WHAT HAPPENS TO HATCHER AND WAGONER'S FORMULA FOR $\pi_0 C(M)$ WHEN THE FIRST POSTNIKOV INVARIANT OF M IS NONTRIVIAL ?

by

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This paper is about algebraic K-theory, homology of rings and diffeomorphisms of smooth manifolds. Our purpose is to correct a discrepancy in a formula of Hatcher and Wagoner relating diffeomorphisms to K_2. The functor K_3 is used to resolve the discrepancy.

We now explain the original formula. Let M be a compact smooth (C^∞) manifold. Then a pseudoisotopy of M (rel ∂M) is defined to be a (self) diffeomorphism of $M \times I$ which fixes $M \times 0 \cup \partial M \times I$ pointwise. ($I = [0,1]$.) The space of all pseudoisotopies of M with the weak C^∞-topology [7] is denoted $C(M)$. In [3] the following statement is made. If $\dim M \geq 7$ then

$$(*) \qquad \pi_0 C(M) \cong Wh_2(\pi_1 M) \oplus Wh_1^+(\pi_1 M;\ \mathbb{Z}_2 \oplus \pi_2 M) \quad .$$

Here Wh_2 is an algebraic K-theory functor related to K_2 and Wh_1^+ is a group homology functor. ($Wh_2(G) = K_2(\mathbb{Z}[G])/\pi_2^s(BG \cup pt)$ and $Wh_1^+(G;A) = H_0(G;A[G])/H_0(G;A)$)

There are two things wrong with Hatcher and Wagoner's formula. (Both mistakes occur in [H] so Wagoner is not responsible.) The first is that the proof assumes that the first Postnikov invariant $k_1(M)$ of M is trivial. When $k_1(M) \neq 0$ the formula is not true and we give a counterexample below. (If $\pi_1 M = \mathbb{Z}_p^3$, $\pi_2 M = \mathbb{Z}_p$ with p odd and $k_1(M)$ is the third exterior power of the generator of $H^1(\mathbb{Z}_p; \mathbb{Z}_p)$ then (*) is false.) The second mistake involves only $\pi_1 M$ and $\mathbb{Z}_2 = \pi_1^s$ = the stable 1-stem. This second mistake is corrected in [9].

The purpose of this paper is to explain how $\pi_0 C(M)$ varies with $k_1(M)$. In

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chapter 8 we examine the proof of [5] in detail and see exactly where the Postnikov invariant assumption $k_1(M) = 0$ is used and we see what happens when $k_1(M) \neq 0$. The variation of $\pi_0 C(M)$ with $k_1(M)$ is seen to be related to $K_3\mathbb{Z}[\pi_1 M]$ and the exact relationship is given by some equation involving the Steinberg group $St(\mathbb{Z}[\pi_1 M])$. In chapter 6 we show how these equations are related to R. K. Dennis's map from algebraic K-theory to Hochschild homology. Thus the correct formula for $\pi_0 C(M)$ is the following exact sequence.

$$K_3\,\mathbb{Z}[\pi_1 M] \xrightarrow{\chi} Wh_1^{+}(\pi_1 M; \mathbb{Z}_2 \oplus \pi_2 M) \longrightarrow \pi_0 C(M) \longrightarrow Wh_2(\pi_1 M) \longrightarrow 0$$

Here χ is the sum of the Grassmann invariant $K_3\,\mathbb{Z}[\pi_1 M] \to Wh_1^{+}(\pi_1 M; \mathbb{Z}_2)$ explained in [9] and the following composition of maps.

$$K_3\,\mathbb{Z}[\pi_1 M] \xrightarrow[h_3]{} H_3(\mathbb{Z}[\pi_1 M]; \mathbb{Z}[\pi_1 M]) \xrightarrow[\kappa[\pi_1 M]\,\cap]{} H_0(\mathbb{Z}[\pi_1 M]; \pi_2 M[\pi_1 M]) \twoheadrightarrow Wh_1^{+}(\pi_1 M; \pi_2 M)$$

Here the first map is Dennis's map (defined in chapter 1), H_* is Hochschild homology and $\kappa[\pi_1 M]$ is an element of $H^3(\mathbb{Z}[\pi_1 M]; \pi_2 M[\pi_1 M])$ related to $\kappa = k_1(M)$. ($\kappa[\pi_1 M] = \ell^3(\kappa)$ is defined in 6.d.) Since R. K. Dennis never published his work we give an exposition of his theory in the first half of the paper.

Here is a list of chapter headings and a brief explanation of what is contained in each chapter. Each chapter also begins with an introduction and a table of contents. (To find the true page numbers add the numbers given to the number of this page.)

Chap. 1. We define Quillen's algebraic K-groups $K_n R$, the Hochschild homology groups

$H_n(R;M)$ and Dennis's map $h_n\colon K_nR \to H_n(R;R)$.

Chap. 2. Let $A \to E \to G$ be an exact sequence of groups where A is abelian. We define a map $b_n\colon H_{n+1}(G,E;\mathbb{Z}) \to H_{n-1}(G;A)$.

Chap. 3. We define the Hochschild cohomology groups $H^n(R;M)$ and the cap product $\cap\colon H^s(R;M) \otimes H_n(R;R) \to H_{n-s}(R;M)$.

Chap. 4. We define the algebraic K-theory of an ideal $I \subset R$ and when $I^2 = I$ we define a map $g_n\colon K_nI \to H_{n-1}(R/I;I)$. The definition involves the map b_n of Ch. 2.

Chap. 5. We discuss the relationship between Loday's product formula for algebraic K-theory and the exterior (shuffle) product in Hochschild homology.

Chap. 6. We do computations in K_1, K_2, K_3 and the Steinberg group in order to get a formula for Dennis's map which may be applied to pseudoisotopy theory.

Chap. 7. We give an example due to R. K. Dennis showing that his map is nontrivial. This later leads to a counterexample for the original formula (*) above.

Chap. 8. We examine Hatcher's argument in [5] to see what happens when $k_1(M) \neq 0$. The topology leads us to an algebraic formula derived in Ch. 6.

Chap. 9. We give a simplified explanation of the counterexample using only K_2 instead of K_3.

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Chapter 1. Dennis's Hochschild homology invariant for algebraic K-theory.

In this chapter we shall define natural transformations $h_n^k\colon K_n^kR = \pi_nBGL_k(R)^+ \to H_n(R;R)$ and show that they are compatible with the stabilization maps $K_n^kR \to K_n^{k+1}R$ induced by the inclusions $GL_k(R) \subset GL_{k+1}(R)$. This will allow us to define a map on the limit $h_n\colon K_nR = \pi_nBGL_\infty(R)^+ \to H_n(R;R)$.

Here are the section titles:

1.a. Definition of $K_n^k R$.

Let R be an associative ring with 1. Let $\mathcal{M}_k(R)$ be the ring of all $k \times k$ matrices with coefficients in R and let $GL_k(R)$ be the group of invertible elements of $\mathcal{M}_k(R)$. Let $E_k(R)$ denote the subgroup of $GL_k(R)$ generated by the elementary matrices denoted by $e_{ij}(r)$, where $i \neq j$ and $r \in R$, which are the matrices with 1's on the diagonal, an r in the ij-position and 0's elsewhere.

If $k \geq 3$ then $E_k(R)$ is perfect, i.e., it is its own commutator subgroup. Thus we may perform the Quillen "plus construction" on the pair $(GL_k(R), E_k(R))$. This gives a space $BGL_k(R)^+$ and a map $p\colon BGL_k(R) \to BGL_k(R)^+$ which is uniquely determined up to homotopy equivalence by the condition that p induces an isomorphism in homology with integral coefficients and $\pi_1 BGL_k(R)^+ = GL_k(R)/E_k(R)$. The unstable K-groups of R for $n \geq 1$, $k \geq 3$ are defined by $K_n^k R = \pi_n BGL_k(R)^+$.

Let $GL(R) = GL_\infty(R)$ be the direct limit of the maps $GL_k(R) \to GL_{k+1}(R)$ given in the standard way by inserting a 1 in the lower right corner. We shall consider these maps to be inclusions. Let the ordinary stable K-groups of R for $n \geq 1$ be given by $K_n R = \pi_n BGL(R)^+ = \varinjlim_k K_n^k R = K_n R$. If $n = 0$ let $K_0 R$ be defined in the standard way [12].

Definition 1.a. (R.K. Dennis)

For $1 \geq n$ and $3 \leq k < \infty$ the Hochschild homology invariant for $K_n^k R$ is defined by the following composition.

$$K_n^k R = \pi_n BGL_k(R)^+ \xrightarrow[H]{} H_n(GL_k(R);\mathbb{Z}) \tag{1}$$

$$\xrightarrow[D_*]{} H_n(GL_k(R);\mathcal{M}_k(R)) \tag{2}$$

$$\xrightarrow{\approx} \mathcal{H}_n(\mathbb{Z}[GL_k(R)];\ \mathcal{M}_k(R)) \qquad (3)$$

$$\longrightarrow \mathcal{H}_n(\mathcal{M}_k(R);\ \mathcal{M}_k(R)) \qquad (4)$$

$$\xrightarrow{\approx} \mathcal{H}_n(R;\ R) \qquad (5)$$

(1) This is the Hurewicz map.

(2) This is the map in homology induced by the coefficient map $D\colon \mathbb{Z} \to \mathcal{M}_k(R)$ determined by $D(1) = I_k$ = the identity matrix.

(3) This is the standard isomorphism between the homology of a group and the Hochschild homology of the corresponding integral group ring. (See 1.d.)

(4) The inclusion $GL_k(R) \subset \mathcal{M}_k(R)$ induces a ring map $\mathbb{Z}[GL_k(R)] \to \mathcal{M}_k(R)$. This is the induced map in Hochschild homology.

(5) This is the standard isomorphism described in 1.e below coming from the fact that R and $\mathcal{M}_k(R)$ are Morita equivalent.

The map described above will be denoted by $h_n^k(R)$ or just $h_n^k\colon K_n^k R \to \mathcal{H}_n(R;R)$. After (1) the map will be denoted with a "+" as follows. $h_n^{k+}(R)$ or $h_n^{k+}\colon H_n(GL_k(R)) \to \mathcal{H}_n(R;R)$.

1.b. Definition of $\mathcal{H}_n(R; M)$.

If M is an R-bimodule and $n \geq 0$ let $C_n(R;M) = \overbrace{R \otimes R \otimes \ldots \otimes R}^{n} \otimes M$ (all tensor products are over $\mathbb{Z}$ unless otherwise stated) and let $d_n\colon C_n(R;M) \to C_{n-1}(R;M)$ be given by the equation:

$$\begin{aligned} d_n(r_1 \otimes \ldots \otimes r_n \otimes m) = {} & r_2 \otimes \ldots \otimes r_n \otimes mr_1 \\ & + \sum_{i=1}^{n-1} (-1)^i\, r_1 \otimes \ldots \otimes r_i r_{i+1} \otimes \ldots \otimes r_n \otimes m \\ & + (-1)^n\, r_1 \otimes \ldots \otimes r_{n-1} \otimes r_n m \end{aligned}$$

Then the n-th <u>Hochschild homology</u> of R with coefficients in M is defined to be the n-th homology group of the complex $(C_*(R;M), d_*)$ and it is denoted by $\mathcal{H}_n(R;M)$. Note that $\mathcal{H}_0(R;M)$ is the cokernel of the map $d_1\colon R \otimes M \to M$ given by $d_1(r \otimes m) = mr - rm$. With this we can define the Hochschild homology invariant for K_0R.

Definition 1.b.1. (Hattori [6] and Stallings [13])

Suppose that P is a finitely generated projective left R-module. Then the Hattori-Stallings trace of P is defined to be

$$\operatorname{tr} P = \operatorname{tr}(1_P \oplus 0_Q) \in H_0(R;R)$$

where Q is an R-module such that $P \oplus Q \cong R^n$ for some finite n and 0_Q denotes the zero map $Q \to Q$.

Proposition 1.b.2. ([6], [13]) The Hattori-Stallings trace of P depends only on P.

Proof: Let $T = 1_P \oplus 0_Q$. Then as an element of R the trace of T depends on the choice of the isomorphism $P \oplus Q \cong R^n$. However the equivalence class of $\operatorname{tr} T$ in $H_0(R;R)$ is easily seen to be well-defined. If Q' is another R-module with $P \oplus Q' \cong R^m$ then we have: $\operatorname{tr}(1_P \oplus 0_Q) = \operatorname{tr}(1_P \oplus 0_Q \oplus 0_{R^m}) = \operatorname{tr}(1_P \oplus 0_Q \oplus 0_P \oplus 0_{Q'}) = \operatorname{tr}(1_P \oplus 0_{R^n} \oplus 0_{Q'}) = \operatorname{tr}(1_P \oplus 0_{Q'})$. □

Definition 1.b.3. Let $h_0(R)$ or simply h_0 denote the natural map $h_0 \colon K_0R \to H_0(R;R)$ given by $h_0([P]-[Q]) = \operatorname{tr} P - \operatorname{tr} Q$. This is well-defined because of the following obvious property of the trace: $\operatorname{tr}(P \oplus P') = \operatorname{tr} P + \operatorname{tr} P'$.

Example 1.b.4. (Hattori) Let G be a finite group and V a finite dimensional representation of G over $\mathbb{C}$. Then V is a projective module over the group ring $\mathbb{C}[G]$ and

$$\operatorname{tr} V = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)\, g$$

where $\chi_V \colon G \to \mathbb{C}^*$ is the character of the representation V. This formula shows that

$$1_{\mathbb{C}} \otimes h_0(\mathbb{C}[G]) \colon \mathbb{C} \otimes K_0\mathbb{C}[G] \longrightarrow H_0(\mathbb{C}[G]; \mathbb{C}[G])$$

is an isomorphism.

1.c. The bar construction for $H_n(G; A)$.

Let G be a group and A, B two left G-modules. Then $A \underset{G}{\otimes} B$ denotes the quotient of $A \otimes B$ by the diagonal left action of G (i.e., $a \otimes b \cong ga \otimes gb$.) Let

$C_n(G)$ be the free left G-module generated by the set G^n and let $d_n: C_n(G) \to C_{n-1}(G)$ be given by

$$\begin{aligned} d_n[g_1, \dots, g_n] = & \; g_1[g_2, \dots, g_n] \\ & + \sum_{i=1}^{n-1} (-1)^i [g_1, \dots, g_i g_{i+1}, \dots, g_n] \\ & + (-1)^n [g_1, \dots, g_{n-1}] . \end{aligned}$$

Then $H_n(G;A)$ is the n-th homology of the complex $(C_*(G) \underset{G}{\otimes} A;\, d_* \otimes 1_A)$. We shall denote $C_n(G) \underset{G}{\otimes} A$ by $C_n(G;A)$.

1.d. Proof that $\mathcal{H}_n(\mathbb{Z}[G]; M) \cong H_n(G; \overline{M})$.

We repeat the standard argument. (See for example [11].)

Theorem 1.d. Let G be a group and M a $\mathbb{Z}[G]$-bimodule. Let $\overline{M}$ be M considered as a left G-module by the conjugation action $g \cdot m = g m g^{-1}$. Then there is a natural isomorphism of chain complexes $f_p: C_p(G;\overline{M}) \xrightarrow{\approx} \mathcal{H}_p(\mathbb{Z}[G]; M)$ given by

$$f_p([g_1, \dots, g_p] \otimes m) = g_1 \otimes \dots \otimes g_p \otimes (g_1 \dots g_p)^{-1} m$$

Proof: One can easily verify that f_p is a chain map. Let $\ell_p: C_p(\mathbb{Z}[G]; M) \to C_p(G;\overline{M})$ be defined by

$$\ell_p\Big(\sum_{i_1} g_{i_1} \otimes \dots \otimes \sum_{i_p} g_{i_p} \otimes m\Big) = \sum_{i_1, \dots, i_p} [g_{i_1}, \dots, g_{i_p}] \otimes g_{i_1} \dots g_{i_p} m$$

Then $\ell_n f_n = 1$ and $f_n \ell_n = 1$. □

1.e. Morita invariance of $\mathcal{H}_n(R; M)$.

Let R and S be two rings with 1. We say that R and S are Morita equivalent if there is an R-S-bimodule A and an S-R-bimodule B such that $A \underset{S}{\otimes} B \cong R$ and $B \underset{R}{\otimes} A \cong S$ as bimodules. This implies that $A \underset{S}{\otimes}$: S-Mod → R-Mod is an isomorphism of categories, its inverse being given by $B \underset{R}{\otimes}$. Similarly $\underset{R}{\otimes} A$: Mod-R → Mod-S and $B \underset{R}{\otimes} - \underset{R}{\otimes} A$: R-Mod-R → S-Mod-S are isomorphisms of categories. Note that A is projective in R-Mod and in Mod-S since it corresponds to S and R in S-Mod and

Mod-R respectively. Similarly B is projective over both rings.

Lemma 1.e.1. Let M be a left R-module. If $M \otimes R$ is considered as an R-bimodule in the standard way then

$$H_i(R;\ M \otimes R) \cong \begin{cases} M & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

Proof: Let $\varepsilon\colon H_0(R;\ M \otimes R) \cong M \otimes R \to M$ be the augmentation map given by $\varepsilon(x \otimes r) = rx$. We can define a chain contraction of the augmented complex by $f_n(r_1 \otimes \ldots \otimes r_n \otimes x \otimes r) = (r \otimes r_1 \otimes \ldots \otimes r_n \otimes x \otimes 1)$ and $\eta(x) = x \otimes 1$. □

Lemma 1.e.2. Let M be a left R-module and A a projective right R-module. Then

$$H_i(R;\ M \otimes A) \cong \begin{cases} A \otimes_R M & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

Proof: $H_*(R;\ M \otimes A)$ is the homology of the complex $C_*(R;\ M \otimes A) \cong A \otimes_R C_*(R;\ M \otimes R)$ where R acts on $C_n(R;\ M \otimes R) = R \otimes \ldots \otimes R \otimes M \otimes R$ by left multiplication on the last (right hand) factor. The lemma follows from 1.e.1 and the exactness of $A \otimes_R$. □

By right-left symmetry we also have:

Lemma 1.e.3. Let M be a right R-module and A a projective left R-module. Then

$$H_i(R;\ A \otimes M) \cong \begin{cases} M \otimes_R A & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$ □

Theorem 1.e.4. Let R, S be rings with 1. Let A be an R-S-bimodule which is projective over both rings and let B be any S-R-bimodule. Then there is a natural sequence of isomorphisms

$$F_n\colon H_n(R;\ A \otimes_S B) \xrightarrow{\ \cong\ } H_n(S;\ B \otimes_R A)$$

which vary functorially with the 4-tuple (R,A,S,B).

Corollary 1.e.5. (R.K. Dennis) Let M be an R-bimodule. Then a chain homotopy equivalence $f^k_*\colon C_*(M_k(R);\ M_k(M)) \to C_*(R;M)$ can be given by the formula

$$f^k_n(X^1 \otimes X^2 \otimes \ldots \otimes X^n \otimes Y) = \sum_{a,b,\ldots,z=1}^{k} (x^1_{ab} \otimes x^2_{bc} \otimes \ldots \otimes x^n_{yz} \otimes y_{za})$$

where x^i_{pq} and y_{pq} are the pq entries of X^i and Y.

Proof: Let R^k be the set of $k \times 1$ matrices with coefficients in R and let M^k be the set of $1 \times k$ matrices with coefficients in M. Then the 4-tuples (R,R,R,M) and $(M_k(R),R^k,R^k,M^k)$ satisfy the conditions of theorem 1.e.4. Furthermore there is a map $G\colon (R,R,R,M) \to (M_k(R),R^k,R,M^k)$ given by

$$G(r,s,t,m) = \left(D(r), \begin{pmatrix} s \\ 0 \\ \vdots \\ 0 \end{pmatrix}, t, (m\,0\,\ldots\,0)\right)$$

where

$$D(r) = \begin{pmatrix} r & & 0 \\ & r & \\ & & \ddots \\ 0 & & & r \end{pmatrix}$$

Thus we get the following commuting square.

$$\begin{array}{ccc} H_n(R;\,M) & \xrightarrow[F_n]{\approx} & H_n(R;\,M) \\ \downarrow G_1 & & \downarrow G_2 \\ H_n(M_k(R);\,M_k(M)) & \xrightarrow[F_n']{\approx} & H_n(R;\,M) \end{array}$$

It is easy to see that G_2 is the identity map (and thus G_1 is an isomorphism) and that G_1 is the map in Hochschild homology induced by the map $g\colon (R,M) \to (M_k(R), M_k(M))$ given by $g(r,m) = (D(r),T(m))$ where $T(m)$ is the $k \times k$ matrix with an m in the upper left corner and 0's elsewhere. Thus at the chain level G_1 is given by $g_n\colon C_n(R;M) \to C_n(M_k(R);\, M_k(M))$, $g_n(r_1 \otimes \ldots \otimes r_n \otimes m) = D(r_1) \otimes \ldots \otimes D(r_n) \otimes T(m)$. Clearly $f_n^k g_n$ is the identity map so $f_\star^k$ must be a chain equivalence. □

Proof of 1.e.4: Let $(C_{pq}, d_\star^1, d_\star^2)$ be the bicomplex given by

$$\begin{aligned} C_{pq} &= \underbrace{R \otimes \ldots \otimes R}_{p-1} \otimes A \otimes \underbrace{S \otimes \ldots \otimes S}_{q-1} \otimes B \\ &= C_{p-1}(R;\, A \otimes \underbrace{S \otimes \ldots \otimes S}_{q-1} \otimes B) \\ &\cong C_{q-1}(S;\, B \otimes \underbrace{R \otimes \ldots \otimes R}_{p-1} \otimes A) \end{aligned}$$

where the isomorphism is given by cyclically permuting the terms. Let $d_{pq}^1\colon C_{pq} \to C_{p-1,q}$ be the boundary map for the complex $C_\star(R;\, A \otimes \overbrace{S \otimes \ldots \otimes S}^{q-1} \otimes B)$ and let $(-1)^{p-1} d_{pq}^2\colon C_{pq} \to C_{p,q-1}$ be the boundary map for the complex $C_\star(S;\, B \otimes \overbrace{R \otimes \ldots \otimes R}^{p-1} \otimes A)$. Then clearly $d^1 d^1 = 0$, $d^2 d^2 = 0$ and $d^1 d^2 + d^2 d^1 = 0$ so we have a bicomplex. Let $X_n = \bigoplus_{p+q=n} C_{pq}$ and $d_n\colon X_n \to X_{n-1}$ be the map given by $d_n = d^1 + d^2$. Since

$(C_{p\star},d^2)$ has only one dimensional homology we have, by 1.e.1, a functorial chain isomorphism $(X_{p+1},d_{p+1}) \cong (H_1(C_{p\star},d^2),d^1)$ given by the composition $X_{p+1} \to C_{p1} \to H_1(C_{p\star},d^2)$ where the first map is projection onto a factor and the second comes from taking $H_1(C_{p\star},d^2) = \text{coker}(d^2_{p2}\colon C_{p2} \to C_{p1})$. Also by 1.e.1 we have a functorial isomorphism $(H_1(C_{p\star},d^2),d^1) \cong (C_{p-1}(R;\, A \underset{S}{\otimes} B),d_\star)$. By the symmetrical argument using 1.e.2 we get a functorial chain isomorphism $(X_{q+1},d_{q+1}) \cong (H_1(C_{\star q},d^1),d^2) \cong (C_{q-1}(S;\, B \underset{R}{\otimes} A),d_\star)$. When we compose one with the inverse of the other we get a functorial isomorphism $H_\star(R;\, A \underset{S}{\otimes} B) \cong H_\star(S;\, B \underset{R}{\otimes} A)$. □

1.f. Stabilization of Dennis's map.

The map h_n^{k+} is represented at the chain level by the following sequence of chain maps whose composition we shall call $\tilde{h}_n^{k+}$.

Formula 1.f.1. (R.K. Dennis)

$$
\begin{array}{llll}
 & [X^1, \ldots, X^n] & \in C_n(GL_k(R); \mathbb{Z}) & \\
\longrightarrow & [X^1, \ldots, X^n] \otimes I_k & \in C_n(GL_k(R); M_k(R)) & \\
\longrightarrow & X^1 \otimes \ldots \otimes X^n \otimes (X^1X^2 \ldots X^n)^{-1} & \in C_n(\mathbb{Z}[GL_k(R)]; M_k(R)) & (1) \\
\longrightarrow & X^1 \otimes \ldots \otimes X^n \otimes (X^1X^2 \ldots X^n)^{-1} & \in C_n(M_k(R); M_k(R)) & \\
\longrightarrow & \sum_{a,\ldots,z=1}^{k} x^1_{ab} \otimes \ldots \otimes x^n_{yz} \otimes y_{za} & \in C_n(R;R) & (2)
\end{array}
$$

(1) This is the isomorphism of 1.d.

(2) This is the map given in 1.e.5.

Theorem 1.f.2. (R.K. Dennis) Let $n \geq 1$. If $j\colon H_n(GL_k(R); \mathbb{Z}) \to H_n(GL_{k+1}(R); \mathbb{Z})$ is the map induced by inclusion then $h_n^{k+} = h_n^{k+1+}j$. Consequently we have a map defined on the limit:

$$h_n^{+}(R)\colon H_n(GL(R); \mathbb{Z}) \to H_n(R;R).$$

Corollary 1.f.3. (R.K. Dennis) If $j'\colon K_n^kR \to K_n^{k+1}R$ is the map induced by the inclusion $(GL_k(R),E_k(R)) \subset (GL_{k+1}(R),E_{k+1}(R))$ then $h_n^k = h_n^{k+1}j'$. Consequently we have a map defined on the limit:

$$h_n(R)\colon K_nR \to H_n(R;R). \qquad \square$$

Proof of 1.f.2: From formula 1.f.1 one can easily see that $(\tilde{h}_n^{k+1+}\tilde{j} - \tilde{h}_n^{k+})[X^1, \ldots, X^n] = 1 \otimes \ldots \otimes 1 \otimes 1 \to C_n(R;R)$ where $\tilde{j}$ is the inclusion map. So if $n \geq 1$ we have $\tilde{h}_n^{k+1+}\tilde{j} - \tilde{h}_n^{k+} = df_n + f_{n-1}d$ where $f_n[X^1, \ldots, X^n] = 1 \otimes \ldots \otimes 1 \in C_{n+1}(R;R)$. $\square$

Proposition 1.f.4. Let M be an R-bimodule. Then the following composition of maps is compatible with stabilization with respect to k. ($n \geq 0$.)

$$H_n(GL_k(R);\, M_k(M)) \xrightarrow{\approx} H_n(\mathbb{Z}[GL_k(R)];\, M_k(M)) \longrightarrow H_n(M_k(R);\, M_k(M)) \xrightarrow{\approx} H_n(R;\, M)\,.$$

Proof: At the chain level the map is given as follows.

$$\begin{array}{llcl} & [X^1, \ldots, X^n] \otimes Y & \in & C_n(GL_k(R);\, M_k(M)) \\ \longrightarrow & X^1 \otimes \ldots \otimes X^n \otimes (X^1 \ldots X^n)^{-1}Y & \in & C_n(M_k(R);\, M_k(M)) \\ \longrightarrow & \sum_{a,\ldots,z=1}^{k} x^1_{ab} \otimes \ldots \otimes x^n_{yz} \otimes z_{za} & \in & C_n(R;\, M) \end{array}$$

After stabilization the $(k+1)$-st column of $(z_{**}) = (X^1 \ldots X^n)^{-1}Y$ is trivial. Thus this chain map strictly commutes with stabilization for all $n \geq 0$. $\square$

1.g. Some splitting theorems.

Take $R = \mathbb{Z}[G]$ and let

$$\ell h_n \colon \quad K_n\mathbb{Z}[G] \longrightarrow H_n(G;\, \mathbb{Z}[G])$$

$$\ell h_n^+\colon H_n(GL(\mathbb{Z}[G]);\mathbb{Z}) \to H_n(G;\, \mathbb{Z}[G])$$

be the maps given by composing h_n and h_n^+ with the isomorphism of 1.d. These maps will be called the Dennis trace maps for K-theory and homology respectively. Let ℓh_n^k and ℓh_n^{k+} be defined analogously.

Theorem 1.g.1. (R.K. Dennis) Let G be any group and $n \geq 1$. Then the map $H_n(G;\, \mathbb{Z}) \to H_n(G;\, \mathbb{Z}[G])$ induced by the coefficient map $\mathbb{Z} \to \mathbb{Z}[G]$ which sends 1 to 1 is equal to the composition $H_n(G;\, \mathbb{Z}) \to H_n(GL(\mathbb{Z}[G]);\mathbb{Z}) \to H_n(G;\, \mathbb{Z}[G])$ where the first map is induced by the inclusion $G \subset \mathbb{Z}[G]^\times = GL_1(\mathbb{Z}[G]) \subset GL(\mathbb{Z}[G])$ and the second map is the Dennis trace map.

Proof: Examine the effect of h_n^+ and ℓ_* at the chain level:

$$\begin{array}{llll} & [g_1, \dots, g_n] \otimes 1 & \in & C_n(G; \mathbb{Z}) \\ \longrightarrow & [g_1, \dots, g_n] \otimes 1 & \in & C_n(GL_1(\mathbb{Z}[G]); \mathbb{Z}) \\ \xrightarrow{\tilde{h}_n^{1+}} & g_1 \otimes \dots \otimes g_n \otimes (g_1 \dots g_n)^{-1} & \in & C_n(\mathbb{Z}[G]; \mathbb{Z}[G]) \\ \xrightarrow{\ell_n} & [g_1, \dots, g_n] \otimes 1 & \in & C_n(G; \mathbb{Z}[G]) \end{array}$$

□

Corollary 1.g.2. (R.K. Dennis) $H_n(G; \mathbb{Z})$ is a direct summand of $H_n(GL_k(\mathbb{Z}[G]); \mathbb{Z})$ for $1 \leq k \leq \infty$ and for all n, G. □

Corollary 1.g.3. Let $n \geq 2$ and let G be a group with $H_i(G; \mathbb{Z}) = 0$ for $0 < i < n$. Then the composition

$$K_n \mathbb{Z}[G] \xrightarrow{\ell h_n} H_n(G; \mathbb{Z}[G]) \xrightarrow{\varepsilon_*} H_n(G; \mathbb{Z})$$

is a split epimorphism if ε_* is the map induced by the augmentation map $\varepsilon: \mathbb{Z}[G] \to \mathbb{Z}$.

Proof: Since G is perfect we can perform the plus construction on BG and we get a map $BG^+ \to BGL(\mathbb{Z}[G])^+$. Now consider the following composition of maps.

$$H_n(G) \cong \pi_n BG^+ \to \pi_n BGL(\mathbb{Z}[G])^+ = K_n \mathbb{Z}[G] \to H_n(GL(\mathbb{Z}[G])) \xrightarrow{\ell h_n^+} H_n(G; \mathbb{Z}[G]) \xrightarrow{\varepsilon_*} H_n(G)$$

The first map is the inverse of the Hurewicz isomorphism. The third map is the Hurewicz map. The composition is the identity by 1.g.1 and the naturality of the Hurewicz map. □

Unfortunately this corollary does not produce any interesting topological examples since the only part of the Dennis trace map which is relevant to pseudoisotopy theory is the quotient map:

$$K_n \mathbb{Z}[G] \longrightarrow \frac{H_n(G; \mathbb{Z}[G])}{H_n(G; \mathbb{Z})} = Wh_{n+1}^+(G; \mathbb{Z})$$

Let $M_k(G)$ be the group of all $k \times k$ monomial matrices with coefficients in G. Then $M_k(G)$ is a subgroup of $GL_k(\mathbb{Z}[G])$ and thus there is an induced map in homology which satisfies the following.

Theorem 1.g.4. The following composition is trivial for all G, n and k.

$$H_n(M_k(G);\mathbb{Z}) \to H_n(GL_k(\mathbb{Z}[G]);\mathbb{Z}) \xrightarrow[\ell h_n^{k+}]{} H_n(G;\mathbb{Z}[G]) \to Wh_{n+1}{}^{+}(G;\mathbb{Z}).$$

Proof: Using 1.f.1 we see that the above composition does the following at the chain level.

$$\begin{array}{lll} & [X^1, \ldots, X^n] & \varepsilon\ C_n(M_k(G);\mathbb{Z}) \\ \longrightarrow & \Sigma\, x^1_{ab} \otimes \ldots \otimes x^n_{yz} \otimes y_{za} & \varepsilon\ C_n(\mathbb{Z}[G];\mathbb{Z}[G]) \\ \longrightarrow & \Sigma\, [x^1_{ab} \otimes \ldots \otimes x^n_{yz}] \otimes 1 & \varepsilon\ C_n(G;\mathbb{Z}[G]) \\ \longrightarrow & 0 & \varepsilon\ C_n(G;\mathbb{Z}[G])/C_n(G;\mathbb{Z}) \end{array}$$

where $y_{za} = ((X^1 \ldots X^n)^{-1})_{za} = (x^1_{ab} \ldots x^n_{yz})^{-1}$ since the matrices are monomial. □

Let $M(G) \subset GL(\mathbb{Z}[G])$ be the union of all the $M_k(G)$'s. Since $M_k(G)'$ is perfect for $k \geq 5$ we can construct $BM(G)^+$.

Corollary 1.g.5. The following composition is trivial.

$$\pi_n BM(G)^+ \to K_n\mathbb{Z}[G] \xrightarrow[\ell h_n]{} H_n(G;\mathbb{Z}[G]) \to Wh_{n+1}{}^{+}(G;\mathbb{Z}). \qquad \square$$

Corollary 1.g.6. $H_n(G;\mathbb{Q})$ is a direct summand of $\mathbb{Q} \otimes K_n\mathbb{Z}[G]$.

Proof: Use the fact that BG and $BM_k(G)$ are rationally equivalent. □

Chapter 2. Homology of group extensions.

Let G be a group and A a left G-module. Let $A \to E \overset{p}{\to} G$ be an extension of A by G. We shall construct a sequence of natural transformations of the following form.

$$b_n\colon H_{n+1}(G,E;\mathbb{Z}) \to H_{n-1}(G;A)$$

The maps arise from the Lyndon spectral sequence of the extension $A \to E \to G$ but knowledge of this is not required.

We shall construct b_n at the chain level using a transversal $t: G \to E$ (a set theoretic right inverse to $p: E \to G$) and show that chain maps arising from different transversals are homotopic. In 2.d and 2.e we show that b_n fits into certain commutative diagrams which will be used later. In 2.a and 2.b we review some basic notation. The section headings are:

2.a. The mapping cone construction.

This is taken from [11] p. 46.

Let $C_{n+1}(G,E;\mathbb{Z}) = C_n(E;\mathbb{Z}) \oplus C_{n+1}(G;\mathbb{Z})$ and define the boundary map by $d(a,b) = (-da, db + p_* a)$. Then $H_*(G,E;\mathbb{Z})$ is the homology of $C_*(G,E;\mathbb{Z})$. The map $\partial: H_{n+1}(G,E;\mathbb{Z}) \to H_n(E;\mathbb{Z})$ in the long exact homology sequence of the pair (G,E) is induced by the chain map given by $\partial(a,b) = (-1)^n a$.

2.b. Factor sets.

Let $t: G \to E$ be a transversal for p. Then every element of E can be written as $(a,g) = a\,t(g)$ where $a \in A$ and $g \in G$. Multiplication is given by $(a,g)(b,h) = (a + gb + f(g,h), gh)$ where $f: G \times G \to A$ is the factor set $f(g,h) = t(g)t(h)t(gh)^{-1}$. Of course f represents a 2-cocycle in $C^2(G;A) = \mathrm{Hom}_G(C_2(G),A)$.

2.c. Definition of b_n.

Let $c_n: C_{n+1}(G,E;\mathbb{Z}) \to C_{n-1}(G;A)$ be defined on the free generators of $C_{n+1}(G,E;\mathbb{Z})$ as follows.

$$c_n([(a_1,g_1), \ldots ,(a_n,g_n)],0) = -[g_2, \ldots ,g_n] \otimes g_1^{-1}a_1$$

$$c_n(0,[g_1, \ldots ,g_{n+1}]) = [g_3, \ldots ,g_{n+1}] \otimes g_2^{-1}g_1^{-1}f(g_1,g_2)$$

A straightforward computation shows that this is a chain map.

Exercise 2.c. Show that c_n induces an isomorphism in homology if $n = 1$ and an epimorphism if $n = 2$.

Suppose that $t'\colon G \to E$ is another transversal for p. Then $(a,g) = at(g) = at(g)t'(g)^{-1}t'(g) = (a - \alpha(g), g)'$ where $\alpha(g) = t'(g)t(g)^{-1}$. Using t' instead of t we get

$$c_n'([(a_1,g_1), \ldots ,(a_n,g_n)],0) = -[g_2, \ldots ,g_n] \otimes g_1^{-1}(a_1 - \alpha(g_1))$$
$$c_n'(0,[g_1, \ldots ,g_{n+1}]) = [g_3, \ldots ,g_n] \otimes g_2^{-1}g_1^{-1}f'(g_1,g_2)$$

where $f'(g_1,g_2) = f(g_1,g_2) + g_1\alpha(g_2) - \alpha(g_1g_2) + \alpha(g_1)$. Let $e_n\colon C_{n+1}(G,E;\mathbb{Z}) \to C_n(G;A)$ be given by $e_n(x,0) = 0$ and $e_n(0,[g_1, \ldots ,g_{n+1}]) = [g_2, \ldots ,g_{n+1}] \otimes g_1^{-1}\alpha(g_1)$. Then a straightforward computation shows that $c_{n+1}' - c_{n+1} = e_{n-1}d + de_n$. Thus c_* induces a functorial map b_* in homology.

2.d. The first diagram.

Let $0 \to A \to B \xrightarrow{q} C \to 0$ be a short exact sequence of left G-modules and let $x \in H^0(G;C)$. Since $H^0(G;C) = \mathrm{Hom}_G(\mathbb{Z}, C)$ we may regard x as an element of C which is fixed under the action of G. Let $\beta^*\colon H^0(G;C) \to H^1(G;A)$ be the connecting homomorphism in cohomology. Then $\beta^*(x)$ is represented by a crossed homomorphism $\alpha\colon G \to A$ given by $\alpha(g) = gy - y$ where y is a fixed element of $q^{-1}(x)$. This produces a homomorphism $f\colon G \to A \tilde{\times} G$ = the semidirect product by $f(g) = (\alpha(g),g)$.

Theorem 2.d. The following diagram commutes up to sign for all n. (Here β_* is the connecting homomorphism in homology and the isomorphism is given by the boundary map.)

$$\begin{array}{ccccc}
H_n(G;\mathbb{Z}) & & \xrightarrow{\quad x_* \quad} & & H_n(G;C) \\
\downarrow{\scriptstyle f_*} & & (-1)^n & & \downarrow{\scriptstyle \beta_*} \\
\dfrac{H_n(A \tilde{\times} G;\mathbb{Z})}{H_n(G;\mathbb{Z})} & \xrightarrow{\approx} & H_{n+1}(G, A \tilde{\times} G;\mathbb{Z}) & \xrightarrow[b_n]{} & H_{n-1}(G;A)
\end{array}$$

Proof: Let $e\colon C_{n-1}(G;\mathbb{Z}) \to C_{n-1}(G;B)$ be given by $e[g_1, \ldots ,g_{n-1}] = [g_1, \ldots ,g_{n-1}] \otimes y$. Then a chain map representing β_*x_* is given as follows. Take $[g_1, \ldots ,g_n] \in$

$C_n(G;\mathbb{Z})$. Map this to $[g_1, \dots, g_n] \otimes x \in C_n(G;C)$, pull it back to $[g_1, \dots, g_n] \otimes y \in C_n(G;B)$ then take $d([g_1, \dots, g_n] \otimes y) \in C_{n-1}(G;B)$. To get this into $C_{n-1}(G;A)$ subtract $ed[g_1, \dots, g_n]$. Computation will show that the result is $[g_2, \dots, g_n] \otimes (g_1^{-1}y - y) = -[g_2, \dots, g_n] \otimes g_1^{-1}\alpha(g_1) = c_n([(\alpha(g_1),g_1), \dots ,(\alpha(g_n),g_n)],0) = (-1)^n c_n(f_*[g_1, \dots, g_n],0)$. □

2.e. The second diagram.

Let $0 \to A \to B \to C \to 0$ be a short exact sequence of G-modules and let $B \to E \to G$ be an extension of B by G. Let $F = E/A$.

Theorem 2.e.1. The following diagram commutes up to sign for all n.

$$\begin{array}{ccccc} H_{n+1}(G,F;\mathbb{Z}) & \xrightarrow{b_n} & H_{n-1}(G;C) & & \\ \partial \downarrow & & (-1)^{n+1} & \searrow^{\delta} & H_{n-2}(G;A) \\ H_n(F,E;\mathbb{Z}) & \xrightarrow[b_{n-1}]{} & H_{n-2}(F;A) & \nearrow_{p_*} & \end{array}$$

∂ = the boundary map for the triple (G,F,E).

δ = the connecting homomorphism corresponding to the coefficient sequence $A \to B \to C$.

p_* = the map induced by the projection $p: F \to G$.

Proof: In this proof group multiplication will be written additively and the letters a, b, c, g with various indices will indicate elements of A, B, C, G respectively. Let $t: G \to F$, $u: G \to E$, $r: C \to B$ be transversals which send 0 to 0. Then each element of F can be written uniquely in the form $(c,g) = c + t(g)$. Let $s: F \to E$ be defined by $s(c,g) = r(c) + u(g)$. Then $r = s|C$ and $u = st$. Thus the formula for s can be written as

$$(*) \qquad s(c,g) = s(c) + st(g).$$

Every element of E can be written uniquely in the form $(a,(c,g)) = a + s(c,g) = a + s(c) + st(g)$. If we think of E as an extension of B by G then each element of E has the unique form $(b,g) = b + u(g) = b + st(g)$. But elements of B can be written in the unique form $(a,c) = a + r(c) = a + s(c)$ so $((a,c),g) = a + s(c) + st(g) = (a,(c,g))$. Let (a,c,g) denote either of these.

Let $f_t\colon G \times G \to C$, $f_{st}\colon G \times G \to B$, $f_s\colon F \times F \to A$ be the factor sets corresponding to the indicated transversals. That is, $f_t(g_1,g_2) = t(g_1) + t(g_2) - t(g_1+g_2)$, $f_{st}(g_1,g_2) = st(g_1) + st(g_2) - st(g_1+g_2)$ and $f_s((c_1,g_1),(c_2,g_2)) = s(c_1,g_1) + s(c_2,g_2) - s((c_1,g_1) + (c_2,g_2))$. The following formulas can easily be verified.

$$f_{st}(g_1,g_2) = (f_s(t(g_1),t(g_2)),f_t(g_1,g_2))$$

$$(\dagger) \qquad f_s((c_1,g_1),(c_2,g_2)) = s(c_1) + g_1 s(c_2) - s(c_1 + g_2 c_2 + f_t(g_1,g_2)) + f_{st}(g_1,g_2)$$

We are now ready to prove the theorem. The proof is based on the following well-known fact.

<u>Lemma 2.e.2</u>. A map between two short exact sequences of chain complexes will produce maps in homology compatible with the corresponding connecting homomorphisms. Thus a map between long exact sequences will result. □

Take the following map of short exact sequences of chain complexes where Y_* and k_* are defined below.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_*(F,E;\mathbb{Z}) & \xrightarrow{\alpha} & Y_* & \xrightarrow{\beta} & C_*(G,F;\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p_*c_*} & & \downarrow{\scriptstyle k_*} & & \downarrow{\scriptstyle c_*} & & \\
0 & \longrightarrow & C_{*-2}(G;A) & \longrightarrow & C_{*-2}(G;B) & \longrightarrow & C_{*-2}(G;C) & \longrightarrow & 0
\end{array}$$

$Y_{n+1} = C_{n+1}(F,E;\mathbb{Z}) \oplus C_{n+1}(G,F;\mathbb{Z}) = C_n(E;\mathbb{Z}) \oplus C_{n+1}(F;\mathbb{Z}) \oplus C_n(F;\mathbb{Z}) \oplus C_{n+1}(G;\mathbb{Z})$

$d\colon Y_{n+1} \to Y_n$ is given by $d(w,x,y,z) = (-dw, dx + \overline{w} - y, -dy, dz + \overline{y})$ where $(\overline{\ \ })$ means image in the appropriate group.

α is the inclusion of the first factor.

β is the projection onto the second factor.

From the formula for d it should be clear that α, β are chain maps and that the connecting homorphism is $(-1)^{n+1}$ times the boundary map $\partial\colon H_{n+1}(G,F;\mathbb{Z}) \to H_n(F,E;\mathbb{Z})$. Thus we need only define a chain map $k_*\colon Y_* \to C_{*-2}(G;B)$ which makes the diagram commute. Let k_{n+1} be defined on the generators of Y_{n+1} as follows.

$$k_{n+1}([(a_1,c_1,g_1), \ldots ,(a_n,c_n,g_n)],0,0,0) = -[g_2, \ldots ,g_n] \otimes g_1^{-1}\overline{a}_1$$

$$k_{n+1}(0,[(c_1,g_1), \ldots ,(c_{n+1},g_{n+1})],0,0) = [g_3, \ldots ,g_{n+1}] \otimes g_2^{-1}g_1^{-1}f_s((c_1,g_1),(c_2,g_2))$$

$k_{n+1}(0,0,[(c_1,g_1), \ldots ,(c_n,g_n)],0) = -[g_2, \ldots ,g_n] \otimes g_1^{-1}s(c_1)$

$k_{n+1}(0,0,0,[g_1, \ldots ,g_{n+1}]) = [g_3, \ldots ,g_{n+1}] \otimes g_2^{-1}g_1^{-1}f_{st}(g_1,g_2)$

Calculation will show (using equation †) that $k_\star$ is a chain map. Compatibility with $p_\star c_\star$ and $c_\star$ is trivial. □

Chapter 3. Hochschild cohomology.

We define Hochschild cohomology and review its basic properties. The section headings are:

3.a. Definition of $H^n(R;M)$.

If R is a ring with 1 let $F_R: \mathbb{Z}\text{-Mod} \to R\text{-Mod-}R$ be the functor given by $F_R(A) = R \otimes A \otimes R$. Then we have a natural isomorphism $\mathrm{Hom}_{\mathbb{Z}}(A,M) \xrightarrow{\psi} \mathrm{Hom}_{R-R}(F_R,M)$ given by $\psi(f)(r \otimes a \otimes s) = rf(a)s$ and thus F_R is adjoint to the forgetful functor $R\text{-Mod-}R \to \mathbb{Z}\text{-Mod}$.

Let $C_n(R) = R^{\otimes(n+2)} = F_R(R^{\otimes n})$ be the tensor product of $n + 2$ copies of R if $n \geq 0$ and $C_n(R) = 0$ if $n < 0$. Let $d_n: C_n(R) \to C_{n-1}(R)$ be given by

$$d_n(r_0 \otimes \ldots \otimes r_{n+1}) = \sum_{i=0}^{n} (-1)^i r_0 \otimes \ldots \otimes r_i r_{i+1} \otimes \ldots \otimes r_{n+1} .$$

Let $C^n(R;M) = \mathrm{Hom}_{R-R}(C_n(R),M) \cong \mathrm{Hom}_{\mathbb{Z}}(R^{\otimes n},M)$. Then the n-th cohomology of the cochain complex $(C^\star(R;M),d^\star)$ will be denote by $H^n(R;M)$ and will be called the n-th <u>Hochschild cohomology</u> group of R with coefficients in M.

The standard interpretations of H^0, H^1 and H^2 are as follows. (From [11])

3.b. Interpretation of $H^2(R;M)$.

An additive map $f\colon R \otimes R \to M$ is a 2-cocycle or linear factor set if it satisfies the condition that for all $r, s, t \in R$ we have

$$rf(s \otimes t) - f(rs \otimes t) + f(r \otimes st) - f(r \otimes s)t = 0 .$$

A linear factor set f can be used to construct a ring $M \oplus_f R$ as follows. The additive structure of $M \oplus_f R$ will be the same as that of $M \oplus R$ but the multiplication will be given as follows.

$$(m,r)(n,s) = (ms + rn + f(r \otimes s), rs)$$

The ring $M \oplus_f R$ has the following properties.

(1) Projection onto the second factor is a ring map $M \oplus_f R \to R$.

(2) The kernel of this map is a square zero ideal $(M^2 = 0)$ isomorphic to M as an R-bimodule.

(3) Any ring extension $E \to R$ whose kernel is a square zero ideal isomorphic to M is equivalent to an extension $M \oplus_f R$ for some 2-cocycle f provided that $E \to R$ splits as a map of $\mathbb{Z}$-modules.

(4) The extensions $M \oplus_f R$ and $M \oplus_{f'} R$ are equivalent if and only if $[f] = [f']$ in $H^2(R;M)$.

(5) Every cohomology class contains a normalized 2-cocycle f satisfying the condition that $f(r \otimes s) = 0$ if either r or s is 1. Thus $M \oplus_f R$ contains the unit $(0,1)$.

3.c. Interpretation of $H^1(R;M)$.

If $f = 0$ then $M \oplus_f R$ is called the semidirect sum and denoted $M \oplus R$. Ring homomorphisms $R \to M \oplus R$ which are sections of the projection map $M \oplus R \to R$ correspond to elements of $H^1(R;M)$. An additive map $f\colon R \to M$ is a 1-cocycle or linear crossed homomorphism if it satisfies the condition that $rf(s) = f(r)s$ for all r, s

in R. The cocycle is normalized if $f(1) = 0$. A map f satisfying these conditions can be used to construct a ring homomorphism $g: R \to M \oplus R$ given by $g(r) = (f(r),r)$. The map g sends 1 to 1 since f is normalized.

3.d. Interpretation of $H^0(R;M)$.

We have the easy isomorphism $H^0(R;M) \cong \mathrm{Hom}_{R-R}(R;M) \cong \{x \in M \mid rx = xr \text{ for all } r \in R\}$.

3.e. Long exact sequences.

Let β: $0 \to L \to M \to N \to 0$ be a short exact sequence of R-bimodules. If we assume that R is projective as a $\mathbb{Z}$-module or that β splits over $\mathbb{Z}$ then the induced sequences

$$0 \to C_*(R;L) \to C_*(R;M) \to C_*(R;N) \to 0$$
$$0 \to C^*(R;L) \to C^*(R;M) \to C^*(R;N) \to 0$$

will be exact sequences of chain and cochain complexes which will lead to the expected long exact sequences in Hochschild homology and cohomology. The connecting homomorphisms in homology and cohomology in the long exact sequences for β will be called β_* and β^*. Since these will appear often we make the following assumption.

Assumption 3.e.1. For the rest of this paper rings represented by the letter "R" will be assumed to be projective as $\mathbb{Z}$-modules.

Remark 3.e.2. Under the above condition we have

$$H_n(R;M) \cong \mathrm{Tor}_n^{R \otimes R^{op}}(R;M) \quad \text{and}$$
$$H^n(R;M) \cong \mathrm{Ext}^n_{R \otimes R^{op}}(R;M)$$

since $C_*(R)$ is a projective $R \otimes R^{op}$ resolution of R.

3.f. Cap product.

Let $s \leq n$ and let M be an R-bimodule. The cap product

$$\cap: H^s(R;M) \otimes H_n(R;R) \to H_{n-s}(R;M)$$

will be defined at the chain level as follows. Suppose that $z = \sum_i r_i^1 \otimes \ldots \otimes r_i^{n+1}$ $\varepsilon\ C_n(R;R)$ and $f: C_s(R) \to M$ is an s-cochain. Then let $f \cap z \ \varepsilon\ C_{n-s}(R;M)$ be defined as follows.

<u>Formula 3.f.1.</u> $\quad f \cap z = (-1)^e \sum_i r_i^1 \otimes \ldots \otimes r_i^{n-s} \otimes f(1 \otimes r_i^{n-s+1} \otimes \ldots \otimes r_i^{n+1})$

where $e = \frac{s^2 - s}{2} + ns.$

One can easily verify that

$$d_{n-s}(f \cap z) = (d_s^* f) \cap z + (-1)^s f \cap d_n z$$

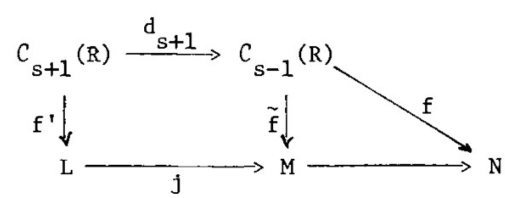
and thus 3.f.1 induces a map in homology.

Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules. Let β_* and β^* be the connecting homomorphisms in Hochschild homology and cohomology induced by this sequence.

<u>Theorem 3.f.2.</u> Let $\kappa\ \varepsilon\ H^s(R;N)$ and $x\ \varepsilon\ H_n(R;R)$. Then

$$\beta_*(\kappa \cap x) = (\beta^* \kappa) \cap x.$$

<u>Proof:</u> Let z be an n-cycle representing x and f an s-cocycle representing κ. Then f lifts to a map $\tilde{f}$ which induces a map f' as indicated below. By definition of β^*, f' represents $\beta^*\kappa$.



We now apply the diagram chase defining β_* to the cycle $f \cap z$. This cycle pulls back to the chain $\tilde{f} \cap z$ which maps down to $d_{n-s+1}(\tilde{f} \cap z) = (d_{s+1}^* \tilde{f}) \cap z = \tilde{f} d_{s+1} \cap z = jf' \cap z$ which is the inclusion map applied to $f' \cap z$. □

Together with h_n the cap product induces a map on K-theory as follows.

$$H^s(R;M) \otimes K_n R \xrightarrow[1 \cap h_n]{} H_{n-s}(R;M).$$

We will be particularly interested in the case $n = s$.

3.g. Proof that $H^n(\mathbb{Z}[G]; M) \cong H^n(G;\overline{M})$.

Theorem 3.g. Under the same conditions as 1.d we have a natural chain isomorphism $f^p\colon C^p(\mathbb{Z}[G]; M) \to C^p(G;\overline{M})$ given by

$$f^p(h)(g_0[g_1, \ldots ,g_p]) = h(g_0 \otimes g_1 \otimes \ldots \otimes g_p \otimes (g_0 \ldots g_p)^{-1}).$$

Proof: One can easily verify that f^p is a cochain map. Let $\overline{\ell}^p\colon C^p(G;M) \to C^p(\mathbb{Z}[G]; M)$ be defined by

$$\overline{\ell}^p(h)\left(\sum_{i_0} g_{i_0} \otimes \ldots \otimes \sum_{i_{p+1}} g_{i_{p+1}} \right) = \sum_{i_0, \ldots ,i_{p+1}} g_{i_0} h[g_{i_1}, \ldots ,g_{i_p}] \, g_{i_1} \cdots g_{i_{p+1}}$$

One can easily verify that $\overline{\ell}^p$ is the inverse of f^p. □

Chapter 4. K-theory of square zero ideals.

In this chapter we discuss the relationship between Hochschild homology and the K-theory of square zero ideal. We define a map $g_n\colon K_nI \to H_{n-1}(E/I;I)$ analogous to the map h_n if I is a square zero ideal in E. The definition uses the map b_n of chapter 2. Here are the section headings.

4.a. Definition of K_n of an ideal.

Let E be any ring with 1 and $M \subset E$ a two sided ideal. If $n = 0$, K_0M will be as defined by Milnor in ([12], p.33.) For $1 \leq n$ and $3 \leq k \leq \infty$ let K_n^kM be defined by

$$K^k_n M = \pi_{n+1}(BGL_k(E/M)^+, BGL_k(E)^+)$$

where $BGL_k(E)^+$ maps to $BGL_k(E/M)^+$ in the obvious way. We shall use the notation $K^k_n(M;E)$ when E needs to be specified. When $k = \infty$ it will be deleted from the notation.

By the long exact homotopy sequence of a pair we have a long exact sequence of the following form for $n \geq 1$.

$$\ldots \to K^k_{n+1}(E/M) \to K^k_n M \to K^k_n E \to K^k_n(E/M) \to \ldots$$

Using [12], lemma 4.1 this sequence can be extended to K_0 in the case $k = \infty$.

4.b. The Hochschild homology invariant for the K-theory of a square zero ideal.

Suppose now that $M^2 = 0$ and $3 \leq k < \infty$. Then the natural homomorphism $GL_k(E) \to GL_k(E/M)$ is surjective and its kernel is isomorphic to the additive group $\mathcal{M}_k(M)$. Let $g^k_n(M)$ or simply g^k_n be defined as the following composition.

$$\begin{aligned} K^k_n M = \pi_{n+1}(BGL_k(E/M)^+, BGL_k(E)^+) &\xrightarrow{\text{Hurewicz}} H_{n+1}(GL_k(E/M), GL_k(E); \mathbb{Z}) \\ &\xrightarrow{b_n} H_{n-1}(GL_k(E/M); \mathcal{M}_k(M)) \\ &\xrightarrow{\approx} \mathcal{H}_{n-1}(\mathbb{Z}[GL_k(E/M)]; \mathcal{M}_k(M)) \\ &\longrightarrow \mathcal{H}_{n-1}(\mathcal{M}_k(E/M); \mathcal{M}_k(M)) \\ &\xrightarrow{\approx} \mathcal{H}_{n-1}(E/M;M) \end{aligned}$$

After the Hurewicz map this composition will be called $g^{k+}_n\colon H_{n+1}(GL_k(E/M), GL_k(E); \mathbb{Z}) \to \mathcal{H}_{n-1}(E/M;M)$. By 1.f.4 these maps are compatible with stabilization and thus they induce maps in the limit

$$g_n\colon \quad K_n(M;E) \longrightarrow \mathcal{H}_{n-1}(E/M;M) \quad \text{and}$$

$$g_n^+\colon H_{n+1}(GL(E/M), GL(E); \mathbb{Z}) \to \mathcal{H}_{n-1}(E/M;M)$$

The map g^{k+}_n is given at the chain level as follows.

Formula 4.b. Let $t\colon E/M \to E$ be a transversal. Then very element of $GL_k(E)$ can be represented uniquely as a pair $(A,X) \in \mathcal{M}_k(M) \times GL_k(E/M)$ by $(A,X) = (I + A)t_*(X)$.

With this notation the map g_n^{k+} is given on $C_{n+1}(GL_k(E/M), GL_k(E))$ as follows.

$$\begin{array}{lll}
(1) & ([(A^1,X^1), \ldots, (A^n,X^n)],0) & \varepsilon\ C_n(GL_k(E);\mathbb{Z}) \\
\longrightarrow & -[X^2, \ldots, X^n] \otimes (X^1)^{-1}A^1 & \varepsilon\ C_{n-1}(GL_k(E/M); M_k(M)) \\
\longrightarrow & -X^2 \otimes \ldots \otimes X^n \otimes (X^1 \ldots X^n)^{-1}A^1 & \varepsilon\ C_{n-1}(\mathbb{Z}[GL_k(E/M)]; M_k(M)) \\
\longrightarrow & -X^2 \otimes \ldots \otimes X^n \otimes (X^1 \ldots X^n)^{-1}A^1 & \varepsilon\ C_{n-1}(M_k(E/M); M_k(M)) \\
\longrightarrow & -\sum_{a,\ldots,z=1}^{k} x^2_{ab} \otimes \ldots \otimes x^n_{yz} \otimes y_{za} & \varepsilon\ C_{n-1}(E/M;M)
\end{array}$$

$$\begin{array}{lll}
(2) & (0,[X^1, \ldots, X^{n+1}]) & \varepsilon\ C_{n+1}(GL_k(E/M);\mathbb{Z}) \\
\longrightarrow & [X^3, \ldots, X^{n+1}] \otimes (X^1X^2)^{-1}f(X^1,X^2) & \varepsilon\ C_{n-1}(GL_k(E/M); M_k(M)) \\
\longrightarrow & X^3 \otimes \ldots \otimes X^{n+1} \otimes (X^1 \ldots X^{n+1})^{-1}f(X^1,X^2) & \varepsilon\ C_{n-1}(M_k(E/M); M_k(M)) \\
\longrightarrow & \Sigma\, x^3_{ab} \otimes \ldots \otimes x^{n+1}_{yz} \otimes z_{za} & \varepsilon\ C_{n-1}(E/M);M)
\end{array}$$

where $(y_{**}) = (X^1 \ldots X^n)^{-1}A^1$, $(z_{**}) = (X^1 \ldots X^{n+1})^{-1}f(X^1,X^2)$ and $f(X^1,X^2) = t_*(X^1)t_*(X^2)t_*(X^1X^2)^{-1}$

4.c. Long exact sequences.

Let $M \subset E$ be a square zero ideal. Then M is an E/M-bimodule and any submodule L of M is an ideal in E. Let $N = M/L$. Then N is a square zero ideal in E/L and the long exact sequence for the triple $(BGL_k(E/M)^+, BGL_k(E/L)^+, BGL_k(E)^+)$ produces the following.

Theorem 4.c.1. There is a long exact sequence of the following form.

$$\ldots \to K^k_n L \xrightarrow{a_*} K^k_n M \xrightarrow{b_*} K^k_n N \xrightarrow{d} K^k_{n-1} L \to \ldots \to K^k_1 N$$

where a_*, b_* are induced by the ring maps $a: (E,L) \to (E,M)$ and $b: (E,M) \to (E/L,M/L)$ and d is the following composition

$$K^k_n N \xrightarrow{c_*} K^k_n(E/L) \xrightarrow{\partial} K^k_{n-1} L$$

where c_* is part of the long exact K-theory sequence for $(E/L,N)$ and ∂ is the boundary map for the long exact sequence for (E,L).

When $k = \infty$ this sequence extends to K_0 but this is not important here.

<u>Theorem 4.c.2</u>. Let $R = E/M$ be projective as a $\mathbb{Z}$-module. Then we have a map of long exact sequences of the following form which commutes up to sign. (The left two squares commute and the right hand square commutes up to the sign $(-1)^{n-1}$.)

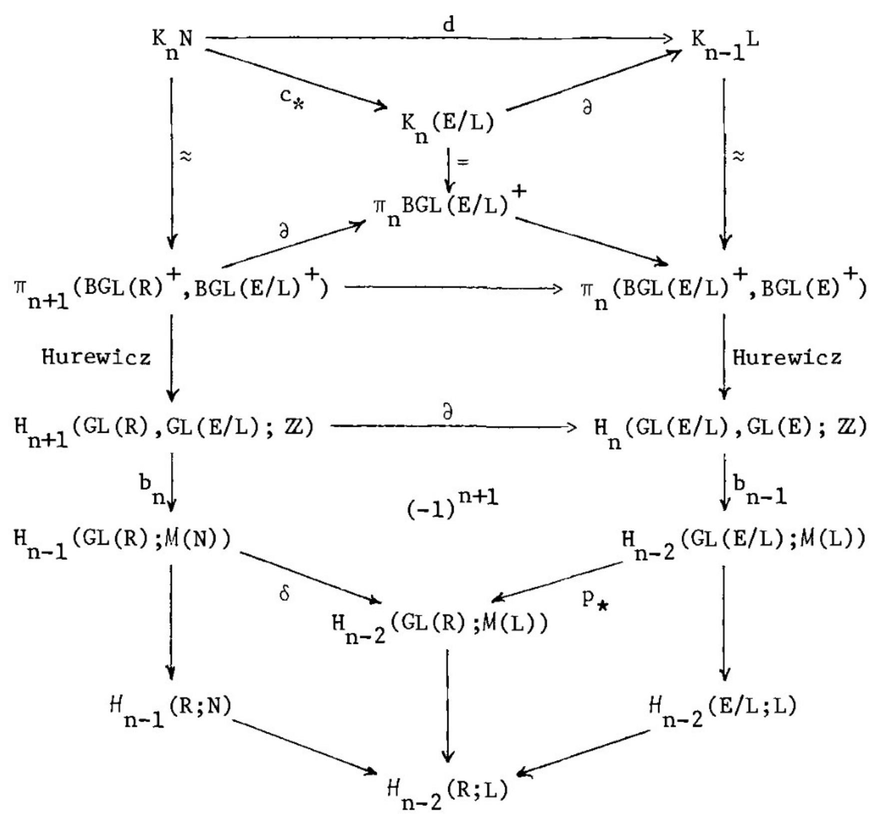
$$\begin{array}{ccccccccc}
\cdots \longrightarrow & K_n^k L & \xrightarrow{a_*} & K_n^k M & \xrightarrow{b_*} & K_n^k N & \xrightarrow{d} & K_{n-1}^k L & \longrightarrow \cdots \\
& \downarrow f_n & & \downarrow g_n(M) & & \downarrow g_n(N) & (-1)^{n-1} & \downarrow f_{n-1} & \\
\cdots \to & \mathcal{H}_{n-1}(R;L) & \to & \mathcal{H}_{n-1}(R;M) & \to & \mathcal{H}_{n-1}(R;N) & \to & \mathcal{H}_{n-2}(R;L) & \to \cdots
\end{array}$$

where f_n is the following composition.

$$K_n^k L \xrightarrow{g_n^k(L)} \mathcal{H}_{n-1}(E/L;L) \to \mathcal{H}_{n-1}(R;L)$$

Here the second map is defined since $M/L = N = \ker(E/L \to R)$ acts trivially on L.

<u>Proof</u>: The left two squares commute by naturality of g_n. To see the skew-commutativity of the right hand square embed it in the following diagram in which every face commutes except the pentagon which commutes up to sign by 2.e.1.



□

4.d. $H^1(R;L) \otimes K_nR \to H_{n-1}(R;L)$.

Let $\beta\colon 0 \to L \to M \overset{p}{\to} N \to 0$ be a short exact sequence of R-bimodules and let $x \in H^0(R;N) \cong \mathrm{Hom}_{R-R}(R,N)$. We can regard x as an element of N such that $rx = xr$ for all $r \in R$. Then $\beta^*(x)$ is represented by a crossed homomorphism $\alpha\colon R \to L$ given be $\alpha(r) = ry - yr$ where y is a fixed element of $p^{-1}(x)$. This produces a ring homomorphism $f\colon R \to L \oplus R$ by $f(r) = (\alpha(r), r)$.

Theorem 4.d. The following diagram commutes up to sign for all n.

$$\begin{array}{ccc}
K_n^k R & \xrightarrow{\quad h_n^k \quad} & H_n(R;R) \\
\downarrow f_* & & \downarrow x_* \\
K_n^k(L \oplus R) & (-1)^n & H_n(R;N) \\
\downarrow \gamma & & \downarrow \beta_* \\
K_n^k L & \xrightarrow[\quad g_n^k \quad]{} & H_{n-1}(R;L)
\end{array}$$

Here $\gamma = \partial^{-1}q$:

$$\gamma\colon K_n^k(L \oplus R) \xrightarrow{q} \frac{K_n^k(L \oplus R)}{K_n^k R} \xleftarrow[\approx]{\partial} K_n^k L$$

Proof: By naturality of the Hurewicz map the following diagram commutes if all the horizontal arrows are Hurewicz maps.

$$\begin{array}{ccc}
K_n^k R & \longrightarrow & H_n(GL_k(R);\mathbb{Z}) \\
\downarrow f_* & & \downarrow \\
K_n^k(L \oplus R) & \longrightarrow & H_n(GL_k(L \oplus R);\mathbb{Z}) \\
\downarrow q & & \downarrow \\
\dfrac{K_n^k(L \oplus R)}{K_n^k R} & \longrightarrow & \dfrac{H_n(GL_k(L \oplus R);\mathbb{Z})}{H_n(GL_k(R);\mathbb{Z})} \\
\approx \uparrow \partial & & \approx \uparrow \partial \\
K_n^k L & \longrightarrow & H_{n+1}(GL_k(R), GL_k(L \oplus R);\mathbb{Z})
\end{array}$$

To complete the proof it suffices to show that the following diagram commutes up to sign for all $k < \infty$.

$$\begin{array}{ccccc}
H_n(GL_k(R);\mathbb{Z}) & \xrightarrow{D_*} & H_n(GL_k(R);M_k(R)) & \longrightarrow & H_n(R;R) \\
\downarrow f_* & \searrow X_* \quad (1) & \downarrow x'_* & (2) & \downarrow x_* \\
\dfrac{H_n(GL_k(L \oplus R);\mathbb{Z})}{H_n(GL_k(R);\mathbb{Z})} & & H_n(GL_k(R);M_k(N)) & \longrightarrow & H_n(R;N) \\
\approx \downarrow \partial^{-1} & (3) & \downarrow \bar{\beta}_* & (4) & \downarrow \beta_* \\
H_{n+1}(GL_k(R),GL_k(L \oplus R);\mathbb{Z}) & \xrightarrow[b_n]{} & H_{n-1}(GL_k(R);M_k(L)) & \longrightarrow & H_{n-1}(R;L)
\end{array}$$

(1) In this triangle X_* is the map induced by the linear coefficient map $X: \mathbb{Z} \to M_k(N)$ given by $X(1) = D(x) = xI$. The triangle commutes because it is induced from a commuting triangle of coefficients.

(2) The horizontal maps are natural.

(3) This part commutes up to the sign $(-1)^n$ as a special case of 2.d since $GL_k(L \oplus R) \cong M_k(L) \tilde{\times} GL_k(R)$.

(4) The commutativity of this square follows from 2.e.2 and the proof of 1.f.4.□

Note that $\beta_* x_*: H_n(R;R) \to H_{n-1}(R;L)$ is the same as the cap product $[\alpha] \cap$ by 3.f.2.

4.e. $\underline{H^2(R;A) \otimes K_nR \to H_{n-2}(R;A)}$.

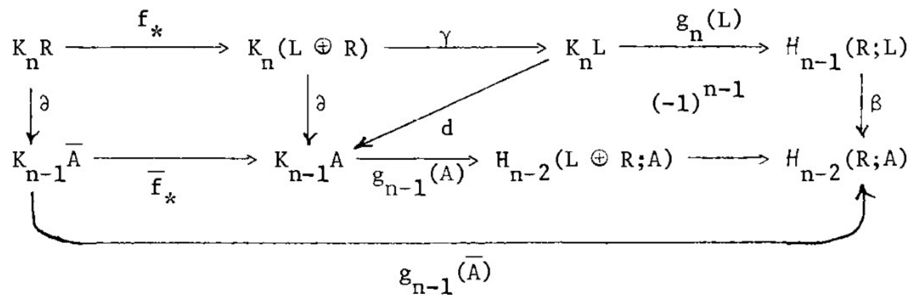
Let $0 \to A \to B \to M \xrightarrow{p} N \to 0$ be an exact sequence of R-bimodules and let $L = \ker p = \operatorname{coker}(A \to B)$. Take $x \in H^0(R;N)$ and $\beta_1^*(x) \in H^1(R;L)$ as in 4.d. Now take $z = \beta_2^*\beta_1^*(x) \in H^2(R;A)$ where β_2^* is the connecting homomorphism for the coefficient sequence $0 \to A \to B \xrightarrow{q} L \to 0$. Then z is represented by a linear factor set $\psi: R \otimes R \to A$ given as follows. Let $\bar{\alpha}: R \to B$ be an additive lifting of the crossed homomorphism $\alpha: R \to L$. Thus $q\bar{\alpha} = \alpha$. Then let $\psi(r \otimes s) = r\bar{\alpha}(s) - \bar{\alpha}(rs) + \bar{\alpha}(r)s$. Let $\bar{A}$ be A considered as an ideal in the twisted extension $A \oplus_\psi R = \bar{A} \oplus_\psi R$.

<u>Theorem 4.e.</u> The following diagram commutes up to sign for all n.

$$\begin{array}{ccc}
K_n^k R & \xrightarrow{h_n^k} & H_n(R;R) \\
\partial \downarrow & (-1)^{n+1} & \downarrow [\psi] \cap \\
K_{n-1}^k \bar{A} & \xrightarrow[g_{n-1}^k]{} & H_{n-2}(R;A)
\end{array}$$

<u>Proof:</u> Let $f: R \to L \oplus R$ be as in 5.d. Then $\bar{A} \oplus_\psi R$ is the pull-back along f

of $B \oplus R$. A formula for the induced ring map $\overline{f}: \overline{A} \oplus_{\psi} R \to B \oplus R$ is given by $\overline{f}(a,r) = (a + \overline{\alpha}(r), r)$. Now consider the following diagram.



The square commutes by naturality of ∂. The triangle commutes by the definition of d. The quadrilateral with $(-1)^{n-1}$ commutes up to that sign by 4.c.2. The bottom rectangle commutes by naturality of g_{n-1}. Now apply 4.d to the top row of arrows and use 3.f.2 to show that $[\psi] \cap = \beta_{2*}\beta_{1*}x_*$. □

Chapter 5. Products in K-theory and Hochschild homology.

In this chapter we review the basic notions of exterior product in algebraic K-theory and homology. The section headings are:

5.a. Definition of $\mu: K_nR \otimes K_mS \to K_{n+m}(R \otimes S)$ for $n, m \geq 1$.

We review the definition of the multiplication in K-theory as given by Loday in [10].

Let R, S be rings with 1, $p, q \geq 3$ and $\phi: R^p \otimes S^q \xrightarrow{\approx} (R \otimes S)^{pq}$ an isomorphism of $R \otimes S$-modules. Let f^i_{pq}, $i = 0, 1, 2$ be the group homomorphisms

$$f^i_{pq}\colon GL_p(R) \times GL_q(S) \to GL(R \otimes S)$$

given by $f^0_{pq}(A,B) = A \otimes B$, $f^1_{pq}(A,B) = A \otimes I_q$, $f^2_{pq}(A,B) = I_p \otimes B$. The homomorphisms f^i_{pq} induce maps $f^i_{pq\star}$ in homology.

$$f^i_{pq\star}\colon H_n(GL_p(R) \times GL_q(S); \mathbb{Z}) \to H_n(GL(R \otimes S); \mathbb{Z})$$

They also induce maps f^{i+}_{pq} of plus constructions.

$$f^{i+}_{pq}\colon BGL_p(R)^+ \times BGL_q(S)^+ \to BGL(R \otimes S)^+$$

Using the fact that $BGL(R \otimes S)^+$ is an infinite loop space we may construct the map $\gamma_{pq} = f^{0+}_{pq} - f^{1+}_{pq} - f^{2+}_{pq}$.

$$\gamma_{pq}\colon BGL_p(R)^+ \times BGL_q(S)^+ \to BGL(R \otimes S)^+$$

Theorem 5.a.1. (Loday [10]) γ_{pq} is null homotopic on $BGL_p(R)^+ \vee BGL_q(S)^+$ and thus induces a map on the smash product.

$$\bar{\gamma}_{pq}\colon BGL_p(R)^+ \wedge BGL_q(S)^+ \to BGL(R \otimes S)^+$$

Furthermore $\bar{\gamma}_{pq}$ is compatible with stabilization of p and q. □

Definition 5.a.2. Let $\mu\colon K_nR \otimes K_mS \to K_{n+m}(R \otimes S)$ be the map in homotopy induced by $\bar{\gamma}_{\infty\infty} = \varinjlim \bar{\gamma}_{pq}$.

See [10] for the proof that μ is bilinear.

5.b. Shuffle product.

By naturality of the Hurewicz homomorphism the following diagram commutes.

$$\begin{array}{ccc} K^p_n R \otimes K^q_m S & \xrightarrow{\text{Hurewicz}} & H_n(GL_p(R); \mathbb{Z}) \otimes H_m(GL_q(S); \mathbb{Z}) \\ \downarrow \mu & & \downarrow s \\ K_{n+m}(R \otimes S) & \xrightarrow{\text{Hurewicz}} & H_{n+m}(GL(R \otimes S); \mathbb{Z}) \end{array}$$

The map s is the composition $H_n(GL_p(R); \mathbb{Z}) \otimes H_m(GL_q(S); \mathbb{Z}) \to H_{n+m}(GL_p(R) \times GL_q(S); \mathbb{Z}) \to H_{n+m}(GL(R \otimes S); \mathbb{Z})$ where the first map is the exterior product in homology and the second map is $f^0_{pq*} - f^1_{pq*} - f^2_{pq*}$. Thus s is the sum of three maps $s = s^0 - s^1 - s^2$. However $s^1 = s^2 = 0$ when $n, m \geq 1$ so $s = s^0$.

The exterior product is given at the chain level by the <u>shuffle product</u>. (See [11] p, 243.) Thus s is given at the chain level by a map

$$s\colon C_n(GL_p(R);\mathbb{Z}) \otimes C_m(GL_q(S);\mathbb{Z}) \to C_{n+m}(GL(R\otimes S);\mathbb{Z})$$

given by

$$s([X^1, \ldots, X^n] \otimes [Y^1, \ldots, Y^m]) = [X^1 \otimes I_q, \ldots, X^n \otimes I_q, I_p \otimes Y^1, \ldots, I_p \otimes Y^m]$$
$$\pm \text{ other shuffles.}$$

5.c. <u>The shuffle product in Hochschild homology</u>.

Let $\bar{s}\colon C_n(R;A) \otimes C_m(S;B) \to C_{n+m}(R\otimes S; A\otimes B)$ be given by

$$\bar{s}((r_1 \otimes \ldots \otimes r_n \otimes a) \otimes (s_1 \otimes \ldots \otimes s_m \otimes b))$$
$$= (r_1 \otimes 1) \otimes \ldots \otimes (r_n \otimes 1) \otimes (1 \otimes s_1) \otimes \ldots \otimes (1 \otimes s_m) \otimes (a \otimes b)$$
$$\pm \text{ other shuffles.}$$

<u>Exercise 5.c.1</u>. Show that this is a chain map.

<u>Exercise 5.c.2</u>. Using formula 1.f.1 show that h_n^+ commutes with shuffle product.

<u>Corollary 5.c.3</u>. The map h_n commutes with products. This means that the following diagram commutes for $n, m \geq 1$.

$$\begin{array}{ccc}
K_nR \otimes K_mS & \xrightarrow{h_n(R)\otimes h_m(S)} & H_n(R;R) \otimes H_m(S;S) \\
\downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \bar{s}_*} \\
K_{n+m}(R\otimes S) & \xrightarrow{h_{n+m}(R\otimes S)} & H_{n+m}(R\otimes S; R\otimes S)
\end{array}$$

<u>Exercise 5.c.4</u>. Let $M \subset R$ be a square zero ideal. Define the unlabled map below and show that the diagram commutes.

$$\begin{array}{ccc}
K_nM \otimes K_mS & \xrightarrow{g_n(M)\otimes h_n(S)} & H_{n-1}(R/M;M) \otimes H_m(S;S) \\
\downarrow & & \downarrow{\scriptstyle \bar{s}_*} \\
K_{n+m}(M\otimes S) & \xrightarrow{g_{n+m}} & H_{n+m-1}(\frac{R\otimes S}{M\otimes S}; M\otimes S)
\end{array}$$

5.d. Definition of $\mu\colon K_0R \otimes K_nS \to K_n(R \otimes S)$.

If $n = 0$ then $\mu\colon K_0R \otimes K_0S \to K_0(R \otimes S)$ is given by $\mu([P] \otimes [Q]) = [P \otimes Q]$ where P, Q are projective modules over R, S.

Let $n \geq 1$ and let P be a finitely generated projective left R-module. We shall define multiplication by P, $\mu_P\colon K_nS \to K_n(R \otimes S)$. Let Q be an R-module such that $P \oplus Q \cong R^m$ for some m. Let $\gamma_P^k\colon GL_k(S) \to GL_{mk}(R \otimes S)$ be given by $\gamma_P^k(X) = (1_P \otimes X) \oplus (1_Q \otimes I_k)$. Then γ_P^k is a homomorphism which takes $E_k(S)$ to $E_{mk}(R \otimes S)$. Thus we have an induced map $\overline{\gamma}_P^k\colon K_n^kS \to K_n(R \otimes S)$. Since γ_P^k commutes up to conjugation with stabilization, $\overline{\gamma}_P^k$ strictly commutes with stabilization. The map $\mu([P] \otimes X) = \overline{\gamma}_P(X)$ is clearly bilinear.

5.e. The product formula for h_n.

Let $\tau\colon K_nR \to K_{n+1}R[t,t^{-1}]$ be left multiplication by $\{t\} \in K_1\mathbb{Z}[t,t^{-1}]$. Let $\overline{R}$ be R considered as a bimodule over itself. Then there is a surjective ring map $p\colon R[t,t^{-1}] \to \overline{R} \oplus R$ given by $p(\Sigma r_it^i) = \Sigma(ir_i, r_i)$,

Theorem 5.e. The following diagram commutes up to sign for all $n \geq 0$. Here γ is as in 4.d.

$$\begin{array}{ccccccc}
K_nR & \xrightarrow{\ \tau\ } & K_{n+1}R[t,t^{-1}] & \xrightarrow{\ p_*\ } & K_{n+1}(\overline{R} \oplus R) & \xrightarrow{\ \gamma\ } & K_{n+1}\overline{R} \\
\Big\downarrow{\scriptstyle h_n(R)} & & & (-1)^n & & & \Big\downarrow{\scriptstyle g_{n+1}(\overline{R})} \\
H_n(R;R) & & & \xrightarrow{\quad = \quad} & & & H_n(R;R)
\end{array}$$

Remark: This proves that Dennis's trace map h_n is the higher dimensional analogue of the Hattori-Stallings trace map h_0.

Proof: (We give the proof only in the case $n \geq 1$.) First apply the Hurewicz map to the top row. The result is the following: $H_nGL(R) \to H_1GL(\mathbb{Z}[t,t^{-1}]) \otimes H_nGL(R) \to H_{n+1}GL(R[t,t^{-1}]) \to H_{n+1}GL(\overline{R} \oplus R) \to H_{n+2}(GL(R), GL(\overline{R} \oplus R))$. Thus it suffices to show that the following square commutes up to sign. (There is no room on this page. Take the diagram (*) on the next page and let $k = \infty$.) By the stabilization results of 1.f this is equivalent to showing that the unstable version of (*) commutes up to sign.

$$(*) \qquad \begin{array}{ccc} H_n GL_k(R) & \xrightarrow{\quad\sigma\quad} & H_{n+2}(GL_k(R), GL_k(\bar{R} \oplus R)) \\ \Big\downarrow h_n^{k+} & & \Big\downarrow g_{n+1}^{k+} \\ \mathcal{H}_n(R;R) & \xrightarrow{\quad = \quad} & \mathcal{H}_n(R;R) \end{array}$$

Here σ is given at the chain level as follows.

$$\begin{array}{lr}
[X^1, \ldots, X^n] & \varepsilon \;\; C_n GL_k(R) \\
\downarrow & \downarrow \\
t \otimes [X^1, \ldots, X^n] & \varepsilon \;\; C_1 GL_k(\mathbb{Z}[t,t^{-1}]) \otimes C_n GL_k(R) \\
\downarrow & \downarrow \\
\sum_{i=0}^{n} (-1)^i [X^1 \otimes 1, \ldots, X^i \otimes 1, I_k \otimes t, X^{i+1} \otimes 1, \ldots, X^n \otimes 1] + & \\
& \varepsilon \;\; C_{n+1} GL_k(R[t,t^{-1}]) \\
\sum_{i=0}^{n} (-1)^{i+1} [X^1 \otimes 1, \ldots, X^i \otimes 1, I_k \otimes 1, X^{i+1} \otimes 1, \ldots, X^n \otimes 1] & \Big\downarrow \\
\downarrow & \\
\sum_{i=0}^{n} (-1)^i [(0,X^1), \ldots, (0,X^i), (I_k, I_k), (0,X^{i+1}), \ldots, (0,X^n)] + & \\
& \varepsilon \;\; C_{n+1} GL_k(\bar{R} \oplus R) \\
\sum_{i=0}^{n} (-1)^{i+1} [(0,X^1), \ldots, (0,X^i), (0,I_k), (0,X^{i+1}), \ldots, (0,X^n)] & \downarrow \\
\downarrow & \\
(-1)^{n+1}(\text{same}, 0) & \varepsilon \;\; C_{n+2}(GL_k(R), GL_k(\bar{R} \oplus R))
\end{array}$$

Now apply c_{n+1} to the last item. We get

$$(-1)^n [X^1, \ldots, X^n] \otimes I_k \;\; \varepsilon \;\; C_n(GL_k(R); M_k(R)) \qquad \square$$

5.f. The second form of the product formula for h_n.

The image of $\{t\} \in K_1 \mathbb{Z}[t,t^{-1}]$ in $\mathcal{H}_1(\mathbb{Z}[t,t^{-1}], \mathbb{Z}[t,t^{-1}])$ is the homology class of $t \otimes 1 \in \mathcal{H}_1(\mathbb{Z}[t,t^{-1}], \mathbb{Z}[t,t^{-1}])$. Let $t \times : \mathcal{H}_n(R;R) \to \mathcal{H}_{n+1}(R[t,t^{-1}]; R[t,t^{-1}])$ be left multiplication by $[t \otimes 1]$. Then as a special case of 5.c.3 we have the following commutative diagram for $n \geq 1$. (The diagram also commutes if $n = 0$.)

$$(5.f.1) \qquad \begin{array}{ccc} K_n R & \xrightarrow{\quad h_n(R) \quad} & \mathcal{H}_n(R;R) \\ \Big\downarrow \tau & & \Big\downarrow t \times \\ K_{n+1} R[t,t^{-1}] & \xrightarrow{h_n(R[t,t^{-1}])} & \mathcal{H}_{n+1}(R[t,t^{-1}]; R[t,t^{-1}]) \end{array}$$

Let T denote the free multiplicative group generated by t. Thus $R[T] = R[t,t^{-1}]$. Then $H^1(T; \mathbb{Z}) \cong \mathbb{Z}$ is generated by the cohomology class of the 1-cocycle

$z \in C^1(T;\mathbb{Z})$ given by $z[t^n] = n$. The push-out of the sequence $C_1(T) \to C_0(T) \to \mathbb{Z} \to 0$ along the map $z\colon C_1(T) \to \mathbb{Z}$ is a sequence $\beta\colon\ \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ of T-modules where T acts on $\mathbb{Z} \oplus \mathbb{Z}$ by $t(a,b) = (a+b,b)$. One can easily verify that $[z] = \beta^*(1)$ where $1 \in H^0(T;\mathbb{Z})$ is the unit of the cohomology ring $H^*(T;\mathbb{Z})$. (1 is the cohomology class of the augmentation map $\mathbb{Z}[T] \to \mathbb{Z}$.)

Consider β as a sequence of $\mathbb{Z}[T]$-bimodules by making the right action of T trivial. Apply $\otimes R[T]$ to β and we get a sequence of $R[T]$-bimodules

$$\overline{\beta}\colon\ 0 \to R[T] \to R[T] \oplus R[T] \to R[T] \to 0$$

where the right action of T is the standard one and the left action of T on $R[T] \oplus R[T]$ is given by $t(a,b) = (ta+tb,tb)$. Computation shows that the connecting homomorphism $\overline{\beta}^*\colon \mathcal{H}^0(R[T];R[T]) \to \mathcal{H}^1(R[T];R[T])$ sends the unit 1 to the cohomology class $[\alpha]$ of the linear crossed homomorphism $\alpha\colon R[T] \to R[T]$ given by $\alpha(\Sigma r_i t^i) = \Sigma i r_i t^i$. By 3.f.2 this means that the homology connecting homomorphism $\overline{\beta}_*\colon \mathcal{H}_{n+1}(R[T];R[T]) \to \mathcal{H}_n(R[T];R[T])$ given by the sequence $\overline{\beta}$ is cap product with $[\alpha]$.

We shall now compute the composition

$$\mathcal{H}_n(R;R) \xrightarrow[t\,\times]{} \mathcal{H}_{n+1}(R[T];R[T]) \xrightarrow[{[\alpha]\,\cap}]{} \mathcal{H}_{n+1}(R[T];R[T])$$

at the chain level. Let $r_1 \otimes \dots \otimes r_n \otimes r$ be a generator of $\mathcal{H}_n(R;R)$. Then by the shuffle product formula we get

$$\begin{aligned} t \times (r_1 \otimes \dots \otimes r_n \otimes r) &= (t \otimes 1) \times (r_1 \otimes \dots \otimes r_n \otimes r) \\ &= \sum_{i=0}^{n} (-1)^i r_1 \otimes \dots \otimes r_i \otimes t \otimes r_{i+1} \otimes \dots \otimes r_n \otimes r \,. \end{aligned}$$

If $s_1 \otimes \dots \otimes s_{n+1} \otimes s$ is a generator of $C_{n+1}(R[T];R[T])$ then the cap product formula 3.f.1 gives us

$$[\alpha] \cap (s_1 \otimes \dots \otimes s_{n+1} \otimes s) = (-1)^{n+1} s_1 \otimes \dots \otimes s_n \otimes \alpha(s_{n+1})s \,.$$

Since $R \subset R[T]$ is the kernel of α we get

$$(5.f.2) \qquad [\alpha] \cap (t \times (r_1 \otimes \dots \otimes r_n \otimes r)) = -r_1 \otimes \dots \otimes r_n \otimes tr$$

Combining 5.f.1 and 5.f.2 we get:

Theorem 5.f.3. The following diagram anticommutes for all $n \geq 0$ where $\bar{t}_*$ is induced by the R-bimodule morphism $\bar{t}: R \to R[T]$ given by $\bar{t}(r) = tr$.

$$\begin{array}{ccccc} K_{n+1}R[T] & & \xrightarrow{\quad h_{n+1} \quad} & & \mathcal{H}_{n+1}(R[T];R[T]) \\ \uparrow \tau & & (-1) & & \downarrow [\alpha] \cap \\ K_nR & \xrightarrow[h_n]{} & \mathcal{H}_n(R;R) & \xrightarrow[\bar{t}_*]{} & \mathcal{H}_{n+1}(R[T];R[T]) \end{array} \qquad \square$$

Remark 5.f.4. The augmentation map $\varepsilon: R[T] \to R$ is a left inverse for $\bar{t}$ and thus $\bar{t}_*$ is a split monomorphism.

Exercise 5.f.5. Derive 5.e from 5.f.3, 4.d and 5.f.4.

Chapter 6. Computations using the Steinberg group.

In this chapter we do computations in low dimensions (≤ 3) using the Steinberg group and Gersten's theorem that $K_3E \cong H_3St(E)$. The section titles are:

6.a. Definition of $St(E)$.

If E is a ring let $St(E)$ be the group given by generators and relations as follows. $St(E)$ is generated by symbols $x_{ij}(r)$ where $i \neq j$ are distinct positive integers and $r \in E$. The relations among the symbols $x_{ij}(r)$ are given as follows.

(0) $x_{ij}(r)x_{ij}(s) = x_{ij}(r + s)$

(1) $[x_{ij}(r),x_{k\ell}(s)] = 1$ if $i \neq \ell$ and $j \neq k$

(2) $[x_{ij}(r),x_{jk}(s)] = x_{ik}(rs)$

Let ϕ: $St(E) \to GL(E)$ be the homomorphism given by $\phi(x_{ij}(r)) = e_{ij}(r)$.

In the long exact sequence of the pair $(BGL(E)^+,BGL(E))$ we can make the following identifications functorially. (This follows easily from [2].)

$$\begin{array}{ccccccccccc} 0 \to & \pi_2(BGL(E)^+) & \to & \pi_2(BGL(E)^+,BGL(E)) & \to & \pi_1(BGL(E)) & \to & \pi_1(BGL(E)^+) & \to 0 \\ & \downarrow = & & \downarrow \approx \psi & & \downarrow = & & \downarrow = & \\ 0 \longrightarrow & K_2E & \longrightarrow & St(E) & \longrightarrow & GL(E) & \longrightarrow & K_1E & \longrightarrow 0 \end{array}$$

Also $K_3E \cong H_3St(E)$ and the isomorphism is given by

$$K_3E = \pi_3BGL(E)^+ \xrightarrow[\phi_\star^{-1}]{\approx} \pi_3BSt(E)^+ \xrightarrow[\text{Hurewicz}]{\approx} H_3St(E)$$

In the special case $E = \mathbb{Z}[G]$ we shall take the following presentation of $St(E)$.

Generators: $x_{ij}(u)$ where $i \neq j$ are natural numbers and $u \in G$.

Relations: (I) $[x_{ij}(u),x_{k\ell}(v)] = 1$ if $i \neq \ell$ and $j \neq k$.

(II) $[x_{ij}(u),x_{jk}(v)]x_{ik}(uv)^{-1} = 1$

Notation: We usually write $x_{ij}(-u)$ for $x_{ij}(u)^{-1}$.

These generators and their inverses will be called <u>elementary operations</u>.

6.b. <u>Milnor's map $K_2E/M \to K_1M$.</u>

If X is a connected space let $X_{(n)}$ represent the n-coskeleton of X, that is, $X \to X_{(n)}$ is terminal among $(n + 1)$-connected maps. Then we have a fibration: $BSt(E) \to BGL(E) \to BGL(E)^+_{(2)}$. If $M \subset E$ is a square zero ideal we have the following square of fibrations.

$$\begin{array}{ccccc} F_2 & \longrightarrow & BM(M) & \longrightarrow & F_1 \\ \downarrow & & \downarrow & & \downarrow \\ BSt(E) & \longrightarrow & BGL(E) & \longrightarrow & BGL(E)^+_{(2)} \\ \downarrow & & \downarrow & & \downarrow \\ BSt(E/M) & \longrightarrow & BGL(E/M) & \longrightarrow & BGL(E/M)_{(2)} \end{array} \qquad (1)$$

Here F_1 and F_2 are the inducated homotopy fibers. Note that F_2 is connected since $p_*: St(E) \to St(E/M)$ is onto. Thus $\mathcal{M}(M) \to \pi_1 F_1 = K_1 M$ is onto and the kernel is the image of the kernel of p_*. The long exact homotopy sequence for the horizontal fibrations look like this:

$$
\begin{array}{ccccccccccc}
 & & & & & & 1 & & & & \\
 & & & & & & \downarrow & & & & \\
 & & & & \ker p_* & \longrightarrow & \mathcal{M}(M) & \xrightarrow{t} & K_1M & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & K_2E & \longrightarrow & St(E) & \longrightarrow & GL(E) & \longrightarrow & K_1E & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & K_2E/M & \longrightarrow & St(E/M) & \longrightarrow & GL(E/M) & \longrightarrow & K_1E/M & \longrightarrow & 1 \\
 & & & & \downarrow & & & & & & \\
 & & & & 1 & & & & & &
\end{array} \tag{2}
$$

Milnor's definition of $\partial: K_2E/M \to K_1M$ comes from chasing this diagram as in the snake lemma. ([12], p.54.) It is easy for any topologist to see that this map is exactly the negative of the map given by the long exact sequence of the fibration $F_1 \to BGL(E)^+_{(2)} \to BGL(E/M)^+_{(2)}$. Thus Milnor's map is the negative of the one we are using.

6.c. $K_1M \cong \mathcal{H}_0(E/M;M)$.

Theorem 6.c.1. Let $M \subset E$ be a square zero ideal. Then $g_1(M): K_1M \to \mathcal{H}_0(E/M;M)$ is an isomorphism.

Corollary 6.c.2. The map $g_2(M): K_2M \to \mathcal{H}_1(R;M)$ is an epimorphism if $E = M \oplus R$ and R is projective as a $\mathbb{Z}$-module.

Proof: Let P be a projective R-bimodule which maps onto M and let L be the kernel of the epimorphism $P \to M$. Considering L as an ideal in $P \oplus R$ we have a map of long exact sequences as in 4.c.

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & K_2M & \longrightarrow & K_1L & \longrightarrow & K_1P & \longrightarrow & \cdots \\
 & & {\scriptstyle g_2(M)}\downarrow & (-1) & \downarrow{\scriptstyle \approx} & & \downarrow{\scriptstyle \approx} & & \\
\cdots & \longrightarrow & \mathcal{H}_1(R;M) & \longrightarrow & \mathcal{H}_0(R;L) & \longrightarrow & \mathcal{H}_0(R;P) & \longrightarrow & \cdots
\end{array}
$$

Here $\mathcal{H}_0(R;L) \cong \mathcal{H}_0(M \oplus R;L)$ since M acts trivially on L. The commutativivy of this diagram up to sign proves the corollary. □

Proof of 6.c.1: It is an easy exercise to show that K_1M is isomorphic to $H_0(E/M;M)$ and that t in 6.b.2 is given by the trace. What is not immediate is why $g_1(M)$ gives this isomorphism. So take $A \in M(M)$. Then $A \in M_k(M)$ for some finite $k \geq 3$. We shall use formula 4.b.1.

$$\begin{array}{lll} & [A] & \in C_1(M_k(M);\mathbb{Z}) \\ \longrightarrow & ([(A,I_k)],0) & \in C_1(GL_k(E);\mathbb{Z}) \\ \xrightarrow[4.b.1]{} & -\mathrm{tr}\,A & \in C_0(E/M;M) \end{array}$$

Thus according to our sign conventions $g_1(M)t(A) = -\mathrm{tr}\,A$. □

6.d. Integral group rings.

We shall now restrict to the case $R = \mathbb{Z}[G]$. If M is a left G-module then $M[G] = M \otimes \mathbb{Z}[G]$ is given the structure of a $\mathbb{Z}[G]$-bimodule by $g(a \otimes b)h = ga \otimes gbh$. A natural homomorphism $\ell^r\colon H^r(G;M) \to H^r(\mathbb{Z}[G];M[G])$ can be defined as follows.

An element $\kappa \in H^r(G;M)$ is represented by an r-fold extension of left G-modules:

$$(*) \qquad 0 \to M \to E_r \to \dots \to E_1 \to \mathbb{Z} \to 0$$

Tensoring this with $\mathbb{Z}[G]$ we get an r-fold extension of $\mathbb{Z}[G]$-bimodules:

$$(**) \qquad 0 \to M[G] \to E_r[G] \to \dots \to E_1[G] \to \mathbb{Z}[G] \to 0$$

which represents an element $\ell^r(\kappa) = \kappa[G] \in \mathrm{Ext}^r_{\mathbb{Z}[G]-\mathbb{Z}[G]}(\mathbb{Z}[G];M[G]) \cong H^r(\mathbb{Z}[G];M[G])$.

Theorem 6.d.1. The map ℓ^r is a homomorphism given at the cochain level by linearly extending the following formula. (Compare this with 3.g.)

$$\ell^r(f)(g_0 \otimes \dots \otimes g_{r+1}) = g_0 f[g_1,\dots,g_r] \otimes g_0 g_1 \cdots g_{r+1}.$$

Proof: If $f\colon C_r(G) \to M$ is an r-cochain then $\ell^r(f)$ is the following composition.

$$C_r(\mathbb{Z}[G]) \xrightarrow{\approx} C_r(G) \otimes \mathbb{Z}[G] \xrightarrow{f[G]} M[G]$$

where the chain isomorphism is given by

$$g_0 \otimes \dots \otimes g_{r+1} \longrightarrow g_1[g_1, \dots, g_r] \otimes g_0 g_1 \dots g_{r+1}.$$

If f is a cocycle representing κ then there is a map of chain complexes from the augmented complex $\dots \to C_0(G) \to \mathbb{Z}$ to (*) above which is f on $C_r(G)$ and 1 on $\mathbb{Z}$. Tensoring everything with $\mathbb{Z}[G]$ and composing with the chain isomorphism given above we get a map of chain complexes:

$$\begin{array}{ccccccc} \dots \to C_r(\mathbb{Z}[G]) & \to \dots \to & C_0(\mathbb{Z}[G]) & \to & \mathbb{Z}[G] & \to 0 \\ \downarrow \ell^r(f) & & \downarrow & & \downarrow = & \\ 0 \longrightarrow M[G] & \longrightarrow \dots \longrightarrow & E_1[G] & \longrightarrow & \mathbb{Z}[G] & \to 0 \end{array}$$

This shows that $\ell^r(f)$ represents $\kappa[G]$. □

We shall sometimes denote the cochain $\ell^r(f)$ by f_ℓ.

<u>Definition 6.d.2</u>. Let χ_n^r be defined by the following composition and let $\chi_n = \chi_n^n$.

$$\begin{aligned} H^r(G;M) \otimes K_n \mathbb{Z}[G] &\xrightarrow[\ell^r \otimes h_n]{} H^r(\mathbb{Z}[G];M[G]) \otimes H_n(\mathbb{Z}[G];\mathbb{Z}[G]) \\ &\xrightarrow[\cap]{} H_{n-r}(\mathbb{Z}[G];M[G]) \\ &\xrightarrow[\approx]{} H_{n-r}(G;M[G]) \end{aligned}$$

where G acts on $M[G]$ by conjugation.

6.e. <u>A formula for χ_2: $H^2(G;M) \otimes K_2 \mathbb{Z}[G] \to H_0(G;M[G])$.</u>

An element of $H^2(G;M)$ is represented by a factor set $f: G \times G \to M$. We shall assume that f is normalized which is equivalent to saying that it comes from a transversal which preserves 1. This implies that $f(1,g) = f(g,1) = 0$ for all $g \in G$. By 6.d.1 the linearized version f_ℓ: $\mathbb{Z}[G] \otimes \mathbb{Z}[G] \to M[G]$ is given by $f_\ell(g_1 \otimes g_2) = f(g_1,g_2) \otimes g_1 g_2$.

<u>Formula 6.e.1</u>. Let $x \in K_2 \mathbb{Z}[G]$ be represented by a product of elementary operations $x = x^1 x^2 \dots x^n \in \ker \phi \subset St(\mathbb{Z}[G])$. Then

$$\chi_2([f] \otimes x) = -\sum_{k,p} f_\ell(r^k_{pi_k} \otimes u_k) s^k_{j_k p}$$

where $x^k = x_{i_k j_k}(u_k)$, $u_k \in \pm G$ and $(r^k_{pq}) = \phi(x^1 \dots x^{k-1}) = (s^k_{qp})^{-1}$.

Proof: The factor set f_ℓ determines an extension $E = M[G] \oplus_{f_\ell} \mathbb{Z}[G]$. By theorem 4.e $\chi_2([f] \otimes x) = -g^k_1(M[G])\partial x$ where ∂ is the boundary map $K_2\mathbb{Z}[G] \to K_1M[G]$ for the extension E. If we use Milnor's formula for ∂ and use trace instead of $g^k_1(M[G])$ $g^k_1(M[G])$ the sign will change twice and thus we get

$$\chi_2([f] \otimes x) = -\mathrm{tr}\ \partial_{Milnor}x.$$

To compute $\mathrm{tr}\ \partial_{Milnor}x$ we chase x around the diagram 6.b.2 as follows.

	$x = x_{i_1j_1}(u_1) \dots$	$\in K_2\mathbb{Z}[G] \subset St(\mathbb{Z}[G])$
$\leftarrow$	$x_{i_1j_1}(0,u_1) \dots$	$\in St(E)$
$\rightarrow$	$e_{i_1j_1}(0,u_1) \dots$	$\in GL(E)$
$\leftarrow$	same	$\in M(M[G]) + I$

$$\rightarrow \quad \mathrm{tr}(e_{i_1j_1}(0,u_1) \dots e_{i_nj_n}(0,u_n) - I) \in H_0(G;M[G])$$

where I is the identity matrix. To compute this last term we need to following.

Formula 6.e.2. Let $x^1, \dots, x^n$ be a sequence of elementary operations of the form $x^k = x_{i_kj_k}(0,u_k) \in St(E)$. Let ψ: $St(E) \to GL(\mathbb{Z}[G])$ be the obvious map. Then the pq-entry of the matrix $\phi(x^1 \dots x^n) \in GL(E)$ is given by

$$(*) \qquad t_{pq} = \Big(\sum_{k=1}^{n} f_\ell(r^k_{pi_k} \otimes u_k)s^k_{j_kq}, r^{n+1}_{pq}\Big)$$

where
$$(r^k_{**}) = \psi(x^1 \dots x^{k-1})$$
$$(s^k_{**}) = \psi(x^{k+1} \dots x^n)$$

Proof: By induction on n. Let $n = 1$. Then $\phi(x^1) = (0,\psi(x^1))$ and $t_{pq} = (0,r^1_{pq})$ for all p, q since $f_\ell(1 \otimes u_1) = 0$.

Now suppose that the formula holds for n. Let $x^{n+1} = x_{ab}(0,v)$. Then the pq-entry of $\phi(x^1 \dots x^{n+1})$ is given by

$$\bar{t}_{pq} = \begin{cases} t_{pq} & \text{if } q \neq b \\ t_{pb} + t_{pa}(0,v) & \text{if } q = b \end{cases}$$

Using (*) for t_{pa} we see that

$$(**)\qquad t_{pa}(0,v) = (\sum_{k=1}^{n} f_\ell(r^k_{pi_k} \otimes u_k)s^k_{j_k a}v + f_\ell(r^{n+1}_{pa} \otimes v), r^{n+1}_{pa}v)$$

Now look at the right hand side of (*). When n increases to $n+1$ we acquire one extra summand $(f_\ell(r^{n+1}_{pa} \otimes v)\bar{s}^{n+1}_{bq}, 0)$ where $\bar{s}^{n+1}_{**}$ is the identity matrix. The second coordinate changes from r^{n+1}_{pq} to

$$r^{n+2}_{pq} = \begin{cases} r^{n+1}_{pq} & \text{if } q \neq b \\ r^{n+1}_{pb} + r^{n+1}_{pa}v & \text{if } q = b \end{cases}$$

Each $s^k_{j_k q}$ changes to

$$\bar{s}^k_{j_k q} = \begin{cases} s^k_{j_k q} & \text{if } q \neq b \\ s^k_{j_k b} + s^k_{j_k a}v & \text{if } q = b \end{cases}$$

This agrees with the three things in (**) so the formula is proved. □

Formula 6.e.2 can now be used to compute $tr(e_{i_1 j_1}(0,u_1) \ldots e_{i_n j_n}(0,u_n) - I)$ and we get formula 6.e.1. □

If $a = \sum_k u_k$, $u_k \in \pm G$, then note that $x_{ij}(a) = \prod_k x_{ij}(u_k)$ is a well-defined element of $St(\mathbb{Z}[G])$.

Theorem 6.e.3. Formula 6.e.1 is valid even if the condition $u_k \in \pm G$ is replaced by the condition $u_k \in \mathbb{Z}[G]$.

Proof: If $x^k = x_{ij}(a+b)$ we shall show that the formula is unchanged when x^k is replaced by $x_{ij}(a)x_{ij}(b)$. Only one term changes: $-\sum_p f_\ell(r^k_{pi} \otimes (a+b))s^k_{jp}$ will change to $-\sum_p f_\ell(r^k_{pi} \otimes a)s^k_{jp} - \sum_p f_\ell(\bar{r}^k_{pi} \otimes b)\bar{s}^k_{jp}$ where $(\bar{r}^k_{**}) = (r^k_{**})e_{ij}(a)$ and $(\bar{s}^k_{**}) = e_{ij}(-a)(s^k_{**})$. Thus $\bar{r}^k_{pi} = r^k_{pi}$ and $\bar{s}^k_{jp} = s^k_{jp}$. The term has not changed. □

6.f. The group P(G).

Let G be a group. If Z is any set let $\mathbb{Z}[G]\langle Z\rangle$ be the free left G-module generated by the elements of Z enclosed in square brackets. If G is given by generators and relations by $G = \langle X|Y\rangle$ then an exact sequence of the following kind can be constructed.

$$\mathbb{Z}[G]\langle Y\rangle \xrightarrow[d_2]{} \mathbb{Z}[G]\langle X\rangle \xrightarrow[d_1]{} \mathbb{Z}[G] \xrightarrow[\varepsilon]{} \mathbb{Z} \longrightarrow 0$$

where ε is the augmentation map, d_1 is given by $d_1[x] = x - 1$ and d_2 is given by $d_2[x_1 \ldots x_n] = [x_1] + x_1[x_2] + \ldots + x_1 \ldots x_{n-1}[x_n]$ with the convention that $[x^{-1}] = -x^{-1}[x]$. This is the beginning of a $\mathbb{Z}[G]$-free resolution of $\mathbb{Z}$. The bar constuction is an example of such a resolution where $X = G$ and the elements $[x_1,x_2]$ of G^2 correspond to the relations $(x_1)(x_2)(x_1x_2)^{-1} = 1$.

Proposition 6.f. Let $P(G) = \ker d_2$ and let M be any G-module. Then the following sequences are exact.

(1) $\quad 0 \to H_3G \to H_0(G;P(G)) \xrightarrow{d_{2*}} H_0(G; \mathbb{Z}[G]\langle Y\rangle)$

(2) $\quad \mathrm{Hom}_G(\mathbb{Z}[G]\langle Y\rangle;M) \xrightarrow[d_2^*]{} \mathrm{Hom}_G(P(G);M) \to H^3(G;M) \to 0$

Elements of $\mathrm{Hom}_G(P(G);M)$ will be called presentation cocycles. Elements of $P(G)$ can be represented pictorially [9] but this is not important here.

6.g. A formula for the map $\ell^3 \cap h_3^+ \colon H^3(G;M) \otimes H_3GL(\mathbb{Z}[G]) \to H_0(G;M[G])$.

An element of $H^3(G;M)$ can be represented by a normalized 3-cocycle $f\colon G \times G \times G \to M$. This means that $f(g_1,g_2,g_3) = 0$ if one of the g_i's is 1. Let $f_\ell\colon \mathbb{Z}[G] \otimes \mathbb{Z}[G] \otimes \mathbb{Z}[G] \to M[G]$ be the linearization of f given by $f(g_1 \otimes g_2 \otimes g_3) = f(g_1,g_2,g_3) \otimes g_1g_2g_3$. Combining formula 1.f.1 for h_3^+ and 3.f.1 for the cap product, the formula for $\ell^3 \cap h_3^+$ at the chain level is as follows.

$$f \otimes [g^1,g^2,g^3] \qquad \in C^3(G;M) \otimes C_3(GL_k(\mathbb{Z}[G]);\mathbb{Z})$$

$$\xrightarrow{\ell^3 \cap h_3^{k+}} f_\ell \otimes \sum_{a,b,c,d} r^1_{ab} \otimes r^2_{bc} \otimes r^3_{cd} \otimes s_{da} \in C^3(\mathbb{Z}[G];M[G]) \otimes C_3(\mathbb{Z}[G];\mathbb{Z}[G])$$

$$\longrightarrow \quad (*) \quad \sum_{a,b,c,d} f_\ell(r^1_{ab} \otimes r^2_{bc} \otimes r^3_{cd})s_{da} \qquad \in C_0(\mathbb{Z}[G];M[G]) \cong M[G]$$

where r^i_{**} are the entries of the matrix g^i, s_{**} are the entries of $(g^1g^2g^3)^{-1}$ and $(-1)^e = (-1)^{12} = 1$.

The formula $(*)$ can be used to define a map $F\colon C_3(GL(\mathbb{Z}[G])) \to M[G]$ by $F(g^0[g^1,g^2,g^3]) = (*)$, i.e. ignore g^0. Since $f_\ell \cap h_3^+$ is a chain map up to sign the following composition is trivial.

$$C_4(GL(E)) \xrightarrow{d_4} C_3(GL(E)) \xrightarrow{F} M[G] \longrightarrow H_0(G;M[G])$$

Consequently we get an induced map $F_P\colon P(GL(E)) \to H_0(G;M[G])$ where $P(GL(E))$ is the kernel of $d_2\colon C_2(GL(E)) \to C_1(GL(E))$ and $E = \mathbb{Z}[G]$.

Proposition 6.g. F_P is the restriction of the additive map $F_C\colon C_2(GL(\mathbb{Z}[G])) \to H_0(G;M[G])$ given on the additive generators of $C_2(GL(\mathbb{Z}[G])$ as follows.

$$(**) \qquad F_C(g^1[g^2,g^3]) = -\sum_{a,b,c,d} f\,(r^1_{ab} \otimes r^2_{bc} \otimes r^3_{cd})s_{da}$$

where r^i_{**} are the entries of g^i and s_{**} are the entries of $(g^1g^2g^3)^{-1}$.

Proof: Just compute:

$$\begin{aligned}
F_C d_3(g^0[g^1g^2g^3]) &= -\Sigma\, f_\ell(r^0_{ab}r^1_{bc} \otimes r^2_{cd} \otimes r^3_{de})s_{ea} \\
&\quad +\Sigma\, f_\ell(r^0_{ab} \otimes r^1_{bc}r^2_{cd} \otimes r^3_{de})s_{ea} \\
&\quad -\Sigma\, f_\ell(r^0_{ab} \otimes r^1_{bc} \otimes r^2_{cd}r^3_{de})s_{ea} \\
&\quad +\Sigma\, f_\ell(r^0_{ab} \otimes r^1_{bc} \otimes r^2_{cd})r^3_{de}s_{ea} \\
&= \Sigma\, r^0_{ab}f_\ell(r^1_{bc} \otimes r^2_{cd} \otimes r^3_{de})s_{ea} \qquad \text{since } f_\ell \text{ is a cocycle} \\
&= \Sigma\, f_\ell(r^1_{bc} \otimes r^2_{cd} \otimes r^3_{de})s_{ea}r^0_{ab} \\
&= F(g^0[g^1,g^2,g^3])
\end{aligned}$$

□

6.h. A formula for $\chi_3\colon H^3(G;M) \otimes K_3\,\mathbb{Z}[G] \to H_0(G;M[G])$.

Let $K_3\,\mathbb{Z}[G]$ be identified with $H_3St(\mathbb{Z}[G])$ in such a way that the Hurewicz map $K_3\,\mathbb{Z}[G] \to H_3GL(\mathbb{Z}[G])$ corresponds to the map in homology induced by ϕ. Let f and f_ℓ be as above. We shall derive a formula for the cohomology class $\chi_f = \chi_3([f],-) \in H^3(St(\mathbb{Z}[G]); H_0(G;M[G]))$.

If $x \in H_3St(\mathbb{Z}[G])$ then $\chi_3([f],x) = f_\ell \cap h_3^{+}\phi_*(x)$ where ϕ_* is the map on H_3 induced by ϕ. Thus we will derive a formula for ϕ_* and then apply the formulas in 6.f above. We first choose a map of partial resolutions as follows. (Next page) Here ϕ_0 is the ring map induced by ϕ, $\phi_1[x] = [\phi(x)]$ if $x \in X_{St}$, $\phi_2[[a,b]] = [\phi(a),\phi(b)] - [\phi(b),\phi(a)]$ and $\phi_2[[a,b]c^{-1}] = [\phi(a),\phi(b)] - [\phi(c),\phi(b)] - [\phi(bc),\phi(a)]$.

$$\begin{array}{ccccccccc}
\mathbb{Z}[St]\langle Y_{St}\rangle & \xrightarrow{d_2} & \mathbb{Z}[St]\langle X_{St}\rangle & \xrightarrow{d_1} & \mathbb{Z}[St] & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle =} & & \\
\mathbb{Z}[GL]\langle GL \times GL\rangle & \xrightarrow[d_2]{} & \mathbb{Z}[GL]\langle GL\rangle & \xrightarrow[d_1]{} & \mathbb{Z}[GL] & \longrightarrow & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

One can easily check that this diagram commutes. Let $\phi_3\colon P(St) \to P(GL)$ be the map induced by ϕ_2 on the kernel of the d_2's. The cohomology class χ_f is represented by the presentation cocycle $\chi_f^P = F_P\phi_3$ which is the restriction to $P(St(\mathbb{Z}[G]))$ of the additive map $\chi_f^C = F_C\phi_2\colon \mathbb{Z}[St]\langle Y_{St}\rangle \to H_0(G;M[G])$ given on the additive generators of $\mathbb{Z}[St]\langle Y_{St}\rangle$ as follows.

<u>Formula 6.h.</u> (1) $\chi_f^C(t[y]) = 0$ if $y = [x_{ij}(u), x_{k\ell}(v)]$

(2) $\chi_f^C(t[[x_{ij}(u),x_{jk}(v)]x_{ik}(-uv)]) = -\sum_p f\,(r_{pi} \otimes u \otimes v)s_{kp}$

where $(r_{**}) = \phi(t) = (s_{**})^{-1}$.

<u>Proof:</u> (1) $\chi_f^C(t[[x,y]]) = F_C(\phi(t)[\phi(x),\phi(y)] - \phi(t)[\phi(y),\phi(x)] =$

$$-\sum\{f_\ell(r^1_{ab} \otimes r^2_{bc} \otimes r^3_{cd})s_{da} - f_\ell(r^1_{ab} \otimes r^3_{bc} \otimes r^2_{cd})s_{da}\} = 0$$

since each summand is zero. (Both terms between the braces are zero unless $b = c = d$ in which case they are still zero since f_ℓ is normalized.)

(2) $\chi_f^C(t[[x_{ij}(u),x_{jk}(v)]x_{ik}(-uv)])$

$$= F_C(\phi(t)[e_{ij}(u),e_{jk}(v)] - \phi(t)[e_{ik}(uv),e_{jk}(v)] - \phi(t)[e_{jk}(v)e_{ik}(uv),e_{ij}(u)])$$

$$= -\sum_p f_\ell(r_{pi} \otimes u \otimes v)s'_{kp} + 0 + 0$$

where $(r_{**}) = \phi(t)$ and $(s'_{**}) = e_{jk}(-v)e_{ij}(-u)\phi(t)^{-1}$. But $s'_{kp} = s_{kp}$ where $(s_{**}) = \phi(t)^{-1}$ since $k \neq i$. □

6.i. <u>The product formula.</u>

Let $\beta\colon 0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$ be the short exact sequence of $\mathbb{Z}[T]$-modules given in 5.f. Let β_0 be the same sequence considered as a sequence of $G \times T$-modules, G acting trivially. Then take $\overline{\beta}_0 = \beta_0[G \times T] = \beta_0 \otimes \mathbb{Z}[G \times T]$. By 5.f.3 we have the following anticommuting diagram.

(a)
$$\begin{array}{ccccc} K_2\,\mathbb{Z}[G] & \xrightarrow{h_2} & H_2(\mathbb{Z}[G];\,\mathbb{Z}[G]) & \searrow^{\bar{t}_*} & \\ \downarrow{\tau} & & & & \\ K_3\,\mathbb{Z}[G\times T] & \xrightarrow{h_3} & H_3(\mathbb{Z}[G\times T];\,\mathbb{Z}[G\times T]) & \xrightarrow{\bar{\beta}_{0*}} & H_2(\mathbb{Z}[G\times T];\,\mathbb{Z}[G\times T]) \end{array}$$

Suppose that $\kappa \in H^2(G;M)$. Then κ is represented by a two-fold extension of G-modules as follows.

$$0 \to M \to E_2 \to E_1 \to \mathbb{Z} \to 0$$

If N is the kernel of $E_1 \to \mathbb{Z}$ then this breaks up into two short exact sequences:

$$\beta_2': \quad 0 \to M \to E_2 \to N \to 0$$
$$\beta_1': \quad 0 \to N \to E_1 \to \mathbb{Z} \to 0$$

and we have that $\kappa = \beta_2'^{*}\beta_1'^{*}(1)$. Let β_1, β_2 be the same sequences considered as $G \times T$ -modules, the action of T being trivial. This produces the following commutative diagram where $\bar{\beta}_i = \beta_i[G \times T]$ and $\bar{\beta}_i' = \beta_i'[G]$.

(b)
$$\begin{array}{ccc} H_2(\mathbb{Z}[G];\,\mathbb{Z}[G]) & \xrightarrow{\bar{\beta}_{2*}'\bar{\beta}_{1*}'} & H_0(\mathbb{Z}[G];M[G]) \\ \downarrow{\bar{t}_*} & & \downarrow{\bar{t}_*} \\ H_2(\mathbb{Z}[G\times T];\,\mathbb{Z}[G\times T]) & \xrightarrow{\bar{\beta}_{2*}\bar{\beta}_{1*}} & H_0(\mathbb{Z}[G\times T];M[G\times T]) \end{array}$$

Combine the two diagrams (a) and (b) and we get a larger diagram which produces the following equation for any $x \in K_2\,\mathbb{Z}[G]$.

(c)
$$\bar{t}_*\bar{\beta}_{2*}'\bar{\beta}_{1*}'h_2(x) = -\bar{\beta}_{2*}\bar{\beta}_{1*}\bar{\beta}_{0*}h_3\tau(x)$$

This can be rewritten as follows.

<u>Formula 6.i.</u>
$$\bar{t}_*\chi_2(\kappa,x) = -\chi_3([z]\times\kappa,\tau(x))$$

where $[z]$ is the generator of $H^1(T;\,\mathbb{Z})$ given in 5.f and "$\times$" is the exterior product in group cohomology.

<u>Proof</u>: By definition of χ_2 we have $\bar{t}_*\chi_2(\kappa,x) = \bar{t}_*(\ell^2(\kappa) \cap h_2(x)) = \bar{t}_*(\ell^2(\beta_2'^{*}\beta_1'^{*}(1)) \cap h_2(x)) = \bar{t}_*\bar{\beta}_2'^{*}\bar{\beta}_1'^{*}(1 \cap h_2(x)) = \bar{t}_*\bar{\beta}_2'^{*}\bar{\beta}_1'^{*}h_2(x) = -\bar{\beta}_{2*}\bar{\beta}_{1*}\bar{\beta}_{0*}h_3\tau(x) = -\bar{\beta}_{2*}\bar{\beta}_{1*}\bar{\beta}_{0*}(1 \cap h_3\tau(x)) = -\ell^3(\beta_2^{*}\beta_1^{*}\beta_0^{*}(1)) \cap h_3\tau(x) = -\chi_3(\beta_2^{*}\beta_1^{*}\beta_0^{*}(1),h_3\tau(x))$. Thus we need only show that $\beta_2^{*}\beta_1^{*}\beta_0^{*}(1) = \kappa \times [z]$. But $\beta_2^{*}\beta_1^{*}$ is cup product with

$\kappa \times 1$ and $\beta_0^*(1) = 1 \times [z]$. □

Chapter 7. The algebraic counterexample.

Any nonzero element of $Wh_1^+(G;A) = H_0(G;A[G])/H_0(G;A)$ which is in the image of $H^3(G;A) \otimes K_3\mathbb{Z}[G]$ under the map $\chi_3^+ = p\chi_3$ where p is the projection of $H_0(G;A[G])$ onto $Wh_1^+(G;A)$ is a counterexample to the original statement of Hatcher and Wagoner. In 7.a we give an example due to R.K. Dennis of a nontrivial element in the image of χ_2^+ and in 7.b we use the product formula for h_n to get an example for χ_3^+.

7.a. The example in $K_2\mathbb{Z}[G]$.

Let p be an odd integer ≥ 3. Let $G = \mathbb{Z}_p \times \mathbb{Z}_p = \langle x,y | x^p, y^p, [x,y]\rangle$. Let $A = \Lambda^2 G \cong \mathbb{Z}_p$ considered as a left G-module with the trivial action. Let $f: G \times G \to A$ be the factor set given by $f(a,b) = a \wedge b$. Then f represents a cohomology class $\kappa = [f] \in H^2(G;A)$. Let $a = x(y-1) \in \mathbb{Z}[G]$ and $b = 1 + y + y^2 + \ldots + y^{p-1}$ $\in \mathbb{Z}[G]$. Then $ab = 0$ and so $[e_{12}(a), e_{21}(b)] = 1$ in $GL(\mathbb{Z}[G])$. Consequently $[x_{12}(a), x_{21}(b)] \in \ker \phi \cong K_2\mathbb{Z}[G]$.

Theorem 7.a. $\chi_2^+([f],[x_{12}(a),x_{21}(b)]) \neq 0$. In fact it is equal to $2\sum_{i=1}^{p} g \otimes xy^i$ where $g = x \wedge y$ is the generator of $\Lambda^2 G$.

Proof: We use theorem 6.e.3 to compute $\chi_2([f], x_{12}(a)x_{21}(b)x_{12}(-a)x_{21}(-b))$. The formula gives this as the sum of the following four terms.

(1) $$-\sum_q f_\ell(r^1_{q1} \otimes a)s^1_{2q} = 0$$

(2) $$-\sum_q f_\ell(r^2_{q2} \otimes b)s^2_{1q} = -f_\ell(a \otimes b) = -f_\ell((xy - x) \otimes (1 + y + \ldots + y^{p-1})) =$$
$$\sum_{i=1}^{p} [f_\ell(x \otimes y^{i+1}) - f_\ell(xy \otimes y^i)] = \sum_{i=1}^{p} [(x \wedge y^{i+1}) \otimes xy^{i+1} - (x \wedge y^i) \otimes xy^{i+1}]$$
$$= \sum_{i=1}^{p} (x \wedge y) \otimes xy^{i+1}$$

(3) $$-\sum_q f_\ell(r^3_{q1} \otimes a)s^3_{2q} = -f_\ell(b \otimes a) = -f_\ell(a \otimes b) = \sum_{i=1}^{p} (x \wedge y) \otimes xy^{i+1}$$

(4) $$-\sum_q f_\ell(r^4_{q2} \otimes (-b))s^4_{1q} = 0$$

Thus $\chi_2([f],[x_{12}(a),x_{21}(b)]) = 2\sum_{i=1}^{p} (x \wedge y) \otimes xy^{i+1} \neq 0$ in $Wh_1{}^+(G;A)$. □

The element $[x_{12}(a),x_{21}(b)]$ is also denoted by $\langle a,b\rangle$.

7.b. An example in $K_3[G \times T]$.

Let $\bar{t}_*{}^+$ be the map induced by $\bar{t}_*$ as indicated in the following diagram.

$$\begin{array}{ccccccc} H_0(G;A) & \longrightarrow & H_0(G;A[G]) & \longrightarrow & Wh_1{}^+(G;A) & \to & 0 \\ \downarrow \approx & & \downarrow \bar{t}_* & & \downarrow \bar{t}_*{}^+ & & \\ H_0(G \times T;A) & \longrightarrow & H_0(G \times T;A[G \times T]) & \longrightarrow & Wh_1{}^+(G \times T;A) & \to & 0 \end{array}$$

Then by formula 6.i we have

(1) $$\chi_3{}^+([z] \times \kappa, \tau(x)) = -\bar{t}_*{}^+\chi_2{}^+(\kappa,x)$$

This formula together with 7.a gives the following.

(2) $$\chi_3{}^+([z] \times [f],\{t\}\cdot\langle a,b\rangle) = 2\sum_{i=1}^{p} (x \wedge y) \otimes xy^i t$$

Another version of this example is as follows. Let $G = \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ and let $\xi^3 \in H^3(G;\mathbb{Z}_p)$ be the third exterior power of the generator $\xi \in H^1(\mathbb{Z}_p;\mathbb{Z}_p)$. If p is odd and ≥ 3 then $\chi_3{}^+(\xi^3,\langle a,b\rangle\cdot\{z\}) = 2\sum_{i=1}^{p} xy^i z \in \mathbb{Z}_p[G]/\mathbb{Z}_p[1] \cong Wh_1{}^+(G;\mathbb{Z}_p)$. By applying symmetries the image of $\chi_3{}^+(\xi^3,-)$ is seen to be very large.

Chapter 8. Application to pseudoisotopies.

8.a. Introduction.

Let M be a compact smooth (C^∞) manifold of dimension ≥ 6. Let $G = \pi_1 M$ and $A = \pi_2 M$ and let $\kappa = k_1(M) \in H^3(G;A)$ be the first Postnikov invariant of M. Let $C(M) = \mathrm{Diff}(M \times I \text{ rel } M \times 0 \cup \partial M \times 0)$ be the pseudoisotopy space of M. What Hatcher and Wagoner actually proved in [3] is the following.

Theorem 8.a.1. (Hatcher-Wagoner) There is a natural exact sequence as follows.

$$Wh_1^+(G;A) \oplus Wh_1^+(G;\mathbb{Z}_2) \xrightarrow{j_1 + j_2} \pi_0 C(M) \xrightarrow{\lambda} Wh_2(G) \to 0$$

Furthermore if the first Postnikov invariant of M is trivial and a section $\sigma: M_{(1)} \to M_{(2)}$ of the fibration $B^2A \to M_{(2)} \to M_{(1)} = BG$ is chosen then a retraction $\sigma_*: \pi_0 C(M) \to Wh_1^+(G;A)$ of j_1 is obtained. Here $Wh_2(G) = K_2\mathbb{Z}[G]/\pi_2 BM(G)^+$.

In this chapter we shall examine Hatcher's construction of σ_* (as explained in [5] and in personal communications to the author.) We shall see exactly where the assumption that $k_1(M) = 0$ is used. Furthermore we shall see shat happens when $\kappa = k_1(M)$ is nonzero. This is the purpose of the entire paper. We shall prove:

Theorem 8.a.2. There is a natural exact sequence as follows.

$$K_3\mathbb{Z}[G] \xrightarrow{\chi_3^+(\kappa,-) + \chi_G} Wh_1^+(G;A) \oplus Wh_1^+(G;\mathbb{Z}_2) \xrightarrow{j} \pi_0 C(M) \xrightarrow{\lambda} Wh_2(G) \to 0$$

where $\chi_G: K_3\mathbb{Z}[G] \to Wh_1^+(G;\mathbb{Z}_2)$ is the Grassmann invariant.

Strictly speaking the Grassmann invariant [9] comes first in logical order since it depends only on $G = \pi_1 M$. However the reader should have no trouble following the arguments here without knowledge of the Grassmann invariant.

We shall also investigate to what extent the map σ_* depends on the choice of $\sigma: M_{(1)} \to M_{(2)}$ and we shall prove:

Theorem 8.a.3. Suppose that $\sigma^1, \sigma^2: M_{(1)} \to M_{(2)}$ are two sections of the map $M_{(2)} \to M_{(1)}$ and suppose that σ^1 and σ^2 differ by the difference class $\upsilon \in H^2(G;A)$. Then the retractions $\sigma^i_*: \pi_0 C(M) \to Wh_1^+(G;A)$ differ by the function $\chi_2^{Wh}(\upsilon,\lambda(-))$ where $\chi_2^{Wh}: H^2(G;A) \otimes Wh_2(G) \to Wh_1^+(G;A)$ is the map induced by χ_2^+ using 1.g.5.

Here is the outline of the rest of this chapter:

8.b. Review of Cerf theory.

In this section we review briefly the function theory approach to pseudoisotopies. This is often called "Cerf theory" after its originator [1] although many good ideas were added in [3].

If M is a compact smooth manifold and $I = [0,1]$, let $F(M)$ denote the space of all smooth maps $M \times I \to I$ which agree with the projection map in a neighborhood of $\partial(M \times I)$. Let $E(M)$ denote the subspace of $F(M)$ consisting of functions without critical points. When these spaces are equipped with the weak topology [7] $F(M)$ is contractible and $E(M) \simeq C(M)$. Thus $\pi_0 C(M) \cong \pi_1(F(M),E(M))$. Consequently an element of $\pi_0 C(M)$ is represented by a one-parameter family of functions $f_t: M \times I \to I$, $t \in I$. And $[f_t] = 0$ in $\pi_0 C(M)$ if there is an admissible deformation of f_t which eliminates all the singularities.

Using jet transversality one sees the following. Transversality of the first derivative to 0 implies that the singular set is a smooth 1-manifold. Transversality of the second derivative to the stratification of the set of $n \times n$ matrices according to rank says that the singular points are all nondegenerate except for a finite number of cancellation (birth-death) points.

It is shown in [4] that f_t can be deformed do that all critical points lie in the two middle indices i and $i+1$ where $i = [m/2]$ if $m = \dim M \geq 6$. To study f_t further we must choose a gradient-like vector field v_t for f_t. This means that $v_t = -\text{grad}\, f_t$ with respect to some metric on $M \times I$ which varies with t. We shall assume that $-v_t(x)$ is the standard unit vector $(0,1)$ in the I direction if x is near $\partial(M \times I)$.

If the trajectory of v_t goes from a Morse critical point of index a down to a Morse point of index b we shall call this trajectory an a/b intersection. Cerf showed that trajectories leading up and down from birth-death points can be made disjoint from all other critical points. Transversality shows that there are no $i/i+1$ intersections, there are only a finite number of $i+1/i+1$ intersections (these are called handle additions) and for a fixed t there are only a finite number of $i+1/i$ intersections called incidence points. There are also i/i handle additions but these can be eliminated by sliding them up to $i+1/i+1$ handle additions.

If $x \in M \times I$ is a Morse point of f_t then the stable sphere of x is the collection of all trajectories of v_t leading away from x. The stable sphere of x can be equipped with a topology and a smooth structure in an obvious way so that it forms a sphere of dimension one less than the index of x. Similarly the unstable sphere of s is the collection of all trajectories of v_t leading into x.

Suppose now that (f_t,v_t) has no handle additions. Then an element of Wh_1+ $Wh_1^+(G;A \oplus \mathbb{Z}_p)$ can be associated to (f_t,v_t) in the following way. The condition of no handle additions implies that each critical point set component S of f_t has exactly two degenerate critical points, one birth and one death point. Also S is null homotopic in $X = M \times I \times I$. The incidence points between the upper and lower index Morse point arcs in S form a framed 1-manifold J_S (the framing is explained in [9] and is not important in this paper) and a choice of null homotopies of S in X produces a map $J_S \to X^{S^1}$ which sends J_S to $X_0^{S^1}$ = the component of X^{S^1} containing the trivial loop. Summing over all S we get an element of $\Omega_1^{fr}(X^{S^1}, X_0^{S^1})$ $\cong Wh_1^+(G;A \oplus \mathbb{Z}_2)$. Ignoring the framing of J_S we get an element of $Wh_1^+(G;A)$. It is easy to see that this is an invariant of (f_t,v_t) independent of the choice of null homotopies of S in X. It is also easy to see that any element of $Wh_1^+(G;A \oplus \mathbb{Z}_2)$ is the invariant of some family (f_t,v_t) without handle additions. Also any two such families with the same $Wh_1^+(G;A \oplus \mathbb{Z}_2)$ invariant represent the same element of $\pi_0C(M)$. This is the definition of the map $j_1 + j_2$ in 8.a.1. Given $\sigma\colon BG^2 \to X$ we shall show how the $Wh_1^+(G;A)$ summand of this invariant can be extended to more general families thus resulting in a map $\sigma_*\colon \pi_0C(M) \to Wh_1^+(G;A)$.

8.c. The section $\sigma\colon BG^2 \to X$.

Take $X = M \times I \times I$ as before and consider the cannonical map $p\colon X \to X_{(1)} \simeq BG$ of X into its 1-coskeleton. Let BG be the standard CW-complex corresponding to the normalized bar construction for G. Then there exists a homotopy right inverse $\sigma: BG^2 \to X$ of p over the 2-skeleton BG^2 of BG. If σ is extendable to BG^3 then the composition $BG^2 \to X \to X_{(2)}$ is uniquely extendable to BG (up to homotopy.) And conversely any map $BG \to X_{(2)}$ which induces the identity on π_1 is homotopic to a map derived in this way. The specification of the map $\sigma: BG^2 \to X$ is equivalent to making the following choices.

(1) Choose a base point $* \in X$.

(2) For each $u \in G$ choose a loop $\sigma_1(u): (\Delta^1,\partial) \to (X,*)$ representing $u \in \pi_1 X$. Take $\sigma_1(1)$ to be the trivial loop.

(3) For every pair $u, v \in G$ choose a map $\sigma_2(u,v): \Delta^2 \to X$ whose faces are $d_0 = \sigma_1(u)$, $d_1 = \sigma_1(uv)$ and $d_2 = \sigma_1(v)$. Let $\sigma_2(u,1) = \sigma_1(u)s_0$ and $\sigma_2(1,v) = \sigma_1(v)s_1$ where $s_j: \Delta^2 \to \Delta^1$ is the j-th degeneracy map.

Now consider a generic one-parameter family of functions $f_t: M \times I \to I$, $t \in I$, together with a gradient-like vector field v_t. We shall assume that (f_t,v_t) is a <u>lens-shaped family</u> which means that it satisfies the following three conditions.

(a) f_t has critical points only in the middle two indices i, $i + 1$, $i = [\dim M/2]$.

(b) (f_t,v_t) has only $i+1/i+1$ handle additions, i.e., it has no i/i handle additions.

(c) The third condition will be explained later.

Given any triple (σ,f_t,v_t) where $\sigma: BG^2 \to X$ is extendable to BG^3 and (f_t,v_t) is a lens-shaped family we shall associate an element of $Wh_1^+(G;A)$. Since every element of $\pi_0 C(M)$ is represented by a lens-shaped family we will get a map $\sigma_*: \pi_0 C(M) \to Wh_1^+(G;A)$. The argument will be to make more choices which we will call "c" and then to define $\sigma_*(f_t,v_t,c)$ and show that it is independent of the choice of c.

8.d. <u>Choices needed to define $\sigma_*: \pi_0 C(M) \to Wh_1^+(G;A)$.</u>

To define $\sigma_*(f_t,v_t)$ we must make the following choices.

(1) Choose a path from each birth point to the base point $* \in X$.

(2) Number the birth points $1, 2, \ldots, k$.

(3) Choose an orientation for the negative eigenspace of $D^2(f_t)_x$ at each Morse point $(x,t) \in (M \times I) \times I$. These orientations should vary continuously along a path of Morse points.

With these choices we may define an incidence matrix $r(t) = (r_{pq}(t)) \in GL_k(\mathbb{Z}[G])$ $GL_k(\mathbb{Z}[G])$ for all but a finite number of critical t's in I. For each critical t we may associate an elementary operation $x(t) = x_{pq}(u)$, $u \in \pm G$, which has the property that $r(t + \varepsilon) = r(t - \varepsilon)e_{pq}(u)$ for small $\varepsilon > 0$. We may perform the choices in (3) so that $r(0)$ is the identity matrix. We will also assume that $r(1)$ is the identity matrix. <u>This is the third condition for (f_t,v_t) to be lens-shaped.</u> Let $t_1 < t_2 < \ldots < t_n$ represent the critical t's. Then the condition $r(1) = I_k$ implies that the product $x(t_1) \ldots x(t_n)$ considered as an element of $St(\mathbb{Z}[G])$ is in the kernel of the map $\phi\colon St(\mathbb{Z}[G]) \to GL(\mathbb{Z}[G])$ and thus represents an element of $K_2\,\mathbb{Z}[G]$. The image of this element in $Wh_2(G)$ is $(-1)^{i+1}\lambda([f])$. The lens-shaped assumption also implies that we can make the following choice.

(4) Choose a path $\gamma(z)$ from each critical point z to the base point of X so that $\gamma(z)$ varies continuously with z and agrees with choice (1) when z is a birth point.

Let t_j be a critical parameter value with $x(t_j) = x_{pq}(\varepsilon u)$, $\varepsilon = \pm 1$, $u \in G$. This means that a trajectory ξ_j of the vector field v_{t_j} goes from z_q down to z_p where z_s is the s-th upper index Morse point. It also implies that $u = [\gamma(z_p)\xi_j\gamma(z_q)^{-1}]$.

(5) Choose a homotopy from $\gamma(z_p)\xi_j\gamma(z_q)^{-1}$ to $\sigma_1(u)$, or equivalently choose a homotopy from $\xi_j\gamma(z_q)^{-1}\sigma_1(u)^{-1}$ to $\gamma(z_p)^{-1}$.

Let c represent the choices (1) - (5) above. We shall define $\sigma_*(f_t,v_t,c) \in Wh_1^+(G;A)$ and we shall show that it does not depend on the choice of c assuming that σ is extendable to BG^3. Under the same assumption we show that $\sigma_*(f_t,v_t)$ is invariant under homotopies of f_t and v_t. Our detailed proof will also show what happens when σ is not extendable to BG^3.

8.e. Definition of $\sigma_*(f_t, v_t, c)$.

We may assume that all birth points occur at the same t-coordinate, say a, and that all death points occur at the t-coordinate b. Let y_p denote the p-th lower index Morse point of f_t, $t \in (a,b)$. The unstable sphere of y_p will be called the p-th unstable sphere. Taking the union over all $t \in I' = [a + \delta, b - \delta]$ where $\delta > 0$ is infinitesmal we get a 1-parameter family of spheres which we call the p-th unstable cylinder. This cylinder will be identified with $S_p^{m-i} \times I' = S^{m-i} \times I' \times p \subset S^{m-i} \times I' \times \{1,2, \ldots ,k\}$.

Let $J_{pq} \subset S_p^{m-i} \times I'$ be the closure of the set of all trajectories going from z_q down to y_p. These are the pq-incidence points. By transversality J_{pq} is a compact 1-manifold with boundary, the boundary being a finite set of points with t-coordinates equal to one of $t_0, t_1, \ldots , t_{n+1}$ where $t_0 = a + \delta$, $t_{n+1} = b - \delta$ and the other t_j's are as before. Let J_{pq} be the set of all points in J_{pq} with t-coordinate equal to $t_0, \ldots , t_{n+1}$. By transversality we may assume that J_{pq} is a finite set.

For $x \in J_{pq} - \delta J_{pq}$ let $u(x) \in G$ be the homotopy class $u(x) = [\gamma(y_p)\xi(x) \gamma(z_q)^{-1}]$. If x is not a critical point of the projection map $J_{pq} \to I'$ then there is a sign $\varepsilon(x) = \pm 1$ such that x contributes the monomial $\varepsilon(x)u(x)$ to the pq-entry of the incidence matrix $r(x) = r(t(x))$ where $t(x)$ is the I'-coordinate of x. If we interpret $\varepsilon(x)$ as an orientation for $J_{pq} - \delta J_{pq}$ by associating the tangent vector $\varepsilon(x)dt$ to x we get an orientation which varies continuously on all of $J_{pq} - \delta J_{pq}$.

Let $\mu(x) \in \mathbb{Z}[G]$ be the qp-entry of the matrix $s(x) = r(x)^{-1}$. If $w \in G$ let $\mu(x,w)$ be the coefficient of w in $u(x)\mu(x)$, i.e.,

$$u(x)\mu(x) = \sum_{w \in G} \mu(x,w)w.$$

Also let $\Omega_w X$ be the w-component of ΩX. Define a continuous function $\Psi^w: J_{pq} - \delta J_{pq} \to \Omega_w X$ for each $w \in G$ by the following equation.

$$\Psi^w(x) = \gamma(y_p)\xi(x)\gamma(z_q)^{-1}\sigma_1(u(x))^{-1}\sigma_1(w) = \Psi^1(x)\sigma_1(w)$$

Now consider $J_{pq} - \delta J_{pq}$ as a disjoint union of open intervals and compactify it to get a disjoint union of closed intervals (attach a point at each end.) Call this compactification $\tilde{J}_{pq}$. If $x \in \delta J_{pq}$ there are at most two points in $\tilde{J}_{pq}$ corresponding to x. These will be called x_L, x_R where x_L is the limit in $\tilde{J}_{pq}$ of points in $J_{pq} - \delta J_{pq}$ converging to x from the left (i.e., smaller t-values) and x_R is the right limit. Note that the functions ξ, Ψ^w extend uniquely to continuous function on all of J_{pq} and thus induce continuous functions on $\tilde{J}_{pq}$. Also the locally constant functions r, s, μ extend uniquely to locally constant functions on $\tilde{J}_{pq}$.

Since $A[G] \cong H_1(\Omega X)$ maps onto $H_0(G;A[G]) \cong H_1(X^{S^1})$ which maps onto $Wh_1^+(G;A)$ we may define $\sigma_*(f_t,v_t,c)$ as the image in $Wh_1^+(G;A)$ of the homology class in $H_1(X^{S^1})$ given by

$$(*) \qquad (-1)^i \Sigma\, \mu(x,w)\overline{\Psi}^w$$

where the summation runs over all w in G, all components of $\tilde{J}_{pq}$ and all p, q. The maps $\overline{\Psi}^w$ are derived from Ψ^w as follows.

Let $\overline{J}_{pq}$ be the mapping cylinder of the inclusion map $\partial\tilde{J}_{pq} \to \tilde{J}_{pq}$. Thus $\overline{J}_{pq}$ is equal to $\tilde{J}_{pq}$ with an interval adjoined to each boundary point. The map $\overline{\Psi}^w: \overline{J}_{pq} \to \Omega_w X$ will be an extension of the map $\Psi^w: J_{pq} \to \Omega_w X$. The extension will be defined by choosing a homotopy $H(x,w)$ from $\Psi^w(x)$ to $\sigma_1(w)$ for each $x \in \partial\tilde{J}_{pq}$.

We divide the elements of δJ_{pq} into three classes. First there are the points $x \in \delta J_{pq}$ which lie in the interior of J_{pq}. To each of these points is associated the left and right limits x_L, $x_R \in \partial\tilde{J}_{pq}$. These <u>interior points</u> have t-coordinate equal to one of the critical values t_j for $1 \leq j \leq n$. The second class of points in δJ_{pq} are those that occur at $t = t_0$ and $t = t_{n+1}$. These will be called <u>end points</u>. The remaining points of δJ_{pq} lie in the third class. These will be called the <u>T-points</u>.

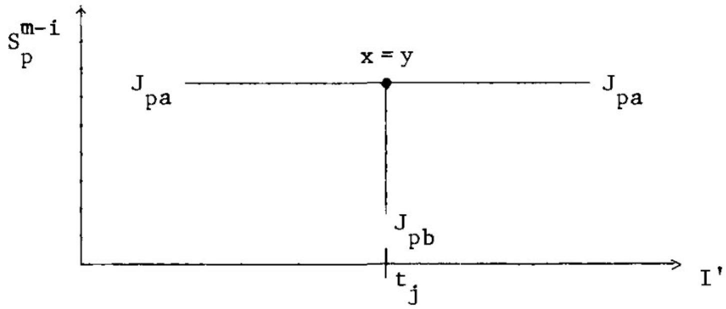
Let $x \in \delta J_{pq}$ be an interior point and take $w \in G$. Then the homotopy $H(x,w) = H(x_L,w) = H(x_R,w)$ is chosen as follows.

(1) Choose a null homotopy of the loop $\Psi^1(x) = \gamma(y_p)\xi(x)\gamma(z_q)^{-1}\sigma_1(u(x))^{-1}$.

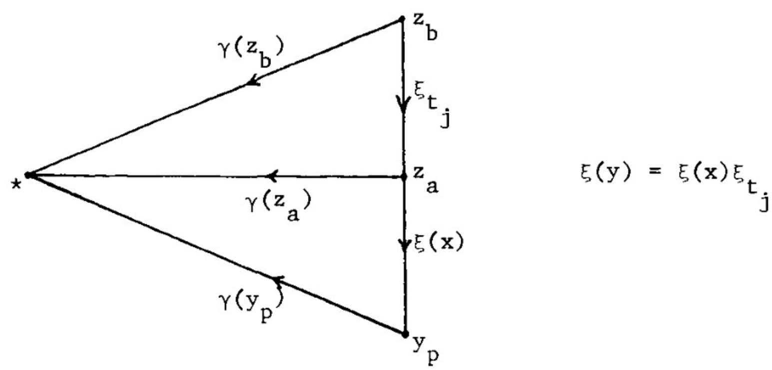
(2) For each $w \in G$ multiply this homotopy by $\sigma_1(w)$. This gives a homotopy from $\Psi^w(x)$ to $\sigma_1(w)$. Let this be $H(x,w)$.

Suppose that x occurs at the critical t-value t_j and $x(t_j) = x_{ab}(\varepsilon v)$. If $q \neq a$ then $\mu(x_L) = \mu(x_R)$ and thus the choice in (1) above does not affect the homology class of (*). If $q = a$ then $x \in \delta J_{pa} \subset S_p^{m-i} \times I'$ is equal to a T-point $y \in \delta J_{pb}$. The points x and y are related by the equation $\xi(y) = \xi(x)\xi_{t_j}$ as indicated in the following drawings.

(3)



(4)



Let $H(y,w)$ be the composition of the following homotopies.

(5)
$$\begin{aligned}\psi^w(y) &= \gamma(y_p)\xi(y)\gamma(z_b)^{-1}\sigma_1(u(y))^{-1}\sigma_1(w)\\ &= \gamma(y_p)\xi(x)\xi_j\gamma(z_b)^{-1}\sigma_1(u(x)v)^{-1}\sigma_1(w)\\ &\underset{\alpha}{\simeq} \gamma(y_p)\xi(x)\xi_j\gamma(z_b)^{-1}\sigma_1(v)^{-1}\sigma_1(u(x))\sigma_1(w)\\ &\underset{\beta}{\simeq} \gamma(y_p)\xi(x)\gamma(z_a)^{-1}\sigma_1(u(x))^{-1}\sigma_1(w)\\ &\simeq \sigma_1(w) \qquad \text{(by } H(x,w))\end{aligned}$$

where α is the inverse of the homotopy $\sigma_2(u(x),v)$ and β is the homotopy given in 8.d.5. These two homotopies do not depend on the choice of H. Since $u(y) = u(x)v$ and $\mu(x_L) = \mu(x_R) + \varepsilon v\mu(y)$ we have $\mu(x_L,w) = \mu(x_R,w) + \varepsilon\mu(y,w)$ for all $w \in G$. This implies that the sum (*) is independent of the choice of $H(x,w)$. (The sign ε is taken care of by the orientation of $\overline{J}_{pq}$.)

All T-points are derived in the above manner and all end points are irrelevant as we shall now show. If $x \in \delta J_{pq}$ is an end point we must have $p = q$, $u(x) = 1$, $r(x) = I_k$, $s(x) = I_k$ and $\mu(x) = 1$. Thus $\mu(x,w) = 0$ for all $w \neq 1$ in G. In the homology class (*) these points and their components in J_{pq} are irrelevant since they lie in the kernel of the map $H_0(G;A[G]) \to Wh_1{}^+(G;A)$. This completes the definition of $\sigma_*(f_t,v_t,c)$.

8.f. $\sigma_*(f_t,v_t,c)$ is independent of the choice of c.

Lemma 8.f.1. $\sigma_*(f_t,v_t,c)$ is independent of the choice made in 8.d.5.

Proof: Let j be fixed and suppose that $x(t_j) = x_{ab}(\varepsilon v)$. Suppose that the homotopy in 8.d.5 is altered by the element $\alpha \in A = \pi_2 X$. Then the only thing that changes in the definition of $\sigma_*(f_t,v_t,c)$ is the homotopy β in 8.e.5. This changes by $u(x)\cdot\alpha \in A$. This means that the homology class 8.e(*) changes by $(-1)^i \varepsilon\varepsilon(x)\mu(y,w)$ $u(x)\cdot\alpha \otimes w$ for each $x \in J_{*a}$ at $t = t_j$ and each $w \in G$. The total change can be computed as follows where the summation is over all $x \in J_{*a}$ and all $w \in G$.

$$(-1)^i \sum_{x,w} \varepsilon\varepsilon(x)\mu(y,w)u(x)\cdot\alpha \otimes w$$

$$= (-1)^i \sum_{x,w} \varepsilon\varepsilon(x)\mu(y,w)v^{-1}\cdot\alpha \otimes u(y)^{-1}wu(x)v \qquad \text{(conjugation by } u(y) = u(x)v)$$

$$= (-1)^i \sum_{x,w} \varepsilon v^{-1}\cdot\alpha \otimes [\varepsilon(x)\mu(y,w)u(y)^{-1}wu(x)]v$$

$$= (-1)^i \varepsilon v^{-1}\cdot\alpha \otimes [\Sigma_p \Sigma_{x \in J_{pa}} \varepsilon(x)\mu(y)u(x)]v \qquad \text{(by definition of } \mu(y,w))$$

$$= (-1)^i \varepsilon v^{-1}\cdot\alpha \otimes [\Sigma_p s_{bp} r_{pa}]v \qquad \text{(since } \mu(y) = s_{bp} \text{ and } r_{pa} = \Sigma_{x \in J_{pa}} \varepsilon(x)u(x))$$

$$= 0 \qquad \text{(since } s = r^{-1} \text{ and } a \neq b) \qquad \square$$

Lemma 8.f.2. $\sigma_*(f_t,v_t,c)$ is independent of the choices made in 8.d.1 and 8.d.4 provided that σ is extendable to BG^3.

Proof: Note first that in 8.e we did not use $\gamma(z)$ for z in a neighborhood of the birth-death points. So suppose γ' is another choice of paths homotopic to γ on the set of Morse points of f_t. This homotopy composed with the homotopy 8.d.5 gives a homotopy from $\gamma'(z_p)\xi_j\gamma'(z_q)^{-1}$ to $\sigma_1(u)$. Using this choice of 8.d.5 the homology class 8.e(*) is the same as before.

Now suppose that γ' differs from γ on the p-th component by $u_p \in G$. By the comments in the previous paragraph we may assume that $\gamma'(z_q) = \sigma_1(u_q)\gamma(z_q)$ and $\gamma'(y_p) = \sigma_1(u_p)\gamma(y_p)$. Thus for every x we have

$$r_{pq}(x)' = u_p r_{pq}(x) u_q^{-1}$$

$$\mu(x) = s_{qp}(x)' = u_q s_{qp}(x) u_p^{-1} = u_q \mu(x) u_p^{-1}$$

$$u(x)' = u_p u(x) u_q^{-1}$$

$$u(x)'\mu(x)' = u_p u(x)\mu(x) u_p^{-1} = \sum_w \mu(x,w) u_p w u_p^{-1}$$

$$\mu(x,w)' = \mu(x, u_p^{-1} w u_p)$$

$$(3) \qquad \Psi^w(x)' = \sigma_1(u_p)\gamma(y_p)\xi(x)\gamma(z_q)^{-1}\sigma_1(u_q)^{-1}\sigma_1(u_p u(x) u_q^{-1})^{-1}\sigma_1(w)$$

We shall show that if $\mu(x,w)'[\overline{\Psi}^w(x)']$ (with x varying in some component of $\overline{J}_{pq}$) is one term in the new homology class then it is equal to the term in the original class given by $\mu(x,\underline{w})u_p[\overline{\Psi}^{\underline{w}}(x)]u_p^{-1}$ where $\underline{w} = u_p^{-1} w u_p$. Thus the invariant will be unchanged in $Wh_1^+(G;A)$.

The homology class $u_p[\overline{\Psi}^{\underline{w}}(x)]u_p^{-1}$ is represented by the loop $\sigma_1(u_p)\overline{\Psi}^{\underline{w}}(x)$ $\sigma_1(u_p)^{-1}$. This loop contains the path

$$(4) \qquad \sigma_1(u_p)\Psi^{\underline{w}}(x)\sigma_1(u_p)^{-1} = \sigma_1(u_p)\gamma(y_p)\xi(x)\gamma(z_q)^{-1}\sigma_1(u(x))^{-1}\sigma_1(\underline{w})\sigma_1(u_p)^{-1}$$

together with homotopies of this path (when $x \in \partial\tilde{J}_{pq}$) to the path $\sigma_1(u_p)\sigma_1(\underline{w})\sigma_1(u_p)^{-1}$. To find these homotopies first choose a null homotopy of $\Psi^1(x) = \gamma(y_p)\xi(x)\gamma(z_q)^{-1}$ $\sigma_1(u(x))^{-1}$ for each $x \in \partial\tilde{J}_{pq}$ as in 8.e.1. This gives rise to the following null homotopy of $\Psi^1(x)'$ for the same boundary point x.

$$\begin{aligned} & \sigma_1(u_p)\gamma(y_p)\xi(x)\gamma(z_q)^{-1}\sigma_1(u_q)^{-1}\sigma_1(u_p u(x) u_q^{-1})^{-1} \\ (5) \qquad \simeq\ & \sigma_1(u_p)\gamma(y_p)\xi(x)\gamma(z_q)^{-1}\sigma_1(u(x))^{-1}\sigma_1(u_p)^{-1} \\ \simeq\ & \sigma_1(u_p)\sigma_1(u_p)^{-1} \qquad \text{(by the null homotopy of } \Psi^1(x)) \\ \simeq\ & 0 \end{aligned}$$

where the homotopy (5) is given by the homotopy $\sigma_1(u_q)^{-1}\sigma_1(u_p u(x) u_q^{-1})^{-1} \simeq$

$\sigma_1(u(x))^{-1}\sigma_1(u_p)^{-1}$ given uniquely up to homotopy by σ_2.

In this paragraph we shall prove that $\mu(x,w)'[\overline{\Psi}^w(x)'] = \mu(x,\underline{w})u_p[\overline{\Psi}^{\underline{w}}(x)]u_p^{-1}$ in the case in which both boundary points are interior in J_{pq}. For this we choose any homotopy from $\sigma_1(\underline{w}) = \sigma_1(u_p^{-1}wu_p)$ to $\sigma_1(u_p)^{-1}\sigma_1(w)\sigma_1(u_p)$. This gives a homotopy of $\sigma_1(u_p)\sigma_1(\underline{w})\sigma_1(u_p)^{-1}$ to $\sigma_1(w)$. Using this homotopy we may deform the loop $\sigma_1(u_p)\overline{\Psi}^{\underline{w}}(x)\sigma_1(u_p)^{-1}$ in X^{S^1} to the following new loop. The path (4) is changed to the path $\sigma_1(u_p)\gamma(y_p)\xi(x)\gamma(z_q)^{-1}\sigma_1(u(x))^{-1}\sigma_1(u_p)^{-1}\sigma_1(w)$. The boundary points have the homotopies to $\sigma_1(w)$ given by first contracting $\Psi^1(x) = \gamma(y_p)\xi(x)\gamma(z_q)^{-1}\sigma_1(u(x))^{-1}$ and then cancelling $\sigma_1(u_p)\sigma_1(u_p)^{-1}$. This new loop is homotopic to the loop $\overline{\Psi}^w(x)'$ in the following way. Along the path $\Psi^w(x)'$ perform the homotopy used in (5) above.

$$\begin{aligned} &\sigma_1(u_p)\gamma(y_p)\xi(x)\gamma(z_q)^{-1}\sigma_1(u_q)^{-1}\sigma_1(u_pu(x)u_q^{-1})^{-1}\sigma_1(w) \\ \simeq\ &\sigma_1(u_p)\gamma(y_p)\xi(x)\gamma(z_q)^{-1}\sigma_1(u(x))^{-1}\sigma_1(u_p)^{-1}\sigma_1(w). \end{aligned}$$

This extends to the boundary points in an obvious way thus concluding our proof for interior points.

We now consider the T-points. Given the choices 8.d.5 for γ we can make the following choices of homotopies for γ': $\sigma_1(u_a)\gamma(z_a)\xi_j\gamma(z_b)^{-1}\sigma_1(u_b)^{-1} \simeq \sigma_1(u_a)\sigma_1(u)\sigma_1(u_b)^{-1} \simeq \sigma_1(u_auu_b^{-1})$. The last homotopy is given uniquely up to homotopy by σ_2 because of the existence of σ_3. Examining the two loops $\overline{\Psi}^w(y)'$ and $\sigma_1(u_p)\overline{\Psi}^{\underline{w}}(y)\sigma_1(u_p)^{-1}$ we see that σ_2 gives a homotopy between them:

$$\begin{aligned} \Psi^w(y)' &= \sigma_1(u_p)\gamma(y_p)\xi(y)\gamma(z_b)^{-1}\sigma_1(u_b)^{-1}\sigma_1(u_pu(y)u_b^{-1})^{-1}\sigma_1(w) \\ \sigma_1(u_p)\Psi^{\underline{w}}(y)\sigma_1(u_p)^{-1} &= \sigma_1(u_p)\gamma(y_p)\xi(y)\gamma(z_b)^{-1}\sigma_1(u(y))^{-1}\sigma_1(u_p^{-1}wu_p)\sigma_1(u_p)^{-1} \quad \square \end{aligned}$$

<u>Lemma 8.f.6</u>. $\sigma_*(f_t,v_t,c)$ is independent of the choices made in 8.d.2 and 8.d.3.

<u>Proof</u>: A change in the numbering (8.d.2) or the orientations (8.d.3) has no affect on the geometric choices (the paths and the homotopies.) The only things that change are the matrices which change by conjugation. This has no affect on the homology class 8.e(*). □

We have proved the following.

<u>Theorem 8.f.7</u>. If σ is extendable to BG^3 then $\sigma_*(f_t,v_t,c)$ is independent of c.□

8.g. The difference obstruction.

In this section we shall consider f_t and v_t to be fixed and consider how $\sigma_*(f_t,v_t)$ varies with the choice of σ. Thus let σ, σ': $BG^3 \to X$ be two map which induce the identity map on π_1. Since σ and σ' are homotopic when restricted to BG^1 we may assume that they are actually equal on BG^1. Thus the choices in 8.c.1 and 8.c.2 are uniquely chosen and two choices σ_2 and σ_2' are given for the map σ_2 of 8.c.3. Using the standard construction of the difference cochain, the maps σ_2 and σ_2' produce a map $\sigma_2' - \sigma_2 = \tau(u,v)$: $S^2 \to X$ for every pair u, v in G such that the homotopy class $h(u,v) = [\tau(u,v)] \in A = \pi_2X$ gives a normalized factor set h: $G \times G \to A$. Let h_ℓ: $\mathbb{Z}[G] \otimes \mathbb{Z}[G] \to A[G]$ represent the corresponding linear factor set.

Theorem 8.g. $\sigma_*'(f_t,v_t) - \sigma_*(f_t,v_t) = \chi_2^{Wh}([h],\lambda(f_t,v_t))$

Proof: Let c represent the choices 8.d.1 - 5 for σ. Then c is also a suitable set of choices for σ' since $\sigma_1' = \sigma_1$. The only difference between $\sigma_*(f_t,v_t,c)$ and $\sigma_*'(f_t,v_t,c)$ lies in the homotopy 8.e.5. which changes by $h(u(x),v)$. Therefore the change in the homology class 8.e(*) is given as follows where the summation is over all $x \in J_{*a}$, all $w \in G$ and $j = 1, 2, \ldots, n$. (Note that a, b, ε, v depend on j since $x(t_j) = x_{ab}(\varepsilon v)$.)

$$
\begin{aligned}
& (-1)^i \sum_{x,w,j} \varepsilon\varepsilon(x)\mu(y,w)h(u(x),v)w \\
= \quad & (-1)^i \sum_{x,w,j} h_\ell(\varepsilon(x)u(x) \otimes \varepsilon v)\mu(y,w)v^{-1}u(x)^{-1}w \\
= \quad & (-1)^i \sum_{x,j} h_\ell(\varepsilon(x)u(x) \otimes \varepsilon v)s_{bp} \\
= \quad & (-1)^i \sum_{j,p} h_\ell(r_{pa} \otimes \varepsilon v)s_{bp} \\
= \quad & (-1)^{i+1}\chi_2^{Wh}([h],(-1)^{i+1}\lambda[f_t]) \\
= \quad & \chi_2^{Wh}([h],\lambda[f_t]).
\end{aligned}
$$

We used formula 6.e.1 and the fact that $x(t_1) \ldots x(t_n) = (-1)^{i+1}\lambda[f_t]$. □

This completes the proof of 8.a.3 modulo the invariance of $\sigma_*(f_t,v_t)$.

8.h. <u>Invariance of $\sigma_*(f_t, v_t)$ under lens-shaped homotopies of (f_t, v_t).</u>

A <u>lens-shaped homotopy</u> of (f_t, v_t) will mean a two parameter family of functions and gradient-like vector fields (f_{ts}, v_{ts}) satisfying the following conditions.

(0) There is a finite set $S \subset \text{int } I$ such that if s_0 is a fixed element of $I - S$ then (f_{ts_0}, v_{ts_0}) is a lens shaped family.

If s_0 is a critical value of s, i.e., an element of S then one of the following four things happens at s_0. ( <=> $\emptyset$)

(1) A component of the critical set of f_{ts} is either cancelled or created as s goes from $s_0 - \varepsilon$ to $s_0 + \varepsilon$.

(2) A pair of consecutive handle additions of the form $x_{ab}(v)$ and $x_{ab}(-v)$ is either cancelled or created as the value of s passes s_0.

(3) Two commuting handle additions pass each other.

(4) Two noncommuting handle additions of the form $x_{ab}(u)$ and $x_{bc}(v)$ pass each other and a third handle addition $x_{ac}(uv)$ is either created or destroyed depending on which of the two original handle additions started on the left.

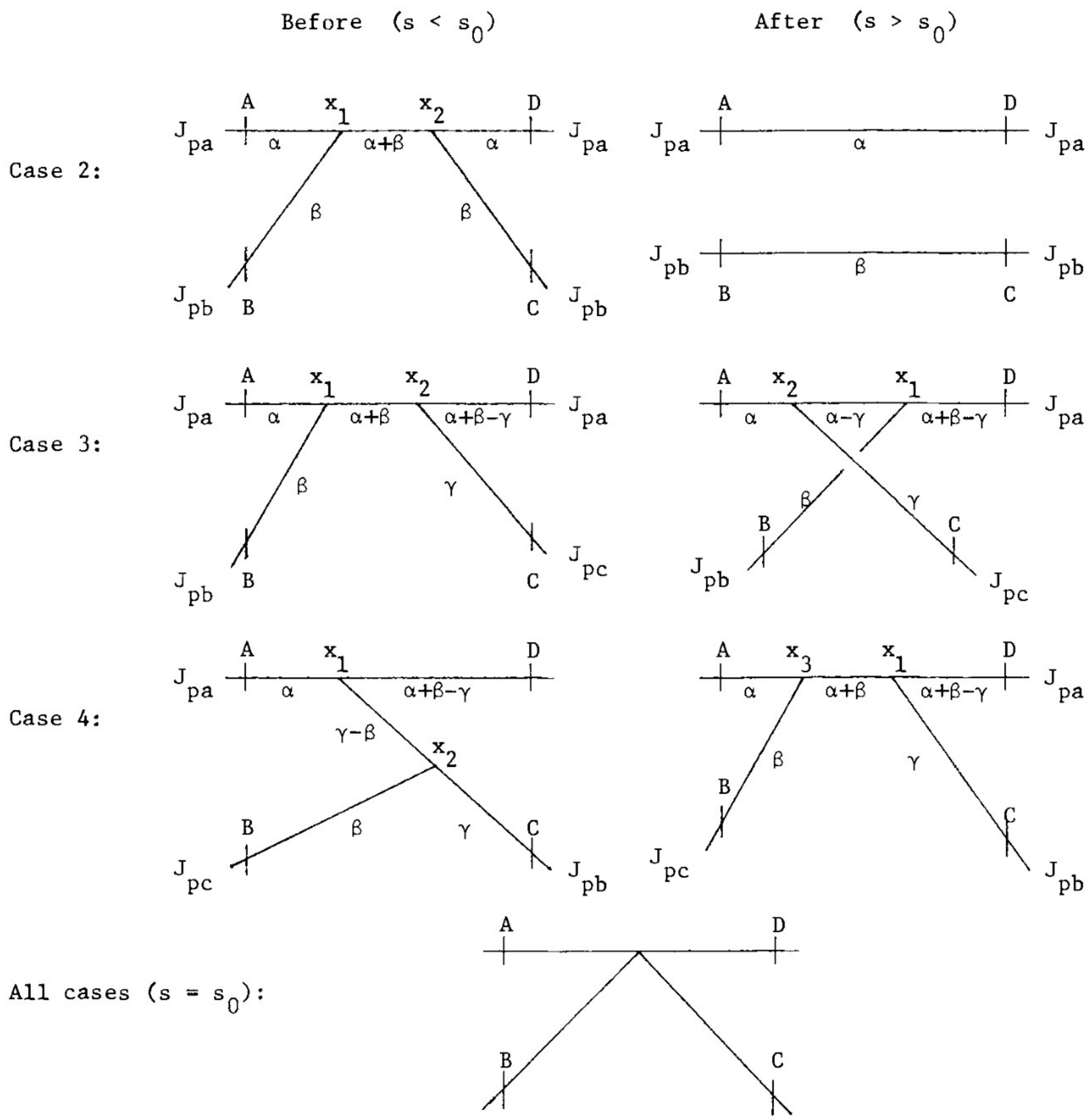
<u>Theorem 8.h.5</u>. If $\sigma\colon BG^2 \to X$ is fixed and extendable to BG^3 and if (f_{ts}, v_{ts}) is a lens-shaped homotopy then $\sigma_*(f_{t0}, v_{t0}) = \sigma_*(f_{t1}, v_{t1})$.

<u>Proof</u>: The proof is by case by case analysis. It is not difficult to see that $\sigma_*(f_{ts}, v_{ts})$ does not change as s varies between two consecutive elements of S. We will show that $\sigma_*(f_{ts}, v_{ts})$ does not change at any of the four types of critical values of s.

In case (1) the value of $\sigma_*(f_{ts}, v_{ts})$ does not change because a lens-shaped critical point set component on the verge of being cancelled will have only incidence points x with $u(x) = 1$.

In case (2) it can be seen that the $Wh_1^+(G;A)$ invariant does not change if the choices 8.d.5 for the two consecutive handle additions $x_{ab}(v)$ and $x_{ab}(-v)$ are chosen compatibly. The interesting part of the verification of this case is illustrated in 8.h.6. The line segment $\overline{AD}$ is broken at the two points x_1 and x_2.

Figure 8.h.6. These are subsets of the unstable cylinder $S_p^{m-i} \times I'$ before and after the deformations of cases 2, 3, 4. The Greek letters indicate values of $\varepsilon(x)\mu(x,w)$.



The function Ψ^w is continuous on $\overline{AD}$ but $\overline{\Psi}^w$ is given by taking paths from $\Psi^w(x_1)$ and $\Psi^w(x_2)$ to $\sigma_1(w)$. The paths are identical for the right and left limits at both points so the α contribution to the invariant 8.e(*) is not changed by the deformation. The function Ψ^w is not continuous on the path Bx_1x_2C but the compatible choice of 8.d.5 implies that the composition of the three loops $\overline{\Psi}^w$ (component of B), $\overline{\Psi}^w(\overline{x_1x_2})$, $\overline{\Psi}^w$ (component of C) is homotopic to the loop $\overline{\Psi}^w$ (component of B and C) in the "after" picture. Thus the β contribution does not change.

In case (3) the $Wh_1^+(G;A)$ invariant does not change if the choices 8.d.5 for the two commuting homotopies are chosen in such a way that they vary continuously with the deformation parameter s. The verification is trivial except when the two handle additions are of the form $x_{ab}(u)$ and $x_{ac}(v)$. This case is illustrated in 8.h.6.3 above. The α contribution to 8.e(*) is unchanged by the deformation for the same reason as in case (2). The β and γ contributions are unchanged by the continuous choice of 8.d.5.

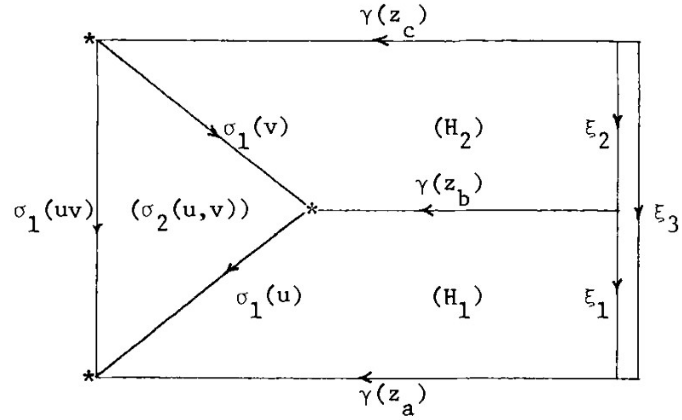
Case (4) is the final and most important case. This is the only part of the proof which requires the assumption that σ be extendable to BG^3. Suppose that the two handle additions are $x_{ab}(u)$ on the left and $x_{bc}(v)$ on the right (i.e., the latter has a larger t-value.) When they cross the handle addition $x_{ac}(uv)$ is created. Let the homotopies in 8.d.5 for the handle additions $x_{ab}(u)$ and $x_{bc}(v)$ be chosen continuously with respect to s and let them be denoted by H_1 and H_2. Then all the contributions to the sum 8.e(*) are unchanged by the deformation except for those involving the new handle addition $x_{ac}(uv)$. This situation is illustrated in 8.h.6.4.

One choice of 8.d.5 for the new handle addition can be given as follows. Let ξ_1, ξ_2 and $\xi_3 = \xi_1\xi_2$ be the trajectories corresponding to the handle additions $x_{ab}(u)$, $x_{bc}(v)$, $x_{ac}(uv)$ respectively. Take a homotopy from $\gamma(z_a)\xi_3\gamma(z_c)^{-1}$ to $\sigma_1(uv)$ as follows. (See also figure 8.h.7.)

$$\begin{aligned}
\gamma(z_a)\xi_3\gamma(z_c)^{-1} &= \gamma(z_a)\xi_1\xi_2\gamma(z_c)^{-1} \\
&\simeq \gamma(z_a)\xi_1\gamma(z_b)^{-1}\sigma_1(v) && \text{(by } H_2) \\
&\simeq \sigma_1(u)\sigma_1(v) && \text{(by } H_1) \\
&\simeq \sigma_1(uv) && \text{(by } \sigma_2(u,v))
\end{aligned}$$

Look now at figure 8.h.6.4. The α contribution to 8.e(*) is unchanged for the reason given in case (2). The γ contribution is unchanged by continuity of the choice of H_1. Thus the β contribution is the only thing that might change. In the "before" picture of 8.h.6.4 the β contribution is Bx_2x_1D. In the "after" picture it is Bx_3x_1D. We will use the notation y_1, y_2, y_3 to indicate x_1, x_2, x_3 considered as limit points of J_{pb}, J_{pc}, J_{pc} respectively. By continuity we then have the following equalities at $s = s_0$: $x_1 = x_3$, $x_2 = y_1$, $y_2 = y_3$, $\xi(y_1) = \xi(x_1)\xi_1$ and

Figure 8.h.7.



$\xi(y_2) = \xi(x_1)\xi_3 = \xi(x_1)\xi_1\xi_2 = \xi(y_1)\xi_2$.

We now examine the "before" picture. Let $x = u(x_1)$ and choose a homotopy H_0 from $\gamma(y_p)\xi(x_1)\gamma(z_a)^{-1}$ to $\sigma_1(x)$. We also use H_0 to indicate the corresponding null homotopy of $\overline{\Psi}^1(x_1) = \gamma(y_p)\xi(x_1)\gamma(z_a)^{-1}\sigma_1(x)^{-1}$. This gives a homotopy from $\overline{\Psi}^w(x_1) = \overline{\Psi}^1(x_1)\sigma_1(w)$ to $\sigma_1(w)$ for any $w \in G$, thus connecting $\overline{\Psi}^w(\overline{x_1 D})$ to $\sigma_1(w)$ in the prescribed way.

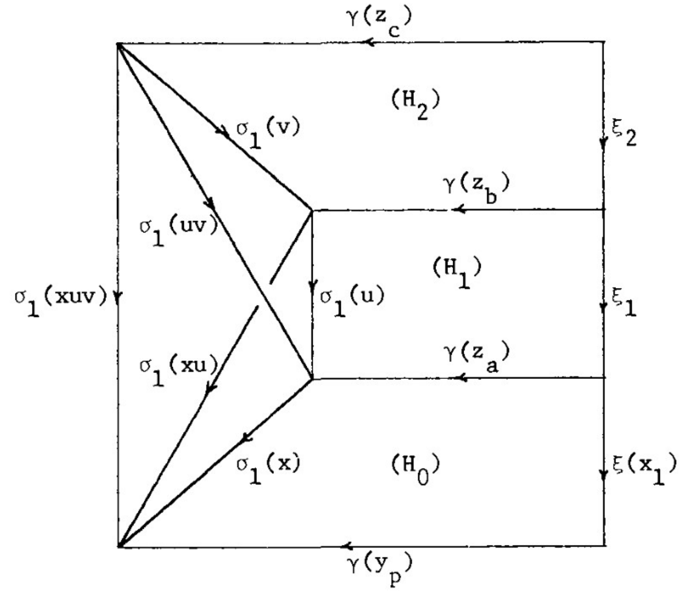
The prescribed method (8.e.5) of connecting $\Psi^w(y_1)$ to $\sigma_1(w)$ is given by taking the following null homotopy of $\Psi^1(y_1)$. (See figure 8.h.8.)

$$\begin{aligned}
\Psi^1(y_1) &= \gamma(y_p)\xi(y_1)\gamma(z_b)^{-1}\sigma_1(xu)^{-1} \\
&= \gamma(y_p)\xi(x_1)\xi_1\gamma(z_b)^{-1}\sigma_1(xu)^{-1} \\
&\simeq \gamma(y_p)\xi(x_1)\xi_1\gamma(z_b)^{-1}\sigma_1(u)^{-1}\sigma_1(x)^{-1} && \text{(by } \sigma_2(x,u)) \\
&\simeq \gamma(y_p)\xi(x_1)\gamma(z_a)^{-1}\sigma_1(x)^{-1} && \text{(by } H_1) \\
&\simeq \ * && \text{(by } H_0)
\end{aligned}$$

This will be taken as the null homotopy of $\Psi^1(x_2) = \Psi^1(y_1)$.

The prescribed method of connecting $\Psi^w(y_2)$ to $\sigma_1(w)$ is given by taking the null homotopy of $\Psi^1(y_2) = \gamma(y_p)\xi(x_1)\xi_1\xi_2\gamma(z_c)^{-1}\sigma_1(xuv)^{-1}$ given by H_0, H_1, H_2 and $\sigma_2(x,u)$, $\sigma_2(xu,v)$ which represent the back faces of the tetrahedron in 8.h.8. In the "after" picture the null homotopy of $\Psi^1(y_3) = \Psi^1(y_2)$ is given by H_0, H_1, H_2 and $\sigma_2(u,v)$, $\sigma_2(x,uv)$ which represent the two front faces of the tetrahedron in 8.h.8. Assuming that σ is extendable to BG^3 these two homotopies are homotopic thus completing the proof of 8.h.5. □

Figure 8.h.8.



8.i. Proof of the main theorem.

Suppose now that $\sigma\colon BG^2 \to X$ is not extendable to BG^3. Then to each triple $(x,u,v) \in G^3$ we get a mapping $\partial\Delta^3 \to X$ given by σ_1 on the 1-simplices and σ_2 on the 2-simplices as indicated by the tetrahedron in 8.h.8. Taking the upper left corner as the base point this gives an element of $A = \pi_2 X$ which we denote by $h(x,u,v)$. By the standard argument h is a normalized 3-cocycle representing the first Postnikov invariant of X: $[h] = k_1(X) \in H^3(G;A)$. (With the appropriate sign and orientation conventions.)

Consider the proof of 8.h.5. The β contribution in 8.h.6.4 changes from the back null homotopy of 8.h.8 to the front null homotopy. This means that the β contribution changes by $h(x,u,v) \otimes w$. Since this counts $(-1)^i\varepsilon(y_2)\mu(y_2,w)$ times in the sum 8.e(*) the change in this sum is given by

$$\begin{aligned}
&(-1)^i \sum_{p,w,x_1 \in J_{pa}} \varepsilon(y_2)\mu(y_2,w)h(u(x_1),u,v) \otimes w \\
&= (-1)^i \sum_{p,x_1} \varepsilon(y_2)h(u(x_1),u,v) \otimes u(y_2)\mu(y_2) \\
&= (-1)^i \sum_{p,x_1} \varepsilon(x_1)h(u(x_1),u,v) \otimes u(x_1)uvs_{cp} \\
&= (-1)^i \sum_{p,x_1} h_\ell(\varepsilon(x_1)u(x_1) \otimes u \otimes v)s_{cp} \\
&= (-1)^i \sum_{p} h_\ell(r_{pa} \otimes u \otimes v)s_{cp}
\end{aligned}$$

where $h_\ell = h[G]$ is the linearization of h defined in 6.d.

Proof of 8.a.2: Take an element (x,y) in the kernel of $j_1 + j_2\colon Wh_1^+(G;A) \oplus Wh_1^+(G;\mathbb{Z}_2) \to \pi_0 C(M)$. Then this element is represented by a lens shaped family (f_t,v_t) with no handle additions. Since $[f_t]$ is trivial in $\pi_0 C(M)$ there is a two parameter family (f_{ts},v_{ts}) which is a null homotopy of (f_t,v_t). By [8] we may assume that (f_{ts},v_{ts}) is a lens-shaped null homotopy. The handle addition pattern of (f_{ts},v_{ts}) represents an element P of $K_3\mathbb{Z}[G]$ in a way described in [7]. The computation above together with formula 6.h shows that $x = (-1)^i \chi_3^+(k_1(M),P)$. The result of [9] is that $y = (-1)^i \chi_G(P)$.

Conversely, given any element P of $K_3\mathbb{Z}[G]$ one can construct a lens-shaped homotopy (f_{ts},v_{ts}) which has a handle addition pattern corresponding to P and thus represents a lens shaped null homotopy of a family (f_t,v_t) representing $(-1)^i$ times $(\chi_3^+(k_1(M),P),\chi_G(P))$ in $Wh_1^+(G;A) \oplus Wh_1^+(G;\mathbb{Z}_2)$. □

8.j. $\sigma_*\colon \pi_0 C(M) \to Wh_1^+(G;A)$ is well-defined.

In this section we complete the proof of 8.a.1 (and thus of 8.a.3) by showing that $\sigma_*(f_t,v_t)$ depends only on the homotopy class of f_t and thus σ_* is a well-defined retraction of j_1. The argument is more of less the same as the one given in [5] with the additional assumption that σ_2 is extendable to BG^3.

Suppose that (f_{t0},v_{t0}) and (f_{t1},v_{t1}) are two lens-shaped families for $M \times I$ representing the same element of $\pi_0 C(M)$. Then we may assume that there is a homotopy (f_{ts},v_{ts}), $s \in I$, from (f_{t0},v_{t0}) to (f_{t1},v_{t1}) and a finite critical set $S \subset I$ such that for a fixed $s \notin S$ the family (f_{ts},v_{ts}) is generic and satisfies the following two conditions. First f_t has critical points only in the middle indices i, $i+1$. Second (f_{ts},v_{ts}) has only $i{+}1/i{+}1$ handle additions. The justification for this assumption will be explained later.

Let $s_0 \in S$. Then at $s = s_0$ one of the following things happens.

(1)-(4) Same as 8.h(1)-(4).

(5) A death and birth point are cancelled. ($> <$ $\Rightarrow$ $\asymp$)

(6) Two Morse points are cancelled at t_0 creating a death point at $t_0 - \varepsilon$ and

a birth point at $t_0 + \varepsilon$. (this is the reverse of deformation 5.)

In order for deformations 8.h(1)-(4) to make complete sense here one must make the choices 8.d.1 - 3. These choices may be taken to vary continuously through the first four deformations but they are obviously discontinuous for deformations 5,6.

To show that $\sigma_*(f_{t0},v_{t0}) = \sigma_*(f_{t1},v_{t1})$ we must define $\sigma_*(f_{ts},v_{ts})$ for every s not in the critical set S and show that it is the same for all s. To define $\sigma_*(f_{ts},v_{ts})$ one must make the choices 8.d.1 - 5 for the family (f_{ts},v_{ts}) with the modification that $\gamma(z)$ need not be continuous at birth-death points. The definition of $\sigma_*(f_{ts},v_{ts},c_s)$ works as before except for the proof of 8.f.2 which must be modified to allow the $\gamma(y)$'s and $\gamma(z)$'s to vary independently. This can be accomplished by taking the same proof and changing the notation: wherever the symbol "u_q" appears replace it by "v_q".

The proof of 8.h.5 works here to show that $\sigma_*(f_{ts},v_{ts})$ does not change at deformations of type 1 - 4. For deformation 6 take one choice of c for $s < s_0$ and extend it through s_0 in a continuous manner except for the numbering (8.d.2). The invariant $\sigma_*(f_{ts},v_{ts})$ is obviously unchanged. By symmetry deformation 5 does not change the invariant.

We now give a brief explanation for the assumptions made about the deformation (f_{ts},v_{ts}). It was shown in [8] that if two lens-shaped families (f_{t0},v_{t0}) and (f_{t1},v_{t1}) were homotopic and had the same $K_2\mathbb{Z}[G]$ invariant then suitable "suspensions" of both of them were homotopic by a lens-shaped homotopy. If the K_2-invariants are not the same the difference between the two K_2-invariants must be an element, say X, of $\Omega_2^{fr}(BG)$ since f_{t0} and f_{t1} have the same $Wh_2(G)$ invariant. However, elements of $\Omega_2^{fr}(BG)$ are very easy to describe and given any such element, say Y, one can easily construct a deformation of (f_{t0},v_{t0}) to a lens-shaped family $(f_{t\frac{1}{2}},v_{t\frac{1}{2}})$ whose K_2-invariant differs from that of (f_{t0},v_{t0}) by Y. Take $Y = X$ and we have (after suspension) that $(f_{t\frac{1}{2}},v_{t\frac{1}{2}})$ is lens-shaped homotopic to (f_{t1},v_{t1}).

Chapter 9. A topological version of the counterexample.

Theorem 8.a.2 together with the algebraic counterexample in 7.b produces a counterexample to the original statement of Hatcher and Wagoner. However the proofs involved the functor K_3 and its relationship to two parameter families of functions. For the sake of finding a counterexample this is not necessary. We construct in this chapter a counterexample which uses only K_2, the elementary example in 7.a and 8.g.

9.a. Construction of the manifold M.

In 7.a we gave an example of a group G, a G-module A on which G acts trivially and elements $\kappa \in H^2(G;A)$ and $s \in K_2\mathbb{Z}[G]$ such that $\chi_2^+(\kappa,s) \neq 0$ in $Wh_1^+(G;A)$. In our example G and A also satisfy the condition that BG and $B^2A = K(A,2)$ are 2-equivalent to finite 3-complexes. Given any G, A, κ, s satisfying these conditions we shall construct a compact manifold M with $\pi_1 M = G \times \mathbb{Z}$, $\pi_2 M = A$, $k_1(M) = \kappa \times \xi$ (ξ being the generator of $H^1(\mathbb{Z};\mathbb{Z})$) and show using 8.g that the image of $\chi_2^+(\kappa,s)$ in $Wh_1^+(G \times \mathbb{Z}; A)$ represents zero in $\pi_0 C(M)$.

Let $h\colon BG \times B^2A \to BG \times B^2A$ be the map given by $h(x,y) = (x,\mu(\bar{\kappa}(x),y))$ where $\bar{\kappa}\colon BG \to B^2A$ is the classifying map for κ and μ is the H-space multiplication map for $B^2A \simeq \Omega K(A,3)$. Let X be a finite 3-complex which is 2-equivalent to $BG \times B^2A$. Then there is a cellular map $H\colon X \to X$ corresponding to h.

Let Y be the quotient of $X \times I$ by the identification $(x,1) \sim (H(x),0)$. By embedding Y in a finite Euclidean space and taking a closed regular neighborhood we will get a compact manifold M which looks like a product near $X \times 1/2$. More precisely M contains a codimension one submanifold N such that both N and M - N are homotopy equivalent to N and N has a tubular neighborhood which we identify with $N \times [-1,2]$. Let * be the base point of N.

9.b. Construction of a null homotopy of $\chi_2^+(\kappa,s)$ in $\pi_0 C(M)$.

If $f\colon M \times I \to I$ is a smooth map the <u>support</u> of f will mean the closure of the set of all $x \in M$ such that $f(x,y) \neq y$ for some $y \in I$. Note that if W is a codimension-0 submanifold of M then every element f of $\mathcal{F}(W)$ extends uniquely to an element $\bar{f}$ of $\mathcal{F}(M)$ with support in W. If (f_t,v_t) is a family of functions with vector fields on $M \times I$ the <u>support</u> of (f_t, v_t) will mean the closure of the set of all points $x \in M$ such that for some $y \in I$ and $t \in I$ we have either $f_t(x,y) \neq y$ or $v_t(x,y) \neq (0,-1)$.

Let (f_t^1, v_t^1, c^1) be a lens-shaped family on $N \times [-1,0] \times I$ together with a choice c^1 for 8.d.1 - 5 such that the handle additions of (f_t^1, v_t^1, c^1) are x_1, $x_2, \ldots, x_n$ with product equal to s in $St(\mathbb{Z}[G])$. Let (f_t^2, v_t^2, c^2) be a lens-shaped family on $N \times [1,2] \times I$ together with a choice c^2 for 8.d.1 - 5 such that the handle additions on (f_t^2, v_t^2, c^2) are $x_n^{-1}, \ldots, x_1^{-1}$ with product s^{-1} in $St(\mathbb{Z}[G])$. We shall assume that the critical points of f_t^1 occur at $t < 1/2$ and that those of f_t^2 occur at $t > 1/2$.

Let $\sigma^1 \colon BG^2 \to N = N \times 0$ be any map which is extendable to BG^3 and which induces the identity on π_1. Then we have

<u>Lemma 9.b.1</u>. $\sigma^1{}_*(\tilde{f}_t^2, \tilde{v}_t^2) = \sigma^1{}_*(\overline{f_t^2}, \overline{v_t^2}) \pm \chi_2{}^+(\kappa, s)$ where $(\tilde{f}_t^2, \tilde{v}_t^2)$ and $(\overline{f_t^2}, \overline{v_t^2})$ are the extensions of (f_t^2, v_t^2) to $M - N \times (0,1)$ and $N \times [-1,2]$ respectively.

<u>Proof</u>: Let $(\overline{\overline{f_t^2}}, \overline{\overline{v_t^2}})$ be the extension of (f_t^2, v_t^2) to all of M. Let $\sigma^2 \colon BG^2 \times S^1 \to M$ be a map which extends σ^1 and which induces the identity on π_1. Then we have

$$\sigma^1{}_*(\overline{f_t^2}, \overline{v_t^2}) = \overline{\sigma^2{}_*(\overline{\overline{f_t^2}}, \overline{\overline{v_t^2}}, \bar{\gamma})}$$

$$\sigma^1{}_*(\tilde{f}_t^2, \tilde{v}_t^2) = \sigma^2{}_*(\overline{\overline{f_t^2}}, \overline{\overline{v_t^2}}, \tilde{\gamma})$$

if the paths $\bar{\gamma}$ are chosen to lie in $N \times [-1,2] \times I \times I$ and the paths $\tilde{\gamma}$ are chosen to lie in the complement of $N \times (0,1) \times I \times I$. (Since σ^2 is not extendable to $BG^3 \times S^1$, $\sigma^2{}_*$ depends on the choice of γ but not on the other choices in c as we showed in 8.f.)

Now take the base point of M and move it around $\sigma_1^2(S^1)$. The affect is to conjugate all terms by the generator of $\mathbb{Z}$. Since this commutes with everything the terms are unchanged. However $\bar{\gamma}$ changes to $\tilde{\gamma}$ and σ^2 changes to its conjugate σ^3 which differs from σ^2 by the difference obstruction $\kappa \times 1 \in H^2(G \times \mathbb{Z}; A)$. Thus by 8.g we have

$$\sigma^2{}_*(f_t^2, v_t^2, \bar{\gamma}) = \sigma^3{}_*(f_t^2, v_t^2, \tilde{\gamma}) = \sigma^2{}_*(f_t^2, v_t^2, \tilde{\gamma}) \pm \chi_2{}^+(\kappa, s) \qquad \square$$

Theorem 9.b.2. There exist two lens-shaped families without handle additions on $M \times I$ which are homotopic to each other but have different $Wh_1^+(G \times \mathbb{Z};A)$ invariants.

Proof: Let (f_t^3, v_t^3) be the lens-shaped family on $M \times I$ given by

$$(f_t^3(x,y), v_t^3(x,y)) = \begin{cases} (y,(0,-1)) & \text{if } x \notin N \times ([-1,0] \cup [1,2]) \\ (f_{2t}^1(x,y), v_{2t}^1(x,y)) & \text{if } x \in N \times [-1,0], \quad t \leq 1/2 \\ (f_1^1(x,y), v_1^1(x,y)) & \text{if } x \in N \times [-1,0], \quad t \geq 1/2 \\ (y,(0,-1)) & \text{if } x \in N \times [1,2], \quad t \leq 1/2 \\ (f_{2t-1}^2(x,y), v_{2t-1}^2(x,y)) & \text{if } x \in N \times [1,2], \quad t \geq 1/2 \end{cases}$$

We make the assumption that $(f_t^1(x,y), v_t^1(x,y))$ is independent of t for t sufficiently close to 1 and that $(f_t^2(x,y), v_t^2(x,y)) = (y,(0,-1))$ for t sufficiently close to 0. The two families of the theorem will both be homotopic to (f_t^3, v_t^3).

Let (f_t^4, v_t^4) be the restriction of (f_t^3, v_t^3) to $N \times [-1,2]$. We shall deform (f_t^4, v_t^4) to a lens-shaped family without handle additions. Let $p_1, p_2, \ldots, p_k$ be the death points of (f_t^4, v_t^4) with $t < 1/2$ where the numbering is given by c^1. Let $q_1, \ldots, q_k$ be the birth points of (f_t^4, v_t^4) with $t > 1/2$ numbered according to c^2. Let all the critical points be oriented (8.d.3) according to c^1 and c^2. Now perform the deformation which for each $a = 1, 2, \ldots, k$ cancels the birth point q_a with the death point p_a along a path homotopic to $\gamma_2(q_a)^{-1}\gamma_1(p_a)$ making sure the orientations agree. The resulting family which we shall call (f_t^5, v_t^5, c^5) will have handle additions $x_1, x_2, \ldots, x_n, x_n^{-1}, \ldots, x_1^{-1}$ which can be cancelled in pairs. This gives a family (f_t^6, v_t^6) without handle additions with $Wh_1^+(G;A)$ invariant equal to $\sigma^1{}_*(f_t^6, v_t^6) = \sigma^1{}_*(f_t^5, v_t^5)$ (by 8.h.5 case 1.) $= \sigma^1{}_*(f_t^1, v_t^1) + \sigma^1{}_*(\overline{f_t^2}, \overline{v_t^2})$.

Similarly let (f_t^7, v_t^7) be the restriction of (f_t^3, v_t^3) to $M - N \times (0,1)$. For each $a = 1, 2, \ldots, k$ cancel q_a with p_a along a path homotopic to $\gamma_2(q_a)^{-1}\xi^{-1}$ $\gamma_1(p_a)$ making sure the orientations agree, where ξ is a loop representing the positive generator of $\mathbb{Z}$. The resulting family which we call (f_t^8, v_t^8, c^8) will have handle additions $x_1, x_2, \ldots, x_n, x_n^{-1}, \ldots, x_1^{-1}$ since $\mathbb{Z}$ commutes with G in $G \times \mathbb{Z}$. Of course c^8 is the extension of c^1 but not of c^2. Cancel the handle additions as before and we get a family (f_t^9, v_t^9) without handle additions with $Wh_1{}^+(G;A)$ invariant equal to $\sigma^1{}_*(f_t^9, v_t^9) = \sigma^1{}_*(f_t^8, v_t^8)$ (by 8.h.5 case 1) $=$ $\sigma^1{}_*(f_t^1, v_t^1) + \sigma^1{}_*(\tilde{f}_t^2, \tilde{v}_t^2) = \sigma^1{}_*(f_t^1, v_t^1) + \sigma^1{}_*(\overline{f}_t^2, \overline{v}_t^2) \pm \chi_2{}^+(\kappa, s)$ (by the lemma.) □

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