THE KERVAIRE–MILNOR INVARIANT IN THE STABLE CLASSIFICATION OF SPIN 4-MANIFOLDS

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Abstract. We consider the rôle of the Kervaire–Milnor invariant in the classification of closed, connected, spin 4-manifolds up to stabilisation by connected sums with copies of $S^2 \times S^2$. This stable classification is detected by a spin bordism group over the classifying space $B\pi$ of the fundamental group. Part of the computation of this bordism group via an Atiyah-Hirzebruch spectral sequence is determined by a collection of codimension two Arf invariants. We show that these Arf invariants can be computed by the Kervaire–Milnor invariant evaluated on certain elements of π_2. In particular this yields a new stable classification of spin 4-manifolds with 2-dimensional fundamental groups.

1. Introduction

Two smooth, closed, connected, oriented 4-manifolds M and N are called *stably diffeomorphic* if there exist integers $m, n \in \mathbb{N}_0$ such that

$$M\#^m(S^2 \times S^2) \cong N\#^n(S^2 \times S^2).$$

We require that the diffeomorphism respects orientations, and we will always assume without comment that manifolds are *smooth and connected*. Note that we allow $m \neq n$, but $n - m = (\chi(M) - \chi(N))/2$, so by only considering 4-manifolds with the same Euler characteristic one can enforce $m = n$.

The problem of giving algebraic invariants that determine whether two 4-manifolds are stably diffeomorphic is the *stable classification problem*. For example, the isometry class of the equivariant intersection form on the second homotopy group, up to stabilisation by hyperbolic forms, is such an invariant. The stable classification of simply-connected 4-manifolds is detected by the signature of this intersection form.

Given an immersed sphere S in M with vanishing self-intersection number, the *Kervaire–Milnor invariant* arises as a secondary obstruction to homotoping S to an embedding. Our aim in this article is to explain its rôle in the stable classification problem for spin 4-manifolds. There is an additional condition, namely that S is $\mathbb{RP}^2$*-characteristic*, under which the Kervaire–Milnor invariant $\tau(S) \in \mathbb{Z}/2$ is well-defined. We defer the precise definitions to Section 1.1.

For any closed, oriented 4-manifold M with $\pi = \pi_1(M)$ and classifying map $f\colon M \to B\pi$, the radical $\operatorname{Rad}(\lambda_M) \subseteq \pi_2(M)$ of the intersection form λ_M, i.e. the kernel of the adjoint $\lambda_M^{\mathrm{ad}}\colon \pi_2(M) \to \pi_2(M)^*$, is isomorphic to $H^2(\pi; \mathbb{Z}\pi)$ via $PD \circ f^*\colon H^2(\pi;\mathbb{Z}\pi) \to H_2(M;\mathbb{Z}\pi) \xrightarrow{\cong} \pi_2(M)$ [HKT09, Corollary 3.2]. Also, let $\mathrm{red}_2\colon H^2(\pi;\mathbb{Z}\pi) \to H^2(\pi;\mathbb{Z}/2)$ be the map induced by reduction modulo two on the coefficients, and define $\mathrm{Sq} := (\mathrm{Sq}^2 \circ \mathrm{red}_2)\colon H^2(\pi;\mathbb{Z}\pi) \to H^4(\pi;\mathbb{Z}/2)$.

By restricting to the radical, $[S] \in \operatorname{Rad}(\lambda_M)$, we guarantee vanishing self-intersection number. For spin M we will show that restricting further to $PD \circ f^*(\ker \mathrm{Sq}) \subseteq \operatorname{Rad}(\lambda_M)$ ensures $\mathbb{RP}^2$-characteristic, and thus that τ gives rise to a stable diffeomorphism invariant of spin 4-manifolds, as follows.

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Theorem 1.1. *Let M be a closed, spin 4-manifold with a classifying map $f\colon M \to B\pi$.*

(i) For every element $x \in \ker \mathrm{Sq} \subseteq H^2(\pi;\mathbb{Z}\pi)$, its image $PD \circ f^*(x) \in \pi_2(M)$ has trivial self-intersection number and is $\mathbb{RP}^2$-characteristic, so the Kervaire–Milnor invariant

$$\tau(PD \circ f^*(x)) \in \mathbb{Z}/2$$

is well-defined.

(ii) The induced map $\tau_{M,f}\colon \ker \mathrm{Sq} \to \mathbb{Z}/2$ factors through $\mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} \ker \mathrm{Sq}$.

(iii) If a closed, spin 4-manifold N is stably diffeomorphic to M, then for some classifying map $g\colon N \to B\pi$ we have $\tau_{M,f} = \tau_{N,g}$.

In Theorem 1.5 below, we explain how the stable diffeomorphism invariant from Theorem 1.1 appears in the general stable classification programme, in the case that f factors through a 2-dimensional complex. Theorem 1.5 requires some more background in order to state, so for now we present its application in the case of geometrically 2-dimensional fundamental groups. A group π is (geometrically) d-dimensional if d is the least integer for which the classifying space $B\pi$ admits a finite d-dimensional CW-complex model. Note that for a 2-dimensional group $H^4(\pi;\mathbb{Z}/2) = 0$ so $\ker \mathrm{Sq} = H^2(\pi;\mathbb{Z}\pi)$.

Theorem 1.2. *Let π be a 2-dimensional group, let M and N be closed, spin 4-manifolds, and let $f\colon M \to B\pi$ be a classifying map.*

(i) The map $\tau_{M,f}$ is a homomorphism, i.e. an element of $\mathrm{Hom}_{\mathbb{Z}\pi}(H^2(\pi;\mathbb{Z}\pi),\mathbb{Z}/2)$.

(ii) The 4-manifolds M and N are stably diffeomorphic if and only if
  (a) the signatures of M and N are equal, and
  (b) the Kervaire–Milnor invariants $\tau_{M,f}$ and $\tau_{N,g}$ coincide for some choice of classifying map $g\colon N \to B\pi$.

Hambleton, Kreck, and the third author [HKT09] previously classified 4-manifolds with 2-dimensional fundamental groups, up to s-cobordism, in terms of the equivariant intersection form. They also assumed that the assembly map $H_4(B\pi;\mathbb{L}\langle 1\rangle_\bullet) \to L_4(\mathbb{Z}\pi)$ is injective, which in particular holds whenever the Farrell–Jones conjecture holds for π. While Theorem 1.2 only concerns the stable classification, it has the advantage that to apply it one only needs to compute a relatively small number of Kervaire–Milnor invariants, compared with computing the entire intersection form.

By Theorem 1.2, the map $\tau_{M,f}$ is a homomorphism if π is 2-dimensional. In Theorem 1.5, we will see that this is also the case whenever f factors through a 2-dimensional complex. But in general the following question remains open.

Question 1.3. *Is the map $\tau_{M,f}\colon \ker \mathrm{Sq} \to \mathbb{Z}/2$ always a homomorphism?*

Example 1.4. Let Σ be a closed, oriented surface with positive genus and suppose $\pi \cong \pi_1(\Sigma)$. Then the radical of λ_M is isomorphic to $H^2(\pi;\mathbb{Z}\pi) \cong H^2(\Sigma;\mathbb{Z}\pi) \cong H_0(\Sigma;\mathbb{Z}\pi) \cong \mathbb{Z}$. In this case our classification is particularly efficient since it requires the computation of just a single Kervaire–Milnor invariant $\tau(S)$, where $[S]$ generates $\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathrm{Rad}(\lambda_M) \cong \mathbb{Z}/2$. In particular, $\tau_{M,f}$ is independent of the choice of f.

Among closed, smooth 4-manifolds with $\pi_1(M) \cong \pi_1(\Sigma)$ and signature zero, there are two stable diffeomorphism classes. The class with trivial τ_M is represented by $M = \Sigma \times S^2$ where the radical $\mathrm{Rad}(\lambda_M) = \pi_2(M) \cong \mathbb{Z}$ is generated by an embedded sphere $\{\mathrm{pt}\} \times S^2$. The second stable diffeomorphism class is represented by a 4-manifold M' constructed from $\Sigma \times T^2$ by performing surgery on framed circles representing a dual pair of generators of $\pi_1(T^2) \cong \mathbb{Z}^2$, where the framing of the circles is “twisted”. The generator of $\mathrm{Rad}(\lambda_{M'}) \cong \mathbb{Z}$ cannot be represented by an embedding, even stably.

As mentioned above, Theorem 1.2 follows from our main technical theorem, Theorem 1.5, which we will explain below. First we review the definition of the Kervaire–Milnor invariant and the reformulation of the stable diffeomorphism question into bordism theory by Kreck [Kre99, Theorem C].

1.1. **Review of the Kervaire–Milnor invariant.** The Kervaire–Milnor invariant appeared previously in [FQ90], [Sto94], and [ST01], following a closely related invariant defined in [FK78, Mat78]. A version of this invariant was used by Freedman and Quinn to detect the Kirby–Siebenmann obstruction to smoothing the topological tangent bundle of a simply-connected topological 4-manifold with odd intersection form, in particular detecting the difference between $\mathbb{CP}^2$ and its star partner $*\mathbb{CP}^2$. This is somewhat orthogonal to the appearance of this invariant in the stable classification of spin 4-manifolds, since for spin topological 4-manifolds, it follows from Rochlin's theorem that the Kirby–Siebenmann invariant is computed as the signature divided by 8, and then modulo two.

To recall the definition of the Kervaire–Milnor invariant, let M be a smooth, spin 4-manifold with fundamental group π and suppose that $x \in \pi_2(M) \cong H_2(M;\mathbb{Z}\pi)$ satisfies $\lambda_M(x,x) = 0$, where

$$\begin{aligned}\lambda_M \colon \pi_2(M) \times \pi_2(M) &\to \mathbb{Z}\pi \\ (x,y) &\mapsto \langle PD^{-1}(y), x\rangle\end{aligned}$$

is the equivariant intersection form. Then we can represent x by a generic immersion $S\colon S^2 \looparrowright M$ whose double points can be paired up by immersed Whitney discs $\{W_i\}$, which by boundary twisting and pushing down can be chosen to be disjointly embedded, framed, and to intersect S transversely. Then the *Kervaire–Milnor invariant* of S is the count

$$\tau(S;\{W_i\}) := \sum_i |\mathring{W}_i \pitchfork S| \mod 2.$$

Suppose that x is $\mathbb{RP}^2$-*characteristic*, meaning that for every map $R\colon \mathbb{RP}^2 \to M$,

$$\lambda_2(\mathrm{red}_2(x), R) = 0,$$

where $\mathrm{red}_2\colon \pi_2(M) \cong H_2(M;\mathbb{Z}\pi) \to H_2(M;\mathbb{Z}/2)$ is reduction modulo two, and $\lambda_2\colon H_2(M;\mathbb{Z}/2) \times H_2(M;\mathbb{Z}/2) \to \mathbb{Z}/2$ is the mod 2 intersection pairing. Then $\tau(S;\{W_i\})$ is well-defined [ST01] on the homotopy class $x \in \pi_2(M)$, independent of the choices of S and the $\{W_i\}$, and so we write

$$\tau_M(x) := \tau(S;\{W_i\}) \in \mathbb{Z}/2.$$

In Section 2 we will give more details on the Kervaire–Milnor invariant, as well as relating it with an equivalent definition that is used in the proof of our theorems.

1.2. **Review of the stable classification via spin bordism.** Kreck [Kre99, Theorem C] showed that two closed, spin 4-manifolds with fundamental group π are stably diffeomorphic if and only if there are choices of spin structures and identifications of the fundamental groups with π, giving rise to equal elements in the bordism group $\Omega_4^{\mathrm{Spin}}(B\pi)$. To understand this group of bordism classes of pairs (M,c), where M is a closed 4-manifold with spin structure and $c\colon M \to B\pi$ classifies the universal cover, we consider the Atiyah-Hirzebruch spectral sequence (AHSS) computing $\Omega_4^{\mathrm{Spin}}(B\pi)$, starting with the E^2 page

$$E^2_{p,q} = H_p(B\pi;\Omega_q^{\mathrm{Spin}}).$$

The AHSS gives rise to a filtration whose iterated graded quotients are

$$\mathbb{Z} \cong \Omega_4^{\mathrm{Spin}} \underbrace{\subseteq}_{E^\infty_{2,2}} F_2 \underbrace{\subseteq}_{E^\infty_{3,1}} F_3 \underbrace{\subseteq}_{E^\infty_{4,0}} \Omega_4^{\mathrm{Spin}}(\pi).$$

The first isomorphism is determined by the signature divided by 16. The signature extends to the entire group $\Omega_4^{\mathrm{Spin}}(B\pi)$ and so we reduce our study to $\widetilde{\Omega}_4^{\mathrm{Spin}}(B\pi)$, the kernel of the signature map.

The AHSS then reduces to a shorter filtration

$$E_{2,2}^\infty \underbrace{\subseteq}_{E_{3,1}^\infty} F \underbrace{\subseteq}_{E_{4,0}^\infty} \widetilde{\Omega}_4^{\mathrm{Spin}}(B\pi),$$

where the subgroup F consists of bordism classes represented by signature zero 4-manifolds M with spin structure such that $c\colon M \to (B\pi)^{(3)}$ lands in the 3-skeleton of the classifying space $B\pi$. Similarly, the smallest filtration term $E_{2,2}^\infty$ is represented by elements (M,c) with $c\colon M \to (B\pi)^{(2)}$. Since the $E_{p,q}^2$ term of the spectral sequence is $H_p(\pi;\Omega_q^{\mathrm{Spin}})$, the $E_{p,q}^\infty$-terms are as follows:

$$\begin{aligned}
E_{2,2} &:= E_{2,2}^\infty = H_2(\pi;\mathbb{Z}/2)/\operatorname{im}(d_2,d_3);\\
E_{3,1} &:= E_{3,1}^\infty = H_3(\pi;\mathbb{Z}/2)/\operatorname{im}(d_2);\\
E_{4,0} &:= E_{4,0}^\infty = \ker(d_2,d_3) \subseteq H_4(\pi;\mathbb{Z}).
\end{aligned}$$

Moreover, by [Tei92, Theorem 3.1.3] the d_2 differentials are given by the dual homomorphisms $\mathrm{Sq}_2\colon H_{i+2}(\pi;\mathbb{Z}/2) \to H_i(\pi;\mathbb{Z}/2)$ to the Steenrod squares $\mathrm{Sq}^2\colon H^i(\pi;\mathbb{Z}/2) \to H^{i+2}(\pi;\mathbb{Z}/2)$, precomposed with the homomorphism induced by reduction modulo 2, $\mathrm{red}_2\colon H_{i+2}(\pi;\mathbb{Z}) \to H_{i+2}(\pi;\mathbb{Z}/2)$, where appropriate. Following [Tei92], we obtain the primary invariant $\mathfrak{pri}(M) = c_*[M] \in E_{4,0}$, the secondary invariant $\mathfrak{sec}(M) \in E_{3,1}$ and the tertiary invariant $\mathfrak{ter}(M) \in E_{2,2}$.

1.3. **Relating the tertiary and Kervaire–Milnor invariants.** We studied the primary invariant in [KPT18], and we studied the secondary and tertiary invariants in [KPT20], building on [KLPT17]. In [KPT18] and [KPT20], we gave criteria which can decide whether $(M,c) \in E_{2,2} \subseteq \widetilde{\Omega}_4^{\mathrm{Spin}}(B\pi)$, that is whether (M,c) is bordant to (M',c') such that $c'\colon M' \to B\pi$ factors through the 2-skeleton $B\pi^{(2)} \subseteq B\pi$. Our main theorem, stated next, says that assuming there is such a bordism, one can compute the tertiary invariant $\mathfrak{ter}(M')$ using the Kervaire–Milnor invariant.

One can compute $\mathfrak{ter}(M')$ via a collection of codimension two Arf invariants. The difficulty with this in practice is that one need to first find a homotopy of the classifying map to the 2-skeleton of $B\pi$, take precise inverse images of regular points, and compute with spin structures. Since the Kervaire–Milnor invariant is well-defined on homotopy classes, and does not depend on the choice of spin structure, it represents a computational improvement.

Theorem 1.5. *Let M be a closed, smooth, spin 4-manifold with fundamental group π. Suppose that there is a map $f\colon M \to K$ to a 2-complex K that is an isomorphism on fundamental groups. Let $i\colon K \to B\pi$ be a classifying map.*

(i) For each $\varphi \in \ker(\mathrm{Sq}^2\colon H^2(\pi;\mathbb{Z}/2) \to H^4(\pi;\mathbb{Z}/2))$, there exists a lift $\widehat{\varphi} \in H^2(K;\mathbb{Z}\pi)$ of $i^*(\varphi) \in H^2(K;\mathbb{Z}/2)$.

(ii) The element $PD(f^*(\widehat{\varphi})) \in H_2(M;\mathbb{Z}\pi) \cong \pi_2(M)$ is $\mathbb{RP}^2$-characteristic and has trivial self-intersection number, so that the Kervaire-Milnor invariant $\tau(PD \circ f^*(\widehat{\varphi})) \in \mathbb{Z}/2$ is well-defined.

(iii) The map

$$\begin{aligned}
\tau_{M,f}\colon\ \ker \mathrm{Sq}^2 &\to \mathbb{Z}/2\\
\varphi &\mapsto \tau(PD \circ f^*(\widehat{\varphi}))
\end{aligned}$$

is well-defined (i.e. is independent of the choice of $\widehat{\varphi}$) and is a homomorphism.

(iv) Under the map

$$\mathrm{Hom}(\ker \mathrm{Sq}^2, \mathbb{Z}/2) \xrightarrow{\cong} H_2(\pi;\mathbb{Z}/2)/\operatorname{im}\mathrm{Sq}_2 \to H_2(\pi;\mathbb{Z}/2)/\operatorname{im}(d_2,d_3),$$

$\tau_{M,f}$ is sent to $\mathfrak{ter}(M)$. In particular the image of $\tau_{M,f}$ under this map is independent of the choices of f and K.

Proof of Theorem 1.2 assuming Theorem 1.5. For π a 2-dimensional group, take $K = B\pi$, $f\colon M \to B\pi$ as in Theorem 1.2, and note that $E_{3,1} = E_{4,0} = 0$ and $\ker \mathrm{Sq}^2 = H^2(\pi;\mathbb{Z}/2)$. Given $x \in \mathrm{Rad}(\lambda_M) = PD \circ f^*(H^2(\pi;\mathbb{Z}\pi))$ take $\widehat{\varphi} \in H^2(\pi;\mathbb{Z}\pi)$ to be a preimage. Setting $\varphi \in \ker \mathrm{Sq}^2$ to be the image of $\widehat{\varphi}$ under reduction from $\mathbb{Z}\pi$ to $\mathbb{Z}/2$ coefficients, we have that $x = PD(f^*(\widehat{\varphi}))$. Theorem 1.2 now follows from Theorem 1.5 (ii), (iii), and (iv), our analysis of the AHSS above, and [Kre99, Theorem C]. □

Organisation of the paper. In Section 2 we provide more details on the Kervaire–Milnor invariant of immersed spheres, and give an alternative equivalent description in terms of π_1-trivial immersed surfaces. In Section 3 we explain how the Arf invariant arises in the Atiyah-Hirzebruch spectral sequence computation of spin bordism. In Section 4 we show Theorem 1.5 by comparing the Kervaire–Milnor invariant with the Arf invariant.

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2. The Kervaire–Milnor invariant

Let M be a smooth, closed, oriented, spin 4-manifold. In this section, all surfaces are assumed to be the images of generic maps into M, meaning the maps are immersions, and all intersections and self-intersections are transverse double points. In particular there are no triple points. Moreover the boundary, if nonempty, is assumed to be embedded.

Let Σ be a generically immersed sphere in M with $\lambda_M([\Sigma],[\Sigma]) = 0$. Generalising Freedman and Kirby [FK78, p. 93], Guillou-Marin [GM80], Matsumoto [Mat78], and Freedman–Quinn [FQ90, Definition 10.8], Schneiderman and the third author [ST01] defined an invariant $\widetilde{\tau}(\Sigma)$ with values in a quotient of $\mathbb{Z}[\pi \times \pi]$. Which quotients of $\mathbb{Z}[\pi \times \pi]$ one can take in order to get an invariant of the homotopy class of Σ depends on the intersection numbers of Σ with other immersed surfaces in M.

Assuming that Σ is $\mathbb{RP}^2$-characteristic, the image of $\widetilde{\tau}(\Sigma)$ under the augmentation and reduction modulo two map $\mathbb{Z}[\pi \times \pi] \to \mathbb{Z}/2$ is a well-defined invariant of the homotopy class of Σ. Following the nomenclature of Freedman–Quinn, we call this image the Kervaire–Milnor invariant $\tau(\Sigma) \in \mathbb{Z}/2$ of $[\Sigma] \in \pi_2(M)$.

We will define $\tau(\Sigma)$ carefully in Section 2.1. Then in Section 2.2 we will extend the definition to π_1-trivial generically immersed closed, oriented surfaces in M.

2.1. The τ invariant for generically immersed spheres.

As before, let M be a smooth, closed, oriented, spin 4-manifold, and fix an identification $\pi_1(M) \cong \pi$.

Definition 2.1. [Self-intersection number][Wal99, Chapter 5], [FQ90, Section 1.7]. Let $x \in \pi_2(M)$. Since M is spin, we can represent x by a generically immersed sphere whose normal bundle has even Euler number. Add cusp homotopies in a small open set to make the Euler number of the normal bundle zero. Call the resulting sphere Σ. Now count the self intersections of Σ with sign and group elements. The attribution of signs uses the orientation of M. The group element is the image in $\pi_1(M)$

of a double point loop associated to the self-intersection point, with some choice of orientation of the double point loop. This count gives rise to an element

$$\mu(x) \in \mathbb{Z}\pi/\{g \sim g^{-1}\}.$$

The self-intersection number is valued in a quotient group of the $\mathbb{Z}\pi$-module $\mathbb{Z}\pi$ since there is no canonical way to decide whether to associate g or g^{-1} to a given double point of Σ. The number $\mu(x)$ is a well-defined invariant of the homotopy class of x.

Remark 2.2. Our normalisation of $\mu(x)$ at $1 \in \pi$ implies that $\lambda_M(x,x) = \mu(x) + \overline{\mu(x)}$, which can be useful for a relation to quadratic L-theory. It works the Euler number $e(\nu\Sigma)$ is even for every Σ, or equivalently if $w_2(\widetilde{M}) = 0$. Using cusp homotopies it is always possible to change Σ so that the self-intersection number of Σ at 1 is trivial, even if the universal covering of M is not spin. This gives an element $\mu'(x) \in \mathbb{Z}\pi/g \sim g^{-1}$ which again only depends on the homotopy class of x. Using this convention, $\mu'(x)$ is an obstruction to representing x by an embedded sphere. In the setting of this paper, our 4-manifolds are spin and we usually assume that $\lambda_M(x,x) = 0$. In that case, the two conventions agree and $\mu(x) = \mu'(x) = 0$.

The following lemma is rather useful, in that it tells us that it is enough to consider the equivariant intersection pairing in order to find spheres with vanishing self-intersection number.

Lemma 2.3. *For a closed, oriented, spin 4-manifold M, if $\lambda_M(x,x) = 0$ for $x \in \pi_2(M)$, then $\mu(x) = 0$.*

Proof. Using a representative as in Definition 2.1, we can assume that $0 = \lambda_M(x,x) = \mu(x) + \overline{\mu(x)}$. Suppose that $\sum_g n_g g \in \mathbb{Z}\pi$ is a lift of $\mu(x)$. Then $\mu(x) + \overline{\mu(x)} = 0$ implies that $n_g + n_{g^{-1}} = 0$ for every $g \in \pi$. If $g = g^{-1}$ then we immediately see $n_g = 0$. For the remaining group elements, in the value group of μ we have $g \sim g^{-1}$, so $n_g g + n_{g^{-1}} g^{-1} = (n_g + n_{g^{-1}})g = 0 \cdot g = 0$. Sum over a set of representatives for the subsets $\{g, g^{-1}\}$ with $g \neq g^{-1}$, to obtain $\mu(x) = 0$. □

Denote the map given by augmentation composed with reduction modulo 2 by $\mathrm{red}_2 \colon \mathbb{Z}\pi \xrightarrow{\varepsilon} \mathbb{Z} \to \mathbb{Z}/2$. We abuse notation and also use $\mathrm{red}_2 \colon \pi_2(M) = H_2(M;\mathbb{Z}\pi) \to H_2(M;\mathbb{Z}/2)$ to denote the induced map on homology.

Definition 2.4. Let $\lambda_2 \colon H_2(M;\mathbb{Z}/2) \times H_2(M;\mathbb{Z}/2) \to \mathbb{Z}/2$ be the $\mathbb{Z}/2$-valued intersection pairing.

(1) An element $\alpha \in \pi_2(M)$ is called S^2-*characteristic* if $\mathrm{red}_2(\lambda(\alpha,\beta)) = 0 \in \mathbb{Z}/2$ for all $\beta \in \pi_2(M)$. Let $\mathcal{SC} \subseteq \pi_2(M)$ denote the subset of S^2-characteristic elements α with $\mu(\alpha) = 0$.
(2) An element $\alpha \in \pi_2(M)$ is called $\mathbb{RP}^2$-*characteristic* if $\lambda_2(\mathrm{red}_2(\alpha), [R])) = 0 \in \mathbb{Z}/2$ for every map $R \colon \mathbb{RP}^2 \to M$. Let $\mathcal{RC} \subseteq \pi_2(M)$ denote the subset of $\mathbb{RP}^2$-characteristic elements α with $\mu(\alpha) = 0$.

Lemma 2.5. *An $\mathbb{RP}^2$-characteristic sphere $\alpha \in \pi_2(M)$ is S^2-characteristic. Moreover if $\pi_1(M)$ has no elements of order two, then α is S^2-characteristic if and only if it is $\mathbb{RP}^2$-characteristic.*

Proof. A generic immersion $f \colon S^2 \looparrowright M$ determines a map $\mathbb{RP}^2 \to \mathbb{RP}^2/\mathbb{R}P^1 = S^2 \xrightarrow{f} M$, which can be perturbed to a generic map of $\mathbb{RP}^2$ with the same intersection behaviour with α as the original S^2. Thus $\mathbb{RP}^2$-characteristic implies S^2-characteristic.

On the other hand, if no element of $\pi_1(M)$ has order 2, then for every generic immersion R of $\mathbb{RP}^2$, the induced map $\pi_1(\mathbb{RP}^2) \to \pi_1(M)$ is the zero map. Therefore R is homotopic to a map that factors as $\mathbb{RP}^2 \to \mathbb{RP}^2/\mathbb{R}P^1 = S^2 \xrightarrow{f} M$, and intersections with $f(S^2)$ agree with intersections with R. It follows that S^2-characteristic implies $\mathbb{RP}^2$-characteristic. □

Let $S\colon S^2 \looparrowright M$ be a generically immersed 2-sphere with vanishing self-intersection number $\mu(S) = 0$. Then the self-intersection points of S can be paired up so that each pair consists of two points having oppositely signed but equal group elements associated to their double point loops. Therefore, one can choose a Whitney disc W_i for each pair of self-intersections, and arrange that all the boundary arcs are disjoint. The normal bundle to the disc W_i has a unique framing, and the Whitney framing of the normal bundle of W_i restricted to ∂W_i differs from the restriction of the disc framing by an integer $n_i \in \mathbb{Z}$. (The Whitney framing is determined by a section of the normal bundle $\nu_{W_i}|_{\partial W_i}$ that lies in $TS^2 \cap \nu_{W_i}$ along one boundary arc of ∂W_i and lies in $\nu_{S^2} \cap \nu_{W_i}$ along the other boundary arc.)

Definition 2.6. If S is $\mathbb{RP}^2$-characteristic, then

$$\tau(S) := \sum_i |\mathring{W}_i \pitchfork S| + n_i \mod 2.$$

Lemma 2.7. *The expression $\tau(S)$ is independent of the choice of pairings of double points, sheet choices and Whitney arcs, and Whitney discs. Moreover, $\tau(S)$ only depends on the regular homotopy class of the generic immersion.*

Proof. See [ST01, Theorem 1] for a proof that this number is well-defined. A key observation here is due to Stong [Sto94]. We make a couple of remarks on how to translate the version in [ST01] to the current version. First note that in the formulation of [ST01], as mentioned above the intersections were decorated with a pair of fundamental group elements, to give an invariant in a quotient of $\mathbb{Z}[\pi \times \pi]$ by certain relations. Since we consider the augmentation followed by the reduction modulo two, all but the last relation given in [ST01, Theorem 1] are vacuous. In addition their last relation is irrelevant because we consider $\mathbb{RP}^2$-characteristic elements. Secondly, the formulation of Schneidermann-Teichner requires that Whitney discs be framed, whereas we do not, and include the framing coefficient as part of the definition. However by boundary twisting [FQ90, Section 1.3], one can alter n_i to be zero at the cost of introducing $|n_i|$ intersection points in $\mathring{W}_i \pitchfork S$. □

We fix a regular homotopy class within the homotopy class by the requirement that the Euler number be zero. Thus τ becomes well-defined on $\mathcal{RC} \subset \pi_2(M)$. So we have defined a map

$$\tau\colon \mathcal{RC} \to \mathbb{Z}/2.$$

Remark 2.8. If S is not S^2-characteristic then $\tau(S)$ is not well-defined, since adding a sphere that intersects S in an odd number of points to one of the Whitney discs would change the sum in the definition of τ by one.

If S is S^2-characteristic but not $\mathbb{RP}^2$-characteristic, then $\tau(S)$ is also not well-defined, as observed by Stong [Sto94]. In this case, a change in choice of Whitney arcs, in the presence of 2-torsion in $\pi_1(M)$, can also change $\tau(S)$.

2.2. **The τ invariant for π_1-trivial generically immersed surfaces.** In this subsection we introduce the following extension of the τ invariant, which is defined on $\mathbb{RP}^2$-characteristic, π_1-trivial, generically immersed surfaces instead of on $\mathbb{RP}^2$-characteristic generically immersed spheres. We will not need the full version of this invariant, only the embedded version. But we anticipate that the full version might be useful in the future, so we include it here, as it requires little extra work.

We call a generically immersed, closed, oriented surface $F\colon \Sigma \looparrowright M$ a π_1*-trivial surface* if $F_*\colon \pi_1(\Sigma) \to \pi_1(M)$ is the trivial map. As before let M be a smooth, closed, oriented, spin 4-manifold together with an identification $\pi_1(M) \cong \pi$.

Definition 2.9. A π_1-trivial generically immersed surface $F\colon \Sigma \looparrowright M$ is said to be $\mathbb{RP}^2$-*characteristic* if it intersects every generically immersed $\mathbb{RP}^2$ in general position in an even number of points i.e.

if the element of $\pi_2(M)$ determined by F via the Hurewicz isomorphism $H_2(M;\mathbb{Z}\pi) \cong \pi_2(M)$ is $\mathbb{RP}^2$-characteristic.

A π_1-trivial $\mathbb{RP}^2$-characteristic generically immersed surface F has a self-intersection number $\mu(F) \in \mathbb{Z}\pi/\{g \sim g^{-1} \mid g \in \pi\}$ defined as follows. Add local kinks to F until its normal bundle is trivial; this is possible since F is S^2-characteristic by Lemma 2.5. Now count self-intersection points of the generically immersed surface with group elements and sign. We use π_1-triviality to see that the associated group elements do not depend on the choice of double point loop on F used to compute it.

Let $F\colon \Sigma \looparrowright M$ be a generically immersed π_1-trivial surface with $\mu(F) = 0$, and let α be an embedded circle in F. The circle α bounds a disc C in M, since F is π_1-trivial. The normal direction of α in F gives a section of the normal bundle of C at the boundary α. Therefore, the relative Euler number $e(C)$ of the normal bundle of C is a well-defined integer. We define

$$\varpi(\alpha) := \#(\mathring{C} \pitchfork F) + e(C) \mod 2,$$

where $\#(C \pitchfork F)$ is the number of transverse intersections between the interiors of C and F.

Lemma 2.10. *If F is S^2-characteristic, then the count $\varpi(\alpha)$ does not depend on the choice of C.*

Proof. Let C and C' be two choices of discs with boundary α, and let $\varpi_C(\alpha)$ and $\varpi_{C'}(\alpha)$ temporarily denote the count made using C and C' respectively. Perform boundary twists [FQ90, Section 1.3] in order to arrange that C and C' are framed with respect to their boundaries, i.e. $e(C) = e(C') = 0$. Boundary twists do not change the counts $\varpi_C(\alpha)$ and $\varpi_{C'}(\alpha)$, since a boundary twist changes the relative Euler number of the disc by one and produces a single new intersection between the disc being twisted and F. Now rotate C' near α so that the union of C and C' is a generically immersed 2-sphere. Since F is S^2-characteristic, we have

$$0 = \mathrm{red}_2\lambda(F, C \cup_\alpha C') = \varpi_C(\alpha) + \varpi_{C'}(\alpha) \in \mathbb{Z}/2$$

as desired. □

Consider a hyperbolic basis of $H_1(F;\mathbb{Z})$ represented by embedded circles $a_1, \dots, a_n, b_1, \dots, b_n$ that are disjoint from each other except that a_i intersects b_i transversely in a single point.

Since $\mu(F) = 0$, all double points of F can be paired up by generically immersed Whitney discs $W_1, \dots, W_m \hookrightarrow M$ whose boundary arcs on F are disjoint from each other, the a_i, and the b_i. Let n_j again denote the framing coefficient of the Whitney discs discussed in Section 2.1. Then define:

$$\tau(F) := \sum_{i=1}^{n} \varpi(a_i)\varpi(b_i) + \sum_{j=1}^{m} |\mathring{W}_j \pitchfork F| + n_j \mod 2.$$

Remark 2.11. Note that in the case that F has genus zero, this reduces to the τ invariant of the previous subsection since the first sum vanishes. Also note that in the case of an embedded surface, only the first summand appears, and again the definition simplifies. The restriction to the case that F is an embedding is similar to the version of τ from [FK78]. In this case it is the Arf invariant of the quadratic form given by φ.

Next we will show that $\tau(F)$ is independent of the choice of basis $\{a_i, b_i\}$, as well as the choice of Whitney discs W_j.

Lemma 2.12. *If F is $\mathbb{RP}^2$-characteristic, then the expression $\tau(F) \in \mathbb{Z}/2$ is independent of the choices of a_i, b_i, C and W_j made in its definition. Moreover, $\tau(F)$ only depends on the regular homotopy class of the generic immersion.*

Proof. We already showed in Lemma 2.10 that $\tau(F)$ is independent of the choices of the discs C.

Choose a path from each component of F to the base point of M. Since these paths are 1-dimensional we can choose them so that the interiors of the paths do not intersect F. Since F is π_1-trivial, it lifts to a generically immersed surface in $\widetilde{M}$, and hence defines an element of $H_2(\widetilde{M};\mathbb{Z}) \cong \pi_2(M)$. The strategy is to relate $\tau(F)$ to $\tau(S)$ for $S \in \pi_2(M)$, and use that $\tau(S)$ is well-defined by [ST01].

Choose generic null-homotopies $C_i\colon D^2 \to M$ for a_i and $C_i'\colon D^2 \to M$ for b_i. As in the proof of Lemma 2.10, perform boundary twists to arrange $e(C_i) = e(C_i') = 0$. Again, this does not change $\varpi(\alpha)$. We can turn F into a generically immersed 2-sphere S by performing surgeries along all the a_i, and gluing in two parallel copies of each of the C_i in place of a neighbourhood νa_i of a_i.

(i) Each intersection $C_i \pitchfork F$ yields a pair of cancelling self-intersections of S paired by a Whitney disc constructed from (a parallel copy of) C_i' union a band. A schematic is shown in Figure 1.
(ii) Each self-intersection of C_i yields two pairs of cancelling self-intersections of S, each with generically immersed Whitney disc a parallel copy of C_i', union a band. A schematic is shown in Figure 2.

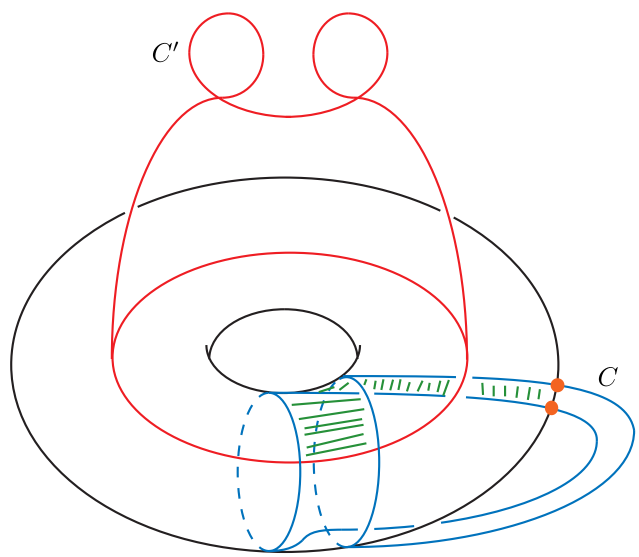

FIGURE 1. A schematic of a genus one surface F with a cap C' attached to the longitude, two parallel copies of a cap C attached to the meridian, each of which intersect F in a single point. A band is shown that, together with the cap C', forms a Whitney disc pairing the two self-intersection points of the sphere obtained from surgery on F using C.

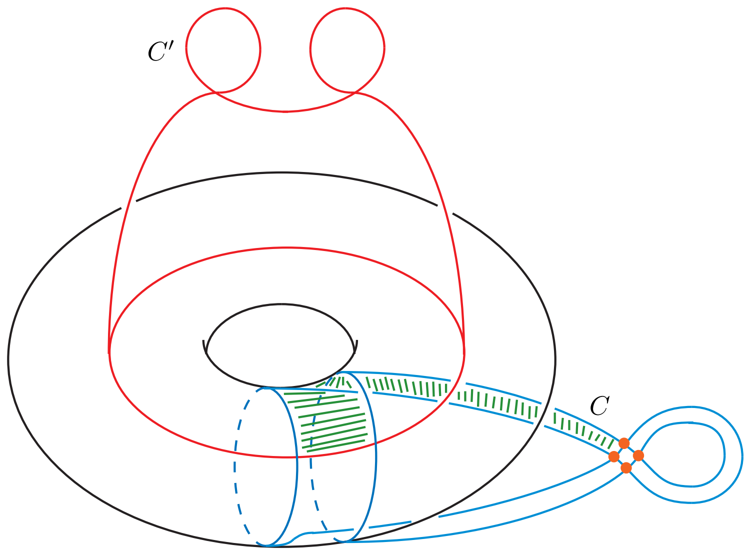

FIGURE 2. A schematic of a genus one surface F with a cap C' attached to the longitude and two parallel copies of a cap C attached to the meridian. The cap C has a single self-intersection points, which gives rise to four self-intersection points of the sphere resulting from surgery on F using C. For one pair of these four points, a band is shown, that together with the cap C', forms a Whitney disc pairing these two self-intersection points.

The boundary arcs of the new Whitney discs are disjoint from the boundary arcs of the old Whitney discs. Thus modulo two we see that

$$\begin{aligned}\tau(S) &= \sum_{i=1}^{n}\Big(\#(\mathring{C}_i \pitchfork F)\#(\mathring{C}'_i \pitchfork F) + 2\#(\mathring{C}_i \pitchfork \mathring{C}_i)\#(\mathring{C}'_i \pitchfork F)\Big) + \sum_{j=1}^{m}|\mathring{W}_j \pitchfork F| + n_j \\ &= \sum_{i=1}^{n}\varpi(a_i)\varpi(b_i) + \sum_{j=1}^{m}|\mathring{W}_j \pitchfork F| + n_j = \tau(F).\end{aligned}$$

The first summand of the first summand corresponds to (i) and the second summand corresponds to (ii).

Note that S and F determine the same element of $\pi_2(M)$, with the right choice of basing paths, since S and F determine the same element of $H_2(\widetilde{M};\mathbb{Z})$, which in turn holds because the difference $[S]-[F]$ bounds the trace of the surgery along the a_i. It follows that S is $\mathbb{RP}^2$-characteristic. We know that the number $\tau(S)$ only depends on the homotopy class of S by [ST01]. But the homotopy class of S is determined by the regular homotopy class of the generic immersion F and does not depend on the choices of a_i, b_i, C_i, C'_i, W_j. Hence the fact that $\tau(S)$ is well-defined implies that $\tau(F)$ is too. $\square$

3. The Arf invariant in the stable classification

We will prove Theorem 1.5 by comparing the Kervaire–Milnor invariant to a codimension two Arf invariant that arises in the Atiyah-Hirzebruch spectral sequence for $\Omega_4^{\mathrm{Spin}}(B\pi)$. Let us explain how this Arf invariant appears.

Let M be a closed, smooth, oriented, spin 4-manifold, where the spin structure will be fixed from now on. Let K be a 2-complex with fundamental group π and let $i\colon K \to B\pi$ be the classifying map. Let $f\colon M \to K$ be a map that is an isomorphism on fundamental groups.

Denote the barycentres of the 2-cells $\{e_i^2\}_{i\in I}$ of K by $\{b_i^2\}_{i\in I}$. Denote the regular preimage of the barycentre $b_i^2 \in K$ by $N_i^f \subseteq M$. We can consider $[N_i^f] \in \Omega_2^{\mathrm{Spin}}$ since the normal bundle of N_i^f in M is trivialised as a pullback of the normal bundle of b_i^2 in e_i^2, and hence N_i^f inherits a spin structure from M. The next lemma is well-known; see [KLPT17, Lemma 2.5] for a proof.

Lemma 3.1. *The homomorphism $\Omega_4^{\mathrm{Spin}}(K) \to H_2(K;\Omega_2^{\mathrm{Spin}})$ from the Atiyah–Hirzebruch spectral sequence coincides with the map*

$$\begin{array}{rcl} \Omega_4^{\mathrm{Spin}}(K) & \to & H_2(K;\Omega_2^{\mathrm{Spin}}) \\ {[f\colon M \to K]} & \mapsto & \Big[\sum\limits_{i\in I}[N_i^f]\cdot e_i^2\Big]. \end{array}$$

Moreover this maps to $\mathfrak{ter}(M)$ under

$$H_2(K;\Omega_2^{\mathrm{Spin}}) \cong H_2(K;\mathbb{Z}/2) \to H_2(\pi;\mathbb{Z}/2) \to H_2(\pi;\mathbb{Z}/2)/\operatorname{im}(d_2,d_3).$$

Remark 3.2. The homomorphism in the statement of the lemma $\Omega_4^{\mathrm{Spin}}(K) \to H_2(K;\Omega_2^{\mathrm{Spin}})$ arises as follows. The abutment of the Atiyah-Hirzebruch spectral sequence $\Omega_4^{\mathrm{Spin}}(K) = F_{4,0}$ maps to its quotient by the first filtration step $F_{2,2}$ that differs from $F_{n,0}$. This term is indeed $F_{2,2}$, since the homology of K vanishes in degrees greater than 2, thus $E_{s,t}^2 = E_{s,t}^\infty = 0$ for all $s > 2$. Moreover no differentials have image in $E_{2,2}^2$, so $E_{2,2}^\infty \subseteq E_{2,2}^2$, and the composition

$$\Omega_4^{\mathrm{Spin}}(K) = F_{4,0} \to F_{4,0}/F_{2,2} \xrightarrow{\cong} E_{2,2}^\infty \to E_{2,2}^2 = H_2(K;\Omega_2^{\mathrm{Spin}}).$$

gives the desired map.

We will need a slight variation of Lemma 3.1. For $\varphi \in H^2(K;\mathbb{Z}/2)$, represent φ by a map $K \to S^2 \subseteq K(\mathbb{Z}/2,2)$ and let $F_\varphi^f \subseteq M$ be a regular preimage of a point $s \in S^2$ under $\varphi\circ f\colon M \to S^2$. As before, a framing of the normal bundle of s in S^2 induces a framing of the normal bundle of F_φ^f in M, and since M is spin, we obtain a spin structure on F_φ^f. Thus we can again consider $[F_\varphi^f]$ in Ω_2^{Spin}.

Lemma 3.3. *The composition*

$$\Omega_4^{\mathrm{Spin}}(K) \to H_2(K;\mathbb{Z}/2) \xrightarrow{x\mapsto\langle -,x\rangle} \operatorname{Hom}_{\mathbb{Z}/2}(H^2(K;\mathbb{Z}/2),\mathbb{Z}/2)$$

maps $[f\colon M \to K]$ to $\big(\varphi \mapsto [F_\varphi^f] \in \Omega_2^{\mathrm{Spin}} \cong \mathbb{Z}/2\big)$.

Proof. By a homotopy of f we can assume that that the b_i^2 are regular points. The first map sends

$$\begin{gathered} \Omega_4^{\mathrm{Spin}}(K) \to H_2(K;\mathbb{Z}/2) \\ {[f\colon M \to K]} \mapsto \sum_{i\in I}[N_i^f]\cdot e_i^2 \end{gathered}$$

with $N_i^f := f^{-1}(b_i^2)$ as in Lemma 3.1. Let $p \in S^2$ be a basepoint and let $s \in S^2$ be antipodal to p. As above, given $\varphi \in H^2(K;\mathbb{Z}/2)$, we represent φ by a map $K \to S^2 \subseteq K(\mathbb{Z}/2,2)$. We can choose the representative $\varphi\colon K \to S^2$ so that for each 2-cell e_i^2 of K, either: (i) $\varphi|_{\overline{e}_i^2}$ sends the whole closed 2-cell to p, or (ii) $\varphi|_{\overline{e}_i^2}$ factors as the quotient map followed by a homeomorphism $\overline{e}_i^2 \to \overline{e}_i^2/\partial\overline{e}_i^2 \cong S^2$, with $\varphi(\partial\overline{e}_i^2) = p$, $\varphi(b_i^2) = s$, and such that $b_i^2 \in e_i^2$ is a regular preimage of $s \in S^2$. Let $E(\varphi) \subseteq I$ be the subset of indices corresponding to the cells for which the latter option (ii) holds. Then $\sqcup_{i\in E(\varphi)} b_i^2$ is a regular preimage of $s \in S^2$ under φ.

Let $F_\varphi^f := (\varphi\circ f)^{-1}(\{s\})$, as above. As usual, $[F_\varphi^f]$ does not depend on the choice of a representative for φ since different choices give spin bordant surfaces. Hence $F_\varphi^f = \sqcup_{i\in E(\varphi)} N_i^f$. Then

$$\big(\varphi \mapsto [F_\varphi^f] = \sum_{i\in E(\varphi)} [N_i^f] \in \Omega_2^{\mathrm{Spin}} \cong \mathbb{Z}/2\big) \in \mathrm{Hom}_{\mathbb{Z}/2}(H^2(K;\mathbb{Z}/2), \mathbb{Z}/2)$$

is the image of $\sum_{i\in I}[N_i^f] \cdot e_i^2$ under the evaluation map $H_2(K;\mathbb{Z}/2) \to \mathrm{Hom}_{\mathbb{Z}/2}(H^2(K;\mathbb{Z}/2), \mathbb{Z}/2)$, as needed. □

For our comparison of the Kervaire–Milnor invariant with the codimension 2 Arf invariant, we need to recall the definition of the Arf invariant $\mathrm{Arf}\colon \Omega_2^{\mathrm{Spin}} \xrightarrow{\cong} \mathbb{Z}/2$. Let F be spin surface. One defines a quadratic refinement of the $\mathbb{Z}/2$ intersection form of F, $\Upsilon\colon H_1(F;\mathbb{Z}/2) \to \mathbb{Z}/2$ as follows. Represent $[\alpha] \in H_1(F;\mathbb{Z}/2)$ by a simple closed curve α in F. Since the normal bundle ν_α^F of α in F is one dimensional, the normal bundle ν_α^F has a canonical framing, where the choice of the direction comes from the orientations. Therefore, together with the spin structure on F, this determines a spin structure on α, so we may consider it as an element of Ω_1^{Spin}. We define $\Upsilon([\alpha]) = 0$ if and only if α is spin null-bordant. Then $\mathrm{Arf}(F)$ is defined to be the Arf invariant of the quadratic form $(H_1(F;\mathbb{Z}/2), \lambda_F, \Upsilon)$.

4. Proof of Theorems 1.1 and 1.5

First let us recall the setup of Theorem 1.5. Let M be a closed, smooth, oriented, spin 4-manifold with fundamental group π, and suppose that there is a map $f\colon M \to K$ to a finite 2-complex K that is an isomorphism on fundamental groups. Let $i\colon K \to B\pi$ be a classifying map.

We start by proving Theorem 1.5 (i); it follows immediately from the next lemma. Let $\mathrm{red}_2\colon \mathbb{Z}\pi \to \mathbb{Z}/2$ be the $\mathbb{Z}\pi$-module homomorphism given by augmentation followed by reduction modulo 2.

Lemma 4.1. *For any 2-complex K with fundamental group π, the map $H^2(K;\mathbb{Z}\pi) \to H^2(K;\mathbb{Z}/2)$ induced by red_2 is surjective.*

Proof. This follows from Bockstein sequence associated with $0 \to \ker(\mathrm{red}_2) \to \mathbb{Z}\pi \xrightarrow{\mathrm{red}_2} \mathbb{Z}/2 \to 0$, using that $H^3(K;\ker(\mathrm{red}_2)) = 0$ since K is 2-dimensional. □

We move on to the proof of Theorem 1.5 (ii). It states: *The element $PD(f^*(\widehat{\varphi})) \in H_2(M;\mathbb{Z}\pi) \cong \pi_2(M)$ is $\mathbb{RP}^2$-characteristic and has trivial self-intersection number, so that the Kervaire-Milnor invariant $\tau(PD\circ f^*(\widehat{\varphi})) \in \mathbb{Z}/2$ is well-defined.*

Let $M^{(3)}$ be the 3-skeleton of M for some chosen handle decomposition. The following lemma is more general than needed in the paper, since it starts with $g\colon M^{(3)} \to K$ only defined on the 3-skeleton, but this generalization will be useful in [KPT20].

Lemma 4.2. *Let $g\colon M^{(3)} \to K$ be a map that is an isomorphism on fundamental groups. Let $j\colon M^{(3)} \to M$ be the inclusion of the 3-skeleton. For every $\varphi \in \ker \mathrm{Sq}^2 \subseteq H^2(B\pi;\mathbb{Z}/2)$ and every lift $\widehat{\varphi} \in H^2(K;\mathbb{Z}\pi)$ of $i^*\varphi \in H^2(K;\mathbb{Z}/2)$, the element*

$$(PD\circ(j^*)^{-1}\circ g^*)(\widehat{\varphi}) \in H_2(M;\mathbb{Z}\pi) \cong \pi_2(M)$$

is $\mathbb{RP}^2$*-characteristic.*

In the lemma above we used the following sequence of maps:

$$\widehat{\varphi} \in H^2(K;\mathbb{Z}\pi) \xrightarrow{g^*} H^2(M^{(3)};\mathbb{Z}\pi) \xrightarrow{(j^*)^{-1}} H^2(M;\mathbb{Z}\pi) \xrightarrow{PD} H_2(M;\mathbb{Z}\pi) \cong \pi_2(M).$$

Proof. Fix a map $\beta\colon \mathbb{RP}^2 \to M$. In addition, let $\mathrm{red}_2\colon \mathbb{Z}\pi \to \mathbb{Z} \to \mathbb{Z}/2$ be the ring homomorphism given by the composition of the augmentation and reduction modulo two. Let $c\colon M \to B\pi$ be a classifying map such that $i_* \circ g_* = c_* \circ j_*\colon \pi_1(M^{(3)}) \to \pi_1(B\pi) = \pi$, from which it follows that $i \circ g \sim c \circ j$. The following equations prove that $PD((j^*)^{-1}g^*\widehat{\varphi}) \in \pi_2(M)$ is $\mathbb{RP}^2$-characteristic. We will give justification for each of the equalities afterwards.

$$\begin{array}{cccc} & \lambda_2(\mathrm{red}_2(PD((j^*)^{-1}g^*(\widehat{\varphi}))), \beta_*[\mathbb{RP}^2]) & = & \lambda_2(\beta_*[\mathbb{RP}^2], \mathrm{red}_2(PD((j^*)^{-1}g^*(\widehat{\varphi})))) \\ = & \langle \mathrm{red}_2((j^*)^{-1}g^*(\widehat{\varphi})), \beta_*[\mathbb{RP}^2]\rangle & = & \langle (j^*)^{-1}g^*\mathrm{red}_2(\widehat{\varphi}), \beta_*[\mathbb{RP}^2]\rangle \\ = & \langle (j^*)^{-1}g^*i^*(\varphi), \beta_*[\mathbb{RP}^2]\rangle & = & \langle (j^*)^{-1}j^*c^*(\varphi), \beta_*[\mathbb{RP}^2]\rangle \\ = & \langle c^*(\varphi), \beta_*[\mathbb{RP}^2]\rangle & = & \langle \beta^*c^*(\varphi), [\mathbb{RP}^2]\rangle. \end{array}$$

The first equation uses symmetry of λ_2. The second equation is the algebraic definition of the intersection form. The third equation uses the naturality of the reduction mod 2. The fourth equation uses that, by definition of $\widehat{\varphi}$, $\mathrm{red}_2(\widehat{\varphi}) = i^*\varphi$. The fifth equation uses that $i \circ g \sim c \circ j$. The last equality uses the naturality of the evaluation.

Using obstruction theory and the fact that $B\pi$ is aspherical, the map $c \circ \beta\colon \mathbb{RP}^2 \to B\pi$ extends to a map $\beta'\colon \mathbb{RP}^\infty \to B\pi$. Now assume for a contradiction that $\langle \beta^*c^*(\varphi), [\mathbb{RP}^2]\rangle$ is nontrivial. Then $\beta^*c^*(\varphi) \in H^2(\mathbb{RP}^2;\mathbb{Z}/2)$ is nontrivial and hence also $(\beta')^*(\varphi) \in H^2(\mathbb{RP}^\infty;\mathbb{Z}/2)$ is nontrivial. In this case, also $\mathrm{Sq}^2((\beta')^*(\varphi)) = (\beta')^*\mathrm{Sq}^2(\varphi) \in H^4(\mathbb{RP}^\infty;\mathbb{Z}/2)$ has to be nontrivial. But we assumed that φ lies in the kernel of Sq^2. Hence $\langle \beta^*c^*(\varphi), [\mathbb{RP}^2]\rangle$ has to be trivial, as desired. □

As explained in Section 2.1, the Kervaire–Milnor invariant is well-defined on $\mathbb{RP}^2$-characteristic spheres with vanishing self-intersection number, so Theorem 1.5 (ii) follows from the next two lemmas.

Lemma 4.3. *The element* $PD(f^*(\widehat{\varphi})) \in H_2(M;\mathbb{Z}\pi) = \pi_2(M)$ *is* $\mathbb{RP}^2$*-characteristic.*

Proof. This follows directly from Lemma 4.2 applied to the composition $f \circ j\colon M^{(3)} \to K$, where as in that lemma $j\colon M^{(3)} \to M$ is the inclusion of the 3-skeleton. □

Lemma 4.4. *For every* $x, y \in H^2(K;\mathbb{Z}\pi)$*, we have that* $\lambda(PD(f^*(x)), PD(f^*(y))) = 0$*. In particular, the self-intersection number* $\mu(PD(f^*(x))) = 0$*.*

Proof. Since K is 2-dimensional we have $f_*([M]) = 0$, and thus

$$\begin{aligned} \lambda(PD(f^*(x)), PD(f^*(y))) &= \langle f^*(y), PD(f^*(x))\rangle = \langle y, f_*(f^*(x) \cap [M])\rangle \\ &= \langle y, x \cap f_*([M])\rangle = \langle y, 0\rangle = 0. \end{aligned}$$

The last sentence follows from Lemma 2.3. □

This completes the proof of Theorem 1.5 (ii). The methods used in its proof also allow us to prove Theorem 1.1. Recall that we have to show: *(i) For every element* $x \in \ker \mathrm{Sq}$*,* $PD \circ f^*(x) \in \pi_2(M)$ *has trivial self-intersection number and is* $\mathbb{RP}^2$*-characteristic. (ii) The induced map* $\tau_{M,f}\colon \ker \mathrm{Sq} \to \mathbb{Z}/2$ *factors through* $\mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} \ker \mathrm{Sq}$*. (iii)* $\tau_{M,f}$*, up to the action of* $\mathrm{Aut}(\pi)$ *on the choice of* f*, is a stable diffeomorphism invariant.*

Proof of Theorem 1.1. By Lemma 4.4 every element in the radical $\operatorname{Rad}(\lambda_M)$ has trivial self-intersection number. We will now show that every element in $PD \circ f^*(\ker \operatorname{Sq})$ is also $\mathbb{RP}^2$-characteristic. Let $x \in \ker \operatorname{Sq} \subseteq H^2(\pi;\mathbb{Z}\pi)$ and fix a map $\beta\colon \mathbb{RP}^2 \to M$. As in the proof of Lemma 4.2,

$$\lambda_2(\operatorname{red}_2(PD(f^*(x))), \beta_*[\mathbb{RP}^2]) = \langle \operatorname{red}_2(f^*(x)), \beta_*[\mathbb{RP}^2]\rangle = \langle \operatorname{red}_2(\beta^* f^*(x)), [\mathbb{RP}^2]\rangle.$$

Let $\beta'\colon \mathbb{RP}^\infty \to B\pi$ be an extension of $f \circ \beta$. We continue to follow the pattern of the proof of Lemma 4.2. Assume for a contradiction that $\langle \operatorname{red}_2(\beta^* f^*(x)), [\mathbb{RP}^2]\rangle$ is nontrivial. Then $\operatorname{red}_2(\beta^* f^*(x)) \neq 0 \in H^2(\mathbb{RP}^2;\mathbb{Z}/2)$, so $\operatorname{red}_2 \circ (\beta')^*(x)$ is nontrivial in $H^2(\mathbb{RP}^\infty;\mathbb{Z}/2)$. Therefore

$$(\beta')^*(\operatorname{Sq}(x)) = (\beta')^*(\operatorname{Sq}^2 \circ \operatorname{red}_2(x)) = \operatorname{Sq}^2 \circ \operatorname{red}_2((\beta')^*(x)) \in H^4(\mathbb{RP}^\infty;\mathbb{Z}/2)$$

is also nontrivial. This contradicts that $x \in \ker(\operatorname{Sq})$. We deduce that $\langle \operatorname{red}_2(\beta^* f^* x), [\mathbb{RP}^2]\rangle$ vanishes, so that $PD \circ f^*(x)$ is $\mathbb{RP}^2$-characteristic as desired, proving (i).

Since stabilisation does not change the value of τ, it follows that $\tau_{M,f}$ is a stable diffeomorphism invariant, up to the choice of f, as asserted in (iii).

It remains to show (ii), that $\tau_{M,f}$ factors through $\mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} \ker \operatorname{Sq}$. Assume $[x] = [y] \in \mathbb{Z}/2 \otimes_{\mathbb{Z}\pi} \ker \operatorname{Sq}$. Then there are $z_i \in \ker \operatorname{Sq}$ and $\kappa_i \in \ker(\operatorname{red}_2\colon \mathbb{Z}\pi \to \mathbb{Z}/2)$ such that $x = y + \sum_{i=1}^n \kappa_i z_i$. As $\mu(y) = \mu(z_i) = \lambda(y, z_i) = 0$, it follows from [KLPT17, Lemma 8.3] that $\tau(y) = \tau(y + \sum_{i=1}^n \kappa_i z_i) = \tau(x)$. □

Now we continue with the proof of Theorem 1.5. We have proven Theorem 1.5 (ii), and so we may define $\tau(PD \circ f^*(\widehat{\varphi})) \in \mathbb{Z}/2$. We will prove Theorem 1.5 (iii) and (iv) by comparing the Kervaire–Milnor invariant to the Arf invariant.

Recall that we have to show the following.

(iii) The map $\tau_{M,f}\colon \ker \operatorname{Sq}^2 \to \mathbb{Z}/2;\ \varphi \mapsto \tau(PD \circ f^*(\widehat{\varphi}))$ *is a well-defined homomorphism.*

(iv) Under the map $\operatorname{Hom}(\ker \operatorname{Sq}^2, \mathbb{Z}/2) \xrightarrow{\cong} H_2(\pi;\mathbb{Z}/2)/\operatorname{im} \operatorname{Sq}_2 \to H_2(\pi;\mathbb{Z}/2)/\operatorname{im}(d_2, d_3)$, $\tau_{M,f}$ *is sent to* $\mathfrak{ter}(M)$.

Remark 4.5. Let us explain the strategy of the rest of the proof. It follows from Lemmas 3.1 and 3.3 that computing the Arf invariant $\operatorname{Arf}(f^* i^*(\varphi))$ gives rise to a well-defined homomorphism in $\operatorname{Hom}(\ker \operatorname{Sq}^2, \mathbb{Z}/2)$ that determines $\mathfrak{ter}(M)$. Thus, to show (iii) and (iv), we shall prove that the Arf invariant $\operatorname{Arf}(f^* i^*(\varphi))$ coincides with the Kervaire–Milnor invariant $\tau(PD(f^*(\widehat{\varphi})))$, where as before $\widehat{\varphi} \in H^2(K;\mathbb{Z}\pi)$ is a lift of $i^*\varphi \in H^2(K;\mathbb{Z}/2)$. For this we will use the description of the Kervaire–Milnor invariant for π_1-trivial (embedded) surfaces from Section 2.2.

For a π_1-trivial, closed, oriented, generically immersed surface $F\colon \Sigma \looparrowright M$, the definition of $\tau(F)$ uses a quadratic refinement $\varpi\colon H_1(F;\mathbb{Z}/2) \to \mathbb{Z}/2$ of the $\mathbb{Z}/2$-intersection form of F that uses, for each curve on F, a relative Euler number and a count of intersections.

For the specific F that arise in our situation, which will be embedded, we will show that after picking a correctly framed disc, the relative Euler number in the definition of ϖ agrees with Υ. (Recall that Υ is the quadratic refinement we use for computing the Arf invariant.) Then we will show that for such a disc the intersection component of the quadratic refinement ϖ is always even, so does not contribute. It will follow that ϖ and Υ coincide, and therefore the Arf invariant and the Kervaire–Milnor invariant of F agree.

Next, we want to produce, for each $\widehat{\varphi} \in H^2(K;\mathbb{Z}\pi)$, a suitable surface F on which to compare the two invariants. Since $\widetilde{K}$ is 2-dimensional, an element $\widehat{\varphi} \in H^2(K;\mathbb{Z}\pi) \cong H^2_{cs}(\widetilde{K};\mathbb{Z})$ can be represented as a map $\widehat{\varphi}\colon \widetilde{K} \to S^2$ with compact support (i.e. the closure of the inverse image of $S^2 \setminus \{*\}$ is compact). This follows from the fact that $\widetilde{K}$ is 2-dimensional, and the definition of cohomology with compact support as a colimit of $H^2(\widetilde{K}, \widetilde{K} \setminus L)$ over compact subsets $L \subseteq \widetilde{K}$.

Let $\widetilde{f}\colon \widetilde{M} \to \widetilde{K}$ be a lift of the map $f\colon M \to K$. Let $x \in S^2$ be a regular value of $\widehat{\varphi} \circ \widetilde{f}\colon \widetilde{M} \to S^2$. Then, as we will prove in the next lemma,

$$F := (\widehat{\varphi} \circ \widetilde{f})^{-1}(x) \subseteq \widetilde{M}$$

represents

$$PD(\widetilde{f}^*(\widehat{\varphi})) \in H_2(\widetilde{M}; \mathbb{Z}).$$

Furthermore, F is an embedded surface, and since $\pi_1(\widetilde{M}) = 0$, F is π_1-trivial.

We can perturb the map $\widehat{\varphi}\colon \widetilde{K} \to S^2$ so that $(\widehat{\varphi})^{-1}(x) \subseteq \widetilde{K}$ is a finite (coming from compact support) discrete set, which satisfies that no two points of $(\widehat{\varphi})^{-1}(x)$ have the same image under $\widetilde{K} \to K$. Then the image of F under $\widetilde{M} \to M$ is still an embedded π_1-trivial surface, which we again denote by $F \in H_2(M; \mathbb{Z}\pi)$. By Lemma 4.3, F is $\mathbb{RP}^2$-characteristic. By Lemma 4.4, $\mu(F) = 0$.

Lemma 4.6. *The inverse image of $x \in S^2$ is a representative for $PD(\widetilde{f}^*(\widehat{\varphi})) \in H_2(\widetilde{M}; \mathbb{Z})$.*

Proof. For Y a compact 4-manifold, a cohomology class y in $H^2(Y, \partial Y; \mathbb{Z})$ is represented by a map $f_y\colon (Y, \partial Y) \to (\mathbb{C}P^2, *)$, and the inverse image of $\mathbb{C}P^1 \subset \mathbb{C}P^2$ is the Poincaré dual to the original class y.

We may take the compact sets L in the colimit defining cohomology with compact support to be codimension zero manifolds with boundary in $\widetilde{M}$. Then for Poincaré duality, we have

$$\begin{aligned} H^2_{cs}(\widetilde{M}) &= \operatorname*{colim}_L H^2(\widetilde{M}, \widetilde{M} \setminus L) \cong \operatorname*{colim}_L H^2(L, \partial L) \\ &\cong \operatorname*{colim}_L H_2(L) \cong H_2(\widetilde{M}). \end{aligned}$$

where L belongs to the collection of compact subsets of $\widetilde{M}$ ordered by inclusions. Note that π acts on the compact subsets via the deck transformations. Tracing these isomorphisms, it follows from the corresponding fact for compact manifolds that the inverse image of $\mathbb{C}P^1 \subseteq \mathbb{C}P^2$ of the map $\widehat{\varphi} \circ \widetilde{f}\colon \widetilde{M} \to \mathbb{C}P^2$ is the Poincaré dual to $f^*(\widehat{\varphi})$. Then observe that, since the map $\widehat{\varphi} \circ \widetilde{f}\colon \widetilde{M} \to \mathbb{C}P^2$ factors through S^2, the inverse image of a generic $\mathbb{C}P^1$ is the inverse image of a point in S^2. This completes the proof that the inverse image of $x \in S^2$ in $\widetilde{M}$ is a representative for $PD(\widetilde{f}^*(\widehat{\varphi}))$. □

Definition 4.7. Let $(v_1, v_2) \in T_xS^2 \oplus T_xS^2$ be a framing of the point x. For each simple closed curve α in F we pick a generically immersed disc C_α in M with boundary α such that the image of the normal vector of $S^1 \subseteq D^2$ in $T_{C_\alpha(y)}M \cong T_{C_\alpha(y)}F \oplus \nu_F^M|_{C_\alpha(y)}$ agrees with $(0, (\widehat{\varphi} \circ f)^* v_1)$ for every $y \in S^1$. We can construct such a disc C_α by taking an annulus $S^1 \times I \subseteq M$ with $S^1 \times \{0\} = \alpha$ and such that a nonvanishing section of $\nu_{S^1 \times \{0\}}^{S^1 \times I}$ agreed with $(0, (\widehat{\varphi} \circ f)^* v_1)$. Then cap off $S^1 \times \{1\}$ with the trace of a null-homotopy in M. We say that C_α is an *f-cap*.

Lemma 4.8. *Let α be a simple closed curve on F and let C_α be an f-cap. The spin bordism class of α, as an element of $\Omega_1^{\mathrm{Spin}} \cong \mathbb{Z}/2$, is equal to the relative Euler number $e(C_\alpha)$ of C_α.*

Proof. The bundle ν_α^F is 1-dimensional, and thus we obtain a canonical framing w from the orientation. The framing $(w, (\widehat{\varphi} \circ f)^* v_1, (\widehat{\varphi} \circ f)^* v_2)$ of ν_α^M together with the spin structure on $\nu_M^{\mathbb{R}^\infty}|_\alpha$ determines the spin structure of α.

As $\nu_M^{\mathbb{R}^\infty}|_\alpha$ extends over C_α and $\nu_\alpha^{C_\alpha}$ agrees with $(\widehat{\varphi} \circ f)^* v_1$, α is spin null bordant if and only if the framing $(w, (\widehat{\varphi} \circ f)^* v_2)$ stably extends over C_α. Since $\nu_{C_\alpha}^M$ is 2-dimensional, the normal vector w extends over C_α if and only if $(w, (\widehat{\varphi} \circ f)^* v_2)$ extends over C_α. Thus $(w, (\widehat{\varphi} \circ f)^* v_2)$ can stably be extended over C_α if and only if the relative Euler number $e(C_\alpha)$ is even, so is zero modulo 2. This completes the proof of the lemma. □

We have one final lemma for the proof of Theorem 1.5.

Lemma 4.9. *Let α be a simple closed curve on F and let C_α be an f-cap. The interior of the image of C_α intersects F transversely in an even number of points.*

Proof. The image of the boundary S^1 under $f \circ C_\alpha \colon D^2 \to K$ is a point. Thus $f \circ C_\alpha$ factors as $f \circ C_\alpha \colon D^2 \to S^2 \xrightarrow{j} K$, where j is defined by this factorisation.

Recall that we have a map $\widehat{\varphi} \colon \widetilde{K} \to S^2$ representing $\widehat{\varphi} \in H^2_{cs}(\widetilde{K};\mathbb{Z}) \cong H^2(K;\mathbb{Z}\pi)$ that lifts $i^*\varphi \in H^2(K;\mathbb{Z}/2)$, where $i\colon K \to B\pi$ is the inclusion of the 2-skeleton. Let $p\colon \widetilde{K} \to K$ be the projection and define a map $\psi\colon K \to S^2$ that sends the points $p((\widehat{\varphi})^{-1}(x))$ to x and sends everything outside a small neighbourhood of these points to the base point of S^2. Then $\psi\colon K \to S^2$ composed with the inclusion $\varrho\colon S^2 \to K(\mathbb{Z}/2,2)$ represents $i^*\varphi \in H^2(K;\mathbb{Z}/2)$.

Since the normal bundle of $S^1 \subseteq D^2$ under $dC_\alpha \colon TD^2 \to TM$ agrees with the direction of $(\widehat{\varphi}\circ f)^* v_1$, and F is the preimage of the points $p((\widehat{\varphi})^{-1}(x))$, the mapping degree of $\psi \circ j\colon S^2 \to S^2$ agrees with the number of transverse intersections of C_α with F.

Compose $\psi \circ j\colon S^2 \to S^2$ with the inclusion of the 2-skeleton $\varrho\colon S^2 \to K(\mathbb{Z}/2,2)$. The map $\varrho\circ\psi\circ j\colon S^2 \to K(\mathbb{Z}/2,2)$ factors through $B\pi$ by definition of $\psi = i^*\varphi$, and therefore is null homotopic, since $B\pi$ is aspherical. It follows that the mapping degree of $\psi\circ j$ is even, which proves the lemma. □

Proof of Theorem 1.5. As already mentioned, (ii) follows directly from Lemma 4.3 and Lemma 4.4. Since, by Lemmas 3.1 and 3.3, $\mathfrak{ter}(M)$ can be computed using the codimension two Arf invariant and the latter determines a homomorphism, to prove (iii) and (iv) it suffices to show $\mathrm{Arf}(F) = \tau(F)$ for the surface F defined above Lemma 4.6. The Arf invariant of F depends only on the relative Euler numbers of the f-caps by Lemma 4.8, whereas the τ invariant depends on the relative Euler number *and* the intersections of the form $C \pitchfork F$. Lemma 4.9 shows that the latter do not contribute. Therefore we have $\mathrm{Arf}(F) = \tau(F)$, as desired, which completes the proof. □

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