FAKE TORI

Wu-Chung Hsiang and Julius L. Shaneson

A basic problem in geometric topology is the classification of piecewise linear (PL) manifolds of a given homotopy type. This paper considers this problem for one of the simplest of all homotopy types, that of the n-dimensional torus $T^n = S^1 \times \cdots \times S^1$.

Theorem A. *Let* F *be a free abelian group of rank n. Let* $GL(\mathsf{F})[\cong GL(n, \mathsf{Z})]$ *be the group of automorphisms of* F. *Then for* $n \geq 5$, *the set of* PL *equivalence classes of* PL *manifolds of the homotopy type of* T^n *is in one-to-one correspondence with the set of orbits of* $(\Lambda^{n-3}\mathsf{F}) \otimes \mathsf{Z}_2$ *under the natural action of* $GL(\mathsf{F})$. *The standard torus corresponds to the zero element under this action.*

For example, there exist several nonstandard 5-dimensional tori. We call a nonstandard PL torus a *fake torus*. Results concerning the smooth fake tori can also be obtained, but we do not consider them here.

The somewhat surprising conclusion of Theorem A is perhaps of interest as an application of the tools that have been developed during the past few years for the analysis of nonsimply connected manifolds. The methods involved in the proof of Theorem A also apply to questions posed by Kirby in relation to the annulus conjecture, the triangulation problem for topological manifolds, and the Hauptvermutung. For example, a given fake torus can be covered by those tori which are not as counterfeit. In particular, we have

Theorem B. *For* $n \geq 5$ *every fake torus is covered by the standard torus.*

Kirby [18] has shown that Theorem B implies the stable homeomorphisms conjecture for dimension greater than four. Thus this conjecture and its important corollary, the annulus conjecture, are true in dimensions greater than four.

On the other hand, coverings of fake tori corresponding to subgroups of the fundamental group of odd index are still fake. This fact has been used to show that the Hauptvermutung for manifolds is false ([19] and [36]).

Kirby also originally posed the following question:

QUESTION $S(k, n)$. Let $h: D^k \times T^n \to W^{n+k}$ be a homeomorphism of PL manifolds whose restriction to the boundary is PL, where $D^k = k$-disk. Is there a PL homeomorphism $f: D^k \times T^n \to W^{n+k}$ that agrees with h on the boundary?

Kirby [17] (also see [19]) showed that an affirmative answer to $S(k, n)$ would imply the Hauptvermutung for PL manifolds and the existence of combinatorial triangulations for topological manifolds. In fact, it would suffice to have an affirmative answer to the weaker question $\tilde{S}(k, n)$, in which the conclusion is asked to be true only after passage to finite covers.

Theorem C. *Let* $k + n \geq 5$. *For* $k \geq 4$, $S(k, n)$ *has an affirmative answer. The answers to* $\tilde{S}(1, n)$ *and* $\tilde{S}(2, n)$ *are affirmative. For* $\tilde{S}(3, n)$ *there is at most a* Z_2*-obstruction.*

Of course, it follows from the disproof of the Hauptvermutung that $\tilde{S}(3, n)$ is false. Thus there is an essentially unique counterexample to $\tilde{S}(3, n)$.

We actually give a result formally stronger than Theorem C in that it applies when $h = D^k \times T^n \to W^{k+n}$ is only a homotopy equivalence that is PL on the boundary. Theorem C plays a role in the work of Kirby–Siebenmann [19] and Lashof–Rothenberg [23] on triangulation of manifolds and the Hauptvermutung. Using this work one can improve upon, reformulate, and generalize the results of this paper. For example, one can show that every topological manifold of the homotopy type of T^n, $n \geq 5$, is homeomorphic to T^n [48].

The main theorems in this paper were gotten independently by C. T. C. Wall [47] and A. Casson. We announced the results in [14].

The paper is organized as follows: First we review the splitting theorem of [13] and its application to Wall's surgery obstruction groups given in [35]. Then, via Sullivan's reformulation of the Browder–Novikov theory, which we also review, we prove Theorems A and B and related results in detail. Next, we study manifolds of the homotopy type of $S^k \times T^n$, S^k the standard k-sphere. As a consequence of this study, we derive Theorem C in the case $k + n \geq 6$. We also show in a final section how to derive Theorem C (and a theoretically stronger result concerning homotopy equivalence as well as homeomorphism) via the relative Browder–Novikov–Sullivan theory (this point of view is due to Wall) and we indicate a comparison of the two approaches.

1. A Splitting Theorem for Manifolds with $\pi_1 = G \times \mathsf{Z}$

Let G be a finitely presented group and let Z denote the group of integers (written multiplicatively). In this section we review the splitting theorem of

[13] for the special case of orientable manifolds with fundamental group $G \times Z$.

Let $Z(G)$ be the integral group ring of G. Let $C(G)$ denote the reduced Grothendieck group on pairs $[P, \nu]$, P a projective (right) $Z(G)$-module, and ν a nilpotent endomorphism of P. Addition in $C(G)$ is by exact sequences: If

$$0 \to (P_1, \nu_1) \to (P, \nu) \to (P_2, \nu_2) \to 0$$

is exact, we say

$$[P, \nu] = [P_1, \nu_1] + [P_2, \nu_2].$$

The "forgetting functor" that forgets the nilpotent endomorphism induces a split epimorphism

$$C(G) \to \tilde{K}_0(G),$$

where $\tilde{K}_0(G)$ is the projective class group of G. Let $\tilde{C}(G)$ be the kernel of this epimorphism. $\tilde{C}(G)$ here is just $\tilde{C}(G, \text{id})$ of [13]. Let Wh(G) be the Whitehead group G, with conjugation $x \to x^*$ induced by group inversion.

Theorem 1.1 ([3]). *There is a direct decomposition*

$$\text{Wh}(G \times Z) = \text{Wh}(G) \oplus \tilde{K}_0(G) \oplus \tilde{C}(G) \oplus \tilde{C}(G).$$

REMARKS. 1. The factor Wh(G) is induced by the inclusion.

2. The conjugation of Wh$(G \times Z)$ leaves the factors Wh(G), $\tilde{K}_0(G)$ invariant and interchanges the two copies of $\tilde{C}(G)$.

3. There is a generalization to twisted extensions [13].

By projecting on, say, the first copy of $\tilde{C}(G)$ and using the decomposition $C(G) = \tilde{C}(G) \oplus \tilde{K}_0(G)$, we get a map $p : \text{Wh}(G \times Z) \to C(G)$.

Let Q^n be a smooth or PL manifold of dimension n, $n \geq 6$. Let $L \subseteq Q$ be a (locally flat) submanifold of codimension 1, with $\partial L = \partial Q \cap L$, ∂L meeting ∂Q transversely. Suppose $\pi_1 L = G$, $\pi_1 Q = G \times Z$, and the map induced by inclusion is the natural inclusion of G into $G \times Z$. Suppose given a homotopy equivalence

$$\varphi : (M, \partial M) \to (Q, \partial Q),$$

transverse to L and ∂L. Assume in addition that φ restricts to a homotopy equivalence of $(\partial M, \varphi^{-1}(\partial L))$ with $(\partial Q, \partial L)$; in this case we say $\varphi|\partial M$ is split along ∂L. We say φ is *splittable* along L if φ is homotopic relative to ∂M to ψ such that

1. ψ is transverse to L; and
2. if $N = \psi^{-1}(L)$, $\psi : (M, N) \to (Q, L)$ is a homotopy equivalence.

Theorem 1.2 ([13]). *The map φ is splittable along L if and only if $p(\tau(\varphi)) = 0$, where $\tau(\varphi) \in \text{Wh}(G \times Z)$ is the torsion of φ.*

REMARKS. 1. Actually one only needs $(Q, \partial Q)$ to be a finite Poincaré pair of formal dimension n and $(L, \partial L) \subseteq (Q, \partial Q)$ a codimension one sub-Poincaré pair of formal dimension $(n - 1)$ with a product neighborhood.

2. It seems that the definition of p involved an arbitrary choice between the two copies of $\tilde{C}(G)$ in the decomposition of Wh$(G \times Z)$. But if $\varphi|\partial M$ is split along ∂L as in Theorem 1.2, then it is not hard to see that

$$\tau(\varphi) = (-1)^{n+1}\tau(\varphi)^* + \xi$$

with $\xi \in \text{Wh}(G)$. Thus the component of $p(\tau(\varphi))$ in $\tilde{C}(G)$ changes by a sign at most if we change the choice involved in the definition of p.

Proposition 1.3. *If φ is splittable along L, then $\tau(\varphi)$ is in the image of* Wh(G) *under the map induced by the inclusion.*

The proof of this fact is elementary. In fact, suppose that φ is split. Let $\bar{\varphi} = \varphi|N : N \to L$ and let $\varphi_L : M_N \to Q_L$ be obtained by splitting along N and L. Then

$$\tau(\varphi) = i_*(\tau(\bar{\varphi}) - \tau(\varphi_L)).$$

Corollary 1.4. *φ is splittable if and only if $\tau(\varphi)$ comes from* Wh(G) *under the map induced from the inclusion. In particular, φ is splittable if $\tau(\varphi) = 0$.*

2. Künneth Formula for Surgery Obstructions

Let X^n be a PL manifold, $n \geq 5$. For simplicity, assume that X is orientable. Let ν be a vector bundle over X of the same fiber homotopy type as the stable normal bundle of X. Let $B_n(X, \nu)$ denote the set of cobordism classes of triples (M, φ, F), where $\varphi : (M, \partial M) \to (X, \partial X)$ is a map of degree 1 with $\varphi|\partial M : \partial M \to \partial X$ a simple homotopy equivalence; and where F is a stable trivialization of $\tau(M) \oplus \varphi^*\nu$, $\tau(M)$ the tangent bundle of M. The notion of cobordism is the obvious one; in particular, we require an s-cobordism of boundaries. If we insist only that $\varphi|\partial M : \partial M \to \partial X$ be a homotopy equivalence and require only an h-cobordism of boundaries in the definition of cobordism, we get a bordism set $B_n^h(X, \nu)$.

According to [46], there is a map

$$\theta : B_n(X, \nu) \to L_n(\pi_1 X)$$

such that $\theta(M, \varphi, F) = 0$ if and only if (M, φ, F) is cobordant to (N, ψ, G) with ψ a simple homotopy equivalence. Here $L_n(\pi_1 X)$ is an abelian group which depends functorially on $\pi_1 X$. We use the same symbol for the analogous map $\theta : B_n^h(X, \nu) \to L_n^h(\pi_1 X)$; this map vanishes on a class which contains just a homotopy equivalence. We have $L_n = L_{n+4}$, $L_n^h = L_{n+4}^h$, and if CP^2 is the complex projective space and H is the standard framing of $\tau(CP^2) \oplus$

$\nu(CP^2)$, where $\nu(CP^2)$ is the stable normal bundle of CP^2, then, by [46],

$$\theta(M, \varphi, F) = \theta(M \times CP^2, \varphi \times \mathrm{id}, F \times H).$$

(In the simply connected case this formula is due to Sullivan.)

Suppose $X = L \times S^1$, with $\pi_1 L = G$, and $n \geq 5$ or $n \geq 6$ if $\partial L \neq \emptyset$. Let (M, φ, F) represent an element of $B_n(X, \nu)$. By Corollary 1.4, $\varphi|\partial M$ is splittable along $\partial L = \partial L \times (\mathrm{pt}) \subseteq \partial L \times S^1$; let us therefore assume that $\varphi|\partial M$ is split. We may also assume that φ is transverse to L. Let $N = \varphi^{-1}L$. Let $\psi = \varphi|N : N \to L$. Then we put

$$\alpha_L(M, \varphi, F) = \theta(N, \psi, F|N)$$

an element of $L^h_{n-1}(G)$. According to [35], α_L is a well-defined cobordism invariant which vanishes on classes containing a simple homotopy equivalence.

Suppose that $L^{n-1} = K \times I$, K closed, $n \geq 7$ and $I = [0, 1]$ the unit interval. Then by [46] every element of $L_n(G \times \mathrm{Z})$ can be realized as $\theta(M, \varphi, F)$, where (M, φ, F) represents an element of $B_n(K \times I \times S^1, \nu)$ with $\varphi|\partial_- M$ a PL equivalence of $\partial_- M$ with $K \times 0 \times S^1$. We put

$$\alpha(K)[\theta(M, \varphi, F)] = \alpha_{K \times I}(M, \varphi, F).$$

Let $i : G \to G \times \mathrm{Z}$ be the inclusion.

Theorem 2.1 ([35]). *The relation $\alpha(K)$ is a well-defined homomorphism and the following sequence is exact and splits:*

$$0 \to L_n(G) \xrightarrow{i_*} L_n(G \times \mathrm{Z}) \xrightarrow{\alpha(K)} L^h_{n-1}(G) \to 0.$$

A splitting is given by taking products with a circle.

The proof of this theorem uses Corollary 1.4. We really need this result only for $G = \mathrm{Z}^n$, the free abelian group on n generators; for this case the result was given in [33]. In fact, since $\mathrm{Wh}(\mathrm{Z}^n) = 0$ by [3], we will never need to worry about the torsion.

Proposition 2.2. $\alpha(K) = \alpha(K \times CP^2)$.

This is immediate from the definitions.

We will also use the well-known computation of surgery obstruction for the simply connected case in terms of the index in dimension $4k$, the Arf–Kervaire invariant in dimension $4k + 2$, and zero in other dimensions. (See [5] and [16].) Given (M, φ, F) representing an element of $B_n(X, \nu)$, one can always define the $\frac{1}{8}$(index), Arf–Kervaire, or zero invariants, for $n \equiv 0 \pmod 4$, $n \equiv 2 \pmod 4$, or $n \equiv 1 \pmod 2$, respectively. These are cobordism invariants; in fact, they are the image in $L_n(e)$ of $\theta(M, \varphi, F)$ under the natural map of $L_n(\pi_1 X)$ onto $L_n(e)$. By abuse of language, we call these invariants "simply connected surgery obstructions." They satisfy the product formula of Sullivan (see [5]). We write $I(M, \varphi, F)$ for the index obstruction.

3. Browder–Novikov–Sullivan Theory

Let M^n be an orientable PL manifold. Then by $ht(M, \partial M)$ we denote the equivalence classes of (simple) homotopy triangulations of M relative to the boundary. A (simple) homotopy triangulation is a (simple) homotopy equivalence $h:(K, \partial K) \to (M, \partial M)$ of PL manifolds with $h|\partial K : \partial K \to \partial M$ a PL homeomorphism. The triangulation h is equivalent to $h' : (K', \partial K') \to (M, \partial M)$ if there exists a PL homeomorphism $f : (K, \partial K) \to (K', \partial K')$ with $h' \circ f$ homotopic to h relative to ∂K. If $\partial M = \emptyset$, we write $ht(M)$ for $ht(M, \partial M)$. The distinction between homotopy and simple homotopy triangulations will not arise in the cases we consider.

Let $h : (K, \partial K) \to (M, \partial M)$ be a homotopy triangulation. Let g be a homotopy inverse with $(g|\partial M) = (h|\partial K)^{-1}$. Let N be a large integer, and let $\mathrm{R}^N = N$-dimensional euclidean space. Let $\bar{g} = M \to K \times \mathrm{R}^N$ be an embedding that approximates $(g, 0)$ and equals $(g, 0)$ on ∂M. Let v be the normal bundle of the embedding $\bar{g}$. $v|\partial M$ is obviously trivial. By engulfing or the PL weak h-cobordism theorem [15], we can find a PL equivalence $c : E(v) \to K \times \mathrm{R}^N$ whose restriction to the zero section agrees with $\bar{g}$ and whose restriction to $E(v|\partial M)$ agrees with $g \times$ (identity). The composite $(h \times \mathrm{id}) \circ c$ is a fiber homotopy trivialization of v that is PL on $v|\partial M$. Let $\mathrm{G}/\mathrm{P}L$ be the classifying space for fiber homotopy trivializations of PL bundles ([4] and [30]). Then $(h \times \mathrm{id}) \circ c$ determines an element $\eta(h)$ in $[M/\partial M; \mathrm{G}/\mathrm{P}L]$, the group of homotopy classes of maps of $M/\partial M$ into the H-space $\mathrm{G}/\mathrm{P}L$. This is Sullivan's definition of "normal invariant" or "characteristic $\mathrm{G}/\mathrm{P}L$-bundle" of h. It depends only on the equivalence class of h, and so defines a map

$$\eta : ht(M, \partial M) \to [M/\partial M; \mathrm{G}/\mathrm{P}L].$$

Let $\xi \in [M/\partial M, \mathrm{G}/\mathrm{P}L]$. Then we can define a surgery obstruction $S_M(\xi) \in L_n(\pi_1 M)$. Choose a representative fiber homotopy trivialization

$$\begin{array}{ccc} E(v) & \xrightarrow{t} & M \times \mathrm{R}^N \\ & {\scriptstyle p_v}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & M & \end{array}$$

for ξ, with $t|E(v|\partial M)$ an equivalence of PL bundles. Since t is a proper map, we can change it by a homotopy relative to the boundary to a new map t, transverse to $M \times 0$. Let $K = (t_1)^{-1}(M)$. Let $h = p_v|K$. Via t we get a trivialization of the normal bundle of K in $E(v)$. The tangent bundle of $E(v)$ is $p_v^*(v \oplus \tau(M))$. So the trivialization determines a stable bundle map of $\nu(K)$, the tangent bundle of K, to $v \oplus \tau(M)$, covering h. So if $\nu(\xi)$ is a bundle stably equivalent to $-(v \oplus \tau(M))$, then we have a stable framing F of $\tau(M) \oplus h^*\nu(\xi)$. Thus (M, h, F) represents an element of $B_n(M, \nu(\xi))$; call it

$\beta(\xi)$. It is easy to see that $\beta(\xi)$ really is well defined in terms of ξ. We put

$$S_M(\xi) = \theta(\beta(\xi)).$$

When no confusion is possible, we write $S = S_M$. To get a definition in low dimensions, we put $S_M(\xi) = \theta(\beta(\xi) \times CP^2)$, if $\dim M \leq 5$. This is consistent with the above definition.

Proposition 3.1 ([44]; see also [46]). $S^{-1}(0) = \operatorname{Im} \eta$, $n \geq 5$. *For all* n, $S \circ \eta = 0$.

Now G/PL is an H-space, so $[M/\partial M, \mathrm{G}/PL]$ is an abelian group. Unfortunately, S is not in general a homomorphism. But suppose $M = L \times D^k$ $(k \geq 1)$. Then $M/\partial M = \Sigma^k(L_+)$, where L_+ is the union of L with a disjoint point. So in this case $[M/\partial M, \mathrm{G}/PL]$ also receives a group structure from the co-H-space structure of $\Sigma^k(L_+)$. It is a standard result of algebraic topology that the two structures coincide. Using the structure from $\Sigma^k L_+$, it is easy to show the following:

Proposition 3.2 ([46])

$$S : [\Sigma^k L_+, \mathrm{G}/PL] \to L_n(\pi_1 M)$$

is a homomorphism.

Next, $L_{n+1}(\pi_1 M)$ acts on $ht(M, \partial M)$. To see this, let (W, φ, F) represent an element of $B_n(M \times I, v)$, v some bundle over M. We assume $\partial W = \partial_- W \cup \partial_0 W \cup \partial_+ W$, with $h_1 = \varphi|\partial_- W : \partial_- W \to M \times 0$ and $h_2 = \varphi_2|\partial_+ W : \partial_+ W \to M \times 1$ simple homotopy equivalences, with $\partial_0 W = \partial(\partial_- W) \times I$ and $\varphi|\partial_0 W = (h_1|\partial(\partial_- W)) \times \mathrm{id}$. Let $[h_i]$ be the class of h_i in $ht(M, \partial M)$. Then we define

$$\theta(W, \varphi, F)[h_1] = [h_2].$$

Using the realization theorem of [46] (or 1.1 of [35]) and the addition theorem (1.4 of [35]), it is easy to see that this really defines an action; i.e., $(\alpha + \beta)x = \alpha(\beta x)$.

The following is straightforward:

Proposition 3.3. *Let* $x, y \in ht(M, \partial M)$. *Let* $n \geq 5$. *Then* $\eta(x) = \eta(y)$ *if and only if* x *and* y *are in the same orbit under the action of* $L_{n+1}(\pi_1 M)$.

Finally, we define $\partial : L_{n+1}(\pi_1 M) \to ht(M, \partial M)$ by $\partial\gamma = \gamma[\mathrm{id}]$.

4. Normal Invariants for Fake Tori

Theorem 4.1. *For all* n, $\eta : ht(T^n) \to [T^n, \mathrm{G}/PL]$ *is trivial.*

Corollary 4.2. *If* τ^n *is a* PL *manifold of the homotopy type of* T^n, τ^n *is stably parallelizable.*

Proof. The composite $ht(T^n) \xrightarrow{\eta} [T^n, \mathrm{G}/PL] \to [T^n, BPL]$, the second induced by the natural map, is trivial. But this carries the class of a homotopy triangulation $h : \tau^n \to T^n$ to the classifying map for the stable normal bundle of τ^n.

Corollary 4.3. *If* τ^n *is a* PL *manifold of homotopy type of* T^n, τ^n *is smoothable,* $n \geq 5$.

Proof. $\partial : L_{n+1}(\pi_1 T^n) \to ht(T^n)$ is onto, and the surgery obstructions have smooth realizations.

REMARK. Actually, smooth fake tori are (smoothly) stably parallelizable. For $[T^n, BO] \to [T^n, BPL]$ is monomorphic. This can be seen as a consequence of the facts that $\pi_i(BO) \to \pi_i(BPL)$ is monomorphic for all i, BO and BPL are infinite loop spaces, and T^n suspends to a wedge of spheres.

Theorem 4.1 actually follows from the "characteristic variety theorem" of Sullivan [44], the fact that the product of a characteristic variety with CP^2 is still a characteristic variety, and Theorem 1.2. One can also use a part of the proof of the characteristic variety theorem to compute the map $S : [T^n, \mathrm{G}/PL] \to L_n(\pi_1 T^n)$. The Hirzebruch classes $l_i \in H^i_4(\mathrm{G}/PL; Q)$ and the Kervaire classes $k_i \in H^{4i+2}(\mathrm{G}/PL; Z_2)$, defined by Sullivan, are involved. However, the extreme regularity of T^n allows one to give an inductive proof of Theorem 4.1 that does not require such deep knowledge of the homotopy theory of G/PL. We give such a proof. The possibility of an inductive proof was suggested to us by J. Levine.

We show by induction on n that if $\xi \in [T^n, \mathrm{G}/PL]$, $S(\xi) = 0$ only if $\xi = 0$. For $n = 1$, this is trivial, since $\pi_1(\mathrm{G}/PL) = 0$. So let $n \geq 2$, and suppose that the result is true for $n - 1$. Let $T^{n-1} \subseteq T^n$ be a standard subtorus (i.e., one obtained by holding one coordinate fixed). Let $i : T^{n-1} \subseteq T^n$ be the inclusion. Let $\pi : T^n \times CP^2 \to T^n$ and $\pi_1 : T^{n-1} \times CP^2 \to T^{n-1}$ be the standard projections. Let $\xi \in [T^n, \mathrm{G}/PL]$ be an element with $S_{T^n}(\xi) = 0$ in $L_n(\pi_1 T^n)$. Then $S_{T^n \times CP^2}\pi^*(\xi) = 0$. Hence there exists a homotopy equivalence $h : W \to T^n \times CP^2$ with $\eta(h) = \pi^*\xi$. By Theorem 1.2, h is splittable along $T^{n-1} \times CP^2$. Let us assume that h is split and let $Q = h^{-1}(T^{n-1} \times CP^2)$. Let $f = h|Q : Q \to T^{n-1} \times CP^2$. Then it is easy to see that

$$\eta(f) = (i \times 1)^*\pi^*(\xi).$$

But $\pi \circ (i \times 1) = i \circ \pi_1$. Hence $S_{T^{n-1} \times CP^2}(\pi_1^* i^*(\xi)) = 0$. By periodicity of surgery obstructions or by definition in low dimensions, this means that $S_{T^{n-1}}(i^*\xi) = 0$. Hence, by the inductive hypothesis, $i^*\xi = 0$.

Thus the restriction of ξ to every standard subtorus is trivial. But G/PL is an infinite loop space [4]; hence there is a space Y such that, as sets,

$$[T^n, G/PL] = [T^n, \Omega^2 Y] = [\Sigma^2 T^n, Y].$$

Moreover, $\Sigma^2 T^n$ is a one-point union of spheres, up to homotopy. It now follows easily that ξ is trivial on the $(n-1)$-skeleton of T^n.

Thus if $C : T^n \to S^n$ is the map that collapses the $(n-1)$-skeleton, then $\xi = C^*\mu$, $\mu \in \pi_n(G/PL)$. It is not hard to see that the following diagram commutes:

$$\begin{array}{ccc} [T^n, G/PL] & \xrightarrow{S} & L_n(\pi_n T^n) \\ \uparrow{\scriptstyle C^*} & & \uparrow \\ [S^n, G/PL] & \xrightarrow{S} & L_n(e) \end{array}$$

The unlabeled map is induced by the inclusion of the trivial group $\{e\}$. But the lower S is an isomorphism in all dimensions $n \neq 4$; for $n = 4$, it is still a monomorphism ([31] and [44]). By general nonsense, the unlabeled map is monomorphic. Hence $\mu = 0$ and so $\xi = 0$. This completes the inductive step and proves Theorem 4.1.

5. Classification of $ht(T^n)$

We identify Z^n with $\pi_1 T^n$. Fixing a standard orientation of S^1, the various circles in $T^n = S^1 \times \cdots \times S^1$ determine a basis $t_1, \ldots, t_n$ for Z^n. This notation is fixed through Section 9.

Proposition 5.1. *The set $ht(T^n)$ is in one-to-one correspondence with the quotient of $L_{n+1}(Z^n)$ by the subgroup of elements acting trivially on* [id] (i.e., *by the subgroup of x with $\partial(x) =$* [id]).

Proof. Immediate from Theorem 4.1 and Proposition 3.2.

Let $J \subseteq \{1, \ldots, n\}$. Let $|J|$ denote the number of elements of J. Corresponding to each J there is a standard subtorus $T(J) \subseteq T^n$, where $T(J) = \{(x_1, \ldots, x_n) | x_i = \text{the base point of } S^1 \text{ for } i \notin J\}$. If $H = \{1, \ldots, n\} - J$, we have a canonical decomposition $T^n = T(J) \times T(H)$.

For each J with $m = |J| \equiv 1 \pmod 2$, we are going to define an element $\xi(J, n) \in L_{n+1}(Z^n)$. When no confusion is possible, we write $\xi(J) = \xi(J, n)$. First consider the case $m \geq 5$. Choose a generator of the cyclic group $L_{m+1}(e)$; call it 1. [This is Z if $m + 1 \equiv 0 \pmod 4$, Z_2 if $m + 1 = 2 \pmod 4$.] Let (M, h, F) represent an element of $B_{m+1}(D^{m+1}, \varepsilon)$, ε the trivial bundle. By the realization theorem (plumbing) of Kervaire–Milnor for simply connected surgery [5], we can take $\theta(M, h, F)$ to be the chosen generator. Note that by the generalized Poincaré conjecture, ∂M is PL homeomorphic to S^m.

Let K be obtained from $T(J) \times I$ and M by taking a boundary connected sum along $T(J) \times 1$, the upper boundary of $T(J) \times I$. We write $K = (T(J) \times I) \amalg M$. (The notation is due to Browder [5].) Similarly, we can take connected sums of maps and framings to get $f : K \to (T(J) \times I) \amalg D^{m+1} = T(J) \times I$ and E, a stable framing of $\tau(K) \oplus f^*v$, v the stable normal bundle of $T(J) \times I$. We write

$$(K, f, E) = (T(J) \times I) \amalg (M, h, F).$$

Using the definitions of even-dimensional nonsimply connected surgery obstructions in [45], it is not hard to see that $\theta(K, f, E)$ is the image of the chosen generator of $L_{m+1}(e)$ in $L_{m+1}(\pi_1(T(J))$ under the map induced by inclusion. Let $H = \{1, \ldots, n\} - J$. Let D be the standard framing of $\tau(T(H)) \oplus \nu(T(H))$. Define

$$\xi(J) = \theta(K \times T(H), f \times 1, E \times D).$$

Next consider the case $|J| = m = 1$. There is an element $(S^1 \times S^1, h, F)$, representing a class in $B_2(S^2, \varepsilon)$ with $\theta(S^1 \times S^1, h, F) \neq 0$. The existence of this class follows from the Thom–Pontrjagin construction applied to the nonzero element of $\pi_2(G)$. Alternatively, it is easy to construct the "wrong framing" F on $S^1 \times S^1$ and show that it gives a nonzero Kervaire invariant. To define $\xi(T)$ in this case, we proceed as above, except that we take a connected sum with $T(J) \times I$ in the interior. We omit the details.

Finally, let $|J| = M = 3$. We cannot make the same definition, essentially because there is no almost parallelizable PL manifold of index 8. Instead, let (M, h, F) represent an element of $B_8(D^8, \varepsilon)$ with $\theta(M, h, F)$ a generator of $L_8(e)$. Let

$$(K, f, E) = (T(J) \times I \times CP^2) \amalg (M, h, F).$$

Let $\xi(J) = \theta(K \times T(H), f \times 1, E \times D)$ where $H = \{1, \ldots, n\} - J$ again. Then $\xi(J)$ is an element of $L_{m+4+1}(Z^n) = L_{m+1}(Z^n)$.

Of course, one can propose a similar definition for all the $\xi(J)$. Using the periodicity of (simply connected) surgery obstructions it is not hard to see that this would change nothing.

Similarly, one can show that to obtain $2\xi(J)$, $|J| = 3$, one begins with (M, h, F) representing an element of $B_4(D^4, \varepsilon)$ with $\theta(M, h, F)$ twice a generator of $L_4(e)$, takes the boundary connected sum with $T(J) \times I$, takes the product of the result with the complementary $T(H)$, and evaluates the surgery obstruction of the result. The existence of (M, h, F) follows easily from the existence of an almost-parallelizable smooth or PL manifold of index 16 [29].

Lemma 5.2. *Every element of $L_{n+1}(Z^n)$ has a unique expression $\Sigma\beta(J)\xi(J)$, where the sum is over $\varnothing \neq J \subseteq \{1, \ldots, n\}$ with $|J| \equiv 1 \pmod 2$, and where $\beta(J) \in Z$ if $|J| \equiv -1 \pmod 4$ and $\beta(J) \in Z_2$ if $|J| \equiv 1 \pmod 4$.*

For $|J| \neq 3$, it is apparent that $\xi(J)$ acts trivially on $[\mathrm{id}] \in ht(T^n)$. For $|J| = 3$, it is clear from the paragraph preceding Lemma 5.2 that $2\xi(J)$ acts trivially.

Lemma 5.3. *The subgroup of* $L_{n+1}(\mathbf{Z}^n)$ *generated by* $\{\xi(J) \mid |J| \neq 3\} \cup \{2\xi(J) \mid |J| = 3\}$ *is precisely the kernel of* ∂ (i.e., *the subgroup of elements acting trivially on* [id]).

We will prove these two lemmas in the next section. From these we see the following immediately:

Theorem 5.4. $\partial : L_{n+1}(\mathbf{Z}^n) \to ht(T^n)$ *induces a one-to-one correspondence of* $\Lambda^{n-3}\mathbf{Z}^n \otimes \mathbf{Z}_2$ *with* $ht(T^n)$.

We will discuss this correspondence in more detail in Sections 6 and 7. Here we note that if $h : \tau^n \to T^n$ represents a nonzero element of $ht(T^n)$, then τ^n cannot be PL homeomorphic to T^n.

6. Proof of Lemmas

Suppose $J \subseteq H \subseteq \{1,\ldots, n\}$ with $m = |J| \equiv 1 \pmod 2$. Let $n \geq 5$ from now on. Let $k = |H|$. Then exactly as for $\xi(J, n)$, we define $\xi(J, H) \in L_{k+1}(\pi_1(TH))$; we take the product with a complementary torus to $T(J)$ in $T(H)$ rather than in T^n. In particular, if $H = \{1,\ldots, n\}$, $\xi(J, H) = \xi(J, n)$. If $H = J$, $\xi(J, J)$ is the image of a generator of $L_{m+1}(e)$ under the map induced by inclusion.

Suppose $J \subseteq H \subseteq \{1,\ldots, n\}$ with $|J| = |H| - 1$. Let $m = |J|$. Then, by Theorem 2.1, we have a well-defined map

$$\alpha(T(J) \times CP^2) : L_{m+6}(\pi_1(T(H))) \to L_{m+5}(\pi_1(T(J))).$$

We write $\alpha(T(J) \times CP^2) = \alpha(J, H)$, which we view as a map of $L_{m+2}(\pi_1(T(H))$ onto $L_{m+1}(\pi_1(T(J))$. For $m \geq 6$, we can define $\alpha(T(J))$, and, by Proposition 2.2,

$$\alpha(T(J)) = \alpha(T(J) \times CP^2) = \alpha(J, H).$$

To simplify the notation, we write $\pi_1(J)$ for $\pi_1(T(J))$, for example.

For arbitrary $J \subseteq H \subseteq \{1,\ldots, n\}$, choose the unique sequence

$$J = J_0 \subset J_1 \subset \cdots \subset J_k = H$$

with $|J_i| = |J_{i-1}| + 1$ and with $\max(J_i - J) < \max(J_{i+1} - J)$. Define

$$\alpha(J, H) = \alpha(J_0, J_1) \circ \cdots \circ \alpha(J_{k-1}, J_k) = \prod_{i=1}^{k} \alpha(J_{i-1}, J_i).$$

We put $\alpha(J, J) =$ identity, and if $H = \{1,\ldots, n\}$, we write $\alpha(J,H) = \alpha(J, n)$. When no confusion is possible, we write $\alpha(J)$ for $\alpha(J, n)$.

The order of the deletion to get from H to J really is not relevant to the definition of $\alpha(J, H)$. We will show this algebraically in Section 8, the only place we will need to use this fact. One could also prove it geometrically, straight from the definitions.

By $w(J)$ we denote the natural projection of $L_{|J|+1}(\pi_1(J))$ on $L_{|J|+1}(e)$.

Lemma 6.1. *For* $J \subseteq K \subseteq H$, $|J| \equiv 1 \pmod 2$,

$$\alpha(K, H)\xi(J, H) = \xi(J, K).$$

Proof. Immediate from the definitions.

Lemma 6.2. *For* $K \subseteq H$, $J \subseteq H$ *but* $K \not\supseteq J$, $|J| \equiv 1 \pmod 2$,

$$\alpha(K, H)\xi(J, H) = 0.$$

Proof. Let $K = K_0 \subset \cdots \subset K_k = H$ be as in the definition of $\alpha(K, H)$. Let i be the first integer such that $K_{i+1} \supseteq J$. Then $\alpha(K, H) = \alpha(K, K_i) \circ \alpha(K_i, K_{i+1}) \circ \alpha(K_{i+1}, H)$. By Lemma 6.1, $\alpha(K_{i+1}, H)\xi(J, H) = \xi(J, K_{i+1})$. It suffices to show that $\alpha(K_i, K_{i+1}) \circ \xi(J, K_{i+1}) = 0$.

So we may as well take $K_{i+1} = H$ and suppose that K is obtained from J by deleting the largest element. Let L be obtained from J by deleting the same element. Recall that $\xi(J, J)$ can be defined as $\theta(W, \varphi, E)$, where $(W, \varphi, E) = (M, h, F) \amalg (T(J) \times I \times CP^2)$, (M, h, F) representing an element of $B_{|J|+5}(D^{|J|+5}, \tau)$ with $\theta(M, h, F)$ a generator of $L_{|J|+5}(e)$. We can take the boundary connected sum along a disk that misses $T(L) \times I \times CP^2 \subseteq T(J) \times I \times CP^2$. Thus we have $T(L) \times I \times CP^2 \subseteq W$, φ transverse to $T(L) \times I \times CP^2$, $\varphi^{-1}(T(L) \times I \times CP^2) = T(L) \times I \times CP^2$, and $\varphi | T(L) \times I \times CP^2$ the identity, which is a homotopy equivalence. But once we set the thing up this way, the result is a simple consequence of the definitions.

Let $\delta(H, J) = 0$ for $H \neq J$. $\delta(J, J) = 1 \in \mathbf{Z}$ if $|H| \equiv -1 \pmod 4$, $\delta(J, J) = 1 \in \mathbf{Z}_2$ if $|J| \equiv 1 \pmod 4$, $\delta(J, J) = 0$ for $|J| \equiv 0 \pmod 2$.

Proposition 6.3. *Let* $|J| \equiv 1 \pmod 2$. *Then*

$$w(H)\alpha(H, n)\xi(J, n) = \delta(H, J).$$

Proof. $w(J)\alpha(J, n)\xi(J, n) = w(J)\xi(J, J) = 1$. [Recall that 1 is the chosen generator of $L_{|J|+1}(e)$.] If $J \not\subseteq H$, then $\alpha(H, n)\xi(J, n) = 0$ by Lemma 6.2. Suppose that $J \subsetneq H$. Then $\alpha(H, n)\xi(J, n) = \xi(J, H)$. But as the index of a torus is zero, it follows from the product formula for simply connected surgery obstruction [5] that $w(H)\xi(J, H) = 0$.

REMARK. It is convenient, but not necessary, for us to use the case of Proposition 6.3 that follows from the product formula for simply connected surgery obstructions. For example, using induction on $|J|$, the reader can easily rewrite the following proof of Lemma 5.2 to avoid using Proposition 6.3. One can also prove 6.3 using 1.2 of [35].

Proof of Lemma 5.2. Take the free abelian group on subsets $J \subseteq \{1,..., n\}$ with $|J| \equiv 1 \pmod 2$, and introduce the relation $2J = 0$ if $|J| \equiv 1 \pmod 4$. Let A be the resulting abelian group. Define $\rho : L_{n+1}(Z^n) \to A$ by

$$\rho(\xi) = \Sigma(w(J)\alpha(J, n)\xi)J.$$

By Proposition 6.3, ρ is an epimorphism. By Theorem 2.1, applied inductively, $L_{n+1}(Z^n)$ and A are abstractly isomorphic finitely generated abelian groups. Hence ρ is an isomorphism. Clearly this implies Lemma 5.2.

REMARK. Using Lemma 5.2 and the fact that $\xi(J), |J| \neq 3$, and $2\xi(J), |J| = 3$, act trivially, we can get an upper-bound estimate on the size of $ht(T^n)$. We could then pass to a study of the finite covering spaces of the *possible* fake tori, using methods similar to those in Section 9 below, and prove Theorem B. Thus the annulus conjecture does not involve Lemma 5.3 and so does not use Rohlin's theorem [29], which is crucial to the proof of Lemma 5.3. A similar remark will apply to Theorem C and its consequences for the triangulation of manifolds and the Hauptvermutung.

Proof of Lemma 5.3. Suppose $\xi = \Sigma a(J)\xi(J)$ acts trivially on $[\mathrm{id}] \in ht(T^n)$. The sum is as in Lemma 5.2. We want to show that $|J| = 3$ implies $a(J) = 0$. Suppose for a particular H with $|H| = 3$, $a(H) \neq 0$. Even multiples of $\xi(H)$ act trivially, so we may as well assume $a(H) = 1$. Then we have $w(H)\alpha(H)\xi = 1$. Let us interpret this fact geometrically and derive a contradiction to Rohlin's theorem [29].

Let (W, φ, F) represent an element $B_{n+1}(T^n \times I, \varepsilon)$ with $\theta(W, \varphi, F) = \xi$ and with $\varphi|\partial_- W : \partial_- W \to T^n \times 0$ a PL equivalence. Since we are supposing ξ acts trivially on [id], we may also assume that $\varphi|\partial_+ W : \partial_+ W \to T^n \times 1$ is a PL equivalence. But then $\varphi|\partial_- W$ and $\varphi|\partial_+ W$ are obviously transverse to the various subtori of $T^n \times 0$ and $T^n \times 1$, and the restrictions to inverse images are, in fact, PL homeomorphisms. As a result, we can successively peel off the circles complementary to $T(H) \times I$ without invoking the splitting theorem, 1.2. In particular, there is no necessity to take the product with CP^2 in order to raise the dimensions enough to ensure the applicability of this theorem.

Thus we obtain a triple (P, f, E) representing an element of $B_4(T(H) \times I, \varepsilon)$ with the following properties:

1. $f|\partial_- P : \partial_- P \to T(H) \times 0$ and $f|\partial_+ P : \partial_+ P \to T(H) \times 1$ are PL equivalences;
2. $\theta((P, f, E) \times CP^2) = \alpha(H)\xi \in L_8(\pi_1(H))$.

But we have

$$I((P, f, E) \times CP^2) = w(H)\theta((P, f, E) \times CP^2).$$

So

$$I((P, f E) \times CP^2) = w(H)\alpha(H)\xi = a(H) = 1.$$

By periodicity of simply connected surgery obstructions, $I(P, f, E) = 1$, the generator of $L_4(e)$. Identify $T(H)$ with T^3. We can assume after modification of f, if necessary, that $f : P \to T^3 \times I$ induces a π_1 isomorphism [45]. Let $K_2(P)$ be the kernel of $f_* : H_2(P) \to H_2(T^3 \times I)$. Then the intersection numbers define a nonsingular bilinear form $B_f(x, y)$ on $K_2(P)$, and $I(P, f, E) = \frac{1}{8}$ Index B_f ([5] and [45]). So Index $B_f = 8$.

View T^3 as $T^2 \times S^1$ and glue copies of $T^2 \times D^2$ along $\partial_- P$ and $\partial_+ P$ using $f|\partial_- P$ and $f|\partial_+ P$, respectively, for identifications of boundaries. Let W be the manifold obtained. The union of f and the identity on the copies of $T^2 \times D^2$ defines a map

$$g : W \to (T^2 \times D^2) \cup (T^3 \times I) \cup (T^2 \times D^2)$$
$$= T^2 \times S^2.$$

It is obvious from Mayer–Vietoris sequences that $K_2(P)$ is isomorphic with $K_2(W)$ under the map induced by inclusion $P \subseteq W$. It is also clear that $B_f(x, y) = B_g(x, y)$. So Index B_f = Index $B_g = 8$. But it is easy to see ([5], for example) that Index B_g = Index W − Index$(T^2 \times S^2)$. Clearly, Index$(T^2 \times S^2) = 0$. So W has index 8.

Using excision and the Künneth formula, it is easy to see that $H^1(W, P; Z_2) = 0$ and that $\delta : H^1(P; Z_2) \to H^2(W, P; Z_2)$ is onto. Hence $H^i(W; Z_2) \to H^i(P; Z_2)$ is monomorphic for $i = 1, 2$. Let τ be the tangent bundle of W; then $\tau|P$, the tangent bundle of P, is trivial. So $w^1(\tau|P)$ and $w^2(\tau|P)$, the first and second Stiefel–Whitney classes, vanish. Hence $w^1(W) = w^2(W) = 0$ also, by naturality of Stiefel–Whitney classes.

So finally we have produced a closed, orientable PL 4-dimensional manifold W of index 8 and with vanishing second Stiefel–Whitney class. Since $\Gamma_1 = \Gamma_2 = \Gamma_3 = 0$, W is smoothable [21]; so we get a smooth, almost parallelizable, closed 4-dimensional manifold of index 8. This contradicts the theorem of Rohlin [29]. So ξ cannot act trivially on [id]. This completes the proof of Lemma 5.3.

7. Geometric Description of Invariants

In this section we give a geometric description for the complete set of invariants for $ht(T^n)$, $n \geq 5$. Let x be an element of $ht(T^n)$. Let $J \subseteq \{1,..., n\}$, $|J| = 3$, and choose the sequence $J = J_0 \subset \cdots \subset J_k = \{1,..., n\}$ as in the beginning of Section 6. Let (W, φ, F) represent an element of $B_{n+1}(T^n \times I, \varepsilon)$, with $\varphi|\partial_- W : \partial_- W \to T^n \times 0$ a PL equivalence and $\varphi|\partial_+ W : \partial_+ W \to T^n \times 1$ representing x; this is possible by Theorem 4.1.

Consider $(W, \varphi, F) \times CP^2$. Let $\bar{f}_k = \mathrm{id} \times \varphi$. Then by Theorem 1.2 and the homotopy extension property, $\bar{f}_k$ is homotopic relative to $\partial_- W$ to f_k

such that $f_k|\partial_+ W$ transverse to $T(J_{k-1}) \times 1 \times CP^2$, with the restriction of f_k inducing a homotopy equivalence of $(f_k|\partial_+ W)^{-1}(T(J_{k-1}) \times 1 \times CP^2)$ with $T(J_{k-1}) \times 1 \times CP^2$. In addition, we can assume that f_k is transverse to $T(J_{k-1}) \times I \times CP^2$. Let $W_{k-1} = f_k^{-1}(T(J_{k-1}) \times I \times CP^2)$, let $f_{k-1} : W_{k-1} \to T(J_{k-1}) \times I \times CP^2$ be the restriction of f_k, and let F_{k-1} be the restriction of $F \times D$, where D is a standard framing of $\tau(CP^2) \oplus \nu(CP^2)$. Clearly, we can apply the same procedure to $(W_{k-1}, f_{k-1}, F_{k-1})$. Eventually, we get (W_0, f_0, F_0), representing an element of $B_8(T(J) \times I \times CP^2, \varepsilon \times \nu(CP^2))$. Let $\lambda_J(x)$ be the reduction mod 2 of the index invariant $I(W_0, f_0, F_0)$.

Theorem 7.1. *The invariants λ_J, $J \subseteq \{1,..., n\}$, are well defined and are a full set of invariants for $ht(T^n)$. Each possible collection of integers* mod 2 *indexed by $J \subseteq \{1,..., n\}$* with $|J| = 3$ *is realized as $\{\lambda_J(x)\}$,* for *some $x \in ht(T^n)$.*

Proof. Let $\xi = \theta(W, \varphi, F)$. Then, clearly

$$\lambda_J(x) = w(J)\alpha(J)\xi \pmod 2.$$

So, if $\xi = \Sigma a(H)\xi(H)$, the sum being as in Lemma 5.2, $a(J)$ reduced mod 2 is just $\lambda_J(x)$. The result follows.

These invariants satisfy a certain naturality property. Consider, for example, the standard inclusion $T^{n-1} \subseteq T^n$, $n \geq 6$. Let $h : M \to T^n$ be a homotopy equivalence representing $x \in ht(T^n)$. By Theorem 1.2 we may assume that h is transverse to T^{n-1} and that if $\tau^{n-1} = h^{-1}(T^{n-1})$, $h|\tau^{n-1} : \tau^{n-1} \to T^{n-1}$ is a homotopy equivalence. Let $y \in ht(T^{n-1})$ be the class of this homotopy equivalence. Then the following is clear:

Proposition 7.2. *Let $J \subseteq \{1,..., n-1\}$, with $|J| = 3$. Then $\lambda_J(y) = 0$ if and only if $\lambda_J(x) = 0$.*

We also remark at this point that one can use methods similar to the above to get an upper bound on the number of h-cobordism classes of homotopy triangulations of T^4. There are at most four.

One cannot find an analogue of Proposition 7.2 for higher codimensions. But with a little care one can show that for $n \geq 10$, $h : \tau^h \to T^n$ is equivalent in $ht(T^n)$ to

$$f \times g : \tau^p \times \tau^q \to T^p \times T^q \qquad \text{for } p + q = n,\ p \geq 5,\ q \geq 5,$$

if and only if $\lambda_J(h) \neq 0$ only when $J \subseteq \{1,..., p\}$ or $J \subseteq \{p+1,..., n\}$. The invariants of f and g can be determined as in Proposition 7.2. The proof involves taking products with CP^2 several times and the fact that $S^1 \times CP^2$ has no nontrivial homotopy triangulations [35]. We leave the details as an exercise.

8. Proof of Theorem A

In [35] we state Theorem 2.1 for a closed manifold K, but a similar result holds for a manifold with boundary. The same proof goes through; one simply has to keep track of some extra portion of the boundary on which nothing ever happens. If K is an orientable manifold of dimension ≥ 5, with $\pi_1 K = G$, and if $l : G \to G \times \mathbf{Z}$ is the inclusion, then using the geometric interpretation [35, lemma 5.2] of the map of surgery obstruction groups induced by l, it is not hard to see that the following diagram commutes:

$$\begin{array}{ccc} L_{n+3}(G \times \mathbf{Z}) & \xrightarrow{(l \times 1)_*} & L_{n+3}(G \times \mathbf{Z} \times \mathbf{Z}) \\ \downarrow{\scriptstyle \alpha(K \times I)} & & \downarrow{\scriptstyle \alpha(K \times S^1)} \\ L_{n+2}(G) & \xrightarrow{l_*} & L_{n+2}(G \times \mathbf{Z}). \end{array}$$

From now on in this section let $n \geq 5$, and let $\mathbf{Z}^n = \pi_1 T^n$ be written additively for notational convenience with basis $t_1,..., t_n$ determined by the respective circles, and let $\mathbf{Z}(\{t_j\})_{j \in J} = \mathbf{Z}(J)$ be the subgroup generated by $\{t_j | j \in J\}$. Using the above diagram and results of Section 6, it is not hard to prove the following:

Lemma 8.1. *If $|J| = 3$, $J_1 \supset J$, $|J_1| = 4$, where $J_1 \subseteq \{1,..., n\}$, then $\xi(J, J_1)$ is the image of a generator of $L_5(\mathbf{Z}(J_1 - J))$ under the map induced by inclusion. If $J_2 \supset J_1$, $|J_2| = 5$, $J_2 \subseteq \{1,..., n\}$, then $\xi(J, J_2)$ is the image of an infinite cyclic generator of $L_6(\mathbf{Z}(J_2 - J))$ under the map induced by inclusion.*

The lemma also follows from the naturality of inclusion induced by the maps with respect to taking products with a circle. (This naturality also seems to require a geometric proof.)

Proposition 8.2. *Let $J \subseteq \{1,..., n\}$. Let $J = J_0 \subset \cdots \subset J_k = \{1,..., n\}$ with $|J_i| - 1 = |J_{i-1}|$. Then*

$$\alpha(J, n) = \alpha(J_0, J_1) \circ \cdots \circ \alpha(J_{k-1}, J_k).$$

Proof. By Lemma 5.2 it suffices to show that both sides have the same value on $\xi(H, n)$, all $H \subseteq \{1,..., n\}$, with $|H| \equiv 1 \pmod 2$. If $H \subseteq J$, we apply Lemma 6.1 to see that both sides yield $\xi(H, J)$. If $H \not\subset J$, choose the first i with $J_{i+1} \supset H$. Then by Lemmas 6.1 and 6.2,

$$\left[\prod \alpha(J_q, J_{q+1})\right]\xi(H, n) = \left[\prod_{q<i} \alpha(J_q, J_{q+1})\right]\alpha(J_i, J_{i+1})\xi(H, J).$$

But $\alpha(J_i, J_{i+1})\xi(H, J_{i+1}) = 0$ by Lemma 6.2. So in this case, both sides yield zero on $\xi(H, n)$.

Now we start the proof of Theorem A. Let $GL(n, \mathbf{Z})$ be the group of automorphisms of $\mathbf{Z}^n$. This group acts on $ht(T^n)$ as follows: for each $U \in GL(n, \mathbf{Z})$, let $\bar{U} : T^n \to T^n$ be a PL equivalence with the map induced by $\bar{U}$ on $\pi_1(T^n)$ equal to U. This determines $\bar{U}$ up to a homotopy. Given $h : M \to T^n$, a homotopy equivalence of PL manifolds, define the action of U on $[h]$ by $U[h] = [\bar{U} \circ h]$.

Every element of $ht(T^n)$ determines a PL equivalence class of manifolds of the homotopy type of T^n. Since any homotopy equivalence of T^n with itself is homotopic to a PL homeomorphism, two elements of $ht(T^n)$ are in the same orbit under $GL(n, \mathbf{Z})$ if and only if they determine the same class of manifolds.

For $J \subseteq \{1,..., n\}$, let $H = \{1,..., n\} - J$. Assume that $|J| = 3$. Let $H = \{i_1,..., i_{n-3}\}$ and let

$$t^J = t_H = (t_{i_1} \wedge \cdots \wedge t_{i_{n-3}}) \otimes 1 \in (\Lambda^{n-3}\mathbf{Z}^n) \otimes \mathbf{Z}_2.$$

For each $x \in ht(T^n)$, let

$$\lambda(x) = \sum_{|J|=3} \lambda_J(x)t^J.$$

This defines a bijection $\lambda : ht(T^n) \to (\Lambda^{n-3}\mathbf{Z}^n) \otimes \mathbf{Z}_2$, by Theorem 7.1.

Now $GL(n, \mathbf{Z})$ acts naturally on $(\Lambda^{n-3}\mathbf{Z}^n) \otimes \mathbf{Z}_2$. For example, $U(t_{i_1} \wedge \cdots \wedge t_{i_{n-3}}) = Ut_{i_1} \wedge \cdots \wedge Ut_{i_{n-3}}$. So to prove Theorem A, we need to show the following:

Theorem 8.3. *λ is $GL(n, \mathbf{Z})$-equivariant; i.e.,*

$$\lambda(Ux) = U\lambda(x) \qquad \text{for } x \in ht(T^n) \text{ and } U \in GL(n, \mathbf{Z}).$$

Now $GL(n, \mathbf{Z})$ acts on $L_{n+1}(\mathbf{Z}^n)$ by induced transformations; let U_* be the transformation induced by U. Suppose (W, φ, F) represents an element of $B_{n+1}(T^n \times I, \varepsilon)$; then it is not hard to see that

$$\theta(W, (\bar{U} \times 1) \circ \varphi, F) = U_*\theta(W, \varphi, F).$$

Hence $\partial : L_{n+1}(\mathbf{Z}^n) \to ht(T^n)$ is $GL(n, \mathbf{Z})$-equivariant.

We use the basis $t_1,..., t_n$ to represent elements of $GL(n, \mathbf{Z})$ by matrices. Every element of $GL(n, \mathbf{Z})$ is a product of a diagonal matrix and elementary matrices $I + aE_{ij} (i \neq j)$, where I is the identity matrix and E_{ij} is the matrix with 1 in the (i, j)th place and zeros elsewhere. But $(I + aE_{ij})(I + bE_{ij}) = I + (a + b)E_{ij}$. So elements of $GL(n, \mathbf{Z})$ are all products of a diagonal matrix and elementary matrices $I \pm E_{ij}$. Hence it suffices to prove equivalence of λ with respect to these elements of $GL(n, \mathbf{Z})$.

First consider $V \in GL(n, \mathbf{Z})$, $V(t_i) = \pm t_i$. Then V acts trivially on $(\Lambda^{n-3}\mathbf{Z}^n) \otimes \mathbf{Z}_2$. So we have to show, $|J| = 3$, $x \in ht(T^n)$, that $\lambda_J(Vx) = \lambda_J(x)$. Let $x = \partial\xi$. But for $\bar{V}$, we can take $\bar{V}(x_1,..., x_n) = (x_1^{\pm 1},..., x_n^{\pm 1})$, the inverse being taken in the field $\mathbf{C}$ of complex numbers. Then it is clear that the only effect of $\bar{V}$ is a possible change of orientations, so that we have $w(J)\alpha(J, n)V_*\xi = \pm w(J)\alpha(J, n)\xi$. Hence $\lambda_J(x) = \lambda_J(Vx)$.

Before proceeding further, we state a lemma. The proof is straightforward and so is omitted.

Lemma 8.4. *Let $J \subseteq H \subseteq \{1,..., n\}$. Let S be an automorphism of $\pi_1(H) = \pi_1(T(H))$. Assume that $S(t_i) = t_i$ for $i \in H - J$ and $S(\pi_1(J)) \subseteq \pi_1(J)$. Then if $\xi \in L_{|H|+1}(\pi_1(H))$,*

$$\alpha(J, H)S_*\xi = (S|\pi_1(J))_*\alpha(J, H)\xi.$$

Now consider $U \in GL(n, \mathbf{Z})$ with $U(t_j) = t_j$ for $i \neq j$ and $U(t_i) = t_i + \varepsilon t_k$, $k \neq i$, $\varepsilon = \pm 1$. We want to show that $\lambda(Ux) = U\lambda(x)$, all $x \in ht(T^n)$. But we have

$$\lambda(U(\partial(\xi + \eta))) = \lambda(\partial(U_*\xi + U_*\eta)) = \lambda(\partial U_*\xi) + \lambda(\partial U_*\eta)$$
$$= \lambda(U(\partial\xi)) + \lambda(U(\partial\eta)).$$

This follows from the fact that ∂ is $GL(n, \mathbf{Z})$-equivariant and that λ_H is induced by homomorphisms of $L_{n+1}(\mathbf{Z}^n)$ onto $\mathbf{Z}_2$, namely $w(H)\alpha(H, n)$ followed by reduction mod 2. (See the proof of Theorem 7.1.) Hence it suffices to consider $x = \partial\xi(J)$, $|J| = 3$, $J \subseteq \{1,..., n\}$, and to show that $\lambda(Ux) = U\lambda(x)$ for this x. Note that $\lambda(x) = t^J$, and recall that $\lambda_H(\partial\xi)$ is the mod 2 reduction of $w(H)\alpha(H)(\xi)$.

Case 1. $i \in J$, $k \in J$. Then $U\lambda(x) = Ut^J = t^J = \lambda(x)$; so we must show that $x = Ux$; i.e., $\lambda_H(Ux) = \lambda_H(x)$ for all H with $|H| = 3$.

Suppose that $H \neq J$. Let $H = H_0 \subset H_1 \subset \cdots \subset H_{n-3} = \{1,..., n\}$ with $|H_i| = |H_{i-1}| + 1$. Suppose first that $i \in H$. If $k \in H$ also, then by Lemmas 8.4 and 6.2,

$$\alpha(H)U_*\xi(J) = (U|\pi_1(H))_*\alpha(H)\xi(J) = 0.$$

If $k \notin H$, take $H_1 = H \cup \{k\}$. Then

$$\alpha(H)U_*\xi(J) = \alpha(H, H_1)\alpha(H_1)U_*\xi(J)$$
$$= \alpha(H, H_1)(V|\pi_1(H_1))_*\alpha(H_1)\xi(J).$$

The last expression vanishes unless $H_1 \supset J$, in which case it is $\alpha(H, H_1)(U|\pi_1(H_1))\xi(J, H_1)$. Let $H_1 = J \cup \{l\}$ in this case. Then, by Lemma 8.1, $\xi(J, H_1)$ is in the image of $L_5(\mathbf{Z}(t_l))$ in $\pi_1(H_1)$. But $U(t_l) = t_l$, so by naturality (since L is a functor),

$$(U(\pi_1(H_1)))_*\xi(J, H_1) = \xi(J, H_1).$$

But $\alpha(H, H_1)\xi(J, H_1) = 0$. So if $i \in H \neq J$, $\alpha(H)\xi(J) = 0$. Now suppose that $i \notin H$. If $k \in H$, take $H_1 = H \cup \{i\}$. Say $H_1 \geq J$. The image of $L_5(\pi_1(H))$ in $L_5(\pi_1(H_1))$ is (pointwise) fixed under $(U|\pi_1(H_1))_*$, by naturality. But $\xi(J, H_1)$

is in this image; this follows either from Lemma 8.1 or from Theorem 2.1 and the fact that $\alpha(H, H_1)\xi(J, H_1) = 0$. So $(U|\pi_1(H_1))_*\xi(J, H_1) = \xi(J, H_1)$. But then $\alpha(H)U_*\xi(J) = \alpha(H, H_1)(U|\pi_1(H_1))_*\xi(J, H_1) = 0$. If $H_1 \neq J$, $\alpha(H)U_*\xi(J) = \alpha(H, H_1)(U|\pi_1(H_1))_*\xi(J) = 0$ also.

Suppose that $i \notin K$ and $k \notin H$. Then we take $H_1 = H \cup \{k\}$ and $H_2 = H_1 \cup \{i\}$ and argue similarly to show $\alpha(H)U_*\xi(J) = 0$.

So, for all $H \neq J$, J in case 1, $\lambda_H(Ux) = 0$. Since $Ux \neq [\mathrm{id}]$, we must have $\lambda_J(Ux) = 1$. Hence $\lambda(Ux) = \lambda(x)$, so $Ux = x$.

Case 2. $i \in J$, $k \notin J$. Say that $H \neq J$. If $k \in H$, then

$$\alpha(H)U_*\xi(J) = (U|\pi_1(H))\alpha(H)\xi(J) = 0.$$

If $K \notin H$, let $H_1 = H \cup \{k\}$ and argue similarly to show that $\alpha(H_1)U_*\xi(J) = 0$ and hence $\alpha(H)U_*\xi(J) = 0$. As in case 1, this shows that $\lambda(Ux) = \lambda(x)$ and so $Ux = x$. Clearly $U\lambda(x) = \lambda(x)$. This concludes case 2.

Case 3. $i \notin J$, $k \notin J$. Say that $H \neq J$. Let $H_1 = H \cup \{i, k\}$. Then $H_1 \neq J$, so $\alpha(H_1)\xi(J) = 0$. So

$$0 = (U|\pi_1(H_1))_*\alpha(H_1)\xi(J) = \alpha(H_1)U_*\xi(J).$$

So

$$\alpha(H)U_*\xi(J) = \alpha(H, H_1)\alpha(H_1)U_*\xi(J) = 0.$$

So

$$\lambda(Ux) = \lambda(x).$$

Clearly $U\lambda(x) = \lambda(x)$. So this concludes case 3.

Case 4. $i \notin J$, $k \notin J$. This is the hardest of the four cases. Let $J = \{k, l, m\}$, $H = \{i, l, m\}$. Then clearly we have $U\lambda(x) = Ut^J = t^J + t^H$. So we have to show that $\lambda_H(Ux) = \lambda_J(Ux) = 1$, and all the other invariants vanish for Ux. Let $K = \{k, i, m\}$, $L = \{k, l, i\}$. We first study $\lambda_K(Ux)$, $\lambda_L(Ux)$, $\lambda_J(Ux)$, and $\lambda_H(Ux)$. Let $J_1 = J \cup \{i\}$. Let $V = U|\pi_1(J_1)$. We have

$$\alpha(K, J_1)V_*\xi(J, J_1) = V_*\alpha(K, J_1)\xi(J, J_1) = 0.$$

It follows easily from this, using Lemma 8.4 again (as we have in earlier cases) that $\lambda_K(Ux) = 0$. Similarly, $\lambda_H(Ux) = 0$. By Lemma 5.2 this shows that

$$V_*(\xi(J, J_1)) = \gamma\xi(J, J_1) + \delta\xi(H, J_1)\ (\text{mod torsion}).$$

The following diagram commutes:

$$\begin{array}{ccc} L_5(\mathsf{Z}(t_i)) & \xrightarrow{h_*} & L_5(\pi_1(J_1)) \\ \downarrow{\scriptstyle (V|\mathsf{Z}(t_i))_*} & & \downarrow{\scriptstyle V_*} \\ L_5(\mathsf{Z}(t_i + \varepsilon t_k)) & \xrightarrow{h_*} & L_5(\pi_1(J_1)). \end{array}$$

Here h is inclusion, and the upper horizontal map is also induced by inclusion. By Lemma 8.1 the upper map carries an infinite cyclic generator to $\xi(J, J_1)$. Let $m : \mathsf{Z}(t_k) \to \mathsf{Z}(t_i + \varepsilon t_k)$ be given by $m(t_k) = t_i + \varepsilon t_k$. Let $\pi : \pi_1(J_1) \to \mathsf{Z}(t_k)$ be the natural projection. Then $\pi_* H_* m_* = (\pm\mathrm{id})_*$; i.e., the following commutes:

$$\begin{array}{ccc} L_5(\mathsf{Z}(t_i + \varepsilon t_k)) & \xrightarrow{h_*} & L_5(\pi_1(J_1)) \\ \uparrow{\scriptstyle m_*} & & \downarrow{\scriptstyle \pi_*} \\ L_5(\mathsf{Z}(t_k)) & \xrightarrow{(\pm\mathrm{id})_*} & L_5(\mathsf{Z}(t_k)). \end{array}$$

From these squares we see that $\pi_* V_*\xi(J, J_1)$ must be an infinite cyclic generator of rank one group $L_5(\mathsf{Z}(t_k))$.

Now we assert that $\pi_*\xi(J, J_1) = 0$. First $\xi(J, J_1)$ is in the image of the map $L_5(\mathsf{Z}(t_i)) \to L_5(\pi_1(T(J_1)))$ induced by inclusion; hence $\pi_*\xi(J, J_1)$ is in the image of the map $L_5(\mathsf{Z}(t_i)) \to L_5(\mathsf{Z}(t_k))$ induced by the trivial map. This map factors through $L_5(e) = 0$, because L is a functor. Hence it is trivial.

On the other hand, it follows from Lemma 8.1 that $\pi_*\xi(H, J_1)$ is an infinite generator. Thus we conclude that $\delta = \pm 1$. Similarly, $\gamma = \pm 1$.

By Lemma 8.4 and what we have just proved,

$$\alpha(J_1)U_*\xi(J) = \pm\xi(J, J_1) \pm \xi(H, J_1)(\text{mod torsion}).$$

Since $W(H)\alpha(H, J_1)$ and $W(J)\alpha(J, J_1)$ vanish on torsion, we can apply each of these to the above equation to conclude that $\lambda_J(Ux) = \lambda_H(Ux) = 1$.

Suppose that $|I| = 3$ and $I \not\subset J_1$. If $i \in I$ and $k \in I$, then using Lemma 8.4 $\alpha(I)U_*\xi(J) = (U|\pi_1(I))_*\alpha(I)\xi(J) = 0$. If $i \in I$ and $k \notin I$, let $I_1 = I \cup \{k\}$. Since $I_1 \not\supset J$, $\alpha(I_1)U_*\xi(J) = 0$, using Lemma 8.4 as usual, and so $\alpha(I)U_*\xi(J) = 0$. Note that in this case $I_1 \supset J$ is impossible, since then we would have $I_1 = J$, and so $I \subset J_1$. So $\lambda_I(Ux) = 0$ in this case.

Suppose $i \notin I$ and $k \in I$. Then let $I_1 = I \cup \{i\}$. Then $I_1 \not\supset J$, as $i \notin J$ and $I \neq J$. So in this case we argue as before, using Lemmas 8.4 and 6.2 to show that $\alpha(I)U_*\xi(J) = 0$, and so $\lambda_I(Ux) = 0$.

Say that $i \notin I$ and $k \notin I$. Let $K = I \cup \{i, k\}$, $K_1 = K - \{i\}$. Let $V = U|\pi_1(K)$. As usual, using Lemma 8.4, we want to show that

$$\alpha(I, K)V_*\xi(J, K) = 0;$$

this implies that $\lambda_I(Ux) = 0$.

If $K_1 \not\supset J$, $\xi(J, K)$ is in the image of $L_6(\pi_1(K_1))$ under inclusion, by Lemma 8.1, for example; by naturality this image is invariant under V_*. Since $\alpha(I, K) = \alpha(I, K)\alpha(K, K)$ vanishes in this image, the result follows.

Suppose $K_1 \supset J$. Write $J = \{k, l, m\}$ as before. Let $\{s\} = I - I \cap J$; then $\{l, m, s\} = I$. We have a commutative diagram

$$\begin{array}{ccc} L_6(\mathbf{Z}(t_s, t_i)) & \xrightarrow{h} & L_6(\pi_1(K)) \\ \downarrow{\scriptstyle V_*} & & \downarrow{\scriptstyle V_*} \\ L_6(\mathbf{Z}(t_s, t_i + \varepsilon t_k)) & \xrightarrow{h_*} & L_6(\pi_1(K)); \end{array}$$

h = inclusion and the unlabeled map is induced by inclusion. So $V_*\xi(J, K)$ is the image of an infinite cyclic generator under h_*. [Recall that, by Theorem 2.1, $L_6(\mathbf{Z} \times \mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}_2$.]

Now, in general, if $\mu, \rho \in K$ and $\pi(\mu, \rho) : \pi_1(K) \to \mathbf{Z}(t_\mu, t_\rho)$ is the natural projection and $i(\mu, \rho)$ the natural inclusion [so $\pi(\mu, \rho) i(\mu, \rho) = \mathrm{id}$], then for maps induced on L_6 of the appropriate groups we have

$$\pi(\mu, \rho)_* \circ i(\mu', \rho')_* \equiv 0 \text{ (mod torsion)}$$

unless $\{\mu, \rho\} = \{\mu', \rho'\}$; for if $\{\mu, \rho\} \neq \{\mu', \rho'\}$, this composite factors through $L_6(\mathbf{Z}(t_\mu))$, $L_6(\mathbf{Z}(t_\rho))$, or $L_6(e)$, all of which are $\mathbf{Z}_2$. So for any $P \subseteq K$, $|P| = 3$, $\pi(\mu, \rho)_*\xi(P, K)$ is an element of order 2 unless $P \cup \{\mu, \rho\} = K$, in which case it is an infinite cyclic generator, by Lemma 8.1.

On the other hand, $\pi(s, i) \circ h$ is an isomorphism; hence $\pi(s, i)_* V_*(\xi(J, K))$ is an infinite cyclic generator. Similarly, so is $\pi(s, k)(V_*\xi(J, K))$. If $\{\mu, \rho\} \neq \{s, i\}$ or $\{s, k\}$, then $\pi(\mu, \rho) \circ h$ factors through $\mathbf{Z}$ and so $\pi(\mu, \rho)V_*\xi(J, K)$ will have order 2, as $L_6(\mathbf{Z}) = \mathbf{Z}_2$. Thus modulo elements of order 2,

$$V_*\xi(J, K) = \pm\xi(J, K) \pm \xi(H, K),$$

where $H = \{i, l, m\}$. So this shows that $\alpha(I, K)V_*\xi(J, K) = 0$, as $I \neq J$ and $I \neq H$. So $\lambda_I(Ux) = 0$. Incidently, this reproves $\lambda_H(Ux) = \lambda_J(Ux) = 1$, in a somewhat more complicated way.

This concludes case 4. We see that $\lambda_I(Ux) = 0$ if $I = J$ or H and $\lambda_H(Ux) = \lambda_J(Ux) = 1$; this is what is needed.

So Theorem 8.3 and Theorem A are proved.

9. Finite Coverings of Fake Tori

Let X^{n-1}, $n \geq 6$, be a closed connected orientable manifold. Let V be the normal bundle of $X \times I$, and let $\tilde{B}_n(X \times I, v) = \tilde{B}_n^s(X \times I, v)$ and $\tilde{B}_n^h(X \times I, v)$ be as in the first paragraph of sec. 5 of [35]. Let $p : Y \to X$ be a finite covering map, with Y connected. The differential $dp : \tau(Y) \to \tau(X)$ is a bundle map covering p and so there is a natural identification of the normal bundle of $Y \times I$ with $w = (p \times 1)^*v$. Given (M, φ, F) representing an element of $B_n(X \times I, v)$, let $\varphi^*(p) : E(\varphi^*(p)) \to M$ be the covering induced by φ from p, let $\hat{\varphi} : E(\varphi^*(p)) \to Y$ be the natural covering map covering φ, and let $\varphi^*(F)$ be the induced framing of $\tau(E(\varphi^*(p)) \oplus \hat{\varphi}^*W$. Then sending (M, φ, F) to $(E(\varphi^*(p)), \hat{\varphi}, \varphi^*(F))$ induces a map

$$p^\# : \tilde{B}_n(X \times I, v) \to \tilde{B}_n(Y \times I, w),$$

and it is easy to see that there is an induced homomorphism

$$p^! : L_n(\pi_1 X) \to L_n(\pi_1 Y)$$

with $p^!\theta(x) = \theta(p^\# x)$ for $x \in \tilde{B}_n(X \times I, v)$. Similarly, we define

$$p^! : L_n^h(\pi_1 X) \to L_n^h(\pi_1 Y).$$

REMARK. It appears that $p^!$ depends in an essential way only upon the image of $\pi_1 Y$ in $\pi_1 X$ under the map induced by p. But we do not try to prove this here.

Suppose, for example, that $X = L \times S^1$, $\pi_1 L = G$. Let $p(x, y) = (x, y^m)$, $y \in S^1 \subseteq \mathbf{C}$ = the complex numbers. Then we have

Proposition 9.1. *The following commutes:*

$$\begin{array}{ccc} L_n(G \times \mathbf{Z}) & \xrightarrow{\alpha(L)} & L_{n-1}^h(G) \\ \downarrow{\scriptstyle p^!} & \nearrow{\scriptstyle \alpha(L)} & \\ L_n(G \times \mathbf{Z}) & & \end{array}$$

The proof is a very simple application of the definitions and is omitted.

Let j be the inclusion of G in $G \times \mathbf{Z}$. Then we have

Lemma 9.2. *Let* $p : L \times S^1 \to L \times S^1$ *be* $p(x, y) = (x, y^m)$. *Then*

$$p^! J_*(\xi) = mj_*(\xi).$$

Proof. Let (Q, ψ, E) represent an element of $\tilde{B}_n(L \times I \times I, v|L) \times I \times I)$ with $\theta(Q, \psi, E) = \xi$. By theorems 5.8 and 6.5 of [46], we can take $\partial Q = (L \times I \times 0) \cup (L \times \partial I \times I) \cup \partial_+ Q$ with $\psi(\partial_+ Q) \subseteq L \times I \times I$ with $\psi|(\partial Q - \partial_+ Q)$ the identity, and with E compatible with the natural identification of $L \times 0 \times I$ and $L \times 1 \times I$. Gluing up these parts of the boundary by this identification, we get (M, φ, F), representing an element of $\tilde{B}(L \times S^1 \times I, v \times I)$. By lemma 5.2 of [35],

$$\theta(M, \varphi, F) = j_*(\xi).$$

(Here we identify $S^1 = I/\partial I$, using the map $t \to e^{2\pi i t}$.)

Now we construct $p^{\#}(M, \varphi, F)$ explicitly. Let (Q_i, ψ_i, E_i), $1 \leq i \leq m$, $\psi_i = Q_1 \to L \times I \times I$, be m disjoint copies of (Q, ψ, E). We have, for example,

$$\partial Q_i = (L_i \times I \times 0) \cup (L_i \times \partial I \times I) \cup \partial_+ Q_i.$$

Let $t_i = L \times I \times I \to L \times I \times I$ be given by

$$t_i(x, a, b) = (x, a, (b + (i - 1))/m), \qquad 1 \leq i \leq m.$$

Then let P be the union of $Q_1, \ldots, Q_m$ with $L_i \times 1 \times I$ identified with $L_{i+1} \times 0 \times I$ in the obvious way for $1 \leq i \leq m - 1$. Let $f = f_1\psi_1 \cup \cdots \cup t_m\psi_m$, and let $H = E_1 \cup \cdots \cup E_m$. Then

$$\partial P = (L \times I \times 0) \cup (L_1 \times 0 \times I) \cup (L_m \times 1 \times I) \cup \partial_+ P,$$

and it is not hard to see that $p^{\#}(M, \varphi, F)$ can be obtained from gluing $L_1 \times 0 \times I$ and $L_m \times 1 \times I$ by the natural identification. So by lemma 5.2 of [35],

$$p^! j_*(\xi) = j_* \theta(P, f, H).$$

By the additive property of surgery obstructions [46],

$$\theta(P, f, H) = m\xi.$$

So

$$p^! j_*(\xi) = j_*(m\xi) = m j_*(\xi).$$

Now suppose that $p : T^n \to T^n$ is a finite covering map, $n \geq 5$. Then if $h : M \to T^n$ is a homotopy triangulation, we can take induced covering $\hat{M}$ and the induced map of coverings $\hat{h}$ covering h to get a new homotopy triangulation. This induces a map of $ht(T^n)$ to itself; we denote this by

$$p^! : ht(T^n) \to ht(T^n).$$

We want to study the effect of $p^!$ with respect to the invariants λ_J.

Lemma 9.3. *The following diagram commutes:*

$$\begin{array}{ccc} L_{n+1}(\pi_1 T^n) & \xrightarrow{\partial} & ht(T^n) \\ \downarrow p^! & & \downarrow p^! \\ L_{n+1}(\pi_1 T^n) & \xrightarrow{\partial} & ht(T^n). \end{array}$$

Proof. Obvious.

Let us study the effect of the simplest type of covering transformation,

$$p(x_1, \ldots, x_n) = (x_1, \ldots, x_i^m, \ldots, x_n),$$

for i fixed, $x_j \in S^1 \subseteq \mathbf{C}$, $1 \leq j \leq n$. Let $J \subseteq \{1, \ldots, n\}$, $|J| = 3$. If $i \notin J$, then it is clear from the definitions (or from Lemma 9.1 applied inductively) that $p^! \xi(J) = \xi(J)$. If $i \in J$, let $H = \{1, \ldots, n\} - \{i\}$. Then $\alpha(H)\xi(J) = 0$ by Lemma 6.2, so by Theorem 2.1 $\xi(J)$ is in the image of $L_{n+1}(\pi_1(H))$ under the map induced by inclusion. Hence, by Lemma 9.3, $p^! \xi(J) = m\xi(J)$. So we have

$$\lambda_H(p^!(\partial\xi(J))) = \begin{cases} 0 & H \neq J, \\ 1 & H = J, i \notin J, \\ m \,(\mathrm{mod}\ 2) & H = J, i \in J. \end{cases}$$

Now, let us define $\lambda^* : ht(T^n) \to (\Lambda^3 \mathbf{Z}^n) \otimes \mathbf{Z}_2$ as follows: If $J = \{i_1, i_2, i_3\}$, let $t_J = (t_{i_1} \wedge t_{i_2} \wedge t_{i_3}) \otimes 1$ and set $\lambda^*(x) = \Sigma_{|J|=3} \lambda_J(x) t_J$. Let

$$\Lambda^*(p) : (\Lambda^3 \mathbf{Z}^n) \otimes \mathbf{Z}_2 \to (\Lambda^3 \mathbf{Z}^n) \otimes \mathbf{Z}_2$$

be induced by the map sending t_j to t_j for $j \neq i$ and t_i to mt_i. Then we have shown that

$$\lambda^*(p^! x) = {}^*(p)\lambda^*(x).$$

Suppose that $p(x_1, \ldots, x_n) = (x_1^{m_1}, \ldots, x_n^{m_n})$. Then $p = p_1 \circ \cdots \circ p_n$, where $p_i(x_1, \ldots, x_n) = (x_1, \ldots, x_i^{m_i}, \ldots, x_n)$. Then $p^! = p_n^! \circ \cdots \circ p_1^!$. Let $\Lambda^*(p) = \Lambda^*(p_n) \circ \cdots \circ \Lambda^*(p_1)$, the map induced by $t_i \to m_i t_i$. Then it follows that we have

Theorem 9.4. *For* $p(x_1, \ldots, x_n) = (x_1^{m_1}, \ldots, x_n^{m_n})$,

$$\lambda^*(p^! x) = \Lambda^*(p)\lambda^*(x).$$

Every covering map $q : T^n \to T^n$ can be written as $q = U \circ p$ where $U \in GL(n, \mathbf{Z})$ and p is as in Theorem 9.4. But $U^! : ht(T^n) \to ht(T^n)$ is just $U^! x = U^{-1} x$. Let ${}^t U$ denote the transpose of U with respect to the basis $t_1, \ldots, t_n$. It acts on $\Lambda^3 \mathbf{Z}^n \otimes \mathbf{Z}_2$ naturally.

Lemma 9.5. $\lambda^*(U^{-1}x) = {}^t U \lambda^*(x)$.

Proof. Let $D : (\Lambda^{n-3} \mathbf{Z}^n) \otimes \mathbf{Z}_2 \to (\Lambda^3 \mathbf{Z}^n) \otimes \mathbf{Z}_2$ be the standard duality map. Then $\lambda^*(x) = D\lambda(x)$. On the other hand, for $y \in (\Lambda^{n-3} \mathbf{Z}^n) \otimes \mathbf{Z}_2$, $DU^{-1}y = {}^t U D y$. This follows from an easy calculation. So

$$\lambda^*(U^{-1}x) = DU^{-1}\lambda(x) = {}^t U D\lambda(x) = {}^t U \lambda^*(x).$$

Theorem 9.6. *Let* $q = U \circ p$ *be as above. Then*

$$\lambda^*(q^! x) = \Lambda^*(p)^t U \lambda^*(x).$$

Suppose we view t_i as a class of $H_1(T^n; Z)$. Then if $x \in ht(T^n)$, we can view $\lambda(x) \in H_{n-3}(T^n; Z_2)$. Then the dual class, $\lambda^*(x)$, becomes an element of $H^3(T^n : Z_2)$ and we can reformulate Theorem 9.6 as follows.

Theorem 9.7. *Let $q : T^n \to T^n$ be a covering map. Then the following diagram commutes:*

$$\begin{array}{ccc} ht(T^n) & \xrightarrow{\lambda^*} & H^3(T^n; Z_2) \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q^*} \\ ht(T^n) & \xrightarrow{\lambda^*} & H^3(T^n; Z_2). \end{array}$$

This formulation is due to Wall. Since every finite covering space of T^n is PL homeomorphic to T^n, this covers all finite covering spaces of fake tori.

Theorem B. *Every fake torus τ^n is covered by T^n, $n \geq 5$.*

Proof. Apply Theorem 9.4 to $p(x_1,..., x_n) = (x_1^2,..., x_n^2)$.

Corollary 9.8. *Let $p(x_1,..., x_n) = (x_1^{m_1},..., x_n^{m_n})$ with $m_1,..., m_n$ odd, $n \geq 5$. Then for $x \in ht(T^n)$, $p^! x = x$.*

Proof. $\Lambda^*(p) =$ identity.

Corollary 9.8 (essentially) is crucial for Siebenmann's proof that there is a fake torus homeomorphic to the standard torus [36].

10. Manifolds Tangentially Cobordant to $S^k \times T^n$

In this section we indicate how to use our methods to obtain some results on manifolds of the homotopy of $S^k \times T^n$, $k \geq 2$. We have the sequence

$$L_{n+k+1}(Z^n) \xrightarrow{\partial} ht(S^k \times T^n) \to [S^k \times T^n, G/PL] \xrightarrow{S} L_{n+k}(Z^n).$$

In this case, η is nontrivial. For example, any composite $S^2 \times T^n \to T^n \to G/PL$, where the first map is the natural projection, is the normal invariant of some homotopy equivalence.

However, we can use the same ideas as for the torus to determine the map ∂. By [44], every homeomorphism $h : M \to S^k \times T^n$, $k \times n \geq 5$, represents an element in the image of ∂. If $h : M \to S^k \times T^n$ is any homotopy equivalence representing an element in the image of ∂, we say that M is *tangentially cobordant to* $S^k \times T^n$.

For $J \subseteq \{1,..., n\}$ with $|J| = m = k + 1 \pmod 2$ we define classes $\xi(J) = \xi(J, k, n) \in L_{n+k+1}(Z^n)$ analogously to what we did in Section 5. We take the image of a generator of $L_{m+k+1}(e)$ in $L_{m+k+1}(\pi_1(J)) = L_{m+k+5}(\pi_1(J))$ and realize it as $\theta(W, \varphi, F)$, where (W, φ, F) represents an element of $B_{m+k+5}(S^k \times T(J) \times I \times CP^2, V)$, V the normal bundle of $S^k \times T(J) \times I \times CP^2$, and $\varphi(\partial_- W : \partial_- W \to S^k \times T(J) \times 0 \times CP^2$ a PL equivalence.

Let $H = \{1,..., n\} - J$. We take

$$\xi(J) = \theta[(W, \varphi, F) \times T(H)].$$

If k is odd, we must allow $J = \varnothing$; let $T(\varnothing)$ be a point.

As in Section 5, we need not take product with CP^2 if $|J| + 1 > 4$ or $|J| + k + 1 = 2$. This implies that $\partial(\xi(J)) = [\mathrm{id}]$ for such J. For $|J| + k + 1 = 4$, we still have, as in Section 5, $\partial(2\xi(J)) = [\mathrm{id}]$.

We can also introduce the maps

$$\alpha(J) = \alpha(J, k, n) : L_{n+k+1}(Z^n) \to L_{m+k+1}(\pi_1(J))$$

for $|J| = m$, and the projections

$$w(J) : L_{m+k+1}(\pi_1(J)) \to L_{m+k+1}(e).$$

We have the formula $w(H)\alpha(H)\xi(J) = \delta(H, J)$, as in Proposition 6.3. Again we may have $H = \varnothing$; let $\pi_1(H) = \{e\}$. From this, we can prove

Lemma 10.1. *Every element of $L_{n+k+1}(Z^n)$ has a unique expression $\Sigma a(J)\xi(J)$, the sum over $J \subseteq \{1,..., n\}$ with $|J| \equiv 1 + k \pmod 2$, where $a(J) \in Z$ if $|J| \equiv -1 + k \pmod 4$ and $a(J) \in Z_2$ if $|J| = 1 + k \pmod 4$.*

Theorem 10.2. *For $k > 3$, $\partial : L_{n+k+1}(Z^n) \to ht(S^k \times T^n)$ is trivial.*

Proof. $|J| + k + 1 > 4$ for all $J \subseteq \{1,..., n\}$, so $\xi(J)$ acts trivially on [id]. By Lemma 10.1, this implies that $L_{n+k+1}(Z^n)$ acts trivially on [id]; i.e., ∂ is trivial.

Now let $k = 2$ or 3. Then using Rohlin's theorem [29], we can prove

Lemma 10.3. *The subgroup $\{\xi | \partial\xi = [\mathrm{id}]\}$ is generated by $\{\xi(J) | k + |J| \neq 3\} \cup \{2\xi(J) | k + |J| = 3\}$. In fact, if $h : M \to S^k \times T^n$ represents $\partial\xi(J)$ with $k + |J| = 3$, then M is not PL equivalent to $S^k \times T^n$.*

The first statement is proved exactly as Lemma 5.3. To prove the second statement one needs the following:

Sublemma 10.4. *Let p and q be integers, $p \geq 1$, $q \geq 1$. Let $f : S^p \times T^q \to S^p \times T^q$ be a homotopy equivalence. Let $T^{q-1} \subseteq T^q$ be a standard subtorus. Then f is homotopic to g, transverse to $S^p \times T^{q-1}$, so that $g^{-1}(S^p \times T^{q-1})$ is PL equivalent to $S^p \times T^{q-1}$.*

Proof. Let $\pi : S^p \times T^q \to S^1$ be the standard projection with fiber $S^p \times T^{q-1}$. Since $S^1 = K(Z, 1)$, π represents an element $z \in H^1(S^p \times T^q; Z)$, and $\pi \circ f$ represents f^*z. Let $d : T^q \to T^q$ be a PL equivalence with $(1 \times d)^*z = f^*z$. Then $\pi \circ f$ is homotopic to $\pi \circ d$. By the covering homotopy property for the fibration π, there is a map g, homotopic to f, so that $\pi \circ g = \pi \circ d$. It is obvious that g satisfies the conclusion of the sublemma.

Assuming now that M in Lemma 10.3 were PL equivalent to $S^k \times T^n$, we use Sublemma 10.4 to peel off circles and argue as in Section 6 to derive a contradiction to Rohlin's theorem [29]. We did not need a lemma like Sublemma 10.4 in the torus case because any homotopy equivalence of T^n with itself is homotopic to a PL equivalence.

For $k = 3$, the only invariant of the image of $\partial : L_{n+4}(Z^n \to ht(S^3 \times T^n))$ is

$$\lambda_{\varnothing}(\xi) \equiv w(\varnothing)\alpha(\varnothing)\xi \bmod 2.$$

So we have the following, except for the last sentence.

Theorem 10.5. *For $k = 3$, $n \geq 2$, the image of ∂ has two elements. There is a unique manifold M, up to* PL *equivalence, that is, tangentially cobordant to $S^3 \times T^n$. Every finite cover of M is* PL *homeomorphic to M.*

The final sentence of Theorem 10.5 follows from Proposition 9.1 and the fact that $\lambda_{\varnothing}$ detects M.

REMARKS. 1. M^{n+3}, $n \geq 3$, as in Theorem 10.5, let $K^{n+2} \subseteq M^{n+3}$ be a fiber of a PL fibration of M over a circle [12]. Then it is not hard to see that K is the unique manifold tangentially cobordant but not PL equivalent to $S^3 \times T^{n-1}$ and that $M = K \times S^1$ (cf. Proposition 7.2).

2. If there is a fake 3-sphere Σ^3 with $\Sigma^3 = \partial W^4$ and $\operatorname{Index}(W^4) = 8$, then the unique manifold of dimension $(n + 3)$ of Theorem 10.5 is $\Sigma^3 \times T^n$.

3. It follows from the classification theorem of Kirky–Siebenmann that the manifolds of Theorem 10.5 are homeomorphic to $S^3 \times T^n$. In fact, this follows from [19] and the disproof of the Hauptvermutung [36].

As for the fake tori, we can also define invariants λ_i for the elements of the image of $\partial : L_{n+3}(Z^n) \to ht(S^2 \times T^n)$, $n \geq 3$. We let $\lambda_i(\partial\xi)$ be the mod 2 reduction of $w(\{t_i\})\alpha(\{t_i\})\xi$. Let $t_1, \ldots, t_n$ be the standard basis of $\pi_1(S^2 \times T^n)$ and let $t^{(i)} = (t_{i_1} \wedge \cdots \wedge t_{i_{n-1}}) \otimes 1 \in \Lambda^{n-1}Z^n \otimes Z_2$. For $x \in \operatorname{Im}\partial$, let $\lambda(x) = \Sigma_{i=1}^{n}\lambda_i(x)t^{(1)}$. As in the torus case, we have

Theorem 10.6. *$\lambda : \operatorname{Im}\partial \to (\Lambda^{n-1}Z^n) \otimes Z_2$ is a bijection.*

We can define an action of $GL(n, Z)$ on $ht(S^2 \times T^n)$ as follows: If $U \in GL(n, Z)$, let $\bar{U} : T^n \to T^n$ induce U on the fundamental group and set $U[h] = [(1 \times \bar{U}) \circ h]$, $h : M \to S^2 \times T^n$ a homotopy equivalence. Then it is clear that $U(\partial\xi) = \partial(U_*\xi)$, so $\operatorname{Im}\partial$ is invariant. As in Section 8, one can show:

Theorem 10.7. *λ is $GL(n, Z)$-invariant.*

Corollary 10.8. *There is a unique manifold M tangentially cobordant but not* PL *equivalent to $S^2 \times T^n$.*

Proof. The natural action of $GL(n, Z)$ on $(\Lambda^{n-1}Z^n) \otimes Z_2$ is transitive on nonzero elements.

As in Section 9, we can define a dual invariant $\lambda^* : \operatorname{Im}\partial \to (\Lambda^1 Z^n) \otimes Z_2 = (Z^2)^n$. If we view t_i as homology classes, then $\lambda(x) \in H_{n+1}(S^2 \times T^n; Z_2)$ and $\lambda^*(x) \in H^1(S^2 \times T^n; Z_2)$ and we have

Theorem 10.9. *Let $p : S^2 \times T^n \to S^2 \times T^n$ be a covering map. Then the following diagram commutes:*

$$\begin{array}{ccc} \operatorname{Im}\partial & \xrightarrow{\lambda^*} & H^1(S^2 \times T^n; Z_2) \\ \uparrow{\scriptstyle p^!} & & \uparrow{\scriptstyle p^*} \\ \operatorname{Im}\partial & \xrightarrow{\lambda^*} & H^1(S^2 \times T^n; Z_2). \end{array}$$

Finally, using the technique involved in Theorem 10.2, one can show the following, using [44].

Theorem 10.10. *Let P^q, $q \geq 4$, be a simply connected closed* PL *manifold. Assume that $h : M \to P^q \times T^n$, $q + n \geq 5$, has a trivial normal invariant. Then h is homotopic to a* PL *equivalence. In particular, if $H^4(P^q \times T^n; 2)$ has no 2-torsion and h is a homeomorphism, then h is homotopic to a* PL *equivalence.*

REMARK. The second author has proved (unpublished) that any manifold of the homotopy type of $P^4 \times S^1$ is PL equivalent to it.

Corollary 10.11. *If M is tangentially cobordant to $S^k \times T^n$, $k = 2, 3$, $k + n \geq 3$, then M embeds in R^{n+k+3} with trivial normal bundle.*

Using product formulas for nonsimply connected surgery obstructions, we can show that $M \times S^2$ is PL equivalent $S^k \times T^n \times S^2$.

11. Application to Questions $S(k, n)$

Let $h : D^k \times T^n \to W$ be a homeomorphism that is PL on the boundary. The question $S(k, n)$ asks that there be a PL homeomorphism of $D^k \times T^n$ with W that agrees with h on the boundary. The question $\tilde{S}(k, n)$ asks for the same conclusion after passage to some finite covering spaces. We study these statements as an application of Section 10, and we prove Theorem C of the introduction for $k + n \geq 6$.

Let us begin with the case $k \geq 5$. We can assume that h is PL on a collar neighborhood of the boundary. Let $B \subseteq \operatorname{Int} D^k$ be a small disk of dimension k with $B \times T^k$ lying inside this collar. Then $k = h|\operatorname{cl}(D^k \times T^n - B \times T^n)$ is a homeomorphism, PL on the boundary, onto $W - h(\operatorname{Int} B \times T^n)$. By the s-cobordism theorem (see [5]), this manifold is PL homeomorphic to $\partial B \times T^n \times I$ via a PL homeomorphism φ with $\varphi(x) = (h^{-1}(x), 0)$ for $x \in h(\partial B \times T^n)$. The domain of k is PL homeomorphic to $\partial B \times T^n \times I$ via a PL equivalence φ_1 carrying $\partial B \times T^n$ to $\partial B \times T^n \times 0$ in the standard way. Then if we identify $S^{k-1} = \partial B$,

$$f = \varphi k \varphi_1^{-1} : S^{k-1} \times T^n \times I \to S^{k-1} \times T^n \times I$$

is a homeomorphism that is the identity on $S^{k-1} \times T^n \times 0$ and PL on a neighborhood of $S^{k-1} \times T^n \times 1$. We are going to try to show that f is PL pseudoisotopic to the identity; i.e., there is a PL homeomorphism of

$S^{k-1} \times T^n \times I$ with itself extending $f|S^{k-1} \times T^n \times \partial I$. Let M be the mapping torus of $f^{-1}|S^{k-1} \times T^n \times 1$; i.e., let M be obtained from $S^{k-1} \times T^n \times I$ by identifying $(x, 0)$ with $f^{-1}(x, 1)$ for $x \in S^{k-1} \times T^n$. Then M is a PL manifold and f determines a homeomorphism

$$g : M \to S^{k-1} \times T^n \times S^1.$$

Up to this point our constructions all go through for a homotopy equivalence h that is PL on the boundary. But then g would necessarily be only a homotopy equivalence. Since g is a homeomorphism, $\eta(g) = 0$ by [44] and so by Theorem 10.2 or 10.10 g is homotopic to a PL equivalence

$$d : M \to S^{k-1} \times T^n \times S^1.$$

The projection of $S^k \times T^n \times I$ on I induces a PL fibration $\pi : M \to S^1$. Let $\pi_1 : S^{k-1} \times T^n \times S^1 \to S^1$ be the projection on the last factor. Then $\pi_1 \circ g$ is homotopic to π and so $\pi_1 \circ d$ is also homotopic to π. By a trick of Browder [8], we can assume, after a pseudoisotopy of d, if necessary, that for a preassigned base point $*$ of S^1, $(\pi_1 \circ d)^{-1}(*) = \pi^{-1}(*)$. View S^1 as $I/\partial I$ and let $(*) = \{\partial I\}$. Browder's trick works here because Farrell's fibering theorem [12] and the S-cobordism theorem are valid for a free abelian fundamental group. (The trick is given in the proof of lemma 2 of [8]. Lemma 2 of [8] is itself slightly misstated.)

Now split along the fibers of π and π_1; then d determines a PL homeomorphism

$$\psi : S^{k-1} \times T^n \times I \to S^{k-1} \times T^n \times I$$

with $\psi(f^{-1}(x), 1) = (x, 0)$. Let $\psi_0 = \psi|S^{k-1} \times T^n \times 0$. Then $f|S^{k-1} \times T^n \times 1 = (\psi_0^{-1} \times \text{id}) \circ (\psi|S^{k-1} \times T^n \times 1)$. So $F = (\psi_0^{-1} \times \text{id}) \circ \psi$ is a PL homeomorphism of $S^{k-1} \times T^n \times I$ with itself with $F|S^{k-1} \times T^n \times \partial I = f|S^{k-1} \times T^n \times \partial I$. The PL homeomorphism asked for in $S(k, n)$ is then just $(\varphi^{-1} F \varphi_1) \cup (h|B \times T^n)$.

So this proves: $S(k, n)$ has an affirmative answer for $k \geq 5$.

For $k \leq 4$, the argument could break down only because the map $g : M \to S^{k-1} \times T^n \times S^1$ may not be homotopic to a PL equivalence.

Suppose $k = 4$. By construction M is fibered over S^1 with fiber $S^3 \times T^n$. So by Remark 1 after Theorem 10.5, M is PL equivalent to $S^3 \times T^n \times S^1$ and hence g must be homotopic to a PL homeomorphism. So $S(k, n)$ has an affirmative answer for $k = 4$. Suppose $k = 2$. Then M is a possibly fake torus. Consider the invariant $\lambda^*(x) \in H^3(T^{n+2}; \mathbf{Z}_2)$, x the class of g in $ht(S^1 \times T^n \times S^1) = ht(T^{n+2})$. Let $p : T^{n+2} \to T^{n+2}$ be $p(x_1, x_2, \ldots, x_{n+1}, x_{n+2}) = (x_1, x_2^2, \ldots, x_{n+1}^2, x_{n+2})$. (Recall that $x_i \in S^1 \subseteq \mathbf{C}$.) Then $p^*\lambda^*(x) = 0$, by Theorem 9.7. Hence $\lambda^*(p^t x) = [\text{id}]$ and so $p^t x$ is the trivial homotopy triangulation.

Thus if $\hat{g} : \hat{M} \to S^1 \times T^n \times S^1$ is the induced map on the induced cover $\hat{M}$ of M, g is homotopic to a PL homeomorphism. Since we used a covering map p of the form $1 \times q \times 1$, $q : T^n \to T^n$ a covering, it is easy to see from our argument above that $\tilde{S}(2, n)$ *has an affirmative answer.*

We leave the case $k = 1$ as an exercise.

Suppose $k = 3$. We have a homeomorphism

$$q : M \to S^2 \times T^n \times S^1.$$

We have the invariants $\lambda_i(x)$, $x \in ht(S^2 \times T^n \times S^1)$, the class of g, $1 \leq i \leq n + 1$. By construction $g^{-1}(S^2 \times T^n \times (*))$ is PL equivalent with $S^2 \times T^n$, and the restriction of g is a PL equivalence. Hence $\lambda_i(x) = 0$ for $1 \leq i \leq n$. (Compare Proposition 7.2.) So we have an obstruction $\lambda_{n+1}(x) \in \mathbf{Z}_2$ for $S(k, n)$. It is easy to see that this obstruction depends only upon the homeomorphism $h : D^3 \times T^n \to W$ and not upon any of the intervening constructions. If we start with a covering map $\hat{h} : D^3 \times T^n \to \hat{W}$ for h, and perform the same constructions we get $\hat{g} : \hat{M} \to S^2 \times T^n \times S^1$ covering g. But the construction does not involve the last S^1 and so, by Theorem 10.9, $\lambda_{n+1}(\hat{h}) = \lambda_{n+1}(h)$. *Hence there is a* $\mathbf{Z}_2$-*obstruction to* $\tilde{S}(k, n)$, $k = 3$. This proves Theorem C for $k + n \geq 6$.

(By Kirby–Siebenmann [19] and Siebenmann [36], we know that $\tilde{S}(3, n)$ cannot have an affirmative answer.)

Theorem 11.1. *Let* $f : S^{k-1} \times T^n \to S^{k-1} \times T^n$, $k + n \geq 6$, *be a* PL *homeomorphism that is topologically pseudoisotopic to the identity. If* $k \geq 4$, *then* f *is* PL *pseudoisotopic to the identity. If* $k \leq 2$, f *is covered by a map of finite covers that* PL *pseudoisotopic to the identity. For* $k = 3$, f^2 *is* PL *pseudoisotropic to the identity.*

Proof. We have seen everything except the last statement. For $k = 3$, let $F : S^2 \times T^n \times I \to S^2 \times T^n \times I$ be a topological pseudoisotopy of f with the identity. Let M be the mapping torus of f^{-1}. Then F induces a homeomorphism

$$g : M \to S^2 \times T^n \times S^1,$$

and we saw that $\lambda_i([g]) = 0$ if $i \neq n + 1$. Let $p(x, y, z) = (x, y, z^2)$, $x \in S^2$, $y \in T^n$, $z \in S^1 \subseteq \mathbf{C}$. Then, by Theorem 10.9 $p^t[g] = 0$. This implies the conclusion.

12. Another Approach to Theorem C

In this section we outline a slightly different approach to Theorem C.

Let $f : W \to S^{k-1} \times T^n \times I$, $k + n \geq 5$ be a homotopy equivalence of PL manifolds which restricts to a PL homeomorphism of boundaries. Then, as in Section 11, f determines a homotopy equivalence $g : M \to S^{k-1} \times T^n \times S^1$, M obtained from W by identifying $f^{-1}(x, 0)$ with $f^{-1}(x, 1)$ for $x \in S^{k-1} \times T^n$. It is easy to see that this determines a map

$$\begin{aligned} \mathscr{M} : ht(S^{k-1} \times T^n \times I, S^{k-1} \times T^n \times \partial I) \\ \to ht(S^{k-1} \times T^n \times S^1). \end{aligned}$$

We write $ht(S^{k-1} \times T^n \times I, \partial)$ for the domain of $\mathscr{M}$. Let $\rho : S^{k-1} \times T^n \times S^1 \to S^{k-1} \times T^n \times I/S^{k-1} \times T^n \times \partial I = \Sigma(S^{k-1} \times T^n)_+$ be the natural quotient map. Let $j : \mathsf{Z}^n \subseteq \mathsf{Z}^{n+1}$ be the natural inclusion on the first n factors. Then one can prove:

Theorem 12.1. *The following diagram commutes:*

$$\begin{array}{ccccc} L_{n+k+1}(\mathsf{Z}^n) & \xrightarrow{\partial} & ht(S^{k-1} \times T^n \times I, \partial) & \xrightarrow{\eta} & [\Sigma(S^{k-1} \times T^n)_+, \mathsf{G/PL}] \\ \downarrow j_* & & \downarrow \mathscr{M} & & \downarrow \rho^* \\ L_{n+k+1}(\mathsf{Z}^{n+1}) & \xrightarrow{\partial} & ht(S^{k-1} \times T^{n+1}) & \xrightarrow{\eta} & [S^{k-1} \times T^{n+1}, \mathsf{G/PL}] \end{array}$$

$$\begin{array}{l} \xrightarrow{S} L_{n+k}(\mathsf{Z}^n) \\ \quad \downarrow j_* \\ \xrightarrow{S} L_{n+k}(\mathsf{Z}^{n+1}). \end{array}$$

Further, ρ^* and $\mathscr{M}$ are one-to-one.

Now, given $f : W \to S^{k-1} \times T^n \times I$, representing an element of $ht(S^{k-1} \times T^n \times I, \partial)$, we can use $(f|\partial W)^{-1}$ to attach $D^k \times T^n$ to the lower boundary of W by identifying $f^{-1}(x, 0)$ with x. Then if we call the result Q, $G = f \cup \mathrm{id} : Q \to D^k \times T^n$ represents an element of $ht(D^k \times T^n, \partial)$. This defines

$$\mathfrak{F} : ht(S^{k-1} \times T^n \times I, \partial) \to ht(D^k \times T^n, \partial).$$

In Section 11, we essentially saw that $\mathfrak{F}$ was onto, and we approached Theorem C by studying a lift to $ht(S^{k-1} \times T^n \times I, \partial)$ of element of $ht(D^k \times T^n, \partial)$ represented by a homeomorphism.

There is a natural quotient map

$$\begin{aligned} \pi : \Sigma^k T^n_+ = D^k \times T^n/S^{k-1} \times T^n \to \Sigma(S^{k-1} \times T^n)_+ \\ = S^{k-1} \times T^n \times I/S^{k-1} \times T^n \times \partial I \end{aligned}$$

obtained by viewing $S^{k-1} \times T^n \times I$ as a boundary collar of $D^k \times T^n$; π just collapses $D^k \times T^n \to (\mathrm{Int}\, S^{k-1} \times T^n \times I)$ to a point. One can show:

Theorem 12.2. *The following diagram commutes:*

$$\begin{array}{ccccccc} & \overset{\partial}{\nearrow} & ht(D^k \times T^n, \partial) & \xrightarrow{\partial} & [\Sigma^k T^n_+, \mathsf{G/PL}] & \overset{S}{\searrow} & \\ L_{n+k+1}(\mathsf{Z}^n) & & \uparrow \mathfrak{F} & & \uparrow \pi^* & & L_{n+k}(\mathsf{Z}^n) \\ & \underset{\partial}{\searrow} & ht(S^{k-1} \times T^n \times I, \partial) & \xrightarrow{\eta} & [\Sigma(S^{k-1} \times T^n)_+, \mathsf{G/PL}]^s. & \underset{S}{\nearrow} & \end{array}$$

$\mathfrak{F}|\mathrm{Im}\,\partial$ is a monomorphism.

REMARK. The last statement follows from the fact that, loosely speaking, the surgery problems can be pushed into any part of the manifold which carries the fundamental group. This is the "local character" of surgery obstructions.

Using Section 10 and these two theorems, it is easy to analyze $\partial : L_{n+k+1}(\mathsf{Z}^n) \to ht(D^k \times T^n, \partial)$. Using the fact that $\Sigma^k T^n_+$ is a one-point union of spheres, it follows easily for $k \geq 1$ that the homomorphism S is monomorphic. Hence $\eta = 0$. Thus we see that Theorem C is true for $k + n = 5$ and that it remains true if we replace the homeomorphism $h : D^k \times T^n \to W$ of $S(k, n)$ with a homotopy equivalence that induces a PL homeomorphism of boundaries. It follows from [36] and [19], however, that this does not, in fact, add any generality.

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