}_2)$ is the restriction of $\pi_1(f)$, so it is an isomorphism. Since $\pi_j(i_2)$ is an isomorphism for $j > 1$ and f is a homotopy automorphism (hence $\pi_j(f)$ is an isomorphism), $\pi_j(\bar{f}_2)$ is also an isomorphism. Therefore $\bar{f}_2$ is a homotopy automorphism too. □

Definition 6.4. Let $A_{2k}(m) \leq \operatorname{Aut}(G)$ denote the subgroup of matrices $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$ such that $c^k \equiv \pm 1 \bmod m$.

Theorem 6.5. *We have* $\operatorname{Im}(\pi_1 \colon \operatorname{hAut}(S^1 \times L) \to \operatorname{Aut}(G)) = A_{2k}(m)$.

Proof. Suppose that $f \colon S^1 \times L \to S^1 \times L$ is a homotopy automorphism with $\pi_1(f) = \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$. By Lemma 6.3 there is a homotopy automorphism $\bar{f}_2 \colon L \to L$ such that $f \circ i_2 \simeq i_2 \circ \bar{f}_2$. This implies that $\pi_1(\bar{f}_2) = c$, so by Lemma 6.1 $c^k \equiv \pm 1 \bmod m$. Therefore $\pi_1(f) \in A_{2k}(m)$.

Let $c \in C_m^\times$ be such that $c^k \equiv \pm 1 \bmod m$. By Lemma 6.1 there is a homotopy automorphism $f \colon L \to L$ such that $\pi_1(f) = c$. Then $\operatorname{Id}_{S^1} \times f$ is a homotopy automorphism of $S^1 \times L$ such that $\pi_1(\operatorname{Id}_{S^1} \times f) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & c \end{smallmatrix}\right)$. Matrices of this form, together with the matrices realised in Lemma 6.2, generate $A_{2k}(m)$. Therefore $A_{2k}(m) \subseteq \operatorname{Im}(\pi_1 \colon \operatorname{hAut}(S^1 \times L) \to \operatorname{Aut}(G))$. □

6.2. **The Whitehead torsion.** Next we prove that every homotopy automorphism of $S^1 \times L$ is simple.

Theorem 6.6. *If $f \in \mathrm{hAut}(S^1 \times L)$, then $\tau(f) = 0$.*

Remark 6.7. When m is odd, this also follows from results of Khan and Hsiang-Jahren. By [Kha17, Corollary 6.2], $\mathrm{hAut}(S^1 \times L)$ is generated by maps of the form $f \times \mathrm{Id}_L$, $\mathrm{Id}_{S^1} \times g$ and r_h, where $f \in \mathrm{hAut}(S^1)$, $g \in \mathrm{hAut}(L)$, $h\colon S^1 \to \mathrm{Map}(L)$ and $r_h(x,y) = (x, h(x)(y))$. By Proposition 2.32, generators of the first two types have vanishing Whitehead torsion. If m is odd, then it follows from [HJ83, Proposition 3.1] that $\tau(r_h) = 0$ (see also [Kha17, Corollary 6.4]).

The proof consists of two parts, which we will prove in the following two lemmas. For a CW complex X and an integer $n \geq 0$, let $\mathrm{sk}_n(X)$ denote the n-skeleton of X. We will endow $S^1 \times L$ with a standard CW decomposition, that we recall during the next proof.

Lemma 6.8. *If $g\colon S^1 \times L \to S^1 \times L$ is a homotopy automorphism such that $g|_{L \cup \mathrm{sk}_{2k-2}(S^1 \times L)} = \mathrm{Id}$, then $\tau(g) = 0$.*

Proof. We will denote the generators of the components of G by $\alpha \in C_\infty$ and $\beta \in C_m$. Let $\Sigma := \sum_{j=0}^{m-1} \beta^j \in \mathbb{Z}C_m \subseteq \mathbb{Z}G$ be the norm element of $\mathbb{Z}C_m$.

The lens space $L = L_{2k-1}(m; q_1, \dots, q_k)$ has a CW decomposition with one cell E_j in each dimension $0 \leq j \leq 2k-1$ (see e.g. [Coh73, §28]). The corresponding cellular chain complex (C_j^L, d_j^L) with $\mathbb{Z}C_m$ coefficients has $C_j^L \cong \mathbb{Z}C_m$ for every j and

$$d_j^L = \begin{cases} \Sigma & \text{if } j \text{ is even} \\ \beta^{r_h} - 1 & \text{if } j = 2h-1 \end{cases}$$

where $r_h q_h \equiv 1 \bmod m$.

The product $M = S^1 \times L$ has a CW decomposition containing the cells of L, plus a $(j+1)$-cell E'_{j+1}, the product of E_j and the 1-cell of S^1, for every $0 \leq j \leq 2k-1$. Its chain complex (C_j^M, d_j^M) with $\mathbb{Z}G$ coefficients is the tensor product of (C_j^L, d_j^L) and the chain complex of S^1 with $\mathbb{Z}C_\infty$ coefficients, $\mathbb{Z}C_\infty \xrightarrow{\alpha-1} \mathbb{Z}C_\infty$. It has $C_0^M \cong C_{2k}^M \cong \mathbb{Z}G$ and $C_j^M \cong \mathbb{Z}G^2$ for every $1 \leq j \leq 2k-1$ (with $\mathbb{Z}G \oplus \{0\}$ corresponding to E_j and $\{0\} \oplus \mathbb{Z}G$ corresponding to E'_j). The differentials are

$$d_1^M = \begin{pmatrix} \beta^{r_1} - 1 \\ \alpha - 1 \end{pmatrix}, \quad d_{2h-1}^M = \begin{pmatrix} \beta^{r_h} - 1 & 0 \\ \alpha - 1 & \Sigma \end{pmatrix} \quad \text{for } h > 1,$$

$$d_{2k}^M = (1-\alpha, \beta^{r_k} - 1), \quad d_{2h}^M = \begin{pmatrix} \Sigma & 0 \\ 1-\alpha & \beta^{r_h} - 1 \end{pmatrix} \quad \text{for } h < k.$$

We write elements of $\mathbb{Z}G^j$ as row vectors, so a homomorphism $\mathbb{Z}G^j \to \mathbb{Z}G^h$ is multiplication on the right by a $j \times h$ matrix.

We can assume that g is cellular, let $g_j\colon C_j^M \to C_j^M$ denote the induced chain map. Since $g|_{L \cup \mathrm{sk}_{2k-2}(L \times S^1)} = \mathrm{Id}$, we have $g_j = \mathrm{Id}$ for every $j \leq 2k-2$ and $g_{2k-1}|_{\mathbb{Z}G \oplus \{0\}} = \mathrm{Id}$.

We investigate the map $g_{2k-1}\colon C_{2k-1}^M \to C_{2k-1}^M$. It is represented by a 2×2 matrix with entries in $\mathbb{Z}G$, with respect to the basis corresponding to E_{2k-1} and E'_{2k-1}. Since $g_{2k-1}|_{\mathbb{Z}G \oplus \{0\}} = \mathrm{Id}$, the matrix is $\left(\begin{smallmatrix} 1 & 0 \\ x & y \end{smallmatrix}\right)$ for some $x, y \in \mathbb{Z}G$. To compute x and y, we consider the difference between g and Id on E'_{2k-1}. Since $g|_{\partial E'_{2k-1}} = \mathrm{Id}$, the cell E'_{2k-1} and $g(E'_{2k-1})$ together form a map $S^{2k-1} \to S^1 \times L$. This map represents an element of $\pi_{2k-1}(S^1 \times L) \cong \pi_{2k-1}(\mathbb{R} \times S^{2k-1}) \cong \mathbb{Z}$, so its homotopy class is an integer multiple of that of the inclusion $S^{2k-1} \to \mathbb{R} \times S^{2k-1}$, composed with the projection $\mathbb{R} \times S^{2k-1} \to S^1 \times L$. The image of the fundamental class of S^{2k-1} under this composition is represented by the chain $\Sigma \in \mathbb{Z}G \oplus \{0\} < \mathbb{Z}G^2 \cong C_{2k-1}^M$. So E'_{2k-1} and $g(E'_{2k-1})$ differ by some integer multiple of $\Sigma \in \mathbb{Z}G \oplus \{0\}$. Therefore (by changing g on E'_{2k-1} and E'_{2k}, to replace it with another cellular map which is homotopic to it) we can assume that $y = 1$ and $x = a\Sigma$ for some $a \in \mathbb{Z}$. That is, $g_{2k-1} = \left(\begin{smallmatrix} 1 & 0 \\ a\Sigma & 1 \end{smallmatrix}\right)$.

Since (g_j) is a chain map, we have $d_{2k}^M \circ g_{2k} = g_{2k-1} \circ d_{2k}^M$. When converted to matrices this yields

$$g_{2k}(1-\alpha, \beta^{r_k} - 1) = (1-\alpha, \beta^{r_k} - 1)\left(\begin{smallmatrix} 1 & 0 \\ a\Sigma & 1 \end{smallmatrix}\right) = (1-\alpha, \beta^{r_k} - 1),$$

since $\beta^{r_k}\Sigma = \Sigma$. Looking at the first coordinate, we have $g_{2k}(1-\alpha) = (1-\alpha)$. Multiplication by $(1-\alpha)$ is an injective map $\mathbb{Z}G \to \mathbb{Z}G$, so we deduce that $g_{2k} = 1$.

Therefore g_j is an isomorphism for every j. By Lemma 2.18 we can compute the torsion via the formula $\tau(g) = \sum_{j=0}^{2k}(-1)^j[g_j]$. Since g_{2k-1} is an elementary matrix and $g_j = \mathrm{Id}$ for $j \neq 2k-1$, this implies that $\tau(g) = 0$. □

Lemma 6.9. *For every homotopy automorphism $f\colon S^1 \times L \to S^1 \times L$ there is another homotopy automorphism g such that $\tau(f) = \tau(g)$ and $g\big|_{L\cup \mathrm{sk}_{2k-2}(L\times S^1)} = \mathrm{Id}$.*

Proof. By Lemma 6.3 there is a homotopy automorphism $\bar{f}_2\colon L \to L$ such that $f \circ i_2 \simeq i_2 \circ \bar{f}_2$. Let $\overline{h}\colon L \to L$ be the homotopy inverse of $\bar{f}_2$, and let $h := \mathrm{Id}_{S^1} \times \overline{h}\colon S^1 \times L \to S^1 \times L$. Note that $h \circ i_2 = i_2 \circ \overline{h}\colon L \to S^1 \times L$. By Corollary 2.33 $\tau(h) = 0$. Let $f' = f \circ h$. Then by Proposition 2.31, we have that $\tau(f') = \tau(f)$. Moreover, combining the above observations we have

$$f' \circ i_2 = f \circ h \circ i_2 = f \circ i_2 \circ \overline{h} \simeq i_2 \circ \bar{f}_2 \circ \overline{h} \simeq i_2.$$

This means that after changing f' by a homotopy we can assume that $f'\big|_L = \mathrm{Id}$.

The embedding $i_1\colon S^1 \to S^1 \times L$ represents $(1,0) \in \pi_1(S^1 \times L) \cong C_\infty \times C_m$. Since $\pi_1(f')$ is an automorphism of $C_\infty \times C_m$, $\pi_1(f')^{-1}(1,0) = (a,b)$ for some $a \in \{\pm 1\}$, $b \in C_m$. It follows from Lemma 6.2 that there is a diffeomorphism $h \in \mathrm{Diff}(S^1 \times L)$ such that $\pi_1(h)(1,0) = (a,b)$. Let $f'' = f' \circ h$. Then $\tau(f'') = \tau(f')$ by Proposition 2.31 and Chapman's Theorem 2.6. Moreover $\pi_1(f'')(1,0) = (1,0)$, i.e. $f'' \circ i_1 \simeq i_1$. The diffeomorphisms constructed in Lemma 6.2 keep L pointwise fixed, so $f''\big|_L = \mathrm{Id}$, and after applying a homotopy we can also assume $f''\big|_{S^1} = \mathrm{Id}$.

Finally, we recursively construct maps f''_j for every $1 \leq j \leq 2k-2$ such that $f'' \simeq f''_j$ rel $S^1 \vee L$ and $f''_j\big|_{L\cup \mathrm{sk}_j(L\times S^1)} = \mathrm{Id}$. We start with $f''_1 = f''$. If f''_{j-1} is already defined, we consider the single j-cell E'_j in $S^1 \times L - L$. Since $f''_{j-1}\big|_{\partial E'_j} = \mathrm{Id}$, the cell E'_j and $f''_{j-1}(E'_j)$ together form a map $S^j \to S^1 \times L$. This map is nullhomotopic (because $\pi_j(S^1 \times L) \cong \pi_j(\mathbb{R} \times S^{2k-1}) = 0$ for $2 \leq j \leq 2k-2$), so $f''_{j-1}\big|_{E'_j}$ is homotopic to $\mathrm{Id}_{E'_j}$ rel $\partial E'_j$. We can extend this homotopy to a homotopy between f''_{j-1} and an f''_j with $f''_j\big|_{L\cup \mathrm{sk}_j(L\times S^1)} = \mathrm{Id}$. Therefore f''_j can be defined for every $1 \leq j \leq 2k-2$.

To complete the proof of Lemma 6.9, we take $g = f''_{2k-2}\colon S^1 \times L \to S^1 \times L$. □

Proof of Theorem 6.6. Consider a homotopy equivalence $f\colon S^1 \times L \to S^1 \times L$. By Lemma 6.9, we can replace f by a homotopy equivalence $g\colon S^1 \times L \to S^1 \times L$ such that $\tau(f) = \tau(g)$ and $g\big|_{L\cup \mathrm{sk}_{2k-2}(L\times S^1)} = \mathrm{Id}$. Then g satisfies the hypotheses of Lemma 6.8, and so $\tau(g) = 0$. Therefore $\tau(f) = 0$, and f is a simple homotopy equivalence, as desired. □

7. The proof of Theorems B, C, and D

We can now combine the results of the previous sections to prove Theorems B, C, and D. Let $n = 2k \geq 4$ be an even integer and fix a category CAT satisfying Hypothesis 1.1.

Recall the definition of the sets $\mathcal{M}^h_s(M)$, $\mathcal{M}^{\mathrm{hCob}}_s(M)$ and $\mathcal{M}^h_{s,\mathrm{hCob}}(M)$ from Definition 4.10. Also recall the definition of $A_{2k}(m)$ (Definition 6.4): $A_{2k}(m) \leq \mathrm{Aut}(C_\infty \times C_m)$ denotes the subgroup of matrices $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$ such that $c^k \equiv \pm 1 \bmod m$. Note that $A_{2k}(m)$ acts on $\mathrm{Wh}(C_\infty \times C_m)$ and on the subgroups $\mathcal{I}_n(C_\infty \times C_m)$ and $\mathcal{J}_n(C_\infty \times C_m)$.

Proposition 7.1. *Let $m \geq 2$ and $q_1, \dots, q_k$ such that $\gcd(m, q_j) = 1$. Let $L = L_{2k-1}(m; q_1, \dots, q_k)$ be an $(n-1)$-dimensional lens space with $\pi_1(L) \cong C_m$. Let $G := C_\infty \times C_m \cong \pi_1(S^1 \times L)$. Then there are bijections of pointed sets $\mathcal{M}^h_s(S^1 \times L) \cong \mathcal{J}_n(G)/A_{2k}(m)$, $\mathcal{M}^{\mathrm{hCob}}_s(S^1 \times L) \cong \mathcal{I}_n(G)/A_{2k}(m)$ and $\mathcal{M}^h_{s,\mathrm{hCob}}(S^1 \times L) \cong \widehat{H}^{n+1}(C_2; \mathrm{Wh}(G))/A_{2k}(m)$.*

Proof. Firstly note that, since $C_\infty \times C_m$ is polycyclic, it is a good group [FQ90, KOPR21], so Hypothesis 1.1 is satisfied. Moreover, since the map ψ is surjective for $G = C_\infty \times C_m$, $w \equiv 1$ and n even by Proposition 3.12, the vertical maps in the diagram of Theorem 4.11 (c) are bijections if $M = S^1 \times L$. Therefore $\mathcal{M}^h_s(S^1 \times L) \cong \mathcal{J}_n(G)/\operatorname{hAut}(S^1 \times L)$, $\mathcal{M}^{\mathrm{hCob}}_s(S^1 \times L) \cong q(\mathcal{I}_n(G))$, and $\mathcal{M}^h_{s,\mathrm{hCob}}(S^1 \times L) \cong \widehat{H}^{n+1}(C_2; \mathrm{Wh}(G))/\operatorname{hAut}(S^1 \times L)$.

By Theorem 6.6, every homotopy automorphism of $S^1 \times L$ is simple, therefore the action of $\operatorname{hAut}(S^1 \times L)$ on $\mathrm{Wh}(G)$ factors through the action of $\mathrm{Aut}(G)$ (see Definition 4.3 and Remark 4.18).

By Theorem 6.5, the image of $\pi_1\colon \operatorname{hAut}(S^1\times L)\to \operatorname{Aut}(G)$ is $A_{2k}(m)$. In particular, the orbits of the action of $\operatorname{hAut}(S^1\times L)$ and $A_{2k}(m)$ on $\mathcal{J}_n(G)$, $\mathcal{I}_n(G)$, and $\widehat{H}^{n+1}(C_2;\operatorname{Wh}(G))$ coincide. This implies that $\mathcal{J}_n(G)/\operatorname{hAut}(S^1\times L)=\mathcal{J}_n(G)/A_{2k}(m)$, $q(\mathcal{I}_n(G))=\mathcal{I}_n(G)/A_{2k}(m)$. Hence, as required, we have $\widehat{H}^{n+1}(C_2;\operatorname{Wh}(G))/\operatorname{hAut}(S^1\times L)=\widehat{H}^{n+1}(C_2;\operatorname{Wh}(G))/A_{2k}(m)$. □

In particular Proposition 7.1 implies that $|\mathcal{M}^h_s(S^1\times L)|$, $|\mathcal{M}^{\mathrm{hCob}}_s(S^1\times L)|$ and $|\mathcal{M}^h_{s,\mathrm{hCob}}(S^1\times L)|$ are independent of the choice of the q_j and of CAT, and only depend on n and m. This proves part (a) of Theorems B, C, and D.

From now on, we will write M^n_m for $S^1\times L$, where L is any $(n-1)$-dimensional lens space with $\pi_1(L)\cong C_m$.

Lemma 7.2. *Let $n=2k\geq 4$ be an even integer, $m\geq 2$ and $G=C_\infty\times C_m$. Then the following hold.*

(a) $|\mathcal{M}^h_s(M^n_m)|=1$ *if and only if* $\mathcal{J}_n(G)=0$.
(b) $|\mathcal{M}^h_s(M^n_m)|=\infty$ *if and only if* $|\mathcal{J}_n(G)|=\infty$.
(c) *If $\mathcal{M}^h_s(M^n_m)$ is finite, then* $\frac{|\mathcal{J}_n(G)|}{2m^2}<|\mathcal{M}^h_s(M^n_m)|\leq|\mathcal{J}_n(G)|$.
(d) $|\mathcal{M}^{\mathrm{hCob}}_s(M^n_m)|=1$ *if and only if* $\mathcal{I}_n(G)=0$.
(e) $|\mathcal{M}^{\mathrm{hCob}}_s(M^n_m)|=\infty$ *if and only if* $|\mathcal{I}_n(G)|=\infty$.
(f) *If $\mathcal{M}^{\mathrm{hCob}}_s(M^n_m)$ is finite, then* $\frac{|\mathcal{I}_n(G)|}{2m^2}<|\mathcal{M}^{\mathrm{hCob}}_s(M^n_m)|\leq|\mathcal{I}_n(G)|$.
(g) $|\mathcal{M}^h_{s,\mathrm{hCob}}(M^n_m)|=1$ *if and only if* $\widehat{H}^{n+1}(C_2;\operatorname{Wh}(G))=0$.
(h) $|\mathcal{M}^h_{s,\mathrm{hCob}}(M^n_m)|=\infty$ *if and only if* $|\widehat{H}^{n+1}(C_2;\operatorname{Wh}(G))|=\infty$.
(i) *If $\mathcal{M}^h_{s,\mathrm{hCob}}(M^n_m)$ is finite, then* $\frac{|\widehat{H}^{n+1}(C_2;\operatorname{Wh}(G))|}{2m^2}<|\mathcal{M}^h_{s,\mathrm{hCob}}(M^n_m)|\leq|\widehat{H}^{n+1}(C_2;\operatorname{Wh}(G))|$.

Proof. (a) By Proposition 7.1, $|\mathcal{M}^h_s(M^n_m)|=|\mathcal{J}_n(G)/A_{2k}(m)|$. The group $A_{2k}(m)\leq\operatorname{Aut}(G)$ acts on $\operatorname{Wh}(G)$, and hence on $\mathcal{J}_n(G)$, by automorphisms. So $0\in\mathcal{J}_n(G)$ is a fixed point of this action, and $|\mathcal{J}_n(G)/A_{2k}(m)|=1$ if and only if $\mathcal{J}_n(G)=0$.

(b) $\operatorname{Aut}(G)$, and hence its subgroup $A_{2k}(m)$, is finite. Hence $|\mathcal{J}_n(G)/A_{2k}(m)|=\infty$ if and only if $|\mathcal{J}_n(G)|=\infty$.

(c) It is easy to see that $\frac{|\mathcal{J}_n(G)|}{|A_{2k}(m)|}\leq|\mathcal{J}_n(G)/A_{2k}(m)|\leq|\mathcal{J}_n(G)|$. We have $|A_{2k}(m)|\leq|\operatorname{Aut}(G)|\leq 2m(m-1)<2m^2$ since elements of $\operatorname{Aut}(G)$ can be represented as matrices of the form $\left(\begin{smallmatrix} a & b\\ 0 & c\end{smallmatrix}\right)$, where $a\in\{\pm1\}$, $b\in C_m$, $c\in C_m^\times$.

The proofs of parts (d), (e), and (f) (resp. parts (g), (h) and (i)) are entirely analogous to those of parts (a), (b), and (c) respectively, and so will be omitted for brevity. □

Theorem 7.3. *Let $n=2k\geq 4$ be an even integer and $m\geq 2$. Then the following hold.*

(a) $|\mathcal{M}^h_s(M^n_m)|=1$ *if and only if* $m\in\{2,3,5,6,7,10,11,13,14,17,19\}$.
(b) $|\mathcal{M}^h_s(M^n_m)|=\infty$ *if and only if m is not square-free.*
(c) $|\mathcal{M}^h_s(M^n_m)|\to\infty$ *as* $m\to\infty$ *(uniformly in n).*
(d) $|\mathcal{M}^{\mathrm{hCob}}_s(M^n_m)|=1$ *if and only if* $m\in\{2,3,5,6,7,10,11,13,14,15,17,19,29\}$.
(e) $|\mathcal{M}^{\mathrm{hCob}}_s(M^n_m)|=\infty$ *if and only if m is not square-free.*
(f) $|\mathcal{M}^{\mathrm{hCob}}_s(M^n_m)|\to\infty$ *as* $m\to\infty$ *(uniformly in n).*
(g) *There are infinitely many m such that* $|\mathcal{M}^h_{s,\mathrm{hCob}}(M^n_m)|=1$ *for every n.*
(h) $|\mathcal{M}^h_{s,\mathrm{hCob}}(M^n_m)|$ *is finite for every n and m.*
(i) $\limsup\limits_{m\to\infty}\Big(\inf\limits_{n}|\mathcal{M}^h_{s,\mathrm{hCob}}(M^n_m)|\Big)=\infty$.

Proof. (a), (b), (d), (e), (g), and (h) follow immediately from Lemma 7.2 and Theorems 5.14, 5.15, and 5.17.

(c) For every m, it follows from Lemma 7.2 (c) that $|\mathcal{M}^h_s(M^n_m)|>\frac{|\mathcal{J}_n(C_\infty\times C_m)|}{2m^2}$ for every n. By Theorem 5.14 (iii) $\frac{|\mathcal{J}_n(C_\infty\times C_m)|}{2m^2}\to\infty$ as $m\to\infty$. Item (f), can be proved similarly using Theorem 5.15 (iii).

(i) For every m, it follows from Lemma 7.2 (i) that $\inf_n|\mathcal{M}^h_{s,\mathrm{hCob}}(M^n_m)|>\frac{|\widehat{H}^{n+1}(C_2;\operatorname{Wh}(C_\infty\times C_m))|}{2m^2}$. By Theorem 5.17 (iii), we see that $\frac{1}{2m^2}\sup_{l\leq m}|\widehat{H}^{n+1}(C_2;\operatorname{Wh}(C_\infty\times C_l))|\to\infty$ as $m\to\infty$. This implies that $\sup_{l\leq m}\frac{|\widehat{H}^{n+1}(C_2;\operatorname{Wh}(C_\infty\times C_l))|}{2l^2}\to\infty$ as $m\to\infty$. So the map $m\mapsto\frac{|\widehat{H}^{n+1}(C_2;\operatorname{Wh}(C_\infty\times C_m))|}{2m^2}$ is unbounded. Therefore $\limsup_{m\to\infty}\frac{|\widehat{H}^{n+1}(C_2;\operatorname{Wh}(C_\infty\times C_m))|}{2m^2}=\infty$. □

This completes the proof of the remaining parts of Theorems B, C, and D, subject to the proofs of Theorems 11.1 to 11.3 which will be postponed until Section 11. We can now deduce:

Theorem 7.4. *There exists an infinite collection of orientable,* CAT *n-manifolds that are all homotopy equivalent to one another but are pairwise not simple homotopy equivalent.*

Proof. By Theorem 7.3 (b), if m is not square-free (e.g. if $m = 4$), then $\mathcal{M}^h_s(M^n_m)$ is infinite. Note that $M^n_m = S^1 \times L$ is orientable, so the same is true for every n-manifold homotopy equivalent to it. Therefore we can get a suitable infinite collection by choosing a representative from each element of $\mathcal{M}^h_s(M^n_m)$. □

This completes the proof of Theorem A. Note that this does not depend on the results postponed to Section 11. In particular, if m is not square-free, then $|NK_1(\mathbb{Z}C_m)| = \infty$ (Theorem 1.5) which implies $|\mathcal{J}_n(C_\infty \times C_m)| = \infty$ (Lemma 5.12) and so $|\mathcal{M}^h_s(M^n_m)| = \infty$ (Lemma 7.2 (b)).

Part 3. The involution on $\widetilde{K}_0(\mathbb{Z}C_m)$

The aim of this part will be to prove Theorems 11.1, 11.2, and 11.3, which are key ingredients in the proofs of Theorems 5.14, 5.15, and 5.17 respectively. In Sections 8 and 9, we will recall the necessary background on class groups, Tate cohomology, and $\mathbb{Z}C_2$-modules. The main technical heart of this part will be Section 10 where we will investigate the involution on $\widetilde{K}_0(\mathbb{Z}C_m)$ and prove general results which allow it to be computed. In Section 11, we make use of the results in Section 10 to prove Theorems 11.1, 11.2, and 11.3. Throughout, we assume that all modules are left modules.

8. Locally free class groups

We will now recall the theory of locally free class groups for orders in semisimple $\mathbb{Q}$-algebras. Good references for this material are [Swa80, Section 1-3] and [CR87, Section 49A & 50E].

8.1. Definitions and properties.

Recall that for a ring A, a nonzero A-module is *simple* if it contains no simple A-submodules other than itself and 0, and is *semisimple* if isomorphic as an A-module to a direct sum of its simple A-submodules. We say that a ring A is *simple* (resp. *semisimple*) if A, viewed as an A-module, is simple (resp. semisimple). For a field K, a *K-algebra* is a ring A for which K is a subring of the centre $Z(A)$. A K-algebra is *finite-dimensional* if it is finite-dimensional as a K-vector space.

Let A be a finite-dimensional semisimple $\mathbb{Q}$-algebra. An *order* in A is a subring $\Lambda \subseteq A$ that is finitely generated as an abelian group and which has $\mathbb{Q} \cdot \Lambda = A$. For example, let G be a finite group. Then $A = \mathbb{Q}G$ is a finite dimensional $\mathbb{Q}$-algebra which is semisimple by Maschke's theorem in representation theory, and $\Lambda = \mathbb{Z}G$ is an order in A.

From now on, fix an order Λ in a finite-dimensional semisimple $\mathbb{Q}$-algebra A. For a prime p, let $\Lambda_p = \mathbb{Z}_p \otimes_{\mathbb{Z}} \Lambda$ and let $A_p = \mathbb{Q}_p \otimes_{\mathbb{Q}} A$ denote the p-adic completions of Λ and A.

Definition 8.1. A Λ-module M is *locally free* if $M_p = \mathbb{Z}_p \otimes_{\mathbb{Z}} M$ is a free Λ_p-module for all primes p.

The following is [Swa80, Lemma 2.1]. Note that the converse need not hold, i.e. there exist orders Λ and projective Λ-modules that are not locally free [Swa80, p. 156].

Proposition 8.2. *If M is a locally free Λ-module, then M is projective.*

We say that two locally free Λ-modules M and N are *stably isomorphic*, written $M \cong_{\mathrm{st}} N$, if there exists $r, s \geq 0$ such that $M \oplus \Lambda^r \cong N \oplus \Lambda^s$ are isomorphic as Λ-modules.

Definition 8.3. Define the *locally free class group* $C(\Lambda)$ to be the set of equivalence classes of locally free Λ-modules up to stable isomorphism. This is an abelian group under direct sum (since the direct sum of locally free modules is locally free).

It follows that $C(\Lambda) \leq \widetilde{K}_0(\Lambda)$ is a subgroup, where $\widetilde{K}_0$ is the 0th reduced algebraic K-group as defined in Section 5.1. It is a consequence of the Jordan-Zassenhaus theorem that $C(\Lambda)$ is finite [CR87, Remark 49.11 (ii)].

We will now specialise to the case where $\Lambda = \mathbb{Z}G$ for G a finite group. In contrast to the situation for general orders, we have the following [Swa80, p. 156]. By Proposition 8.2, this implies that a $\mathbb{Z}G$-module is projective if and only if it is locally free.

Proposition 8.4. *Let G be a finite group.*

(i) *If M is a projective $\mathbb{Z}G$-module, then M is locally free.*
(ii) *There is an isomorphism of abelian groups $\widetilde{K}_0(\mathbb{Z}G) \cong C(\mathbb{Z}G)$. In particular, $\widetilde{K}_0(\mathbb{Z}G)$ is finite.*

Finally we note the following which relates locally free class groups to ideal class groups (see [Rei75, Section 35]).

Proposition 8.5. *Let $K/\mathbb{Q}$ be a finite field extension. Then $C(\mathcal{O}_K)$ coincides with the ideal class group of $\mathcal{O}_K$.*

8.2. **Kernel groups.** Let A be a finite-dimensional semisimple $\mathbb{Q}$-algebra A. An order in A is said to be *maximal* if it is not properly contained in another other order in A. Since every finite field extension of $\mathbb{Q}$ is separable, A is a separable algebra and so every order in A is contained in a maximal order [Swa70, Proposition 5.1].

Let Λ be an order in A and let Γ be a maximal order in A containing Λ. The inclusion map $i\colon \Lambda \hookrightarrow \Gamma$ induces a map $i_*\colon C(\Lambda) \to C(\Gamma)$ given by extension of scalars $[M] \mapsto [\Gamma \otimes_\Lambda M]$ which is necessarily surjective by [CR87, Theorem 49.25].

Definition 8.6. Define the *kernel group* $D(\Lambda)$ to be the kernel of the map $i_*\colon C(\Lambda) \to C(\Gamma)$. This is often also referred to as the defect group.

The group $C(\Gamma)$ does not depend on the choice of maximal order in A; in fact, if Γ_1, Γ_2 are maximal orders in A, then $C(\Gamma_1) \cong C(\Gamma_2)$ [CR87, Theorem 49.32]. Furthermore, the kernel group $D(\Lambda)$ does not depend on the choice of maximal order. This can be seen from the fact that it can be defined without reference to a maximal order: if M is a locally free Λ-module, then $[M] \in D(\Lambda)$ if and only if there exists a finitely generated Λ-module X such that $M \oplus X \cong \Lambda^n \oplus X$ for some n [CR87, Proposition 49.34].

In particular, we have a well-defined exact sequence of abelian groups:

$$0 \to D(\Lambda) \to C(\Lambda) \to C(\Gamma) \to 0$$

where Γ can be taken to be any maximal order in A containing Λ.

8.3. **The idèlic approach to locally free class groups.** Let Λ be an order in a finite-dimensional semisimple $\mathbb{Q}$-algebra A.

Definition 8.7. Define the *idèle group*

$$J(A) = \{(\alpha_p) \in \prod_p A_p^\times \mid \alpha_p \in \Lambda_p^\times \text{ for all but finitely many } p\} \subseteq \prod_p A_p^\times.$$

As a subgroup of $A_p^\times$, this is independent of the choice of order Λ [CR87, p. 218]. Every class in $C(\Lambda)$ is represented by a locally free Λ-module $M \subseteq A$ [CR87, p. 218]. For each p, there exists $\alpha_p \in A_p$ such that $M_p = \Lambda_p \alpha_p \subseteq A_p$. For all but finitely many p, $M_p \cong \Lambda_p$ and so $\alpha_p \in \Lambda_p^\times$. In particular, $\alpha = (\alpha_p) \in J(A)$. Conversely, given an idèle $\alpha \in J(A)$, we have that $\Lambda\alpha = A \cap \bigcap_p \Lambda_p \alpha_p \subseteq A$ is a locally free Λ-ideal. Let $\alpha, \beta \in J(A)$. Then $\Lambda\alpha \cong \Lambda\beta$ as Λ-modules if and only if $\beta \in U(\Lambda) \cdot \alpha \cdot A^\times$ where $A^\times \subseteq J(A)$ by sending $a \in A^\times$ to $\alpha_p = 1 \otimes_{\mathbb{Q}} a$ for all p, and $U(\Lambda) = \{(\alpha_p) \in \prod_p A_p^\times \mid \alpha_p \in \Lambda_p^\times \text{ for all } p\} \subseteq J(A)$ [CR87, 49.6]. Furthermore, we have that $\Lambda\alpha \oplus \Lambda\beta \cong \Lambda \oplus \Lambda\alpha\beta$ [CR87, 49.8]. This leads to the following.

Proposition 8.8. *There is a surjective group homomorphism*

$$[\Lambda\,\cdot\,]\colon J(A) \twoheadrightarrow C(\Lambda), \quad \alpha \mapsto [\Lambda\alpha].$$

Remark 8.9. Whilst we will not make use of it in this article, we note that this leads to the formula

$$C(\Lambda) \cong \frac{J(A)}{J_0(A) \cdot A^\times \cdot U(\Lambda)}$$

for the locally free class group, where $J_0(A) = \{x \in J(A) \mid \mathrm{nr}(x) = 1\}$ and $\mathrm{nr}\colon J(A) \to J(Z(A))$ is induced by the reduced norm. This is due to Fröhlich [Frö75] (see also [CR87, Theorem 49.22]).

8.4. **Involutions on locally free class groups.** Let Λ be an order in a finite-dimensional semisimple $\mathbb{Q}$-algebra A. Suppose further that A is a ring equipped with an involution $\bar{\cdot}\colon A \to A$ which restricts to Λ, i.e. the map $\bar{\cdot}$ is an involution on A as an abelian group which satisfies $\overline{xy} = \overline{y}\cdot\overline{x}$ for all $x, y \in A$ and $\overline{x} \in \Lambda$ for all $x \in \Lambda$. For example, if G is a finite group and $w\colon G \to \{\pm 1\}$ is an orientation character, then $A = \mathbb{Q}G$ has an involution given by $\sum_{i=1}^{k} n_i g_i \mapsto \sum_{i=1}^{k} w(g_i) n_i g_i^{-1}$ for $n_i \in \mathbb{Z}$ and $g_i \in G$ which restricts to $\Lambda = \mathbb{Z}G$ (see Section 2.2). Given a (left) Λ-module M, we now have the notion of a dual (left) Λ-module M^* (see Definition 2.23).

We now note the following properties of the dual of locally free modules. This follows from the fact that, by Proposition 8.2, locally free modules are projective and hence reflexive.

Lemma 8.10. *Let M be a locally free Λ-module. Then:*

(i) *M^* is locally free;*
(ii) *The evaluation map* $\mathrm{ev} : M \to M^{**}$, $m \mapsto (f \mapsto f(m))$ *is an isomorphism of Λ-modules.*

We can use this to define an involution on the locally free class group.

Definition 8.11. There is an involution of abelian groups:

$$*\colon C(\Lambda) \to C(\Lambda), \quad [M] \mapsto -[M^*].$$

That is, $*$ is a group homomorphism such that $*^2 = \mathrm{Id}_{C(\Lambda)}$.

In the case $\Lambda = \mathbb{Z}G$, we have that $\widetilde{K}_0(\mathbb{Z}G) \cong C(\mathbb{Z}G)$ and the involution above coincides with the standard involution on $\widetilde{K}_0$ as defined in Section 5.1.

We will now explore some properties of this involution. In what follows we refer to [CR87, pp. 275-6]. This deals only with the case $\Lambda = \mathbb{Z}G$, though the arguments there apply to the more general setting described above.

It is an immediate consequence of the alternative description of the kernel group given in Section 8.2 that, if $[M] \in D(\Lambda)$, then $[M^*] \in D(\Lambda)$ [CR87, p. 275]. In particular, the involution $*$ restricts to $D(\Lambda)$. This implies that, if $i\colon \Lambda \hookrightarrow \Gamma$ for Γ a maximal order in A, then $*$ induces an involution on $C(\Gamma)$ via the map $i_*\colon C(\Lambda) \twoheadrightarrow C(\Gamma)$, and this coincides with the involution on $C(\Gamma)$ coming from the fact that Γ is an order in A.

Recall that an involution on an abelian group is the same structure as a $\mathbb{Z}C_2$-module, where the C_2-action is given by the involution. In particular, we have shown the following.

Proposition 8.12. *There is a short exact sequence of $\mathbb{Z}C_2$-modules*

$$0 \to D(\Lambda) \to C(\Lambda) \to C(\Gamma) \to 0$$

where $D(\Lambda)$, $C(\Lambda)$, and $C(\Gamma)$ are $\mathbb{Z}C_2$-modules under the involutions described above.

We will conclude this section by noting that the idèlic approach to class groups gives a different way to define an involution on $C(\Lambda)$. The involution $\bar{\cdot}\colon A \to A$ induces involutions on A_p for each p and so on $J(A)$. It can be shown that the involution fixes the subgroups $A^\times$, $U(\Lambda)$ and $J_0(A)$ and so induces an involution on $C(\Lambda)$ given by $[\Lambda\alpha] \mapsto [\Lambda\overline{\alpha}]$ (see [CR87, p. 274]).

The following is proven in [CR87, p. 274].

Proposition 8.13. *The involution on $C(\Lambda)$ induced by the involution on $J(A)$ is the standard involution $*\colon C(\Lambda) \to C(\Lambda)$, $[M] \mapsto -[M^*]$.*

This gives an alternate way to understand the standard involution on $\widetilde{K}_0(\mathbb{Z}G)$. We make further use of this description the following section.

9. Tate cohomology and $\mathbb{Z}C_2$-modules

In this section, we will recall some basic facts about $\mathbb{Z}C_2$-modules which will be used throughout the proofs of Theorems 11.1, 11.2, and 11.3. We will also explain how the methods of Tate cohomology can be applied to $\mathbb{Z}C_2$-modules in preparation for the proof of Theorem 11.3.

9.1. **Tate cohomology.** The following can be found in [Bro94, VI.4].

Definition 9.1. Given a finite group G and a $\mathbb{Z}G$-module A, the *Tate cohomology groups* $\widehat{H}^n(G;A)$ for $n \in \mathbb{Z}$ are defined as follows. Let $A^G = \{x \in A \mid g \cdot x = x \text{ for all } g \in G\}$ be the invariants, let $A_G = A/\langle g \cdot x - x \mid g \in G, x \in A\rangle$ be the coinvariants and let $N \colon A_G \to A^G$ be the norm map $x \mapsto \sum_{g \in G} g \cdot x$ which is a well-defined homomorphism of abelian groups. Then define:

$$\widehat{H}^n(G;A) = \begin{cases} H^n(G;A), & \text{if } n \geq 1 \\ \operatorname{coker}(N \colon A_G \to A^G), & \text{if } n = 0 \\ \ker(N \colon A_G \to A^G), & \text{if } n = -1 \\ H_{-n-1}(G;A), & \text{if } n \leq -2, \end{cases}$$

where H^n, H_{-n-1} denote the usual group cohomology and homology groups.

We now recall the following basic properties. The first can be found in [Bro94, VI.5.1], the second follows from the first since functoriality means that α_* is split whenever α is, the third is [CE56, XII.2.5], and the fourth is [CE56, XII.2.7].

Proposition 9.2. *Let G be a finite group.*

(i) *Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be a short exact sequence of $\mathbb{Z}G$-modules. Then there is a long exact sequence of Tate cohomology groups:*

$$\cdots \to \widehat{H}^{n-1}(G;C) \xrightarrow{\partial} \widehat{H}^n(G;A) \xrightarrow{\alpha_*} \widehat{H}^n(G;B) \xrightarrow{\beta_*} \widehat{H}^n(G;C) \xrightarrow{\partial} \widehat{H}^{n+1}(G;A) \to \cdots$$

(ii) *Let A, B be $\mathbb{Z}G$-modules. Then $\widehat{H}^n(G;A \oplus B) \cong \widehat{H}^n(G;A) \oplus \widehat{H}^n(G;B)$ for all $n \in \mathbb{Z}$.*

(iii) *Let A be a $\mathbb{Z}G$-module. Then $|G| \cdot \widehat{H}^n(G;A) = 0$, i.e. $|G| \cdot x = 0$ for all $x \in \widehat{H}^n(G;A)$.*

(iv) *Let A be a finite $\mathbb{Z}G$-module and suppose $(|G|, |A|) = 1$. Then $\widehat{H}^n(G;A) = 0$ for all $n \in \mathbb{Z}$.*

9.2. **Tate cohomology and the structure of $\mathbb{Z}C_2$-modules.** Let A be a $\mathbb{Z}C_2$-module or, equivalently, an abelian group with an involution $\bar{\cdot} \colon A \to A$. In order to prove Theorems 11.1 to 11.3, we would like to find techniques to determine the following groups associated to A:

$$\{x \in A \mid x = (-1)^n \overline{x}\}, \quad \{x + (-1)^n \overline{x} \mid x \in A\}, \quad \frac{\{x \in A \mid x = (-1)^n \overline{x}\}}{\{x + (-1)^n \overline{x} \mid x \in A\}}.$$

We are especially interested in the case n odd (i.e. $n = 1$) but we will consider both cases. Note that, for the groups on the left, the notation $A^- = \{x \in A \mid x = -\overline{x}\}$ and $A^+ = \{x \in A \mid x = \overline{x}\}$ is often used since they are the (-1) and $(+1)$-eigenspaces of the involution action. If $2 \in A$ is invertible, then $A \cong A^+ \oplus A^-$ but this need not hold in general.

The following lemma will suffice for the study of the first two classes of groups. Part (i) follows from the fact that $A \mapsto A^{C_2}$ is a left-exact functor where A is given the altered involution $x \mapsto (-1)^n \overline{x}$, and part (ii) is immediate.

Lemma 9.3. *Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of $\mathbb{Z}C_2$-modules and let $n \in \mathbb{Z}$.*

(i) *Then α, β induce an exact sequence of abelian groups:*

$$0 \to \{x \in A \mid \overline{x} = (-1)^n x\} \xrightarrow{\alpha} \{x \in B \mid \overline{x} = (-1)^n x\} \xrightarrow{\beta} \{x \in C \mid \overline{x} = (-1)^n x\}.$$

(ii) *There are injective and surjective maps induced by α, β:*

$$\{x + (-1)^n \overline{x} \mid x \in A\} \overset{\alpha}{\hookrightarrow} \{x + (-1)^n \overline{x} \mid x \in B\} \overset{\beta}{\twoheadrightarrow} \{x + (-1)^n \overline{x} \mid x \in C\}.$$

Remark 9.4. For each n, the right map in (i) need not be surjective. For example, take $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$ where $\mathbb{Z}/2$ has the trivial involution and $\mathbb{Z}$ has the involution $x \mapsto (-1)^n x$. For each n, the sequence (ii) need not be exact in the middle. For example, take $0 \to \mathbb{Z}/2 \to \mathbb{Z} \oplus \mathbb{Z}/2 \to \mathbb{Z} \to 0$ where $\mathbb{Z}/2$ has the trivial involution, $\mathbb{Z} \oplus \mathbb{Z}/2$ has involution $(x,y) \mapsto ((-1)^{n+1}x, x+y)$ and $\mathbb{Z}$ has the involution $x \mapsto (-1)^n x$.

The third class of groups can be studied using Tate cohomology due to the following. This is standard (see, for example, [CE56, p. 251]) but we will include a proof here for convenience.

Proposition 9.5. *Let A be a $\mathbb{Z}C_2$-module and let $n \in \mathbb{Z}$. Then*
$$\widehat{H}^n(C_2; A) \cong \frac{\{x \in A \mid x = (-1)^n \overline{x}\}}{\{x + (-1)^n \overline{x} \mid x \in A\}}.$$

Proof. Since C_2 has 2-periodic Tate cohomology group (see, for example, [Bro94, VI.9.2]), it suffices to compute $\widehat{H}^0(C_2; A)$ and $\widehat{H}^{-1}(C_2; A)$. We have $A^{C_2} = \{x \in A \mid x = \overline{x}\}$ and $A_{C_2} = A/\{x - \overline{x} \mid x \in A\}$ and so the norm map is given by
$$N\colon \frac{A}{\{x - \overline{x} \mid x \in A\}} \to \{x \in A \mid x = \overline{x}\}, \quad x \mapsto x + \overline{x}.$$
The result follows since $\widehat{H}^{-1}(C_2; A) = \ker(N)$, $\widehat{H}^0(C_2; A) = \operatorname{coker}(N)$. □

Remark 9.6. This shows that, for a group G and $w\colon G \to \{\pm 1\}$ an orientation character, we have
$$\mathcal{J}_n(G, w)/\mathcal{I}_n(G, w) \cong \widehat{H}^{n+1}(C_2; \mathrm{Wh}(G, w))$$
which was noted already in Section 2.2.

We now recall a series of special facts about the Tate cohomology of C_2. The first is a consequence of Proposition 9.2 (i) and the fact that finite cyclic groups have 2-periodic cohomology (see [Ser79, p. 133]). Note that this also applies for C_2 replaced by an arbitrary finite cyclic group.

Proposition 9.7. *Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be a short exact sequence of $\mathbb{Z}C_2$-modules. Then there is a 6-periodic exact sequence of abelian groups:*
$$\begin{array}{ccccc} \widehat{H}^1(C_2; A) & \xrightarrow{\alpha_*} & \widehat{H}^1(C_2; B) & \xrightarrow{\beta_*} & \widehat{H}^1(C_2; C) \\ \partial \uparrow & & & & \downarrow \partial \\ \widehat{H}^0(C_2; C) & \xleftarrow[\beta_*]{} & \widehat{H}^0(C_2; B) & \xleftarrow[\alpha_*]{} & \widehat{H}^0(C_2; A). \end{array}$$

We will next consider results which apply only in the case where A is finite $\mathbb{Z}G$-module, i.e. a $\mathbb{Z}G$-module whose underlying abelian group is finite. This is a consequence of the theory of Herbrand quotients (see [Ser79, Chapter VIII, Section 4]) though we will include a direct proof here for convenience.

Proposition 9.8. *Let A be a finite $\mathbb{Z}C_2$-module. Then there exists $d \geq 0$ such that*
$$\widehat{H}^n(C_2; A) \cong (\mathbb{Z}/2)^d$$
for all $n \in \mathbb{Z}$. In particular, $|\widehat{H}^n(G; A)|$ is independent of $n \in \mathbb{Z}$.

Proof. First note that, since A is finite, so is $\widehat{H}^n(C_2; A)$ by Proposition 9.5. Next note that there are exact sequences of finite abelian groups (see, for example, [Ser79, p. 134]):
$$0 \to \{x \in A \mid \overline{x} = x\} \to A \to A \to \frac{A}{\{x - \overline{x} \mid x \in A\}} \to 0$$
$$0 \to \widehat{H}^1(C_2; A) \to \frac{A}{\{x - \overline{x} \mid x \in A\}} \to \{x \in A \mid \overline{x} = x\} \to \widehat{H}^0(C_2; A) \to 0$$
where the middle maps are $x \mapsto x + \overline{x}$.

The first implies that $|\{x \in A \mid \overline{x} = x\}| = |A/\{x - \overline{x} \mid x \in A\}|$ and, by combining this with the second, we get that $|\widehat{H}^1(C_2; A)| = |\widehat{H}^0(C_2; A)|$. Hence, since $\widehat{H}^n(C_2; A)$ is 2-periodic (see Proposition 9.5), we get that $|\widehat{H}^n(C_2; A)|$ is independent of $n \in \mathbb{Z}$.

Next recall that, by Proposition 9.2 (iii), we have $2 \cdot \widehat{H}^n(C_2; A) = 0$, i.e. every element in $\widehat{H}^n(C_2; A)$ has order at most two. This implies that $\widehat{H}^n(C_2; A) \cong (\mathbb{Z}/2)^{d_n}$ for some $d_n \geq 0$. Since $|\widehat{H}^n(C_2; A)|$ is independent of n, this implies that d_n is also independent of n. □

For a finite abelian group A and a prime p, let $A_{(p)} = \{x \in A \mid p^n \cdot x = 0 \text{ for some } n \geq 1\}$ denote the p-primary component of A. This coincides with the Sylow p-subgroup of A and we use this notation since $A_{(p)} \cong \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} A$ where $\mathbb{Z}_{(p)}$ denotes the localisation of $\mathbb{Z}$ at $S = \mathbb{Z} \setminus (p)$.

Proposition 9.9. *Let A be a finite $\mathbb{Z}C_2$-module and let $n \in \mathbb{Z}$. Then*
$$\widehat{H}^n(C_2; A) \cong \widehat{H}^n(C_2; A_{(2)}).$$

In order to prove this, we will use the following standard fact about localisations.

Lemma 9.10. *Let A be a finite $\mathbb{Z}C_2$-module. For p prime, $A_{(p)} \leq A$ is a $\mathbb{Z}C_2$-submodule and $A \cong \bigoplus_{p||A|} A_{(p)}$ as $\mathbb{Z}C_2$-modules.*

Proof of Proposition 9.9. By Lemma 9.10 and Proposition 9.2 (ii), we get that

$$\widehat{H}^n(C_2; A) \cong \bigoplus_{p||A|} \widehat{H}^1(C_2; A_{(p)}).$$

If $p \neq 2$, then $|A_{(p)}|$ is odd and so $\widehat{H}^1(C_2; A_{(p)}) = 0$ by Proposition 9.2 (iv). This gives that $\widehat{H}^n(C_2; A) \cong \widehat{H}^n(C_2; A_{(2)})$, as required. □

10. Computing the involution on $\widetilde{K}_0(\mathbb{Z}C_m)$

The aim of this section will be to investigate the involution on $\widetilde{K}_0(\mathbb{Z}C_m)$ in preparation for the proofs of Theorems 11.1, 11.2, and 11.3 in Section 11.

We will view $\widetilde{K}_0(\mathbb{Z}C_m)$ as a $\mathbb{Z}C_2$-module with the C_2-action coming from the standard involution on $\widetilde{K}_0$ as defined in Section 5.1. We saw in Section 8.1 that $\widetilde{K}_0(\mathbb{Z}C_m) \cong C(\mathbb{Z}C_m)$ is an isomorphism of $\mathbb{Z}C_2$-modules where $C(\mathbb{Z}C_m)$ denotes the locally free class group, and in particular it is finite. Our basic approach for computing $C(\mathbb{Z}C_m)$ will be to use the following short exact sequence of $\mathbb{Z}C_2$-modules established in Section 8.4

$$0 \to D(\mathbb{Z}C_m) \to C(\mathbb{Z}C_m) \to C(\Gamma_m) \to 0$$

where Γ_m is a maximal order in $\mathbb{Q}C_m$ containing $\mathbb{Z}C_m$ and $D(\mathbb{Z}C_m)$ has the induced involution.

The plan for this section is as follows. In Section 10.1, we relate the involution on $C(\Gamma_m)$ to the involution on $C(\mathbb{Z}[\zeta_d])$ induced by conjugation. In Section 10.2, we study the conjugation action on $C(\mathbb{Z}[\zeta_d])$ and its relation to the class numbers $h_d = |C(\mathbb{Z}[\zeta_d])|$. In Section 10.3, we survey results on divisibility of class numbers as well as make minor extensions (see Proposition 10.7 (ii)). In Section 10.4, we investigate the involution on $D(\mathbb{Z}C_m)$.

10.1. The induced involution on the maximal order.

Let $m \geq 2$ and let $C_m = \langle x \mid x^m \rangle$. Then there is an isomorphism of $\mathbb{Q}$-algebras:

$$\mathbb{Q}C_m \cong \prod_{d|m} \mathbb{Q}(\zeta_d), \quad x \mapsto \prod_{d|m} (\zeta_d)$$

where $\zeta_d = e^{2\pi i/d}$ denotes a dth primitive root of unity. Since $\mathcal{O}_{\mathbb{Q}(\zeta_d)} = \mathbb{Z}[\zeta_d]$, it follows that $\Gamma_m = \prod_{d|m} \mathbb{Z}[\zeta_d]$ is a maximal order in $\mathbb{Q}C_m$. The image of $\mathbb{Z}C_m$ under the isomorphism above is contained in Γ_m and so Γ_m contains $\mathbb{Z}C_m$. In fact, Γ_m is the unique maximal order in $\mathbb{Q}C_m$ containing $\mathbb{Z}C_m$ [CR87, p. 243]. This implies that there is an isomorphism of abelian groups

$$C(\Gamma_m) \cong \bigoplus_{d|m} C(\mathbb{Z}[\zeta_d]),$$

where, as noted in Proposition 8.5, the locally free class group $C(\mathbb{Z}[\zeta_d])$ coincides with the ideal class group of $\mathbb{Z}[\zeta_d]$.

For an integer $d \geq 1$, let $\bar{\cdot}\colon C(\mathbb{Z}[\zeta_d]) \to C(\mathbb{Z}[\zeta_d])$ denote the map induced by conjugation, i.e. if $\sigma\colon \mathbb{Z}[\zeta_d] \to \mathbb{Z}[\zeta_d]$ is the ring homomorphism generated by $\zeta_d \mapsto \zeta_d^{-1}$, then $\bar{\cdot} = \sigma_*$ is the induced map on $C(\mathbb{Z}[\zeta_d])$. We will now compute the induced involution on $\bigoplus_{d|m} C(\mathbb{Z}[\zeta_d])$. This was shown in the case where m is prime in [Rei68] (see also [CR87, p. 275]).

Proposition 10.1. *Let $i\colon \mathbb{Z}C_m \hookrightarrow \Gamma_m$, $x \mapsto \prod_{d|m}(\zeta_d)$ and let $i_*\colon C(\mathbb{Z}C_m) \to \bigoplus_{d|m} C(\mathbb{Z}[\zeta_d])$ denote the induced map. Then, under i_*, the standard involution on $C(\mathbb{Z}C_m)$ induces the conjugation map on each $C(\mathbb{Z}[\zeta_d])$.*

This means that, if $x \in C(\mathbb{Z}C_m)$ and $i_*(x) = \prod_{d|m} x_d$, then $i_*(c(x)) = \prod_{d|m} c_d(x_d)$ where $c\colon C(\mathbb{Z}C_m) \to C(\mathbb{Z}C_m)$ is the standard involution and $c_d\colon C(\mathbb{Z}[\zeta_d]) \to C(\mathbb{Z}[\zeta_d])$ is induced by conjugation.

Proof. For each $d \mid m$, let $i^{(d)} \colon \mathbb{Z}C_m \to \mathbb{Z}[\zeta_d]$, $x \mapsto \zeta_d$. It suffices to prove that, under the map $i_*^{(d)} \colon C(\mathbb{Z}C_m) \to C(\mathbb{Z}[\zeta_d])$, the involution on $C(\mathbb{Z}C_m)$ induces conjugation on $C(\mathbb{Z}[\zeta_d])$. By Proposition 8.13, the standard involution on $C(\mathbb{Z}C_m)$ is induced by the involution on the idèle group $J(\mathbb{Q}C_m)$. Note that $i_*^{(d)}$ is induced by the map $J(i^{(d)}) \colon J(\mathbb{Q}C_m) \to J(\mathbb{Q}(\zeta_d))$. Under the map $i^{(d)}$, the involution on $\mathbb{Q}C_m$ induces conjugation on $\mathbb{Q}(\zeta_d)$. In particular, the involution on $C(\mathbb{Z}[\zeta_d])$ induced by $i_*^{(d)}$ coincides with the involution induced by conjugation on $J(\mathbb{Q}(\zeta_d))$. The result now follows since, if a locally free $\mathbb{Z}[\zeta_d]$-ideal $M = (x_1, \dots, x_n) \subseteq \mathbb{Q}(\zeta_d)$ is represented by $\alpha \in J(\mathbb{Q}(\zeta_d))$, then $\overline{M} = (\overline{x}_1, \dots, \overline{x}_n) \subseteq \mathbb{Q}(\zeta_d)$ is represented by $\overline{\alpha} \in J(\mathbb{Q}(\zeta_d))$. □

In summary, we have shown that there is a short exact sequence of $\mathbb{Z}C_2$-modules

$$0 \to D(\mathbb{Z}C_m) \to C(\mathbb{Z}C_m) \to \bigoplus_{d \mid m} C(\mathbb{Z}[\zeta_d]) \to 0,$$

where $C(\mathbb{Z}C_m)$ has the standard involution, $D(\mathbb{Z}C_m)$ has the induced involution, and each $C(\mathbb{Z}[\zeta_d])$ has the involution induced by conjugation.

10.2. **Ideal class groups of cyclotomic fields.** For every integer $m \geq 2$, let $\lambda_m = \zeta_m + \zeta_m^{-1}$. Let $i \colon \mathbb{Z}[\lambda_m] \hookrightarrow \mathbb{Z}[\zeta_m]$ denote inclusion and recall that the map $i_* \colon C(\mathbb{Z}[\lambda_m]) \to C(\mathbb{Z}[\zeta_m])$ is injective [Lan78, Theorem 4.2]. Furthermore, the norm map gives a surjection $N \colon C(\mathbb{Z}[\zeta_m]) \to C(\mathbb{Z}[\lambda_m])$ such that the composition

$$C(\mathbb{Z}[\zeta_m]) \xrightarrow{N} C(\mathbb{Z}[\lambda_m]) \xrightarrow{i_*} C(\mathbb{Z}[\zeta_m])$$

is the map $x \mapsto x + \overline{x}$ (see, for example, [Lan78, pp. 83-4]). By viewing $C(\mathbb{Z}[\zeta_d])$ and $C(\mathbb{Z}[\lambda_m])$ as $\mathbb{Z}C_2$-modules under the conjugation action, the maps i_* and N are $\mathbb{Z}C_2$-module homomorphisms. Note that the conjugation action induces the identify on $C(\mathbb{Z}[\lambda_m])$.

This has the following useful consequences. Recall that, for A a $\mathbb{Z}C_2$-module, we defined $A^- = \{x \in A \mid x = -\overline{x}\}$ and $A^+ = \{x \in A \mid x = \overline{x}\}$.

Lemma 10.2.
(i) *The map i_* induces an isomorphism $C(\mathbb{Z}[\lambda_m]) \cong \{x + \overline{x} \mid x \in C(\mathbb{Z}[\zeta_m])\}$.*
(ii) *There is a short exact sequence of $\mathbb{Z}C_2$-modules:*

$$0 \to C(\mathbb{Z}[\zeta_m])^- \to C(\mathbb{Z}[\zeta_m]) \xrightarrow{N} C(\mathbb{Z}[\lambda_m]) \to 0.$$

Remark 10.3. Since $\mathbb{Q}(\lambda_m)$ is the maximal real subfield of $\mathbb{Q}(\zeta_m)$, it is often written as $\mathbb{Q}(\zeta_m)^+$. However, whilst $C(\mathbb{Z}[\lambda_m]) \subseteq C(\mathbb{Z}[\zeta_m])^+$ is a subgroup, these groups are not equal in general. For example, if $m = 29$, then $C(\mathbb{Z}[\lambda_{29}]) = 0$ and $C(\mathbb{Z}[\zeta_{29}])^+ \cong (\mathbb{Z}/2)^3$.

Proof. (i) Since N is surjective, we have

$$\mathrm{Im}(i_*) = \mathrm{Im}(i_* \circ N) = \{x + \overline{x} \mid x \in C(\mathbb{Z}[\zeta_m])\}.$$

(ii) Since i_* is injective, we have

$$\ker(N) = \ker(i_* \circ N) = \{x \in C(\mathbb{Z}[\zeta_m]) \mid x + \overline{x} = 0\}.$$ □

In order to set up later applications, we will now use Lemma 10.2 to obtain information about each of the following groups:

$$\{x \in C(\mathbb{Z}[\zeta_m]) \mid x = -\overline{x}\},\ \{x - \overline{x} \mid x \in C(\mathbb{Z}[\zeta_m])\},\ \underbrace{\frac{\{x \in C(\mathbb{Z}[\zeta_m]) \mid x = -\overline{x}\}}{\{x - \overline{x} \mid x \in C(\mathbb{Z}[\zeta_m])\}}}_{\cong \widehat{H}^1(C_2; C(\mathbb{Z}[\zeta_m]))}.$$

Since $C(\mathbb{Z}[\zeta_m])$ is a finite abelian group (see, for example, Section 8.1), we can define the *class number* of the cyclotomic integers to be $h_m = |C(\mathbb{Z}[\zeta_m])|$. By Lemma 10.2 (ii), we have that $h_m = h_m^- h_m^+$ where $h_m^- = |C(\mathbb{Z}[\zeta_m])^-|$ and $h_m^+ = |C(\mathbb{Z}[\lambda_m])|$. We refer to h_m^- as the *minus part* of the class number and h_m^+ as the *plus part* of the class number respectively.

For an integer m, let $\mathrm{odd}(m)$ denote the odd part of m, i.e. $\mathrm{odd}(m)$ is the unique odd integer r such that $m = 2^k r$ for some k.

Proposition 10.4. *There are subgroups:*
(i) $C(\mathbb{Z}[\zeta_m])^- \leq \{x \in C(\mathbb{Z}[\zeta_m]) \mid x = -\overline{x}\}$;

(ii) $2 \cdot C(\mathbb{Z}[\zeta_m])^- \leq \{x - \overline{x} \mid x \in C(\mathbb{Z}[\zeta_m])\}$.

In particular, h_m^- *divides* $|\{x \in C(\mathbb{Z}[\zeta_m]) \mid x = -\overline{x}\}|$ *and* $\mathrm{odd}(h_m^-)$ *divides* $|\{x - \overline{x} \mid x \in C(\mathbb{Z}[\zeta_m])\}|$.

Proof. Parts (i) and (ii) each follows from Lemma 9.3, Lemma 10.2 (ii) and the fact that $\overline{x} = -x$ for all $x \in C(\mathbb{Z}[\zeta_m])^-$. For the last part note that, since $C(\mathbb{Z}[\zeta_m])^-$ is a finite abelian group, we have $C(\mathbb{Z}[\zeta_m])^- \cong A \oplus B$ where $|A|$ is even and $|B| = \mathrm{odd}(h_m^-)$ is odd. Since B is odd, $2 \cdot B = B$ and so

$$B \leq 2 \cdot A \oplus B = 2 \cdot C(\mathbb{Z}[\zeta_m])^- \leq \{x - \overline{x} \mid x \in C(\mathbb{Z}[\zeta_m])\}.$$

Then use that $\{x - \overline{x} \mid x \in C(\mathbb{Z}[\zeta_m])\}$ has a subgroup of size $\mathrm{odd}(h_m^-)$. □

10.3. **Divisibility of class numbers of cyclotomic fields.** The aim of this section will be to survey results on the divisibility of the class numbers h_m and h_m^-. The most basic divisibility results are that, for $n \mid m$, we have $h_n \mid h_m$ [Was97, p. 205] and $h_n^- \mid h_m^-$ [MM76, Lemma 5]. Motivated by Proposition 10.4, we will now pursue divisibility results of two distinct types. We will start by considering $\mathrm{odd}(h_m^-)$, i.e. the unique odd integer r such that $h_m^- = 2^k r$ for some k, and we then consider the parity of h_m^-.

Recall the following theorem of Masley-Montgomery [MM76] (see also [Was97, p. 205]). In anticipation of its application in the proofs of Theorems 11.1 and 11.2, we will separate out the case where m is square-free.

Proposition 10.5. *The integers* $m \geq 2$ *for which* $h_m^- = 1$ *are as follows.*

(i) *If* m *is square-free, then*

$$m = \begin{cases} p, & \text{where } p \in \{2,3,5,7,11,13,17,19\} \\ 2p, & \text{where } p \in \{3,5,7,11,13,17,19\} \\ pq \text{ or } 2pq, & \text{where } (p,q) \in \{(3,5),(3,7),(3,11),(5,7)\}. \end{cases}$$

(ii) *If* m *is not square-free, then*

$$m \in \{4,8,9,12,16,18,20,24,25,27,28,32,36,40,44,45,48,50,54,60,84,90\}.$$

Furthermore, $h_m^- = 1$ *if and only if* $h_m = 1$.

The proof of Proposition 10.5 is based on the fact that $h_m^- \to \infty$ as $m \to \infty$. We will now establish lower bounds on the growth rate of h_m^-, and hence on h_m since $h_m \geq h_m^-$. Let $\varphi(m)$ denote Euler's totient function.

Proposition 10.6. *There exists a constant* $C > 0$ *such that, for all* $m \geq 1$, *we have:*

$$h_m^- \geq e^{C\frac{m \log m}{\log\log m}}.$$

In particular, $h_m^- \to \infty$ *super-exponentially in* m.

Proof. It is shown in [Was97, Theorem 4.20] that $\log h_m^- / (\frac{1}{4}\varphi(m)\log m) \to 1$ as $m \to \infty$. This implies that $\log h_m^- \geq C_0 \varphi(m) \log m$ for some $C_0 > 0$. The result now follows from the fact that $\varphi(m) \geq m/2\log\log m$ for m sufficiently large [HW54, Theorem 328]. □

The following result gives the analogue of Proposition 10.5 for $\mathrm{odd}(h_m^-)$. Note that $h_m^- = 1$ implies $\mathrm{odd}(h_m^-) = 1$, so we need not consider these m since they are classified in Proposition 10.5. This was established by Horie [Hor89, Theorems 2 and 3] and builds on Friedman's theorem from Iwasawa theory [Fri81] and the Brauer-Siegel theorem for abelian fields [Uch71].

Proposition 10.7. *The complete list of* $m \geq 2$ *for which* $\mathrm{odd}(h_m^-) = 1$ *and* $h_m^- \neq 1$ *is as follows.*

(i) *If* m *is square-free, then* $m \in \{29, 39, 58, 65, 78, 130\}$.

(ii) *If* m *is not square-free, then* $m \in \{56, 68, 120\}$.

Furthermore, $\mathrm{odd}(h_m^-) = 1$ *if and only if* $\mathrm{odd}(h_m) = 1$.

In [Hor89, Theorem 1], Horie also showed that $\mathrm{odd}(h_m^-) \to \infty$ as $m \to \infty$ but gave no bound on the growth rate. In fact, we have the following result analogous to Proposition 10.6.

Proposition 10.8. *There exists a constant* $C > 0$ *such that, for all* $m \geq 1$, *we have:*

$$\mathrm{odd}(h_m^-) \geq e^{C\frac{m \log m}{\log\log m}}.$$

In particular, $\mathrm{odd}(h_m^-) \to \infty$ *super-exponentially in* m.

We will prove this by tracing through Horie's proof of [Hor89, Theorem 1]. Recall that an abelian field K is a finite Galois extension $K/\mathbb{Q}$ with $\mathrm{Gal}(K/\mathbb{Q})$ abelian and we can assume that $K \subseteq \mathbb{C}$. For an abelian field K with maximal real subfield K^+, let $h_K = |C(\mathcal{O}_K)|$, $h_K^+ = |C(\mathcal{O}_{K^+})|$ and $h_K^- = h_K/h_K^+$ (which is an integer). Let $\mathrm{disc}(K)$ denote the discriminant of a number field K.

Proof. We will use that, if L/K is an extension of abelian fields, then $\mathrm{odd}(h_K^-) \mid \mathrm{odd}(h_L^-)$ [Hor89, Lemma 1]. For each abelian field K, let K' denote the maximal subfield of K with degree a power of 2. By the fundamental theorem of Galois theory and the fact that a finite abelian group A has a subgroup of order d for all $d \mid |A|$, we get that $|K'/\mathbb{Q}|$ is the highest power of 2 dividing $|K/\mathbb{Q}|$. For an integer $n \geq 1$, let $\mathcal{A}_n = \{m \mid \mathrm{odd}(h_m^-) \leq n\}$ and $\mathcal{B}_n = \{|\mathbb{Q}(\zeta_m)'/\mathbb{Q}| \mid n \in \mathcal{A}_n\}$.

We will begin by finding a bound for $\sup(\mathcal{B}_n)$. Let $K = \mathbb{Q}(\zeta_m)'$ for some $m \in \mathcal{A}_n$. Then $\mathbb{Q}(\zeta_m)/K$ is an extension of abelian fields and so $\mathrm{odd}(h_K^-) \mid h_m^-$ and so $\mathrm{odd}(h_K^-) \leq n$. Since $|K/\mathbb{Q}|$ is a power of 2, h_K must be odd [Was97] and so $h_K^- = \mathrm{odd}(h_K^-) \leq n$. Furthermore, note that K is imaginary unless $K = \mathbb{Q}$ [Hor89, p. 468].

It follows from the proof of Theorem 1 and Proposition 1 in [Uch71] that $\frac{|K/\mathbb{Q}|}{\log|\mathrm{disc}(K)|}$ is uniformly bounded across imaginary abelian fields K, where $\mathrm{disc}(K)$ denotes the discriminant, and that $h_K^- \geq |\mathrm{disc}(K)|^6$ for all but finitely many imaginary abelian fields K. This implies that there exists a constant $C > 0$ such that $|K/\mathbb{Q}| \leq C\log(h_K^-)$ for all imaginary abelian fields K. Hence, if $K = \mathbb{Q}(\zeta_m)'$ for $m \in \mathcal{A}_n$, then $|K/\mathbb{Q}| \leq C\log n$. This implies that $\sup(\mathcal{B}_n) \leq C\log n$. It also follows that there are only finitely many fields of the form $\mathbb{Q}(\zeta_m)'$ for $m \in \mathcal{A}_n$.

We now aim to find a bound for $\sup(\mathcal{A}_n)$. First let S denote the set of primes which are ramified in some field $\mathbb{Q}(\zeta_m)'$ for $m \in \mathcal{A}_n$. This is finite since there are finitely many such fields, and coincides with the primes which are ramified in $\mathbb{Q}(\zeta_m)$ for some $m \in \mathcal{A}_n$ [Hor89, p. 468]. Let $S = \{p_1, \dots, p_s\}$ for distinct primes p_i. By [Hor89, p. 469] there exists a cyclotomic field $L = \mathbb{Q}(\zeta_\ell)$ such that $L \subseteq \mathbb{Q}(\zeta_m) \subseteq L_\infty$ where L_∞ is the basic $\mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_s}$-extension over L. Furthermore, we have $|\mathbb{Q}(\zeta_m)/L| = \prod_{i=1}^s p_i^{n(p_i)}$ for some $n(p_i) \geq 1$ and so, in the notation of [Fri81], we can write $\mathbb{Q}(\zeta_m) = L_N$ where $N = (n_1, \dots, n_s)$. Let $e_N^{(2)}$ denote the highest power of 2 dividing h_m. By [Fri81, Theorem B], we have that $e_N^{(2)} = A \cdot n(2) + B$ for all but finitely many N, where $A, B \geq 0$ are integers that do not depend on N. This implies that there exists a constant $C' > 0$ such that $e_N^{(2)} \leq C' \cdot n(2)$ for all N.

Let $K = \mathbb{Q}(\zeta_m)'$. Then $|\mathbb{Q}(\zeta_m)/\mathbb{Q}| = 2^r t$ for some $r \geq 0$ and t odd, where $2^r = |K/\mathbb{Q}| \leq C\log n$ by the bound on $\log(\mathcal{B}_n)$. Since $|\mathbb{Q}(\zeta_m)/L| \mid |\mathbb{Q}(\zeta_m)/\mathbb{Q}|$, we have that $2^{n(2)} \leq 2^r \leq C\log n$. Hence $h_m^-/\mathrm{odd}(h_m^-) \leq h_m/\mathrm{odd}(h_m) = 2^{e_N^{(2)}} \leq (C\log n)^{C'}$. Since $m \in \mathcal{A}_n$, we have $\mathrm{odd}(h_m^-) \leq n$. This gives that $h_m^- \leq a(\log n)^b n$ for some constants $a, b > 0$ and so, for any $\varepsilon > 0$, we have that $h_m^- \leq an^{1+\varepsilon}$. Combining this with Proposition 10.6 gives that $\log(an^{1+\varepsilon}) \geq C_0 \frac{m\log m}{\log\log m}$ for some $C_0 > 0$, which implies that $\log n \geq C\frac{m\log m}{\log\log m}$ for some $C > 0$.

Finally, fix $m \geq 2$. Then $m \in \mathcal{A}_n$ where $n = \mathrm{odd}(h_m^-)$, and so gives

$$\log(\mathrm{odd}(h_m^-)) = \log n \geq C\frac{m\log m}{\log\log m},$$

which is the required bound. □

We will now consider the parity of h_m and h_m^-. Whilst we will not explicitly make use of it, we will record the following basic observation which dates back to Kummer (see [Has52, Satz 45], [Yos98, Remark 1]).

Lemma 10.9. *Let $m \geq 2$. Then h_m is odd if and only if h_m^- is odd.*

We now state more detailed results in the case where m is a prime power.

Lemma 10.10. *Let p be a prime such that $p \leq 509$ and let $n \geq 1$. Then:*

(i) *h_p is odd if and only if*

$$p \notin \{29, 113, 163, 197, 239, 277, 311, 337, 349, 373, 397, 421, 463, 491\}.$$

(ii) *h_{p^n} is odd if and only if h_p is odd.*

Proof. (i) This was proven by Schoof [Sch98, Table 4.4].

(ii) The case $p = 2$ is proven in [Has52, Satz 36'], but the original result is attributed to an 1886 article of Weber [Web86]. See also [Yos98, p. 2590]. For $n \geq 1$ and $p \leq 509$ an odd prime, it was shown by Ichimura–Nakajima that h_{p^n}/h_p is odd [IN12, Theorem 1 (II)]. The result follows. $\square$

Remark 10.11. Prior to the results of Ichimura–Nakajima, it was shown by Washington that h_{p^n}/h_p is odd for $p = 3, 5$ [Was75]. Note that both results of Washington and Ichimura–Nakajima all depend on Iwasawa theory. As far as we are aware, this is an essential ingredient in all known proofs that there exists an odd prime p such that h_{p^n} is odd for all $n \geq 1$.

10.4. **Kernel groups of $\mathbb{Z}C_m$.** The aim of this section will be to determine the involution on $D(\mathbb{Z}C_m)$ which is induced by the involution on $C(\mathbb{Z}C_m)$ (see Proposition 8.12).

We will begin with the following classical result due to Rim [Rim59, Theorem 6.24] (see also [CR87, Theorem 50.2]). Recall that, if p is a prime and Γ_p is the maximal order in $\mathbb{Q}C_p$ containing $\mathbb{Z}C_p$, then $\Gamma_p \cong \mathbb{Z} \times \mathbb{Z}[\zeta_p]$ and so $C(\Gamma_p) \cong C(\mathbb{Z}[\zeta_p])$ since $\mathbb{Z}$ is a PID.

Lemma 10.12. *Let p be a prime. Then the map $\mathbb{Z}C_p \to \mathbb{Z}[\zeta_p]$, $x \mapsto \zeta_p$ induces an isomorphism $C(\mathbb{Z}C_p) \cong C(\mathbb{Z}[\zeta_p])$. In particular, $D(\mathbb{Z}C_p) = 0$.*

We will now determine $D(\mathbb{Z}C_m)$, as well as its involution, in the case where m is square-free. This will be used in the proofs of Theorems 11.1 and 11.2. As usual, we will view an abelian group with involution as a $\mathbb{Z}C_2$-module.

Let $\pi_1 = 1$, let $\pi_p = 1 - \zeta_p$ for a prime p and, more generally, let $\pi_m = \prod_{p \mid m} \pi_p$ for an integer $m \geq 2$. Note that $\mathbb{Z}[\zeta_m]/\pi_m \cong \bigoplus_{p \mid m} \mathbb{Z}[\zeta_m]/\pi_p$ since the π_p are coprime (see, for example, [CR87, p. 249]). Let $\Psi_m \colon \mathbb{Z}[\zeta_m]^\times \to (\mathbb{Z}[\zeta_m]/\pi_m)^\times$ be the natural map.

Definition 10.13. For a square-free integer $m \geq 2$, define

$$V_m = \operatorname{coker}(\Psi_m \colon \mathbb{Z}[\zeta_m]^\times \to (\mathbb{Z}[\zeta_m]/\pi_m)^\times).$$

We will view this as a $\mathbb{Z}C_2$-module with the involution induced by the conjugation map

$$\bar{\cdot} \colon (\mathbb{Z}[\zeta_m]/\pi_m)^\times \to (\mathbb{Z}[\zeta_m]/\pi_m)^\times, \quad \zeta_m \mapsto \zeta_m^{-1}.$$

Note that, if p is prime, then $\mathbb{Z}[\zeta_p]/\pi_p \cong \mathbb{F}_p$. We can then see that $\Psi_p \colon \mathbb{Z}[\zeta_p]^\times \to \mathbb{F}_p^\times$ is surjective by considering the cyclotomic units $1 + \zeta_p + \cdots + \zeta_p^{i-1} \in \mathbb{Z}[\zeta_p]^\times$ for $(i, p) = 1$. In particular, $V_p = 1$. We will now show how $D(\mathbb{Z}C_m)$ is related to V_d for $d \mid m$.

Lemma 10.14. *Let $m \geq 2$ be a square-free integer. Let $d_1, \dots, d_n$ be the distinct nontrivial positive divisors of m, ordered such that d_{i+1} has at least as many prime factors as d_i (so d_1 is prime and $d_n = m$). Then there is a chain of $\mathbb{Z}C_2$-modules*

$$1 = A_0 \leq \cdots \leq A_n = D(\mathbb{Z}C_m)$$

such that $A_i/A_{i-1} \cong V_{d_{n-i+1}}$ for $1 \leq i \leq n$.

It is proven in [CR87, Theorem 50.6] that $|D(\mathbb{Z}C_m)| = \prod_{d \mid n} |V_d|$. Our proof will involve following the argument given there, and extending it to determine the group structure and the involution.

Proof. First recall that $\Gamma = \bigoplus_{d \mid m} \mathbb{Z}[\zeta_d]$ is the unique maximal order in $\mathbb{Q}C_m$ which contains $\mathbb{Z}C_m$. This implies that there is a pullback square:

$$\begin{array}{ccc} \mathbb{Z}C_m & \xrightarrow{i_2} & \Gamma \\ \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle j_2} \\ \mathbb{Z}C_m/m\Gamma & \xrightarrow{j_1} & \Gamma/m\Gamma. \end{array}$$

By [CR87, p. 246] this induces an exact sequence

$$(\mathbb{Z}C_m/m\Gamma)^\times \oplus \Gamma^\times \xrightarrow{(j_1, j_2)} (\Gamma/m\Gamma)^\times \xrightarrow{\partial} D(\mathbb{Z}C_m) \to 0$$

where $\partial \colon u \mapsto M(u)$ where

$$M(u) = \{(x, y) \in (\mathbb{Z}C_m/m\Gamma) \times \Gamma \mid j_1(x) = j_2(y)u \in \Gamma/m\Gamma\}$$

is the $\mathbb{Z}C_m$-module with action $\lambda \cdot (x, y) = (i_1(\lambda)x, i_2(\lambda)y)$ for $\lambda \in \mathbb{Z}C_m$. We now claim that the conjugation map on $(\Gamma/m\Gamma)^\times$ induces the involution on $D(\mathbb{Z}C_m)$. First note that, by Proposition 8.13, the involution on $D(\mathbb{Z}C_m)$ is induced by the natural involution $x \mapsto x^{-1}$ on the idèle group $J(\mathbb{Q}C_m) \subseteq \prod_p \mathbb{Q}_p C_m$. For all primes p, we have $\mathbb{Z}_p C_m \subseteq \Gamma_p \oplus (\mathbb{Z}C_m/m\Gamma)_p \subseteq \mathbb{Q}_p C_m$. If $M(u) = \mathbb{Z}C_m \alpha$ for $\alpha = (\alpha_p) \in J(\mathbb{Q}C_m)$, then [CR87, Exercise 53.1] implies that

$$\alpha_p = \begin{cases} (1,1) \in \Gamma_p \oplus (\mathbb{Z}C_m/m\Gamma)_p, & \text{if } (p,m) = 1 \\ (u_p, 1) \in \Gamma_p \oplus (\mathbb{Z}C_m/m\Gamma)_p, & \text{if } (p,m) \neq 1, \end{cases}$$

where $u_p \in \Gamma_p$ is any element such that $j_2(u_p) = [u] \in (\Gamma/m\Gamma)_p$. By the same argument as in the proof of Proposition 10.1, the involution on $J(\mathbb{Q}C_m)$ induces an involution on $J(\mathbb{Q}(\zeta_d))$ which coincides with the involution induced by conjugation. In particular, the involution maps $\alpha_p \mapsto (1,1)$ or $(\overline{u}_p, 1)$ where $\overline{\cdot}\colon \Gamma_p \to \Gamma_p$ is induced by conjugation on Γ. In particular, this coincides with the involution induced by conjugation on $(\Gamma/m\Gamma)^\times$.

By [CR87, Lemma 50.7], j_1 can be replaced by the map $\alpha\colon (\mathbb{Z}C_m/m\mathbb{Z}C_m)^\times \to (\Gamma/m\Gamma)^\times$. By [CR87, Lemma 50.8], $\operatorname{coker}(\alpha) \cong \bigoplus_{d|m} (\mathbb{Z}[\zeta_d]/\pi_d)^\times$ and it follows from the proof that conjugation map on $(\Gamma/m\Gamma)^\times$ induces conjugation on $(\mathbb{Z}[\zeta_d]/\pi_d)^\times$ for each $d \mid m$. If $\gamma\colon \Gamma^\times \to \operatorname{coker}(\alpha)$ is the map induced by j_2, then we obtain an exact sequence

$$\bigoplus_{d|m} \mathbb{Z}[\zeta_d]^\times \xrightarrow{\gamma} \bigoplus_{d|m} (\mathbb{Z}[\zeta_d]/\pi_d)^\times \xrightarrow{\overline{\partial}} D(\mathbb{Z}C_m) \to 0.$$

Let $d_1, \dots, d_n$ be the ordered sequence of divisors of m. By [CR87, p. 250] we have that

$$\gamma\,|_{\mathbb{Z}[\zeta_{d_i}]^\times} : \mathbb{Z}[\zeta_{d_i}]^\times \to (\mathbb{Z}[\zeta_{d_i}]/\pi_{d_i})^\times \oplus \bigoplus_{p \mid \frac{m}{d_i}} (\mathbb{Z}[\zeta_{pd_i}]/\pi_p)^\times \subseteq \bigoplus_{j \geq i} (\mathbb{Z}[\zeta_{d_j}]/\pi_{d_j})^\times$$

given by $x \mapsto (x, x^{-1}, \dots, x^{-1})$, where the last inclusion comes from the fact that $\mathbb{Z}[\zeta_d]/\pi_p \subseteq \bigoplus_{q|d} \mathbb{Z}[\zeta_d]/\pi_q \cong \mathbb{Z}[\zeta_d]/\pi_d$ for primes q since the π_q are pairwise coprime in $\mathbb{Z}[\zeta_d]$. If $x \in \mathbb{Z}[\zeta_k]^\times$, then $x^{-1} \in (\mathbb{Z}[\zeta_k]/\pi_p)^\times \subseteq (\mathbb{Z}[\zeta_{pk}]/\pi_p)^\times$.

For $1 \leq i \leq n$, this shows that γ restricts to a map $\gamma_i\colon \bigoplus_{j \geq i} \mathbb{Z}[\zeta_{d_j}]^\times \to \bigoplus_{j \geq i} (\mathbb{Z}[\zeta_{d_j}]/\pi_j)^\times$ where $\gamma = \gamma_1$. By a mild generalisation of [CR87, Exercise 50.2], this implies that there is an exact sequence induced by the projection map:

$$1 \to \operatorname{coker}(\gamma'_{i+1}\colon W \to \bigoplus_{j \geq i+1} (\mathbb{Z}[\zeta_{d_j}]/\pi_j)^\times) \to \operatorname{coker}(\gamma_i) \to \operatorname{coker}(\Psi_{d_i}) \to 1,$$

where $W = \{x \in \mathbb{Z}[\zeta_{d_i}]^\times \mid x \equiv 1 \bmod \pi_{d_i}\} \oplus \bigoplus_{j \geq i+1} \mathbb{Z}[\zeta_{d_j}]^\times$. By [CR87, p. 252], we have $\operatorname{Im}(\gamma'_{i+1}) = \operatorname{Im}(\gamma_{i+1})$ and so $\operatorname{coker}(\gamma'_{i+1}) = \operatorname{coker}(\gamma_{i+1})$. Let $A_i = \operatorname{coker}(\gamma_{n-i+1})$ for $1 \leq i \leq n$ and $A_0 = 1$. Then we have that $1 = A_0 \leq \cdots \leq A_n = \operatorname{coker}(\gamma) = D(\mathbb{Z}C_m)$such that there are isomorphisms $A_i/A_{i-1} \cong \operatorname{coker}(\Psi_{d_{n-i+1}})$ as abelian groups.

Since the involution on $D(\mathbb{Z}C_m)$ is induced by conjugation on $(\Gamma/m\Gamma)^\times$, it follows that it restricts to A_i where it acts via conjugation on the $\mathbb{Z}[\zeta_d]$. Hence, with respect to the involution, $A_i \leq D(\mathbb{Z}C_m)$ is a $\mathbb{Z}C_2$-modules and the chain $A_0 \leq \cdots \leq A_n$ is a chain of $\mathbb{Z}C_2$-modules. Under the abelian group isomorphism

$$A_i/A_{i-1} \cong \operatorname{coker}(\Psi_{d_{n-i+1}} \colon \mathbb{Z}[\zeta_{d_{n-i+1}}]^\times \to (\mathbb{Z}[\zeta_{d_{n-i+1}}]/\pi_{d_{n-i+1}})^\times),$$

the involution on $\operatorname{coker}(\Psi_{d_{n-i+1}})$ induced by the involution on A_i coincides with the involution induced by conjugation on $(\mathbb{Z}[\zeta_{d_{n-i+1}}]/\pi_{d_{n-i+1}})^\times$. Hence there is an isomorphism of $\mathbb{Z}C_2$-modules $A_i/A_{i-1} \cong V_{n-i+1}$, as required. ☐

Let $m \geq 2$ be a square-free integer. We will now give a method for analysing the involution on $V_m = \operatorname{coker}(\Psi_m)$. For a field $\mathbb{F}$ and $\alpha_1, \dots, \alpha_n \in \mathbb{C}$, we will write $\mathbb{F}[\alpha_1, \dots, \alpha_n]$ to denote $\mathbb{F} \otimes_{\mathbb{Z}} \mathbb{Z}[\alpha_1, \dots, \alpha_n]$. Note that this may not be a field and so need not coincide with $\mathbb{F}(\alpha_1, \dots, \alpha_n)$.

First note that, as described above, we have that

$$\Psi_m \colon \mathbb{Z}[\zeta_m]^\times \to \bigoplus_{p|m} (\mathbb{Z}[\zeta_m]/\pi_p)^\times,$$

where p ranges over the prime factors of m. If $p \mid m$ then, since m is square-free, we have $m = pk$ where $(k,p) = 1$ and so we can take $\zeta_m = \zeta_p \cdot \zeta_k$. This implies that $\mathbb{Z}[\zeta_m] = \mathbb{Z}[\zeta_p, \zeta_k]$ and so

$$\mathbb{Z}[\zeta_m]/\pi_p \cong \mathbb{Z}[\zeta_p, \zeta_k]/\pi_p \cong \mathbb{F}_p[\zeta_k].$$

Let $m = p_1 \cdots p_n$ for distinct primes p_i. Then Ψ_m can be written as

$$\Psi_m \colon \mathbb{Z}[\zeta_m]^\times \to \bigoplus_{i=1}^n \mathbb{F}_{p_i}[\zeta_{m/p_i}]^\times,$$

where $\zeta_m = \prod_j \zeta_{p_j}$, $\zeta_{m/p_i} = \prod_{j \neq i} \zeta_{p_j}$ and the map $\mathbb{Z}[\zeta_m]^\times \to \mathbb{F}_{p_i}[\zeta_{m/p_i}]^\times$ is the map sending $\zeta_{p_i} \mapsto 1$. This motivates the following definition.

Definition 10.15. For a square-free integer $m \geq 2$, define

$$\widetilde{V}_m \cong \operatorname{coker}\left(\Psi_m^+ \colon \mathbb{Z}[\zeta_m]^\times \to \bigoplus_{i=1}^n \frac{\mathbb{F}_{p_i}[\zeta_{m/p_i}]^\times}{\mathbb{F}_{p_i}[\lambda_{m/p_i}]^\times}\right),$$

where $\mathbb{F}_{p_i}[\lambda_{m/p_i}]^\times \leq \mathbb{F}_{p_i}[\zeta_{m/p_i}]^\times$ is induced by the inclusion $\mathbb{Z}[\lambda_{m/p_i}] \leq \mathbb{Z}[\zeta_{m/p_i}]$ and Ψ_m^+ is the composition of Ψ_m with the quotient maps $\mathbb{F}_{p_i}[\zeta_{m/p_i}]^\times \twoheadrightarrow \mathbb{F}_{p_i}[\zeta_{m/p_i}]^\times / \mathbb{F}_{p_i}[\lambda_{m/p_i}]^\times$.

We will view this as a $\mathbb{Z}C_2$-module with the involution induced by the conjugation map

$$\bar{\cdot} \colon \mathbb{F}_{p_i}[\zeta_{m/p_i}]^\times \to \mathbb{F}_{p_i}[\zeta_{m/p_i}]^\times, \quad \zeta_{m/p_i} \mapsto \zeta_{m/p_i}^{-1}.$$

Lemma 10.16. *Let $m \geq 2$ be a square-free integer.*

(i) *There is a surjective $\mathbb{Z}C_2$-module homomorphism $V_m \twoheadrightarrow \widetilde{V}_m$.*
(ii) *$\widetilde{V}_m = \widetilde{V}_m^-$, i.e. if $x \in \widetilde{V}_m$, then $\overline{x} = -x \in \widetilde{V}_m$.*

Proof. (i) For $1 \leq i \leq n$, we let $f_i = \operatorname{Id}_{\mathbb{F}_{p_i}} \otimes_{\mathbb{Z}} \iota_i$, where $\iota_i \colon \mathbb{Z}[\lambda_{m/p_i}] \hookrightarrow \mathbb{Z}[\zeta_{m/p_i}]$ is the natural inclusion map. Then the surjective homomorphism $V_m \twoheadrightarrow \widetilde{V}_m$ is induced by noting that

$$\begin{aligned}\widetilde{V}_m &\cong \operatorname{coker}((\Psi_m, f_1, \dots, f_n) \colon \mathbb{Z}[\zeta_m]^\times \oplus \bigoplus_{i=1}^n \mathbb{F}_{p_i}[\lambda_{m/p_i}]^\times \to \bigoplus_{i=1}^n \mathbb{F}_{p_i}[\zeta_{m/p_i}]^\times) \\ &\cong \operatorname{coker}((f_1, \dots, f_n) \colon \bigoplus_{i=1}^n \mathbb{F}_{p_i}[\lambda_{m/p_i}]^\times \to V_m).\end{aligned}$$

(ii) We first claim that, if p is prime and n is an integer, then $\alpha \in \mathbb{F}_p[\zeta_n]$ implies $\alpha \cdot \overline{\alpha} \in \mathbb{F}_p[\lambda_n]$. Let $\beta = \alpha \cdot \overline{\alpha}$. Since $\mathbb{Z}[\zeta_n]$ has integral basis $\{\zeta_n^i\}_{i=0}^{n-1}$, we can write $\beta = \sum_{i=0}^{n-1} a_i \otimes \zeta_n^i$ for $a_i \in \mathbb{F}_p$. Note that $\overline{\beta} = \beta$ which implies that

$$\textstyle\sum_{i=0}^{n-1} a_i \otimes \zeta_n^i = \sum_{i=0}^{n-1} a_{n-i} \otimes \zeta_n^i \in \mathbb{F}_p[\zeta_n].$$

Since $\mathbb{F}_p[\zeta_n] = \mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{Z}[\zeta_n] \cong \mathbb{F}_p^n$ as an abelian group, we get that $a_n = a_{n-i} \in \mathbb{F}_p$ for all i and so

$$\beta = a_0 + \textstyle\sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} a_i \otimes (\zeta_n^i + \zeta_n^{-i}) + \varepsilon \in \mathbb{F}_p[\lambda_n],$$

where $\varepsilon = -a_{n/2}$ for n even and $\varepsilon = 0$ for n odd.

Finally, let $f \colon V_m \to \widetilde{V}_m$ be the map described above and let $\alpha = [(\alpha_1, \dots \alpha_n)]$, where $\alpha_i \in \mathbb{F}_{p_i}[\zeta_{d/p_i}]^\times$. We have shown that $\alpha_i \cdot \overline{\alpha}_i \in \mathbb{F}_{p_i}[\lambda_{d/p_i}]^\times$ for all i and so

$$f(\alpha) \cdot f(\overline{\alpha}) = [(\alpha_1 \cdot \overline{\alpha}_1, \dots, \alpha_n \cdot \overline{\alpha}_n)] = [(1, \dots, 1)] = 1$$

and $f(\overline{\alpha}) = f(\alpha)^{-1}$. Since f is surjective and induces the involution on $\widetilde{V}_m$, this implies that $\overline{x} = -x$ for all $x \in \widetilde{V}_m$, where we now write the inverse as $-x$ rather than x^{-1} since $\widetilde{V}_m$ is an abelian group. □

We will now deduce the following, which is the main result of this section. Note that this is analogous to Proposition 10.4 which applied in the case of ideal class groups. Recall that, for an integer m, we let $\operatorname{odd}(m)$ denote the unique odd integer r such that $m = 2^k r$ for some k.

Proposition 10.17. *Let $m \geq 2$ be a square-free integer. Let $d_1, \dots, d_n$ be the distinct nontrivial positive divisors of m, ordered such that d_{i+1} has at least as many prime factors as d_i, and let A_i be the $\mathbb{Z}C_2$-modules defined in Lemma 10.14. Then there is a chain of abelian subgroups*

$$1 = \{x - \overline{x} \mid x \in A_0\} \leq \cdots \leq \{x - \overline{x} \mid x \in A_n\} = \{x - \overline{x} \mid x \in D(\mathbb{Z}C_m)\}$$

and, for each $1 \leq i \leq n$, there are surjective group homomorphisms

$$\{x - \overline{x} \mid x \in A_i\} / \{x - \overline{x} \mid x \in A_{i-1}\} \twoheadrightarrow \{x - \overline{x} \mid x \in V_{d_{n-i+1}}\} \twoheadrightarrow 2 \cdot \widetilde{V}_{d_{n-i+1}}.$$

In particular, $\prod_{d \mid m} \operatorname{odd}(|\widetilde{V}_d|)$ divides $|\{x - \overline{x} \mid x \in D(\mathbb{Z}C_m)\}|$.

Proof. The chain of abelian subgroups follows direct from Lemma 10.14 and Lemma 9.3 (ii). Let $1 \le i \le n$. By Lemma 10.14, there is a short exact sequence of $\mathbb{Z}C_2$-modules

$$0 \to A_{i-1} \to A_i \to V_{d_{n-i+1}} \to 0.$$

By Lemma 9.3 (ii), there are injective and surjective maps:

$$\{x - \overline{x} \mid x \in A_{i-1}\} \hookrightarrow \{x - \overline{x} \mid x \in A_i\} \twoheadrightarrow \{x - \overline{x} \mid x \in V_{d_{n-i+1}}\}.$$

Since the composition is necessarily the zero map, this gives the first surjective homomorphism. The second is a direct consequence of both parts of Lemma 10.16 as well as Lemma 9.3 (ii) again.

For the last part, the two statements we have proved so far imply that

$$|\{x - \overline{x} \mid x \in D(\mathbb{Z}C_m)\}| = \prod_{i=1}^{n} |\{x - \overline{x} \mid x \in A_i\}/\{x - \overline{x} \mid x \in A_{i-1}\}|$$

and $|2 \cdot \widetilde{V}_{d_{n-i+1}}|$ divides $|\{x - \overline{x} \mid x \in A_i\}/\{x - \overline{x} \mid x \in A_{i-1}\}|$ for all $1 \le i \le n$. It now suffices to note that $\mathrm{odd}(|\widetilde{V}_{d_{n-i+1}}|)$ divides $|2 \cdot \widetilde{V}_{d_{n-i+1}}|$ by the same argument as in the proof of Proposition 10.4. □

Remark 10.18. To obtain a complete analogue of Proposition 10.4, it would be desirable to also obtain bounds on $|\{x \in D(\mathbb{Z}C_m) \mid x = -\overline{x}\}|$. However, unlike Proposition 10.4, the bounds we obtain in Proposition 10.17 are obtained by subquotients rather than just subgroups. We therefore cannot apply Lemma 9.3 (i) since the final map in the sequence need not be surjective in general (see Remark 9.4). As we shall see in Section 11, it is possible to circumvent the need for such bounds in the proof of Theorem 11.1.

We now conclude this section with a result which holds in the case that m is not square-free. Firstly an analogue of Lemma 10.14 holds in the case that m is a prime power, by Kervaire–Murthy [KM77, Theorem 1.2]. For brevity, we will not state this result here. We will instead make do with the following consequence of their result in the case that $p = 2$ which will be used in the proof of Theorem 11.3. Note that $V_{2^{n+1}}$ is directly analogous to the $\mathbb{Z}C_2$-modules V_m defined in Definition 10.13 when m is square-free.

Proposition 10.19. *Let $n \ge 1$. Then there exists an exact sequence of $\mathbb{Z}C_2$-modules*

$$0 \to V_{2^{n+1}} \to D(\mathbb{Z}C_{2^{n+1}}) \to D(\mathbb{Z}C_{2^n}) \to 0,$$

where $V_{2^{n+1}} = \bigoplus_{i=1}^{n-2} (\mathbb{Z}/2^i)^{2^{n-i-2}}$ with the involution acting by negation.

Proof. This is a consequence of results of Kervaire–Murthy [KM77]. In [KM77, p. 419] they show that there is an exact sequence of $\mathbb{Z}C_2$-modules

$$0 \to V_{2^{n+1}} \to \widetilde{K}_0(\mathbb{Z}C_{2^{n+1}}) \xrightarrow{\alpha} \widetilde{K}_0(\mathbb{Z}C_{2^n}) \oplus \widetilde{K}_0(\mathbb{Z}[\zeta_{2^{n+1}}]) \to 0$$

where α is induced by the natural map of rings $\mathbb{Z}C_{2^{n+1}} \to \mathbb{Z}C_{2^n} \times \mathbb{Z}[\zeta_{2^{n+1}}]$. Since the maximal order $\mathbb{Z}C_{2^n} \subseteq \Gamma_{2^n} \subseteq \mathbb{Q}C_{2^n}$ is given by $\Gamma_{2^n} = \bigoplus_{i=1}^{n} \mathbb{Z}[\zeta_{2^i}]$, we get that

$$\ker(\alpha) \cong \ker(\beta \colon D(\mathbb{Z}C_{2^{n+1}}) \to D(\mathbb{Z}C_{2^n}))$$

where β is the $\mathbb{Z}C_2$-module homomorphism induced by map $\mathbb{Z}C_{2^{n+1}} \to \mathbb{Z}C_{2^n}$. This gives an exact sequence of the required form. It follows from [KM77, Theorem 1.1] that $V_{2^{n+1}}$ is as described. □

10.5. **Divisibility and lower bounds for kernel groups.** The aim of this section will be to establish divisibility results for $|D(\mathbb{Z}C_m)|$ and $\mathrm{odd}(|\widetilde{V}_m|)$. These results are necessary for determining the involution on $D(\mathbb{Z}C_m)$ in an analogous way to how divisibility results for class numbers h_m were necessary for determining the involution on $C(\mathbb{Z}[\zeta_m])$ (see Section 10.3). The results on $\mathrm{odd}(|\widetilde{V}_m|)$ are motivated by Proposition 10.17.

We begin by recalling the following, which is [CR87, Theorem 50.18].

Proposition 10.20. *If p is a prime and G is a finite p-group, then $D(\mathbb{Z}G)$ is an abelian p-group. In particular, if $p \neq 2$, then $|D(\mathbb{Z}G)|$ is odd.*

We will now find conditions on $m \geq 2$ square-free for which $\operatorname{odd}(|\widetilde{V}_m|) \neq 1$. Our strategy is motivated by the bounds d such that $d \mid |D(\mathbb{Z}C_m)|$ which were obtained by Cassou-Noguès in [CN72, CN74]. In particular, our argument shows that these bounds actually give factors of $|\widetilde{V}_m|$. Recall that $\widetilde{V}_m = \operatorname{coker}(\Psi_m^+)$, where

$$\Psi_m^+ \colon \mathbb{Z}[\zeta_m]^\times \to \bigoplus_{i=1}^n \frac{\mathbb{F}_{p_i}[\zeta_{m/p_i}]^\times}{\mathbb{F}_{p_i}[\lambda_{m/p_i}]^\times}$$

is the map defined in Definition 10.15.

Lemma 10.21. *Let $m \geq 2$ be an integer, let p be a prime such that $p \nmid m$, let $f_p = \operatorname{ord}_m(p)$ denote the order of p in $\mathbb{Z}/m$ and let $g_p = \varphi(m)/2f_p$.*

(i) *The inclusion $\mathbb{Z}[\lambda_m] \subseteq \mathbb{Z}[\zeta_m]$ induces inclusion $\mathbb{F}_p[\lambda_m] \subseteq \mathbb{F}_p[\zeta_m]$.*

(ii) *If $m \geq 3$, then $|\mathbb{F}_p[\zeta_m]^\times| = (p^{f_p} - 1)^{g_p}$.*

(iii) *If $m \geq 3$, then* $|\mathbb{F}_p[\lambda_m]^\times| = \begin{cases} (p^{\frac{f_p}{2}} - 1)^{g_p}, & \text{if } f_p \text{ is even} \\ (p^{f_p} - 1)^{\frac{g_p}{2}}, & \text{if } f_p \text{ is odd.} \end{cases}$

If $m = 2$, then $\zeta_2, \lambda_2 \in \mathbb{Z}$ and so $|\mathbb{F}_p[\zeta_2]^\times| = |\mathbb{F}_p[\lambda_2]^\times| = p - 1$.

Proof. The proofs of (ii) and (iii) are analogous. We will prove (iii) only as it is more complicated. First note that $\mathbb{Q}(\lambda_m)/\mathbb{Q}$ is a Galois extension and p is unramified in $\mathbb{Q}(\lambda_m)/\mathbb{Q}$ since it is unramified in $\mathbb{Q}(\zeta_m)/\mathbb{Q}$ (see, for example, [Was97, Lemma 15.48]). This implies that $p \cdot \mathbb{Z}[\lambda_m] = \mathcal{P}_1 \cdots \mathcal{P}_g$ for some $g \geq 1$, where the $\mathcal{P}_i \subseteq \mathbb{Z}[\lambda_m]$ are distinct prime ideals. The $\mathcal{P}_i$ coincide under the Galois action and the $\mathbb{Z}[\lambda_m]/\mathcal{P}_i$ all coincide with the splitting field $\mathbb{F}_p(\lambda_m)$ and so

$$\mathbb{F}_p[\lambda_m] \cong \mathbb{Z}[\lambda_m]/(p) \cong \mathbb{F}_p(\lambda_m)^g$$

which implies that $\mathbb{F}_p[\lambda_m]^\times \cong (\mathbb{F}_p(\lambda_m)^\times)^g$.

Let $f = [\mathbb{F}_p(\lambda_m) : \mathbb{F}_p]$. Since $\operatorname{Gal}(\mathbb{F}_p(\lambda_m)/\mathbb{F}_p)$ is generated by the Frobenius element $\operatorname{Frob}_p \colon x \mapsto x^p$, we get that f is the smallest positive integer such that $\operatorname{Frob}_p^f = \operatorname{Id}_{\mathbb{F}_p(\lambda_m)}$. Note that $\operatorname{Frob}_p^f(\lambda_m) = \zeta_m^{p^f} + \zeta_m^{-p^f}$. This implies that $\operatorname{Frob}_p^f = \operatorname{Id}_{\mathbb{F}_p(\lambda_m)}$ if and only if $\zeta_m^{p^f} = \zeta_m^{\pm 1}$ and so f is the order of p in $(\mathbb{Z}/m)^\times/\{\pm 1\}$. Since $[\mathbb{Q}(\lambda_m) : \mathbb{Q}] = \varphi(m)/2$, we have that $|\mathbb{F}_p[\lambda_m]| = p^{\varphi(m)/2}$. Since $|\mathbb{F}_p(\lambda_m)^g| = p^{fg}$, this gives that $g = \varphi(m)/2f$. Hence we have $|\mathbb{F}_p[\lambda_m]^\times| = |\mathbb{F}_p(\lambda_m)^\times|^g = (p^f - 1)^g$. Note that $f = f_p$ if and only if f_p is odd, and otherwise $f = f_p/2$.

Finally, (i) follows by comparing the expressions for $\mathbb{F}_p[\lambda_m]$ and $\mathbb{F}_p[\zeta_m]$ as products of fields. □

We will now use Lemma 10.21 to obtain bounds on $|\widetilde{V}_{pq}|$ for p, q distinct odd primes, and for $|\widetilde{V}_{2p}|$, where p is an odd prime respectively.

Proposition 10.22. *Let p and q be distinct odd primes. Let $f_p = \operatorname{ord}_q(p)$, $g_p = (q-1)/2f_p$, $f_q = \operatorname{ord}_p(q)$ and $g_q = (p-1)/2f_q$. Define*

$$c_{pq} = \begin{cases} \frac{1}{2pq}(p^{\frac{f_p}{2}} + 1)^{g_p}(q^{\frac{f_q}{2}} + 1)^{g_q}, & \text{if } f_p, f_q \text{ are even} \\ \frac{1}{2pq}(p^{\frac{f_p}{2}} + 1)^{g_p}(q^{f_q} - 1)^{\frac{g_q}{2}}, & \text{if } f_p \text{ is even and } f_q \text{ is odd} \\ \frac{1}{2pq}(p^{f_p} - 1)^{\frac{g_p}{2}}(q^{\frac{f_q}{2}} + 1)^{g_q}, & \text{if } f_p \text{ is odd and } f_q \text{ is even} \\ \frac{1}{2pq}(p^{f_p} - 1)^{\frac{g_p}{2}}(q^{f_q} - 1)^{\frac{g_q}{2}}, & \text{if } f_p, f_q \text{ are odd.} \end{cases}$$

Then $c_{pq} \mid |\widetilde{V}_{pq}|$. In particular, if $\operatorname{odd}(c_{pq}) \neq 1$, then $\operatorname{odd}(|\widetilde{V}_{pq}|) \neq 1$.

Proof. Let $E = \langle \zeta_{pq}, \mathbb{Z}[\lambda_{pq}]^\times \rangle \leq \mathbb{Z}[\zeta_{pq}]^\times$. By [Was97, Corollary 4.13], E has index two with $\mathbb{Z}[\zeta_{pq}]^\times/E \cong \mathbb{Z}/2$ generated by $1 - \zeta_{pq}$. Let $\psi_p \colon \mathbb{Z}[\zeta_{pq}]^\times \to \mathbb{F}_p[\zeta_q]^\times$ be the map sending $\zeta_p \mapsto 1$. If $\alpha \in \mathbb{Z}[\lambda_{pq}]^\times$, then $\alpha = \sum_{i=0}^{\frac{pq-1}{2}} a_i(\zeta_{pq}^i + \zeta_{pq}^{-i})$ for some $a_i \in \mathbb{Z}$ and so

$$\psi_p(\alpha) = \textstyle\sum_{i=0}^{\frac{q-1}{2}} \widetilde{a}_i(\zeta_q^i + \zeta_q^{-i}) \in \mathbb{F}_p[\lambda_q]^\times$$

where $\widetilde{a}_i = \sum_{k=0}^{\frac{p-1}{2}} a_{i+kq}$. Hence the composition $\mathbb{Z}[\zeta_{pq}]^\times \xrightarrow{\psi_p} \mathbb{F}_p[\zeta_q]^\times \to \frac{\mathbb{F}_p[\zeta_q]^\times}{\mathbb{F}_p[\lambda_q]^\times}$ is trivial and so

$$\widetilde{V}_{pq} = \operatorname{coker}(\Psi_{pq}^+) = \operatorname{coker}\left(\mathbb{Z}/pq \oplus \mathbb{Z}/2 \to \frac{\mathbb{F}_p[\zeta_q]^\times}{\mathbb{F}_p[\lambda_q]^\times} \oplus \frac{\mathbb{F}_q[\zeta_p]^\times}{\mathbb{F}_q[\lambda_p]^\times}\right)$$

where $1 \in \mathbb{Z}/pq$ maps to $\Psi_{pq}^+(\zeta_{pq})$ and $1 \in \mathbb{Z}/2$ maps to $\Psi_{pq}^+(1-\zeta_{pq})$. In particular, this implies that $|\widetilde{V}_{pq}|$ is divisible by $\frac{1}{2pq} \cdot |\frac{\mathbb{F}_p[\zeta_q]^\times}{\mathbb{F}_p[\lambda_q]^\times}| \cdot |\frac{\mathbb{F}_q[\zeta_p]^\times}{\mathbb{F}_q[\lambda_p]^\times}|$. The result now follows from Lemma 10.21. ☐

Proposition 10.23. *Let p be an odd prime. Let $f_2 = \mathrm{ord}_p(2)$ and $g_2 = (p-1)/2f_2$. Define*

$$c_{2p} = \begin{cases} \frac{1}{p}(2^{\frac{f_2}{2}}+1)^{g_2}, & \text{if } f_2 \text{ is even} \\ \frac{1}{p}(2^{f_2}-1)^{\frac{g_2}{2}}, & \text{if } f_2 \text{ is odd.} \end{cases}$$

Then $c_{2p} \mid |\widetilde{V}_{2p}|$. In particular, if $\mathrm{odd}(c_{2p}) \neq 1$, then $\mathrm{odd}(|\widetilde{V}_{2p}|) \neq 1$.

Proof. Let $\pi_{2p} \colon \mathbb{Z}[\zeta_p]^\times \to \mathbb{F}_2[\zeta_p]^\times$ be reduction mod 2. Since $\zeta_{2p} = -\zeta_p$ and $\zeta_2 = -1$, we have $\Psi_{2p} \colon \mathbb{Z}[\zeta_p]^\times \to \mathbb{F}_2[\zeta_p]^\times \oplus \mathbb{F}_p^\times$. It is shown in [CR87, Theorem 50.14] that projection induces an isomorphism $V_{2p} = \mathrm{coker}(\Psi_{2p}) \cong \mathrm{coker}(\pi_{2p})$. Similarly, if $\pi_{2p}^+ \colon \mathbb{Z}[\zeta_p]^\times \to \frac{\mathbb{F}_2[\zeta_p]^\times}{\mathbb{F}_2[\lambda_p]^\times}$, then $\widetilde{V}_{2p} = \mathrm{coker}(\Psi_{2p}^+) \cong \mathrm{coker}(\pi_{2p}^+)$. By [Was97, Corollary 4.13], $\mathbb{Z}[\zeta_p]^\times = \langle \zeta_p, \mathbb{Z}[\lambda_p]^\times \rangle$. The same argument as Proposition 10.22 implies that $|\mathrm{coker}(\pi_{2p}^+)|$ is divisible by

$$\frac{1}{p} \cdot \left| \frac{\mathbb{F}_p[\zeta_2]^\times}{\mathbb{F}_p[\lambda_2]^\times} \right| \cdot \left| \frac{\mathbb{F}_2[\zeta_p]^\times}{\mathbb{F}_2[\lambda_p]^\times} \right| = \frac{1}{p} \cdot \left| \frac{\mathbb{F}_2[\zeta_p]^\times}{\mathbb{F}_2[\lambda_p]^\times} \right|.$$

The result now follows from Lemma 10.21. ☐

11. Proof of main results on the involution on $\widetilde{K}_0(\mathbb{Z}C_m)$

The aim of this section will be to prove the following three theorems. As we saw previously, these theorems are required to prove Theorems 5.14, 5.15, and 5.17 respectively.

Theorem 11.1. *Let $m \geq 2$ be a square-free integer. Then*

(i) $|\{x \in \widetilde{K}_0(\mathbb{Z}C_m) \mid \overline{x} = -x\}| = 1$ *if and only if* $m \in \{2,3,5,6,7,10,11,13,14,17,19\}$*; and*
(ii) $|\{x \in \widetilde{K}_0(\mathbb{Z}C_m) \mid \overline{x} = -x\}| \to \infty$ *super-exponentially in* m.

Theorem 11.2. *Let $m \geq 2$ be a square-free integer. Then*

(i) $|\{x - \overline{x} \mid x \in \widetilde{K}_0(\mathbb{Z}C_m)\}| = 1$ *if and only if* $m \in \{2,3,5,6,7,10,11,13,14,15,17,19,29\}$*; and*
(ii) $|\{x - \overline{x} \mid x \in \widetilde{K}_0(\mathbb{Z}C_m)\}| \to \infty$ *super-exponentially in* m.

Theorem 11.3. *Let $m \geq 2$ be an integer. Then*

(i) $|\{x \in \widetilde{K}_0(\mathbb{Z}C_m) \mid \overline{x} = -x\}/\{x - \overline{x} \mid x \in \widetilde{K}_0(\mathbb{Z}C_m)\}| = 1$ *for infinitely many m; and*
(ii) $\sup_{n \leq m} |\{x \in \widetilde{K}_0(\mathbb{Z}C_n) \mid \overline{x} = -x\}/\{x - \overline{x} \mid x \in \widetilde{K}_0(\mathbb{Z}C_n)\}| \to \infty$ *exponentially in* m.

11.1. Proof of Theorems 11.1 and 11.2.

The proofs of Theorems 11.1 and 11.2 are best handled together since

$$\{x - \overline{x} \mid x \in \widetilde{K}_0(\mathbb{Z}C_m)\} \leq \{x \in \widetilde{K}_0(\mathbb{Z}C_m) \mid \overline{x} = -x\}.$$

In particular, it suffices to prove the following four statements.

(A1) $|\{x - \overline{x} \mid x \in \widetilde{K}_0(\mathbb{Z}C_m)\}| \to \infty$ super-exponentially in m.
(A2) If $m \notin \{2,3,5,6,7,10,11,13,14,15,17,19,29\}$, then $|\{x - \overline{x} \mid x \in \widetilde{K}_0(\mathbb{Z}C_m)\}| \neq 1$.
(A3) If $m \in \{2,3,5,6,7,10,11,13,14,17,19\}$, then $|\{x \in \widetilde{K}_0(\mathbb{Z}C_m) \mid \overline{x} = -x\}| = 1$.
(A4) If $m \in \{15,29\}$, then $|\{x - \overline{x} \mid x \in \widetilde{K}_0(\mathbb{Z}C_m)\}| = 1$ and $|\{x \in \widetilde{K}_0(\mathbb{Z}C_m) \mid \overline{x} = -x\}| \neq 1$.

To see this, note that (A1) coincides with Theorem 11.2 (ii) and implies Theorem 11.1 (ii). The forwards direction of Theorem 11.1 (i) is implied by (A2) and (A4), and the backwards direction coincides with (A3). The forwards direction of Theorem 11.2 (i) coincides with (A2) and the backwards direction is implied by (A3) and (A4).

Recall from Section 8.1 that $\widetilde{K}_0(\mathbb{Z}C_m) \cong C(\mathbb{Z}C_m)$ is an isomorphism of $\mathbb{Z}C_2$-modules, where $C(\mathbb{Z}C_m)$ denotes the locally free class group. We therefore have the following short exact sequence of $\mathbb{Z}C_2$-modules established in Section 10.1:

$$0 \to D(\mathbb{Z}C_m) \to \widetilde{K}_0(\mathbb{Z}C_m) \to \bigoplus_{d \mid m} C(\mathbb{Z}[\zeta_d]) \to 0,$$

where $D(\mathbb{Z}C_m)$ has the induced involution, and each $C(\mathbb{Z}[\zeta_d])$ has the involution induced by conjugation. Each of statements (A1)–(A4) will be proven via the following lemma.

Lemma 11.4. *Let $m \geq 2$ be an integer. Then*

(i) $|\{x - \overline{x} \mid x \in D(\mathbb{Z}C_m)\}| \leq |\{x - \overline{x} \mid x \in \widetilde{K}_0(\mathbb{Z}C_m)\}|$;

(ii) $\prod_{d \mid m} |\{x - \overline{x} \mid x \in C(\mathbb{Z}[\zeta_d])\}| \leq |\{x - \overline{x} \mid x \in \widetilde{K}_0(\mathbb{Z}C_m)\}|$;

(iii) *if* $|\{x-\overline{x} \mid x \in D(\mathbb{Z}C_m)\}| \neq 1$ *or* $|\{x-\overline{x} \mid x \in C(\mathbb{Z}[\zeta_m])\}| \neq 1$, *then* $|\{x-\overline{x} \mid x \in \widetilde{K}_0(\mathbb{Z}C_m)\}| \neq 1$ *(and so* $|\{x \in \widetilde{K}_0(\mathbb{Z}C_m) \mid \overline{x} = -x\}| \neq 1$*)*;

(iv) *if* $|\{x \in D(\mathbb{Z}C_m) \mid \overline{x} = -x\}| = 1$ *and* $\prod_{d \mid m} |\{x \in C(\mathbb{Z}[\zeta_d]) \mid \overline{x} = -x\}| = 1$, *then* $|\{x \in \widetilde{K}_0(\mathbb{Z}C_m) \mid \overline{x} = -x\}| = 1$ *(and so* $|\{x - \overline{x} \mid x \in \widetilde{K}_0(\mathbb{Z}C_m)\}| = 1$*)*.

Proof. By Lemma 9.3, there is an exact sequence

$$0 \to \{x \in D(\mathbb{Z}C_m) \mid \overline{x} = -x\} \to \{x \in \widetilde{K}_0(\mathbb{Z}C_m) \mid \overline{x} = -x\} \to \bigoplus_{d \mid m} \{x \in C(\mathbb{Z}[\zeta_d]) \mid x = -\overline{x}\}$$

as well as injective and surjective maps

$$\{x - \overline{x} \mid x \in D(\mathbb{Z}C_m)\} \hookrightarrow \{x - \overline{x} \mid x \in \widetilde{K}_0(\mathbb{Z}C_m)\} \twoheadrightarrow \bigoplus_{d \mid m} \{x - \overline{x} \mid x \in C(\mathbb{Z}[\zeta_d])\}.$$

The second sequence implies (i) and (ii), with (iii) as a corollary. The first sequence implies (iv). □

We will now proceed to prove each of statements (A1)-(A4). We will begin with the following which, by Lemma 11.4 (ii), implies (A1).

Proposition 11.5. *We have that* $|\{x - \overline{x} \mid x \in C(\mathbb{Z}[\zeta_m])\}| \to \infty$ *super-exponentially in* m.

Proof. It was shown in Proposition 10.4 that $\operatorname{odd}(h_m^-)$ divides $|\{x - \overline{x} \mid x \in C(\mathbb{Z}[\zeta_m])\}|$. The result now follows from that fact that, by Proposition 10.8, $\operatorname{odd}(h_m^-) \to \infty$ super-exponentially in m. □

We will now prove (A2). Our approach will be to use Lemma 11.4 (iii). In particular, we will begin by classifying the $m \geq 2$ square-free for which $|\{x - \overline{x} \mid x \in C(\mathbb{Z}[\zeta_m])\}| = 1$. We will then determine the subset of these values for which $|\{x - \overline{x} \mid x \in D(\mathbb{Z}C_m)\}| = 1$.

Proposition 11.6. *The complete list of $m \geq 2$ square-free for which $|\{x - \overline{x} \mid x \in C(\mathbb{Z}[\zeta_m])\}| = 1$ is as follows:*

$$m = \begin{cases} p, & \text{where } p \in \{2, 3, 5, 7, 11, 13, 17, 19, 29\} \\ 2p, & \text{where } p \in \{3, 5, 7, 11, 13, 17, 19, 29\} \\ pq \text{ or } 2pq, & \text{where } (p, q) \in \{(3, 5), (3, 7), (3, 11), (5, 7), (3, 13)\}. \end{cases}$$

Proof. First note that $h_m^- = 1$ implies $|\{x - \overline{x} \mid x \in C(\mathbb{Z}[\zeta_m])\}| = 1$. On the other hand, it follows from Proposition 10.4 that $\operatorname{odd}(h_m^-) \neq 1$ implies $|\{x - \overline{x} \mid x \in C(\mathbb{Z}[\zeta_m])\}| \neq 1$. In Proposition 10.7 (i), it is shown that the $m \geq 2$ square-free for which $\operatorname{odd}(h_m^-) = 1$ and $h_m^- \neq 1$ are precisely the $m \in S$, where $S = \{29, 39, 58, 65, 78, 130\}$. It remains to determine for which $m \in S$ we have $|\{x - \overline{x} \mid x \in C(\mathbb{Z}[\zeta_m])\}| = 1$.

Suppose $m \in S$. By [Was97, p. 421], we have $h_m^+ = 1$ and so $C(\mathbb{Z}[\zeta_m]) = C(\mathbb{Z}[\zeta_m])^-$ by Lemma 10.2 (ii). In particular, we have:

$$\{x - \overline{x} \mid x \in C(\mathbb{Z}[\zeta_m])\} = 2 \cdot C(\mathbb{Z}[\zeta_m]).$$

If $m = 29$ or 68, then $C(\mathbb{Z}[\zeta_m]) \cong (\mathbb{Z}/2)^3$ and $2 \cdot C(\mathbb{Z}[\zeta_m]) = 0$ [Was97, p. 412]. If $m = 39$ or 78, then $h_m = 2$ and $C(\mathbb{Z}[\zeta_m]) \cong \mathbb{Z}/2$ and $2 \cdot C(\mathbb{Z}[\zeta_m]) = 0$ [Was97, p. 412]. If $m = 65$ or 130, then [Hor93, Proposition 1 (iv)] gives that $C(\mathbb{Z}[\zeta_m]) \cong (\mathbb{Z}/2)^2 \times (\mathbb{Z}/4)^2$ and $2 \cdot C(\mathbb{Z}[\zeta_m]) \cong (\mathbb{Z}/2)^2$.

Hence we have shown that, for $m \geq 2$ square-free, $|\{x - \overline{x} \mid x \in C(\mathbb{Z}[\zeta_m])\}| = 1$ if and only if $h_m^- = 1$ or $m \in \{29, 39, 58, 78\}$. The result now follows by Proposition 10.5 (i). □

We now prove the following. By Lemma 11.4 (iii), this completes the proof of (A2).

Proposition 11.7. *The complete list of $m \geq 2$ square-free for which $|\{x - \overline{x} \mid x \in C(\mathbb{Z}[\zeta_m])\}| = 1$ and $|\{x - \overline{x} \mid x \in D(\mathbb{Z}C_m)\}| = 1$ is as follows:*

$$m = \begin{cases} p, & \text{where } p \in \{2, 3, 5, 7, 11, 13, 17, 19, 29\} \\ 2p, & \text{where } p \in \{3, 5, 7\}, \\ pq, & \text{where } (p, q) = (3, 5). \end{cases}$$

That is, $m \in \{2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 29\}$.

Before turning to the proof, we will begin by recalling that Cassou-Noguès determined the integers $m \geq 2$ for which $D(\mathbb{Z}C_m) = 0$ [CN74, Theorème 1] (see also [CN72]). The following can be deduced by comparing this with Proposition 11.6.

Lemma 11.8. *Let* $m \geq 2$ *be a square-free integer such that* $|\{x - \overline{x} \mid x \in C(\mathbb{Z}[\zeta_m])\}| = 1$. *Then* $D(\mathbb{Z}C_m) = 0$ *if and only if* m *is prime or* $m = 2p$ *where* $p \in \{3, 5, 7\}$.

Proof of Proposition 11.7. Let $m \in \{2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 29\}$. If $m \neq 15$, then Lemma 11.8 implies that $D(\mathbb{Z}C_m) = 0$ and so $|\{x - \overline{x} \mid x \in D(\mathbb{Z}C_m)\}| = 1$. If $m = 15$, then it is shown in [CN72, p. 48] that $|D(\mathbb{Z}C_{15})| = 2$. This implies that $D(\mathbb{Z}C_{15}) \cong \mathbb{Z}/2$ has the trivial action and so $|\{x - \overline{x} \mid x \in D(\mathbb{Z}C_{15})\}| = 1$.

By Proposition 11.6, the remaining $m \geq 2$ square-free for which $|\{x - \overline{x} \mid x \in C(\mathbb{Z}[\zeta_m])\}| = 1$ are:

$$m = \begin{cases} 2p, & \text{where } p \in \{11, 13, 17, 19, 29\} \\ pq, & \text{where } (p, q) \in \{(3, 7), (3, 11), (5, 7), (3, 13)\} \\ 2pq, & \text{where } (p, q) \in \{(3, 5), (3, 7), (3, 11), (5, 7), (3, 13)\}. \end{cases}$$

By Proposition 10.17, we have that $\prod_{d|m} \text{odd}(|\widetilde{V}_d|)$ divides $|\{x - \overline{x} \mid x \in D(\mathbb{Z}C_m)\}|$, where $\widetilde{V}_d$ is as defined in Definition 10.15. It therefore suffices to prove that, for each m listed above, we have $\text{odd}(|\widetilde{V}_d|) \neq 1$ for some $d \mid m$.

In the case $m = 2p$, the bound c_{2p} from Proposition 10.23 is computed as in the following table.

p	11	13	17	19	29
c_{2p}	3	5	17	27	565

In each case, $\text{odd}(c_{2p}) \neq 1$ and so $\text{odd}(|\widetilde{V}_{2p}|) \neq 1$. Hence $|\{x - \overline{x} \mid x \in D(\mathbb{Z}C_m)\}| \neq 1$ for $m = 2p$ where $p \in \{11, 13, 17, 19, 29\}$.

In the case $m = pq$ for odd primes p, q, the bound c_{pq} from Proposition 10.22 is computed as in the following table.

(p, q)	(3,5)	(3,7)	(3,11)	(5,7)	(3,13)
c_{pq}	2	4	44	90	104

In the cases $(p, q) \in \{(3, 11), (5, 7), (3, 13)\}$, $\text{odd}(c_{pq}) \neq 1$ and so $\text{odd}(|\widetilde{V}_{pq}|) \neq 1$. Hence $|\{x - \overline{x} \mid x \in D(\mathbb{Z}C_m)\}| \neq 1$ for $m = pq$ or $2pq$ where $(p, q) \in \{(3, 11), (5, 7), (3, 13)\}$.

We will deal the the three remaining cases $m = 21, 30$ and 42 directly from the definition of $\widetilde{V}_m$:

$$\widetilde{V}_m \cong \text{coker}\left(\Psi_m^+ \colon \mathbb{Z}[\zeta_m]^\times \to \bigoplus_{i=1}^{n} \frac{\mathbb{F}_{p_i}[\zeta_{m/p_i}]^\times}{\mathbb{F}_{p_i}[\lambda_{m/p_i}]^\times}\right).$$

First suppose that $m = 30$. Then we have

$$\Psi_{30}^+ \colon \mathbb{Z}[\zeta_{15}]^\times \to \frac{\mathbb{F}_2[\zeta_{15}]^\times}{\mathbb{F}_2[\lambda_{15}]^\times} \oplus \frac{\mathbb{F}_3[\zeta_5]^\times}{\mathbb{F}_3[\lambda_5]^\times} \oplus \frac{\mathbb{F}_5[\zeta_3]^\times}{\mathbb{F}_5[\lambda_3]^\times}$$

and $E = \langle \zeta_{15}, \mathbb{Z}[\lambda_{15}]^\times \rangle$ has index two in $\mathbb{Z}[\zeta_{15}]^\times$. By the same argument as Proposition 10.22, we get that $|\widetilde{V}_{30}| = |\text{coker}(\Psi_{30}^+)|$ is divisible by

$$c_{30} = \frac{1}{30} \cdot \left|\frac{\mathbb{F}_2[\zeta_{15}]^\times}{\mathbb{F}_2[\lambda_{15}]^\times}\right| \cdot \left|\frac{\mathbb{F}_3[\zeta_5]^\times}{\mathbb{F}_3[\lambda_5]^\times}\right| \cdot \left|\frac{\mathbb{F}_5[\zeta_3]^\times}{\mathbb{F}_5[\lambda_3]^\times}\right| = c_{15} \cdot \left|\frac{\mathbb{F}_2[\zeta_{15}]^\times}{\mathbb{F}_2[\lambda_{15}]^\times}\right| = 2 \cdot (2^{\frac{f_2}{2}} + 1)^{g_2} = 10$$

since $f_2 = \text{ord}_{15}(2) = 4$ and $g_2 = 1$. Since $\text{odd}(c_{30}) \neq 1$, this implies that $\text{odd}(|\widetilde{V}_{30}|) \neq 1$. Hence $|\{x - \overline{x} \mid x \in D(\mathbb{Z}C_{15})\}| \neq 1$.

We will deal with the remaining cases $m = 21, 42$ by computing the involution on $\widetilde{V}_{21}$ explicitly. This turns out to be necessary since, by Remark 11.12, we have $|D(\mathbb{Z}C_m)| = 4$ in each case and so $\text{odd}(|\widetilde{V}_{21}|) = \text{odd}(|\widetilde{V}_{42}|) = 1$. If $|\{x - \overline{x} \mid x \in D(\mathbb{Z}C_m)\}| = 1$ for $m = 21$ or 42 then, by Proposition 10.17, we would have $|2 \cdot \widetilde{V}_{21}| = 1$. Hence it suffices to prove that $|2 \cdot \widetilde{V}_{21}| \neq 1$.

We now claim that $\widetilde{V}_{21} = \text{coker}(\Psi_{21}^+) \cong \mathbb{Z}/4$, which implies that $2 \cdot \widetilde{V}_{21} \cong \mathbb{Z}/2$. First note that, by the proof of Proposition 10.22, we have that

$$\text{coker}(\Psi_{21}^+) \cong \text{coker}\left(\mathbb{Z}/21 \oplus \mathbb{Z}/2 \to \frac{\mathbb{F}_3[\zeta_7]^\times}{\mathbb{F}_3[\lambda_7]^\times} \oplus \frac{\mathbb{F}_7[\zeta_3]^\times}{\mathbb{F}_7[\lambda_3]^\times}\right)$$

where $1 \in \mathbb{Z}/21$ maps to $\Psi_{21}^{+}(\zeta_{21})$ and $1 \in \mathbb{Z}/2$ maps to $\Psi_{21}^{+}(1-\zeta_{21})$.

Since $g_3 = 1$, $\mathbb{F}_3[\zeta_7] \cong \mathbb{F}_3(\zeta_7)$ is a field and $\mathbb{F}_3[\lambda_7] = \mathbb{F}_3[\lambda_7]$ is a subfield. This implies that $\frac{\mathbb{F}_3[\zeta_7]^\times}{\mathbb{F}_3[\lambda_7]^\times} \cong \frac{\mathbb{Z}/(3^6-1)}{\mathbb{Z}/(3^3-1)} \cong \mathbb{Z}/28$. We have $\mathbb{F}_7[\zeta_3] = \mathbb{Z}[x]/\langle 7, 1+x+x^3\rangle = \mathbb{Z}[x]/\langle 7, (x-2)(x-4)\rangle \cong \mathbb{F}_7 \times \mathbb{F}_7$ where $\zeta_3 \mapsto (2,4)$ and so $\mathbb{F}_7[\zeta_3]^\times \cong (\mathbb{Z}/6)^2$. Since $\mathbb{F}_7[\lambda_3]^\times = \mathbb{F}_7^\times \cong \mathbb{Z}/6$, this implies that $\frac{\mathbb{F}_7[\zeta_3]^\times}{\mathbb{F}_7[\lambda_3]^\times} \cong \mathbb{Z}/6$. Hence $D := \frac{\mathbb{F}_3[\zeta_7]^\times}{\mathbb{F}_3[\lambda_7]^\times} \oplus \frac{\mathbb{F}_7[\zeta_3]^\times}{\mathbb{F}_7[\lambda_3]^\times} \cong \mathbb{Z}/28 \oplus \mathbb{Z}/6$.

Now $\Psi_{21}^{+}(\zeta_{21}) = [(\zeta_7, \zeta_3)]$ where $[\zeta_7] \in \mathbb{Z}/28$ has order 7 and $[\zeta_3] \in \mathbb{Z}/6$ has order 3. This implies that $\operatorname{coker}(\mathbb{Z}/21 \to D) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$. Note that $\Psi_{21}^{+}(1-\zeta_{21}) = [(1-\zeta_7, 1-\zeta_3)]$. The isomorphism $\mathbb{F}_7[\zeta_3]^\times/\mathbb{F}_7[\lambda_3]^\times \to \mathbb{Z}/6$ sends $1-\zeta_3 \mapsto 1-2 = -1$ and so the image of $\Psi_{21}^{+}(1-\zeta_{21})$ in $\operatorname{coker}(\mathbb{Z}/21 \to D) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$ has the form $(*, -1)$. Since it has order two, this implies that $\operatorname{coker}(\Psi_{21}^{+}) \cong \operatorname{coker}(\mathbb{Z}/21 \oplus \mathbb{Z}/2 \to D) \cong \mathbb{Z}/4$ as required. □

We will now prove (A3). Our approach will be to use Lemma 11.4 (iv), and will be analogous to our proof of (A2). In particular, we will begin by classifying the $m \geq 2$ square-free for which $\prod_{d \mid m} |\{x \in C(\mathbb{Z}[\zeta_d]) \mid x = -\overline{x}\}| = 1$. We will then determine the subset of these values for which $|\{x \in D(\mathbb{Z}C_m) \mid x = -\overline{x}\}| = 1$.

Proposition 11.9. *Let $m \geq 2$ be square-free. Then $\prod_{d \mid m} |\{x \in C(\mathbb{Z}[\zeta_d]) \mid x = -\overline{x}\}| = 1$ if and only if $h_m^- = 1$.*

Proof. If $h_m^- = 1$, then $h_d^- = 1$ for all $d \mid m$ and so $\prod_{d \mid m} |\{x \in C(\mathbb{Z}[\zeta_d]) \mid x = -\overline{x}\}| = 1$. Conversely, if $h_m^- \neq 1$ and $m \notin \{29, 39, 58, 78\}$, then Proposition 11.6 implies that $|\{x - \overline{x} \mid x \in C(\mathbb{Z}[\zeta_m])\}| \neq 1$ and so $|\{x \in C(\mathbb{Z}[\zeta_m]) \mid x = -\overline{x}\}| \neq 1$. It now suffices to show that, if $m \in \{29, 39, 58, 78\}$, then $|\{x \in C(\mathbb{Z}[\zeta_m]) \mid x = -\overline{x}\}| \neq 1$.

Suppose $m \in \{29, 39, 58, 78\}$. By Proposition 11.6, we have that $C(\mathbb{Z}[\zeta_m]) = C(\mathbb{Z}[\zeta_m])^-$ and so

$$\{x \in C(\mathbb{Z}[\zeta_m]) \mid x = -\overline{x}\} = \{x \in C(\mathbb{Z}[\zeta_m]) \mid 2x = 0\}.$$

If $m = 29$ or 58, then $C(\mathbb{Z}[\zeta_m]) \cong (\mathbb{Z}/2)^3$ and so $\{x \in C(\mathbb{Z}[\zeta_m]) \mid 2x = 0\} \cong (\mathbb{Z}/2)^3$ [Was97, p. 412]. If $m = 39$ or 78, then $C(\mathbb{Z}[\zeta_m]) \cong \mathbb{Z}/2$ and so $\{x \in C(\mathbb{Z}[\zeta_m]) \mid 2x = 0\} \cong \mathbb{Z}/2$ [Was97, p. 412]. □

We will now prove the following. By Lemma 11.4 (iv), this completes the proof of (A3).

Proposition 11.10. *The complete list of $m \geq 2$ square-free for which*

$$\prod_{d \mid m} |\{x \in C(\mathbb{Z}[\zeta_d]) \mid x = -\overline{x}\}| = 1 \quad \text{and} \quad |\{x \in D(\mathbb{Z}C_m) \mid x = -\overline{x}\}| = 1$$

is as follows:

$$m = \begin{cases} p, & \text{where } p \in \{2,3,5,7,11,13,17,19\} \\ 2p, & \text{where } p \in \{3,5,7\}. \end{cases}$$

That is, $m \in \{2, 3, 5, 6, 7, 10, 11, 13, 14, 17, 19\}$.

Proof. If $|\{x \in D(\mathbb{Z}C_m) \mid x = -\overline{x}\}| = 1$, then $|\{x - \overline{x} \mid x \in D(\mathbb{Z}C_m)\}| = 1$. By Propositions 10.5, 11.7, and 11.9, the $m \geq 2$ square-free for which $\prod_{d \mid m} |\{x \in C(\mathbb{Z}[\zeta_d]) \mid x = -\overline{x}\}| = 1$ and $|\{x - \overline{x} \mid x \in D(\mathbb{Z}C_m)\}| = 1$ are as follows:

$$m = \begin{cases} p, & \text{where } p \in \{2,3,5,7,11,13,17,19\} \\ 2p, & \text{where } p \in \{3,5,7\}, \\ pq, & \text{where } (p,q) = (3,5). \end{cases}$$

If $m = p$ for $p \leq 17$ prime or $m = 2p$ for $p \in \{3, 5, 7\}$, then Lemma 11.8 implies that $D(\mathbb{Z}C_m) = 0$ and so $|\{x \in D(\mathbb{Z}C_m) \mid x = -\overline{x}\}| = 1$. If $m = 15$, then it is shown in [CN72, p. 48] that $|D(\mathbb{Z}C_{15})| = 2$. This implies that $D(\mathbb{Z}C_{15}) \cong \mathbb{Z}/2$ has the trivial action and so $\{x \in D(\mathbb{Z}C_m) \mid x = -\overline{x}\} \cong \mathbb{Z}/2$. □

Finally, we will prove (A4). Note that these results are implied by computations used the proofs of Propositions 11.7 and 11.10, but we will repeat them here for the convenience of the reader.

Proposition 11.11.

(i) $\{x - \overline{x} \mid x \in \widetilde{K}_0(\mathbb{Z}C_{15})\} = 0$ *and* $\{x \in \widetilde{K}_0(\mathbb{Z}C_{15}) \mid x = -\overline{x}\} \cong \mathbb{Z}/2$.

(ii) $\{x - \overline{x} \mid x \in \widetilde{K}_0(\mathbb{Z}C_{29})\} = 0$ *and* $\{x \in \widetilde{K}_0(\mathbb{Z}C_{29}) \mid x = -\overline{x}\} \cong (\mathbb{Z}/2)^3$.

Proof. (i) By [Was97, p. 412], we have $h_{15} = 1$ and so $\widetilde{K}_0(\mathbb{Z}C_{15}) \cong D(\mathbb{Z}C_{15})$. It is shown in [CN72, p. 48] that $|D(\mathbb{Z}C_{15})| = 2$ and so $D(\mathbb{Z}C_{15}) \cong \mathbb{Z}/2$ has the trivial involution. Hence we have $\{x - \overline{x} \mid x \in \widetilde{K}_0(\mathbb{Z}C_{15})\} = 0$ and $\{x \in \widetilde{K}_0(\mathbb{Z}C_{15}) \mid x = -\overline{x}\} \cong \mathbb{Z}/2$.

(ii) By Lemma 11.8, we have $D(\mathbb{Z}C_{29}) = 0$ and so $\widetilde{K}_0(\mathbb{Z}C_{29}) \cong C(\mathbb{Z}[\zeta_{29}])$. By [Was97, p. 412], we have that $C(\mathbb{Z}[\zeta_{29}]) \cong (\mathbb{Z}/2)^3$. We also have $C(\mathbb{Z}[\zeta_{29}])^+ = 1$ [Was97, p. 412] and so, by Lemma 10.2 (ii), $C(\mathbb{Z}[\zeta_{29}]) = C(\mathbb{Z}[\zeta_{29}])^-$ and so has the trivial involution. Hence we have $\{x - \overline{x} \mid x \in \widetilde{K}_0(\mathbb{Z}C_{29})\} = 0$ and $\{x \in \widetilde{K}_0(\mathbb{Z}C_{29}) \mid x = -\overline{x}\} \cong (\mathbb{Z}/2)^3$. □

This completes the proofs of (A1)-(A4) and so, by the discussion at the start of this section, completes the proofs of Theorems 11.1 and 11.2.

Remark 11.12. In order to minimise the possibility of errors, we computed $D(\mathbb{Z}C_m)$ for all relevant m using the algorithm described in [BB06] and implemented in Magma by Werner Bley. We checked that $D(\mathbb{Z}C_m) = 0$ for the m listed in Lemma 11.8, we checked that all the bounds c_m computed in the proof of Proposition 11.7 divide $|D(\mathbb{Z}C_m)|$ and we computed $D(\mathbb{Z}C_{21}) \cong \mathbb{Z}/4$, $D(\mathbb{Z}C_{42}) \cong \mathbb{Z}/2$ and $D(\mathbb{Z}C_{15}) \cong \mathbb{Z}/2$. These computations are all consistent with the calculations above.

11.2. **Proof of Theorem 11.3.** For $m \geq 2$, let

$$A_m := \frac{\{x \in \widetilde{K}_0(\mathbb{Z}C_m) \mid x = -\overline{x}\}}{\{x - \overline{x} \mid x \in \widetilde{K}_0(\mathbb{Z}C_m)\}} \cong \widehat{H}^1(C_2; \widetilde{K}_0(\mathbb{Z}C_m))$$

where the isomorphism comes from Proposition 9.5. Similarly to the proofs of Theorems 11.1 and 11.2, we will begin by noting that it now suffices to prove the following two statements.

(B1) If $n \geq 1$, then $|A_{3^n}| = 1$
(B2) If $n \geq 1$, then $|A_{2^n}| \cdot |A_{2^{n+1}}| \geq 2^{2^{n-2}-1}$.

To see this, note that (B1) directly implies Theorem 11.3 (i). Next, Theorem 11.3 (ii) follows from (B2) since it implies that

$$\begin{aligned}\sup_{n \leq m} |A_n| &\geq \max\{|A_{2^{\lfloor \log_2(m) \rfloor}}|, |A_{2^{\lfloor \log_2(m) \rfloor - 1}}|\} \geq \sqrt{|A_{2^{\lfloor \log_2(m) \rfloor}}| \cdot |A_{2^{\lfloor \log_2(m) \rfloor - 1}}|} \\ &\geq 2^{2^{\lfloor \log_2(m) \rfloor - 3} - 1} \geq 2^{\frac{m}{16} - 1}\end{aligned}$$

which tends to infinity exponentially in m.

We will again make use of the following short exact sequence of $\mathbb{Z}C_2$-modules established in Section 10.1:

$$0 \to D(\mathbb{Z}C_m) \to C(\mathbb{Z}C_m) \to \bigoplus_{d \mid m} C(\mathbb{Z}[\zeta_d]) \to 0,$$

where $D(\mathbb{Z}C_m)$ has the induced involution, and each $C(\mathbb{Z}[\zeta_d])$ has the involution induced by conjugation. Each of statements (B1) and (B2) will be proven via the following lemma.

Lemma 11.13. *Let $m \geq 2$. Then there is a 6-periodic exact sequence of finite abelian groups*

$$\begin{array}{ccccc} \widehat{H}^1(C_2; D(\mathbb{Z}C_m)_{(2)}) & \longrightarrow & A_m & \longrightarrow & \bigoplus_{d \mid m} \widehat{H}^1(C_2; C(\mathbb{Z}[\zeta_d])_{(2)}) \\ \partial \uparrow & & & & \downarrow \partial \\ \bigoplus_{d \mid m} \widehat{H}^0(C_2; C(\mathbb{Z}[\zeta_d])_{(2)}) & \leftarrow & \widehat{H}^0(C_2; C(\mathbb{Z}C_m)_{(2)}) & \longleftarrow & \widehat{H}^0(C_2; D(\mathbb{Z}C_m)_{(2)}). \end{array}$$

Furthermore, we have that:

(i) *If h_m is odd, then $A_m \cong \widehat{H}^1(C_2; D(\mathbb{Z}C_m)_{(2)})$*
(ii) *If $|D(\mathbb{Z}C_m)|$ is odd, then $A_m \cong \bigoplus_{d \mid m} \widehat{H}^1(C_2; C(\mathbb{Z}[\zeta_d])_{(2)})$.*
(iii) *If h_m and $|D(\mathbb{Z}C_m)|$ are both odd, then $A_m = 0$.*

Proof. By Proposition 9.9, we have that $A_m \cong \widehat{H}^1(C_2; C(\mathbb{Z}C_m)_{(2)})$. The short exact sequence stated above induces a short exact sequence on their 2-primary submodules:

$$0 \to D(\mathbb{Z}C_m)_{(2)} \to C(\mathbb{Z}C_m)_{(2)} \to \bigoplus_{d \mid m} C(\mathbb{Z}[\zeta_d])_{(2)} \to 0.$$

This follows since, for example, localisation is an exact functor. The existence of the required 6-periodic exact sequence now follows from Proposition 9.7 and the fact that

$$\widehat{H}^n(C_2; \bigoplus_{d|m} C(\mathbb{Z}[\zeta_d])_{(2)}) \cong \bigoplus_{d|m} \widehat{H}^n(C_2; C(\mathbb{Z}[\zeta_d])_{(2)})$$

for $n \in \mathbb{Z}$ by Proposition 9.2 (ii).

To prove (i), suppose h_m is odd. Then h_d is odd for all $d \mid m$ since $h_d \mid h_m$ [Was97, p. 205]. This implies that $C(\mathbb{Z}[\zeta_d])_{(2)} = 0$ for all $d \mid m$ and so $\bigoplus_{d|m} \widehat{H}^n(C_2; C(\mathbb{Z}[\zeta_d])_{(2)}) = 0$ for all $n \in \mathbb{Z}$. The 6-periodic exact sequence then gives that $A_m \cong \widehat{H}^1(C_2; D(\mathbb{Z}C_m)_{(2)})$.

To prove (ii), suppose $|D(\mathbb{Z}C_m)|$ is odd. Then $D(\mathbb{Z}C_m)_{(2)} = 0$, $\widehat{H}^n(C_2; D(\mathbb{Z}C_m)_{(2)}) = 0$ for all n, so the 6-periodic exact sequence gives that $A_m \cong \bigoplus_{d|m} \widehat{H}^1(C_2; C(\mathbb{Z}[\zeta_d])_{(2)})$. If h_m and $|D(\mathbb{Z}C_m)|$ are odd, then $D(\mathbb{Z}C_m)_{(2)} = 0$ and so $A_m \cong \widehat{H}^1(C_2; D(\mathbb{Z}C_m)_{(2)}) = 0$, which proves (iii). □

We will now prove the following which implies (B1) since $h_3 = 1$ is odd (see, for example, Proposition 10.5). Note that it also implies that $|A_{p^n}| = 1$ for $n \geq 1$ and other primes $p \neq 3$.

Proposition 11.14. *Let $n \geq 1$ and let $p \leq 509$ be an odd prime such that h_p is odd. Then $|A_{p^n}| = 1$.*

Remark 11.15. By Lemma 10.10 (i), this condition holds precisely for the odd primes $p \leq 509$ with

$$p \notin \{29, 113, 163, 197, 239, 277, 311, 337, 349, 373, 397, 421, 463, 491\}.$$

Proof. Since p is an odd prime, Proposition 10.20 implies that $|D(\mathbb{Z}C_{p^n})|$ is odd. Since $p \leq 509$ and h_p is odd, Lemma 10.10 (ii) implies that h_{p^n} is odd. Hence $|A_{p^n}| = 1$ by Lemma 11.13 (iii). □

We will now prove (B2). By the discussion above, this completes the proof of Theorem 11.3.

Proposition 11.16. *If $n \geq 1$, then $|A_{2^n}| \cdot |A_{2^{n+1}}| \geq 2^{2^{n-2}-1}$.*

Proof. By Lemma 10.10 (ii), h_{2^n} is odd and so Lemma 11.13 (i) implies that

$$A_{2^n} \cong \widehat{H}^1(C_2; D(\mathbb{Z}C_{2^n})_{(2)}).$$

By Propositions 9.7 and 10.19, we have a 6-periodic exact sequence of finite abelian groups:

$$\begin{array}{ccccc} \widehat{H}^1(C_2; V_{2^{n+1}}) & \longrightarrow & A_{2^{n+1}} & \longrightarrow & A_{2^n} \\ \partial \uparrow & & & & \downarrow \partial \\ \widehat{H}^0(C_2; D(\mathbb{Z}C_{2^n})) & \longleftarrow & \widehat{H}^0(C_2; D(\mathbb{Z}C_{2^{n+1}})) & \longleftarrow & \widehat{H}^0(C_2; V_{2^{n+1}}). \end{array}$$

Furthermore, by Proposition 9.8, we have that $\widehat{H}^0(C_2; D(\mathbb{Z}C_{2^k})) \cong A_{2^k}$ as abelian groups for all $k \geq 1$, and $\widehat{H}^1(C_2; V_{2^{n+1}}) \cong \widehat{H}^0(C_2; V_{2^{n+1}})$ as abelian groups.

Since the involution on $V_{2^{n+1}}$ acts by negation, we have that

$$\widehat{H}^1(C_2; V_{2^{n+1}}) \cong V_{2^{n+1}}/2V_{2^{n+1}} \cong \bigoplus_{i=1}^{n-2} (\mathbb{Z}/2)^{2^{n-i-2}} \cong (\mathbb{Z}/2)^{2^{n-2}-1}.$$

This implies that the 6-periodic exact sequence restricts to:

$$A_{2^n} \xrightarrow{\alpha} (\mathbb{Z}/2)^{2^{n-2}-1} \xrightarrow{\beta} A_{2^{n+1}}.$$

By the first isomorphism theorem and exactness, we get that:

$$2^{2^{n-2}-1} = |\ker(\beta)| \cdot |\operatorname{Im}(\beta)| = |\operatorname{Im}(\alpha)| \cdot |\operatorname{Im}(\beta)| \leq |A_{2^n}| \cdot |A_{2^{n+1}}|$$

which was the required bound. □

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