SOME REMARKS ON h-COBORDISMS BETWEEN SMOOTH 4-MANIFOLDS

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ABSTRACT. It is not known whether the realisation part of the s-cobordism theorem holds for smooth 4-manifolds, nor whether every pair of smoothly h-cobordant 4-manifolds is also smoothly s-cobordant. We provide some new conditions under which these questions admit a positive answer. We also give conditions under which the 'standard' method to construct an h-cobordism with specified torsion cannot work.

1. INTRODUCTION

A smooth cobordism $W\colon M \rightsquigarrow N$ between compact smooth d-manifolds (possibly with boundary) is an *h-cobordism* if $\partial W = M \cup (\partial M \times I) \cup N$ and the inclusions $M \hookrightarrow W$ and $N \hookrightarrow W$ are homotopy equivalences; in this case we say that M and N are *smoothly h-cobordant*. If moreover both inclusions are simple homotopy equivalences, then $W\colon M \rightsquigarrow N$ is an *s-cobordism*, and we say M and N are *smoothly s-cobordant*.

To each h-cobordism, throughout assumed to be connected and based, we can associate its *torsion* $\tau(W, M) \in \mathrm{Wh}(\pi_1(M))$, taking values in the Whitehead group of the fundamental group [Whi50, §2]. By Milnor's duality formula [Mil66, p. 394] for torsion, $\tau(W, N) = 0$ if and only if $\tau(W, M) = 0$, which by a result of Whitehead [Whi50, §10] in turn holds if and only if W is an s-cobordism. In dimensions $d \geq 5$, the Whitehead torsion induces a bijection [Ker65]

$$\frac{\{\text{smooth } h\text{-cobordisms } W\colon M \rightsquigarrow N\}}{\text{diffeomorphisms fixing } M \cup (\partial M \times I) \text{ pointwise}} \xrightarrow{\cong} \mathrm{Wh}(\pi_1(M)).$$

In dimension $d = 4$, it is well-known that this map is not injective [Don87], and it is an open question whether it is surjective [BKR26, Problem 4.18]. Let us make the latter explicit.

Question 1.1. Given a connected, compact, smooth 4-manifold M with fundamental group $\pi = \pi_1(M)$, can each torsion $\tau \in \mathrm{Wh}(\pi)$ be realised by a smooth h-cobordism?

The following related question was raised explicitly in [KPR22, Question 1.16] and [BKR26, Problem 4.18], and also remains open.

Question 1.2. Is every pair of compact, smooth 4-manifolds that are smoothly h-cobordant also smoothly s-cobordant?

If the Whitehead group $\mathrm{Wh}(\pi)$ vanishes, for example if π is torsion-free and satisfies the K-theoretic Farrell-Jones conjecture [Lue25], then every h-cobordism is an s-cobordism; restricting to 4-manifolds with this fundamental group, both Question 1.1 and 1.2 trivially have positive answers. Finite cyclic groups are the best-known examples of groups for which the Whitehead group can be nontrivial [Mil66, Oli88]. Our first result is that both questions admit a positive answer when certain technical conditions are satisfied, e.g. for π finite cyclic.

Recall that a smooth h-cobordism $W\colon M \rightsquigarrow M'$ is *inertial* if M and M' are diffeomorphic, $\mathrm{SK}_1(\pi) := \ker[K_1(\mathbb{Z}\pi) \to K_1(\mathbb{Q}\pi)]$, and $L^s_5(\mathbb{Z}\pi)$ and $L^h_5(\mathbb{Z}\pi)$ are the Wall surgery obstruction groups, with s- and h-decoration respectively.

Theorem A. *Let M be a smooth, compact, connected, oriented 4-manifold with finite fundamental group $\pi := \pi_1(M)$. Suppose that either*

(i) $\mathrm{SK}_1(\pi) = 0$ *and the forgetful map $L^s_5(\mathbb{Z}\pi) \to L^h_5(\mathbb{Z}\pi)$ is injective;*

(ii) $\mathrm{SK}_1(\pi) = 0$ *and M is* topologically pre-stabilised*, that is, homeomorphic to $X\#(S^2 \times S^2)$ for some closed topological 4-manifold X; or*

2020 *Mathematics Subject Classification.* 57K40, 57R67, 19J10.

(iii) M *is* smoothly pre-stabilised*, that is, diffeomorphic to* $X\#(S^2 \times S^2)$ *for some closed smooth 4-manifold* X*.*

Then, for each $x \in \mathrm{Wh}(\pi)$ *there exists an inertial smooth* h*-cobordism* $W\colon M \rightsquigarrow M'$ *with torsion* $\tau(W, M) = x$*.*

A positive answer to Question 1.1 that produces inertial h-cobordisms, as in Theorem A, leads to a positive answer to Question 1.2.

Corollary B. *If* M *satisfies the hypotheses of Theorem A, then every 4-manifold that is smoothly* h*-cobordant to* M *is also smoothly* s*-cobordant to* M*.*

Example 1.3.

- Case (i) applies if π is a finite cyclic group C_n of order n, since $\mathrm{SK}_1(\pi) = 0$ by [BMS67, Proposition 4.14] and $L_5^s(\mathbb{Z}C_n) = 0$ by [Bak76].
- Case (ii) applies to many more examples of fundamental groups π, since $\mathrm{SK}_1(\pi) = 0$ in for example the following cases:
  - π is abelian and either (a) each Sylow subgroup of π has the form C_{p^n} or $C_p \times C_{p^n}$ for some n, or (b) $\pi = (C_2)^k$ for some k [Oli88, Theorem 14.2 (iii)],
  - π is a dihedral group D_{2n} of order $2n$ [Mag76, Theorem 21.1],
  - π is on the list of [Ush94, Theorem A], e.g. the binary icosahedral group.

Remark 1.4. By the Ranicki–Rothenberg exact sequence of [Sha69, Section 4], the kernel of $L_5^s(\mathbb{Z}\pi) \to L_5^h(\mathbb{Z}\pi)$ is a quotient of the Tate cohomology group $\widehat{H}^6(C_2; \mathrm{Wh}(\pi))$. Since the latter are 2-torsion, this kernel is trivial for example when $L_5^s(\mathbb{Z}\pi)$ is free abelian.

We continue to consider Question 1.1, but now without any consideration of whether the realising h-cobordism is inertial. There is a 'standard' method to try to construct a smooth h-cobordism $W\colon M \rightsquigarrow N$ with a given torsion x. We explain it in detail later, but roughly one represents x by an $(n \times n)$-matrix X with columns X_i, adds n trivial 5-dimensional 2-handles to $M \times I$, and then represents each X_i by an embedded 2-sphere by tubing together parallel copies of capped-off cores of the new 2-handles. Attaching a 3-handle to each such embedded 2-sphere results in a cobordism $W\colon M \rightsquigarrow N$, where $M \hookrightarrow W$ is homotopy equivalence with $\tau(W, M) = [X] \in \mathrm{Wh}(\pi)$. But $N \hookrightarrow W$ is in general only a $\mathbb{Z}\pi$-homology isomorphism, as there is no reason a priori for the embedded 2-spheres representing the X_i to admit immersed dual spheres and hence one loses control on the fundamental group $\pi_1(N)$.

Our second result says that whether this construction can work or not is closely related to an invariant of M. Cohen [Coh77]. To each $x \in \mathrm{Wh}(\pi)$, he associates a number $\dim(x) \in \{0, 2, 3\}$, with $\dim(0) = 0$ and $\dim(x) \in \{2, 3\}$ for $x \neq 0$, and he conjectured that $x \neq 0 \Leftrightarrow \dim(x) = 3$.

Theorem C. *Let* M *be a smooth, compact, connected, oriented 4-manifold with fundamental group* $\pi := \pi_1(M)$*. Then for* $x \in \mathrm{Wh}(\pi)$ *we have:*

(i) if $\dim(x) \in \{0, 2\}$*, there exists an* h*-cobordism* $W\colon M \rightsquigarrow N$ *with* $\tau(W, M) = x$*;*

(ii) if $\dim(x) = 3$*, the standard construction* cannot *yield any* h*-cobordism* $W\colon M \rightsquigarrow N$ *with* $\tau(W, M) = x$*.*

Example 1.5. Rothaus and Magurn found examples of $x \in \mathrm{Wh}(D_{2n})$ with $\dim(x) = 3$ [Rot76, Rot77, Mag81]; see Example 3.4 for more details. By Theorem C one cannot realise these torsions using the standard construction. Theorem A (ii), however, does realise these torsions for topologically pre-stabilised manifolds, as does (iii) for smoothly pre-stabilised 4-manifolds. Since our proof of Theorem A relies on surgery theory and smoothing theory, it is non-constructive. We hope 4-manifold topologists are encouraged to find explicit constructions.

Acknowledgements. The authors would like to thank Ian Hambleton for several helpful remarks, including the suggestion to add case (iii) of Theorem A. The second author thanks Michelle Daher, Daniel Galvin, Daniel Kasprowski, Csaba Nagy, John Nicholson, and Arunima Ray for stimulating conversations on these and related questions. The authors are grateful to the organisers of the Zeeman Centenary conference at the University of Warwick in December 2025, at which some of this work was done. AK acknowledges the support of the Natural Sciences and Engineering Research Council of Canada (NSERC) [funding reference number 512156 and 512250].

Contents

2. Proof of Theorem A and Corollary B

Recall that M is a smooth, compact, connected, oriented 4-manifold with finite fundamental group $\pi := \pi_1(M)$. Since π is finite, it is a good group in the sense of [FQ90], and hence by the topological realisation of Whitehead torsion by h-cobordisms (e.g. [KPR22, Theorem 3.5]), for every $x \in \mathrm{Wh}(\pi)$ there exists a topological h-cobordism $W : M \rightsquigarrow N$ with $\tau(W, M) = x$. We want to show that M and N are homeomorphic. To do this, we deal with case (i) of Theorem A first, in Lemma 2.1, and then we show it for cases (ii) and (iii) simultaneously, in Lemma 2.2. After that we complete the proof of Theorem A in unison for all three cases.

Lemma 2.1. *In case* (i)*, M and N are homeomorphic.*

Proof. Let $g : W \to M$ be a homotopy inverse to the inclusion $\mathrm{inc}_M : M \to W$. By using the collar on M we may assume that g is a weak deformation retract, which in particular means that $g \circ \mathrm{inc}_M = \mathrm{Id}_M$. Let $f := g \circ \mathrm{inc}_N : N \to M$ be the homotopy equivalence $f : N \to M$ induced by W. This has Whitehead torsion $\tau(f) = x - \overline{x}$ (e.g [NNP23, Proposition 2.38]), where $x \mapsto \overline{x}$ is the involution on the Whitehead group. The involution acts trivially on the quotient of $\mathrm{Wh}(\pi)$ by $\mathrm{SK}_1(\pi)$ [Oli88, Corollary 7.5], so using the hypothesis that $\mathrm{SK}_1(\pi) = 0$, the involution acts trivially on $\mathrm{Wh}(\pi)$, and thus $\tau(f) = x - \overline{x} = x - x = 0$, i.e. f is a simple homotopy equivalence.

Take the product of g with a continuous map $h : W \to I$ satisfying $h^{-1}(\{0\}) = M$ and $h^{-1}(\{1\}) = N$, to obtain a homotopy equivalence of pairs $(F, \mathrm{Id}_M \sqcup f) : (W, M \cup N) \to (M \times I, M \cup M)$. This can be considered as a degree one normal map over $M \times I$ by adding normal data as follows: take $G : M \times I \to W$ to be a homotopy inverse to F, and consider the stable vector bundle data

$$\begin{array}{ccc} \nu_W & \longrightarrow & G^*\nu_W \\ \downarrow & & \downarrow \\ W & \xrightarrow{F} & M \times I, \end{array}$$

where ν_W is the stable normal bundle of W. (It may be helpful to observe this is similar to the construction of the map $\eta : \mathcal{S}^s(M \times I, \partial) \to \mathcal{N}(M \times I, \partial)$ in the surgery exact sequence.)

Since the restrictions of this degree one map to the ends are given by $f : N \to M$ and $\mathrm{Id}_M : M \to M$, both of which are simple homotopy equivalences, the simple surgery obstruction $\sigma_s(W, F)$ of (W, F) represents an element in the surgery obstruction group $L_5^s(\mathbb{Z}\pi)$. Since F is a homotopy equivalence, (W, F) has trivial surgery obstruction in $L_5^h(\mathbb{Z}\pi)$, and hence $\sigma_s(W, F)$ lies in the kernel of $L_5^s(\mathbb{Z}\pi) \to L_5^h(\mathbb{Z}\pi)$. As this kernel is trivial by hypothesis (i), $\sigma_s(W, F) = 0$, and so (W, F) is normally bordant relative to the boundary to a simple homotopy equivalence [Wal99, Chapter 6]; the resulting bordism is an s-cobordism. Since π is finite, and finite groups are good, by the s-cobordism theorem [FQ90, Chapter 7] it follows that M and N are homeomorphic. □

Lemma 2.2. *In cases* (ii) *and* (iii)*, M and N are homeomorphic.*

Proof. As M and N are h-cobordant, they are stably homeomorphic [Law78]. Since (ii) and (iii) require there is a homeomorphism $M \cong X\#(S^2 \times S^2)$, the fundamental group is finite, and M as well as N are closed, [HK93, Theorem B] implies that M and N are homeomorphic. □

We have now shown that M and N are homeomorphic in all three cases. We can therefore think of W as an inertial topological h-cobordism $W : M \rightsquigarrow M$ with $\tau(W, M) = x$.

We will need the following in the proof of the next lemma. Given a self-homeomorphism $\varphi : M \to M$ that is the identity on the boundary, let C_φ denote the mapping cylinder of φ, with the evident smooth structure on the boundary. The *Casson–Sullivan invariant* $\mathrm{cs}(\varphi)$ of φ is defined as the image of the Kirby-Siebenmann invariant $\mathrm{ks}(\mathrm{C}_\varphi, \partial \mathrm{C}_\varphi)$ under the isomorphisms

$$H^4(\mathrm{C}_\varphi, \partial\mathrm{C}_\varphi; \mathbb{Z}/2) \xrightarrow[\cong]{\mathrm{PD}} H_1(\mathrm{C}_\varphi; \mathbb{Z}/2) \xleftarrow[\cong]{\mathrm{inc}_*} H_1(M; \mathbb{Z}/2),$$

where $\mathrm{inc}\colon M \to \mathrm{C}_\varphi$ is the inclusion into the mapping cylinder as the domain. In our cases, Galvin showed that it is possible to realise each element of $H_1(M; \mathbb{Z}/2)$ as $\mathrm{cs}(\varphi)$, for some φ. For cases (i) and (ii), apply [Gal24, Theorem 1.7], using that the *Casson–Sullivan realisability criterion* of [Gal24, Definition 4.11] is satisfied whenever $\mathrm{SK}_1(\pi) = 0$, by [Gal24, Proposition 4.13]. For case (iii) we instead use [Gal24, Theorem 1.1], which applies to smoothly pre-stabilised 4-manifolds.

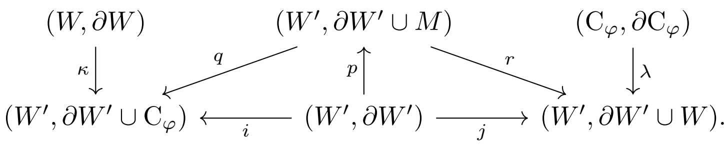
Lemma 2.3. *The given smooth structure on the first M, and some diffeomorphic smooth structure on the second M, extend to a smooth structure on W.*

Proof. Start with the given smooth structure on $\partial W = M \cup_{\partial M} M$. The Kirby–Siebenmann obstruction to smoothing W relative to its boundary is

$$\mathrm{ks}(W, \partial W) \in H^4(W; \partial W; \mathbb{Z}/2).$$

We claim that we can reparametrise the second M by precomposing the embedding $M \to W$ with the inverse of a suitable homeomorphism $\varphi\colon M \to M$ to obtain an h-cobordism $W\colon M \rightsquigarrow M$ with a new smooth structure on its boundary, so that now the obstruction to smoothing W relative to its boundary vanishes. Assuming this, since the dimension of W is at least 5, high-dimensional smoothing theory [KS77, Essay IV] implies that this smooth structure on $M \cup_{\partial M} M$ extends to a smooth structure on W.

We use the Casson–Sullivan realisation described before the lemma, together with a suitable additivity formula for the Kirby–Siebenmann invariant, to prove the claim. Reparametrising using the inverse of homeomorphism $\varphi\colon M \to M$ is equivalent to replacing W with the union $W' \coloneqq W \cup_M \mathrm{C}_\varphi$. Consider the following commutative diagram of maps of pairs:



The maps κ and λ induce isomorphisms on cohomology by excision, and the maps i and j induce isomorphisms on cohomology because W and C_φ are h-cobordisms. By naturality of the Kirby–Siebenman invariant we have that

$$\begin{aligned} &\mathrm{ks}(W', \partial W') = p^* \mathrm{ks}(W', \partial W' \cup M); \quad \mathrm{ks}(W, \partial W) = \kappa^* \mathrm{ks}(W', \partial W' \cup \mathrm{C}_\varphi); \text{ and} \\ &\mathrm{ks}(\mathrm{C}_\varphi, \partial \mathrm{C}_\varphi) = \lambda^* \mathrm{ks}(W', \partial W' \cup W). \end{aligned} \tag{2.4}$$

We therefore have the following in $H^4(W', \partial W'; \mathbb{Z}/2)$. Here the first and last equations use (2.4), the second equation uses naturality of obstruction theory, and the third uses that $i = q \circ p$ and $j = r \circ p$.

$$\begin{aligned} \mathrm{ks}(W', \partial W') &= p^* \mathrm{ks}(W', \partial W' \cup M) \\ &= p^*\big(q^* \mathrm{ks}(W', \partial W' \cup \mathrm{C}_\varphi) + r^* \mathrm{ks}(W', \partial W' \cup W)\big) \\ &= i^* \mathrm{ks}(W', \partial W' \cup \mathrm{C}_\varphi) + j^* \mathrm{ks}(W', \partial W' \cup W) \\ &= i^*(\kappa^*)^{-1} \mathrm{ks}(W, \partial W) + j^*(\lambda^*)^{-1} \mathrm{ks}(\mathrm{C}_\varphi, \partial \mathrm{C}_\varphi). \end{aligned}$$

The fact that $j^*(\lambda^*)^{-1}$ is surjective, together with the aforementioned realisation result for Casson–Sullivan invariants, means that we can choose φ so that $j^*(\lambda^*)^{-1} \mathrm{ks}(\mathrm{C}_\varphi, \partial \mathrm{C}_\varphi) = i^*(\kappa^*)^{-1} \mathrm{ks}(W, \partial W)$, and hence $\mathrm{ks}(W', \partial W') = 0$. To see that we can choose φ appropriately, we compare the images of the invariants under Poincaré duality, in $H_1(W'; \mathbb{Z}/2)$ (a group that does not depend on φ), and note that by Galvin's Casson–Sullivan realisation we can obtain any element of $H_1(M; \mathbb{Z}/2) \cong H_1(\mathrm{C}_\varphi; \mathbb{Z}/2) \cong H_1(W'; \mathbb{Z}/2)$. ◻

Let M' denote M with the second smooth structure arising from Lemma 2.3. By the lemma, the smooth structure on $M \cup M'$ extends to one on W. Thus $W: M \rightsquigarrow M'$ is the desired inertial smooth h-cobordism with Whitehead torsion $x \in \mathrm{Wh}(\pi)$.

To deduce Corollary B, start with a smooth h-cobordism $V: M \rightsquigarrow N$, and let $y \in \mathrm{Wh}(\pi_1(M))$ denote its Whitehead torsion. Let $\theta: \pi_1(M) \to \pi_1(N)$ denote the isomorphism induced by V. Set $x = -\theta_*(y) \in \mathrm{Wh}(\pi_1(N))$, and apply Theorem A to obtain a smooth inertial h-cobordism $W: N \rightsquigarrow N'$ with torsion $-\theta_*(y)$. Then the concatenation $W \circ V: M \rightsquigarrow N'$ has torsion $y + \theta_*^{-1}(-\theta_*(y)) = 0 \in \mathrm{Wh}(\pi_1(M))$, and so is a smooth s-cobordism. Then, since N' is diffeomorphic to N, by glueing on $N \times I$ using the diffeomorphism we deduce that in fact M and N are smoothly s-cobordant, as required.

3. Proof of Theorem C

We explain how an attempt to replicate, in dimension $d = 4$, the 'standard' high-dimensional construction of h-cobordisms with specified torsion, leads to a group-theoretic conjecture of M. Cohen. This discussion goes through in all categories of manifolds—smooth, PL, and topological—so does not shed light on questions specifically concerning smooth manifolds.

Remark 3.1. Freedman and Quinn [FQ90, Section 7.1, p. 102] assert: "The standard construction of h-cobordisms (Rourke and Sanderson [RS82, p. 90]) works in this dimension, and shows that there is an h-cobordism with any given torsion." Combining Theorem C and Example 1.5, this statement is not correct. This minor error was observed before in [HJ18, Lemma 5.8], which also explained how to use [FQ90] to fix it (though they, in turn, erroneously omit the hypothesis that the fundamental group must be good and assert the proof works for smooth manifolds). See the survey [KPR22, Theorem 3.5] for a complete proof of topological realisation of Whitehead torsion by h-cobordisms.

3.1. **A conjecture of M. Cohen.** We start by recalling definitions from [Coh77, §1]. Let $\mathcal{P}$ denote the data of a group π, a free group $F = \langle x_1, \ldots, x_n \rangle$ on n generators, and relators $r_1, \ldots, r_n$ that lie in the smallest normal subgroup N of $\pi * F$ containing $x_1, \ldots, x_n$. Sending x_i to e induces the right surjection, split by the inclusion of π, in the short exact sequence

$$1 \longrightarrow \frac{N}{(r_1, \ldots, r_n)} \longrightarrow \pi(\mathcal{P}) := \frac{\pi * F}{(r_1, \ldots, r_n)} \longrightarrow \pi \longrightarrow 1.$$

Such an extension is said to be *trivial* if the right map $\pi(\mathcal{P}) \to \pi$ is an isomorphism, and *proper* if it is not. The relations r_i can, like every element in N, be written as a product

$$r_i = \prod_{k=1}^{n(i)} g(i,k) x(i,k)^{\varepsilon(i,k)} g(i,k)^{-1},$$

with $n(i) \geq 0$, $g(i,k) \in \pi$, $x(i,k) \in \{x_1, \ldots, x_n\}$, and $\varepsilon(i,k) \in \pm 1$. Define then an $(n \times n)$-matrix $X(\mathcal{P})$ with entries in $\mathbb{Z}\pi$ by

$$X(\mathcal{P})_{ij} := \sum_{x(i,k)=x_j} \varepsilon(i,k) g(i,k).$$

See [Coh77, §4] for more details as well as an alternative description of $X(\mathcal{P})$ in terms of Fox derivatives. The following lemma is a straightforward exercise.

Lemma 3.2. *Given a finitely presented group π and an $(n \times n)$-matrix X with entries in $\mathbb{Z}\pi$, there exists $\mathcal{P}$ such that $X(\mathcal{P}) = X$.*

Following M. Cohen, we say that $\mathcal{P}$ is *admissible* if $X(\mathcal{P})$ is invertible. We are interested specifically in admissible $\mathcal{P}$, because in such cases $X(\mathcal{P})$ represents an element

$$\tau(\mathcal{P}) = [X(\mathcal{P})] \in \mathrm{Wh}(\pi).$$

By [Coh77, Proposition 4.1], if the extension is trivial, i.e. $\pi(\mathcal{P}) \xrightarrow{\cong} \pi$ is an isomorphism, $X(\mathcal{P})$ is invertible and thus the extension is admissible. There exist, however, also admissible proper extensions, as we will see in Example 3.4.

Definition 3.3. For $x \in \mathrm{Wh}(\pi)$, define

$$\dim(x) = \begin{cases} 0 & \text{if } x = 0, \\ 2 & \text{if } x \neq 0 \text{ and there exists a trivial admissible presentation with } \tau(\mathcal{P}) = x, \\ 3 & \text{if } x \neq 0 \text{ and every admissible presentation with } \tau(\mathcal{P}) = x \text{ is proper.} \end{cases}$$

Example 3.4. For D_{2n} the dihedral group with $2n$ elements, there exist elements $x \in \mathrm{Wh}(D_{2n})$ with $\dim(x) = 3$ whenever n does not divide 12. This was proven by Rothaus for n prime [Rot77, Theorem 11] and in general by Magurn [Mag81, Corollary 13 and 15]. An explicit example appears in [Rot76, p. 285]: writing $D_{10} = \langle g, h \mid g^5, h^2, hgh = g^{-1} \rangle$, the unit $(-1 + g - g^2 + g^3 + g^4) + h(1 - 2g + g^2) \in \mathbb{Z}D_{10}$ represents an element $x \in \mathrm{Wh}(D_{10})$ with $\dim(x) = 3$.

There are no known examples of x with $\dim(x) = 2$ and M. Cohen conjectured [Coh77, Conjecture A] the following.

Conjecture 1 (M. Cohen)**.** *For every group π, $\dim(x) = 3$ if and only if $x \neq 0 \in \mathrm{Wh}(\pi)$.*

3.2. **The standard construction.** To begin the proof of Theorem C, let us recall what is arguably the 'standard' method for constructing an h-cobordism with specified torsion; it is alluded to in Remark 3.1 and used in [Hud69, Part 2, §12] and [RS82, p. 82]. Let M be a connected compact smooth 4-manifold with $\pi = \pi_1(M)$ and $x \in \mathrm{Wh}(\pi)$ be represented by $X \in \mathrm{GL}_n(\mathbb{Z}\pi)$. Our candidate $W \colon M \rightsquigarrow N$ will be a concatenation $W := W_1 \circ W_0 \colon M \rightsquigarrow N$ of two cobordisms. The first cobordism is

$$W_0 = (M \times I) \natural (S^2 \times D^3)^{\natural n} \colon M \rightsquigarrow M \# n(S^2 \times S^2)$$

and the second cobordism $W_1 \colon M\#(S^2 \times S^2)^{\# n} \rightsquigarrow N$ we shall construct now. We have $H_2(M\#n(S^2 \times S^2); \mathbb{Z}\pi) \cong H_2(M; \mathbb{Z}\pi) \oplus (\mathbb{Z}\pi \oplus \mathbb{Z}\pi)^n$ where the first copy of $\mathbb{Z}\pi$ in the ith $\mathbb{Z}\pi \oplus \mathbb{Z}\pi$ summand is represented by the embedded 2-sphere $A_i = S^2 \times \{0\}$ and the second by $B_i = \{0\} \times S^2$, intersecting transversely in a single point. For each column X_j of X, we want to construct a 2-sphere C_j with trivialised normal bundle whose collection of $\mathbb{Z}\pi$-equivariant intersection numbers with the 2-spheres A_i is given by the column X_j, by tubing together parallel copies of the B_i. If we do so, attaching 5-dimensional 3-handles along the C_j yields a second cobordism

$$W_1 \colon M\#n(S^2 \times S^2) \rightsquigarrow N,$$

so that $M \hookrightarrow W = W_1 \circ W_0$ is a homotopy equivalence with torsion $\tau(W, M)$ given by x. The difficulty, which does not arise in higher dimensions, is to guarantee that the inclusion $N \to W$ is also a homotopy equivalence; note that by Poincaré–Lefschetz duality and the universal coefficients theorem, it is always a $\mathbb{Z}\pi$-homology equivalence, so this is a question about the induced map on fundamental groups.

To investigate this difficulty, we need prescribe the tubing, and to do so we use an admissible presentation $\mathcal{P}$ with generators $x_1, \ldots, x_n$ and relators $r_1, \ldots, r_n$, satisfying $[X(\mathcal{P})] = [X] \in \mathrm{Wh}(\pi)$, which exists by Lemma 3.2. Recall that we can write

$$r_i = \prod_{k=1}^{n(i)} g(i,k) x(i,k)^{\varepsilon(i,k)} g(i,k)^{-1},$$

where $n(i) \geq 1$ since X is invertible. We may and will assume—by conjugating, taking the inverse, and reordering, if necessary—that $x(i,1) = x_i$, $g(i,1) = e$, and $\varepsilon(i,1) = 1$, for each i. Here we use the following observations.

(1) Replacing r_i by $\gamma r_i^\delta \gamma^{-1}$, for $\delta \in \{\pm 1\}$ and $\gamma \in \pi$, corresponds to multiplying the ith row of $X(\mathcal{P})$ by $\delta\gamma$. This does not affect $[X(\mathcal{P})]$.
(2) Changing the order of the factors $g(i,k)x(i,k)^{\varepsilon(i,k)} g(i,k)^{-1}$ in r_i does not affect $X(\mathcal{P})$, hence does not affect $[X(\mathcal{P})]$.

We start by taking the 2-sphere $B(i,1)$ to be B_i, and for $k \geq 2$. we take $B(i,k)$ to be a disjoint parallel copy of B_i, each contained in the ith copy of $(S^2 \times S^2)^\circ$ in $M\#n(S^2 \times S^2)$ where $(-)^\circ$ denotes that we remove the interior of a 4-disc. Identifying $(S^2 \times S^2) \setminus \cup_{k=1}^{n(i)} B(i,k)$ with

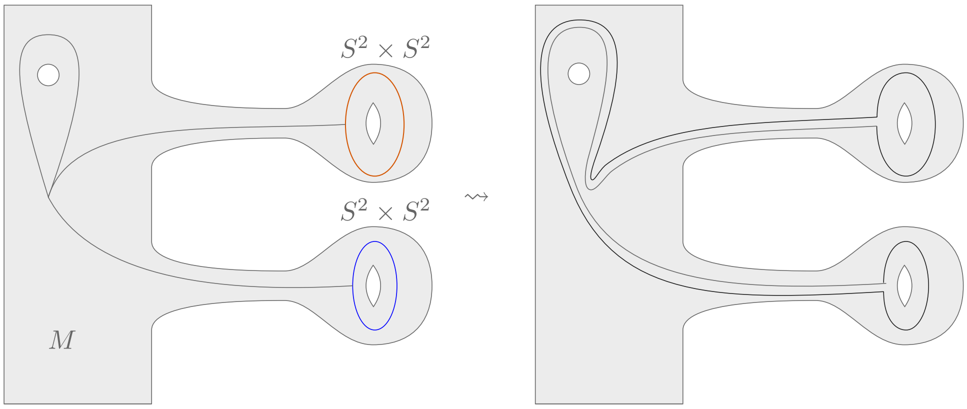


FIGURE 1. A parallel copy of B_1 (in red) is tubed to a parallel copy of B_2 (in blue) along a path given concatenating a preferred path from the former to the basepoint, a loop in M, and a preferred path from the basepoint to the latter. In general we perform many such tubings, possibly involving many parallel copies of the same cores, and also need to specify with which orientation the tubings should be performed.

$S^2 \times (S^2 \setminus \{n(i) \text{ points}\})$ and observing that removing the interior of a 4-disc does not change the fundamental group, we get an isomorphism

$$\pi_1\Big((S^2 \times S^2)^\circ \setminus \cup_{k=1}^{n(i)} B(i,k)\Big) \cong \Big\langle x(i,1), \dots, x(i,n(i)) \,\Big|\, \textstyle\prod_{k=1}^{n(i)} x(i,k) \Big\rangle .$$

This group is free, but it is helpful for the upcoming computation to present it in this manner, as having $n(i)$ generators and a single relation. Using Seifert–van Kampen, this implies

$$\pi_1\Big(M\#n(S^2 \times S^2) \setminus \cup_{i=1}^{n} \cup_{k=1}^{n(i)} B(i,k)\Big) \cong \pi * \mathop{\ast}_{i=1}^{n} \Big\langle x(i,1), \dots, x(i,n(i)) \,\Big|\, \textstyle\prod_{k=1}^{n(i)} x(i,k) \Big\rangle$$

Here the $x(i,k)$ are represented by meridians of the removed 2-spheres, connected to the basepoint in M by specified paths. Next, for $k \geq 2$ we tube $B(i,k)$ to $B(j_{i,k},1)$, where $j_{i,k}$ is defined by $x(i,k) = x_{j_{i,k}}$. For the tubing, use mutually disjoint, embedded paths, given as follows: go from $B(i,k)$ to the basepoint along the specified path, follow the loop $g(i,k)^{-1}$, and then go from the basepoint to $B(j_{i,k},1)$, again using the specified path. The tubing preserves the orientation if $\varepsilon(i,k)$ is 1 and reverses it if $\varepsilon(i,k)$ is -1. The end result is n embedded 2-spheres C_i.

Using Seifert–van Kampen (see e.g. [Boy88, Lemma 9]), each tubing has the effect of adding a relation $x(i,k) = g(i,k)x(j_{i,k},1)^{\varepsilon(i,k)}g(i,k)^{-1}$. Write

$$t(i,k) := g(i,k)x_{j_{i,k}}^{\varepsilon(i,k)}g(i,k)^{-1} = g(i,k)x(j_{i,k},1)^{\varepsilon(i,k)}g(i,k)^{-1} = g(i,k)x(i,k)^{\varepsilon(i,k)}g(i,k)^{-1}.$$

Note that $t(i,1) = x(i,1) = x_1$ and that

$$\prod_{k=1}^{n(i)} t(i,k) = \prod_{k=1}^{n(i)} g(i,k)x(i,k)^{\varepsilon(i,k)}g(i,k)^{-1} = r_i.$$

Let $\langle y_1 \dots y_\ell \mid s_1, \dots, s_m \rangle$ be a presentation of π. Introducing this and the relations coming from tubing, we obtain

$$\pi_1\Big(M\#n(S^2 \times S^2) \setminus \cup_{i=1}^{n} C_i\Big) \cong$$
$$\Big\langle \big(y_a\big)_{a=1}^{\ell}, \big(x(i,k)\big)_{k=1}^{n(i)} \,\Big|\, \big(s_b\big)_{b=1}^{m}, \textstyle\prod_{k=1}^{n(i)} x(i,k), \big(x(i,k) = t(i,k)\big)_{k=2}^{n(i)} \Big\rangle ,$$

where in each generator or relator in which i appears, i ranges from 1 to n. The relations $x(i,k) = t(i,k)$ can be cancelled against the generators $x(i,k)$, for $k = 2, \dots, n(i)$ and $i = 1, \dots, n$.

When doing this, we must substitute using these relations into $\prod_{k=1}^{n(i)} x(i,k)$. This yields

$$\prod_{k=1}^{n(i)} x(i,k) = x(i,1) \cdot \prod_{k=2}^{n(i)} x(i,k) = t(i,1) \cdot \prod_{k=2}^{n(i)} t(i,k) = \prod_{k=1}^{n(i)} t(i,k) = r_i.$$

We obtain

$$\left\langle y_1, \dots, y_\ell, x(i,1), i = 1, \dots, n \,\middle|\, s_1, \dots, s_m, \textstyle\prod_{k=1}^{n(i)} t(i,k) \right\rangle \cong \frac{\pi * \langle x_1, \dots, x_n \rangle}{(r_1, \dots, r_n)}.$$

The latter group is exactly $\pi(\mathcal{P})$. Finally, using Seifert–van Kampen, attaching the copies of $D^3 \times S^1$ from the upper boundaries of the 3-handles does not change the fundamental group, and so the inclusion induces an isomorphism

$$\pi_1(N) \xleftarrow{\cong} \pi_1\Big(M \# n(S^2 \times S^2) \setminus \cup_{i=1}^n C_i\Big) \cong \pi(\mathcal{P}).$$

Tracing through the construction, the map induced by the inclusion

$$\pi_1(N) \cong \pi(\mathcal{P}) \longrightarrow \pi_1(W) \cong \pi$$

is identified under the isomorphisms with the projection given by sending each x_i to e.

Let $x = [X] = [X(\mathcal{P})] \in \mathrm{Wh}(\pi)$. If $\dim(x) = 0, 2$, then it follows that the map $\pi_1(N) \to \pi_1(W)$ induced by the inclusion is an isomorphism, and so W is an h-cobordism. By construction $\tau(W, M) = x$. On the other hand, if $\dim(x) = 3$, then $\pi_1(N) \to \pi_1(W)$ is not an isomorphism, and so W arising from the standard construction fails to be an h-cobordism. This completes the proof of Theorem C.

Example 3.5. For $\pi = C_5 = \langle g \mid g^5 \rangle$ and $x \in \mathrm{Wh}(\pi)$ represented by the unit $1 - g + g^2$, we can assume $\mathcal{P}$ to have a single generator x and a single relation $x(gx^{-1}g^{-1})(g^2xg^{-2})$. Then the above construction has outgoing boundary where $\pi_1(N)$ surjects onto S_5, so is not isomorphic to C_5 [Coh77, Example (4.5)]. That is, $\pi(\mathcal{P})$ is a proper admissible extension. A computer search with GAP among similar admissible extensions of C_5 yielded no trivial examples.

Remark 3.6. We finish by commenting on how far a general h-cobordism is from the standard construction. Standard handle trading techniques yield the following: every h-cobordism $W\colon M \rightsquigarrow N$ can be assumed to only have 2- and 3-handles, with 2-handles necessarily trivially attached, as many 3-handles as 2-handles, and the 3-handles attached simultaneously. Thus it is determined by disjoint embeddings $\varphi_i\colon S^2 \hookrightarrow M \# n(S^2 \times S^2)$ with trivialised normal bundles. However, these need not be isotopic to the type of 2-spheres used in the standard construction. It is not apparent whether it is possible to extract from these 2-spheres an admissible presentation or relate its torsion to $\tau(W, M)$.

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