Lecture on Proper Homotopy Theory

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Abstract

Drawing from Chapters 11, 16 and 17 of Geoghegan [1], we define some proper homotopy invariants of spaces and use them to show that the Whitehead manifold W is not homeomorphic to $\mathbf{R}^3$, though W is an open and contractible 3-manifold.

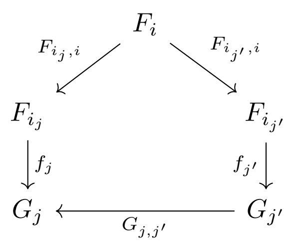
Contents

1 Pro-objects and pro-categories

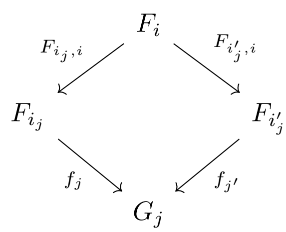
A *directed set* is a preorderd set $(I, \leq)$ in which every pair of elements $x, y \in I$ has an upper bound, i.e., a $z \in I$ such that $z \leq x$ and $z \leq y$. A pre-ordered $(I, \leq)$ may be viewed as a category by letting the elements of I be the objects and setting a unique arrow $x \to y$ whenever $x \leq y$.

Let $\mathbf{C}$ be the category **Group** of groups or the category **Ab** of abelian groups. A *pro-object* of $\mathbf{C}$ is a contravariant functor $F\colon I \to \mathbf{C}$ from some directed set I to $\mathbf{C}$. That is, for each $x \in I$, an object F_x of $\mathbf{C}$ and a morphism $F_{x,y}\colon F_y \to F_x$ whenever $x \leq y$ such that $F_{x,y} \circ F_{y,z} = F_{x,z}$ whenever $x \leq y \leq z$. The $F_{x,y}$ are called the *bonding maps* of F.

We will define a category of pro-objects of $\mathbf{C}$ called the *pro-category of* $\mathbf{C}$ and denoted pro($\mathbf{C}$). Let $F\colon I \to \mathbf{C}$ and $G\colon J \to \mathbf{C}$ be two pro-objects of $\mathbf{C}$. A *morphism* $F \to G$ *in* pro($\mathbf{C}$) is represented by a J-indexed family $\{f_j\colon F_{i_j} \to G_j\}_{j \in J}$ of $\mathbf{C}$-morphisms subject to the following constraint. For each $j \leq j'$ in I, there exists an i in I, with $i \geq i_j$ and $i \geq i_{j'}$, such that the following diagram commutes.



Morphisms of pro($\mathbf{C}$) are equivalence classes of such families of $\mathbf{C}$-morphisms. Specifically, two families $\{f_j\colon F_{i_j} \to G_j\}_{j\in J}$ and $\{f'_j\colon F_{i'_j} \to G_j\}_{j\in J}$ represent the same morphism of pro($\mathbf{C}$) if, for every $j \in J$, there is an $i \in I$, with $i \geq i_j$ and $i \geq i'_j$, such that the following diagram commutes



The composition of pro($\mathbf{C}$)-morphisms is the obvious one and is well defined.

Given a pro-object $F\colon I \to \mathbf{C}$, we can take its categorical limit

$$\varprojlim F = \Big\{(g_i)_i \in \prod_i F_i \ ; F_{i,i'}(g_{i'}) = g_i \text{ for all } i \leq i' \text{ in } I\Big\}$$

in $\mathbf{C}$ (this is often called the inverse limit or projective limit). Using the universal property of the categorical limit we see that $\varprojlim$ is a functor from pro($\mathbf{C}$) to $\mathbf{C}$. In the case $\mathbf{C} = \mathbf{Ab}$, pro($\mathbf{Ab}$) is an abelian category and $\varprojlim$ is left-exact. Its first derived functor is denoted $\varprojlim^1$.

For our purposes we are mostly interested in pro-objects $F\colon \mathbf{N} \to \mathbf{C}$, where $\mathbf{N}$ has its usual ordering. These are called *towers*. We give an explicit description of $\varprojlim^1(F)$ for F a *tower*. Consider the map

$$\begin{aligned} s\colon \prod_n F_n &\to \prod_n F_n \\ (x_n)_n &\mapsto \big(x_n - F_{n,n+1}(x_{n+1})\big)_n \end{aligned}$$

The kernel of s is $\varprojlim F$. The cokernel of s is $\varprojlim^1 F$, i.e.,

$$\varprojlim^1 F = \prod_n F_n / \operatorname{image}(s).$$

This defines a functor $\varprojlim^1 \colon \mathbf{Ab}^{\mathbf{N}} \to \mathbf{Ab}$. We can partially generalize this construction to $\mathbf{Group}^{\mathbf{N}}$ obtaining a functor $\varprojlim^1 \colon \mathbf{Group}^{\mathbf{N}} \to \mathbf{Set}_*$, where $\mathbf{Set}_*$ is the category of based sets. For $F \colon \mathbf{N} \to \mathbf{Group}$, we set

$$\varprojlim^1(F) = \prod_n F_n \Big/ \sim,$$

where $(g_n)_n \sim (g'_n)_n$ if there exists $(h_n)_n$ such that

$$g'_n = h_n g_n F_{n,n+1}(h_{n+1}^{-1}),$$

for all n.

A tower $F \colon \mathbf{N} \to \mathbf{Group}$ is *semi-stable* if, for each n,

$$F_n \supseteq \operatorname{image}(F_{n,n+1}) \supseteq \operatorname{image}(F_{n,n+2}) \supseteq \operatorname{image}(F_{n,n+2}) \supseteq \cdots$$

eventually stabilizes. Semi-stability of F is equivalent to F being pro($\mathbf{Group}$)-isomorphic to a tower $F' \colon \mathbf{N} \to \mathbf{Group}$ with the $F'_{n,m}$ all epimorphisms.

Theorem 1.1. *Let $F \colon \mathbf{N} \to \mathbf{Group}$ be a tower of countable groups. Then F is semi-stable if and only if $\varprojlim^1(F)$ is trivial.*

2 Proper maps and proper homotopy

In this lecture spaces are locally finite, connected simplicial complexes.

A continuous map $f \colon X \to Y$ of spaces is *proper* if the preimage $f^{-1}(K)$ is compact for every compact set $K \subseteq Y$. A *proper homotopy* $F \colon X \times I \to Y$ is a homotopy that is proper. We can define proper homotopy equivalences and proper homotopy types using proper homotopies.

Example 2.1. The real line $\mathbf{R}$ is not proper homotopy equivalent to a point.

3 Ends and strong ends

Let X be a space. We are interested in proper homotopy invariants of X which describe its behaviour "at infinity", i.e., away from arbitrarily large compact subspaces. We begin by discussing "0-dimensional" properties. One such property is the set $\mathscr{S}\mathscr{E}(X)$ of *strong ends* of X. These are the proper homotopy classes of proper maps $[0,\infty) \to X$. This meets our "at infinity" criterion because any such proper map $\omega \colon [0,\infty) \to X$ must eventually escape any compact $K \subseteq X$, i.e., $\omega([n,\infty) \subseteq X \setminus K$ for large enough n. Moreover, ω is proper homotopic to $\omega|_{[n,\infty)}$ for any n.

Example 3.1. The plane $\mathbf{R}^2$ has a single strong end. The line $\mathbf{R}$ has two strong ends. The bi-infinite ladder (i.e. the 1-skeleton of the combinatorial line cross an edge) has infinitely many strong ends.

The last example shows that this notion may be a bit too strong for some purposes. We may choose a coarser equivalence relation than proper homotopy equivalence as follows. We say $\omega\colon [0,\infty) \to X$ and $\omega'\colon [0,\infty) \to X$ determine the same *end* of X if there is a proper map from the infinite ladder (i.e. the 1-skeleton of the combinatorial ray cross an edge) such that ω and ω' factor through its two sides. The resulting equivalence classes are the set $\mathscr{E}(X)$ of ends of X. Another construction giving $\mathscr{E}(X)$ is

$$\mathscr{E}(x) = \varprojlim_{K} \pi_0(X \setminus K)$$

where the limit is taken in **Set**, the K range over the compact subspaces of X and the bonding maps are the inclusions. Because X is a connected, locally finite simplicial complex, we can exhaust it with a filtration of compact subspaces

$$K_0 \subseteq K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$$

with $\cup_n K_n = X$ so, by cofinality, we have

$$\mathscr{E}(X) = \varprojlim_{i} \pi_0(X \setminus K_i).$$

4 Homtopy groups at infinity

We would like to define higher proper homotopy invariants at infinity of X. To do so we need an analog at infinity of the basepoint. Let $\omega\colon [0,\infty) \to X$ be a proper map. Generalizing $\mathscr{S}\mathscr{E}(X)$, we define the *strong nth homotopy group* $\pi_n^e(X,\omega)$ as the set of proper homotopy classes of maps

$$\big(S^n \times [0,\infty), \{*\} \times [0,\infty)\big) \to (X,\omega)$$

with multiplication performed pointwise along $[0,\infty)$ for $n \geq 1$. This is clearly a proper homotopy invariant and we have $\pi_0^e(X,\omega) = \big(\mathscr{S}\mathscr{E}(X), \mathrm{se}(\omega)\big)$.

To generalize $\mathscr{E}(X)$ we will first generalize the pro-object $\pi_0(X \setminus K_i)$. We choose a filtration $(K_n)_n$ of X by compact subspaces and, if necessary, reparameterize ω so that $w(n) \in K_{n+1} \setminus K_n$ (this can always be done by a proper homotopy). The actual choice of K_n does not matter. The *nth homotopy pro-group* $\pi_n^p(X,\omega)\colon \mathbf{N} \to \mathbf{Group}$ of (X,ω) is the pro-object of **Group** with

$$\pi_n^p(X,\omega)_k = \pi_n(X \setminus K_n, \omega(n))$$

where the bonding maps $\pi_n^p(X,\omega)_{k,k'}$ are given by whiskering along $\omega|_{[k,k']}$.

Finally, the *Čech nth homotopy group of* (X,ω) is defined by

$$\check{\pi}_n(X,\omega) = \varprojlim \pi_n^p(X,\omega).$$

Note that $\check{\pi}_0(X,\omega) = \big(\mathscr{E}(X), e(\omega)\big)$.

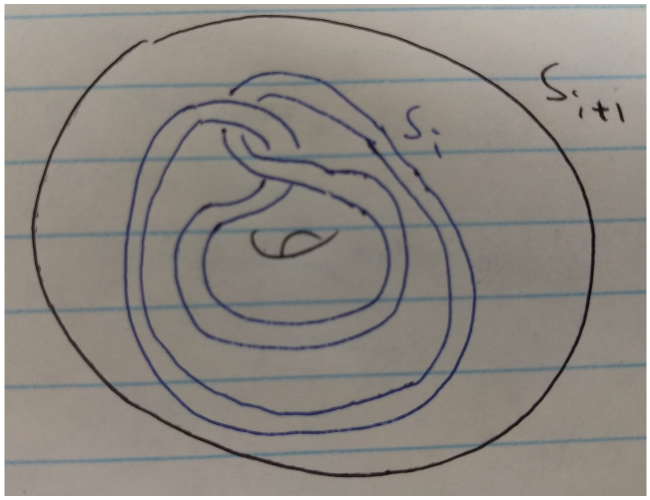


Figure 1: Figure for the definition of the Whitehead manifold.

Example 4.1. Let X' be $[0,\infty)$ a circle wedged at each integer vertex. Let ω be the inclusion of $[0,\infty)$ in X' and let ω' be a proper ray in X' passing once through each edge of X'. Let $X'' = [0,\infty) \times S^1$ and let X be the union of X' and X'' glued along $[0,\infty)$. Then X has one end but $\breve{\pi}_n(X,\omega) = \mathbf{Z}$ but $\breve{\pi}_n(X,\omega')$ is trivial.

The example shows that the Čech homotopy groups, and hence the homotopy pro-groups, are not invariant under a change of ω to some ω' which determines the same end of X. These groups are, however, invariant under a change of baseray within the same *strong end* of X.

Proposition 4.2. *Let $\mathscr{S}\mathscr{E}\big(X, \mathrm{e}(\omega)\big)$ be the set of strong ends of X which determine* $\mathrm{e}(\omega)$. *Then* $\varprojlim^1\big(\pi_1^p(X,\omega)\big) \cong \mathscr{S}\mathscr{E}\big(X, \mathrm{e}(\omega)\big)$.

It follows that if $\varprojlim^1\big(\pi_1^p(X,\omega)\big)$ is trivial then $\pi_n^p(X,\omega)$, and hence, $\breve{\pi}_n(X,\omega)$ are invariant under a change of baseray within the same end. Recall that triviality of $\varprojlim^1\big(\pi_1^p(X,\omega)\big)$ is equivalent to semi-stability of $\pi_1^p(X,\omega)$.

Theorem 4.3. *There is a natural short exact sequence*

$$0 \to \varprojlim^1\big(\pi_{n+1}^p(X,\omega)\big) \to \pi_n^e(X,\omega) \to \breve{\pi}_n(X,\omega) \to 0$$

of based sets when $n = 0$, of groups when $n = 1$ and of abelian groups when $n \geq 2$.

5 The Whitehead manifold

Consider the sequence

$$S_1 \xrightarrow{\varphi_1} S_2 \xrightarrow{\varphi_2} S_3 \xrightarrow{\varphi_3} \cdots$$

where $S_i \cong D^2 \times S^1$ for all i and $\varphi_i \colon S_i \to S_{i+1}$ is the embedding depicted in Figure 1. Note that φ_i is homotopically trivial.

The colimit

$$W = \varinjlim S_i = \bigsqcup_i S_i \big/ \sim$$

($x \sim \varphi_i(x)$ for all $x \in S_i$ and all i) is the *Whitehead manifold*. The Whitehead manifold is an open 3-manifold. It is contractible by Whitehead's theorem since the φ_i are homotopically trivial. However, we will see that W is not homeomorphic to $\mathbf{R}^3$.

Let A_0 be S_1 and let A_i be the closure of $S_{i+1} \setminus S_i$ for $i \geq 1$. Then $A_i \cap A_{i+1} = \partial S_{i+1}$. Set $U_j = \cup_{i=j}^{\infty} A_i$. Set $U_j = \cup_{i=j}^{\infty} A_i$. Then $U_0 = W$ and U_j is the closure of $W \setminus \cup_{i=1}^{j-1} S_j$ for $j \geq 1$. So $U_j \cong U_{j'}$ for all $j, j' \geq 1$.

Choose a base ray ω in W so that $W|_{\mathbf{N}} \subseteq \cup_i \partial S_{i+1}$ and the embedding $W \to W$ sending S_i to S_{i+1} via φ_i restricts to a homeomorphism $\omega|_{[i,i+1]} \to \omega|_{[i+1,i+2]}$. The boundary tori of A_i, $i \geq 1$, π_1-inject into A_i, so

$$\pi_1\big(U_j, \omega(j)\big) \cong \pi_1\big(A_j, \omega(j)\big) *_{\pi_1(\partial S_{j+1})} \pi_1\big(U_{j+1}, w(j+1)\big)$$

with $\pi_1(\partial S_{j+1})$ embedding in both free factors. So the bonding maps of $\pi_1^p(W, \omega)$ are non-epimorphic monomorphisms. Hence $\pi_1^p(W, \omega)$ cannot be pro-epimorphic and so $\pi_1^p(W, \omega)$ is not semi-stable and $\mathscr{S}\mathscr{E}(W)$ is non-trivial. It follows that $W \ncong \mathbf{R}^3$ since $\mathbf{R}^3$ has only one strong end.

References

[1] Ross Geoghegan. *Topological methods in group theory*, volume 243 of *Graduate Texts in Mathematics*. Springer, New York, 2008.