Removing Intersections of Immersed Surfaces in a 4-Manifold

Mark Powell

September 13, 2009

Abstract

We show that all the double points of an immersed sphere in a simply connected 4-manifold W can be removed to find a representative embedded surface for each homology class $H_2(W;\mathbb{Z})$

Theorem. *Let W be a compact, simply connected 4-manifold. Then every homology class in $H_2(W;\mathbb{Z})$ can be represented by an embedded surface.*

For a simply-connected 4-manifold, every class in $H_2(W;\mathbb{Z})$ can be represented by an element of $\pi_2(W)$, every element of which is arbitrarily close to an immersion. We show how to remove these double points.

We have the following local situation around a double point of an immersed surface. Let $p = (0,0,0,0) \in \mathbb{R}^4$ be the point of intersection of two planes $P_1 := \{(x,y,0,0) \,|\, x,y \in \mathbb{R}\} \subseteq \mathbb{R}^4$ and $P_2 := \{(0,0,z,t) \,|\, z,t \in \mathbb{R}\}$. We remove the intersection of the open unit ball with the two planes:

$$(P_1 \cup P_2) \cap \mathring{D}^4$$

where $\mathring{D}^4 = \{(x,y,z,t) \in \mathbb{R}^4 \,|\, x^2+y^2+z^2+t^2 < 1\}$. The boundary of D^4 is $S^3 = \{(x,y,z,t) \in \mathbb{R}^4 \,|\, x^2+y^2+z^2+t^2 = 1\}$. The following lemma is the crucial observation.

Lemma. The intersection of the planes with S^3, $(P_1 \cup P_2) \cap S^3$, is a Hopf link.

Proof. We will be able to see this clearly by considering the slices of $\mathbb{R}^4$ given as t, which we can heuristically think of as the time coordinate, varies. At each $t \in [-1,1]$, we see a subset of $\mathbb{R}^3$ which corresponds to the intersection of S^3 with that $\mathbb{R}^3$-slice of $\mathbb{R}^4$. Within this we see the intersections of P_1 and P_2 with S^3. It is clear that the intersection of each of the planes is individually a circle, the boundary of the unit disc in each plane, which we call C_1 and C_2. We need to see that they are linked.

At $t = 0$, $S^3 \cap \{(x,y,z,t) \in \mathbb{R}^4 \,|\, t = 0\}$ is the unit sphere $\{(x,y,z) \,|\, x^2+y^2+z^2 = 1\}$. C_1 is then the unit circle within this sphere given by $z = 0$. The intersection of C_2 with this sphere is the two points $(0,0,\pm 1)$.

Now, for each $t \in [-1,1]$, the slice of S^3 which we see is the sphere of radius $\sqrt{1-t^2}$. For $t \neq 0$, this sphere is disjoint from C_1. We see two points of C_2 in this sphere, the points $(0,0,\pm\sqrt{1-t^2})$. Of course, for $t = \pm 1$, the sphere has radius 0 so is a single point, which is also a point of C_2.

Putting all this together, we have at $t = 0$ a 2-sphere. Then, as t ranges from 0 to $+1$ and from 0 to -1, this 2-sphere becomes bounded on either side by a 3-ball: in this way we get the whole of S^3, since $S^3 = D^3 \cup_{S^2} D^3$ glued together using the identity map $\mathrm{Id}\colon S^2 \to S^2$. C_1 lives on this equatorial 2-sphere as the equatorial 1-sphere. C_2 lives in each D^3 as the straight line which goes between the two points $(0, 0, \pm 1)$ and passes through the origin. When these two straight lines are glued together they of course form a circle, C_2.

These two circles are easily seen to link once: choose a disc with boundary C_1 and count the transverse intersections of C_2 with this disc algebraically to get ± 1 depending on the choice of orientations, which is handed down from the orientations of the planes P_1 and P_2.

We remark that these two circles are fibres of the Hopf map $S^3 \to S^2$, given by:

$$f\colon S^3 \cong \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} \to \mathbb{CP}^1 \cong S^2$$

$$(z_1, z_2) \mapsto [z_1; z_2].$$

With this map $C_1 = f^{-1}([1; 0])$ and $C_2 = f^{-1}([0; 1])$. ■

Now, the Hopf link in S^3 is the boundary of a Seifert Surface, namely a band or annulus $S^1 \times D^1$ with a full twist in it. We can therefore alter the immersed surface by removing the two open discs which coincide with a neighbourhood $\mathring{D}^4$ of the intersection point p, as described above, and instead replace them with the annulus whose boundary is the Hopf link.

This process, performed at each double point of an immersion, gives us an embedded surface. Since this only changes the surface in the D^4s which are the neighbourhoods of the intersection points, and since $H_2(D^4) \cong 0$, the resulting embedding surface represents the same homology class as the original immersed sphere.

Remark. Another application of the lemma is that a knot with unknotting number 1 has slice genus at most 1: if we follow the isotopy to undo the knot this produces an immersed disc in D^4 with a single double point. Removing this using the lemma gives an embedded surface of genus 1 in D^4.