SMOOTHING 3-MANIFOLDS IN 5-MANIFOLDS

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Abstract. We show that every locally flat topological embedding of a 3-manifold in a smooth 5-manifold is homotopic, by a small homotopy, to a smooth embedding. We deduce that topologically locally flat concordance implies smooth concordance for smooth surfaces in smooth 4-manifolds.

1. Introduction

Let $Y^3 = Y_1 \sqcup \cdots \sqcup Y_m$ be a compact 3-manifold with connected components Y_i, and let N^5 be a compact, connected, smooth 5-manifold. Note that Y and N are possibly nonorientable and can have nonempty boundary. Since Y is 3-dimensional it admits a unique smooth structure up to isotopy [Moi52], [Mun60b, Theorem 6.3], [Whi61, Corollary 1.18].

Theorem A. *Let $f\colon Y \to N$ be a locally flat proper topological embedding that is smooth near ∂Y. Then f is homotopic rel. boundary, via an arbitrarily small homotopy, to a smooth embedding.*

Here *proper* means that $f^{-1}(\partial N) = \partial Y$. It is not possible in general to isotope f to a smooth embedding, so the homotopy in the theorem is necessary. For instance, Lashof [Las71] constructed a locally flat knot $L \cong S^3 \subseteq S^5$ that is not isotopic, in fact not even concordant, to any smooth knot. We will make crucial use of Lashof's knot in our proof of Theorem A.

In the rest of the introduction, we explain an application to concordance of surfaces, then we compare with the situation for codimension two embeddings in other dimensions, before finally outlining our proof of Theorem A.

1.1. Topological concordance implies smooth concordance for surfaces in 4-manifolds.

Let Σ be a closed, smooth surface, possibly disconnected, and possibly nonorientable. We consider a smooth, closed, connected 4-manifold X, again possibly nonorientable, and two smooth submanifolds Σ_0 and Σ_1 in X with $\Sigma_0 \cong \Sigma \cong \Sigma_1$.

Definition 1.1. We say that Σ_0 and Σ_1 are *topologically concordant* (respectively *smoothly concordant*) if there is a locally flat (respectively smooth) submanifold $C \cong \Sigma \times I$, properly embedded in $X \times I$, whose intersection with $X \times \{0,1\}$ is precisely $\Sigma_0 \subseteq X \times \{0\}$ and $\Sigma_1 \subseteq X \times \{1\}$. We call C a *topological concordance* (respectively *smooth concordance*).

Corollary 1.2. *Suppose that C is a topological concordance between $\Sigma_0 \subseteq X \times \{0\}$ and $\Sigma_1 \subseteq X \times \{1\}$. Then the inclusion map $C \to X \times I$ is homotopic rel. $\Sigma_0 \cup \Sigma_1$, via an arbitrarily small homotopy, to an embedding whose image is a smooth concordance between Σ_0 and Σ_1.*

This follows immediately from Theorem A by taking $Y = \Sigma \times I$, $N = X \times I$, and $f\colon Y \to N$ to be an embedding with $C = f(Y)$.

Special cases of Corollary 1.2 were known before. First, Kervaire [Ker65] proved that every 2-knot is slice. This holds in both categories, from which it follows that smooth and topological concordance coincide for 2-knots. Sunukjian [Sun15] proved more generally that homologous connected surfaces in a simply-connected 4-manifold X are both smoothly and topologically concordant. Again, it follows immediately that smooth and topological concordance coincide. Similarly Cha-Kim [CK23, Corollary J] proved that smooth and topological concordance coincide for smoothly embedded spheres with a common smoothly embedded geometrically dual framed sphere.

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Work defining surface concordance obstructions includes [Sto93, Sch19, KM21, ST22, AMY21]. Other than in [Sto93], the authors restricted to the smooth category. Our result implies that one automatically obtains topological concordance obstructions.

1.2. **Comparison with other dimensions.** We start with low dimensions. In dimension 3, every locally flat embedding $Y^1 \subseteq N^3$ is isotopic to a smooth embedding. On the other hand, in dimension 4, the existence of topologically slice knots that are not smoothly slice implies the existence of a locally flat embedding $D^2 \hookrightarrow D^4$ that is not even homotopic rel. boundary to a smooth embedding. There are also examples of closed locally flat surfaces in closed 4-manifolds, in particular in $S^2 \times S^2$, $\mathbb{CP}^2$, and $S^2 \widetilde{\times} S^2$, that cannot be smoothed up to homotopy [Kug84, Rud84, Luo88, LW90].We deal with dimension 5 in this article. The analogue of Theorem A for locally flat embeddings of smooth 4-manifolds embedded in smooth 6-manifolds is open, and we intend to investigate it in future work.

Now we discuss high dimensions. For codimension 2 proper embeddings $f\colon Y^m \to N^{m+2}$, when the dimension m of Y is greater or equal to 5, Schultz [Sch] proved the following cf. [LR65].

Theorem 1.3 (Schultz)**.** *Let $m \geq 5$ and $n > m$. Let N^n be a smooth compact n-manifold, and let Y^m be a compact topological manifold equipped with a smooth structure near ∂Y. Let $f\colon Y \to N$ be a locally flat proper topological embedding that is smooth near ∂Y. Then there is a smooth structure on Y, extending the given smooth structure on ∂Y, such that f is isotopic rel. boundary to a smooth embedding if and only if Y has a topological vector bundle neighbourhood.*

Topological vector bundle neighbourhoods always exist for locally flat codimension 1 and 2 embeddings [Bro62, KS75], so Schultz [Sch] deduced the following result.

Theorem 1.4 (Schultz)**.** *Let $k = 1$ or 2, let $m \geq 5$, and let $n = m+k$. Let N^n be a smooth compact n-manifold, and let Y^m be a compact topological manifold equipped with a smooth structure near ∂Y. Let $f\colon Y \to N$ be a locally flat proper topological embedding that is smooth near ∂Y. Then there is a smooth structure on Y, extending the given smooth structure on ∂Y, such that f is isotopic rel. boundary to a smooth embedding.*

Note that in the statements of Theorems 1.3 and 1.4, Y is not a priori smoothable. The existence of a topological vector bundle neighbourhood for an embedding $f\colon Y \to N$ guarantees a smooth structure on $Y \times \mathbb{R}^p$ for some $p \in \mathbb{N}^*$. Hence, for $m \geq 5$, the Product Structure Theorem [KS77, Essay I] implies that Y is smoothable. The proofs of Theorems 1.3 and 1.4 then proceed by using smoothing theory and the Concordance Implies Isotopy Theorem [KS77, Essay I] for smooth structures on Y.

If one first fixes a smooth structure on Y, then the induced structure on Y that emerges from Schultz's argument need not be isotopic to the fixed one. This is a feature of the problem, not a failure of the proof. In fact, if we fix a smooth structure on Y, in general in high dimensions f is not even homotopic to a smooth embedding. For example, Hsiang-Levine-Szczarba [HLS65] showed that the exotic 16-sphere does not embed smoothly in S^{18}. Certainly it does embed topologically.

The results from [KS77] do not apply in the same way for $Y^3 \subseteq N^5$. Our approach is rather different. We fix a smooth structure on a tubular neighbourhood of $f(Y)$ and try to extend it to all of N. As we will describe next, we face obstructions along the way that will require us in general to modify the embedding by a small homotopy to obtain a smooth embedding of Y in N.

1.3. **Outline of the proof of Theorem A.** For a submanifold K of a manifold X with an open tubular neighbourhood i.e. the image of an embedding of a normal bundle, denote the tubular neighbourhood by νK. Write $\overline{\nu} K$ for the closure of the tubular neighbourhood of K in X, which has the structure of a disc bundle over K. Given a closed subset C of X, *a smooth structure on C* will always mean a smooth structure on an open neighbourhood U of C in X.

The proof of Theorem A breaks naturally into two distinct steps, the outlines of which we shall explain next. For a smooth structure σ on a topological manifold X, we write X_σ to specify that X is equipped with the smooth structure σ. In what follows, we will write N_{std} for N equipped with the given smooth structure.

Step 1: *We show that $f\colon Y\to N$ is homotopic, by a small homotopy, to a smooth embedding $g\colon Y\to N_\sigma$, for some σ.*

We write $M := f(Y)$ for the image of f. The idea of the proof is to consider a standard smooth structure on $\overline{\nu}M$ and on ∂N, and then to try to extend this to all of N. We denote the exterior of M by $W_f := N\setminus \nu M$. Smoothing theory (recapped in Section 2) gives a Kirby-Siebenmann obstruction in $H^4(W_f,\partial W_f;\mathbb{Z}/2)$, that vanishes if and only if the smooth structure on ∂W_f extends to all of W_f. It turns out that this obstruction does not always vanish, but that by taking ambient connected sums of M with copies of Lashof's nonsmoothable 3-knot $L\cong S^3\subseteq S^5$ from [Las71], which we discuss in Section 2.2, we can arrange that this obstruction vanishes. Whence f is homotopic, via a small homotopy, to $g\colon Y\to N$ such that $M' := g(Y)$ is smooth in some smooth structure σ on N, that restricts to the given smooth structure on ∂N.

Step 2: *We show that $g\colon Y\to N_\sigma$ is homotopic, via a small homotopy, to a smooth embedding $g'\colon Y\to N_{\mathrm{std}}$.*

Smoothing theory implies that we can arrange for the smooth structure σ on N and the given smooth structure std to agree away from a tubular neighbourhood νS of a surface $S\subseteq N$. By transversality we can assume that $g(Y)$ intersects S in finitely many points, in a neighbourhood of which $M' := g(Y)$ is smooth in σ but not in std. This reduces the smoothing problem for M' in std to finitely many local problems, which can be resolved using a proof analogous to Kervaire's proof [Ker65, Théorème III.6] that every 2-knot is smoothly slice. Kervaire's result was generalised by Sunukjian [Sun15], who showed that homologous connected surfaces in 1-connected 4-manifolds are smoothly concordant, and it is Sunukjian's arguments that apply in our situation.

Remark 1.5. The changes to f in Steps 1 and 2 can be characterised as topological isotopies, together with adding and removing local knots.

Organisation. In Section 2 we recap smoothing theory, prove lemmas on properties of the Kirby-Siebenmann invariant, and recall Lashof's nonsmoothable 3-knot. We prove Step 1 in Section 3, and we prove Step 2 in Section 4. Then in Section 5 we give conditions under which smoothing up to isotopy is possible.

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2. Smoothing theory

In this section we give a brief recap of smoothing theory, and recall the results we will need. Smoothing theory was developed by Cairns [Cai61], Munkres [Mun60b, Mun65, Mun66, Mun64, Mun60a], Milnor [Mil63, Mil64], Hirsch [Hir63], Hirsch-Mazur [HM74], Lashof-Rothenberg [LR65], and Cerf [Cer59, Cer64b, Cer64c, Cer64a, Cer64d, Cer68], among others. Their goal, which they achieved to a large extent, was to understand which PL manifolds admit smooth structures, and if so how many. The theory was extended around 1970 by Kirby and Siebenmann [KS77] to allow one to start with a topological manifold, provided that one is not trying to understand smooth structures on a 4-manifold. For the purposes of this article, since we work in dimensions four and five, the smooth and PL categories are interchangeable. Since it is more common nowadays to work in the smooth category, we shall also do so.

2.1. **Recap of smoothing theory.** Let X be a topological n-manifold possibly with boundary, let C be a closed subset of X, and let σ be a smooth structure on an open neighbourhood U of C. Let $V\subseteq U$ be a smaller open neighbourhood of C. Denote the set of isotopy classes of smooth structures on X that agree with σ near C by $\mathcal{S}_{\mathrm{Diff}}(X,C,\sigma)$. We write $\mathrm{BTOP}(k)$ for the classifying space for topological $\mathbb{R}^k$ bundles, and $\mathrm{BO}(k)$ for the classifying space for rank k smooth vector bundles. Define $\mathrm{BTOP} := \mathrm{colim}_k \mathrm{BTOP}(k)$ and $\mathrm{BO} := \mathrm{colim}_k \mathrm{BO}(k)$, the corresponding stable bundle classifying spaces. Consider the following diagram, which is induced by the stable classifying maps of the tangent bundle of a neighbourhood U of C and the stable tangent microbundle of X:

$$\begin{tikzcd} & \mathrm{TOP}/\mathrm{O} \arrow[d] \\ V \arrow[r, "T_V"] \arrow[d] & \mathrm{BO} \arrow[d] \\ X \arrow[r, "\tau_X"'] \arrow[ur, dashed] & \mathrm{BTOP} \arrow[d] \\ & \mathrm{B}(\mathrm{TOP}/\mathrm{O}). \end{tikzcd}$$

Smoothing theory implies that for $n \geq 6$ or ($n = 5$ and $\partial X \subseteq C$), isotopy classes of smooth structures on X correspond to lifts $X \to \mathrm{BO}$ of the map $X \to \mathrm{BTOP}$, relative to the fixed lift on the smaller neighbourhood $V \subseteq U$. The vertical sequence is a principal fibration, which implies that such a lift exists if and only if the composite $X \to \mathrm{BTOP} \to \mathrm{B}(\mathrm{TOP}/\mathrm{O})$ is null-homotopic, and that homotopy classes of such lifts correspond to $[(X, V), (\mathrm{TOP}/\mathrm{O}, *)]$, homotopy classes of maps $X \to \mathrm{TOP}/\mathrm{O}$ that send V to the base point.

The main result of smoothing theory, applied to 5-manifolds, reads as follows [KS77, Theorem IV.10.1].

Theorem 2.1. *Let X be a 5-dimensional topological manifold, let C be a closed subset of X with $\partial X \subseteq C$, and fix a smooth structure σ on an open neighbourhood U of C.*

(i) There is an obstruction $\mathrm{ks}(X, C) := \mathrm{ks}(X, C, U, \sigma) \in H^4(X, C; \mathbb{Z}/2)$ that vanishes if and only if X admits a smooth structure extending the given smooth structure on some neighbourhood $V \subseteq U$ of C.

(ii) Given two smooth structures σ and π on X extending the given smooth structure on $U \supseteq C$, there is an obstruction $\mathrm{ks}(\sigma, \pi) \in H^3(X, C; \mathbb{Z}/2)$ that vanishes if and only if there is a neighbourhood $V \subseteq U$ of C such that σ and π are isotopic rel. V, i.e. if there is a homeomorphism $f\colon X \to X$ with $f|_V = \mathrm{Id}$, such that $f^*(\pi) = \sigma$ and such that f is topologically isotopic rel. V to Id_X.

(iii) The Kirby-Siebenmann obstructions $\mathrm{ks}(X, C)$ and $\mathrm{ks}(\sigma, \pi)$ from (i) and (ii) are natural for restriction to open submanifolds of X. More precisely, let W be an open submanifold of X and let $i\colon W \hookrightarrow X$ be the inclusion map. Then $i^*\colon H^4(X, C; \mathbb{Z}/2) \to H^4(W, W \cap C; \mathbb{Z}/2)$ sends $\mathrm{ks}(X, C)$ to $\mathrm{ks}(W, W \cap C)$ and $i^*\colon H^3(X, C; \mathbb{Z}/2) \to H^3(W, W \cap C; \mathbb{Z}/2)$ sends $\mathrm{ks}(\sigma, \pi)$ to $\mathrm{ks}(\sigma|_W, \pi|_W)$.

(iv) Given a smooth structure on some neighbourhood V of C in X, the Kirby-Siebenmann obstruction $\mathrm{ks}(X, C)$ from (i) is natural with respect to restriction to a neighbourhood $V' \subseteq V$ of a closed subset $C' \subseteq C$. That is, the inclusion map $H^4(X, C; \mathbb{Z}/2) \to H^4(X, C'; \mathbb{Z}/2)$ sends $\mathrm{ks}(X, C)$ to $\mathrm{ks}(X, C')$.

Proof. The first three items of the theorem for PL structures instead of smooth structures follows from [KS77, Theorem IV.10.1] and the fact that $\mathrm{TOP}/\mathrm{PL} \simeq K(\mathbb{Z}/2, 3)$ [KS77, Section IV.10.12]. However PL 5-manifolds with smooth boundary admit a unique smooth structure up to isotopy, by smoothing theory and since PL/O is 6-connected [HM74, Mun60a, Cer68, KM63]. Hence it is legitimate to replace PL structures by smooth structures, as we have done.

The final item can be seen from the following diagram.

$$\begin{tikzcd} V' \arrow[r] \arrow[dr] & V \arrow[r, "T_V"] \arrow[d] & \mathrm{BO} \arrow[d] & \\ & X \arrow[r, "\tau_X"'] & \mathrm{BTOP} \arrow[r] & \mathrm{B}(\mathrm{TOP}/\mathrm{O}). \end{tikzcd}$$

The obstructions $\mathrm{ks}(X, C)$ and the obstructions $\mathrm{ks}(X, C')$ are both represented by the map $X \xrightarrow{\tau_X} \mathrm{BTOP} \to \mathrm{B}(\mathrm{TOP}/\mathrm{O})$, and the inclusion induced map sends the former to the latter. $\square$

Next we apply Theorem 2.1 to deduce a naturality result for the Kirby-Siebenmann obstructions, that will be useful for submanifolds with corners.

Let K be a smooth 5-manifold with corners. Suppose the corner set of K, denoted by $\angle K$, separates ∂K into $\partial_1 K$ and $\partial_2 K$. Note that $\partial_1 K$ and $\partial_2 K$ are smooth manifolds with boundary. Fix a smooth structure[1] σ on a neighbourhood U of ∂K. By [Wal16, Proposition 1.5.6], U contains a smooth embedding of $\partial_1 K \times [0,1)$ in K. Let K' denote the result of attaching $\partial_1 K \times [0,1)$ to K by the map $h\colon \partial_1 K \times \{0\} \to \partial_1 K$ given by $h(x,0) = x$, and extend $\sigma|_{\partial_1 K}$ to a product structure along $[0,1)$ in $\partial_1 K \times [0,1)$. Denote the resulting smooth structure on $U' := U \cup \partial_1 K \times [0,1)$ by σ'. Note that U' is a smooth manifold with boundary, i.e. no corners. Apply the same procedure to K' along $\partial K'$ to obtain an unbounded manifold K'' and a smooth structure σ'' on $U'' := U' \cup \partial K' \times [0,1)$. Let $j\colon K \hookrightarrow K''$ be the inclusion map.

Definition 2.2. Let K, K'', and j be as above. Let σ be a smooth structure on a neighbourhood U of ∂K. Define the Kirby-Siebenmann obstruction $\mathrm{ks}(K, \partial K)$ as $j^* \mathrm{ks}(K'', \partial K)$ where $\mathrm{ks}(K'', \partial K) = \mathrm{ks}(K'', \partial K, U'', \sigma'')$ is the obstruction to extending the smooth structure σ'' on U'' to K''.

Let X be a smooth 5-manifold with boundary and let K be a smooth 5-manifold with corners that is a submanifold of X such that the corner set of K separates ∂K into $\partial_1 K := K \cap \partial X$ and $\partial_2 K$ with $\operatorname{Int} \partial_2 K \subseteq \operatorname{Int} X$. (By definition of a submanifold, $\partial_2 K$ intersects ∂X transversely.) Consider a smooth structure σ on a neighbourhood U of $\partial X \cup \partial K$ such that $\partial_2 K \hookrightarrow U$ is smooth; this condition guarantees the existence of a smooth bicollar neighbourhood of $\partial_2 K$ in U, which will be implicitly used in the proof of the next proposition.

Proposition 2.3. *Let $i\colon (K, \partial K) \hookrightarrow (X, \partial X \cup \partial K)$ be the inclusion. The induced map*
$$i^*\colon H^4(X, \partial X \cup \partial K; \mathbb{Z}/2) \to H^4(K, \partial K; \mathbb{Z}/2)$$
sends $\mathrm{ks}(X, \partial X \cup \partial K)$ to $\mathrm{ks}(K, \partial K)$.

Proof. Let X' be the open topological manifold obtained from X by attaching an exterior collar $\partial X \times [0,1)$, where x in ∂X is identified with $(x,0)$ in $\partial X \times [0,1)$. Extend σ, by taking a product structure along $\partial X \times [0,1)$, to $U' := U \cup \partial X \times [0,1)$, which is a neighbourhood of $\partial X \cup \partial K$ in X'. Let σ' be the result smooth structure on U'. Then, by 'absorbing the boundary' [KS77, Proposition IV.2.1], this construction determines a natural bijection $\theta\colon \mathcal{S}_{\mathrm{Diff}}(X, \partial X \cup \partial K, \sigma) \to \mathcal{S}_{\mathrm{Diff}}(X', \partial X \cup \partial K, \sigma')$ and it follows from [KS77, Theorem IV.10.1] and [KS77, Remark IV.10.2] that $\mathrm{ks}(X, \partial X \cup \partial K) \mapsto \mathrm{ks}(X', \partial X \cup \partial K)$ under the isomorphism on $H^4(-, \partial X \cup \partial K; \mathbb{Z}/2)$ induced by the obvious homotopy equivalence $X' \simeq X$. Consider the following diagram:

$$\begin{tikzcd}
H^4(X, \partial X \cup \partial K; \mathbb{Z}/2) \arrow[rr, "\cong"] \arrow[d, "i^*"] & & H^4(X', \partial X \cup \partial K; \mathbb{Z}/2) \arrow[d, "i_1^*"] \\
H^4(K, \partial K; \mathbb{Z}/2) & H^4(K'', \partial K; \mathbb{Z}/2) \arrow[l, "j^*"'] & H^4(X', \partial K; \mathbb{Z}/2). \arrow[l, "g^*"']
\end{tikzcd}$$

The map i_1^* is induced by the inclusion map $i_1\colon (X', \partial K) \to (X', \partial X \cup \partial K)$, j^* is induced by the inclusion $j\colon (K, \partial K) \to (K'', \partial K)$ from Definition 2.2, and g^* is induced by the inclusion $(K'', \partial K) \hookrightarrow (X', \partial K)$. As per the above discussion, $\mathrm{ks}(X, \partial X \cup \partial K)$ is sent to $\mathrm{ks}(X', \partial X \cup \partial K)$ by the top horizontal map. By Theorem 2.1 (iv), $i_1^* \mathrm{ks}(X', \partial X \cup \partial K) = \mathrm{ks}(X', \partial K)$. It follows from Theorem 2.1 (iii) that $g^* \mathrm{ks}(X', \partial K) = \mathrm{ks}(K'', \partial K)$. Finally by Definition 2.2 we have that $j^* \mathrm{ks}(K'', \partial K) = \mathrm{ks}(K, \partial K)$. This concludes the proof that $i^* \mathrm{ks}(X, \partial X \cup \partial K) = \mathrm{ks}(K, \partial K)$. □

In practice, the manifold with corners K will be either a closed tubular neighbourhood $\overline{\nu} f(Y)$ of a locally flat proper embedding $f\colon Y \to N$, or the complement $N \setminus \nu f(Y)$. In fact, let $p\colon \overline{\nu} f(Y) \to f(Y)$ be a disc bundle and denote its boundary sphere bundle by Σ. Then $\overline{\nu} f(Y)$ is a smooth manifold with corners (note that it is not necessarily smooth in N) and $\angle \overline{\nu} f(Y) = p^{-1} \partial f(Y) \cap \Sigma$ separates $\partial \overline{\nu} f(Y)$ in two parts with closures $p^{-1} f(\partial Y)$ and Σ.

We also need the following more detailed characterisation of the obstruction $\mathrm{ks}(\sigma, \pi)$. We write X_σ to denote a topological 5-manifold X equipped with a smooth structure σ, and let π be another smooth structure on X that agrees with σ near ∂X.

[1]Note that this is a *smooth structure with corners*, which means that it is a maximal atlas in which two charts with corners (U, ϕ) and (V, θ) are smoothly compatible if $\phi \circ \theta^{-1}\colon \theta(U \cap V) \to \phi(U \cap V)$ admits a smooth extension to an open neighbourhood of each point. See [Lee13].

Proposition 2.4. *Suppose that $S \subseteq X$ is a closed surface smoothly embedded in* $\operatorname{Int} X_\sigma$ *whose $\mathbb{Z}/2$-fundamental class is Poincaré dual to* $\mathrm{ks}(\sigma,\pi) \in H^3(X, \partial X; \mathbb{Z}/2)$. *Then there is an arbitrarily small isotopy of σ, supported away from S and ∂X, to a smooth structure that agrees with π on $X \setminus S$.*

Proof. Using the inclusion $X \setminus S \to X$ we have a map $H^3(X, \partial X; \mathbb{Z}/2) \to H^3(X \setminus S, \partial X; \mathbb{Z}/2)$. By naturality of Kirby-Siebenmann invariants (Theorem 2.1 (iii)), this sends the Kirby-Siebenmann invariant $\mathrm{ks}(\sigma,\pi)$ to the invariant of the restricted structures $\mathrm{ks}(\sigma|_{X\setminus S}, \pi|_{X\setminus S})$. We will denote restricted structures by $\sigma| := \sigma|_{X\setminus S}$, and similarly for π, from now on. The long exact sequence of the triple $\partial X \subseteq X \setminus S \subseteq X$ gives the top row of the following diagram.

$$\begin{tikzcd}
H^3(X, X\setminus S; \mathbb{Z}/2) \arrow[r] \arrow[d, "\cong"] & H^3(X, \partial X; \mathbb{Z}/2) \arrow[r] \arrow[dd, "\cong"] & H^3(X\setminus S, \partial X; \mathbb{Z}/2) \\
H^3(\overline{\nu} S, \partial \overline{\nu} S; \mathbb{Z}/2) \arrow[d, "\cong"] & & \\
H_2(S; \mathbb{Z}/2) \arrow[r] & H_2(X; \mathbb{Z}/2) &
\end{tikzcd}$$

The vertical isomorphisms are given by combining homotopy invariance of homology, excision, and Poincaré-Lefschetz duality. It follows from the diagram that the Poincaré dual to $[S] \in H_2(X;\mathbb{Z})$, which by hypothesis equals $\mathrm{ks}(\sigma,\pi)$, lies in the kernel of the map $H^3(X, \partial X; \mathbb{Z}/2) \to H^3(X \setminus S, \partial X; \mathbb{Z}/2)$. Thus $\mathrm{ks}(\sigma|, \pi|) = 0 \in H^3(X \setminus S, \partial X; \mathbb{Z}/2)$.

By smoothing theory (Theorem 2.1) there is an isotopy of $\sigma|$ to $\pi|$ on $X \setminus S$ rel. ∂X. That is, we have an isotopy of homeomorphisms $f_t \colon X \setminus S \to X \setminus S$, where $f_0 = \mathrm{Id}$, $f_t|_{\partial X} = \mathrm{Id}_{\partial X}$, and $f_1^*(\pi|) = \sigma|$. To prove the desired result we have to delve into the proof of Theorem 2.1 a little. Such an isotopy is constructed chart by chart, and within each chart via a decomposition into handles. Then the handles are smoothed iteratively using [KS77, Theorem I.3.1]. Let d be a metric on X. We can and shall choose charts $\{U_i\}$ covering $X \setminus S$ to be such that if there exists $x \in U_i$ with $d(x, S) < \varepsilon$, then $\mathrm{diam}(U_i) < \varepsilon/10$. We can also make all charts have diameter smaller than an arbitrarily chosen global positive constant. The construction of f_t guarantees that for all i, if $x \in U_i$, then $f_t(x) \in U_i$ for all $t \in [0,1]$. It follows from this and the fact that we controlled the size of the charts as they approach S that f_t extends continuously to an isotopy $F_t \colon X \to X$ that fixes S pointwise for all $t \in [0,1]$. This gives the desired isotopy of σ to a smooth structure σ' on X such that $\sigma'|_{X\setminus S} = \pi|_{X\setminus S}$, i.e. they agree away from S. Since we controlled the global size of all charts, we can also arrange for the isotopy to be arbitrarily small. □

2.2. **Lashof's nonsmoothable 3-knot.** Lashof [Las71] constructed a locally flat 3-knot $L \cong S^3 \subseteq S^5$ that is not isotopic to any smooth knot. As observed by Kwasik and Vogel [KV84, Kwa87], Lashof's knot bounds a Seifert 4-manifold V in S^5 with $\mathrm{sign}(V)/8 \equiv 1 \mod 2$. We can use this to explain why L is not smoothable. The proof is as follows. If L were smoothable, it would bound a smooth Seifert 4-manifold V', which would be spin by naturality of w_2, and therefore would satisfy $\mathrm{sign}(V')/8 \equiv 0 \mod 2$ by Rochlin's theorem. Since the signature of a Seifert 4-manifold is a knot invariant [Lev69], we arrive at a contradiction and it follows that L cannot be smoothed. Since the signature is a concordance invariant, it follows also that L is not concordant to any smooth 3-knot. Let $E_L := S^5 \setminus \nu L$ be the exterior of L, and equip $\partial E_L \cong S^3 \times S^1$ with a standard smooth structure.

Lemma 2.5. *The Kirby-Siebenmann invariant of E_L satisfies*

$$\mathrm{ks}(E_L, \partial E_L) = 1 \in H^4(E_L, \partial E_L; \mathbb{Z}/2) \cong H_1(E_L; \mathbb{Z}/2) \cong \mathbb{Z}/2.$$

Proof. If $\mathrm{ks}(E_L, \partial E_L)$ were trivial, then by smoothing theory there would be a smooth structure τ on S^5 extending the standard smooth structure on $\overline{\nu} L$. Thus L would be smooth in τ. But in fact there is a unique smooth structure on S^5 up to isotopy [KM63], and hence L would be isotopic to a smooth knot in the standard smooth structure on S^5. Since Lashof proved this is not the case, we deduce that $\mathrm{ks}(E_L, \partial E_L)$ is indeed nontrivial. □

3. Smoothing the complement of an embedding

In this section we prove the following result, which proves Step 1 from Section 1.3.

Proposition 3.1. *Let N be a compact, connected, smooth 5-dimensional manifold with (possibly empty) boundary, let Y be a compact 3-dimensional manifold with (possibly empty) boundary, and let $f\colon Y \to N$ be a locally flat proper topological embedding such that f is smooth near ∂Y. Then f is homotopic rel. boundary, via an arbitrarily small homotopy, to a smooth embedding in some smooth structure σ on N that agrees with the given smooth structure on N near ∂N.*

Let $Y = Y_1 \sqcup \cdots \sqcup Y_m$, where each Y_i is connected. We write $M := f(Y)$ and $M_i := f(Y_i)$. By [KS75], M has a normal vector bundle in N. Let $\nu M \subseteq N$ denote the image of an embedding of the normal bundle. Let $W_f := N \setminus \nu M$, $E_i := \partial \overline{\nu} M_i \cap \partial W_f$, and define $E := \bigcup_{i=1}^m E_i$; see Figure 1.

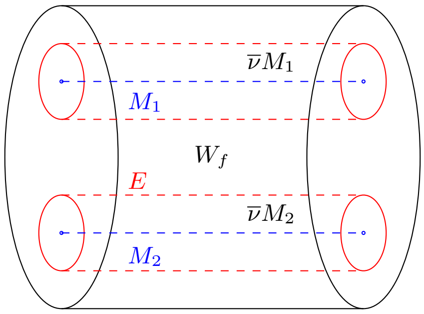

Figure 1. A schematic diagram of N decomposed as $N = W_f \cup_E \overline{\nu} M$, where $M = f(Y)$, showing the case that $Y = Y_1 \sqcup Y_2$ has two connected components with nonempty boundary.

We fix a smooth structure on a neighbourhood of $\partial N \cup E = \partial \overline{\nu} M \cup \partial W_f$. To do this, we use that Y, as a 3-manifold, admits an essentially unique smooth structure. Since $\mathrm{TOP}(2) \simeq \mathrm{O}(2)$, we may assume the normal bundle of M has $\mathrm{O}(2)$ structure group, and hence that the total space of the normal bundle is smooth. The closed tubular neighbourhood $\overline{\nu} M$, which has the structure of a D^2-bundle $\pi\colon \overline{\nu} M \to M$, therefore has the structure of a smooth manifold with corners, with the property that $M \hookrightarrow \overline{\nu} M$ is a smooth map. The corner set gives rise to a decomposition

$$\partial \overline{\nu} M = E \cup \pi^{-1}(\partial M),$$

such that E and $\pi^{-1}(\partial M)$ become smooth 4-manifolds with boundary. We have a smooth structure on a collar neighbourhood of ∂N in N. Next, we choose a topological bicollar neighbourhood of E in N and endow it with a smooth structure that is compatible with the given smooth structure on a neighbourhood of ∂N. Choose the collar neighbourhood of E in $\overline{\nu} M$ to be smooth with respect to the smooth structure on $\overline{\nu} M$. For the outside collar of E into W_f, first consider that given a collar, we obtain a product smooth structure on that collar induced from the smooth structure of E. In a neighbourhood of ∂N, choose the outside collar of E so that the resulting smooth structure is compatible with the smooth structure we already have near ∂N. We can do this because f is smooth near ∂Y. Then extend the collar to the rest of E. Now extend the smooth structure on E to its bicollar as a product structure. Since we chose collars carefully to arrange for the two smooth structures on the bicollar of E and the collar of ∂N to be compatible, we obtain a smooth structure on a neighbourhood of $\partial N \cup E$. We obtain as well a smooth structure on a neighbourhood of ∂W_f in W_f which is the smoothly compatible collar neighbourhood of $\partial N \setminus \mathrm{Int}(\pi^{-1}(\partial M))$ and the part of the bicollar neighbourhood of E that lies in W_f. As such, the neighbourhood of ∂W_f is a smooth manifold with corners, with corner set $\partial \pi^{-1}(\partial M)$.

By Theorem 2.1 we therefore have an obstruction

$$\mathrm{ks}(N, \partial \overline{\nu} M \cup \partial W_f) \in H^4(N, \partial \overline{\nu} M \cup \partial W_f)$$

to extending this smooth structure to all of N. It will be shown in Proposition 3.3 below that $\mathrm{ks}(N, \partial\overline{\nu}M \cup \partial W_f)$ is determined by

$$\mathrm{ks}(\overline{\nu}M, \partial\overline{\nu}M) \in H^4(\overline{\nu}M, \partial\overline{\nu}M) \text{ and } \mathrm{ks}(W_f, \partial W_f) \in H^4(W_f, \partial W_f; \mathbb{Z}/2).$$

Since the smooth structure on E was obtained by restricting a structure on $\overline{\nu}M$, it follows that $\mathrm{ks}(\overline{\nu}M, \partial\overline{\nu}M) = 0$. Thus, $\mathrm{ks}(N, \partial\overline{\nu}M \cup \partial W_f)$ is determined by $\mathrm{ks}(W_f, \partial W_f)$. If the latter vanishes, then so does $\mathrm{ks}(N, \partial\overline{\nu}M \cup \partial W_f)$. We also note that $\mathrm{ks}(N, \partial N) = 0$, because N is a smooth manifold.

Our goal will therefore be to modify f by a small homotopy to arrange for $\mathrm{ks}(W_f, \partial W_f) = 0$. To begin, we record a homology computation in the next lemma.

Lemma 3.2. *The homology of E satisfies $H_1(E; \mathbb{Z}/2) \cong \oplus_{i=1}^m (H_1(M_i; \mathbb{Z}/2) \oplus B_i)$, where B_i is a quotient of $\mathbb{Z}/2$, and may depend on i. If B_i is nontrivial then it is generated by a meridian of M_i.*

Proof. Since $S^1 \hookrightarrow E_i \to M_i$ is a fibration, M_i is path connected, and $\pi_1(M_i)$ acts trivially on $H_*(S^1; \mathbb{Z}/2)$, we will use the Leray-Serre spectral sequence to compute $H_1(E_i; \mathbb{Z}/2)$. We have

$$E^2_{p,q} \cong H_p(M_i; H_q(S^1; \mathbb{Z}/2)) \cong \begin{cases} H_p(M_i; \mathbb{Z}/2) & q = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

and $E^3_{p,q} = E^\infty_{p,q}$. Since the coefficient group is a field, the extension problem is trivial and

$$\begin{aligned} H_1(E_i; \mathbb{Z}/2) &\cong E^\infty_{1,0} \oplus E^\infty_{0,1} \cong H_1(M_i; \mathbb{Z}/2) \oplus \mathbb{Z}/2/\operatorname{Im}(d^2\colon E^2_{2,0} \to E^2_{0,1}) \\ &\cong H_1(M_i; \mathbb{Z}/2) \oplus B_i. \end{aligned}$$

It follows that $H_1(E; \mathbb{Z}/2) \cong \bigoplus_{i=1}^m H_1(M_i; \mathbb{Z}/2) \oplus B_i$. The B_i are quotients of the terms on the E^2-page $H_0(M_i; H_1(S^1; \mathbb{Z}/2)) \cong H_1(S^1; \mathbb{Z}/2)$, and so if B_i is nontrivial it is generated by a meridian to M_i, as asserted. □

Whether or not B_i is trivial depends on the differential d^2. It will not be important for our later proofs whether B_i is nontrivial, and so we do not include an investigation of this.

Let $A \subseteq H^4(W_f, \partial W_f; \mathbb{Z}/2)$ be the subgroup generated by $\{PD^{-1}[\mu_i]\}_{i=1}^m$, where $[\mu_i] \in H_1(W_f; \mathbb{Z}/2)$ is the class represented by a meridian to M_i, and PD denotes the Poincaré-Lefschetz duality isomorphism. That is, writing $\iota\colon E \to W_f$ for the inclusion map, A is by definition the subgroup of $H^4(W_f, \partial W_f; \mathbb{Z}/2)$ Poincaré dual to $\oplus_{i=1}^m \iota(B_i) \subseteq H_1(W_f; \mathbb{Z}/2)$.

Proposition 3.3. *The Kirby-Siebenmann obstruction* $\mathrm{ks}(N, \partial\overline{\nu}M \cup \partial W_f) \in H^4(N, \partial\overline{\nu}M \cup \partial W_f)$ *is determined by* $\mathrm{ks}(W_f, \partial W_f) \in H^4(W_f, \partial W_f; \mathbb{Z}/2)$. *Moreover,* $\mathrm{ks}(W_f, \partial W_f)$ *lies in the subgroup A.*

Proof. All homology and cohomology in this proof will be with $\mathbb{Z}/2$ coefficients, and so to save space we omit them from the notation. Decompose the pair $(N, \partial\overline{\nu}M \cup \partial W_f)$ as

$$(\overline{\nu}M, \partial\overline{\nu}M) \cup (W_f, \partial W_f).$$

The intersections are $\overline{\nu}M \cap W_f = E = \partial\overline{\nu}M \cap \partial W_f$. Consider the relative cohomology Mayer-Vietoris sequence [Hat02, p. 204]:

$$\cdots \to H^{n-1}(E, E) \to H^n(N, \partial\overline{\nu}M \cup \partial W_f) \to H^n(\overline{\nu}M, \partial\overline{\nu}M) \oplus H^n(W_f, \partial W_f) \to H^n(E, E) \to \cdots$$

Taking $n = 4$ and observing that $H^i(E, E) = 0$ for all i, we deduce that

$$H^4(N, \partial\overline{\nu}M \cup \partial W_f) \cong H^4(\overline{\nu}M, \partial\overline{\nu}M) \oplus H^4(W_f, \partial W_f)$$

where this isomorphism has coordinates the two restrictions to $(\overline{\nu}M, \partial\overline{\nu}M)$ and $(W_f, \partial W_f)$. Therefore, by Proposition 2.3 applied twice, once to the inclusion $\overline{\nu}M \hookrightarrow N$ and once to the inclusion $W_f \hookrightarrow N$, this isomorphism sends $\mathrm{ks}(N, \partial\overline{\nu}M \cup \partial W_f) \in H^4(N, \partial\overline{\nu}M \cup \partial W_f)$ to

$$(\mathrm{ks}(\overline{\nu}M, \partial\overline{\nu}M), \mathrm{ks}(W_f, \partial W_f)) = (0, \mathrm{ks}(W_f, \partial W_f)) \in H^4(\overline{\nu}M, \partial\overline{\nu}M) \oplus H^4(W_f, \partial W_f).$$

Here we use that $\mathrm{ks}(\overline{\nu}M, \partial\overline{\nu}M) = 0$, which as mentioned above holds because our chosen smooth structure on $\partial\overline{\nu}M$ was obtained by restricting a structure on $\overline{\nu}M$. This proves the first statement of the proposition.

To prove the second sentence, consider the following diagram:

$$\begin{tikzcd}
H^3(\partial\overline{\nu}M \cup \partial W_f, \partial N) \arrow[r] \arrow[d, "\cong"] & H^4(N, \partial\overline{\nu}M \cup \partial W_f) \arrow[r, "j^*"] \arrow[d, "\cong"] & H^4(N, \partial N) \\
H^3(E, \partial E) \arrow[r] \arrow[d, "\cong"] & H^4(\overline{\nu}M, \partial\overline{\nu}M) \oplus H^4(W_f, \partial W_f) \arrow[d, "\cong"] & \\
H_1(E) \arrow[r, "k"] & H_1(\overline{\nu}M) \oplus H_1(W_f) &
\end{tikzcd}$$

where the upper row is an excerpt from the cohomology long exact sequence of the triple $\partial N \subseteq \partial\overline{\nu}M \cup \partial W_f \subseteq N$, the top left vertical isomorphism is by excision and the bottom vertical isomorphisms use Poincaré–Lefschetz duality. Let $(0, \gamma) \in H_1(\overline{\nu}M) \oplus H_1(W_f)$ be the Poincaré-Lefschetz dual of

$$(\mathrm{ks}(\overline{\nu}M, \partial\overline{\nu}M), \mathrm{ks}(W_f, \partial W_f)) = (0, \mathrm{ks}(W_f, \partial W_f)).$$

Since

$$j^*(\mathrm{ks}(N, \partial\overline{\nu}M \cup \partial W_f)) = \mathrm{ks}(N, \partial N) = 0,$$

by Theorem 2.1 (iv) and the fact that N is smooth, it follows from exactness of the top row and commutativity of the diagram that $(0, \gamma) \in \operatorname{Im} k$. The map $k\colon H_1(E) \to H_1(\overline{\nu}M) \oplus H_1(W_f)$ is induced by the inclusions $\kappa_1\colon E \hookrightarrow \overline{\nu}M$ and $\kappa_2\colon E \hookrightarrow W_f$. By Lemma 3.2,

$$H_1(E) \cong \oplus_{i=1}^m (H_1(M_i) \oplus B_i) \cong H_1(M) \oplus_{i=1}^m B_i.$$

Let

$$(\alpha, \beta_1, \ldots, \beta_m) \in H_1(M) \oplus_{i=1}^m B_i$$

be such that $k(\alpha, \beta_1, \ldots, \beta_m) = (0, \gamma) \in H_1(\overline{\nu}M) \oplus H_1(W_f)$. Note that $\kappa_1|_{B_i} = 0$ and $\kappa_1|_{H_1(M)}$ is an isomorphism. Since $\kappa_1|_{B_i} = 0$ it follows that $\kappa_1(\alpha, \beta_1, \ldots, \beta_m) = \kappa_1|_{H_1(M)}(\alpha)$. Since $\kappa_1(\alpha, \beta_1, \ldots, \beta_m) = 0$, we have that $\kappa_1|_{H_1(M)}(\alpha) = 0$. Using that $\kappa_1|_{H_1(M)}$ is an isomorphism, we deduce that $\alpha = 0$. Thus $PD(\mathrm{ks}(W_f, \partial W_f)) = (0, \beta_1, \ldots, \beta_m) \in \operatorname{Im}(\oplus_{i=1}^m B_i)$ is a sum of meridians of the connected components M_i of M. It follows that $\mathrm{ks}(W_f, \partial W_f)$ lies in A, as desired. □

Let $L \cong S^3 \subseteq S^5$ denote Lashof's non-smoothable 3-knot (see Section 2.2). Write

$$\mathrm{ks}(W_f, \partial W_f) = \sum_{i=1}^m a_i(PD^{-1}[\mu_i]),$$

for $a_i \in \mathbb{Z}/2$ defined by this equality and the stipulation that we take $a_i = 0$ if $PD^{-1}[\mu_i] = 0$ in $H^4(W_f, \partial W_f; \mathbb{Z}/2)$. If $a_i = 1$ then we form a connected sum $M_i \# L$ in an arbitrarily small 5-ball, while if $a_i = 0$ we leave M_i alone. Let $g\colon Y \hookrightarrow N$ denote the resulting embedding. Define

$$\mathcal{I} := \{i \mid a_i = 1\} \subseteq \{1, \ldots, m\}.$$

Let $W_g := N \setminus \nu g(Y)$. Note that

$$W_g \cong W_f \cup_{\sqcup_{\mathcal{I}}(S^1 \times D^3)_i} \bigsqcup_{\mathcal{I}} E_{L_i}.$$

That is, W_f and $\sqcup_i E_{L_i}$ attached along $\sqcup_{\mathcal{I}}(S^1 \times D^3)_i$, where $(S^1 \times 0)_i$ is identified with a meridian to M_i and a meridian to L_i, for each $i \in \mathcal{I}$, and we extend to a tubular neighbourhood $(S^1 \times D^3)_i$ in ∂W_g and ∂E_{L_i} respectively. Also, note that

$$\partial W_g = \left(\partial W_f \setminus \sqcup_{\mathcal{I}}(S^1 \times \mathring{D}^3)_i\right) \cup_{\sqcup_{\mathcal{I}}(S^1 \times S^2)_i} \left(\bigsqcup_{\mathcal{I}} \partial E_{L_i} \setminus \sqcup_{\mathcal{I}}(S^1 \times \mathring{D}^3)_i\right)$$

Hence, $\partial W_g \cup (\sqcup_{\mathcal{I}}(S^1 \times D^3)_i)$ decomposes as $\partial W_f \cup \bigcup_{\mathcal{I}} \partial E_{L_i}$. Figure 2 shows an illustration of W_g when one Lashof knot is attached.

Proposition 3.4. *We have that* $\mathrm{ks}(W_g, \partial W_g) = 0 \in H^4(W_g, \partial W_g; \mathbb{Z}/2)$.

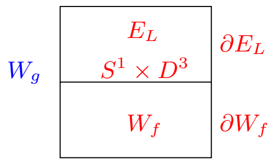

FIGURE 2. A schematic diagram of W_g when one Lashof knot is attached.

Proof. All homology and cohomology in this proof will be with $\mathbb{Z}/2$ coefficients, and so to save space we omit them from the notation. Recall that $\mathrm{ks}(W_f, \partial W_f) = \sum_{\mathcal{I}} PD^{-1}[\mu_i] \in H^4(W_f, \partial W_f)$. From the relative cohomology Mayer-Vietoris sequence of the pair

$$(W_g, \partial W_g \cup (\sqcup_{\mathcal{I}}(S^1 \times D^3)_i)) = (W_f, \partial W_f) \cup (\sqcup_{\mathcal{I}} E_{L_i}, \sqcup_{\mathcal{I}} \partial E_{L_i}),$$

using that $W_f \cap (\sqcup_{\mathcal{I}} E_{L_i}) = \partial W_f \cap (\sqcup_{\mathcal{I}} \partial E_{L_i}) = \sqcup_{\mathcal{I}}(S^1 \times D^3)_i$, we get via an argument similar to that in the proof of Proposition 3.3, that

$$H^4(W_g, \partial W_g \cup (\sqcup_{\mathcal{I}}(S^1 \times D^3)_i)) \cong H^4(W_f, \partial W_f) \oplus_{\mathcal{I}} H^4(E_{L_i}, \partial E_{L_i}).$$

Hence the image of $\mathrm{ks}(W_g, \partial W_g \cup (\sqcup_{\mathcal{I}}(S^1 \times D^3)_i))$ under this isomorphism is

$$(\mathrm{ks}(W_f, \partial W_f), \mathrm{ks}(E_{L_1}, \partial E_{L_1}), \ldots, \mathrm{ks}(E_{L_k}, \partial E_{L_k})) \in H^4(W_f, \partial W_f) \oplus_{\mathcal{I}} H^4(E_{L_i}, \partial E_{L_i}).$$

Recall that by Lemma 2.5 we have that $\mathrm{ks}(E_{L_i}, \partial E_{L_i}) = 1$ for each $i \in \mathcal{I}$, represented by $PD^{-1}[\mu_{L_i}]$, the Poincaré dual to a meridian of L_i.

Consider the following diagram:

$$\begin{tikzcd}
H^3(\partial W_g \cup (\sqcup_{\mathcal{I}}(S^1 \times D^3))_i, \partial W_g) \arrow[r] \arrow[d, "\cong"] & H^4(W_g, \partial W_g \cup (\sqcup_{\mathcal{I}}(S^1 \times D^3)_i)) \arrow[r, "j^*"] \arrow[d, "\cong"] & H^4(W_g, \partial W_g) \arrow[d, "="] \\
H^3(\sqcup_{\mathcal{I}}(S^1 \times D^3)_i, \sqcup_{\mathcal{I}}(S^1 \times S^2)_i) \arrow[r] \arrow[d, "\cong"] & H^4(W_f, \partial W_f) \oplus_{\mathcal{I}} H^4(E_{L_i}, \partial E_{L_i}) \arrow[r] \arrow[d, "\cong"] & H^4(W_g, \partial W_g) \arrow[d, "\cong"] \\
H_1(\sqcup_{\mathcal{I}}(S^1 \times D^3)_i) \cong \oplus_{\mathcal{I}} H_1((S^1 \times 0)_i) \arrow[r, "\Phi"] & H_1(W_f) \oplus_{\mathcal{I}} H_1(E_{L_i}) \arrow[r] & H_1(W_g)
\end{tikzcd}$$

The upper row is an excerpt from the cohomology long exact sequence of the triple

$$\partial W_g \subseteq \partial W_g \cup (\sqcup_{\mathcal{I}}(S^1 \times D^3))_i \subseteq W_g,$$

the top left vertical isomorphism is by excision and the bottom vertical isomorphisms use Poincaré–Lefschetz duality. By naturality of the Kirby-Siebenmann obstruction (Proposition 2.3 applied twice), the upper middle vertical isomorphism sends

$$\kappa := \mathrm{ks}(W_g, \partial W_g \cup (\sqcup_{\mathcal{I}}(S^1 \times D^3)_i))$$

to

$$\big(\mathrm{ks}(W_f, \partial W_f), \mathrm{ks}(E_{L_1}, \partial E_{L_1}), \ldots, \mathrm{ks}(E_{L_k}, \partial E_{L_k})\big) \in H^4(W_f, \partial W_f) \oplus_{\mathcal{I}} H^4(E_{L_i}, \partial E_{L_i}).$$

On the other hand by Theorem 2.1 (iv), $j^*(\kappa) = \mathrm{ks}(W_g, \partial W_g) \in H^4(W_g, \partial W_g)$. By commutativity of the top right square it follows that

$$\big(\mathrm{ks}(W_f, \partial W_f), \mathrm{ks}(E_{L_1}, \partial E_{L_1}), \ldots, \mathrm{ks}(E_{L_k}, \partial E_{L_k})\big)$$

maps under the right hand map of the middle row to

$$\mathrm{ks}(W_g, \partial W_g) \in H^4(W_g, \partial W_g).$$

By commutativity of the bottom right square of the diagram, the Poincaré-Lefschetz dual of the former, $(\sum_{\mathcal{I}}[\mu_i], \sum_{\mathcal{I}}[\mu_{L_i}]) \in H_1(W_f) \oplus_{\mathcal{I}} H_1(E_{L_i})$, is sent to

$$\gamma := PD\big(\mathrm{ks}(W_g, \partial W_g)\big) \in H_1(W_g),$$

the Poincaré-Lefschetz dual of $\mathrm{ks}(W_g, \partial W_g) \in H^4(W_g, \partial W_g)$. Note that the bottom row is the Mayer-Vietoris homology sequence of the decomposition $W_f \cup_{\mathcal{I}} E_{L_i} = W_g$, and for each $i \in \mathcal{I}$

we have $\Phi([S^1 \times 0]_i) = ([\mu_i], [\mu_{L_i}])$, where $[S^1 \times 0]_i$ is the generator of $H_1((S^1 \times 0)_i)$. Hence by linearity,

$$\Phi\big(\sum_{\mathcal{I}}[S^1 \times 0]_i\big) = \big(\sum_{\mathcal{I}}[\mu_i], \sum_{\mathcal{I}}[\mu_{L_i}]\big).$$

Thus $\gamma = 0$ by exactness, and since PD is an isomorphism it follows that $\mathrm{ks}(W_g, \partial W_g) = 0$. □

The main result of this section follows.

Proof of Proposition 3.1. Since $\mathrm{ks}(W_g, \partial W_g) = 0$, we can extend the standard smooth structure on $\partial N \cup \overline{\nu} g(Y)$ to all of N. Call the resulting smooth structure σ. By construction, $g(Y)$ is smooth in σ, and σ agrees with the given smooth structure of N near ∂N. Each connected sum of M_i with L_i can be done arbitrarily close to $M_i = f(Y_i)$, so we can assume that we altered f by an arbitrarily small homotopy. □

4. Comparing with the standard smooth structure on N

Next, we need to compare the smooth structure σ we have just constructed with the given smooth structure std on N. The submanifold $g(Y)$ is smooth in σ, but is a priori not smooth in std. We aim to reduce to a finite collection of local problems, namely neighbourhoods $V_i \subseteq \operatorname{Int} N$ where $g(Y)$ need not be smooth in std. Then we will apply the argument that all 2-knots are smoothly slice [Ker65, Sun15] to further modify $g(Y)$ in each of these neighbourhoods V_i, replacing $g(Y) \cap V_i$ with a slice disc for $g(Y) \cap \partial V_i \cong S^2$ that is smooth in the structure std. Our aim is the following proposition, which proves Step 2 from the introduction. The combination of Proposition 3.1 and Proposition 4.1 proves Theorem A.

Proposition 4.1. *Let N be a compact, connected, smooth 5-dimensional manifold with (possibly empty) boundary, let Y be a compact 3-dimensional manifold with (possibly empty) boundary, and let $g\colon Y \to N_\sigma$ be a smooth embedding for some σ such that σ and std agree near ∂N. Then g is homotopic rel. boundary, via an arbitrarily small homotopy, to a smooth embedding in N_{std}.*

To begin, recall that the structures σ and std correspond via smoothing theory (Theorem 2.1) to two lifts $\sigma, \mathrm{std}\colon N \to \mathrm{BO}$ of $\tau_N\colon N \to \mathrm{BTOP}$. The difference between these lifts gives rise to a map $N \to \mathrm{TOP}/\mathrm{O}$, and whence to an element $\mathrm{ks}(\sigma, \mathrm{std}) \in H^3(N, \partial N; \mathbb{Z}/2) \cong H_2(N; \mathbb{Z}/2)$. In this section we redefine $M := g(Y)$

Lemma 4.2. *The class $PD(\mathrm{ks}(\sigma, \mathrm{std})) \in H_2(N; \mathbb{Z}/2)$ can be represented by a closed surface $S \subseteq \operatorname{Int} N$, which is smoothly embedded in σ and is transverse to $M := g(Y)$.*

Proof. We consider the group $\mathcal{N}_2(N)$ of unoriented surfaces mapping to N, up to bordism. The Atiyah-Hirzebruch spectral sequence for this has E^2-page

$$E^2_{p,q} \cong H_p(N; \mathcal{N}_q).$$

The unoriented bordism groups are given [Tho54] in the range $q \in \{0, 1, 2\}$ by $\mathcal{N}_0 \cong \mathbb{Z}/2 \cong \mathcal{N}_2$, and $\mathcal{N}_1 = 0$. Using that the $q = 1$ row on the E^2-page consists entirely of zeros, we have an exact sequence

$$H_3(N; \mathbb{Z}/2) \xrightarrow{d^3_{3,0}} H_0(N; \mathbb{Z}/2) \to \mathcal{N}_2(N) \to H_2(N; \mathbb{Z}/2) \to 0.$$

In particular every element of $H_2(N; \mathbb{Z}/2)$ lifts to $\mathcal{N}_2(N)$, and so can be represented by a map $h\colon \Sigma \to N$ from some closed surface Σ into N.

By [Hir94, Theorem 2.2.6] we can approximate h by a smooth map in $[N]_\sigma$, and by [Hir94, Theorems 2.2.12 and 2.2.14] we can approximate the result by an embedding, $h'\colon \Sigma \to N$. We write $S := h'(\Sigma)$. Since both S and M are smooth in σ, we apply transversality to complete the proof. □

By Proposition 2.4, by an arbitrarily small isotopy of σ away from S and ∂N, and hence of $M \cap (N \setminus S)$, we can assume that the smooth structures σ and std agree in the complement of the surface S. Replace M and σ by the outcomes of this isotopy.

Let νS denote a smooth open tubular neighbourhood of S in the smooth structure σ. We have that $M \setminus \nu S$ is smooth in $[N \setminus \nu S]_{\mathrm{std}}$. By compactness and transversality, $S \pitchfork M$ consists of finitely many points, $p_1, \ldots, p_n$ say. Moreover, the intersection $M \cap \partial \overline{\nu} S$ consists of a copy of S^2

for each point $p_i \in S \pitchfork M$, which bounds a 3-ball $D_i^3 \subseteq M \cap \overline{\nu}S$ with the centre of D_i^3 equal to p_i. In fact the intersection $M \cap \overline{\nu}S$ comprises exactly $\bigcup_{i=1}^n D_i^3$; the D_i^3 are pairwise disjoint.

Since D_i^3 is locally flat and codimension 2, it has a normal bundle [KS75]. We take a normal bundle of each D_i^3 in $\overline{\nu}S$. We obtain an inclusion of pairs

$$(D_i^3 \times \mathbb{R}^2, S^2 \times \mathbb{R}^2) \subseteq (\overline{\nu}S, \partial\overline{\nu}S).$$

Pull back the smooth structure std to this to obtain

$$V_i := [D_i^3 \times \mathbb{R}^2]_{\mathrm{std}}.$$

This V_i is a smooth manifold that is homeomorphic to $D^3 \times \mathbb{R}^2$, with boundary ∂V_i identified with $S^2 \times \mathbb{R}^2$. In the boundary, $M \cap \partial V_i$ is a 2-sphere T_i that is identified with $S^2 \times \{0\} \subseteq S^2 \times \mathbb{R}^2$.

We remark that $(V_i, \partial V_i)$ may not be diffeomorphic rel. boundary to $(D^3 \times \mathbb{R}^2, S^2 \times \mathbb{R}^2)$. In addition while $M \cap V_i$ is smooth in σ, this need not be the case in std.

Lemma 4.3. *The 2-sphere $T_i \subseteq \partial V_i$ bounds a compact, orientable 3-manifold Z_i smoothly embedded in $V_i = [D_i^3 \times \mathbb{R}^2]_{\mathrm{std}}$.*

Proof. Consider the sequence of maps

$$f\colon S^2 \times D^2 \hookrightarrow S^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \xrightarrow{\cong} S^2 \setminus \{*\} \hookrightarrow S^2 \hookrightarrow \mathbb{CP}^2 \hookrightarrow \mathbb{CP}^\infty.$$

These are given respectively by the inclusion, the projection, the inverse of stereographic projection, the inclusion, identification with $\mathbb{CP}^1$, and the standard inclusion again. Choose another embedding $\mathbb{CP}^1 \to \mathbb{CP}^2$ that intersects our original $\mathbb{CP}^1$ transversely in exactly the image of $\{0\} \in \mathbb{R}^2$ under $\mathbb{R}^2 \xrightarrow{\cong} S^2 \setminus \{*\} \hookrightarrow S^2 \xrightarrow{\mathbb{CP}^1}$. Let $\ell\colon \mathbb{CP}^1 \to \mathbb{CP}^\infty$ denote the composition of this embedding with the inclusion $\mathbb{CP}^2 \hookrightarrow \mathbb{CP}^\infty$. We observe that $f^{-1}(\ell(\mathbb{CP}^1)) = T_i$.

Let $V_i' := D^3 \times D^2 \subseteq D^3 \times \mathbb{R}^2$. We seek an extension of the form:

$$\begin{tikzcd} S^2 \times D^2 \arrow[r, "f"] \arrow[d] & \mathbb{CP}^\infty \\ V_i' \arrow[ur, dashed, "F"'] & \end{tikzcd}$$

Since $\mathbb{CP}^\infty \simeq K(\mathbb{Z}, 2)$, there is a unique obstruction in

$$H^3(D^3 \times D^2, S^2 \times \mathbb{R}^2; \pi_2(\mathbb{CP}^\infty)) \cong H^3(D^3, S^2; \pi_2(\mathbb{CP}^\infty)) \cong H^3(D^3, S^2; \mathbb{Z}) \cong \mathbb{Z}$$

to extending f to F. However the boundary of the D^3 in question is $T_i \subseteq S^2 \times D^2$. The image $f(S^2 \times \{0\})$ is a point in $\mathbb{CP}^\infty$, which represents the trivial element in $\pi_2(\mathbb{CP}^\infty) \cong \mathbb{Z}$. As a result the obstruction cocycle is trivial, and hence the obstruction cohomology class is too. Thus we obtain a map $F\colon V_i' \to \mathbb{CP}^\infty$ as desired.

Using that V_i' is 5-dimensional, homotope F rel. $F|_{S^2 \times D^2} = f$ to a map F' with image in $\mathbb{CP}^2$. We can and shall assume, by perturbing F' further if necessary, that the inverse image of $\ell(\mathbb{CP}^1) \subseteq \mathbb{CP}^2$ lies in the interior of V_i'. Next we perturb F' to be smooth in the smooth structure on the interior of V_i' induced by std, and so that F' is transverse to $\ell(\mathbb{CP}^1)$. By making the perturbation sufficiently small, we can assume that the inverse image still lies in V_i'. The inverse image of $\ell(\mathbb{CP}^1)$ is thus a smooth 3-manifold Z_i in V_i with boundary $S^2 \times \{0\} \subseteq \partial V_i$. As it is the inverse image of a closed set, Z_i is closed, and since $Z_i \subseteq V_i'$ and V_i' is compact, we see that Z_i is compact. Since V_i is orientable, $w_1(Z_i) = w_1(\nu Z_i)$. However $w_1(\nu Z_i)$ is zero because νZ_i can be obtained as the pull back of the normal bundle of $\ell(\mathbb{CP}^1) \subseteq \mathbb{CP}^2$, and $w_1(\nu\mathbb{CP}^1)$ is necessarily trivial since $H^1(\mathbb{CP}^1; \mathbb{Z}/2) = 0$. It follows that $w_1(Z_i) = 0$ and so Z_i is orientable. Then recall that $V_i' \subseteq V_i$, to see that we have constructed the 3-manifold $Z_i \subseteq V_i$ we desire. □

Now we can prove the Proposition 4.1, which is the goal of this section.

Proof of Proposition 4.1. To prove the proposition, it remains to find, for each $i = 1, \ldots, n$, a smooth slice disc $D^3 \subseteq V_i$ with $\partial D^3 = T_i = S^2 \times \{0\} \subseteq S^2 \times \mathbb{R}^2 = \partial V_i$. By Lemma 4.3, for each i we have a smooth, compact, orientable 3-manifold Z_i with $\partial Z_i = T_i$. Since Z_i is orientable and 3-dimensional, it is parallelisable, and thus is in particular spin. We now apply the argument of Sunukjian [Sun15] from his Section 5 and the proof of his Theorem 6.1. As mentioned in the introduction, this is similar to and was inspired by Kervaire's theorem [Ker65] that every 2-knot is slice. For the convenience of the reader we give an outline here.

First perform ambient 1-surgeries on Z_i to arrange that $\pi_1(V_i \setminus Z_i)$ is cyclic. By [Sun15, Proposition 5.1 and Lemma 5.2], there is a spin structure on Z_i such that every spin structure preserving surgery on Z_i can be performed ambiently. Here we use that $\pi_1(V_i \setminus Z_i)$ is cyclic, so that every circle in Z_i bounds an embedded 2-disc whose interior lies in $V_i \setminus Z_i$. Using this spin structure, the union $Z_i \cup D^3$ is a closed, smooth, spin 3-manifold. The group $\Omega_3^{\mathrm{Spin}} = 0$, so $Z_i \cup D^3$ is spin null-bordant. By [Sun15, Lemma 5.4], there is a sequence of spin structure compatible surgeries on circles in Z_i that convert it to D^3. Perform these surgeries ambiently, and obtain a smoothly embedded $D^3 \subseteq [V_i]_{\mathrm{std}}$, as desired, in the restriction to V_i of the smooth structure std. Replacing $M \cap V_i$ with this 3-ball, for each i, yields a smooth embedding $g' \colon Y \hookrightarrow N$ in the smooth structure std. By making the V_i as small as we please, and using that $\pi_3(V_i, \partial V_i) = 0$, we can arrange that we changed g by an arbitrarily small homotopy. □

As mentioned above, Propositions 3.1 and 4.1 combine to complete the proof of Theorem A, noting that in both cases all the modifications we made to the embedding, from f to g to g', consisted of local homotopies or isotopies, in all cases supported outside a neighbourhood of ∂N.

5. Conditions for smoothing up to isotopy

As shown by Lashof's 3-knot [Las71] (Section 2.2), it is not in general possible to isotope a locally flat embedding of a 3-manifold to a smooth embedding. Our main result shows this is possible with an arbitrarily small homotopy. Here we discuss the extent to which smoothing up to isotopy is possible.

As above let $Y = Y_1 \sqcup \cdots \sqcup Y_m$ be a compact 3-manifold with connected components Y_i, and let N be a compact, connected, smooth 5-manifold. We will use the Kirby-Siebenmann invariant $\mathrm{ks}(W_f, \partial W_f) \in H^4(W_f, \partial W_f; \mathbb{Z}/2)$ of the exterior $W_f := N \setminus \nu f(Y)$, and we will use the relative Kirby-Siebenmann invariant $\mathrm{ks}(\sigma, \mathrm{std}) \in H^3(N, \partial N)$ comparing the smooth structure σ on N arising from Step 1 (Proposition 3.1) with the given smooth structure std on N. These invariants were recalled in detail in Section 2. In practice, these invariants are not always easy to evaluate. One way to do this for (i) could be to use the ideas of Kwasik and Vogel [KV84, Kwa87] discussed in Section 2.2 to relate $\mathrm{ks}(W_f, \partial W_f)$ to the signature of an appropriate 4-manifold.

Scholium 5.1. *Let $f \colon Y \to N$ be a locally flat proper topological embedding that is smooth near ∂Y.*

(i) If $\mathrm{ks}(W_f, \partial W_f) = 0$ then there exists a smooth structure σ on N with respect to which f is smooth.

(ii) If in addition $\langle \mathrm{ks}(\sigma, \mathrm{std}), [f(Y_i)] \rangle = 0 \in \mathbb{Z}/2$ for each connected component Y_i of Y, then f is topologically isotopic rel. boundary, via an arbitrarily small isotopy, to a smooth embedding.

Proof. If $\mathrm{ks}(W_f, \partial W_f) = 0$, then Step 1 (Proposition 3.1) can be completed without connect summing with any Lashof knots. We obtain a smooth structure σ on N in which f is smooth, that agrees with the standard smooth structure on N near ∂N. Now suppose $\langle \mathrm{ks}(\sigma, \mathrm{std}), [f(Y_i)] \rangle = 0$ for each $i = 1, \ldots, m$. Let S be an embedded surface Poincaré dual to $\mathrm{ks}(\sigma, \mathrm{std})$ that intersects $f(Y)$ transversely (such an S was produced in Lemma 4.2). The condition implies, by intersection theory, that for each i the count of transverse intersection points between S and $f(Y_i)$ is even. For every i, tube S to itself, along $f(Y_i)$, to obtain a new surface S', in the same $\mathbb{Z}/2$-homology class, $[S] = [S'] \in H_2(N; \mathbb{Z}/2)$, and such that $S' \cap f(Y) = \emptyset$. It then follows from Proposition 2.4 that f is isotopic to a smooth embedding in std. □

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