SMOOTHLY SLICE KNOTS WITH SMOOTHLY NON-APPROXIMABLE TOPOLOGICAL SLICE DISCS

MIN HOON KIM AND MARK POWELL

Abstract. We construct infinitely many smoothly slice knots having topological slice discs that are non-approximable by smooth slice discs.

Let K be a smoothly slice knot in S^3 and let $\mathcal{D}$ be a topological slice disc for K in D^4, that is, $\mathcal{D}$ is a locally flat, topologically embedded disc in D^4 such that $\partial(D^4, \mathcal{D}) = (S^3, K)$. We say that $\mathcal{D}$ is *smoothly non-approximable* if there exists $\varepsilon > 0$ such that for every topological slice disc D that is topologically isotopic rel. boundary to $\mathcal{D}$, the ε-neighbourhood $\nu_\varepsilon(D)$ of D does not contain any smooth slice disc for K.

Theorem. *There exist infinitely many smoothly slice knots K_m in S^3, each of which admits a smoothly non-approximable topological slice disc.*

This answers a refined version of mathoverflow question https://mathoverflow.net/questions/419693, asked by M. Winter.

It is interesting to contrast our theorem with Venema's results in [Ven78, Ven81], which imply that every topological slice disc can be approximated by smoothly embedded disc if the boundary is allowed to move. In particular Venema's smooth approximation could have a different knot type on the boundary, or could even lie in the interior of D^4.

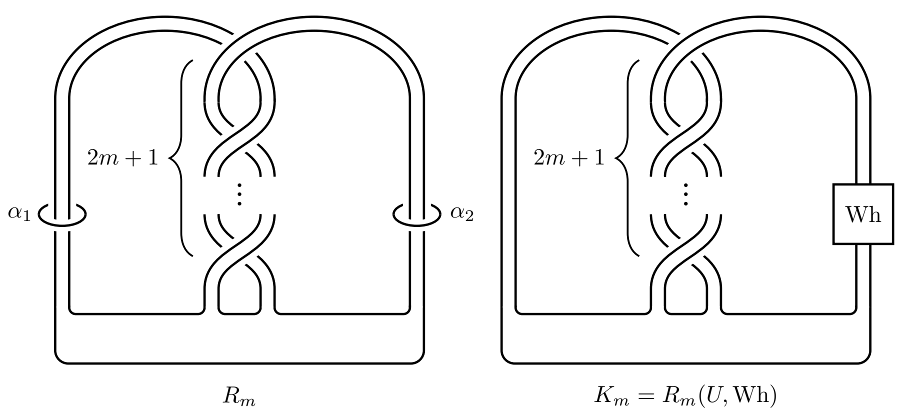

Figure 1. The knots R_m and K_m. There are $2m+1$ crossings between the bands. On the left, the knot R_m has unknots α_1 and α_2 in its complement, each linking one of the bands of the obvious Seifert surface for R_m. Perform a satellite operation on α_2 using Wh to obtain $K_m := R_m(U, \mathrm{Wh})$, as shown on the right.

2020 *Mathematics Subject Classification.* 57K10, 57K18, 57N70, 57R40.
Key words and phrases. Slice discs, Heegaard-Floer invariants.

Proof. For an odd integer $m \geq 1$, let $\mathrm{Wh} := \mathrm{Wh}(T_{2,3})$ denote the positive-clasped zero-twisted Whitehead double of the right-handed trefoil, let U denote the unknot, and let $K_m = R_m(U, \mathrm{Wh})$, as defined in the caption of Figure 1. Since α_1 and α_2 link R_m trivially, the Alexander polynomial is unchanged by the satellites [Sei50], i.e. $\Delta_{K_m}(t) = \Delta_{R_m}(t)$. Also $\Delta_{R_m}(t) = ((m+1)t-m)(mt-(m+1))$. Thus if $m \neq n$, then $\Delta_{K_m}(t) \neq \Delta_{K_n}(t)$, and hence the knots K_m are mutually distinct.

The configuration $(R_m, \alpha_1, \alpha_2)$ is often considered as an operator $R_m(-,-)$ that takes two knots as input, with $R_m(J_1, J_2)$ obtained by performing the satellite operation on α_i with J_i, for $i = 1, 2$. Since taking $J_1 = U$ has no effect, this explains our definition in the caption to Figure 1.

For each m, the knot K_m is smoothly slice, as can be seen by 'cutting' the right hand band in the Seifert surface. From the embedded Morse theory perspective, this corresponds to a saddle move, which results in a 2-component unlink, which can then be capped off by two smooth discs, corresponding to minima, to obtain a smooth slice disc for K_m.

Let $\mathcal{D}_m$ be a topological slice disc for K_m obtained by cutting the left band and capping with two parallel copies of a topological slice disc for Wh, which exists since $\Delta_{\mathrm{Wh}}(t) = 1$ [Fre84]. We claim that $\mathcal{D}_m$ is smoothly non-approximable for each m.

For a contradiction, supposed that for every $\varepsilon > 0$, there exists a topological slice disc D_m for K that is topologically isotopic rel. boundary to $\mathcal{D}_m$, such that the ε-neighbourhood $\nu_\varepsilon(D_m)$ contains a smooth slice disc $D'_{m,\varepsilon}$ for K. Let $N(D_m)$ be an open tubular neighbourhood of the topologically locally flat slice disc D_m, which exists by [FQ90, Theorem 9.3]. Then $D^4 \smallsetminus N(D_m)$ is the exterior of D_m, and $\partial(D^4 \smallsetminus N(D_m)) \cong M_{K_m}$, the 3-manifold obtained by zero-framed surgery on K_m. Choose $\varepsilon > 0$ small enough such that $\nu_\varepsilon(D_m) \subseteq N(D_m)$. We have inclusion maps

$$M_{K_m} \xrightarrow{i_1} D^4 \smallsetminus N(D_m) \xrightarrow{i_2} D^4 \smallsetminus \nu_\varepsilon(D_m) \xrightarrow{i_3} D^4 \smallsetminus D'_{m,\varepsilon}.$$

It follows from Alexander duality that these inclusion maps induce isomorphisms

$$H_1(M_{K_m};\mathbb{Z}) \xrightarrow{\cong} H_1(D^4 \smallsetminus N(D_m);\mathbb{Z}) \xrightarrow{\cong} H_1(D^4 \smallsetminus D'_{m,\varepsilon};\mathbb{Z}).$$

Since $H_1(M_{K_m};\mathbb{Z}) \cong \mathbb{Z}$ we obtain consistent coefficient systems $\pi_1(X) \to \mathbb{Z} \cong \langle t \rangle$ for all $X \in \{M_{K_m}, D^4 \smallsetminus N(D_m), D^4 \smallsetminus D'_{m,\varepsilon}\}$, and can therefore consider homology with $\mathbb{Q}[t^{\pm 1}]$ coefficients. Consider the corresponding composition

$$\iota\colon H_1(M_{K_m};\mathbb{Q}[t^{\pm 1}]) \xrightarrow{(i_1)_*} H_1(D^4 \smallsetminus N(D_m);\mathbb{Q}[t^{\pm 1}]) \xrightarrow{(i_3 \circ i_2)_*} H_1(D^4 \smallsetminus D'_{m,\varepsilon};\mathbb{Q}[t^{\pm 1}]).$$

We need to understand the kernel of this map $\iota := (i_3 \circ i_2 \circ i_1)_*$.

By our description of D_m, $(i_1)_*(\alpha_1) = 0$ and hence $\langle \alpha_1 \rangle \subseteq \ker \iota$, where we abuse notation and use α_1 to also denote the corresponding curve in the complement of K_m. Since $D'_{m,\varepsilon}$ is a slice disc, the kernel of ι is a metaboliser for the Blanchfield form of M_{K_m} (see e.g. [COT03, Theorem 4.4] or [Hil12, Theorem 2.4]). The Blanchfield form is a Hermitian, sesquilinear, nonsingular pairing

$$\mathrm{Bl}\colon H_1(M_{K_m};\mathbb{Q}[t^{\pm 1}]) \times H_1(M_{K_m};\mathbb{Q}[t^{\pm 1}]) \to \mathbb{Q}(t)/\mathbb{Q}[t^{\pm 1}],$$

and we say that $P \subseteq H_1(M_{K_m};\mathbb{Q}[t^{\pm 1}])$ is a metaboliser if $P = P^\perp$. We can compute, using the presentation matrix $tV_m - V_m^T$, where $V_m = \left[\begin{smallmatrix} 0 & m+1 \\ m & 0 \end{smallmatrix}\right]$ is a Seifert matrix for K_m, that

$$H_1(M_{K_m};\mathbb{Q}[t^{\pm 1}]) \cong \frac{\mathbb{Q}[t^{\pm 1}]}{(mt-(m+1))} \oplus \frac{\mathbb{Q}[t^{\pm 1}]}{((m+1)t-m)} \cong \mathbb{Q} \oplus \mathbb{Q}.$$

The first summand is generated by α_1, and the other is generated by α_2.

We claim that $\langle \alpha_1 \rangle = \ker(\iota)$. To see this, we consider $H_1(M_{K_m};\mathbb{Q}[t^{\pm 1}])$ as a $\mathbb{Q}$-vector space. Then $\langle \alpha_1 \rangle \subseteq \ker(\iota) \subseteq H_1(M_{K_m};\mathbb{Q}[t^{\pm 1}])$ implies that $1 \leq \dim_{\mathbb{Q}} \ker(\iota) \leq 2$. However $\ker(\iota)$ is a metaboliser for the Blanchfield form, and the Blanchfield form is nonsingular, so we cannot have $\ker \iota = H_1(M_{K_m};\mathbb{Q}[t^{\pm 1}])$. Hence $\dim_{\mathbb{Q}} \ker(\iota) = 1$. Since $\langle \alpha_1 \rangle \subseteq \ker(\iota)$ and their dimensions as $\mathbb{Q}$-vector spaces are equal, we deduce that $\langle \alpha_1 \rangle = \ker(\iota)$ as claimed.

Now we apply the d-invariant from Heegaard-Floer homology, together with calculations of Cha-Kim [CK21] and Cha [Cha21], to deduce that the smooth disc $D'_{m,\varepsilon}$ cannot exist. For the definition of the d-invariant we refer to Ozsváth–Szabó [OS03, Definition 4.1]. It suffices for our purposes to know that given a rational homology 3-sphere Y and a spin^c structure $\mathfrak{s}$ on Y, the d-invariant is a rational number $d(Y,\mathfrak{s}) \in \mathbb{Q}$.

Let Σ_r be the r-fold cyclic cover of S^3 branched along K_m, which is a rational homology 3-sphere for all prime $r \geq 2$. Then α_1 and α_2 lift to homology classes in $H_1(\Sigma_r;\mathbb{Z})$, say x_1 and x_2 respectively. Since $H_1(\Sigma_r;\mathbb{Z}) \cong H_1(M_{K_m};\mathbb{Z}[t^{\pm 1}])/(t^r-1)$, we have that

$$H_1(\Sigma_r;\mathbb{Z}) \cong \mathbb{Z}/_{(m+1)^r-m^r} \oplus \mathbb{Z}/_{(m+1)^r-m^r},$$

where the summands are generated by x_1 and x_2 respectively. In particular, Σ_r has a unique spin structure $\mathfrak{s}_{\Sigma_r}$ since it is a $\mathbb{Z}/2$-homology 3-sphere.

By [CK21, Lemma 5.2] and [Cha21, p. 17, Assertion], the condition $\langle \alpha_1 \rangle = \ker(\iota)$ implies that there exists a prime r such that the kernel of the inclusion induced map $H_1(\Sigma_r;\mathbb{Z}) \to H_1(V_r;\mathbb{Z})$ is generated by x_1, where V_r is the r-fold cyclic cover of D^4 branched along $D'_{m,\varepsilon}$. Since $D'_{m,\varepsilon}$ is a smooth slice disc, [GRS08, Theorem 1.1] implies that

$$d(\Sigma_r, \mathfrak{s}_{\Sigma_r} + k\widehat{x}_1) = 0$$

for all $k \in \mathbb{Z}$. Here the spin structure $\mathfrak{s}_{\Sigma_r}$ uniquely determines a spin^c structure, and $\widehat{x}_1 := PD^{-1}(x_1) \in H^2(\Sigma_r;\mathbb{Z})$ is the Poincaré dual of x_1. Then the spin^c structure $\mathfrak{s}_{\Sigma_r} + k\widehat{x}_1$ is defined by noting that spin^c structures on Σ_r are a torsor over $H^2(\Sigma_r;\mathbb{Z})$.

We obtain the desired contradiction since there exists an integer $k \in \mathbb{Z}$ such that

$$d(\Sigma_r, \mathfrak{s}_{\Sigma_r} + k\widehat{x}_1) \neq 0,$$

by [CK21, Theorem 5.4] and [Cha21, Lemma 4.1]. This completes the proof. □

Remark. Our examples are topologically doubly slice and smoothly slice, but not smoothly doubly slice. Infinitely many such knots were first constructed by Meier [Mei15], though the earlier d-invariant arguments of Cochran, Harvey, and Horn [CHH13, pp. 2140–1] implicitly show the existence of such a knot. These works do not suffice to prove our theorem, because for a doubly slice knot K each individual slice disc has the property that for each r, the map $H_1(\Sigma_r;\mathbb{Z}) \to H_1(V_r;\mathbb{Z})$ is surjective. Moreover the kernels of these maps, for the two V_r corresponding to each slice disc, are two complementary summands of $H_1(\Sigma_r;\mathbb{Z})$. Then, as in [CHH13, Mei15], one can restrict to a single prime r, and compute the invariants $d(\Sigma_r,\mathfrak{s})$ to obtain a contradiction. In our case, we cannot assume any 'homology ribbon' property, so we cannot apply the same argument. The putative slice disc $D'_{m,\varepsilon}$ and the actual smooth slice disc for K_m from the top of page 2 could a priori share the same kernel. We apply the results of [CK21, Cha21], which circumvent this subtle issue by considering d-invariants of Σ_r for arbitrarily large primes r. Similar d-invariant arguments were applied in Kim-Livingston [KL22] to construct an infinite family of topologically slice knots that are not smoothly concordant to their reverses.

Remark. The fact that an isotopy from $\mathcal{D}$ to D is permitted in the definition of smoothly non-approximable distinguishes our examples from topological slice discs constructed by the following argument, told to us by Robert Gompf. Start with a smooth slice disc Δ for a knot K. For any $\delta > 0$, in the δ-neighbourhood of Δ we may construct a capped tower with one storey T, as in [FQ90], with attaching circle K, such that T does not contain any smooth disc with boundary K. Such a T exists, as shown in e.g. [Gom84]. For some $\varepsilon < \delta$, there is a topological slice disc D for K and $\nu_\varepsilon(D)$ contained within T, with the same attaching circle [Fre82, FQ90]. There cannot be any smooth slice disc for K within $\nu_\varepsilon(D)$, or else there would also be such a smooth slice disc lying within T. This provides an alternative answer to Winter's original question, which did not allow an isotopy.

However, it can be shown that for every $\varepsilon > 0$, D is topologically isotopic to a disc that contains Δ in its ε-neighbourhood, and hence D fails to be smoothly non-approximable in the sense of our definition.

Acknowledgements. We are grateful to Martin Winter for asking the question on mathoverflow that led to this note, and to both Chuck Livingston and an anonymous referee for helpful comments. We are also grateful to Bob Gompf for pointing out an alternative answer to Winter's question discussed in the final remark, which led us to allow an isotopy in the definition of smoothly non-approximable. Min Hoon Kim was partially supported by the Samsung Science and Technology Foundation (SSTF-BA2202-01) and National Research Foundation grant 2021R1C1C1012939. Mark Powell was partially supported by EPSRC New Investigator grant EP/T028335/2.

References

[Cha21] Jae Choon Cha, *Primary decomposition in the smooth concordance group of topologically slice knots*, Forum Math. Sigma **9** (2021), Paper No. e57, 37.

[CHH13] Tim D. Cochran, Shelly Harvey, and Peter Horn, *Filtering smooth concordance classes of topologically slice knots*, Geom. Topol. **17** (2013), no. 4, 2103–2162.

[CK21] Jae Choon Cha and Min Hoon Kim, *The bipolar filtration of topologically slice knots*, Adv. Math. **388** (2021), Paper No. 107868, 32.

[COT03] Tim D. Cochran, Kent E. Orr, and Peter Teichner, *Knot concordance, Whitney towers and L^2-signatures*, Ann. of Math. (2) **157** (2003), no. 2, 433–519.

[FQ90] Michael H. Freedman and Frank Quinn, *Topology of* 4*-manifolds*, Princeton Mathematical Series, vol. 39, Princeton University Press, Princeton, NJ, 1990.

[Fre82] Michael H. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geom. **17** (1982), no. 3, 357–453.

[Fre84] Michael H. Freedman, *The disk theorem for four-dimensional manifolds*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983) (Warsaw), PWN, 1984, pp. 647–663.

[Gom84] Robert E. Gompf, *Infinite family of Casson handles and topological disks*, Topology **23** (1984), no. 4, 395–400.

[GRS08] J. Elisenda Grigsby, Daniel Ruberman, and Sašo Strle, *Knot concordance and Heegaard Floer homology invariants in branched covers*, Geom. Topol. **12** (2008), no. 4, 2249–2275.

[Hil12] Jonathan Hillman, *Algebraic invariants of links*, second ed., Series on Knots and Everything, vol. 52, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.

[KL22] Taehee Kim and Charles Livingston, *Knot reversal acts non-trivially on the concordance group of topologically slice knots*, Selecta Math. (N.S.) **28** (2022), no. 2, Paper No. 38, 17.

[Mei15] Jeffrey Meier, *Distinguishing topologically and smoothly doubly slice knots*, Journal of Topology **8** (2015), no. 2, 315–351.

[OS03] Peter Ozsváth and Zoltán Szabó, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Adv. Math. **173** (2003), no. 2, 179–261.

[Sei50] Herbert Seifert, *On the homology invariants of knots*, Quart. J. Math., Oxford Ser. (2) **1** (1950), 23–32.

[Ven78] Gerard A. Venema, *Approximating disks in 4-space*, Mich. Math. J. **25** (1978), 19–27.

[Ven81] Gerard A. Venema, *Approximating topological surfaces in 4-manifolds*, Transactions of the American Mathematical Society **265** (1981), no. 1, 35–45.

Department of Mathematics, Ewha Womans University, Seoul, Republic of Korea
Email address: minhoonkim@ewha.ac.kr

School of Mathematics and Statistics, University of Glasgow, University Place, Glasgow, G12 8QQ, United Kingdom
Email address: mark.powell@glasgow.ac.uk