Non-concordant links with homology cobordant zero framed surgery manifolds

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Abstract. We use topological surgery theory to give sufficient conditions for the zero framed surgery manifold of a 3-component link to be homology cobordant to the zero framed surgery on the Borromean rings (also known as the 3-torus), via a topological homology cobordism preserving the free homotopy classes of the meridians.

This enables us to give examples of 3-component links with unknotted components and vanishing pairwise linking numbers, such that any two of these links have homology cobordant zero surgeries in the above sense, but the zero surgery manifolds are not homeomorphic. Moreover the links are not concordant to one another, and in fact they can be chosen to be height h but not height $h + 1$ symmetric grope concordant, for each h which is at least three.

1. Introduction

It is well known that the study of homology cobordism of 3-manifolds is essential for understanding the concordance of knots and links: homology cobordism of the *exteriors* of links in S^3 is equivalent to concordance in a homology $S^3 \times I$, and an additional mild normal generation condition for π_1 is equivalent to topological concordance in $S^3 \times I$ (this also holds modulo the 4-dimensional Poincaré conjecture in the smooth case).

We recall the definitions: two m-component links L_0 and L_1 in S^3 are said to be *topologically* (respectively *smoothly*) *concordant* if there exist m locally flat (respectively smoothly embedded) disjoint annuli in $S^3 \times [0, 1]$ cobounded by components of $L_0 \times \{0\}$ and $-L_1 \times \{1\}$. Two 3-manifolds M_0 and M_1 bordered by a 2-manifold Σ, that is endowed with a marking $\mu_i \colon \Sigma \xrightarrow{\cong} \partial M_i$, are *topologically* (respectively *smoothly*) *homology cobordant* if there is a topological (respectively smooth) 4-manifold W with $\partial W = M_0 \sqcup -M_1 \sqcup \Sigma \times [0, 1]/(\mu_0(x) \sim x \times \{0\}, \mu_1(x) \sim x \times \{1\}, x \in \Sigma)$, such that the inclusions $M_i \to W$ ($i = 0, 1$) induce isomorphisms on integral homology groups. In this paper links are oriented, and link exteriors are always bordered by $\bigsqcup_m S^1 \times S^1$ under the zero framing.

In high dimensions, concordance classification results were obtained by studying homology surgery, with the aim of surgeries being to produce a homology cobordism of the exteriors (for example, see [CS74, CS80, LD88]). On the other hand, for knots and links in dimension three, the *zero surgery manifolds* and their 4-dimensional homology cobordisms have been extensively used in the literature in order to understand the structure

peculiar to low dimensions, especially in the topological category. Recall that performing zero framed surgery on a link in S^3 yields a closed 3-manifold, called the zero surgery manifold.

The classical invariants such as the knot signature and Levine's algebraic knot concordance class [Lev69a, Lev69b] are obtained from the zero surgery manifold of a knot, via the Blanchfield form. Also higher order knot concordance obstructions, such as Casson-Gordon invariants [CG78, CG86], and Cochran-Orr-Teichner L^2-signatures [COT03] are obtained from the zero surgery manifold (often together with the homology class of the meridian).

A natural interesting question is whether the homology cobordism class of a zero surgery manifold determines the concordance class of a knot or link or if it determines the homology cobordism class of the exterior.

In this paper we show, in a strong sense involving homotopy of meridians, that the answer is negative for a large class of *links* satisfying a certain nonvanishing condition on Milnor's $\overline{\mu}$-invariants, even in the framework of symmetric grope and Whitney tower generalisations of concordance and homology cobordism in the sense of [COT03, Cha]. Also we employ topological surgery in dimension 4 to give a new construction of homology cobordisms of zero surgery manifolds. Next we state our main theorems, after which we will discuss these aspects further.

Theorem 1.1. *Suppose $h \geq 3$. Then there are infinitely many 3-component links $L_0, L_1,$ ... with vanishing pairwise linking numbers and with unknotted components, satisfying the following for any $i \neq j$.*

  (1) *The zero surgery manifolds M_{L_i} and M_{L_j} are not homeomorphic.*
  (2) *There is a topological homology cobordism between M_{L_i} and M_{L_j} in which the kth meridians of L_i and L_j are freely homotopic for each $k = 1, 2, 3$.*
  (3) *The links L_i and L_j are height h but not height $h+1$ symmetric grope concordant. In particular L_i and L_j are not concordant.*

For a definition of height h symmetric grope concordance, see Definition 4.2. Our links are obtained from the Borromean rings, which will be our L_0 for all h, by performing a satellite construction along a curve lying in the kernel of the inclusion induced map $\pi_1(S^3 \smallsetminus L_0) \to \pi_1(M_{L_0})$.

As a counterpoint to Theorem 1.1, we show that there are infinite families of links with the same non-vanishing Milnor invariants with homeomorphic zero surgery manifolds preserving the homotopy classes of the meridians, but which are not concordant.

The Milnor invariant of an m-component link associated to a multi-index $I = i_1 i_2 \cdots i_r$ with $i_j \in \{1, \ldots, m\}$, as defined in [Mil57], will be denoted by $\overline{\mu}_L(I)$. We denote its length by $|I| := r$. Define $k(m) := \lfloor \log_2(m-1) \rfloor$.

Theorem 1.2. *Let I be a multi-index with non-repeating indices with length $m \geq 2$. For any $h \geq k(m) + 2$ there are infinitely many m-component links $L_0, L_1, \ldots$ with unknotted components, satisfying the following.*

  (1) *The L_i have identical $\overline{\mu}$-invariants: $\overline{\mu}_{L_i}(J) = \overline{\mu}_{L_j}(J)$ for all J. In addition $\overline{\mu}_{L_i}(I) = 1$, and $\overline{\mu}_{L_i}(J) = 0$ for $|J| < |I|$.*
  (2) *There is a homeomorphism between the zero surgery manifolds M_{L_i} and M_{L_j} which preserves the homotopy classes of the meridians.*
  (3) *The links L_i and L_j are height h but not height $h+1$ symmetric grope concordant. In particular L_i and L_j are not concordant.*

The case of $m \geq 3$ should be compared with Theorem 1.1 since then the links L_i have vanishing pairwise linking numbers. To construct such links we start with certain iterated Bing doubles constructed using T. Cochran's algorithm which realise the Milnor invariant required. We then perform satellite operations which affect the concordance class of the link but do not change the homeomorphism type of the zero surgery manifold.

We remark that we could also phrase Theorems 1.1 and 1.2 in terms of symmetric Whitney tower concordance instead of grope concordance.

In the three subsections below, we discuss some features of Theorem 1.1, regarding: (i) the use of topological surgery in dimension 4, (ii) link concordance versus zero surgery homology cobordism, and (iii) link exteriors and the homology surgery approach.

1.1. Topological surgery for 4-dimensional homology cobordism

An interesting aspect of the proof of Theorem 1.1 is that we employ topological surgery in dimension 4 to give a sufficient condition for certain zero surgery manifolds of 3-component links to be homology cobordant. It is well known that topological surgery in dimension 4 is useful for obtaining homology cobordisms (and consequently concordances), although the current state of the art in terms of "good" groups, for which the π_1-null disc lemma is known, is still insufficient for the general case. M. Freedman and F. Quinn showed that knots of Alexander polynomial one are concordant to the unknot [FQ90, Theorem 11.7B]. J. Davis extended the program to show that two component links with Alexander polynomial one are concordant to the Hopf link [Dav06]. The above two cases use topological surgery over fundamental groups $\mathbb{Z}$ and $\mathbb{Z}^2$ respectively. Due to the rarity of good groups for 4-dimensional topological surgery, there are not many other situations where such positive results on knot and link concordance can currently be proven. As another case, S. Friedl and P. Teichner in [FT05] found sufficient conditions for a knot to be homotopy ribbon, and in particular slice, with a certain ribbon group $\mathbb{Z} \ltimes \mathbb{Z}[1/2]$.

We give another instance of the utility of topological surgery for constructing homology cobordisms, using the group $\mathbb{Z}^3$, which is manageable from the point of view of topological surgery in dimension 4. Indeed, our sufficient condition for zero surgery manifolds to be homology cobordant focuses on the Borromean rings as a base link. The zero surgery manifold M_{Bor} of the Borromean rings is the 3-torus $T^3 = S^1 \times S^1 \times S^1$, whose fundamental group is $\mathbb{Z}^3$.

To state our result, we use the following notation: let $\Lambda := \mathbb{Z}[\mathbb{Z}^3] = \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]$. Denote the zero surgery manifold of a link L by M_L as before. For a 3-component link L with vanishing pairwise linking numbers, there is a canonical homotopy class of maps $f_L \colon M_L \to M_{\mathrm{Bor}} = T^3$ which send the homotopy class of the ith meridian of L to that of the Borromean rings, namely the ith circle factor of T^3. After choosing an identification of $\pi_1(T^3) = \mathbb{Z}^3$, we can use this to define the Λ-coefficient homology $H_1(M_L; \Lambda)$. We say that a map $f \colon M_L \to T^3$ is a Λ-*homology equivalence* if f is homotopic to f_L and f induces isomorphisms on $H_*(-; \Lambda)$.

Theorem 1.3. *Suppose L is a 3-component link whose components have trivial Arf invariants and there exists a Λ-homology equivalence $M_L \to T^3$. Then there is a homology cobordism W between M_L and $T^3 = M_{Bor}$ for which the inclusion induced maps $\pi_1(M_L) \to \pi_1(W) \xleftarrow{\cong} \pi_1(T^3)$ are such that the composition from left to right takes meridians to meridians.*

1.2. Link concordance versus zero surgery homology cobordism

We review the general question of whether links with homology cobordant zero surgery manifolds are concordant. The answer to the basic question is easily seen to be no, once one knows of a result of C. Livingston that there are knots not concordant to their reverses [Liv83]. Note that a knot and its reverse have the same zero surgery manifold. This leads us to consider some additional conditions on the homology cobordism, involving the meridians. In what follows meridians are always positively oriented.

First, observe the following: the exteriors of two links are homology cobordant if and only if the zero framed meridians cobound framed annuli disjointly embedded in a homology cobordism of the zero surgery manifolds. (For the if direction, note that the exterior of the framed annuli is a homology cobordism of the link exteriors.) In particular it holds if two links (or knots) are concordant.

Regarding the knot case, in [CFHH], T. Cochran, B. Franklin, M. Hedden and P. Horn considered homology cobordisms of zero surgery manifolds in which the meridians are *homologous*: in the smooth category, they showed that the existence of such a homology cobordism is insufficient for knots to be concordant. In the topological case this is still left unknown.

Concerning a stronger *homotopy* analogue, the following is unknown in both the smooth and topological cases:

Question 1.4. If there is a homology cobordism of zero surgery manifolds of two knots in which the meridians are homotopic, are the knots concordant? Or concordant in a homology $S^3 \times I$?

For the link case, results in the literature give non-concordant examples whose zero surgery manifolds admit a homology cobordism with homotopic meridians. As a generic example in the topological category, consider a 2-component link with linking number one. The zero surgery manifold is a homology 3-sphere, which bounds a contractible topological 4-manifold by [FQ90, Corollary 9.3C]. Taking the connected sum of such 4-manifolds, one obtains the following: *the zero surgery manifolds of any two linking number one 2-component links cobound a simply connected topological homology cobordism.* Note that in this case the meridians are automatically homotopic. There are many linking number one 2-component links which are not concordant, as can be detected, for example, by the multivariable Alexander polynomial [Kaw78, Nak78]. For related in-depth study, the reader is referred to, for instance, [CK99, FP, Cha]. With our respective coauthors, we gave non-concordant linking number one links with two unknotted components, for which abelian invariants such as the multivariable Alexander polynomial are unable to obstruct them from being concordant.

There are other examples, which have knotted components: in [CFHH, end of Section 1], they discuss 2-component linking number zero links with homeomorphic zero surgery manifolds which have non-concordant (knotted) components. These links are obviously not concordant, and it can be seen that the homeomorphisms preserve meridians up to homotopy.

By contrast with the above examples, our links have unknotted components and vanishing pairwise linking numbers. Another feature exhibited by the links of Theorems 1.1 and 1.2 is that a great deal of the subtlety of symmetric grope concordance of links can occur within a single homology cobordism/homeomorphism class of the zero surgeries, even modulo local knot tying.

We remark that all the links of Theorems 1.1 and 1.2 lie in the same 'k-solvequivalence class' for all k in the sense of [CK08, Definition 2.5].

1.3. Link exteriors and the homology surgery approach

Our results serve to underline the philosophy that when investigating the relative problem of whether two links are concordant, and neither of them are the unlink, one should consider obstructions to homology cobordism of the link exteriors viewed as bordered manifolds, rather than to homology cobordism of the zero surgery manifolds, even in low dimensions. This was implemented in, for example [Kaw78, Nak78, Cha] (see also [FP] for a related approach).

Although we stated our results in terms of grope concordance of links, in Theorems 1.1 and 1.2 given above, in fact we show more: the link exteriors are far from being homology cobordant, as measured in terms of Whitney towers. A more detailed discussion is given in Section 5. For the purpose of distinguishing exteriors, we use the amenable Cheeger-Gromov ρ-invariant technology for bordered 3-manifolds (particularly for link exteriors) developed in [Cha], generalising applications of ρ-invariants to concordance and homology cobordism in [COT03, CHL09, CO12].

We will now discuss our results from the viewpoint of the homology surgery approach to link concordance classification, initiated by S. Cappell and J. Shaneson [CS74, CS80] and implemented in high dimensions by J. Le Dimet [LD88] using P. Vogel's homology localisation of spaces [Vog78]. The strategy consists of two parts. Consider the problem of comparing two given link exteriors. First we decide whether the exteriors have the same "Poincaré type," which roughly means that they have homotopy equivalent Vogel homology localisations. If so, there is a common finite target space, into which the exteriors are mapped by homology equivalences rel. boundary. Once this is the case, a surgery problem is defined, and one can try to decide whether homology surgery gives a homology cobordism of the exteriors. The first step is obstructed by homotopy invariants (including Milnor $\overline{\mu}$-invariants in the low dimension). The failure of the second step is measured by surgery obstructions, which are not yet fully formulated in the low dimension (even modulo that 4-dimensional surgery might not work), since the fundamental group plays a more sophisticated central rôle; see [Pow12] for the beginning of an algebraic surgery approach to this problem in the context of knot slicing.

Our examples illustrate that for many Poincaré types, namely those in Theorems 1.1 and 1.2, we get a rich theory of surgery obstructions within each Poincaré type, which is invisible via zero surgery manifolds. We remark that for our links L_i in Theorems 1.1 and 1.2, there is a homology equivalence of the exterior of each L_i into that of a fixed one, say L_1, since we use satellite constructions (see Section 4). It follows that the exteriors have the same Poincaré type in the above sense. In this paper, (parts of the not-yet-fully-formulated) homology surgery obstructions in dimension 4 have their incarnation in Theorem 5.2, the Amenable Signature Theorem.

Organisation of the paper

In Section 2, we explore the implications of the hypothesis that a homology equivalence $f : M_L \to T^3$ as in Theorem 1.3 exists, and we prove Theorem 1.3 in Section 3. In Section 4, we construct links with a given Milnor invariant with non-repeating indices, and perform satellite operations on the links to construct the links of Theorems 1.1 and 1.2, which are height h symmetric grope concordant. In Section 5, we show that none of these links are height $h + 1$ grope concordant to one another.

Acknowledgements

2. Homology type of zero surgery manifolds and the 3-torus

This section discusses the hypotheses of Theorem 1.3. We begin the section by briefly reminding the reader who is familiar with Kirby calculus of a nice way to see the following well known fact.

Lemma 2.1. *The zero surgery manifold of the Borromean rings is homeomorphic to the 3-torus.*

Proof. Place dots on two components of the Borromean rings and a zero near the other. Each component of the Borromean rings is a commutator in the meridians of the other two components, so this is a Kirby diagram for $T^2 \times D^2$, whose boundary is T^3. The 1-handles (dotted circles) can be replaced with zero framed 2-handles without changing the boundary. $\qquad\square$

In the following proposition we expand on the meaning and implications of the condition in Theorem 1.3. Denote the exterior of a link L by $X_L := S^3 \setminus \nu L$ as before.

Proposition 2.2. *Suppose that L is a 3-component link. Then the following are equivalent.*

(1) *There is a Λ-homology equivalence $f \colon M_L \to T^3$.*
(2) *The preferred longitudes generate the link module $H_1(X_L; \Lambda)$.*
(3) *The pairwise linking numbers of L vanish and $H_1(M_L; \Lambda) = 0$.*

Furthermore, (any of) the above conditions imply that L has multi-variable Alexander polynomial $\Delta_L = (t_1 - 1)(t_2 - 1)(t_3 - 1)$, and $\Delta_L = (t_1 - 1)(t_2 - 1)(t_3 - 1)$ implies that the Milnor invariant $\overline{\mu}_L(123)$ is equal to ± 1.

Proof. First we will observe (2) and (3) are equivalent. Longitudes of L represent elements in $H_1(X_L; \Lambda) \cong \pi_1(X_L)^{(1)}/\pi_1(X_L)^{(2)}$ if and only if they are zero in $H_1(X_L; \mathbb{Z}) \cong \mathbb{Z}^3$; that is, the pairwise linking numbers are zero. If this is the case, $H_1(M_L; \Lambda)$ is isomorphic to $H_1(X_L; \Lambda)/\langle \text{longitudes} \rangle$, since M_L is obtained by attaching three 2-handles to E_L along the longitudes and then attaching three 3-handles along the boundary. It follows that longitudes generate $H_1(X_L; \Lambda)$ if and only if $H_1(M_L; \Lambda) = 0$.

Suppose (1) holds. Denote the meridians of L by μ_i ($i = 1, 2, 3$) and the linking number of the ith and jth components by ℓ_{ij}. The ith longitude λ_i, which is homologous to $\sum_{j \neq i} \ell_{ij} \mu_i$, is zero in $H_1(M_L; \mathbb{Z}) \cong H_1(T^3; \mathbb{Z})$. Since $\{f_*([\mu_i])\}$ forms a basis of

$H_1(T^3; \mathbb{Z}) \cong \mathbb{Z}^3$, it follows by linear independence that $\ell_{ij} = 0$ for any i and j. Also, $H_1(M_L; \Lambda) \cong H_1(T^3; \Lambda) = 0$. This shows that (3) holds.

Suppose (3) holds. Start with a map $g \colon \partial X_L = \bigsqcup_3 S^1 \times S^1 \to T^3$ that sends μ_i to the ith S^1 factor and λ_i to a point. Observe that $g_* \colon H_1(\partial X_L; \mathbb{Z}) \to H_1(T^3; \mathbb{Z})$ factors through the inclusion induced map $i_* \colon H_1(\partial X_L; \mathbb{Z}) \to H_1(M_L; \mathbb{Z})$ and the identifications $H_1(M_L; \mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}^3 \xleftarrow{\cong} H_1(T^3; \mathbb{Z})$; this follows from the fact that $H_1(\partial X_L; \mathbb{Z}) \cong \mathbb{Z}^6$ is generated by the μ_i and λ_i and that both g_* and i_* are quotient maps, with their kernels generated by the λ_i. Since T^3 is a $K(\mathbb{Z}^3, 1)$, elementary obstruction theory shows that g extends to a map $f \colon M_L \to T^3$.

Consider the universal coefficient spectral sequence (see e.g. [Lev77, Theorem 2.3]) $E^2_{p,q} = \operatorname{Ext}^p_\Lambda(H_q(M_L; \Lambda), \Lambda) \Rightarrow H^n(M_L; \Lambda)$. We have $E^2_{0,1} = 0$ since $H_1(M_L; \Lambda) = 0$, and $E^2_{1,0} = \operatorname{Ext}^1_\Lambda(\mathbb{Z}, \Lambda) = H^1(T^3; \Lambda) = 0$. It follows that $H^1(M_L; \Lambda) = 0$. By duality, $H_2(M_L; \Lambda) = 0$. Also, $H_3(M_L; \Lambda) = 0$ since the $\mathbb{Z}^3$-cover of M_L is non-compact. Since $H_0(M_L; \Lambda) \cong \mathbb{Z} \cong H_0(T^3; \Lambda)$ and $H_i(T^3; \Lambda) = 0$ for $i > 0$, it follows that f is a Λ-homology equivalence. This completes the proof of the equivalence of (1), (2) and (3).

Suppose (1), (2) and (3) hold. Recall that the scalar multiplication of a loop by t_i in the module $H_1(X_L; \Lambda)$ is defined to be conjugation by μ_i. Since λ_i and μ_i commute as elements of the fundamental group, we have $(t_i - 1)\lambda_i = 0$ in $H_1(X_L; \Lambda)$. From this and (2), it follows that there is an epimorphism of $A := \bigoplus_{i=1}^3 \Lambda/\langle t_i - 1 \rangle$ onto $H_1(X_L; \Lambda)$. Since the zero-th elementary ideal of A is the principal ideal generated by $(t_1 - 1)(t_2 - 1)(t_3 - 1)$, it follows that Δ_L is a factor of $(t_1 - 1)(t_2 - 1)(t_3 - 1)$. We now invoke the Torres condition (see e.g. [Kaw96, Theorem 7.4.1]): $\Delta_L(1, t_2, t_3) = (t_2^{\ell_{12}} t_3^{\ell_{13}} - 1)\Delta_{L'}(t_2, t_3)$ where L' is the sublink of L with the first component deleted and ℓ_{ij} is the pairwise linking number. Since $\ell_{ij} = 0$ by (3), we have $\Delta_L(1, t_2, t_3) = 0$. It follows that $t_1 - 1$ is a factor of Δ_L. Similarly $t_2 - 1$ and $t_3 - 1$ are factors. Therefore $\Delta_L(t_1, t_2, t_3) = (t_1 - 1)(t_2 - 1)(t_3 - 1)$.

To show the last part, suppose that $\Delta_L(t_1, t_2, t_3) = (t_1 - 1)(t_2 - 1)(t_3 - 1)$. By [Kaw96, Proposition 7.3.14], the single-variable Alexander polynomial $\Delta_L(t)$ of L is given by

$$\Delta_L(t) = (t - 1)\Delta_L(t, t, t) = (t - 1)^4 \doteq ((\sqrt{t})^{-1} - \sqrt{t})^4.$$

It follows that L has Conway polynomial $\nabla_L(z) = z^4$, by the standard substitution $z = (\sqrt{t})^{-1} - \sqrt{t}$. In [Coc85, Theorem 5.1], Cochran identified the coefficient of z^4 in $\nabla_L(z)$ with $(\mu_L(123))^2$ for 3-component links with pairwise linking number zero. Applying this to our case, it follows that $\overline{\mu}_L(123) = \pm 1$. □

3. Construction of homology cobordisms using topological surgery

This section gives the proof of Theorem 1.3. The proof will use surgery theory, and will parallel the proof given by Davis in [Dav06] (see also [Hil02, Section 7.6]). We will provide some details in order to fill in where the treatment in [Dav06] was terse, and to convince ourselves that the analogous arguments work in the case of interest.

For the convenience of the reader we restate Theorem 1.3 here.

Theorem 1.3. *Suppose L is a 3-component link whose components have trivial Arf invariants and there exists a Λ-homology equivalence $M_L \to T^3$. Then there is a homology cobordism W between M_L and $T^3 = M_{Bor}$ for which the inclusion induced maps $\pi_1(M_L) \to \pi_1(W) \xleftarrow{\cong} \pi_1(T^3)$ are such that the composition from left to right takes meridians to meridians.*

Remark 3.1. It is an interesting question to determine whether there are extra conditions which can be imposed in order to see that the Arf invariants of the components are forced to vanish by the homological assumptions. In the cases of knots and two component links with Alexander polynomial one, the Arf invariants of the components are automatically trivial. For knots $\Delta_K(-1)$ computes the Arf invariant by [Lev66]. For two component links one observes that $\Delta_L(t, 1)$ and $\Delta_L(1, t)$ give the Alexander polynomials of the components, by the Torres condition, and then applies Levine's theorem. These arguments do not seem to extend to the three component case of current interest.

The proof of Theorem 1.3 will occupy the rest of this section. In order to produce a homology cobordism, we will first show that there is a normal cobordism between normal maps $f\colon M_L \to T^3$ and $\mathrm{Id}\colon T^3 \to T^3$. Interestingly, we can work with smooth manifolds in order to establish the existence of a normal cobordism. This will make arguments which invoke tangent bundles and transversality easier to digest. Only at the end of the proof of Theorem 1.3, where we take connected sums with the E_8-manifold, and where we claim that the vanishing of a surgery obstruction implies that surgery can be done, do we need to leave the realm of smooth manifolds.

Definition 3.2. Let X be an n-dimensional manifold with a vector bundle $\nu \to X$. A degree one normal map (F, b) over X is an n-manifold M with a map $F\colon M \to X$ which induces an isomorphism $F_*\colon H_n(M; \mathbb{Z}) \xrightarrow{\cong} H_n(X; \mathbb{Z})$, together with a stable trivialisation $b\colon TM \oplus F^*\nu \oplus \varepsilon^l \cong \varepsilon^k$.

A degree one normal cobordism (J, e) between normal maps $(F\colon M \to X, b)$ and $(G\colon N \to X, c)$ is an $(n+1)$–dimensional cobordism between M and N with a map $J\colon W \to X \times I$ extending $F\colon M \to X \times \{0\}$ and $G\colon M \to X \times \{1\}$, which induces an isomorphism

$$J_*\colon H_{n+1}(W, \partial W; \mathbb{Z}) \xrightarrow{\cong} H_{n+1}(X \times I, X \times \{0, 1\}; \mathbb{Z}),$$

together with a stable trivialisation $e\colon TW \oplus J^*(\nu \times I) \oplus \varepsilon^{l'} \cong \varepsilon^{k'}$.

For us, let $X = T^3$, and let ν be its tangent bundle. We fix a framing on the stable tangent bundle of the target torus T^3 once and for all. Note that this canonically determines a trivialisation of the tangent bundle of $F^*\nu$, for any map $F\colon M \to X$, by the following diagram, in which the bottom composition is the constant map, denoted $*$, and the top composition is the pull back $F^*\nu$. The middle composition is the induced framing.

$$
\begin{array}{ccccc}
M \times \{0\} & \xrightarrow{F \times \mathrm{Id}} & T^3 \times \{0\} & \xrightarrow{\nu} & BO(n) \\
\downarrow & & \downarrow & & \downarrow \\
M \times I & \xrightarrow{F \times \mathrm{Id}} & T^3 \times I & \longrightarrow & BO \\
\uparrow & & \uparrow & & \uparrow \\
M \times \{1\} & \xrightarrow{F \times \mathrm{Id}} & T^3 \times \{1\} & \xrightarrow{*} & BO(n).
\end{array}
$$

A framing of the tangent bundle of the domain will therefore determine a normal map.

Lemma 3.3. *Let L be a link whose components have trivial* Arf *invariants, and let $f\colon M_L \to T^3$ be a degree one normal map which induces a $\mathbb{Z}$-homology isomorphism and which maps a chosen meridian μ_i to the ith S^1 factor of T^3 for $i = 1, 2, 3$. We can make a homotopy of f and choose a framing on M_L so that $f\colon M_L \to T^3$ and $\mathrm{Id}\colon T^3 \to T^3$ are degree one normal cobordant.*

Proof. We need to show that we can choose a framing on M_L such that the disjoint union $M_L \sqcup -T^3$ represent the trivial element of $\Omega_3^{\text{fr}}(T^3)$. We compute this bordism group:

$$\widetilde{\Omega}_3^{\text{fr}}(T^3) \cong \widetilde{\Omega}_4^{\text{fr}}(\Sigma T^3) \cong \widetilde{\Omega}_4^{\text{fr}}(S^2 \vee S^2 \vee S^2 \vee S^3 \vee S^3 \vee S^3 \vee S^4),$$

with this last isomorphism induced by a homotopy equivalence of spaces. There is a copy of S^{i+1} for each i–cell of T^3, for $i = 1, 2, 3$. To see this homotopy equivalence, we need to see that the attaching maps of the cells are null-homotopic. The suspension of the 1-skeleton of T^3 is $S^2 \vee S^2 \vee S^2$. The Hilton-Milnor theorem [Hil55, Theorem A] computes the homotopy groups of a wedge of spheres. The attaching maps for the 2-cells of T^3 become the attaching maps for the 3-cells of ΣT^3, namely maps in

$$\pi_2(S^2 \vee S^2 \vee S^2) \cong \bigoplus_3 \pi_2(S^2) \cong \bigoplus_3 \mathbb{Z},$$

where the first isomorphism is by the Hilton-Milnor theorem. The commutator attaching maps become trivial in the abelian $\pi_2(S^2)$. Therefore the 3-skeleton of ΣT^3 is a wedge $S^2 \vee S^2 \vee S^2 \vee S^3 \vee S^3 \vee S^3$. The attaching map for the 3-cell of T^3 becomes the attaching map for the 4-cell of ΣT^3, a map in

$$\pi_3(S^2 \vee S^2 \vee S^2 \vee S^3 \vee S^3 \vee S^3) \cong \bigoplus_{1 \leq i \leq 3} \pi_3(S^3) \oplus \bigoplus_3 \pi_3(S^2) \oplus \bigoplus_{1 \leq i < j \leq 3} \pi_3(S^3),$$

again by the Hilton-Milnor theorem, where the last three $\pi_3(S^3)$ summands include into $\pi_3(S^2 \vee S^2 \vee S^2 \vee S^3 \vee S^3 \vee S^3)$ via composition with the Whitehead product: let $f_i \colon S^2 \to S_i^2$ be a generator of $\pi_2(S_i^2)$, where S_i^2 is the ith S^2 component in the wedge. Then the Whitehead product is the homotopy class of the map $[f_i, f_j] \in \pi_3(S_i^2 \vee S_j^2)$, which is the attaching map for the 4-cell in a standard cellular decomposition of $S^2 \times S^2$. Since $\pi_2(S^1) \cong \pi_2(S^1 \vee S^1) \cong 0$, the summands associated to the S^2 components of the wedge do not arise from a suspension. The summands associated to the S^3 components are null-homotopic since the 3-cell of T^3 is attached to each 2-cell twice, once on either side. This completes the explanation of the claimed homotopy equivalence:

$$\Sigma T^3 \simeq S^2 \vee S^2 \vee S^2 \vee S^3 \vee S^3 \vee S^3 \vee S^4.$$

By Mayer-Vietoris, the bordism group $\widetilde{\Omega}_4^{\text{fr}}(S^2 \vee S^2 \vee S^2 \vee S^3 \vee S^3 \vee S^3 \vee S^4)$ is a direct sum

$$\bigoplus_3 \widetilde{\Omega}_4^{\text{fr}}(S^2) \oplus \bigoplus_3 \widetilde{\Omega}_4^{\text{fr}}(S^3) \oplus \widetilde{\Omega}_4^{\text{fr}}(S^4)$$
$$\cong \bigoplus_3 \widetilde{\Omega}_2^{\text{fr}}(S^0) \oplus \bigoplus_3 \widetilde{\Omega}_1^{\text{fr}}(S^0) \oplus \widetilde{\Omega}_0^{\text{fr}}(S^0)$$
$$\cong \bigoplus_3 \Omega_2^{\text{fr}} \oplus \bigoplus_3 \Omega_1^{\text{fr}} \oplus \Omega_0^{\text{fr}}$$
$$\cong \bigoplus_3 \mathbb{Z}_2 \oplus \bigoplus_3 \mathbb{Z}_2 \oplus \mathbb{Z}.$$

Therefore

$$\Omega_3^{\text{fr}}(T^3) \cong \Omega_3^{\text{fr}} \oplus \bigoplus_3 \mathbb{Z}_2 \oplus \bigoplus_3 \mathbb{Z}_2 \oplus \mathbb{Z} \cong \mathbb{Z}_{24} \oplus \bigoplus_3 \mathbb{Z}_2 \oplus \bigoplus_3 \mathbb{Z}_2 \oplus \mathbb{Z}.$$

The isomorphism is given as follows. Let

$$\text{pr}_i \colon T^3 = S^1 \times S^1 \times S^1 \longrightarrow S^1$$

be given by projection onto the ith factor. Similarly let

$$\mathrm{qr}_i \colon T^3 = S^1 \times S^1 \times S^1 \longrightarrow S^1 \times S^1$$

be given by forgetting the ith factor. Let $F \colon M \to T^3$ be an element of $\Omega_3^{\mathrm{fr}}(T^3)$. Making all maps transverse to a point, called $*$, we obtain an 8-tuple:

$$\Big([M], (\mathrm{pr}_1 \circ F)^{-1}(*), (\mathrm{pr}_2 \circ F)^{-1}(*), (\mathrm{pr}_3 \circ F)^{-1}(*), (\mathrm{qr}_1 \circ F)^{-1}(*),$$

$$(\mathrm{qr}_2 \circ F)^{-1}(*), (\mathrm{qr}_3 \circ F)^{-1}(*), F^{-1}(*) \Big)$$

$$\in \Omega_3^{\mathrm{fr}} \oplus \bigoplus_3 \Omega_2^{\mathrm{fr}} \oplus \bigoplus_3 \Omega_1^{\mathrm{fr}} \oplus \Omega_0^{\mathrm{fr}} \cong \mathbb{Z}_{24} \oplus \bigoplus_3 \mathbb{Z}_2 \oplus \bigoplus_3 \mathbb{Z}_2 \oplus \mathbb{Z}.$$

We consider each of the summands in turn.

By choosing the appropriate orientation on M_L, and making the degree one normal maps transverse to a point, it can be arranged so that the disjoint union $f^{-1}(*) \sqcup -\mathrm{Id}^{-1}(*) = \{pt\} \sqcup -\{pt\} = 0 \in \Omega_0^{\mathrm{fr}}$.

As observed in [Dav06, Proof of the Lemma], we can change the framing so that the elements of Ω_1^{fr} agree. First, we change the framing on each of three chosen meridians μ_i to the link components L_i.

Orientable k-plane vector bundles over S^1 are classified by homotopy classes of maps $[S^1, BSO(k)]$. Consider the exact sequence:

$$\pi_2(BSO) \longrightarrow \pi_2(BSO, BSO(k)) \longrightarrow \pi_1(BSO(k)) \xrightarrow{\gamma} \pi_1(BSO).$$

A stably trivial vector bundle over S^1 gives us an element of $\ker(\gamma)$. A choice of trivialisation of the vector bundle gives us a null homotopy and therefore an element of $\pi_2(BSO, BSO(k))$. The possible choices of stable trivialisations, or framings, are indexed by $\pi_2(BSO) \cong \pi_1(SO) \cong \mathbb{Z}_2$.

We can therefore, if necessary, change the framing on each μ_i to be the bounding framing using an element of $\pi_1(SO(2))$ which maps to the non-trivial element of $\pi_1(SO)$. Use the element of $\pi_1(SO(2))$ to change the framing on the normal bundle of μ_i in M_L. We claim that these changes in the framing can be extended to the whole of M_L. To see this, we argue as follows. The dual of the inclusion map $H^1(M_L; \mathbb{Z}) \to H^1(\mu_i; \mathbb{Z})$ is surjective, since each $[\mu_i]$ is a generator of $H_1(M_L; \mathbb{Z})$. The change of framing map $\mu_i \to SO(2)$ represents a homotopy class of maps in $[\mu_i, S^1]$ and therefore an element of $H^1(\mu_i; \mathbb{Z})$. Since this pulls back to an element of $H^1(M_L; \mathbb{Z})$, which can in turn produce a map $M_L \to SO(2)$, the change of framing map can be extended as claimed.

Let $N_i \subset M_L$ be the submanifolds given by $(\mathrm{qr}_i \circ f)^{-1}(*)$, after perturbing f to make $\mathrm{qr}_i \circ f$ transverse to a point. As the inverse image of the ith S^1 factor of T^3 (e.g. $f^{-1}(S^1 \times \{*\} \times \{*\}))$, N_i is a collection of circles. After a homotopy of f, it can be arranged, by the assumption on f, that N_i is a single meridian μ_i, which has the bounding framing and therefore represents the zero element in Ω_1^{fr}. To make this arrangement, it suffices to be able to remove circles N_i whose image in T^3 is null homologous. But in T^3, a null homologous curve is also null homotopic. Therefore we can make a homotopy of f so that N_i misses $S^1 \times \{*\} \times \{*\}$.

After another homotopy, the inverse image $(\mathrm{pr}_i \circ f)^{-1}(*)$ can be arranged to be a capped-off Seifert surface $F_i \cup D^2$, where F_i is a Seifert surface for L_i (possibly with closed connected components). To see this, we again use our assumption that f sends the ith meridian μ_i to the ith circle. This assumption enables us to homotope f so that $\mathrm{pr}_i \circ f$ sends a regular neighbourhood $\mu_i \times D^2$ to S^1 by projection onto the first factor. Then

the inverse image is as desired. A homotopy of f preserves the framed bordism class of $(\mathrm{pr}_i \circ f)^{-1}(*)$, and the class $[F_i \cup D^2] \in \Omega_2^{\mathrm{fr}}$ is determined by the Arf invariant of L_i. By hypothesis, this vanishes.

Finally, again following [Dav06] (see also [FQ90, Proof of Lemma 11.6B]), the framing can be altered in the neighbourhood of a point to change the element $[M] \in \Omega_3^{\mathrm{fr}}$ to the trivial element. We recall the definition of the J homomorphism $J \colon \pi_3(SO) \to \pi_3^S \cong \Omega_3^{\mathrm{fr}}$, for the convenience of the reader, where π_k^S is the kth stable homotopy group of spheres. (Incidentally, $\pi_3(SO) \cong \mathbb{Z}$ and $\pi_3^S \cong \mathbb{Z}_{24}$.) Given $\theta \colon S^3 \to SO$, choose a k sufficiently large so that we can represent θ by a map $\theta \colon S^3 \to SO(k)$. We proceed to construct a map $(J(\theta) \colon S^{k+3} \to S^k) \in \pi_3^S$. So:

$$S^{k+3} = S^3 \times D^k \cup_{S^3 \times D^{k-1}} D^4 \times S^{k-1}.$$

Define a map:

$$\begin{aligned} j(\theta) \colon S^3 \times D^k &\to S^3 \times D^k \\ (x, y) &\mapsto (x, \theta(x)(y)), \end{aligned}$$

since $\theta(x) \in SO(k)$ acts on D^k by identifying D^k with the unit ball in $\mathbb{R}^k$. This map extends to a homeomorphism $j(\theta)$ of $S^3 \times D^k \cup_{S^3 \times D^{k-1}} D^4 \times S^{k-1}$. Form the composition:

$$S^{k+3} = S^3 \times D^k \cup_{S^3 \times D^{k-1}} D^4 \times S^{k-1} \xrightarrow{j(\theta)} S^3 \times D^k \cup_{S^3 \times D^{k-1}} D^4 \times S^{k-1}$$

$$\xrightarrow{col} S^3 \times S^k \xrightarrow{proj_1} S^k,$$

where col is the collapse map which squashes $D^4 \times S^{k-1}$ and $proj_1$ is the projection onto the first factor. This gives an element of π_3^S, which is the image of θ under $J \colon \pi_3(SO) \to \pi_3^S \cong \Omega_3^{\mathrm{fr}}$.

This J homomorphism is onto [Ada66, Example 7.17], so that composing the framing in a neighbourhood D^3 of a point with the choice of map in $\theta \in \pi_3(SO) = [(D^3, \partial D^3), (SO, *)]$ such that $-J(\theta) = [M] \in \Omega_3^{\mathrm{fr}}$, changes the class in Ω_3^{fr} as desired.

This shows the existence of a normal cobordism W'. To see that this is of degree one, note that the map to T^3 which extends over W' can be used to define a map to $T^3 \times I$, by defining a map $g \colon W' \to I$ such that $g(M_L) = \{0\}$ and $g(T^3) = \{1\}$. Now consider the commutative diagram:

$$\begin{array}{ccc} H_4(W', \partial W'; \mathbb{Z}) & \longrightarrow & H_3(\partial W'; \mathbb{Z}) \\ \downarrow & & \downarrow \\ H_4(T^3 \times I, T^3 \times \{0,1\}; \mathbb{Z}) & \longrightarrow & H_3(T^3 \times \{0,1\}; \mathbb{Z}). \end{array}$$

Going right, then down, the fundamental class $[W', \partial W']$ maps to

$$(1, -1) \in H_3(T^3 \times \{0,1\}; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

By commutativity, the relative fundamental class $[W', \partial W']$ must map to a generator of $H_4(T^3 \times I, T^3 \times \{0,1\}; \mathbb{Z})$ □

A Λ-homology equivalence is also an integral homology equivalence, by the following argument. By definition, (see above the statement of Theorem 1.3), a Λ-homology equivalence induces an isomorphism on $H_1(-; \mathbb{Z})$. By duality, we also have an isomorphism on $H_2(-; \mathbb{Z})$. It remains to see that $f \colon M_L \to T^3$ is a degree one map. The assumption that $f_* \colon H_*(M_L; \Lambda) \xrightarrow{\cong} H_*(T^3; \Lambda)$ is an isomorphism implies that the relative homology vanishes: $H_*(T^3, M_L; \Lambda) \cong 0$. The universal coefficient spectral sequence then implies

that $H_*(T^3, M_L; \mathbb{Z}) \cong 0$ since all the E^2 terms $\mathrm{Tor}_p^\Lambda(H_q(T^3, M_L; \Lambda), \mathbb{Z})$ vanish. Therefore a Λ-homology equivalence as in Theorem 1.3 is a degree one map.

Lemma 3.3 then establishes the existence of a choice of stable framing b on M_L such that there is a degree one normal cobordism

$$(F' \colon W' \longrightarrow T^3 \times I, e')$$

between $(f \colon M_L \to T^3, b)$ and $(\mathrm{Id} \colon T^3 \to T^3, c)$. Choosing such a framing, we proceed to apply surgery theory to alter W' into a homology cobordism. Davis' observation in [Dav06] was that the framing on W' is not an intrinsic part of the concordance problem, but rather necessary additional data which is required in order to be able to apply surgery theory. Without the information provided by the self intersection form, it is not possible to obtain algebraic sufficient conditions which ensure that surgery can be performed. Nevertheless, as we shall see, there is a certain amount of freedom in the choice of framing data.

Before giving the proof of Theorem 1.3, we first give the definition of the Wall even dimensional surgery obstruction groups, which we will use in the proof.

Definition 3.4 ([Wal99], Chapter 5). Let A be a ring with involution. A $(-1)^k$-*Hermitian sesquilinear quadratic form* on a free based A-module M is a $(-1)^k$-Hermitian sesquilinear form $\lambda \colon M \times M \to A$ together with a quadratic enhancement. A *quadratic enhancement* of a form $\lambda \colon M \times M \to A$ is a function $\mu \colon M \to A/\{a - (-1)^k \overline{a} \,|\, a \in A\}$ such that

(1) $\lambda(x, x) = \mu(x) + \overline{\mu(x)}$;
(2) $\mu(x + y) - \mu(x) - \mu(y) = \lambda(x, y)$;
(3) $\mu(ax) = a\mu(x)\overline{a}$;

for all $x, y \in M$ and for all $a \in A$.

A hyperbolic quadratic form is a direct sum of standard hyperbolic forms, where the standard hyperbolic form (H, χ, ν) is given by

$$\left(A \oplus A, \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix}, \nu((1,0)^T) = 0 = \nu((0,1)^T) \right).$$

The even dimensional surgery obstruction group $L_{2k}(A)$ is defined to be the Witt group of nonsingular $(-1)^k$-Hermitian sesquilinear quadratic forms on free based A-modules, where addition in the Witt group is by direct sum, and the equivalence class of the hyperbolic forms is the identity element, where the equivalence relation is as follows. Quadratic forms (M, λ, μ) and (M', λ', μ') are said to be equivalent if there are hyperbolic forms (H, χ, ν) and (H', χ', ν') such that there is an isomorphism of forms $(M \oplus H, \lambda \oplus \chi, \mu \oplus \nu) \cong (M' \oplus H', \lambda' \oplus \chi', \mu' \oplus \nu')$. This completes the definition of $L_{2k}(A)$.

For us A will be the group ring $\mathbb{Z}[\pi]$ of some group π; initially π will be $\mathbb{Z}^3$ so that we take $A = \mathbb{Z}[\mathbb{Z}^3] = \Lambda$. We omit the definition of the odd dimensional L-groups since they will only play a peripheral rôle in the proof of Theorem 1.3.

Proof of Theorem 1.3. First, do surgery below the middle dimension [Wal99, Chapter 1] on (W', F', e') to create a normal cobordism $(F \colon W \to T^3 \times I, e)$ which is 2-connected i.e. W is connected and $\pi_1(W) \cong \pi_1(T^3) \cong \mathbb{Z}^3$. The induced map $F_* \colon \pi_2(W) \to \pi_2(T^3 \times I)$ is automatically surjective since T^3 is aspherical.

The Wall surgery obstruction [Wal99, Chapter 5] of the normal cobordism $(F \colon W \to T^3 \times I, e)$ is now defined in $L_4(\mathbb{Z}[\mathbb{Z}^3])$, to be given by the intersection form

$$\lambda_{W'} \colon H_2(W'; \Lambda) \times H_2(W'; \Lambda) \longrightarrow \Lambda,$$

together with the quadratic enhancement

$$\mu\colon H_2(W;\Lambda) \longrightarrow \mathbb{Z}[\mathbb{Z}^3]/\{a = \overline{a}\}$$

defined by counting the self intersections of an immersion of a sphere $S^2 \looparrowright W$ representing an element of $H_2(W;\Lambda) \cong \pi_2(W)$, where the regular homotopy class of the immersion is fixed by the framing e to be the unique class of immersions for which the induced trivialisation of TS^2 extends over the null-homotopy of S^2 in T^3.

The fact that the homology of the boundary $H_j(M_L;\Lambda) \cong H_j(T^3;\Lambda) \cong 0$ for $j = 1,2$, is used crucially here to see that the intersection form λ_W is non-singular, as observed by the surgeon in the "dialogue" of [Dav06].

By [Wal99, Proposition 13B.8], which is based on Shaneson's formula $L_n(\mathbb{Z}[\pi \times \mathbb{Z}]) \cong L_n(\mathbb{Z}[\pi]) \oplus L_{n-1}(\mathbb{Z}[\pi])$, when π has trivial Whitehead group [Sha69], we have that:

$$L_4(\mathbb{Z}[\mathbb{Z}^3]) \cong \bigoplus_{i=0}^{3} \binom{3}{i} L_{4-i}(\mathbb{Z}) \cong L_4(\mathbb{Z}) \oplus \bigoplus_3 L_3(\mathbb{Z}) \oplus \bigoplus_3 L_2(\mathbb{Z}) \oplus L_1(\mathbb{Z})$$

$$\cong L_0(\mathbb{Z}) \oplus \bigoplus_3 L_2(\mathbb{Z}),$$

where the last isomorphism is by periodicity of the L-groups and the fact that the odd dimensional simply connected L-groups vanish. The even dimensional simply-connected L-groups $L_{2k}(\mathbb{Z})$ are computed [KM63], when $k = 0 \bmod 2$, as

$$\begin{array}{rcl} L_0(\mathbb{Z}) & \xrightarrow{\cong} & \mathbb{Z} \\ (M,\lambda,\mu) & \mapsto & \sigma(\mathbb{R} \otimes_{\mathbb{Z}} M, \mathrm{Id} \otimes \lambda)/8, \end{array}$$

while for the dimensions where $k = 1 \bmod 2$ they are computed via

$$\begin{array}{rcl} L_2(\mathbb{Z}) & \xrightarrow{\cong} & \mathbb{Z}_2 \\ (M,\lambda,\mu) & \mapsto & \mathrm{Arf}(\mathbb{Z}_2 \otimes_{\mathbb{Z}} M, \mathrm{Id} \otimes \lambda, \mathrm{Id} \otimes \mu). \end{array}$$

We need to see that we can make further alterations to W in order to make the surgery obstruction vanish.

First, we take the connected sum with $-\sigma(W)/8$ copies of the E_8 manifold, namely the simply connected 4-manifold which is constructed by plumbing disc bundles $D^2 \times D^2$ according to the E_8 lattice. It turns out that the boundary of the resulting 4-manifold is the Poincaré homology sphere. One then caps off with the contractible topological 4-manifold whose boundary is the Poincaré homology sphere [FQ90, Corollary 9.3C]. This produces the E_8 manifold, a closed topological 4-manifold. It has a non-singular intersection form, with a quadratic enhancement induced from a normal map to S^4, and its signature is 8. By a negative copy of this 4 manifold we of course mean the same manifold but with the opposite choice of orientation. By making such a modification to W, we obtain a new normal map, which by abuse of notation we again denote by (W,F,e), for which the obstruction in $L_0(\mathbb{Z})$ is trivial. Note that W s till has fundamental group $\mathbb{Z}^3$ since $\pi_1(E_8 \text{ manifold}) \cong \{1\}$, and moreover ∂W is unchanged.

Next, we may need to alter W again, so that the three Arf invariant obstructions in $L_2(\mathbb{Z})$ vanish. For $i = 1,2,3$, define maps

$$\mathrm{qr}_i\colon T^3 \times I = S^1 \times S^1 \times S^1 \times I \longrightarrow S^1 \times S^1$$

which forget the ith S^1 factor and the I factor. Perform a homotopy of F to ensure that $\mathrm{qr}_i \circ F$ is transverse to $* \in S^1 \times S^1$, and such that $F^{-1}(S^1 \times \{*\} \times \{*\} \times \{\partial I\}) \to S^1 \times \{*\} \times \{*\} \times \{\partial I\}$ is a homotopy equivalence (and similarly with the $*$ terms moved appropriately for $i = 2,3$). This homotopy equivalence was already arranged in the proof

of Lemma 3.3, when we saw that the elements of Ω_1^{fr} can be removed. Let S_i be the surface $(\mathrm{qr}_i \circ F)^{-1}(*)$; each surface has boundary ∂S_i given by the meridian μ_i and the corresponding S^1 factor of T^3.

Let $\mathrm{pr}_i\colon T^3 \times I = S^1 \times S^1 \times S^1 \times I \to S^1 \times I$ be the map which remembers the ith S^1 factor and the I factor. Making F transverse to a point, $(\mathrm{pr}_i \circ F)^{-1}(*)$ is a surface $\Sigma_i \subset W$. Since $F(S_i \cap \Sigma_i)$ is a single point and F is of degree one, we can assume that S_i and Σ_i intersect in a single point. By choosing different points in the I factor, we can ensure that the Σ_i are all distinct.

Now as in [Dav06], for each i with nonzero surgery obstruction in the corresponding $L_2(\mathbb{Z})$ summand of $L_4(\mathbb{Z}[\mathbb{Z}^3])$, remove a neighbourhood $\Sigma_i \times D^2$ of Σ_i and replace it with $\Sigma_i \times \mathrm{cl}(S^1 \times S^1 \smallsetminus D^2)$. That is, replace the D^2 factor with a punctured torus, but define the framing on the torus to be the framing which yields Arf invariant one, that is the Lie framing on both S^1 factors. Since Σ_i is dual to S_i, this adds one to the Arf invariant of the element of $L_2(\mathbb{Z})$ represented by S_i, and so changes the Arf invariant one summands to having Arf invariant zero.

After these alterations we have a normal map $(G'\colon V' \to T^3 \times I, k')$, with vanishing surgery obstruction. Since the fundamental group $\mathbb{Z}^3$ is *good* in the sense of Freedman (poly-cyclic groups are good [FQ90, Theorem 5.1A]), the surgery sequence is exact in the topological category—see [FQ90, Theorem 11.3A]. We can therefore find embedded two spheres representing a half-basis for $\pi_2(G')$, perform surgery, and obtain a topological 4-manifold V which is homotopy equivalent to $T^3 \times I$; in particular, V is a homology cobordism between M_L and T^3.

Moreover, the following diagram commutes.

$$
\begin{array}{ccccc}
\pi_1(M_L) & \longrightarrow & \pi_1(V) & \longleftarrow & \pi_1(T^3) \\
\scriptstyle{f_*}\downarrow & & \cong\downarrow & & \mathrm{Id}\downarrow\scriptstyle{\cong} \\
\pi_1(T^3) & \overset{\cong}{\longrightarrow} & \pi_1(T^3 \times I) & \overset{\cong}{\longleftarrow} & \pi_1(T^3)
\end{array}
$$

Since the meridians μ_i of L are mapped to standard generators of $\pi_1(T^3)$, an easy diagram chase shows that the homotopy classes of the meridians are preserved in the homology cobordism V. $\qquad\square$

4. Construction of links and grope concordance

In this section we give constructions of certain links with a given Milnor invariant, and construct grope concordances, using the methods of [Coc90] and [Cha].

4.1. Iterated Bing doubles with a prescribed Milnor invariant

Let I be a multi-index with non-repeating indices with length $m := |I| \geq 2$. We describe a rooted binary tree $T(m)$ associated to $m \geq 2$, which has m leaves: the right subtree of the root just consists of a single vertex, and the left subtree $T^\dagger(m)$ is the complete binary tree of height $h(m) := \lceil \log_2(m-1) \rceil$ with the rightmost $2(m - h(m) - 1)$ pairs of leaves (and edges ending at these) removed. (By convention, a binary tree is always embedded in a plane with the root on the top.) That is, $T^\dagger(m)$ is a minimal height binary tree with $m-1$ leaves. For example, $T(m)$ for $m = 7$ is shown in Figure 1.

Following the proof of [Coc90, Theorem 7.2], a rooted binary tree T describes a link with components corresponding to the leaves of T. First, a complete binary tree of height
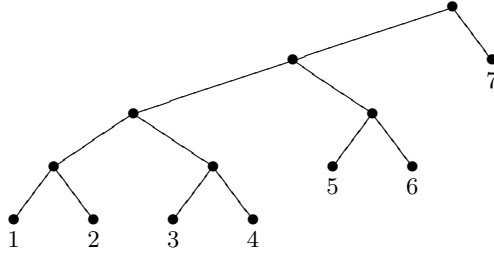
FIGURE 1. The tree $T(m)$ for $m = 7$, labeled with $I = 1234567$.

one is associated to a Hopf link. If T is obtained from T' by attaching two new leaves to a leaf v of T', then the link associated to T is obtained from that of T by Bing doubling the component corresponding to v.

Consider the link described by the tree $T(m)$. Labelling the leaves of $T(m)$ from left to right with the multi-index I (see Figure 1 for $I = 1234567$), the components of the link are ordered. We denote this ordered link by L_I. Then, by [Coc90, Theorem 8.1], the link L_I has $\overline{\mu}_{L_I}(I) = \pm 1$ and $\overline{\mu}_{L_I}(I') = 0$ for $|I'| < |I|$.

4.2. Satellite construction and grope concordance of links

To construct links which are grope concordant, we employ the method of [Cha, Section 4]. We begin by giving the definition of grope concordance. The use of gropes in this context first appeared in [COT03].

Definition 4.1 ([FT95])**.** A *grope* is a pair (2-complex, base circle) of a certain type described below. A grope has a height $h \in \mathbb{N}$. For $h = 1$ a grope is precisely a compact oriented surface Σ with a single boundary component which is the base circle. A grope of height $h + 1$ is defined inductively as follows: let $\{\alpha_i \mid i = 1, \dots, 2 \cdot \text{genus}\}$ be a standard symplectic basis of circles for Σ. Then a grope of height $h + 1$ is formed by attaching gropes of height h to each α_i along the base circles.

An annular grope is defined by replacing the bottom stage surface by a surface with two boundary components.

Definition 4.2 ([Cha], Definition 2.16)**.** Two m-component links L and L' in S^3 are *height n grope concordant* if there are m framed annular gropes G_i of height n, $i = 1, \dots m$, disjointly embedded in $S^3 \times [0, 1]$, with the boundary of G_i the zero framed ith component of $L_i \subset S^3 \times \{0\}$ and $-L_i' \subset S^3 \times \{1\}$.

As mentioned in the introduction we could also phrase our theorems in terms of Whitney towers, but for simplicity of exposition we stick to gropes. See [Cha, Section 2] for an exposition on gropes, Whitney towers, and n-solvable cobordisms (Our Section 5 also contains a limited discussion of n-solvable cobordisms).

We recall that a *capped grope of height k* is a grope of height k together with 2-discs attached along each standard symplectic basis curves of the top layer surfaces. The attached 2-discs are called *caps*, and the grope itself is called the *body*. We always assume that a capped grope embedded in a 4-manifold is framed.

We denote the exterior of a link L by X_L. If L is a link in S^3, η is an unknotted circle in S^3 disjoint from L, and K is a knot, then we denote the satellite link of L with

axis η and companion K by $L(\eta, K)$; this is the image of L under the homeomorphism $X_\eta \cup_\partial X_K \xrightarrow{\approx} S^3$, where the glueing identifies the longitude of η with the meridian of K, and vice versa.

Following [Cha, Definition 4.2], we call (L, η) a *satellite configuration of height k* if L is a link in S^3, η is an unknotted circle in S^3 disjoint from L, and the 0-linking parallel of η in $X_\eta = X_\eta \times \{0\}$ bounds a capped grope of height k embedded in $X_\eta \times [0, 1]$ with body disjoint from $L \times [0, 1]$. The caps should be embedded in $X_\eta \times [0, 1]$ but may intersect $L \times [0, 1]$.

Lemma 4.3 stated below describes how iterated satellite constructions using satellite configurations give us grope concordant links. The setup is as follows. Fix n. (To obtain Theorems 1.1 and 1.2, set $h = n + 2$.) Suppose that (L_0, η) is a satellite configuration of height $k \leq n$. (Later we will use the link L_I described above as L_0.) Suppose that (K_i, α_i) is a satellite configuration of height one, with K_i a slice knot, for $i = 0, \ldots, n - k - 1$. Let J_0^j be the connected sum of N_j copies of the knot described in [CT07, Figure 3.6], where $\{N_j\}$ is an increasing sequence of integers which will be specified later. (Indeed these will be given in terms of the Cheeger-Gromov bound on the ρ-invariants and, for the links of Theorem 1.1, in terms of the Kneser-Haken bound on the number of disjoint non-parallel incompressible surfaces. See Section 5, just before the proof of Proposition 5.3 and Section 4.4, just before Lemma 4.7.) Define $J_i^j := K_{i-1}(\alpha_{i-1}, J_{i-1}^j)$ inductively for $i = 1, \ldots, n - k$. Finally define $L_j := L_0(\eta, J_{n-k}^j)$.

Lemma 4.3 ([Cha, Proposition 4.7]). *The link L_j is height $n + 2$ grope concordant to L_0 for all j.*

Proof. The same as the proof of [Cha, Proposition 4.7], except that L_0 replaces the Hopf link in the last sentence. $\square$

The following observation on the satellite construction is useful.

Lemma 4.4. *If $L' = L(\eta, K)$ is obtained from L by a satellite construction, then L and L' have the same Milnor $\overline{\mu}$-invariants.*

Proof. It is well known that a satellite construction $L' = L(\eta, K)$ comes with an integral homology equivalence $f: (X_{L'}, \partial X_{L'}) \to (X_L, \partial X_L)$ which restricts to a homeomorphism on the boundary preserving longitudes and meridians (see e.g. [Cha10, Proof of Proposition 4.8], [CO13, Lemma 5.3]). As in [CFP, Lemma 2.1], by Stallings [Sta65], it follows that f induces an isomorphism $\pi_1(X_L)/\pi_1(X_L)_q \cong \pi_1(X_{L'})/\pi_1(X_{L'})_q$ that preserves the classes of meridians and longitudes for any q, and consequently L and L' have identical $\overline{\mu}$-invariants. $\square$

4.3. Satellite configuration of iterated Bing doubles

The goal of this section is to construct the examples needed for Theorem 1.2, and to show that they satisfy (1) and (2) of that theorem. Now we consider again the link L_I described in Section 4.1. Recall that $k(m) := \lfloor \log_2(m-1) \rfloor$ where $m = |I|$. Let η be the zero framed longitude of the component of L_I labelled with m, namely the component of the original Hopf link that is never Bing doubled in the construction of L_I.

Lemma 4.5.
   (1) *The pair (L_I, η) is a satellite configuration of height $k(m)$.*
   (2) *The curve η is nonzero in $\pi_1(L_I)/\pi_1(L_I)_m$.*

(3) *For any knot K, the link $L_I(\eta, K)$ has zero surgery manifold homeomorphic to the zero surgery manifold of L_I, via a homeomorphism which preserves the homotopy classes of the meridians.*

We remark that Lemma 4.5 (2) will be used in Section 5.

Proof. Denote $L := L_I$ for this proof.

(1) We go back to the construction of L, and construct the grope as we construct L. We begin with the Hopf link (i.e. $m = 2$), and the curve η as a longitude of L_2. We also begin with a thickened cap $D^2 \times [-1, 1]$, such that $\partial D^2 \times \{0\} = \eta$. This intersects the other component of the Hopf link in a single point.

Every time a component K is Bing doubled in the construction of L, we arrange that one of the clasps lies in $D^2 \times [-1, 1]$, and then replace the thickened cap that intersected K with a genus one capped surface with a single boundary component, whose body surface misses the new Bing doubled components, and such that each cap intersects one of the two new components. See Figure 2, which is somewhat reminiscent of a figure in [FQ90, Chapter 2.1].
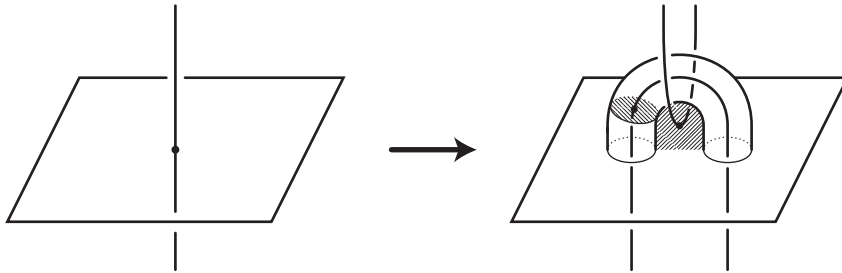

FIGURE 2. Replacing a cap with a capped surface

Since a complete binary tree of height $k(m)$ can be embedded in $T(m)$, we obtain a symmetric embedded capped grope of the required height, with the body lying in the link exterior X_L and the caps intersecting the link transversely.

(2) The nonvanishing of the Milnor invariant $\overline{\mu}_L(I)$ implies that all of the longitudes of L are nontrivial in $\pi_1(X_L)/\pi_1(X_L)_{|I|}$.

(3) A Kirby diagram for the 3-manifold M_L given by zero framed surgery on L can be produced by putting a 0 next to every component of L. If we perform a satellite construction with pattern K and with η as axis, this is equivalent to tying all the strands of L which intersect a disc D, whose boundary is η, in the knot K, with framing zero. In other words, replace the trivial string link in $D \times [0, 1]$ with the string link obtained by taking suitably many parallel copies of K.

But we can make a crossing change of these parallel copies of K at will, by performing handle slides, sliding the parallel strands over the zero framed 2-handle attached along the component parallel to η. This gives a Kirby presentation of a homeomorphic 3-manifold.

By making sufficiently many such crossing changes/handle slides, all the parallel strands which the satellite construction ties in the knot K can be unknotted, recovering the link L. Thus the zero surgery manifolds of the satellite link and the original link are homeomorphic. It is easy to see that the homotopy classes of the meridians of L are preserved under such homeomorphisms. $\square$

Now, let $n \geq k(|I|) = k(m)$. Let L_j be the links obtained by the construction just before Lemma 4.3, using our (L_I, η) as (L_0, η), and using the Stevedore satellite configuration described in [Cha, Figure 6], which for the reader's convenience is shown in Figure 3, as the (K_i, α_i). Then by Lemma 4.3 and Lemma 4.5 (1), the links L_j are height $n + 2$ grope concordant to the link $L_0 = L_I$.
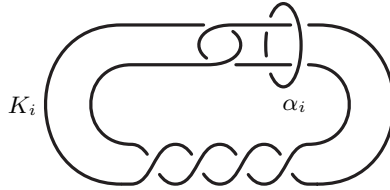

FIGURE 3. Stevedore satellite configuration (K_i, α_i).

Lemma 4.4 shows that the links L_j satisfy Theorem 1.2 (1). They also satisfy Theorem 1.2 (2) by Lemma 4.5 (3). We have also proved, in Lemma 4.3, the first part of Theorem 1.2 (3): the links L_j are mutually height $n + 2$ grope concordant. The second part of Theorem 1.2 (3), namely the failure of the links to be pairwise height $n + 3$ grope concordant, will be shown in Section 5.

4.4. Examples with non-homeomorphic zero surgery manifolds

In order to produce examples satisfying Theorem 1.1 (1), we alter the construction of Sections 4.2 and 4.3 to give examples with non-homeomorphic zero surgery manifolds. We consider the case of $m = 3$ and $I = 123$ only. Then the link $L := L_I$ described in Section 4.1 is the Borromean rings. Let η be the simple closed curve in $S^3 \smallsetminus L$ shown in Figure 4; x, y, and z denote the components of L.
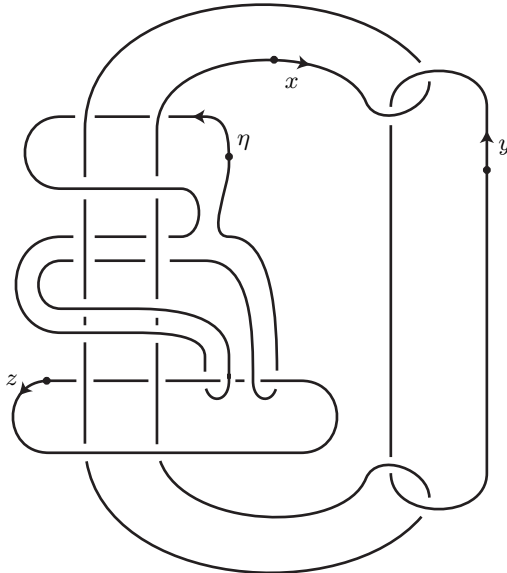

FIGURE 4. A satellite configuration on the Borromean rings

The pair (L, η) also has two of the properties stated in Lemma 4.5, for $m = 3$:

Lemma 4.6.
  (1) *The pair (L, η) is a satellite configuration of height one.*
  (2) *In $\pi_1(X_L)$, $\eta = [x,y][[x,y],x]$, where x, y, and z are the Wirtinger generators corresponding to the dotted arcs in Figure 4. Also, η is nontrivial in $\pi_1(X_L)/\pi_1(X_L)_3$.*

Here $[a,b]$ denotes the commutator $aba^{-1}b^{-1}$.

Proof. (1) Tubing the obvious disc bounded by η along the components of L that intersect it, we obtain a genus two surface V with boundary η which is shown in Figure 5. This is the body of the desired capped grope. The whole capped grope is the body taken together with the four caps shown in Figure 5 as shaded discs.
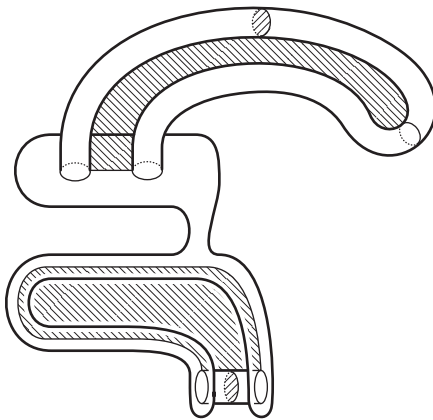

FIGURE 5. The capped grope bounded by η.

(2) The claim that $\eta = [x,y][[x,y],x]$ follows from a straightforward computation in terms of the Wirtinger generators, reading undercrossings of η starting from the dot on η in Figure 4. Since L has vanishing linking number, due to Milnor [Mil57] (see also Stallings [Sta65]), $\pi_1(X_L)/\pi_1(X_L)_3$ is isomorphic to F/F_3 where F is the free group generated by x, y, and z. Consequently, $[[x,y],x] \in \pi_1(X_L)_3$ and $[x,y] \notin \pi_1(X_L)_3$. From this the second conclusion follows. □

As in Section 4.3, we apply the construction described just before Lemma 4.3, using our (L, η) as the seed link (L_0, η) and using the Stevedore satellite configuration described in [Cha, Figure 6] (see our Figure 3) as (K_i, α_i) for $i = 0, \dots, n-2$ as above. Let the resulting links be the L_j. Then by Lemma 4.3, the L_j are height $n+2$ grope concordant to the Borromean rings L, so these satisfy the first part of Theorem 1.1 (3). The second part of Theorem 1.1 (3), on the failure of the links to be pairwise height $n+3$ grope concordant, will be shown in Section 5.

Furthermore, the links L_j satisfies the hypothesis of Theorem 1.3. First note that since our satellite operation does not change the knot type of the components, L_j has unknotted components. In particular the Arf invariants of the components vanish. Recall, from the proof of Lemma 4.4, that there is a homology equivalence $f \colon X_{L_j} \to X_{L_0}$ obtained from the satellite construction $L_j = L_0(\eta, J_{n-1}^j)$; indeed f is obtained by glueing the identity map of $X_{L_0 \sqcup \eta}$ with the standard homology equivalence $(X_{J_{n-1}^j}, \partial X_{J_{n-1}^j}) \to (S^1 \times D^2, S^1 \times S^1)$

along $S^1 \times S^1$. Since our curve $\eta \subset S^3 - L_0$ lies in the commutator subgroup of $\pi_1(S^3 - L_0)$, f is indeed a Λ-homology equivalence $X_{L_j} \to X_{L_0}$, by a Mayer-Vietoris argument. Filling it in with 3 solid tori, we obtain a Λ-homology equivalence $M_{L_j} \to T^3 = M_{L_0}$ as desired. Therefore, by applying Theorem 1.3, it follows that the links L_j satisfy Theorem 1.1 (2).

We need to confirm that the L_j satisfy Theorem 1.1 (1), namely the M_{L_j} are not homeomorphic. The underlying idea is as follows. Recall that L_j is defined by a satellite construction, starting with a knot J_0^j. In many cases, the JSJ pieces of the exterior of J_0^j become parts of the JSJ decomposition of M_{L_j}, so that the M_{L_j} have distinct JSJ decompositions. But a complete proof of this seems to require complicated arguments (a technical issue is that an essential torus might not be parallel to a JSJ torus, because of Seifert fibred pieces). In order to avoid these complications from JSJ decompositions, we will present a simpler argument using only the number of incompressible tori; this is enough for our purpose.

We need the following. The Kneser-Haken finiteness theorem [Hak61] states that for each 3-manifold M, there is a bound, say $C_{KH}(M)$, on the number of disjoint pairwise non-parallel incompressible surfaces that can be embedded in M. Recall that the knot J_0^j used in the construction of the link L_j is a connected sum of N_j knots, where $\{N_j\}$ was an increasing sequence to be specified (see the paragraph before Lemma 4.3). Here is the first requirement on the N_j: we choose the N_j inductively in such a way that $N_j > \max\{C_{KH}(M_{L_k}) \mid k = 0, 1, \ldots, j-1\}$.

Lemma 4.7. *The zero surgery manifolds M_{L_i} and M_{L_j} are not homeomorphic for $i \neq j$.*

Proof. Recall that $M_{L_0} = M_L$ is the 3-torus T^3. Consider $Y := M_L \smallsetminus \nu(\eta)$, where $\nu(\eta)$ is an open tubular neighbourhood of η. For notational convenience, denote the exterior of J_{n-1}^j by $X := X_{J_{n-1}^j}$. The 3-manifold M_{L_j} is obtained by glueing Y and X along their boundaries. Let $T = \partial Y = \partial X$ be the common boundary torus. Note that M_{L_0} can also be described in the same way, using $J_{n-1}^0 :=$ unknot; in this case, the torus T is compressible in M_{L_0} since X is a solid torus.

Claim. *For $j \geq 1$, the torus T is incompressible in Y.*

Using the claim, we will show that the 3-manifolds M_{L_j} are not pairwise homeomorphic. Suppose $j \geq 1$. Since the knot J_{n-1}^j is obtained from an iterated satellite construction with the first stage knot J_0^j a connected sum of N_j nontrivial knots, the exterior X of J_{n-1}^j has at least N_j incompressible tori, including the boundary T. Since $M_{L_j} = Y \cup_T X$ and T is incompressible in Y, it follows that there are N_j non-parallel incompressible tori in M_{L_j}. For any $k < j$, since $N_j > C_{KH}(M_{L_k})$, it follows that M_{L_j} is not homeomorphic to M_{L_k}.

Now, to complete the proof, we will show the claim. If there is an essential curve on T which bounds a disc in Y, then it must be a zero-linking longitude, say η', of η, since the meridian of η is a generator of $H_1(Y \smallsetminus \eta) = \mathbb{Z}^4$. By the following lemma, we have a contradiction. $\square$

Lemma 4.8. *The class of η' is nontrivial in the fundamental group $\pi_1(Y \smallsetminus \eta)$.*

Proof. We consider a Wirtinger presentation of $\pi_1(Y \smallsetminus \eta)$ given as follows: it has 24 generators denoted by $x_1, \ldots, x_{24}$ associated to arcs in Figure 4. Here $(x_1, \ldots, x_{10})$, (x_{11}, x_{12}), $(x_{13}, \ldots, x_{16})$, and $(x_{17}, \ldots, x_{24})$ are those associated to the arcs of the components x, y, y, and η, respectively. In each component, the arc with a dot on it is the first one, and

other arcs are ordered along the orientation. There are 27 relators:

$$x_1 x_{11} \overline{x}_1 \overline{x}_{12},\ x_{11} x_1 \overline{x}_{11} \overline{x}_2,\ x_2 x_{18} \overline{x}_2 \overline{x}_{19},\ x_{19} x_2 \overline{x}_{19} \overline{x}_3,\ x_3 x_{20} \overline{x}_3 \overline{x}_{21},$$

$$x_3 x_{23} \overline{x}_3 \overline{x}_{22},\ x_{22} x_4 \overline{x}_{22} \overline{x}_3,\ x_{21} x_4 \overline{x}_{21} \overline{x}_5,\ x_5 x_{16} \overline{x}_5 \overline{x}_{13},\ x_{13} x_5 \overline{x}_{13} \overline{x}_6,$$

$$x_{11} x_7 \overline{x}_{11} \overline{x}_6,\ x_7 x_{11} \overline{x}_7 \overline{x}_{12},\ x_{13} x_8 \overline{x}_{13} \overline{x}_7,\ x_8 x_{16} \overline{x}_8 \overline{x}_{15},\ x_{21} x_9 \overline{x}_{21} \overline{x}_8,$$

$$x_{22} x_9 \overline{x}_{22} \overline{x}_{10},\ x_{10} x_{23} \overline{x}_{10} \overline{x}_{24},\ x_{10} x_{20} \overline{x}_{10} \overline{x}_{19},\ x_{19} x_1 \overline{x}_{19} \overline{x}_{10},\ x_1 x_{18} \overline{x}_1 \overline{x}_{17},$$

$$x_{15} x_{22} \overline{x}_{15} \overline{x}_{21},\ x_{22} x_{15} \overline{x}_{22} \overline{x}_{14},\ x_{24} x_{13} \overline{x}_{24} \overline{x}_{14},\ x_{13} x_{24} \overline{x}_{13} \overline{x}_{17},$$

$$\overline{x}_{11} \overline{x}_{19} x_{22} \overline{x}_{21} \overline{x}_{13} x_{11} x_{13} x_{21} \overline{x}_{22} x_{19},\ \overline{x}_1 x_7,\ \overline{x}_{24} x_{22} x_8 \overline{x}_5.$$

Indeed, the first 24 are the standard Wirtinger relators for the 4-component link $L \sqcup \eta$ (thus one of these may be omitted), and the last 3 relators arise from the zero surgery performed along L. It is straightforward to read off the curve η':

$$\eta' = x_1 \overline{x}_2 x_{10} \overline{x}_3 x_{15} x_3 \overline{x}_{10} \overline{x}_{13}.$$

We define a representation $\rho\colon \pi_1(Y \smallsetminus \eta) \to \mathrm{SL}(2, \mathbb{Z}_5)$ by mapping the above 24 generators, respectively, to:

$$\begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix},\ \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix},\ \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix},\ \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix},\ \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},\ \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix},\ \begin{bmatrix} 4 & 0 \\ 4 & 4 \end{bmatrix},\ \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix},\ \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix},\ \begin{bmatrix} 4 & 1 \\ 4 & 0 \end{bmatrix},\ \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 4 & 3 \\ 2 & 3 \end{bmatrix},\ \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix},\ \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix},\ \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix},\ \begin{bmatrix} 0 & 1 \\ 4 & 2 \end{bmatrix},\ \begin{bmatrix} 0 & 1 \\ 4 & 2 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ \begin{bmatrix} 3 & 4 \\ 4 & 4 \end{bmatrix},\ \begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix},\ \begin{bmatrix} 4 & 4 \\ 4 & 3 \end{bmatrix},\ \begin{bmatrix} 0 & 4 \\ 1 & 2 \end{bmatrix},\ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

It can be verified that all the relators are sent to the identity, by a straightforward computation. Also, we have that

$$\rho(\eta') = \begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix}$$

is not the identity. (We found the representation ρ using a computer program that performs a brute force search in the representation variety.) This completes the proof. □

5. Grope concordance and amenable signatures

In this section we show that the links described in Sections 4.3 and 4.4 are not height $n + 3$ grope concordant, by using amenable signature obstructions from [Cha]. In fact the amenable signatures we use are obstructions to being n-solvably cobordant, which is a relative analogue for manifolds with boundary, or bordered manifolds, of the notion of n-solvability of [COT03]. For our purpose it suffices to consider the case of link exteriors; an *n-solvable cobordism* between the exteriors X and X' of two links with the same number of components is a 4-manifold W with $\partial W = X \cup_\partial -X'$ satisfying the conditions described in [Cha, Definition 2.8], where the boundary tori of X and X' are identified along the zero framing. Since we do not use the defining condition right now, instead of spelling it out here, we begin with its relationship to grope concordance. The following theorem originates from [COT03, Theorem 8.11], and was given in our context in [Cha].

Theorem 5.1 ([Cha] Theorems 2.16 and 2.13, and Remark 2.11)**.** *If two links are height $n+2$ grope concordant then their exteriors are n-solvably cobordant as bordered 3-manifolds.*

As our key ingredient to detect non-solvably-cobordant 3-manifolds and therefore non-grope-concordant links, we will use the Amenable Signature Theorem, which was first introduced in [CO12] for homology cobordism of closed 3-manifolds and then generalised to n-solvable cobordisms of bordered 3-manifolds in [Cha]. We state a special case which will be sufficient for our purpose. For a closed 3-manifold M and a homomorphism $\phi\colon \pi_1(M) \to G$, denote the von Neumann-Cheeger-Gromov ρ-invariant by $\rho^{(2)}(M, \phi) \in \mathbb{R}$. See e.g. [COT03, Section 5] as well as [CW03, Har08, Cha08, CO12] for definitions and useful properties of $\rho^{(2)}(M, \phi)$. Precise references for the properties that we need will be recalled as we go along.

Theorem 5.2 (A special case of [Cha, Amenable Signature Theorem 3.2]). *Suppose W is an $(n+1)$-solvable cobordism between two bordered 3-manifolds X and X', and G admits a subnormal series*

$$G = G_0 \supset G_1 \supset \cdots \supset G_n \supset G_{n+1} = \{e\}$$

with each quotient G_i/G_{i+1} torsion-free abelian. Then for any $\phi\colon \pi_1(X \cup_\partial -X') \to G$ which factors through $\pi_1(W)$, we have $\rho^{(2)}(X \cup_\partial -X', \phi) = 0$.

Recall that in our construction of the links L_j the knot J_0^j was the connected sum of N_j copies of Cochran-Teichner's knot, say J. Now we proceed to specify the integers N_j. Denote, by $\rho^{(2)}(K) := \int_{S^1} \sigma_K(\omega)\, d\omega$, the integral of the Levine-Tristram signature function over the circle normalized to length one. For M_K the zero surgery on K and $\phi_0\colon \pi_1(M_K) \to \mathbb{Z}$ the abelianisation homomorphism, $\rho^{(2)}(K) = \rho^{(2)}(M_K, \phi_0)$ by [COT04, Proposition 5.1]. We have $\rho^{(2)}(J_0^j) = N_j \rho^{(2)}(J) = 4N_j/3$ by additivity under connected sum and [CT07, Lemma 4.5]. Due to Cheeger and Gromov [CG85], for any closed 3-manifold Y there is a constant $C_Y > 0$ such that $|\rho^{(2)}(Y, \psi)| < C_Y$ for any ψ. From now on we abbreviate $\ell := n - k(m)$. Define

$$R := C_{X_{L_0} \cup_\partial -X_{L_0}} + 2 \sum_{i=0}^{\ell-1} C_{M_{K_i}}.$$

We choose the large integers N_j inductively in such a way that

$$N_j > 3R/4 + \max\{N_k \mid k < j\}.$$

Then we have

$$\rho^{(2)}(J_0^j) > R + \rho^{(2)}(J_0^k)$$

whenever $j > k$. For Theorem 1.1, we make these choices so that the condition in the preamble to Lemma 4.7 relating to the Kneser-Haken bound is simultaneously satisfied.

Now we start the proof that our links L_j are not height $n+3$ grope concordant to one another. Let X and X' be the exteriors of L_j and L_k, respectively. To distinguish them in the notation, we denote the axis curve η in X by η_j, and we denote the corresponding axis curve in X' by η_k.

Recall that $m = |I|$ and that $k(m) = \lfloor \log_2(m-1) \rfloor$. Also note that $k(m) + 1 = \lceil \log_2(m) \rceil$. By Theorem 5.1, it suffices to show the following:

Proposition 5.3. *For $n \geq k(m)$, the bordered 3-manifolds X and X' are not $(n+1)$-solvably cobordant when $j \neq k$.*

By Theorem 5.1, it then follows that our links L_j and L_k are not height $n+3$ grope concordant when $j \neq k$.

Proof. The proof proceeds almost identically to that of [Cha, Theorem 4.8], which combines the Amenable Signature Theorem of [Cha] with a higher order Blanchfield duality argument for a certain 4-dimensional cobordism introduced in [CHL09] (see our W_0 below). So we will give an outline for our case and discuss differences from [Cha, Theorem 4.8].

Suppose W is an $(n+1)$-solvable cobordism with $\partial W = X \cup_\partial -X'$. Similarly to [Cha, Section 4.3] (see the paragraph entitled "Cobordism associated to an iterated satellite construction"), we consider a cobordism V with

$$\partial V = M_{J_0^j} \sqcup -M_{J_0^k} \sqcup M_{K_0} \sqcup -M'_{K_0} \sqcup \cdots \sqcup M_{K_{\ell-1}} \sqcup -M'_{K_{\ell-1}}$$
$$\sqcup (X_{L_0} \cup_\partial -X_{L_0}) \sqcup -(X \cup_\partial -X')$$

which is built by stacking cobordisms associated to satellite constructions [CHL09, p. 1429], where M'_{K_i} is a copy of M_{K_i}, and then construct a cobordism W_0 with

$$\partial W_0 = M_{J_0^j} \sqcup -M_{J_0^k} \sqcup M_{K_0} \sqcup -M'_{K_0} \sqcup \cdots \sqcup M_{K_{\ell-1}} \sqcup -M'_{K_{\ell-1}} \sqcup (X_{L_0} \cup_\partial -X_{L_0})$$

by attaching V to W along $X \cup_\partial -X'$. We omit the detailed construction of V and W_0 but state a couple of useful facts which can be verified as in [Cha, Section 4.3]. Let $\{\mathcal{P}^r G\}$ be the rational derived series of a group G, i.e. $\mathcal{P}^0 G := G$ and $\mathcal{P}^{r+1} G$ is the kernel of $\mathcal{P}^r G \to H_1(\mathcal{P}^r G; \mathbb{Q})$. Let ϕ_0 be the quotient map $\pi_1(W_0) \to G := \pi_1(W_0)/\mathcal{P}^{n+1}\pi_1(W_0)$. Also we denote by ϕ_0 the restrictions of ϕ_0 to the components of ∂W_0 and to $W \subset W_0$, as an abuse of notation. Then we have the following facts.

(1) $\rho^{(2)}(M_{J_0^j}, \phi_0) - \rho^{(2)}(M_{J_0^k}, \phi_0) + \rho^{(2)}(X_{L_0} \underset{\partial}{\cup} -X_{L_0}, \phi_0)$
$$+ \sum_{i=0}^{\ell-1} \rho^{(2)}(M_{K_i}, \phi_0) - \sum_{i=0}^{\ell-1} \rho^{(2)}(M'_{K_i}, \phi_0) = \rho^{(2)}(X \underset{\partial}{\cup} -X', \phi_0).$$

(2) The image of the meridian of J_0^j in $M_{J_0^j} \subset \partial W_0$ under ϕ_0 is a nontrivial element in the torsion-free abelian subgroup $\mathcal{P}^n \pi_1(W)/\mathcal{P}^{n+1}\pi_1(W)$ of G. Similarly for k instead of j.

The proof of (1) is completely identical to that given in [Cha, Section 4.3] (see the paragraphs entitled "Cobordism associated to an iterated satellite construction" and "applications of Amenable Signature Theorem"): briefly, the $\rho^{(2)}$-invariant of ∂W_0, which is the left hand side of (1), is equal to the L^2-signature defect of $W_0 = V \cup_{X \cup_\partial -X'} W$ (this is a standard fact from index theory, or can be taken as the definition of $\rho^{(2)}$). It turns out that V has no contribution to the L^2-signature defect by [CHL09, Lemma 2.4]. So the left hand side of (1) is equal to the L^2-signature defect of W, which is the $\rho^{(2)}$-invariant of ∂W, namely the right hand side of (1).

The proof of (2) is almost identical to that given in [Cha, Theorem 4.10]. Only the following change is required: in the initial step of the inductive argument in [Cha, Theorem 4.10], it was shown that the image of (a parallel copy of) $\eta \subset X \subset \partial W$ is nontrivial under the quotient map $\pi_1(W) \to \pi_1(W)/\mathcal{P}^2 \pi_1(W)$ (see the fourth paragraph of [Cha, Proof of Theorem 4.10]) using a Blanchfield duality argument.

In our case, instead we use Lemma 5.4 below, which is a generalisation of [CFP, Lemma 3.5], to show that the image of η is nontrivial in $\pi_1(W)/\mathcal{P}^{(k(m)+1)}\pi_1(W)$. The argument used in Lemma 5.4 is essentially an application Dwyer's theorem.

Lemma 5.4. *If W is an n-solvable cobordism between two link exteriors (or more generally bordered 3-manifolds) X and X', then the inclusions induce isomorphisms*

$$\pi_1(X)/\pi_1(X)_q \cong \pi_1(W)/\pi_1(W)_q \cong \pi_1(X')/\pi_1(X')_q.$$

for $q \leq 2^n + 1$.

Proof. Recall Dwyer's theorem [Dwy75]: if $f \colon X \to Y$ induces an isomorphism $H_1(X; \mathbb{Z}) \cong H_1(Y; \mathbb{Z})$ and an epimorphism

$$H_2(X; \mathbb{Z}) \longrightarrow H_2(Y; \mathbb{Z}) / \operatorname{Im}\{H_2(Y; \mathbb{Z}[\pi_1(W)/\pi_1(W)_q]) \to H_2(Y; \mathbb{Z})\},$$

then f induces an isomorphism $\pi_1(X)_q/\pi_1(X)_{q+1} \cong \pi_1(Y)_q/\pi_1(Y)_{q+1}$.

In our case, by the definition of an n-solvable cobordism [Cha, Definition 2.8], we have $H_1(X; \mathbb{Z}) \cong H_1(W; \mathbb{Z}) \cong H_1(X'; \mathbb{Z})$, where the isomorphisms are induced by the inclusion maps. Also, by [Cha, Definition 2.8] there are elements $\ell_1, \ldots, \ell_r, d_1, \ldots, d_r$ lying in $H_2(W; \mathbb{Z}[\pi_1(W)/\pi_1(W)^{(n)}])$ such that the images of ℓ_i and d_j generate $H_2(W; \mathbb{Z})$. Since $\pi_1(W)^{(n)}$ is contained in $\pi_1(W)_{2^n}$, the H_2 condition of Dwyer's theorem is satisfied. Therefore it follows that

$$\pi_1(X)_q/\pi_1(X)_{q+1} \cong \pi_1(W)_q/\pi_1(W)_{q+1} \cong \pi_1(X')_q/\pi_1(X')_{q+1}$$

for $q \leq 2^n$ by Dwyer's theorem. From this the desired conclusion follows by the five lemma. $\qquad\square$

Recall that $\eta \subset X$ represents a nontrivial element in $\pi_1(X)/\pi_1(X)_m$ by Lemma 4.5 (2). Since the above isomorphisms preserve longitudes (and meridians), $\eta_j \subset X$ represents a nontrivial element in $\pi_1(W)/\pi_1(W)_m$. Since L_j has vanishing Milnor invariants of length less than $|I| = m$, we have $\pi_1(X)/\pi_1(X)_m \cong F/F_m$, where F is the free group with rank m, by [Mil57, Theorem 4]. Consequently $\pi_1(W)/\pi_1(W)_m$ is torsion free.

We note that for any group π, we have $\pi^{(k(q)+1)} = \pi^{(\lceil \log_2(q) \rceil)} \subseteq \pi_q$. Therefore there is a quotient map $\pi_1(W)/\pi_1(W)^{(k(m)+1)} \to \pi_1(W)/\pi_1(W)_m$, and this map factors through $\pi_1(W)/\mathcal{P}^{(k(m)+1)}\pi_1(W)$ by the definition of $\mathcal{P}^{(k(m)+1)}$ and the fact that the codomain is torsion free. Since η_j is nontrivial in $\pi_1(W)/\pi_1(W)_m$, η_j is also nontrivial in $\pi_1(W)/\mathcal{P}^{(k(m)+1)}\pi_1(W)$. By replacing j with k and X with X' we obtain the corresponding fact for η_k in X'.

To complete the proof of Proposition 5.3, we proceed as in [Cha, Section 4.3]. Observe that for the normal subgroups $G_i := \mathcal{P}^i \pi_1(W_0)/\mathcal{P}^{n+1}\pi_1(W_0)$ of our G, the quotient G_i/G_{i+1} is torsion-free abelian. So by Amenable Signature Theorem 5.2 we have $\rho^{(2)}(X \cup_\partial -X') = 0$. Since the curve η_j represents a nontrivial element in a torsion-free abelian normal subgroup of G, the image of $\pi_1(M_{J_0^j})$ in G under ϕ_0 is the infinite cyclic group. By L^2-induction (see e.g. [CG85, page 8 (2.3)], [COT03, Proposition 5.13]) and [COT04, Proposition 5.1], we have $\rho^{(2)}(M_{J_0^j}, \phi_0) = \rho^{(2)}(J_0^j)$, and similarly for J_0^k. Now combining these two facts with (1) we obtain:

$$(3) \quad \rho^{(2)}(J_0^j) - \rho^{(2)}(J_0^k) + \rho^{(2)}(X_{L_0} \underset{\partial}{\cup} -X_{L_0}, \phi_0)$$

$$+ \sum_{i=0}^{\ell-1} \rho^{(2)}(M_{K_i}, \phi_0) - \sum_{i=0}^{\ell-1} \rho^{(2)}(M'_{K_i}, \phi_0) = 0.$$

Recall that

$$\left| \rho^{(2)}(X_{L_0} \underset{\partial}{\cup} -X_{L_0}, \phi_0) + \sum_{i=0}^{\ell-1} \rho^{(2)}(M_{K_i}, \phi_0) - \sum_{i=0}^{\ell-1} \rho^{(2)}(M'_{K_i}, \phi_0) \right|$$

$$< R := C_{X_{L_0} \cup_\partial -X_{L_0}} + 2 \sum_{i=0}^{\ell-1} C_{M_{K_i}},$$

and in the preamble to Proposition 5.3, we chose N_j in such a way that

$$\left| \rho^{(2)}(J_0^k) - \rho^{(2)}(J_0^j) \right| > R$$

whenever $k \neq j$. Therefore (3) implies that $j = k$. Thus the existence of the $(n+1)$-solvable cobordism W implies that $j = k$, which is the contrapositive of the desired statement. $\square$

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