

SMOOTHING TOPOLOGICAL PSEUDO-ISOTOPIES OF 4-MANIFOLDS

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ABSTRACT. Given a closed, smooth 4-manifold X and self-diffeomorphism f that is topologically pseudo-isotopic to the identity, we study the question of whether f is moreover smoothly pseudo-isotopic to the identity. If the fundamental group of X lies in a certain class, which includes trivial, free, and finite groups of odd order, we show the answer is always affirmative. On the other hand, we produce the first examples of manifolds X and diffeomorphisms f where the answer is negative. Our investigation is motivated by the question, which remains open, of whether there exists a self-diffeomorphism of a closed 4-manifold that is topologically isotopic to the identity, but not stably smoothly isotopic to the identity.

1. INTRODUCTION

A principle in 4-manifold theory says that topological results about smooth 4-manifolds often become true smoothly after stabilisation, meaning after taking connected sum with some number of copies of $S^2 \times S^2$. The most famous instance is the theorem of Wall [Wal64a, Wal64b] and Freedman [Fre82] that homeomorphic closed, smooth, simply connected 4-manifolds are stably diffeomorphic. Similarly Kreck [Kre79], Quinn [Qui86] (with a correction by [GGH⁺23]), and Gabai [Gab22] showed that topologically isotopic diffeomorphisms of such 4-manifolds are stably smoothly isotopic. Further examples of this principle are not confined to the simply connected case e.g. Quinn’s stable s -cobordism theorem [Qui83], Gompf’s extension [Gom84] of the Wall–Freedman result above to all oriented compact 4-manifolds, and Cappell–Shaneson’s stable surgery sequence [CS71]. In the realm of embedded surfaces, Cha–Kim [CK23] proved that topologically embedded surfaces are stably smoothable, and Galvin [Gal24] proved that topologically isotopic surfaces in a simply connected 4-manifold are stably smoothly isotopic.

Lest this long list of results be misconstrued as evidence for a completely general principle, it should be contrasted with exotic phenomena detected by Rochlin’s theorem, or equivalently the Kirby–Siebenmann invariant. For example Kreck [Kre84], Cappell–Shaneson [CS76], and Akbulut [Akb85, Akb84] each constructed an exotic pair of nonorientable 4-manifolds that fail to be stably diffeomorphic. Similarly, Cappell–Shaneson and Galvin’s non-smoothable homeomorphisms [CS71, Gal24] are not stably smoothable.

In this article, we use the Kirby–Siebenmann invariant to investigate the principle in the context of isotopy of diffeomorphisms. We are motivated by the following question.

Question 1.1. Given a pair of self-diffeomorphisms of a closed 4-manifold that are topologically isotopic, are they moreover stably smoothly isotopic?

This remains open, but our results, concerning the intermediate notion of pseudo-isotopy, provide a positive answer in some cases, and indicate that the answer may be negative in general.

1.1. Results. Let X be a smooth, connected, compact 4-manifold, possibly with nonempty boundary, and let $f_0, f_1: X \rightarrow X$ be diffeomorphisms that restrict to the identity on the boundary. A homeomorphism $F: X \times I \rightarrow X \times I$ restricting to f_i on $X \times \{i\}$ and to the identity on $(\partial X) \times [0, 1]$ is called a *topological pseudo-isotopy* from f_0 to f_1 . If F is moreover a diffeomorphism, it is called a *smooth pseudo-isotopy* from f_0 to f_1 .

Given a topological pseudo-isotopy $F: X \times I \rightarrow X \times I$, we can give the boundary of $X \times I \times I$ a new smooth structure as follows. Writing σ for the smooth structure on $X \times I$, endow $X \times I \times \{1\}$ with the smooth structure $F^*\sigma$, and leave the smooth structure alone on the rest of the boundary:

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write $\partial(X \times I \times I)_F$ for this smooth structure. Then $(X \times I \times I, \partial(X \times I \times I)_F)$ is a topological 6-manifold with a smooth structure on its boundary, so has a relative Kirby–Siebenmann invariant

$$\text{KS}(F) := \text{ks}(X \times I \times I, \partial(X \times I \times I)_F) \in H^4(X \times I \times I, \partial(X \times I \times I); \mathbb{Z}/2) \cong_{PD} H_2(X; \mathbb{Z}/2)$$

which obstructs the extension of the smooth structure $\partial(X \times I \times I)_F$ to the whole of $X \times I \times I$. Our first main result is that this invariant also characterises whether F is smoothable.

Theorem A. *Given a topological pseudo-isotopy F from f_0 to f_1 , the obstruction $\text{KS}(F) \in H_2(X; \mathbb{Z}/2)$ vanishes if and only if $F: X \times I \rightarrow X \times I$ is topologically isotopic rel. $\partial(X \times [0, 1])$ to a smooth pseudo-isotopy.*

A result of Gabai [Gab22, Theorem 2.5] states that, for a diffeomorphism, the existence of a smooth pseudo-isotopy to the identity with vanishing primary Hatcher–Wagoner invariants, implies smoothly stably isotopic to the identity. Combining this with work of Galvin–Nonino [GN25] on topological Hatcher–Wagoner invariants, we deduce the following.

Corollary B. *If F is a topological isotopy from f_0 to f_1 with $\text{KS}(F) = 0 \in H_2(X; \mathbb{Z}/2)$, then f_0 and f_1 are smoothly stably isotopic.*

Our next goal is to establish a condition under which diffeomorphisms are smoothly pseudo-isotopic if and only if they are topologically pseudo-isotopic. By Theorem A this is equivalent to showing, for a given pair of diffeomorphisms, that there exists a topological pseudo-isotopy F between them with $\text{KS}(F) = 0$. To formulate our result we will make use of a certain map

$$I_2: H_2(\pi; \mathbb{Z}_{(2)}) \longrightarrow L_6(\mathbb{Z}[\pi])_{(2)},$$

arising from the assembly map in algebraic L -theory. We will describe this map and elucidate some of its properties in Section 5.2 (in particular in Definition 5.8). The following will be proved using methods from surgery theory.

Theorem C. *Let X be an oriented, smooth, connected, compact 4-manifold, write $\pi := \pi_1(X)$, and suppose that*

- (i) *the map $I_2: H_2(\pi; \mathbb{Z}_{(2)}) \rightarrow L_6(\mathbb{Z}[\pi])_{(2)}$ is trivial, and*
- (ii) *$H_1(\pi; \mathbb{Z}_{(2)})$ has no torsion.*

Then a diffeomorphism which is topologically pseudo-isotopic to the identity is smoothly pseudo-isotopic to the identity.

For some $\pi_1(X)$ it was already known that homotopic diffeomorphisms are smoothly pseudo-isotopic, and thus the conclusion of Theorem C was already known for these cases. This is the case for the trivial group, due to Kreck [Kre79, Theorem 1] and more generally for free groups by Krannich–Kupers [KK24, Theorem B]. The following gives additional examples where the hypotheses of Theorem C are satisfied, and hence provides 4-manifolds X where topological pseudo-isotopy of diffeomorphisms implies smooth pseudo-isotopy.

Example 1.2. Any group π with $H_1(\pi; \mathbb{Z}_{(2)})$ torsion-free and $H_2(\pi; \mathbb{Z}_{(2)}) = 0$ satisfies the hypotheses. For example, free groups, any finite group of odd order, or knot groups. Further ad hoc specific examples of this sort can also be constructed. For example, $\pi_1(S^1 \times \Sigma)$ or $\pi_1(\Sigma)$, where Σ is a $\mathbb{Z}_{(2)}$ -homology sphere of some dimension.

If π is a finite group whose 2-Sylow subgroup is abelian or generalised quaternion, then it follows from [Ste77] (see also [TW79, p. 177]) that the map I_2 is trivial. So such groups with $H_1(\pi; \mathbb{Z}_{(2)}) = 0$ satisfy the hypotheses. For example, the alternating group A_5 satisfies this, so $I_2 = 0$ even though $H_2(A_5; \mathbb{Z}_{(2)}) \cong \mathbb{Z}/2 \neq 0$.

Moreover, by functoriality of the assembly map, and hence of I_2 , the assumptions (i) and (ii) are closed under free products, and hence Theorem C holds for $X_1 \# X_2$ if it holds for X_1 and X_2 .

Corollary D. *Let X be an oriented, smooth, connected, compact 4-manifold, write $\pi := \pi_1(X)$. Assume π satisfies the hypotheses (i) and (ii) of Theorem C and also that $\text{Wh}_2(\pi) = 0$. Then a self-diffeomorphism of X that is topologically isotopic to the identity is moreover stably smoothly isotopic to the identity.*

Proof. Consider the topological isotopy as topological pseudo-isotopy. By Theorem C there is a smooth pseudo-isotopy, and then by Gabai’s theorem again [Gab22, Theorem 2.5], which applies since $\text{Wh}_2(\pi) = 0$, there is a stable smooth isotopy. $\square$

Using Corollary D, we can expand the class of groups for which Question 1.1 is known to have a positive answer, which was previously limited to free groups, as discussed above. We note that the conditions of Corollary D are closed under free products of groups.

Example 1.3. It is conjectured that $\text{Wh}_2(\pi) = 0$ for all torsion-free groups [Lüc25, Conjecture 5.22], and this conjecture is known to hold whenever the torsion-free group satisfies the “Farrell–Jones Conjecture for K -theory for torsion-free groups and regular rings”, which itself follows from the full Farrell–Jones conjecture [Lüc25, Theorem 13.65 (i), (ii), (vii)]. For a summary of the many classes of groups known to satisfy these conjectures, see [Lüc25, Theorem 16.1]. In particular, every 3-manifold group satisfies the full Farrell–Jones conjecture [BFL14], and so every classical knot group satisfies the conditions of Corollary D.

Example 1.4. Dunwoody [Dun75] showed that $\text{Wh}_2(\mathbb{Z}/3) = 0$, and so $\pi = \mathbb{Z}/3$ also satisfies the conditions of Corollary D.

Now we consider cases where topological pseudo-isotopy does not imply smooth pseudo-isotopy. After perhaps stabilising the 4-manifold by taking connected sums with $S^2 \times S^2$, there is a plentiful supply of non-smoothable topological pseudo-isotopies between diffeomorphisms, as the following makes precise.

Theorem E. *For each smooth, connected, compact 4-manifold X there is a $g \geq 0$ such that every class $x \in H_2(X \# g(S^2 \times S^2); \mathbb{Z}/2)$ arises as $\text{KS}(F)$ for some topological pseudo-isotopy F from the identity to some diffeomorphism $f: X \# g(S^2 \times S^2) \rightarrow X \# g(S^2 \times S^2)$.*

Combining Theorem E with further methods from surgery theory, we are able to give examples of the following nature.

Theorem F. *There exists a closed, smooth 4-manifold X and a diffeomorphism $f: X \rightarrow X$ such that f is topologically but not smoothly pseudo-isotopic to the identity. Thus f is not smoothly stably isotopic to the identity.*

To prove this, we have to show that $\text{KS}(F) \neq 0$ for every topological pseudo-isotopy F between f and Id . The manifolds X in Theorem F have the form $X = A \# g(S^2 \times S^2)$, where A is an aspherical 4-manifold and g is chosen large enough for Theorem E to apply. The manifolds A we use can be made explicit: see Example 7.3. On the other hand, the diffeomorphisms f we construct are inexplicit. The topological pseudo-isotopies F of f are such that $\text{KS}(F)$ remains nontrivial under the composition

$$H_2(X; \mathbb{Z}/2) \longrightarrow H_2(\pi; \mathbb{Z}/2) \xrightarrow{\delta_*} H_1(\pi; \mathbb{Z}),$$

where δ_* indicates the homology Bockstein for the coefficient sequence $0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0$. In particular, $H_1(\pi; \mathbb{Z}_{(2)})$ has torsion, in contrast with Theorem C.

1.2. Discussion. For a pair of diffeomorphisms of a compact smooth 4-manifold, the following implications hold between the different notions of isotopy rel. boundary.

Smoothly stably isotopic $\Rightarrow$ Smoothly pseudo-isotopic $\Rightarrow$ Top. pseudo-isotopic $\Leftarrow$ Top. isotopic.

The first implication is due to Gabai [Gab22], while the others are immediate from the definitions. Corollary D yields some cases where the first two implications can be reversed and Question 1.1 has a positive answer.

There is not currently a useful theory for improving topological pseudo-isotopies of non-simply connected 4-manifolds to topological isotopies, but if such a theory could be developed, then we would hope it could be used to upgrade our examples from Theorem F and produce diffeomorphisms topologically isotopic to the identity, but not smoothly stably so.

Organisation. In Section 2 we reformulate the main theorems in terms of block automorphisms, which will aid us to structure the proofs. Section 3 contains the proofs of Theorem A and Corollary B. In Section 5 we prove Theorem C, in Section 6 we prove Theorem E, and finally in Section 7 we construct the examples promised in Theorem F.

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Conventions. For pointed topological spaces A and B , the notation $[A, B]$ will indicate the set of pointed continuous maps $A \rightarrow B$, up to pointed homotopy. As a consequence, $[A_+, B]$ denotes unbased homotopy classes of maps from A to B . For spectra L and K , the notation $[L, K]$ will indicate morphisms $L \rightarrow K$, up to homotopy. The sense in which the notation $[-, -]$ is being used will always be clear from the objects in question. For maps of spectra $f, g: L \rightarrow K$, we will write $f - g: L \rightarrow K$ to be any representative map of spectra for the homotopy class $[f] - [g] \in [L, K]$. A map $f: X \rightarrow X$ from a manifold to itself is *rel. boundary* if $f|_{\partial X} = \text{Id}_{\partial X}$.

2. REFORMULATION VIA BLOCK AUTOMORPHISMS

We will now reformulate Theorems A, C, E, and F in a unified framework that will make proofs clearer, and illuminate the structure of the general problem of smoothing pseudo-isotopies.

2.1. Block automorphism groups. Let $\widetilde{\text{Homeo}}_{\partial}^+(X)$ and $\widetilde{\text{Diffeo}}_{\partial}^+(X)$ denote the geometric realisations of the Kan semi-simplicial groups of orientation-preserving block self-homeomorphisms and block self-diffeomorphisms of X rel. boundary, respectively; see e.g. [BLR75, Appendix I, §3] with corrections in the smooth case in [HLLRW21, Section 2.2]. Eliding some technical details regarding collars in the smooth case, the p -simplices consist of homeomorphisms (or diffeomorphisms) $\phi: X \times \Delta^p \rightarrow X \times \Delta^p$ which are the identity on $(\partial X) \times \Delta^p$, preserve orientation, and send $X \times \sigma$ to $X \times \sigma$ for every face $\sigma \subseteq \Delta^p$. In particular, a 0-simplex is precisely a homeomorphism (or diffeomorphism) of X , and a 1-simplex is precisely a topological (or smooth) pseudo-isotopy. Thus the groups of path components,

$$\pi_0(\widetilde{\text{Homeo}}_{\partial}^+(X)) \quad \text{and} \quad \pi_0(\widetilde{\text{Diffeo}}_{\partial}^+(X)),$$

are the groups of orientation-preserving self-homeomorphisms up to topological pseudo-isotopy (or self-diffeomorphisms up to smooth pseudo-isotopy). In other words, they are the *pseudo-mapping class groups*.

There is an analogous Kan semi-simplicial monoid $\widetilde{\text{hAut}}_{\partial}^+(X)$, whose p -simplices are as above but relaxing the condition that ϕ be a homeomorphism, by only asking that it be a homotopy equivalence; see e.g. [Kra22, Section 1.5] for a discussion, which in particular explains why the geometric realisation of this semi-simplicial set is weakly equivalent to the topological space $\text{hAut}_{\partial}^+(X)$ of boundary- and orientation-preserving homotopy automorphisms of X . Further insisting that ϕ be a simple homotopy equivalence defines a Kan semi-simplicial monoid $\widetilde{\text{sAut}}_{\partial}^+(X)$, to which the same discussion applies: the geometric realisation of $\widetilde{\text{sAut}}_{\partial}^+(X)$ corresponds to those path components $\text{sAut}_{\partial}^+(X) \subseteq \text{hAut}_{\partial}^+(X)$ consisting of simple homotopy automorphisms. In this model there are forgetful maps

$$\widetilde{\text{Diffeo}}_{\partial}^+(X) \longrightarrow \widetilde{\text{Homeo}}_{\partial}^+(X) \longrightarrow \widetilde{\text{sAut}}_{\partial}^+(X) \longrightarrow \widetilde{\text{hAut}}_{\partial}^+(X)$$

of Kan semi-simplicial monoids.

2.2. Pseudo-isotopy classes of topological pseudo-isotopies. We wish to study topological pseudo-isotopies from the identity to a diffeomorphism, and this will be clearest if we carefully define a certain set of such.

Definition 2.1. For a smooth 4-manifold X we consider topological pseudo-isotopies $F: X \times I \rightarrow X \times I$ with $F|_{X \times \{0\}} = \text{Id}_X$ and such that $f := F|_{X \times \{1\}}$ is a diffeomorphism, all rel. ∂X . We consider these up to the equivalence relation of a further pseudo-isotopy: define $F_0 \sim F_1$ if there is a homeomorphism G of $X \times I \times I$ which restricts to F_i on $X \times I \times \{i\}$, to the identity on $(X \times \{0\} \times I) \cup ((\partial X) \times I \times I)$, and to a diffeomorphism on $X \times \{1\} \times I$ (which is then a smooth pseudo-isotopy from f_0 to f_1). We write $Q(X)$ for the set of such equivalence classes, and note that it is a group under composition.

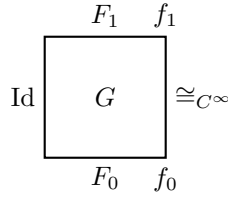


FIGURE 1. Schematic for $X \times I \times I$ with the X direction suppressed. Maps defining the equivalence relation $(f_0, F_0) \sim (f_1, F_1)$ for $Q(X)$ are depicted.

For $[F] \in Q(X)$, we explained in the introduction how to produce a smooth manifold $\partial(X \times I \times I)_F$ by pulling back the smooth structure σ on $X \times I \times \{1\}$ using the homeomorphism F . Then $X \times I \times I$ is a topological 6-manifold with a given smooth structure on its boundary so has a Kirby–Siebenmann invariant

$$\text{KS}(F) := \text{ks}(X \times I \times I, \partial(X \times I \times I)_F) \in H^4(X \times I \times I, \partial(X \times I \times I); \mathbb{Z}/2) \cong_{PD}$$

In the following section, once we describe the details of smoothing theory, we will prove the following.

Lemma 2.2. *The function $[F] \mapsto \text{KS}(F): Q(X) \rightarrow H_2(X; \mathbb{Z}/2)$ is well-defined and is a homomorphism.*

The proof will be given after Lemma 3.2.

2.3. The reformulation. The definition of $Q(X)$ was obtained by spelling out the description of π_0 of the homotopy fibre of the map $\widetilde{\text{Diffeo}}_\partial^+(X) \rightarrow \widetilde{\text{Homeo}}_\partial^+(X)$. Thus there is an exact sequence

$$\pi_1(\widetilde{\text{Diffeo}}_\partial^+(X)) \longrightarrow \pi_1(\widetilde{\text{Homeo}}_\partial^+(X)) \xrightarrow{\alpha} Q(X) \xrightarrow{\beta} \pi_0(\widetilde{\text{Diffeo}}_\partial^+(X)) \xrightarrow{\gamma} \pi_0(\widetilde{\text{Homeo}}_\partial^+(X)). \quad (1)$$

Theorems A, C, E, and F in the introduction can be phrased in terms of the behaviour of this exact sequence, and the map $\text{KS}: Q(X) \rightarrow H_2(X; \mathbb{Z}/2)$, as follows.

Theorem 2.3. *Let X be a smooth, connected, compact 4-manifold.*

- (A) *The homomorphism $\text{KS}: Q(X) \rightarrow H_2(X; \mathbb{Z}/2)$ is injective.*
- (C) *Suppose X is orientable and the fundamental group $\pi = \pi_1(X)$ is such that the map*

$$I_2: H_2(\pi; \mathbb{Z}_{(2)}) \longrightarrow L_6^s(\mathbb{Z}[\pi])_{(2)}$$

specified in Definition 5.8 is trivial. Then the image of the composition

$$\pi_1(\widetilde{\text{Homeo}}_\partial^+(X)) \xrightarrow{\alpha} Q(X) \xrightarrow{\text{KS}} H_2(X; \mathbb{Z}/2)$$

contains the kernel $\ker(H_2(X; \mathbb{Z}/2) \xrightarrow{\delta_} H_1(X; \mathbb{Z}_{(2)}))$ of the Bockstein operator δ_* for the coefficient sequence $0 \rightarrow \mathbb{Z}_{(2)} \xrightarrow{2} \mathbb{Z}_{(2)} \rightarrow \mathbb{Z}/2 \rightarrow 0$.*

- (E) *There is a $g \geq 0$ such that for the manifold $X' := X \# g(S^2 \times S^2)$, the homomorphism $\text{KS}: Q(X') \rightarrow H_2(X'; \mathbb{Z}/2)$ is a bijection.*
- (F) *There exists an X for which γ is not injective, i.e. there is a diffeomorphism of X which is topologically pseudo-isotopic to the identity but not smoothly so.*

Remark 2.4.

- (i) Theorem 2.3 (A) sounds a little weaker than Theorem A, as the latter says that if $\text{KS}(F) = 0$ then F is topologically *isotopic* to a smooth pseudo-isotopy, whereas the former only says that F is topologically *pseudo-isotopic* to a smooth pseudo-isotopy. But in fact they are equivalent, as we will show in Theorem 3.4.
- (ii) Theorem 2.3 (A) and (C) imply Theorem C, as we explain below. Theorem 2.3 (C) is potentially stronger than Theorem C, as it could also be used to analyse particular diffeomorphisms of X when $H_1(\pi; \mathbb{Z}_{(2)})$ has 2-torsion.

In the following sections we will prove the results in the introduction, but we will usually prove them in the form given in Theorem 2.3. For Theorem C we explain the connection next.

Proof of Theorem C assuming Theorem 2.3 (A) and (C). The Postnikov truncation $\mathfrak{t}: X \rightarrow B\pi$ determines a commutative diagram

$$\begin{array}{ccc} H_2(X; \mathbb{Z}/2) & \xrightarrow{\delta_*} & H_1(X; \mathbb{Z}_{(2)}) \\ \downarrow \mathfrak{t}_* & & \cong \downarrow \mathfrak{t}_* \\ H_2(\pi; \mathbb{Z}/2) & \xrightarrow{\delta_*} & H_1(\pi; \mathbb{Z}_{(2)}) \end{array}$$

where surjectivity of $\mathfrak{t}_*$ on H_2 is easily seen by considering a model for $B\pi$ consisting of a cell complex for X with additional cells of dimension 3 or greater attached. The hypothesis that $H_1(\pi; \mathbb{Z}_{(2)})$ has no 2-torsion is equivalent to the lower δ_* being the 0 map. A diagram chase shows this is equivalent to the upper δ_* being the 0 map. By Theorem 2.3 (C), the composition $\text{KS} \circ \alpha$ is then surjective, so in particular KS is surjective. Then by Theorem 2.3 (A), KS is an isomorphism, and so α is surjective and hence γ is injective by exactness of 1. The latter is equivalent to saying that every self-diffeomorphism of X which is topologically pseudo-isotopic to the identity is in fact smoothly pseudo-isotopic to the identity. $\square$

3. PROOF OF THEOREM A AND COROLLARY B

3.1. Smoothing theory. Let Y be a compact d -dimensional topological manifold whose boundary ∂Y is endowed with a fixed smooth structure. The maps classifying the topological tangent microbundle of Y , and the once-stabilised tangent vector bundle of ∂Y , yield the solid arrows in the following commutative diagram, where the right-hand map is modelled by a fibration:

$$\begin{array}{ccc} \partial Y & \xrightarrow{T(\partial Y) \oplus \varepsilon^1} & \text{BO}(d) \\ \downarrow & \nearrow & \downarrow \\ Y & \xrightarrow{\tau_Y} & \text{BTop}(d). \end{array} \quad (2)$$

The fundamental theorem of smoothing theory is then as follows [KS77, Essay V].

Theorem 3.1 (Kirby–Siebenmann). *For $d \geq 5$, the space of smooth structures on Y extending that on ∂Y is homotopy equivalent to the space of dashed maps making both triangles commute.*

In particular, there exists a smooth structure on Y extending that on ∂Y if and only if there exists a dashed map making both triangles commute. This lifting problem may be made more practical by considering the following diagram.

$$\begin{array}{ccccc} \text{BO}(d) & \longrightarrow & \text{BO} & \longrightarrow & \text{E}(\text{Top}/\text{O}) \simeq * \\ \downarrow & & \downarrow & & \downarrow \\ \text{BTop}(d) & \longrightarrow & \text{BTop} & \xrightarrow{\theta} & \text{B}(\text{Top}/\text{O}) \end{array}$$

The right-hand square is formed using the fact that $\text{BO} \rightarrow \text{BTop}$ is an (infinite) loop map so deloops to give $\text{B}(\text{Top}/\text{O})$. The down-right composition consists of two maps in a homotopy fibre sequence, so the square is cartesian, i.e. is a homotopy pull-back square. The left-hand square is formed by stabilisation, and is $(d+1)$ -cartesian [KS77, p. 246 (4)] as long as $d \geq 5$. (Recall we say that a square is n -cartesian if the canonical map from the top left entry to the homotopy pullback of the other three entries is n -connected, or equivalently if the total homotopy fibre is $(n-1)$ -connected.) Pasting these squares with (2), it follows that for $d \geq 5$ there is a smooth structure on Y extending that on ∂Y if and only if there is dashed map in the diagram

$$\begin{array}{ccc} \partial Y & \longrightarrow & \text{E}(\text{Top}/\text{O}) \simeq * \\ \downarrow & \nearrow & \downarrow \\ Y & \xrightarrow{\theta \circ \tau_Y} & \text{B}(\text{Top}/\text{O}) \end{array}$$

making both triangles commute. That is, the solid arrows define a class

$$\text{sm}(Y, \partial Y) \in [Y/\partial Y, \text{B}(\text{Top}/\text{O})]$$

which is trivial if, and for $d \geq 5$ only if, Y admits a smooth structure extending that on ∂Y . There is a map

$$\text{ks}: \text{B}(\text{Top}/\text{O}) \longrightarrow K(\mathbb{Z}/2, 4)$$

which is 8-connected. This follows from [KS77, p. 246 (5)] and the fact that the low-dimensional groups of homotopy spheres satisfy $\Theta_5 = \Theta_6 = 0$. Post-composing $\text{sm}(Y, \partial Y)$ with the map ks gives a class

$$\text{ks}(Y, \partial Y) \in H^4(Y, \partial Y; \mathbb{Z}/2),$$

whose vanishing is necessary, and for $5 \leq d \leq 7$ sufficient, for the existence of a smooth structure on Y extending that on ∂Y . (For all $d \geq 5$ the vanishing of this class is necessary and sufficient for the existence of a PL-structure on Y extending that on ∂Y induced by its smooth structure.)

Finally, we use this discussion to prove Lemma 2.2, and in fact prove something a little stronger. As mentioned above $\text{B}(\text{Top}/\text{O})$ is an infinite loop space. For a based space Z , this determines a group structure on $[Z, \text{B}(\text{Top}/\text{O})]$.

Lemma 3.2. *The function*

$$\text{Sm}: Q(X) \longrightarrow \left[\frac{X \times I \times I}{\partial(X \times I \times I)}, \text{B}(\text{Top}/\text{O}) \right]; \quad [F] \longmapsto \text{sm}(X \times I \times I, \partial(X \times I \times I)_F)$$

is well-defined and is a homomorphism.

Proof. If $[F_0] = [F_1] \in Q(X)$ then by definition there is a homeomorphism G of $X \times I \times I$ which restricts to F_i on $X \times I \times \{i\}$, to the identity on $(X \times \{0\} \times I) \cup ((\partial X) \times I \times I)$, and to a diffeomorphism on $X \times \{1\} \times I$. We endow

$$\partial(X \times I \times I) \times I \subseteq \partial(X \times I \times I \times I)$$

with a smooth structure by giving $X \times I \times \{1\} \times I$ the smooth structure $(X \times I \times \{1\} \times I)_G$ given by pulling the standard one back along G , and giving the rest of $\partial(X \times I \times I) \times I$ the standard smooth structure. One checks that these two rules are compatible where they overlap. This smooth structure restricts to $\partial(X \times I \times I)_{F_i} \times \{i\}$, so the data

$$\begin{array}{ccccccc} \partial(X \times I \times I) \times I & \longrightarrow & \text{BO}(d+3) & \longrightarrow & \text{BO} & \longrightarrow & \text{E}(\text{Top}/\text{O}) \simeq * \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (X \times I \times I) \times I & \longrightarrow & \text{BTop}(d+3) & \longrightarrow & \text{BTop} & \xrightarrow{\theta} & \text{B}(\text{Top}/\text{O}) \end{array}$$

defines a homotopy from $\text{sm}(X \times I \times I, \partial(X \times I \times I)_{F_0})$ to $\text{sm}(X \times I \times I, \partial(X \times I \times I)_{F_1})$.

We show the map is a homomorphism. The group structure on the set $\left[\frac{X \times I \times I}{\partial(X \times I \times I)}, \text{B}(\text{Top}/\text{O}) \right]$ is the group structure given by the infinite loop space $\text{B}(\text{Top}/\text{O})$. The domain is homeomorphic to the suspension $\Sigma^2(X/\partial X)$, hence is a co- H -space, while the codomain is an H -space. It then follows from the Eckmann–Hilton argument [Spa95, p. 43–4], that the group structure arising from the codomain is equivalent to the group structure arising the domain, i.e. from “stacking” in the second (say) interval direction of $X \times I \times I$.

Let $[F]$ and $[F']$ be two elements of $Q(X)$. We observe that the topological manifold with smooth boundary $(X \times I \times I, \partial(X \times I \times I)_{F' \circ F})$ can be obtained up to isomorphism by stacking the topological manifolds with smoothed boundaries

$$(X \times I \times I, \partial(X \times I \times I)_F) \text{ and } (X \times I \times I, (F \times \text{Id}_I)^* \partial(X \times I \times I)_{F'})$$

in the second interval direction. Here, $(F \times \text{Id}_I)^* \partial(X \times I \times I)_{F'}$ indicates the smooth structure obtained by pulling back the smooth structure $\partial(X \times I \times I)_{F'}$, using the restriction of the homeomorphism $F \times \text{Id}_I$, to $\partial(X \times I \times I)$; see Figure 2.

Using this, we see that

$$\begin{aligned} \text{Sm}(F' \circ F) &= \text{sm}(X \times I \times I, \partial(X \times I \times I)_{F' \circ F}) \\ &= \text{sm}((X \times I \times I, \partial(X \times I \times I)_F) \cup (X \times I \times I, (F \times \text{Id}_I)^* \partial(X \times I \times I)_{F'})) \\ &= \text{sm}(X \times I \times I, \partial(X \times I \times I)_F) + \text{sm}(X \times I \times I, (F \times \text{Id}_I)^* \partial(X \times I \times I)_{F'}) \\ &\in \left[\frac{X \times I \times I}{\partial(X \times I \times I)}, \text{B}(\text{Top}/\text{O}) \right]. \end{aligned}$$

$$\begin{array}{ccc}
(F' \circ F)^* \sigma = F^*((F')^* \sigma) & & \\
(F|_{X \times \{0\}} \times \text{Id}_I)^* \sigma = \sigma & \begin{array}{c} \boxed{\begin{array}{c} X \times I \times I \\ F^* \sigma \end{array}} & (F|_{X \times \{1\}} \times \text{Id}_I)^* \sigma = \sigma \\
\sigma & \boxed{\begin{array}{c} X \times I \times I \end{array}} & \sigma \\
& \sigma &
\end{array}$$

FIGURE 2. We denote the standard smooth structure on $X \times I$ by σ . Suppressing the X direction, we indicate the stacking proposed, together with the smooth structures on the various copies of $X \times I$ that make up the boundaries of the two copies of $X \times I \times I$.

The additivity of sm with respect to $\cup$ is due to the Eckmann-Hilton argument explained above. There is an identification of topological manifolds with smoothed boundaries

$$F \times \text{Id}_I : (X \times I \times I, (F \times \text{Id}_I)^* \partial(X \times I \times I)_{F'}) \xrightarrow{\cong} (X \times I \times I, \partial(X \times I \times I)_{F'}),$$

and therefore

$$\begin{aligned}
& \text{sm}(X \times I \times I, \partial(X \times I \times I)_F) + \text{sm}(X \times I \times I, (F \times \text{Id}_I)^* \partial(X \times I \times I)_{F'}) \\
&= \text{sm}(X \times I \times I, \partial(X \times I \times I)_F) + (F \times \text{Id}_I)^* \text{sm}(X \times I \times I, \partial(X \times I \times I)_{F'}) \\
&= \text{Sm}(F) + (F \times \text{Id}_I)^* \text{Sm}(F').
\end{aligned}$$

The map of pairs

$$F : (X \times I, \partial(X \times I)) \longrightarrow (X \times I, \partial(X \times I))$$

is the identity on $X \times \{0\} \cup \partial(X) \times I$ and sends $X \times \{1\}$ into itself, so by Lemma 3.3 below it is homotopic as a map of pairs to the identity. This implies that

$$\text{Sm}(F) + (F \times \text{Id}_I)^* \text{Sm}(F') = \text{Sm}(F) + \text{Sm}(F').$$

We deduce that $\text{Sm}(F' \circ F) = \text{Sm}(F) + \text{Sm}(F') \in \left[\frac{X \times I \times I}{\partial(X \times I \times I)}, \text{B}(\text{Top/O}) \right]$, as required. $\square$

Lemma 3.3. *Let Y be a compact manifold, possibly with boundary, and let*

$$f : (Y \times I, \partial(Y \times I)) \longrightarrow (Y \times I, \partial(Y \times I))$$

be a map of pairs that restricts to the identity on $Y \times \{0\} \cup \partial(Y) \times I$ and sends $Y \times \{1\}$ into itself. Then f is homotopic as a map of pairs to the identity.

Proof. We exhibit a homotopy. Write $f(y, t) = (\gamma(y, t), \tau(y, t))$, where this defines the functions γ and τ . There is a preliminary homotopy given by the formula

$$f_s(y, t) = (\gamma(y, t), (1-s)\tau(y, t) + st),$$

satisfying $f_0 = f$ and $f_1(y, t) = (\gamma(y, t), t)$. This homotopy is fixed on $Y \times \{0\} \cup \partial(Y) \times I$, and if $(y, 1) \in Y \times \{1\}$ so that $f(y, 1) = (\gamma(y, 1), 1)$ then $f_s(y, 1) = (\gamma(y, 1), 1) = f(y, 1) \in \partial(Y \times I)$: thus it is a homotopy which is fixed on $\partial(Y \times I)$.

We make a second homotopy by the formula

$$f_{1,s}(y, t) = (\gamma(y, (1-s)t), t).$$

This satisfies $f_{1,0} = f_1$, and $f_{1,1}(y, t) = (\gamma(y, 0), t) = (y, t)$, so $f_{1,1}$ is the identity. One verifies that it sends $\partial(Y \times I)$ into itself, so gives a homotopy of maps of pairs. $\square$

Proof of Lemma 2.2. Postcompose the function from Lemma 3.2 with

$$\text{ks} : \left[\frac{X \times I \times I}{\partial(X \times I \times I)}, \text{B}(\text{Top/O}) \right] \longrightarrow \left[\frac{X \times I \times I}{\partial(X \times I \times I)}, K(\mathbb{Z}/2, 4) \right] \cong \tilde{H}^4(X \times I \times I, \partial(X \times I \times I); \mathbb{Z}/2), \quad (3)$$

and Poincaré duality. The composition $PD \circ \text{ks} \circ \text{Sm} : Q(X) \rightarrow H_2(X; \mathbb{Z}/2)$ is exactly the function KS in the statement of Lemma 2.2. We need to see that it is a homomorphism. By Lemma 3.2, the function Sm is a homomorphism, and certainly Poincaré duality is. The H -space structures

on $B(\text{Top}/O)$ and $K(\mathbb{Z}/2, 4)$ need not agree a priori, so it is not automatic that the map in (3) is a homomorphism. However, the map $B(\text{Top}/O) \rightarrow K(\mathbb{Z}/2, 4)$ factors through the Postnikov truncation: $B(\text{Top}/O) \rightarrow \tau_{\leq 4}B(\text{Top}/O) \xrightarrow{\sim} K(\mathbb{Z}/2, 4)$. The Postnikov truncation inherits an H -space structure, which follows from obstruction theory (the obstructions to extending the multiplication have coefficients in $\pi_i(\tau_{\leq k}X)$ for $i \geq k+1$, which vanishes). Hence the first map in our factorisation is an H -space map. Again by obstruction theory, Eilenberg–MacLane spaces have a unique H -space structure, so the second map is also an H -space map. (Since the domain has two I factors, we could also have applied the Eckmann–Hilton argument [Spa95, p. 43–4] again.) It follows that ks is also a homomorphism, which completes the proof. $\square$

3.2. Proof of Theorem A. Recall from the introduction that given a closed, smooth, connected 4-manifold X and a homeomorphism $F: X \times I \rightarrow X \times I$ restricting to diffeomorphisms $f_i: X \times \{i\} \rightarrow X \times \{i\}$ for $i = 0, 1$, we can endow the boundary of $X \times I \times I$ with a smooth structure $\partial(X \times I \times I)_F$. Using the discussion in the previous section we define

$$\text{KS}(F) := \text{ks}(X \times I \times I, \partial(X \times I \times I)_F) \in H^4(X \times I \times I, \partial(X \times I \times I)_F; \mathbb{Z}/2) \cong H_2(X; \mathbb{Z}/2).$$

Theorem 3.4. *The following are equivalent:*

- (i) $\text{KS}(F) = 0$;
- (ii) $X \times I \times I$ admits a smooth structure extending $\partial(X \times I \times I)_F$;
- (iii) F is topologically isotopic, rel. boundary, to a smooth pseudo-isotopy;
- (iv) F is topologically pseudo-isotopic, rel. boundary, to a smooth pseudo-isotopy.

Proof. That (i) $\iff$ (ii) is by the discussion in the previous section, using that $\dim(X \times I \times I) = 6$.

We prove that (ii) $\implies$ (iii). If $X \times I \times I$ is endowed with a smooth structure extending $\partial(X \times I \times I)_F$, then it is a concordance rel. boundary from the smooth structure $(X \times [0, 1])_\sigma$ to $(X \times [0, 1])_{F^*\sigma}$. Since $X \times I$ has dimension five and the smooth structures already agree on the 4-dimensional boundary $X \times \{0, 1\}$, we may apply Kirby–Siebenmann’s concordance implies isotopy [KS77, Essay I, Theorem 4.1] for smooth structures, to obtain an isotopy of homeomorphisms $G_t: X \times [0, 1] \rightarrow X \times [0, 1]$, for $t \in [0, 1]$, such that $G_0 = \text{Id}_{X \times [0, 1]}$ and $G_1^*\sigma = F^*\sigma$. Then $F \circ G_1^{-1}: X \times [0, 1] \rightarrow X \times [0, 1]$ is an isotopy from F to $F \circ G_1^{-1}$, and

$$(F \circ G_1^{-1})^*\sigma = (G_1^{-1})^*F^*\sigma = (G_1^{-1})^*G_1^*\sigma = (G_1 \circ G_1^{-1})^*\sigma = \text{Id}^*\sigma = \sigma.$$

Hence $F \circ G_1^{-1}: (X \times [0, 1])_\sigma \rightarrow (X \times [0, 1])_\sigma$ is a diffeomorphism, i.e. a smooth pseudo-isotopy of $X \times [0, 1]$, that is topologically isotopic to F . Thus (iii) holds.

Certainly (iii) $\implies$ (iv). To see that (iv) $\implies$ (i), suppose that $F: X \times I \rightarrow X \times I$ is topologically pseudo-isotopic rel. boundary to a smooth pseudo-isotopy, via a pseudo-isotopy $G: X \times I \times I \rightarrow X \times I \times I$. Let $f_i := F|_{X \times \{i\}}: X \rightarrow X$ be the diffeomorphisms obtained by restricting F , for $i = 0, 1$. Define $F' := F \circ (f_0 \times \text{Id}_I)^{-1}$. This is a topological pseudo-isotopy from Id_X to the diffeomorphism $f_1 \circ f_0: X \rightarrow X$, and hence determines an element $[F'] \in Q(X)$. Moreover $G' := G \circ (f_0 \times \text{Id}_I \times \text{Id}_I)^{-1}$ is a pseudo-isotopy from F' to a diffeomorphism of $X \times I$, i.e. to a smooth pseudo-isotopy. It follows that $[F']$ is equivalent in $Q(X)$ to a smooth pseudo-isotopy $F'' := G'|_{X \times I \times \{1\}}$, and so $\text{KS}(F') = \text{KS}(F'')$ by Lemma 2.2. Since F'' is a diffeomorphism, $\partial(X \times I \times I)_{F''} \cong \partial(X \times I \times I)$, where the latter denotes the standard smooth structure on the boundary. But $\text{ks}(X \times I \times I, \partial(X \times I \times I)) = 0$, so $\text{KS}(F) = \text{KS}(F'') = 0$. $\square$

In particular, the implication (i) $\implies$ (iii) is Theorem A. The implication (i) $\implies$ (iv) is Theorem 2.3 (A), which seems slightly weaker than Theorem A but as we have just seen is in fact not.

3.3. Proof of Corollary B. For a compact smooth d -manifold X , let $\mathcal{P}^{\text{Diff}}(X)$ denote the space of smooth pseudo-isotopies of X , i.e. the set of diffeomorphisms $F: X \times I \rightarrow X \times I$ such that F acts as the identity near $(X \times \{0\}) \cup (\partial X \times I)$, endowed with the Whitney topology. Similarly let $\mathcal{P}^{\text{Top}}(X)$ denote the space of topological pseudo-isotopies of X , with the compact-open topology. For $d \geq 4$, Hatcher–Wagoner [HW73] defined a homomorphism

$$\Sigma^{\text{Diff}}: \pi_0 \mathcal{P}^{\text{Diff}}(X) \longrightarrow \text{Wh}_2(\pi_1(X)).$$

For $d \geq 5$, Burgelea–Lashof–Rothenberg and Pedersen [BLR75, Appendix A] indicated how to extend the definition of Σ^{Diff} to the topological category. The details of the analogous construction

were worked out for $d = 4$ by Galvin–Nonino [GN25, Theorem 1.1], giving rise to a homomorphism $\Sigma^{\text{Top}}: \mathcal{P}^{\text{Top}}(X) \rightarrow \text{Wh}_2(\pi_1(X))$ such that Σ^{Diff} factors through the forgetful map as

$$\Sigma^{\text{Diff}}: \pi_0 \mathcal{P}^{\text{Diff}}(X) \rightarrow \pi_0 \mathcal{P}^{\text{Top}}(X) \xrightarrow{\Sigma^{\text{Top}}} \text{Wh}_2(\pi_1(X)). \quad (4)$$

Finally, recall a theorem of Gabai.

Theorem 3.5 ([Gab22, Theorem 2.5]). *Let $f: X \xrightarrow{\cong} X$ be an orientation-preserving diffeomorphism of a smooth, compact, oriented 4-manifold X . Then f is smoothly stably isotopic to Id_X if and only if f is smoothly pseudo-isotopic to Id_X via a smooth pseudo-isotopy F with vanishing Hatcher–Wagoner obstruction $\Sigma^{\text{Diff}}([F]) = 0 \in \text{Wh}_2(\pi_1(X))$.*

We can now prove Corollary B.

Proof of Corollary B. By Theorem A if F has $\text{KS}(F) = 0$ then it is topologically isotopic rel. $\partial(X \times [0, 1])$ to a smooth pseudo-isotopy F' . If moreover F is a topological isotopy then $[F] = [\text{Id}_X] \in \pi_0 \mathcal{P}^{\text{Top}}(X)$ so $\Sigma^{\text{Top}}([F]) = 0 \in \text{Wh}_2(\pi_1(X))$. By (4), and since $[F'] \mapsto [F]$ under the forgetful map, we deduce that $\Sigma^{\text{Diff}}([F']) = \Sigma^{\text{Top}}([F]) = 0$. By Gabai’s Theorem 3.5, this is equivalent to f being smoothly stably isotopic to Id_X . $\square$

4. DUALITIES AND ORIENTATIONS

We establish notation and recall some fundamental concepts from stable homotopy theory. First, we introduce our notation for the standard spectra we shall invoke. We write $\mathbb{S}$ for the sphere spectrum, and $\mathbb{S}^d := \Sigma^d \mathbb{S}$ for the sphere spectrum shifted by d . Of course, $\mathbb{S}^0 = \mathbb{S}$, and note $\pi_n(\mathbb{S}^d) \cong \pi_{n-d}^s(\text{pt}_+) = \pi_{n-d}^s$. For an abelian group A and $k \in \mathbb{Z}$, we write $\text{HA}[k]$ for the Eilenberg–MacLane spectrum with $\pi_k(\text{HA}[k]) \cong A$ and $\pi_j(\text{HA}[k]) = 0$ for $j \neq k$. Finally let MSTop be the oriented Thom spectrum.

Given spectra X and E , we define the E -theory homology and cohomology, respectively, as

$$E_r(X) := [\mathbb{S}^r, X \wedge E] \quad \text{and} \quad E^r(X) := [X, \Sigma^r E].$$

If Y is a pointed space, we write $E_r(Y) := E_r(\Sigma^\infty Y)$ and $E^r(Y) := E^r(\Sigma^\infty Y)$ for the E -theory homology and cohomology.

4.1. Spanier–Whitehead and Atiyah duality. Spectra A and B are *Spanier–Whitehead dual* if there exists a *duality morphism* (see [Rud98, IV.2.3(a)]), which is a *coevaluation* map $\text{coev}: \mathbb{S} \rightarrow A \wedge B$ satisfying certain properties. The existence of a coevaluation map is equivalent to the existence of an *evaluation map* $\text{ev}: B \wedge A \rightarrow \mathbb{S}$ satisfying certain properties [Rud98, IV.2.6(a)]. Moreover, if A and B Spanier–Whitehead dual then for all spectra C and D coevaluation induces an isomorphism

$$[D \wedge A, C] \xrightarrow{\cong} [D, C \wedge B]; \quad \varphi \mapsto (\varphi \wedge \text{Id}_B) \circ (\text{Id}_D \wedge \text{coev})$$

(see [Rud98, IV.2.5(ii)]).

Given spectra A and B with respective Spanier–Whitehead duals $A^\vee$ and $B^\vee$, the *Spanier–Whitehead dual* of a map $f: A \rightarrow B$ is the map $f^\vee: B^\vee \rightarrow A^\vee$, given by the composition

$$f^\vee: B^\vee \xrightarrow{\cong} \mathbb{S} \wedge B^\vee \xrightarrow{\text{coev} \wedge \text{Id}} A^\vee \wedge A \wedge B^\vee \xrightarrow{\text{Id} \wedge f \wedge \text{Id}} A^\vee \wedge B \wedge B^\vee \xrightarrow{\text{Id} \wedge \text{ev}} A^\vee \wedge \mathbb{S} \xrightarrow{\cong} A^\vee.$$

We will write $\text{SW}(A) = A^\vee$ for the Spanier–Whitehead dual of A , when it exists, and $\text{SW}(f) = f^\vee$ for the Spanier–Whitehead dual of a map between objects which have Spanier–Whitehead duals.

Remark 4.1. If $\text{coev}: \mathbb{S} \rightarrow A \wedge B$ exhibits A and B as Spanier–Whitehead dual, then postcomposing it with the factor switching map $\text{sw}: A \wedge B \rightarrow B \wedge A$ gives a coevaluation map exhibiting B and A as Spanier–Whitehead dual. That is, being in duality is a symmetric property of A and B .

We also note that $\mathbb{S}$ is self-dual, and that if A and B are Spanier–Whitehead dual, then $\Sigma^d A$ and $\Sigma^{-d} B$ are Spanier–Whitehead dual for all d .

Assume X is a d -dimensional topological manifold with possibly nonempty boundary. Recall that the *stable normal microbundle* $\nu_X^{\text{Top}}: X \rightarrow \text{BTop}$ is the map classifying the virtual topological microbundle $[\nu_e^{\text{Top}} - \varepsilon^N]$ of virtual dimension zero, where ν_e^{Top} the normal microbundle of any

locally flat embedding $e: (X, \partial X) \hookrightarrow (D^{d+N}, \partial D^{d+N})$ with $N \gg 0$, with N large enough for normal microbundles to exist [Ste75]. There are Spanier–Whitehead dual pairs

$$\mathrm{SW}(\Sigma^{\infty-d}(X/\partial X)) = \mathrm{Th}(\nu_X^{\mathrm{Top}}) \quad \text{and} \quad \mathrm{SW}(\Sigma^{\infty-d}X_+) = \mathrm{Th}(\nu_X^{\mathrm{Top}})/\mathrm{Th}(\nu_{\partial X}^{\mathrm{Top}});$$

see [Rud98, V.2.3.(i), V.2.14.(a)]. Recall that $X/\partial X = X_+$ if ∂X is empty (this is consistent with the convention that X/A refers to the push out of $\{*\} \leftarrow A \rightarrow X$) and in this case, the two Spanier–Whitehead dualities above agree.

We will describe the duality morphism of the latter. Let $\varepsilon \times \nu_X^{\mathrm{Top}}: X \times X \rightarrow \mathrm{BTop}$ denote the external direct sum of the stable trivial bundle over X and ν_X^{Top} . By definition of the Whitney sum, there is a stable bundle map $\nu_X^{\mathrm{Top}} \simeq \varepsilon \oplus \nu_X^{\mathrm{Top}} \rightarrow \varepsilon \times \nu_X^{\mathrm{Top}}$ covering $\mathrm{diag}: X \rightarrow X \times X$. We may consider this bundle map to cover the diagonal map $(X, \partial X) \rightarrow X \times (X, \partial X)$. Passing to Thom spaces and noting that $\mathrm{Th}(\varepsilon) = \Sigma^\infty X_+$, we obtain the *Thom diagonal*

$$\Delta: \mathrm{Th}(\nu_X^{\mathrm{Top}})/\mathrm{Th}(\nu_{\partial X}^{\mathrm{Top}}) \longrightarrow \Sigma^\infty X_+ \wedge \mathrm{Th}(\nu_X^{\mathrm{Top}})/\mathrm{Th}(\nu_{\partial X}^{\mathrm{Top}}).$$

We then have a map

$$\Sigma^d \xrightarrow{c_X} \mathrm{Th}(\nu_X^{\mathrm{Top}})/\mathrm{Th}(\nu_{\partial X}^{\mathrm{Top}}) \xrightarrow{\Delta} \Sigma^\infty X_+ \wedge \mathrm{Th}(\nu_X^{\mathrm{Top}})/\mathrm{Th}(\nu_{\partial X}^{\mathrm{Top}}),$$

where c_X denotes the Thom collapse map. Desuspending this composition d times gives the desired duality map [Rud98, V.2.3.(i)] (cf. [Bro72, I.1.15]). The duality map for the other claimed Spanier–Whitehead duality is described similarly, but by considering the diagonal as the map of pairs $(X, \emptyset) \rightarrow (X, \partial X) \times X$.

So for each spectrum E and each $r \in \mathbb{Z}$ there are group isomorphisms

$$\begin{aligned} E_r(X/\partial X) &= [\Sigma^r \mathbb{S}, \Sigma^\infty(X/\partial X) \wedge E] = [\mathbb{S}, \Sigma^{\infty-d}(X/\partial X) \wedge \Sigma^{d-r} E] \\ &\cong [\mathrm{Th}(\nu_X^{\mathrm{Top}}), \Sigma^{d-r} E] = E^{d-r}(\mathrm{Th}(\nu_X^{\mathrm{Top}})), \end{aligned}$$

and

$$\begin{aligned} E^r(X/\partial X) &= [\Sigma^\infty(X/\partial X), \Sigma^r E] = [\Sigma^{\infty-d}(X/\partial X), \Sigma^{r-d} E] \\ &\cong [\mathbb{S}, \mathrm{Th}(\nu_X^{\mathrm{Top}}) \wedge \Sigma^{r-d} E] = E_{d-r}(\mathrm{Th}(\nu_X^{\mathrm{Top}})). \end{aligned}$$

These are referred to as *Atiyah duality* [Ati61]. We will denote the map from the latter display, from cohomology to homology, by AD .

We record the following, for later use.

Lemma 4.2. *If X is a d -dimensional compact topological manifold then the Spanier–Whitehead dual of the Thom collapse map $\Sigma^{-d}c_X: \mathbb{S} \rightarrow \Sigma^{-d}(\mathrm{Th}(\nu_X^{\mathrm{Top}})/\mathrm{Th}(\nu_{\partial X}^{\mathrm{Top}}))$ is the map $C_X: \Sigma^\infty X_+ \rightarrow \mathbb{S}$, induced by the constant map $X \rightarrow \mathrm{pt}$, adding basepoints and taking suspension spectra. Moreover, under the Σ – Ω adjunction, the map C_X corresponds to the constant map $X_+ \rightarrow \Omega^\infty \mathbb{S} = \mathrm{colim}_k \Omega^k S^k$ to the class of the identity map.*

Proof. Write $\Delta' := \mathrm{sw} \circ \Delta$ for the Thom diagonal followed by the factor switch map; this will determine the coevaluation map needed for this proof (see Remark 4.1). By definition of the duality, the Spanier–Whitehead dual of the map $C_X: \Sigma^\infty X_+ \rightarrow \mathbb{S}$ is the composition

$$\begin{aligned} \mathbb{S} &\xrightarrow{\simeq} \mathbb{S} \wedge \mathbb{S} \xrightarrow{\Sigma^{-d}c_X \wedge \mathrm{Id}} \Sigma^{-d}(\mathrm{Th}(\nu_X^{\mathrm{Top}})/\mathrm{Th}(\nu_{\partial X}^{\mathrm{Top}})) \wedge \mathbb{S} \xrightarrow{\Delta' \wedge \mathrm{Id}} \Sigma^{-d}(\mathrm{Th}(\nu_X^{\mathrm{Top}})/\mathrm{Th}(\nu_{\partial X}^{\mathrm{Top}})) \wedge \Sigma^\infty X_+ \wedge \mathbb{S} \\ &\xrightarrow{\mathrm{Id} \wedge C_X \wedge \mathrm{Id}} \Sigma^{-d}(\mathrm{Th}(\nu_X^{\mathrm{Top}})/\mathrm{Th}(\nu_{\partial X}^{\mathrm{Top}})) \wedge \mathbb{S} \wedge \mathbb{S} \xrightarrow{\simeq} \Sigma^{-d}(\mathrm{Th}(\nu_X^{\mathrm{Top}})/\mathrm{Th}(\nu_{\partial X}^{\mathrm{Top}})). \end{aligned}$$

Applying the d -fold suspension one obtains

$$\Sigma^d \xrightarrow{c_X} \mathrm{Th}(\nu_X^{\mathrm{Top}})/\mathrm{Th}(\nu_{\partial X}^{\mathrm{Top}}) \xrightarrow{\Delta'} \Sigma^\infty X_+ \wedge \mathrm{Th}(\nu_X^{\mathrm{Top}})/\mathrm{Th}(\nu_{\partial X}^{\mathrm{Top}}) \xrightarrow{\mathrm{Id} \wedge C_X} \mathrm{Th}(\nu_X^{\mathrm{Top}})/\mathrm{Th}(\nu_{\partial X}^{\mathrm{Top}}) \wedge \mathbb{S}.$$

But by definition of the Thom diagonal the composition $(\mathrm{Id} \wedge C_X) \circ \Delta'$ is the identity. Hence $\mathrm{SW}(\Sigma^{-d}c_X) = C_X$, as claimed.

The final assertion follows from adding basepoints to the map $X \rightarrow \mathrm{pt}$ then smashing with S^k to get a map $S^k \wedge X_+ \rightarrow S^k \wedge S^0$, which is the identity on the first factor. Taking adjoints results in a constant map to the identity element $X_+ \rightarrow \Omega^k(S^k \wedge S^0) = \Omega^k(S^k)$. The result follows by taking the colimit. $\square$

4.2. Orientations and fundamental classes. If E is a ring spectrum, and X is a d -dimensional compact topological manifold, then an E -orientation of, or E -theory fundamental class for, X is a class $[X]_E \in E_d(X/\partial X)$ such that for any coordinate chart $\mathbb{R}^d \subseteq X$ with corresponding collapse map

$$c: X/\partial X \rightarrow X/(X - \text{Int}(D^d)) \cong D^d/\partial D^d \cong S^d$$

the class $c_*([X]_E) \in E_d(S^d)$ corresponds under the suspension isomorphism to $\pm 1 \in E_0(S^0)$ [Rud98, V.2.1]. Here $\pm 1 \in E_0(S^0)$ is the image of $\pm 1 \in \pi_0^s(S^0) \cong \mathbb{Z}$ under the map $\pi_0^s(S^0) \rightarrow E_0(S^0)$ induced by the unit map $\eta: \mathbb{S} \rightarrow E$ of the ring spectrum. The notion of E -orientability of a manifold is closely connected to the definition of E -orientability of the stable normal microbundle $\nu_X^{\text{Top}}: X \rightarrow \text{BTop}$ ([Rud98, V.1.12]). Indeed, under Atiyah duality, E -theory fundamental classes $[X]_E \in E_d(X/\partial X)$ correspond to E -theory Thom classes $U_E \in E^0(\text{Th}(\nu_X^{\text{Top}}))$ for ν_X^{Top} (also known as E -orientations for ν_X^{Top}) [Rud98, Corollary V.2.6]. Given a module spectrum F over E , and an E -theory fundamental class $[X]_E$ on the manifold X , there are induced F -theory Poincaré duality isomorphisms

$$[X]_E \frown -: F^r(X/\partial X) \xrightarrow{\cong} F_{d-r}(X_+),$$

which coincide with Atiyah duality for F followed by the homological Thom isomorphism; see [Rud98, V.1.3]) and the second diagram of [Swi02, Theorem 14.41]. Note that in this normalisation the Thom class, and hence the Thom isomorphism, has degree zero.

5. PROOF OF THEOREM C

We will prove Theorem C in the form of Theorem 2.3 (C), so will show that the image of the composition

$$\pi_1(\widetilde{\text{Homeo}}_{\partial}^+(X)) \xrightarrow{\alpha} Q(X) \xrightarrow{\text{ks}} H_2(X; \mathbb{Z}/2)$$

contains the kernel $\ker(H_2(X; \mathbb{Z}/2) \xrightarrow{\delta_*} H_1(X; \mathbb{Z}_{(2)}))$ of the Bockstein operator for the coefficient sequence $0 \rightarrow \mathbb{Z}_{(2)} \xrightarrow{2} \mathbb{Z}_{(2)} \rightarrow \mathbb{Z}/2 \rightarrow 0$. It suffices to do so after precomposing with the connecting map

$$\partial: \pi_2\left(\frac{\widetilde{\text{sAut}}_{\partial}^+(X)}{\widetilde{\text{Homeo}}_{\partial}^+(X)}\right) \rightarrow \pi_1(\widetilde{\text{Homeo}}_{\partial}^+(X))$$

in the long exact sequence of semi-simplicial homotopy groups (see e.g. [GJ09, §1.7] for a definition) associated to the fibre sequence $\widetilde{\text{Homeo}}_{\partial}^+(X) \rightarrow \widetilde{\text{sAut}}_{\partial}^+(X) \rightarrow \frac{\widetilde{\text{sAut}}_{\partial}^+(X)}{\widetilde{\text{Homeo}}_{\partial}^+(X)}$ of Kan semi-simplicial sets. The domain of this connecting map has the advantage that it may be described via surgery theory, as we now explain.

5.1. Geometric surgery. First we briefly recall the *simple structure set*, and describe its relation to domain of ∂ , above. Then we recall the surgery exact sequence of Browder–Novikov–Sullivan–Wall and show how it can be related to the maps α , ks , and Sm .

5.1.1. The structure set. In the following we let $\text{CAT} \in \{\text{Top}, \text{Diff}, \text{PL}\}$ be a category of manifolds. Let Y be a d -dimensional compact CAT manifold with (possibly empty) boundary ∂Y . The basic object of study in surgery theory is the simple CAT -structure set $\mathcal{S}_{\partial}^{\text{CAT}}(Y)$, described as follows.

Definition 5.1. Elements of $\mathcal{S}_{\partial}^{\text{CAT}}(Y)$ are equivalence classes of maps of pairs $(f, \partial f): (M, \partial M) \rightarrow (Y, \partial Y)$ from a CAT manifold such that ∂f is a CAT isomorphism and f is a simple homotopy equivalence. The equivalence relation on such maps is s -cobordism: if there is an s -cobordism W from $(M_0, \partial M_0)$ to $(M_1, \partial M_1)$, trivial on the boundary, and a map $(F, \partial_- F): (W, \partial_- W) \rightarrow (Y, \partial Y)$ restricting to $(f_0, \partial f_0): (M_0, \partial M_0) \rightarrow (Y, \partial Y)$ and $(f_1, \partial f_1): (M_1, \partial M_1) \rightarrow (Y, \partial Y)$ at the ends, then $(f_0, \partial f_0) \sim (f_1, \partial f_1)$.

Alternatively, if $d \geq 5$ (or $d = 4$, $\text{CAT} = \text{Top}$, and $\pi_1(Y)$ is good) then the CAT s -cobordism theorem applies and we can rephrase the equivalence relation as $(f_0, \partial f_0) \sim (f_1, \partial f_1)$ if and only if there is a CAT isomorphism $(\phi, \partial \phi): (M_0, \partial M_0) \rightarrow (M_1, \partial M_1)$ with $(f_1, \partial f_1) \circ (\phi, \partial \phi) \simeq (f_0, \partial f_0)$.

For each $k \geq 0$, we now describe a map

$$\pi_k\left(\frac{\widetilde{\text{sAut}}_{\partial}^+(Y)}{\widetilde{\text{Homeo}}_{\partial}^+(Y)}\right) \rightarrow \mathcal{S}_{\partial}^{\text{Top}}(Y \times D^k),$$

and show it is an injection for $k = 0$ and a bijection for $k > 0$, as long as $k + d \geq 5$. The domain is a (semi)-simplicial homotopy group of a quotient of a semi-simplicial set by a semi-simplicial group. Unravelling definitions shows that elements are represented by simple homotopy equivalences

$$f: Y \times \Delta^k \longrightarrow Y \times \Delta^k$$

which preserve faces, which restrict to the identity on $(\partial Y) \times \Delta^k$, and which restrict to homeomorphisms on $Y \times \partial \Delta^k$ (and are well-defined up to a homotopy which is an isotopy on $Y \times \partial \Delta^k$ and fixed on $(\partial Y) \times \Delta^k$, and precomposing by block homeomorphisms of $Y \times \Delta^k$ which are the identity on $(\partial Y) \times \Delta^k$). Such maps define elements of $\mathcal{S}_\partial^{\text{Top}}(Y \times D^k)$, and this assignment can be shown to be well-defined. To see that it is surjective for $k > 0$, we observe that an element of $\mathcal{S}_\partial^{\text{Top}}(Y \times D^k)$ is represented by a simple homotopy equivalence $\phi: (M, \partial M) \rightarrow (Y \times D^k, \partial(Y \times D^k))$ which is a homeomorphism on the boundary. Identifying $D^k = [0, 1]^k$ we may view M as an h -cobordism from $Y \times [0, 1]^{k-1} \times \{0\}$ to $Y \times [0, 1]^{k-1} \times \{1\}$ rel. boundary, but as M is simple homotopy equivalent to $Y \times [0, 1]^k$ this is in fact an s -cobordism. By the s -cobordism theorem there is a homeomorphism $M \cong Y \times [0, 1]^k$, and so the map ϕ may be considered as a simple homotopy equivalence $\phi: (Y \times D^k, \partial(Y \times D^k)) \rightarrow (Y \times D^k, \partial(Y \times D^k))$ which is a homeomorphism on the boundary, but this is precisely our description of elements coming from $\pi_k(\widetilde{\text{sAut}}_\partial^+(Y)/\widetilde{\text{Homeo}}_\partial^+(Y))$. As usual, injectivity is proved by a relative form of this surjectivity argument.

5.1.2. The surgery exact sequence. The surgery theory of Browder–Novikov–Sullivan–Wall [Wal99, LM24] describes the sets $\mathcal{S}_\partial^{\text{Top}}(Y \times D^k)$ for $d + k \geq 5$ via the surgery exact sequence

$$\begin{aligned} \cdots &\longrightarrow L_{d+3}(\mathbb{Z}[\pi]) \longrightarrow \mathcal{S}_\partial^{\text{Top}}(Y \times I \times I) \longrightarrow \left[\frac{Y \times I \times I}{\partial(Y \times I \times I)}, \text{G/Top} \right] \longrightarrow \\ &\longrightarrow L_{d+2}(\mathbb{Z}[\pi]) \longrightarrow \mathcal{S}_\partial^{\text{Top}}(Y \times I) \longrightarrow \left[\frac{Y \times I}{\partial(Y \times I)}, \text{G/Top} \right] \longrightarrow \\ &\longrightarrow L_{d+1}(\mathbb{Z}[\pi]) \longrightarrow \mathcal{S}_\partial^{\text{Top}}(Y) \xrightarrow{\eta_{\text{Top}}} [Y/\partial Y, \text{G/Top}] \xrightarrow{\sigma} L_d(\mathbb{Z}[\pi]), \end{aligned}$$

where $\pi := \pi_1(Y)$. Exactness is just as for the long exact sequence of homotopy groups for a fibration of pointed spaces: the last three terms are pointed sets, the next three are groups, and the rest are abelian groups, with the maps being homomorphisms to the extent possible (most important for us will be that the top row consists of abelian groups). The L -groups $L_k(\mathbb{Z}[\pi])$ are Wall's (simple) surgery obstruction groups. The topological monoid G is the structure group for stable spherical fibrations, and the space G/Top is defined as the homotopy fibre of the map $\text{BTop} \rightarrow \text{BG}$ induced by the forgetful map.

The stable normal microbundle $\nu_Y^{\text{Top}}: Y \rightarrow \text{BTop}$ is a preferred lift of the *Spivak normal fibration* $\nu_Y^{\text{G}}: Y \rightarrow \text{BG}$. The group $[Y/\partial Y, \text{G/Top}]$ acts freely transitively on the set of lifts of the Spivak normal fibration, relative to a fixed lift on the boundary ∂Y .

Given an element $[(f, \partial f): (M, \partial M) \rightarrow (Y, \partial Y)]$ of the structure set $\mathcal{S}_\partial^{\text{Top}}(Y)$, the homotopy equivalence f determines a stable microbundle $(f^{-1})^* \nu_M^{\text{Top}}: Y \rightarrow \text{BTop}$. As f is a homotopy equivalence, the underlying spherical fibration of $(f^{-1})^* \nu_M^{\text{Top}}$ is canonically homotopic to the Spivak normal fibration $Y \rightarrow \text{BG}$. As f is a homotopy equivalence, the underlying spherical fibrations of $(f^{-1})^* \nu_M^{\text{Top}}$ and ν_Y^{Top} are (canonically) identified. As ∂f is a homeomorphism this identification extends to one of stable microbundles over ∂Y . This gives a *difference class*

$$\eta_{\text{Top}}([(f, \partial f)]) := d((f^{-1})^* \nu_M^{\text{Top}}, \nu_Y^{\text{Top}}) \in [Y/\partial Y, \text{G/Top}].$$

See [Wal99, LM24] for further details on the surgery exact sequence.

5.1.3. Mapping the surgery exact sequence in. For a 4-manifold X we may form the diagram

$$\begin{array}{ccc} \pi_2\left(\frac{\widetilde{\text{sAut}}_\partial^+(X)}{\widetilde{\text{Homeo}}_\partial^+(X)}\right) & \xrightarrow{\cong} \mathcal{S}_\partial^{\text{Top}}(X \times I \times I) & \xrightarrow{\eta_{\text{Top}}} \left[\frac{X \times I \times I}{\partial(X \times I \times I)}, \text{G/Top} \right] \\ \downarrow \partial & & \downarrow \\ \pi_1(\widetilde{\text{Homeo}}_\partial^+(X)) & \xrightarrow{\alpha} Q(X) \xrightarrow{\text{Sm}} \left[\frac{X \times I \times I}{\partial(X \times I \times I)}, \text{B(Top/O)} \right] & \xrightarrow{\text{ks}} \widetilde{H}^4\left(\frac{X \times I \times I}{\partial(X \times I \times I)}; \mathbb{Z}/2\right). \end{array} \quad (5)$$

The right-hand vertical map is induced by the composition $G/\text{Top} \rightarrow B\text{Top} \rightarrow B(\text{Top}/O)$. The left-hand vertical map is the connecting homomorphism from the long exact sequence of the fibration.

Lemma 5.2. *The diagram (5) commutes.*

Proof. Let $\phi: (M, \partial M) \rightarrow (X \times I \times I, \partial(X \times I \times I))$ represent an element of $\mathcal{S}_\partial^{\text{Top}}(X \times I \times I)$, so it is a simple homotopy equivalence which is a homeomorphism on the boundary. In particular, M can be considered as an h -cobordism from $X \times I \times \{0\}$ to $X \times I \times \{1\}$ rel. boundary. By the composition formula its Whitehead torsion satisfies

$$f_*\tau(M, X \times I \times \{0\}) + \tau(\phi) = \tau(X \times I \times I, X \times I \times \{0\}) = 0 \in \text{Wh}(\pi_1(X \times I \times I)),$$

and as ϕ is a simple homotopy equivalence it follows that $\tau(M, X \times I \times \{0\}) = 0$, and thus M is an s -cobordism. The s -cobordism theorem then gives a homeomorphism

$$\psi: M \xrightarrow{\cong} X \times I \times I \quad \text{rel.} \quad X \times I \times \{0\} \cup \partial(X \times I) \times I,$$

which restricts to a homeomorphism $F: X \times I \times \{1\} \xrightarrow{\cong} X \times I \times \{1\}$ rel. boundary. Spelling out the isomorphism and definition of ∂ in the diagram shows that

$$\partial([\phi]) = [F] \in \pi_0(\widetilde{\text{Homeo}}_\partial^+(X \times I)) = \pi_1(\widetilde{\text{Homeo}}_\partial^+(X)).$$

Now $\alpha([F]) = [\text{Id}, F]$, which is just F considered as a topological pseudo-isotopy from the identity to the identity diffeomorphism, so $\text{ks} \circ \text{Sm} \circ \alpha([F]) \in \left[\frac{X \times I \times I}{\partial(X \times I \times I)}, K(\mathbb{Z}/2, 4) \right]$ is given by

$$\begin{array}{ccccccccc} \partial(X \times I \times I)_F & \xrightarrow{T\partial(X \times I \times I)_F \oplus \varepsilon^1} & \text{BO}(6) & \longrightarrow & \text{BO} & \longrightarrow & E(\text{Top}/O) & \longrightarrow & PK(\mathbb{Z}/2, 4) \simeq * \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X \times I \times I & \xrightarrow{\tau_{X \times I \times I}} & B\text{Top}(6) & \longrightarrow & B\text{Top} & \xrightarrow{\theta} & B(\text{Top}/O) & \xrightarrow{\text{ks}} & K(\mathbb{Z}/2, 4). \end{array}$$

Now ψ induces an isomorphism of topological manifolds with smoothed boundary

$$\psi: (M, \partial M) \xrightarrow{\cong} (X \times I \times I, \partial(X \times I \times I)_F)$$

so we have

$$\text{ks} \circ \text{Sm} \circ \alpha([F]) = \text{ks}(X \times I \times I, \partial(X \times I \times I)_F) = (\psi^{-1})^* \text{ks}(M, \partial M) \in \tilde{H}^4\left(\frac{X \times I \times I}{\partial(X \times I \times I)}; \mathbb{Z}/2\right).$$

The composition $\phi \circ \psi^{-1}: (X \times I \times I, \partial(X \times I \times I)) \rightarrow (X \times I \times I, \partial(X \times I \times I))$ is the identity on $X \times I \times \{0\} \cup \partial(X \times I) \times I$ and sends $X \times I \times \{1\}$ into itself, so by Lemma 3.3 it is homotopic as a map of pairs to the identity. It follows that ψ^{-1} is homotopic as map of pairs to ϕ^{-1} , so the right-hand side of the previous equation may be written as $(\phi^{-1})^* \text{ks}(M, \partial M)$.

On the other hand, by definition of the normal invariant map η_{Top} the composition

$$\mathcal{S}_\partial^{\text{Top}}(X \times I \times I) \xrightarrow{\eta_{\text{Top}}} \left[\frac{X \times I \times I}{\partial(X \times I \times I)}, G/\text{Top} \right] \longrightarrow \left[\frac{X \times I \times I}{\partial(X \times I \times I)}, B\text{Top} \right]$$

sends the simple homotopy equivalence $[\phi]$ to the map classifying the stable microbundle $(\phi^{-1})^* \nu_M^{\text{Top}} - \nu_{X \times I \times I}^{\text{Top}}$ together with the trivialisation over $\partial(X \times I \times I)$ given by the fact that ϕ is a homeomorphism on $\partial(X \times I \times I)$. Post-compose with $B\text{Top} \xrightarrow{\theta} B(\text{Top}/O) \xrightarrow{\text{ks}} K(\mathbb{Z}/2, 4)$. Since the Kirby–Siebenmann class is additive for Whitney sum of bundles [KS77, Annex 3 Lemma 15.5], ν_Y^{Top} and the stable tangent microbundle τ_Y^{Top} have the same Kirby–Siebenmann class. We therefore obtain

$$\begin{aligned} \text{ks}((\phi^{-1})^* \nu_M^{\text{Top}} - \nu_{X \times I \times I}^{\text{Top}}) &= (\phi^{-1})^* \text{ks}(\nu_M^{\text{Top}}) - \text{ks}(\nu_{X \times I \times I}^{\text{Top}}) = (\phi^{-1})^* \text{ks}(\tau_M^{\text{Top}}) - \text{ks}(\tau_{X \times I \times I}^{\text{Top}}) \\ &= (\phi^{-1})^* \text{ks}(M, \partial M) - \text{ks}(X \times I \times I, \partial) \\ &= (\phi^{-1})^* \text{ks}(M, \partial M) \in \tilde{H}^4\left(\frac{X \times I \times I}{\partial(X \times I \times I)}; \mathbb{Z}/2\right), \end{aligned}$$

where we have again used additivity, that ν_Y^{Top} and τ_Y^{Top} are stable inverses, the definition of $\text{ks}(Y, \partial Y)$, and that $X \times I \times I$ is smooth so has trivial Kirby–Siebenmann class. This completes the verification that both passages around the diagram yield the same result. $\square$

We give an outline of our strategy for the proof of Theorem 2.3 (C). Recall that our goal is to show that the image of $PD \circ \text{ks} \circ \text{Sm} \circ \alpha \circ \partial$ contains $\ker(\delta_*: H_2(X; \mathbb{Z}/2) \rightarrow H_1(X; \mathbb{Z}/2))$. As the codomain of the right vertical map in diagram (5) is a vector space over $\mathbb{Z}/2$, this map factors through the 2-localisation of its domain. This allows us to augment diagram (5) as follows

$$\begin{array}{ccccc}
\pi_2\left(\frac{\widetilde{\text{sAut}}_{\partial}^+(X)}{\text{Homeo}_{\partial}^+(X)}\right) & \xrightarrow{\cong} & \mathcal{S}_{\partial}^{\text{Top}}(X \times I \times I) & \xrightarrow{\eta_{\text{Top}}} & \left[\frac{X \times I \times I}{\partial(X \times I \times I)}, \text{G/Top}\right] \\
\downarrow \partial & & \downarrow & & \downarrow \\
& & \mathcal{S}_{\partial}^{\text{Top}}(X \times I \times I)_{(2)} & \xrightarrow{\eta_{\text{Top}}} & \left[\frac{X \times I \times I}{\partial(X \times I \times I)}, \text{G/Top}\right]_{(2)} \xrightarrow{\sigma} L_6(\mathbb{Z}[\pi])_{(2)} \\
& & & & \downarrow \\
\pi_1(\widetilde{\text{Homeo}}_{\partial}^+(X)) & \xrightarrow{\alpha} & Q(X) & \xrightarrow{\text{Sm}} & \left[\frac{X \times I \times I}{\partial(X \times I \times I)}, \text{B(Top/O)}\right] \\
& & & & \cong \downarrow \text{ks} \\
& & & & \widetilde{H}^4\left(\frac{X \times I \times I}{\partial(X \times I \times I)}; \mathbb{Z}/2\right) \xrightarrow{PD} H_2(X; \mathbb{Z}/2).
\end{array} \tag{6}$$

The 2-localised row in diagram (6) is meaningful because the surgery exact sequence for $X \times I \times I$ is an exact sequence of abelian groups, and can therefore be localised at 2, which furthermore preserves exactness. Diagram (5), with the right vertical arrow factored through the 2-localisation, is a subdiagram of diagram (6), and so this big square commutes by Lemma 5.2. The smaller square in diagram (6) commutes, by naturality of localisation. So diagram (6) commutes overall.

We will produce, for each $u \in \ker(\delta_*: H_2(X; \mathbb{Z}/2) \rightarrow H_1(X; \mathbb{Z}/2))$, a lift to an element $\xi_u \in \left[\frac{X \times I \times I}{\partial(X \times I \times I)}, \text{G/Top}\right]_{(2)}$ such that $\sigma(\xi_u) = 0$. By exactness we will then obtain an element in $\mathcal{S}_{\partial}^{\text{Top}}(X \times I \times I)_{(2)}$ mapping to u . After multiplying by an odd integer, this lifts to an element $\mathcal{S}_{\partial}^{\text{Top}}(X \times I \times I)$, also mapping to u . It follows that u is in the image of the clockwise composition starting in the top left corner. By commutativity of the big square, we can conclude u lies in the image of $PD \circ \text{ks} \circ \text{Sm} \circ \alpha \circ \partial$. This will complete the proof of Theorem 2.3 (C).

5.2. Ranicki–Sullivan duality and Poincaré duality. To do what we have just described, we will use Ranicki’s algebraic surgery exact sequence; see [Ran92]. In the topological category, this is an exact sequence isomorphic to the geometric surgery exact sequence, and expressed in terms of the $\mathbb{L}$ -spectra. In the algebraic surgery exact sequence, the surgery obstruction map σ from the geometric surgery exact sequence factors through the *Ranicki–Sullivan duality* map, followed by the *algebraic assembly* map, as we shall describe below. The main task of this section is to relate Ranicki–Sullivan duality back to ordinary Poincaré duality.

5.2.1. $\mathbb{L}$ -spectra. Given a ring R with anti-involution (often called an involution in the surgery literature), there are defined spectra $\mathbb{L}^s(R)$ and $\mathbb{L}^q(R)$, respectively the *symmetric* and *quadratic L*-spectra. The reader is referred to [Ran92] for the construction, but note we are using the naming convention of [Lur11], whereas Ranicki writes $\mathbb{L}^{\bullet}(R)$ and $\mathbb{L}_{\bullet}(R)$ respectively for the symmetric and quadratic spectra. Products in L -theory (see [Ran92, Appendix B], [Ran80]) endow $\mathbb{L}^s(R)$ with the structure of a ring spectrum and $\mathbb{L}^q(R)$ with the structure of an $\mathbb{L}^s(R)$ -module spectrum. We will denote by $\mathbb{L}^s = \tau_{\geq 0}\mathbb{L}^s(\mathbb{Z})$, the 0-connective cover of $\mathbb{L}^s(\mathbb{Z})$, by $\mathbb{L}^q = \tau_{\geq 0}\mathbb{L}^q(\mathbb{Z})$ the 0-connective cover of $\mathbb{L}^q(\mathbb{Z})$, and by $\mathbb{L}^q\langle 1 \rangle = \tau_{\geq 1}\mathbb{L}^q(\mathbb{Z})$ the 1-connective cover of $\mathbb{L}^q(\mathbb{Z})$; $\mathbb{L}^s$ is again a ring spectrum and $\mathbb{L}^q$ and $\mathbb{L}^q\langle 1 \rangle$ are $\mathbb{L}^s$ -module spectra. We will continue with the notation $E_r(-)$ and $E^r(-)$ to refer to the corresponding (co)homology theories for $E = \mathbb{L}^s, \mathbb{L}^q, \mathbb{L}^q\langle 1 \rangle$, where Ranicki uses the notation $H_r(-; E)$ and $H^r(-; E)$.

5.2.2. The symmetric L -theory fundamental class. There is a morphism $\sigma: \text{MSTop} \rightarrow \mathbb{L}^s$ of ring spectra, referred to as the *Ranicki orientation*; see [Ran92, §16]. For an oriented topological d -manifold Y , write $\nu_Y^{\text{Top}}: Y \rightarrow \text{BStop}$ for the oriented stable normal microbundle of the pair $(Y, \partial Y)$. We have an induced map of spectra $\text{Th}(\nu_Y^{\text{Top}}) \rightarrow \text{MSTop}$, and the composition $\text{Th}(\nu_Y^{\text{Top}}) \rightarrow \text{MSTop} \xrightarrow{\sigma} \mathbb{L}^s$ results in a $\mathbb{L}^s$ -theory Thom class $U_{\mathbb{L}^s} \in (\mathbb{L}^s)^0(\text{Th}(\nu_Y^{\text{Top}}))$; see [Ran92, §16]. Under

Atiyah duality $(\mathbb{L}^s)^0(\mathrm{Th}(\nu_Y^{\mathrm{Top}})) \cong (\mathbb{L}^s)_d(Y/\partial Y)$ we therefore obtain an $\mathbb{L}^s$ -theory fundamental class $[Y]_{\mathbb{L}^s} \in (\mathbb{L}^s)_d(Y/\partial Y)$.

5.2.3. The surgery obstruction and assembly. In [Ran79], Ranicki constructed a homotopy equivalence $G/\mathrm{Top} \simeq \Omega^\infty \mathbb{L}^q \langle 1 \rangle$, which we use to identify these two spaces from now on. In particular, if Y is a topological d -manifold as above then we have an identification

$$D: [Y/\partial Y, G/\mathrm{Top}] \cong [Y/\partial Y, \Omega^\infty \mathbb{L}^q \langle 1 \rangle] \cong [\Sigma^\infty(Y/\partial Y), \mathbb{L}^q \langle 1 \rangle] \xrightarrow{[Y]_{\mathbb{L}^s} \cap -} [\mathbb{S}^d, \Sigma^\infty Y_+ \wedge \mathbb{L}^q \langle 1 \rangle], \quad (7)$$

given by Poincaré duality in $\mathbb{L}^q \langle 1 \rangle$ -theory (this is the duality often referred to as *Ranicki–Sullivan duality*). Note that the first two terms refer to maps of spaces, while the last two refer to maps of spectra. Under this identification, the surgery obstruction map from [Wal99, §3] factors as

$$\begin{array}{ccc} [Y/\partial Y, G/\mathrm{Top}] & \xrightarrow{\sigma} & L_d(\mathbb{Z}[\pi]) \\ \cong \downarrow D & & \uparrow A_\pi \\ [\mathbb{S}^d, \Sigma^\infty Y_+ \wedge \mathbb{L}^q \langle 1 \rangle] & \longrightarrow & [\mathbb{S}^d, \Sigma^\infty Y_+ \wedge \mathbb{L}^q] \xrightarrow{\mathfrak{t}_*} [\mathbb{S}^d, \Sigma^\infty B\pi_+ \wedge \mathbb{L}^q] \end{array}$$

where the first lower map is induced by the canonical map $\mathbb{L}^q \langle 1 \rangle \rightarrow \mathbb{L}^q$ given by the connective cover, the second is induced by the Postnikov truncation map $\mathfrak{t}: Y \rightarrow B\pi$, and the third map A_π is the assembly map in quadratic L -theory, defined in [Ran79]. This is shown in [HMTW88, Appendix 2, Theorem 2], where it is credited to Quinn [Qui70] and Ranicki [Ran79, Ran92], with a contribution by Nicas [Nic82].

Our objective for the remainder of this subsection is prove Lemma 5.6, below, where we relate Ranicki–Sullivan duality back to ordinary Poincaré duality.

5.2.4. Splitting L -theory spectra. The 2-localised spectra $\mathbb{L}_{(2)}^s$ and $\mathbb{L}^q \langle 1 \rangle_{(2)}$ are generalised Eilenberg–MacLane spectra; see [TW79, §2]. Moreover, Taylor and Williams [TW79] write down *specific* equivalences of $\mathrm{H}\mathbb{Z}$ -module spectra

$$\binom{l^s}{r} : \mathbb{L}_{(2)}^s \xrightarrow{\simeq} \bigoplus_{i \geq 0} \mathrm{H}\mathbb{Z}_{(2)}[4i] \oplus \bigoplus_{j \geq 0} \mathrm{H}\mathbb{Z}/2[4j+1], \quad (8)$$

and

$$\binom{l^q}{k} : \mathbb{L}^q \langle 1 \rangle_{(2)} \xrightarrow{\simeq} \bigoplus_{i \geq 1} \mathrm{H}\mathbb{Z}_{(2)}[4i] \oplus \bigoplus_{j \geq 0} \mathrm{H}\mathbb{Z}/2[4j+2]. \quad (9)$$

Lemma 5.3. *Under the identification $G/\mathrm{Top} \simeq \Omega^\infty \mathbb{L}^q \langle 1 \rangle$ and with respect to the Taylor–Williams splitting (9), the cohomology class $G/\mathrm{Top} \rightarrow B\mathrm{Top} \rightarrow B(\mathrm{Top}/O) \xrightarrow{\mathrm{ks}} K(\mathbb{Z}/2, 4)$ is given by $\mathrm{red}_2(l_4^q) + k_2 \smile k_2$. That is, the following diagram commutes*

$$\begin{array}{ccc} G/\mathrm{Top} & \xrightarrow{\quad} & \Omega^\infty \mathbb{L}^q \langle 1 \rangle_{(2)} \\ \downarrow & & \downarrow (9) \quad l_4^q \times k_2 \\ K(\mathbb{Z}/2, 4) & \xleftarrow{\mathrm{red}_2(-) + - \smile -} & K(\mathbb{Z}_{(2)}, 4) \times K(\mathbb{Z}/2, 2). \end{array}$$

This is surely known in some form, but we were unable to find a proof. We give one here that manages to avoid the specifics of the definitions of l_4^q or k_2 .

Proof. Recall that $\mathrm{Top}/\mathrm{PL} \simeq K(\mathbb{Z}/2, 3)$, and that the composition in the statement of the lemma is the map u in the fibre sequence $G/\mathrm{PL} \rightarrow G/\mathrm{Top} \xrightarrow{u} B(\mathrm{Top}/\mathrm{PL}) \simeq K(\mathbb{Z}/2, 4)$. With respect to the decomposition

$$(G/\mathrm{Top})_{(2)} \simeq \Omega^\infty \mathbb{L}^q \langle 1 \rangle_{(2)} \simeq \prod_{i \geq 1} K(\mathbb{Z}_{(2)}, 4i) \times \prod_{j \geq 0} K(\mathbb{Z}/2, 4j+2)$$

induced by the maps l_{4i}^q and k_{4j+2} from (9), we have

$$u \in H^4(G/\mathrm{Top}; \mathbb{Z}/2) \cong \mathbb{Z}/2\{\mathrm{red}_2(l_4^q), k_2 \smile k_2\}.$$

To see this isomorphism, first note that since $G/\text{Top} \rightarrow G/\text{Top}_{(2)}$ induces an isomorphism on homology with $\mathbb{Z}_{(2)}$ -coefficients (by definition of localisation), the same map induces an isomorphism on (co)homology with any $\mathbb{Z}_{(2)}$ -module coefficients by universal coefficients. Since $\mathbb{Z}/2$ is a $\mathbb{Z}_{(2)}$ -module, we have an isomorphism $H^4(G/\text{Top}; \mathbb{Z}/2) \cong H^4(G/\text{Top}_{(2)}; \mathbb{Z}/2)$. In addition, the map $G/\text{Top}_{(2)} \rightarrow K(\mathbb{Z}_{(2)}, 4) \times K(\mathbb{Z}/2, 2)$ is 6-connected, so the latter group $H^4(G/\text{Top}_{(2)}; \mathbb{Z}/2)$ is isomorphic to

$$H^4(K(\mathbb{Z}_{(2)}, 4) \times K(\mathbb{Z}/2, 2); \mathbb{Z}/2) \cong \mathbb{Z}/2\{\text{red}_2(l_4^q), k_2 \smile k_2\},$$

as claimed.

It follows that $u = A \cdot \text{red}_2(l_4^q) + B \cdot k_2 \smile k_2$ for some $A, B \in \mathbb{Z}/2$. In these terms, the 5-truncation of the 2-localised G/PL is described as a homotopy pullback

$$\begin{array}{ccc} \tau_{\leq 5}(G/\text{PL})_{(2)} & \xrightarrow{l_4^q} & K(\mathbb{Z}_{(2)}, 4) \\ \downarrow k_2 & & \downarrow A \cdot \text{red}_2 \\ K(\mathbb{Z}/2, 2) & \xrightarrow{B \cdot \text{Sq}^2} & K(\mathbb{Z}/2, 4). \end{array} \quad (10)$$

Now Sullivan has shown (see e.g. [MM79, Theorem 4.8]) that $\tau_{\leq 5}(G/\text{PL})_{(2)}$ is homotopy equivalent to the homotopy fibre of $\delta_* \circ \text{Sq}^2: K(\mathbb{Z}/2, 2) \rightarrow K(\mathbb{Z}_{(2)}, 5)$, where $\delta_*: K(\mathbb{Z}/2, 4) \rightarrow K(\mathbb{Z}_{(2)}, 5)$ denotes the map classifying the cohomology Bockstein. From this we first see that $A = 1$, otherwise the pullback (10) would give the wrong π_3 . Given that $A = 1$, the diagram (10) expresses $\tau_{\leq 5}(G/\text{PL})_{(2)}$ as the homotopy fibre of $B \cdot \delta_* \circ \text{Sq}^2$, because we can extend the diagram downwards as follows, such that the right-hand column is a fibration sequence corresponding to the Bockstein long exact sequence.

$$\begin{array}{ccc} \tau_{\leq 5}(G/\text{PL})_{(2)} & \xrightarrow{l_4^q} & K(\mathbb{Z}_{(2)}, 4) \\ \downarrow k_2 & & \downarrow \text{red}_2 \\ K(\mathbb{Z}/2, 2) & \xrightarrow{B \cdot \text{Sq}^2} & K(\mathbb{Z}/2, 4) \\ \downarrow B \cdot \delta_* \circ \text{Sq}^2 & & \downarrow \delta_* \\ K(\mathbb{Z}_{(2)}, 5) & \xrightarrow{=} & K(\mathbb{Z}_{(2)}, 5) \end{array}$$

If $B = 0$ then we would have $\tau_{\leq 5}(G/\text{PL})_{(2)} \simeq K(\mathbb{Z}/2, 2) \times K(\mathbb{Z}_{(2)}, 4)$. If this were true then the fibration sequence $K(\mathbb{Z}_{(2)}, 4) \rightarrow \tau_{\leq 5}(G/\text{PL})_{(2)} \xrightarrow{f} K(\mathbb{Z}/2, 2) \xrightarrow{\delta_* \circ \text{Sq}^2} K(\mathbb{Z}_{(2)}, 5)$ arising from Sullivan's theorem would admit a splitting $i_1: K(\mathbb{Z}/2, 2) \rightarrow \tau_{\leq 5}(G/\text{PL})_{(2)}$ with $f \circ i_1 \simeq \text{Id}$, whence we have homotopies of maps

$$* = * \circ i_1 \simeq (\delta_* \circ \text{Sq}^2) \circ f \circ i_1 \simeq (\delta_* \circ \text{Sq}^2) \circ \text{Id} = \delta_* \circ \text{Sq}^2: K(\mathbb{Z}/2, 2) \rightarrow K(\mathbb{Z}_{(2)}, 5).$$

Here $*$ denotes a constant map. This contradicts the nontriviality of the cohomology operation $\delta_* \circ \text{Sq}^2$. Thus we must have $B = 1$. $\square$

Remark 5.4. An alternative proof that $A = B = 1$ could be made that proceeds by evaluating on degree one normal maps $E_8 \rightarrow S^4$ and $*\mathbb{C}P^2 \rightarrow \mathbb{C}P^2$.

Recall that for a space X and a double loop space or spectrum Y , localising induces an isomorphism of abelian groups $[X, Y_{(2)}] = [X, Y]_{(2)}$. We will use this without further comment.

Via $G/\text{Top} \simeq \Omega^\infty \mathbb{L}^q \langle 1 \rangle$, the Taylor–Williams splitting (9) of $\mathbb{L}^q \langle 1 \rangle_{(2)}$ gives an identification

$$[Y/\partial Y, G/\text{Top}]_{(2)} \cong [\Sigma^\infty(Y/\partial Y), \mathbb{L}^q \langle 1 \rangle_{(2)}] \cong \bigoplus_{i \geq 1} H^{4i}(Y, \partial Y; \mathbb{Z}_{(2)}) \oplus \bigoplus_{j \geq 0} H^{4j+2}(Y, \partial Y; \mathbb{Z}/2), \quad (11)$$

and using (ordinary) Poincaré duality, with respect to the given $\mathbb{Z}$ -orientation of Y , we can identify the latter with

$$\bigoplus_{i \geq 1} H_{d-4i}(Y; \mathbb{Z}_{(2)}) \oplus \bigoplus_{j \geq 0} H_{d-4j-2}(Y; \mathbb{Z}/2). \quad (12)$$

On the other hand, the Taylor–Williams splitting (9) of $\mathbb{L}^q \langle 1 \rangle_{(2)}$ also gives an identification

$$[Y/\partial Y, G/\text{Top}]_{(2)} \xrightarrow{\cong} [\mathbb{S}^d, \Sigma^\infty Y_+ \wedge \mathbb{L}^q \langle 1 \rangle_{(2)}] \cong \bigoplus_{i \geq 1} H_{d-4i}(Y; \mathbb{Z}_{(2)}) \oplus \bigoplus_{j \geq 0} H_{d-4j-2}(Y; \mathbb{Z}/2). \quad (13)$$

The two identifications of $[Y/\partial Y, G/\text{Top}]_{(2)}$ with (12) are not the same. Their difference is measured by the homology classes which correspond to $[Y]_{\mathbb{L}^s}$ under the Taylor–Williams splitting (8) of $\mathbb{L}_{(2)}^s$, because (13) used capping with $[Y]_{\mathbb{L}^s}$, whereas passing from (11) to (12) uses capping with the ordinary homology fundamental class $[Y] = [Y]_{\text{HZ}}$. These are described in the next lemma. For its statement, we let

$$\ell = 1 + \ell_1 + \ell_2 + \cdots \in \bigoplus_{i \geq 0} H^{4i}(\text{BStoP}; \mathbb{Z}_{(2)})$$

denote the Morgan–Sullivan class [MS74], with $\ell_i \in H^{4i}(\text{BStoP}; \mathbb{Z}_{(2)})$, and we write

$$\mathbb{V} := \sum_{j \geq 0} V_{2j} \smile \text{Sq}^1(V_{2j}) \in \bigoplus_{j \geq 0} H^{4j+1}(\text{BStoP}; \mathbb{Z}/2),$$

where $V_k \in H^k(\text{BStoP}; \mathbb{Z}/2)$ denotes the k^{th} Wu class.

Lemma 5.5. *Under the Taylor–Williams identification (8), the 2-local fundamental class $[Y]_{\mathbb{L}_{(2)}^s} \in (\mathbb{L}_{(2)}^s)_d(Y/\partial Y)$ is given by*

$$([Y] \smile ((\nu_Y^{\text{Top}})^* \ell), [Y] \smile ((\nu_Y^{\text{Top}})^* \mathbb{V})) \in \bigoplus_{i \geq 0} H_{d-4i}(Y, \partial Y; \mathbb{Z}_{(2)}) \oplus \bigoplus_{j \geq 0} H_{d-(4j+1)}(Y, \partial Y; \mathbb{Z}/2).$$

Proof. By [TW79, (1.9)] the pullbacks under the Ranicki orientation

$$\sigma^* l^s \in \bigoplus_{i \geq 0} H^{4i}(\text{MStoP}; \mathbb{Z}_{(2)}) \quad \text{and} \quad \sigma^* r \in \bigoplus_{j \geq 0} H^{4j+1}(\text{MStoP}; \mathbb{Z}/2)$$

are the images under the Thom isomorphism of the classes ℓ and $\mathbb{V}$ respectively. Pulling these back along $\text{Th}(\nu_Y^{\text{Top}}) \rightarrow \text{MStoP}$ we obtain the identities

$$\begin{aligned} l_*^s U_{\mathbb{L}_{(2)}^s} &= ((\nu_Y^{\text{Top}})^* \ell) \smile U_{\text{HZ}_{(2)}} \in \bigoplus_{i \geq 0} H^{4i}(\text{Th}(\nu_Y^{\text{Top}}); \mathbb{Z}_{(2)}) \\ r_* U_{\mathbb{L}_{(2)}^s} &= ((\nu_Y^{\text{Top}})^* \mathbb{V}) \smile U_{\text{HZ}_{(2)}} \in \bigoplus_{j \geq 0} H^{4j+1}(\text{Th}(\nu_Y^{\text{Top}}); \mathbb{Z}/2). \end{aligned}$$

Applying Atiyah duality (AD), which may be written as the composition of the inverse of the Thom isomorphism followed by Poincaré duality, we obtain the identities

$$\begin{aligned} AD(l_*^s U_{\mathbb{L}_{(2)}^s}) &= [Y] \smile ((\nu_Y^{\text{Top}})^* \ell) \in \bigoplus_{i \geq 0} H_{d-4i}(Y, \partial Y; \mathbb{Z}_{(2)}) \\ AD(r_* U_{\mathbb{L}_{(2)}^s}) &= [Y] \smile ((\nu_Y^{\text{Top}})^* \mathbb{V}) \in \bigoplus_{j \geq 0} H_{d-(4j+1)}(Y, \partial Y; \mathbb{Z}/2). \end{aligned}$$

By definition of the fundamental class $[Y]_{\mathbb{L}_{(2)}^s} \in (\mathbb{L}_{(2)}^s)_d(Y/\partial Y)$ as the image of the generalised Thom class $U_{\mathbb{L}_{(2)}^s} \in (\mathbb{L}_{(2)}^s)^0(\text{Th}(\nu_Y^{\text{Top}}))$ under Atiyah duality, we have identities $AD(l_*^s U_{\mathbb{L}_{(2)}^s}) = l_*^s AD(U_{\mathbb{L}_{(2)}^s}) = l_*^s [Y]_{\mathbb{L}_{(2)}^s} \in \bigoplus_{i \geq 0} H_{d-4i}(Y, \partial Y; \mathbb{Z}_{(2)})$. Similarly, for the other case we have identities $AD(r_* U_{\mathbb{L}_{(2)}^s}) = r_* AD(U_{\mathbb{L}_{(2)}^s}) = r_* [Y]_{\mathbb{L}_{(2)}^s} \in \bigoplus_{j \geq 0} H_{d-(4j+1)}(Y, \partial Y; \mathbb{Z}/2)$. Combining these with the previous display yields the statement of the lemma. $\square$

The two identifications of $[Y/\partial Y, G/\text{Top}]_{(2)}$ with (12) discussed above are related as follows.

Lemma 5.6. *The diagram*

$$\begin{array}{ccc}
(\mathbb{L}^q\langle 1 \rangle_{(2)})^0(Y/\partial Y) & \xleftarrow{\cong} [Y/\partial Y, G/\text{Top}]_{(2)} & \xrightarrow{\cong} (\mathbb{L}^q\langle 1 \rangle_{(2)})_d(Y) \\
\downarrow = & & \downarrow = \\
[\Sigma^\infty(Y/\partial Y), \mathbb{L}^q\langle 1 \rangle_{(2)}] & \xrightarrow{[Y]_{\mathbb{L}^s_{(2)}} \frown -} & [\mathbb{S}^d, \Sigma^\infty Y_+ \wedge \mathbb{L}^q\langle 1 \rangle_{(2)}] \\
\cong \downarrow TW(11) & & \downarrow \cong TW(13) \\
\bigoplus_{i \geq 1} H^{4i}(Y, \partial Y; \mathbb{Z}_{(2)}) & & \\
\oplus \bigoplus_{j \geq 0} H^{4j+2}(Y, \partial Y; \mathbb{Z}/2) & & \\
\cong \downarrow \Phi & & \\
\bigoplus_{i \geq 1} H^{4i}(Y, \partial Y; \mathbb{Z}_{(2)}) & \xrightarrow{[Y] \frown -} & \bigoplus_{i \geq 1} H_{d-4i}(Y; \mathbb{Z}_{(2)}) \\
\oplus \bigoplus_{j \geq 0} H^{4j+2}(Y, \partial Y; \mathbb{Z}/2) & \xrightarrow{\cong} & \oplus \bigoplus_{j \geq 0} H_{d-4j-2}(Y; \mathbb{Z}/2)
\end{array}$$

commutes, where the map Φ is given by

$$(u, v) \mapsto ((\nu_Y^{\text{Top}})^* \ell \smile u, (\nu_Y^{\text{Top}})^* \ell \smile v + \delta^*((\nu_Y^{\text{Top}})^* \mathbb{V} \smile v))$$

for δ^* the cohomology Bockstein associated to the coefficient sequence $0 \rightarrow \mathbb{Z}_{(2)} \xrightarrow{2} \mathbb{Z}_{(2)} \rightarrow \mathbb{Z}/2 \rightarrow 0$. Here $TW(11)$ and $TW(13)$ are the maps denoted $\cong_{TW}$ in (11) and (13) respectively.

Proof. The top square commutes by the definition of the map D from (7).

Write $m: \mathbb{L}_{(2)}^s \wedge \mathbb{L}^q\langle 1 \rangle_{(2)} \rightarrow \mathbb{L}^q\langle 1 \rangle_{(2)}$ for the $\mathbb{L}_{(2)}^s$ -module structure map. According to [TW79, (1.13)], the spectrum cohomology classes l^q , k , l^s and r defining the identifications (8) and (9) behave as follows under pullback along the map m :

$$m^* l^q = l^s \wedge l^q + \delta^*(r \wedge k) \quad \text{and} \quad m^* k = l^s \wedge k.$$

By (9), the spectrum $\mathbb{L}^q\langle 1 \rangle_{(2)}$ is homotopy equivalent to a coproduct of Eilenberg–MacLane spectra via the maps l^q and r , so these formulae fully determine the homotopy class of the map m .

Using the formula for $[Y]_{\mathbb{L}_{(2)}^s}$ under the splitting (8) given by Lemma 5.5, and this description of the map m , we may derive a formula for the cap product map

$$[Y]_{\mathbb{L}_{(2)}^s} \frown -: [\Sigma^\infty(Y/\partial Y), \mathbb{L}^q\langle 1 \rangle_{(2)}] \xrightarrow{\cong} [\mathbb{S}^d, \Sigma^\infty Y_+ \wedge \mathbb{L}^q\langle 1 \rangle_{(2)}]$$

under the identifications (11) and (13). Namely, it is given by sending

$$(u, v) \in \bigoplus_{i \geq 1} H^{4i}(Y, \partial Y; \mathbb{Z}_{(2)}) \oplus \bigoplus_{j \geq 0} H^{4j+2}(Y, \partial Y; \mathbb{Z}/2)$$

to

$$\begin{aligned}
& ([Y] \frown ((\nu_Y^{\text{Top}})^* \ell \smile u), [Y] \frown ((\nu_Y^{\text{Top}})^* \ell \smile v) + [Y] \frown \delta^*((\nu_Y^{\text{Top}})^* \mathbb{V} \smile v)) \\
& \in \bigoplus_{i \geq 1} H_{d-4i}(Y; \mathbb{Z}_{(2)}) \oplus \bigoplus_{j \geq 0} H_{d-4j-2}(Y; \mathbb{Z}/2).
\end{aligned}$$

This describes the composition $TW(13) \circ ([Y]_{\mathbb{L}_{(2)}^s} \frown -) \circ (TW(11))^{-1}$ in the diagram, which agrees with the composition $([Y] \frown -) \circ \Phi$ by the definition of Φ . $\square$

5.3. Proof of Theorem 2.3 (C). Now we specialise the discussion of the previous sections to the case at hand, with $Y = X \times I \times I$ for an oriented 4-manifold X . We may consider the diagram

$$\begin{array}{ccc}
 H_2(X; \mathbb{Z}/2) & \xleftarrow{\text{ks}} & \left[\frac{X \times I \times I}{\partial(X \times I \times I)}, G/\text{Top} \right]_{(2)} \xrightarrow{\sigma} L_6(\mathbb{Z}[\pi])_{(2)} \\
 & \cong \downarrow D & \uparrow A_\pi \\
 & [\mathbb{S}^6, \Sigma^\infty X_+ \wedge \mathbb{L}^q \langle 1 \rangle]_{(2)} \rightarrow [\mathbb{S}^6, \Sigma^\infty X_+ \wedge \mathbb{L}^q]_{(2)} \xrightarrow{t_*} [\mathbb{S}^6, \Sigma^\infty B\pi_+ \wedge \mathbb{L}^q]_{(2)} \\
 & \cong \downarrow TW & \cong \downarrow TW \\
 & H_0(X; \mathbb{Z}/2) & H_0(\pi; \mathbb{Z}/2) \\
 & \oplus H_2(X; \mathbb{Z}_{(2)}) & \oplus H_2(\pi; \mathbb{Z}_{(2)}) \\
 & \oplus H_4(X; \mathbb{Z}/2) & \oplus H_4(\pi; \mathbb{Z}/2) \\
 & & \oplus H_6(\pi; \mathbb{Z}_{(2)})
 \end{array} \quad (14)$$

$\xrightarrow{t_* \oplus 0}$

Ψ

We use that $X \times I \times I \simeq X$ to simplify notation here. The top rectangle comes from Subsection 5.2.3. For the bottom rectangle we applied TW to the $\mathbb{L}^q$ homology of both $\Sigma^\infty X_+$, and $\Sigma^\infty B\pi_+$, using naturality with respect to the truncation map t and the change of homology theory $\mathbb{L}^q \langle 1 \rangle \rightarrow \mathbb{L}^q$ to see that the rectangle commutes. The map Ψ , defined so as to make the left-hand triangle commute, can be evaluated by applying Lemma 5.3 and Lemma 5.6.

Lemma 5.7. *We have $\Psi(a, b, c) = \text{red}_2(b)$.*

Proof. We consider the diagram of Lemma 5.6 for $Y = X \times I \times I$. This has

$$(\nu_Y^{\text{Top}})^* \ell = 1 + (\nu_Y^{\text{Top}})^* \ell_1 \in H^0(Y, \partial Y; \mathbb{Z}_{(2)}) \oplus H^4(Y, \partial Y; \mathbb{Z}_{(2)})$$

and

$$(\nu_Y^{\text{Top}})^*(\mathbb{V}) = (\nu_Y^{\text{Top}})^*\left(\sum_{j \geq 0} V_{2j} \smile \text{Sq}^1(V_{2j})\right) = (\nu_Y^{\text{Top}})^*(V_2 \smile \text{Sq}^1(V_2)) \in H^5(Y, \partial Y; \mathbb{Z}/2)$$

for degree reasons, namely that $\text{Sq}^1(V_0) = 0$ and $H^k(Y, \partial Y; \mathbb{Z}_{(2)}) = 0$ for $k > 6$. Thus the isomorphism

$$\begin{aligned}
 \Phi: H^2(Y, \partial Y; \mathbb{Z}/2) \oplus H^4(Y, \partial Y; \mathbb{Z}_{(2)}) \oplus H^6(Y, \partial Y; \mathbb{Z}/2) \\
 \longrightarrow H^2(Y, \partial Y; \mathbb{Z}/2) \oplus H^4(Y, \partial Y; \mathbb{Z}_{(2)}) \oplus H^6(Y, \partial Y; \mathbb{Z}/2)
 \end{aligned}$$

is given by $(u, v, w) \mapsto (u, v, w + (\nu_Y^{\text{Top}})^* \ell_1 \smile u)$.

Start with a class

$$(a, b, c) = ([Y] \frown z, [Y] \frown y, [Y] \frown x) \in H_0(Y; \mathbb{Z}/2) \oplus H_2(Y; \mathbb{Z}_{(2)}) \oplus H_4(Y; \mathbb{Z}/2)$$

in the bottom-left corner of the diagram (14), which equals the bottom-right corner of the diagram of Lemma 5.6 (in this case we take just $i = 1$ and $j = 0, 1$). Here (x, y, z) in the codomain of Φ are uniquely determined by Poincaré duality. We wish to pass from here anticlockwise around the diagram from Lemma 5.6 to the domain of Φ , from which we can evaluate the map ks using Lemma 5.3. By Lemma 5.6, we can instead pass clockwise around the diagram from that lemma, i.e. apply the isomorphism $\Phi^{-1} \circ ([Y] \frown -)^{-1}$. We obtain the class

$$(x, y, z - (\nu_Y^{\text{Top}})^* \ell_1 \smile x) \in H^2(Y, \partial Y; \mathbb{Z}/2) \oplus H^4(Y, \partial Y; \mathbb{Z}_{(2)}) \oplus H^6(Y, \partial Y; \mathbb{Z}/2).$$

By Lemma 5.3 the Kirby–Siebenmann class ks is given by $\text{red}_2(l_4^q) + k_2 \smile k_2$, so is $\text{red}_2(y) + x \smile x \in H^4(Y, \partial Y; \mathbb{Z}/2)$. But as $Y/\partial Y \cong (X/\partial X) \wedge S^2$ is a reduced suspension, the cup product $x \smile x$ vanishes. Thus under Poincaré duality the Kirby–Siebenmann class is $[Y] \frown \text{red}_2(y) = \text{red}_2([Y] \frown y) = \text{red}_2(b)$, so Ψ has the claimed description. $\square$

We can now describe the map I_2 to which the statement of Theorem C and Theorem 2.3 (C) refers, and then complete the argument.

Definition 5.8. The map $I_2: H_2(\pi; \mathbb{Z}_{(2)}) \rightarrow L_6^q(\mathbb{Z}[\pi])_{(2)}$ is by definition the restriction of the right-hand column $A_\pi \circ TW^{-1}$ of the diagram at the start of Section 5.3 to the summand $H_2(\pi; \mathbb{Z}_{(2)})$.

Proof of Theorem 2.3 (C). By the strategy described at the end of Section 5.1, after diagram (6), it suffices to produce, for each $u \in \ker(\delta_*: H_2(X; \mathbb{Z}/2) \rightarrow H_1(X; \mathbb{Z}/2))$, a lift $\xi_u \in [\frac{X \times I \times I}{\partial(X \times I \times I)}, G/\text{Top}]_{(2)}$ along ks such that $\sigma(\xi_u) = 0$. We do so as follows. Given such a class u , exactness of the Bockstein sequence

$$H_2(X; \mathbb{Z}/2) \xrightarrow{\text{red}_2} H_2(X; \mathbb{Z}/2) \xrightarrow{\delta_*} H_1(X; \mathbb{Z}/2) \xrightarrow{2} H_1(X; \mathbb{Z}/2)$$

shows that we may choose a $u' \in H_2(X; \mathbb{Z}/2)$ which reduces modulo 2 to u . We then consider the element

$$(0, u', 0) \in H_0(X; \mathbb{Z}/2) \oplus H_2(X; \mathbb{Z}/2) \oplus H_4(X; \mathbb{Z}/2)$$

in the bottom-left corner of the previous diagram (14). Then $(0, u', 0)$ corresponds under the vertical isomorphism $D^{-1} \circ TW^{-1}$ to an element

$$\xi_u \in [\frac{X \times I \times I}{\partial(X \times I \times I)}, G/\text{Top}]_{(2)}$$

which satisfies that $\text{ks}(\xi_u) = u$, by the description of Ψ in Lemma 5.7. Under our assumption that the map $I_2: H_2(\pi; \mathbb{Z}/2) \rightarrow L_6(\mathbb{Z}[\pi])_{(2)}$ is zero, the element $(0, u', 0)$ maps to zero in $L_6(\mathbb{Z}[\pi])_{(2)}$ by going anticlockwise around the diagram (14), and therefore $\sigma(\xi_u) = 0$ as required. This completes the proof of Theorem 2.3 (C) and hence of Theorem C. $\square$

6. PROOF OF THEOREM E

We recall that Theorem E states the following. For each smooth, connected, compact 4-manifold X there is a $g \geq 0$ such that every class $x \in H_2(X \# g(S^2 \times S^2); \mathbb{Z}/2)$ arises as $\text{KS}(F)$ for some topological pseudo-isotopy F from the identity to some diffeomorphism $f: X \# g(S^2 \times S^2) \rightarrow X \# g(S^2 \times S^2)$.

Proof of Theorem E. Without loss of generality we may suppose that X has nonempty boundary, by removing an open disc.

Given a class $x \in H_2(X; \mathbb{Z}/2)$, we use Poincaré duality and a suspension isomorphism to consider it as an element of

$$H^3(X \times I, \partial; \mathbb{Z}/2) = [\frac{X \times I}{\partial}, \text{Top}/\text{O}],$$

using the fact that the Postnikov 6-type of Top/O is a $K(\mathbb{Z}/2, 3)$. Thus x gives a smooth structure $(X \times I)_{\Sigma(x)}$ on the 5-manifold $X \times I$ relative to the standard smoothing of the boundary, by the discussion in Section 3.1. As $X = X \times \{0\} \hookrightarrow X \times I$ is a simple homotopy equivalence, the smooth 5-manifold $(X \times I)_{\Sigma(x)}$ is an s -cobordism. By the $(S^2 \times S^2)$ -stable s -cobordism theorem [Qui83], there is a $g \geq 0$ such that the stabilisation of the s -cobordism $(X \times I)_{\Sigma(x)}$ obtained by gluing on the trivial smooth cobordism on $(\partial X \times I) \# g(S^2 \times S^2)$ is a trivial cobordism starting at

$$X' := X \cup_{\partial X} ((\partial X \times I) \# g(S^2 \times S^2)) \cong X \# g(S^2 \times S^2).$$

This yields maps

$$X' \times I \xrightarrow{\cong_{\text{Top}}} (X' \times I)_{\Sigma(x)} \xrightarrow{\cong_{\text{Diff}}} X' \times I.$$

The first is just the identity homeomorphism between two copies of $X' \times I$, while the second map is the diffeomorphism arising from the $(S^2 \times S^2)$ -stable s -cobordism theorem.

This yields a self-homeomorphism of $X' \times I$ that is the identity on $X' \times \{0\}$ and that restricts on $X' \times \{1\}$ to a diffeomorphism f . This is the same as a topological pseudo-isotopy F from the identity to f . By construction, this data has

$$\text{KS}(F) = (x, 0) \in H_2(X; \mathbb{Z}/2) \oplus H_2(g(S^2 \times S^2); \mathbb{Z}/2) = H_2(X'; \mathbb{Z}/2).$$

By choosing g large enough, since $H_2(X; \mathbb{Z}/2)$ is finite we may arrange that all of $H_2(X; \mathbb{Z}/2) \subseteq H_2(X', \mathbb{Z}/2)$ lies in the image of $\text{KS}: Q(X') \rightarrow H_2(X'; \mathbb{Z}/2)$.

For the summand $H_2(g(S^2 \times S^2); \mathbb{Z}/2) \subseteq H_2(X'; \mathbb{Z}/2)$, we use that $W_{g,1} := g(S^2 \times S^2) \setminus \text{Int}(D^4)$ is simply-connected and so the hypotheses of Theorem C are satisfied for this manifold (alternatively, the conclusion of Theorem C for simply connected manifolds was already a consequence of [Kre79, Theorem 1]). It follows that

$$\pi_1(\widehat{\text{Homeo}}_0^+(W_{g,1})) \xrightarrow{\alpha} Q(W_{g,1}) \xrightarrow{\text{ks}} H_2(W_{g,1}; \mathbb{Z}/2)$$

is surjective. Plugging such topological pseudo-isotopies of $W_{g,1}$ into X' shows that $H_2(g(S^2 \times S^2); \mathbb{Z}/2) \subseteq H_2(X'; \mathbb{Z}/2)$ also lies in the image of KS for the manifold X' . Since KS is a homomorphism by Lemma 2.2, this completes the proof of Theorem E. $\square$

7. PROOF OF THEOREM F

We begin the proof of Theorem F by defining the 4-manifold X appearing in the statement of Theorem F. Then we outline the strategy of the proof, motivating the conditions the manifold X will have been constructed to satisfy. The strategy involves choosing judicious maps into PL normal invariant sets, and the commutativity of a diagram. The rest of this section, and the rest of the proof of Theorem F, will comprise the proof that this diagram commutes.

7.1. The 4-manifold X . Let A be a closed smooth 4-manifold, and consider the following list of conditions.

- (i) A is aspherical,
- (ii) A is stably framable,
- (iii) the map $\pi_1(\text{Diffeo}^+(A)) \rightarrow \pi_1(\text{hAut}^+(A))$ is surjective,
- (iv) $H_1(\pi_1(A); \mathbb{Z})$ has an element of order 2.

Remark 7.1. We are interested in the higher homotopy groups of various automorphism groups based at the identity. As homotopy automorphisms which are homotopic to the identity are simple, we may freely interchange hAut^+ and sAut^+ , and similarly for hAut and sAut . By the same token homeomorphisms in the identity component preserve orientation, so we may also freely interchange Homeo and Homeo^+ , and so on.

Remark 7.2. For an aspherical space A , there is an isomorphism $\pi_1(\text{hAut}(A)) \cong Z(\pi_1(A))$, where the latter denotes the centre [Got65, Corollary I.13]. This can be argued by picking a point $x \in A$ and considering the map $\text{ev}_x: \text{hAut}(A) \rightarrow A$, given by evaluating the automorphism at x . The homotopy fibre of this map is the space of based homotopy equivalences $\text{hAut}_x(A)$. As A is aspherical, we have $\pi_0(\text{hAut}_x(A)) \cong \text{Aut}(\pi_1(A, x))$ and $\pi_k(\text{hAut}_x(A)) = 0$ for all $k \geq 0$. The long exact sequence of the fibration thus has a portion

$$0 \longrightarrow \pi_1(\text{hAut}(A), \text{Id}) \longrightarrow \pi_1(A, x) \longrightarrow \pi_0(\text{hAut}_x(A)) \cong \text{Aut}(\pi_1(A, x)).$$

An element $g \in \pi_1(A, x)$ is sent to the map $\gamma \mapsto g\gamma g^{-1} \in \text{Aut}(\pi_1(A, x))$ in this sequence, and thus $\pi_1(A) \rightarrow \pi_0(\text{hAut}_x(A))$ has kernel $Z(\pi_1(A))$. For further details we refer to [Got65].

The next example shows that there exist many 4-manifolds A satisfying these conditions.

Example 7.3. Let Σ be a closed, oriented hyperbolic 3-manifold with 2-torsion in $H_1(\Sigma; \mathbb{Z})$. Such manifolds Σ are abundant; for example, take any hyperbolic knot $K \subseteq S^3$ and perform n -surgery on S^3 along K , where n is even and $n/1$ is not one of the finite number of *exceptional* slopes p/q for $S^3 \setminus \nu K$ which produce a non-hyperbolic filling. A concrete example is given by performing 6-surgery on the figure eight knot [Thu97, Theorem 4.7].

The manifold $A := \Sigma \times S^1$ satisfies conditions (i), (ii), (iii), and (iv). Indeed, condition (i) holds because Σ is hyperbolic, and thus aspherical, so $\Sigma \times S^1$ is a product of aspherical spaces, thus also aspherical. Condition (ii) holds as oriented 3-manifolds are framable, and so A is a product of framable manifolds. For condition (iii) we will argue that $\pi_1(\Sigma)$ has trivial centre, so that the centre of $\pi_1(A) \cong \pi_1(\Sigma) \times \pi_1(S^1)$ is $\mathbb{Z}$ generated by the circle factor. But the circle action on $\Sigma \times S^1$ is smooth and generates this factor in $\pi_1(\text{hAut}(A)) \cong Z(\pi_1(A))$ (recall Remark 7.2), showing that (iii) is satisfied. Condition (iv) holds as Σ was chosen to have 2-torsion in $H_1(\Sigma; \mathbb{Z})$.

It remains to see that the centre of the group $\pi_1(\Sigma)$ is trivial. For a contradiction, let g be a nontrivial element of the centre $Z(\pi_1(\Sigma))$. Since $\pi_1(\Sigma)$ is torsion-free the subgroup $\langle g \rangle$ is isomorphic to $\mathbb{Z}$. By [BH99, Corollary III.Γ.3.10 (2), p. 462], $\langle g \rangle$ has finite index in its centraliser $C_{\pi_1(\Sigma)}(g)$. Since g is central, $C_{\pi_1(\Sigma)}(g) = \pi_1(\Sigma)$, so $\pi_1(\Sigma)$ has a subgroup $\mathbb{Z}$ of finite index. The corresponding finite cover $\tilde{\Sigma}$ is a closed, orientable, aspherical 3-manifold and so $\mathbb{Z} \cong H_3(\tilde{\Sigma}; \mathbb{Z}) \cong H_3(B\pi_1(\tilde{\Sigma}); \mathbb{Z}) \cong H_3(B\mathbb{Z}; \mathbb{Z}) = 0$, which is a contradiction.

Definition 7.4. Choose a closed smooth 4-manifold A satisfying conditions (i), (ii), (iii), and (iv), above. Define a closed smooth 4-manifold

$$X := A \# g(S^2 \times S^2)$$

with g is chosen large enough that $\text{ks}: Q(X) \rightarrow H_2(X; \mathbb{Z}/2)$ is an isomorphism; such a g exists by Theorem E. Note that X inherits properties (ii) and (iv) from A .

In this case the Postnikov truncation map

$$\mathfrak{t}: X \longrightarrow B\pi \simeq A$$

may be modelled as the map which collapses $g(S^2 \times S^2) \setminus \text{Int}(D^4) \subseteq X$, so it has degree ± 1 .

7.2. Outline of proof. Let X and A be instances of the closed, oriented, connected 4-manifolds from Section 7.1. We consider the following diagram, whose terms will be defined below. After that, we will prove the theorem assuming commutativity. Then the majority of the rest of this section will be devoted to proving that it commutes.

$$\begin{array}{ccccccc}
 \pi_1(\widetilde{\text{Homeo}}^+(X)) & \xrightarrow{\alpha} & Q(X) & \xrightarrow{\beta} & \pi_0(\widetilde{\text{Diffeo}}^+(X)) & \xrightarrow{\gamma} & \pi_0(\widetilde{\text{Homeo}}^+(X)) \\
 \downarrow & & \text{Sm} \downarrow \cong & & \downarrow \cong & & \downarrow \cong \text{KS} \\
 (17) \quad [\frac{X \times I \times I}{\partial}, \text{B}(\text{Top/O})] & \xrightarrow{\cong} & [X_+, \Omega(\text{Top/O})] & & & & \\
 \downarrow & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
 \pi_1(\widetilde{\text{hAut}}^+(X)) & \xrightarrow{\eta} & [X_+, \Omega(\text{G/PL})] & \longleftarrow & [X_+, \Omega(\text{Top/PL})] & \xrightarrow{\cong} & H_2(X; \mathbb{Z}/2) \\
 \downarrow \mathfrak{t}_* & & \downarrow \mathfrak{t}_! & & \downarrow \mathfrak{t}_! & & \downarrow \mathfrak{t}_* \\
 \pi_1(\widetilde{\text{hAut}}^+(A)) & \xrightarrow{\eta} & [A_+, \Omega(\text{G/PL})] & \longleftarrow & [A_+, \Omega(\text{Top/PL})] & \xrightarrow{\cong} & H_2(A; \mathbb{Z}/2) \\
 \uparrow & & \uparrow 0 & & \uparrow j & & \downarrow \delta_* \\
 \pi_1(\text{Diffeo}^+(A)) & & & & \ker \left(\begin{array}{c} H_1(A; \mathbb{Z}) \xrightarrow{\cdot 2} \\ H_1(A; \mathbb{Z}) \end{array} \right) & &
 \end{array} \tag{15}$$

- The top row is the exact sequence of groups from (1).
- The diagram consists of groups and homomorphisms, and in fact all but possibly $\pi_0(\widetilde{\text{Diffeo}}^+(X))$ and $\pi_0(\widetilde{\text{Homeo}}^+(X))$ are abelian groups. For this we consider the spaces BO , BPL , BTop , BG as infinite loop spaces via Whitney sum [BV73]. The spaces Top/O , Top/PL and G/PL inherit compatible infinite loop space structures coming from taking fibres of the infinite loop maps $\text{BO} \rightarrow \text{BPL} \rightarrow \text{BTop} \rightarrow \text{BG}$.
- The upper map η is defined by the composition

$$\pi_1(\widetilde{\text{hAut}}^+(X)) \cong \pi_0(\widetilde{\text{hAut}}^+(X \times I)) \longrightarrow \mathcal{S}_\partial^{\text{PL}}(X \times I) \xrightarrow{\eta_{\text{PL}}} [\frac{X \times I}{\partial}, \text{G/PL}] \cong [X_+, \Omega(\text{G/PL})],$$

where η_{PL} is the PL normal invariant. The lower map η is defined similarly, with A in place of X . Using the PL surgery sequence for manifolds of the form $Y \times I$ justifies that these normal invariant groups are abelian and that η is indeed a homomorphism.

- We use that $\text{Top/PL} \simeq K(\mathbb{Z}/2, 3)$ to obtain $\Omega(\text{Top/PL}) \simeq K(\mathbb{Z}/2, 2)$. Thus we identify $[Y_+, \Omega(\text{Top/PL})] \cong H^2(Y; \mathbb{Z}/2)$ for $Y \in \{X, A\}$. With respect to this identification the maps PD correspond to ordinary Poincaré duality.
- By our choice of g in the definition of X , and Theorem E, the maps KS and Sm are surjective. They are injective (without need to stabilise) by Theorem A, and hence are isomorphisms.
- The right-most map $\mathfrak{t}_*$ is induced by the Postnikov truncation $\mathfrak{t}: X \rightarrow B\pi \simeq A$. This is easily seen to be surjective from the fact that there exists a model for $B\pi$ arising from taking a cell complex for X and then adding cells of dimension at least three.
- By functoriality of Postnikov truncation, every homotopy automorphism of X induces a homotopy automorphism of its truncation $B\pi \simeq A$, yielding the map $\mathfrak{t}_*: \pi_1(\widetilde{\text{hAut}}(X)) \rightarrow \pi_1(\widetilde{\text{hAut}}(A))$ on the left.
- The map δ_* is the homology Bockstein for the coefficient sequence $0 \rightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0$. The fact that δ_* is surjective follows from the Bockstein long exact sequence.
- The isomorphism in the second row is given by the composition

$$[\frac{X \times I \times I}{\partial}, \text{B}(\text{Top/O})] \cong [\Sigma^2 X_+, \text{B}(\text{Top/O})] \cong [X_+, \Omega^2 \text{B}(\text{Top/O})] \cong [X_+, \Omega(\text{Top/O})].$$

- In Lemma 7.5 below, the map j in the diagram will be defined and proven to be injective, and the lower right triangle will be proven to commute.
- We discuss the bottom left triangle. Recall from Section 2.1 that the geometric realisation of $\widetilde{\text{hAut}}^+(A)$ is homotopy equivalent to $\text{hAut}^+(A)$. Combining with property (iii) of the manifold A , this implies the map $\pi_1(\text{Diffeo}^+(A)) \rightarrow \pi_1(\widetilde{\text{hAut}}^+(A))$ is surjective. An element in the image of the up-right composition in the bottom left triangle is the PL normal invariant of a diffeomorphism of $A \times I$, and is thus trivial, showing that the bottom left triangle commutes.
- The region involving the map KS commutes by definition of the map KS .
- The maps $\mathfrak{t}_!$ are defined as umkehr maps using Poincaré duality. This exploits the fact that stably framed manifolds are $\mathbb{S}$ -oriented and thus satisfy Poincaré duality in E -theory for any spectrum E . This is discussed in detail in Section 7.3, below. Commutativity of the central square involving the maps $\mathfrak{t}_!$, and the right-hand square involving $\mathfrak{t}_!$ and $\mathfrak{t}_*$, is discussed in Section 7.3, the latter in Lemma 7.9.
- The L-shaped region labelled (17) will be shown to commute in Section 7.4.
- The square labelled (19) will be shown to commute in Section 7.5. This will require several pages and takes up the majority of the rest of the proof.

Lemma 7.5. *For any closed 4-manifold Y , the natural map $[Y_+, \Omega(\text{Top/PL})] \rightarrow [Y_+, \Omega(\text{G/PL})]$ factors as*

$$[Y_+, \Omega(\text{Top/PL})] \xrightarrow{PD, \cong} H_2(Y; \mathbb{Z}/2) \xrightarrow{\delta_*} \ker(H_1(Y; \mathbb{Z}) \xrightarrow{\cdot 2} H_1(Y; \mathbb{Z})) \xrightarrow{j} [Y_+, \Omega(\text{G/PL})].$$

where as above δ_* denotes the homology Bockstein for the sequence $0 \rightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0$.

Proof. The natural map $[\Sigma Y_+, \text{Top/PL}] \rightarrow [\Sigma Y_+, \text{G/PL}]$ factors as

$$[\Sigma Y_+, \text{Top/PL}] \cong H^3(\Sigma Y_+; \mathbb{Z}/2) \rightarrow H^3(\Sigma Y_+; \mathbb{Z}/2)/H^3(\Sigma Y_+; \mathbb{Z}) \hookrightarrow [Y_+, \Omega(\text{G/PL})],$$

by [KS77, Annex C, Theorem 15.1]. Next we use the identification $H^3(\Sigma Y_+; \mathbb{Z}/2) \cong H^2(Y; \mathbb{Z}/2)$, and by the Bockstein sequence for $0 \rightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0$, we identify

$$H^3(\Sigma Y_+; \mathbb{Z}/2)/H^3(\Sigma Y_+; \mathbb{Z}) \cong H^2(Y; \mathbb{Z}/2)/H^3(Y; \mathbb{Z}) \cong \text{Im}(\delta^*),$$

where $\delta^*: H^2(Y; \mathbb{Z}/2) \rightarrow H^3(Y; \mathbb{Z})$ is the cohomology Bockstein. There is thus an exact sequence

$$[Y_+, \Omega(\text{Top/PL})] \cong H^2(Y; \mathbb{Z}/2) \xrightarrow{\delta^*} \text{Im}(\delta^*) \hookrightarrow [Y_+, \Omega(\text{G/PL})].$$

By naturality of the Bockstein exact sequence under Poincaré duality, we have an identification $\text{Im}(\delta^*: H^2(Y; \mathbb{Z}/2) \rightarrow H^3(Y; \mathbb{Z})) \cong \text{Im}(\delta_*: H_2(Y; \mathbb{Z}/2) \rightarrow H_1(Y; \mathbb{Z}))$, and hence an exact sequence

$$[Y_+, \Omega(\text{Top/PL})] \cong H_2(Y; \mathbb{Z}/2) \xrightarrow{\delta_*} \text{Im}(\delta_*) \hookrightarrow [Y_+, \Omega(\text{G/PL})].$$

Finally, $\text{Im}(\delta_*) = \ker(H_1(Y; \mathbb{Z}) \xrightarrow{\cdot 2} H_1(Y; \mathbb{Z}))$ by the homology Bockstein long exact sequence. $\square$

Proof of Theorem F assuming that (15) commutes. Recall that we constructed X in Section 7.1. Theorem F is the statement that the map γ in (15) is not injective. This is equivalent to the statement that there exists $F \in Q(X)$ with $\beta(F)$ nontrivial.

As $H_1(\pi; \mathbb{Z}) \cong H_1(A; \mathbb{Z})$ has an element of order two, it follows that $\ker(H_1(A; \mathbb{Z}) \xrightarrow{\cdot 2} H_1(A; \mathbb{Z}))$ is nontrivial. As indicated in the diagram, the composition

$$\delta_* \circ \mathfrak{t}_* \circ \text{KS}: Q(X) \longrightarrow \ker(H_1(A; \mathbb{Z}) \xrightarrow{\cdot 2} H_1(A; \mathbb{Z}))$$

is surjective. Choose a topological pseudo-isotopy $F: X \times I \rightarrow X \times I$ in $Q(X)$ with $\delta_* \circ \mathfrak{t}_* \circ \text{KS}(F) \neq 0$. We need to argue that its image under β is nontrivial. Supposing instead that $\beta(F)$ is trivial, we deduce that F lies in the image of α , and hence that the clockwise composition $\delta_* \circ \mathfrak{t}_* \circ \text{KS} \circ \alpha$ is nontrivial. Since j is injective, moreover the composition

$$j \circ \delta_* \circ \mathfrak{t}_* \circ \text{KS} \circ \alpha: \pi_1(\widetilde{\text{Homeo}}^+(X)) \longrightarrow [A_+, \Omega(\text{G/PL})]$$

is nontrivial. By commutativity of the diagram (15), this is equal to the anti-clockwise route

$$\pi_1(\widetilde{\text{Homeo}}^+(X)) \longrightarrow \pi_1(\widetilde{\text{hAut}}^+(X)) \xrightarrow{\mathfrak{t}_*} \pi_1(\widetilde{\text{hAut}}^+(A)) \xrightarrow{\eta} [A_+, \Omega(\text{G/PL})].$$

But commutativity of the bottom left triangle and surjectivity of $\pi_1(\text{Diffeo}^+(A)) \twoheadrightarrow \pi_1(\widetilde{\text{hAut}}^+(A))$ implies that the bottom map η is trivial, and hence the anti-clockwise route is trivial. We obtain a contradiction, and so $\beta(F) = F|_{X \times \{1\}} : X \rightarrow X$ is our desired diffeomorphism topologically but not smoothly pseudo-isotopic to the identity. $\square$

7.3. Stably framed manifolds are $\mathbb{S}$ -oriented. Recall that for the ring spectrum $\mathbb{S}$, the corresponding homology theory is stable homotopy $\pi_*^s(-)$. We remind the reader again that in stable homotopy theory, homology theories are reduced by default, so for example the stable stem π_*^s is given by $\pi_*^s(S^0) = \pi_*^s(\text{pt}_+)$.

We will now explain that stably framed manifolds have fundamental classes with respect to the sphere spectrum $\mathbb{S}$ and so satisfy Poincaré duality with respect to any generalised (co)homology theory, by the discussion in Section 4.2.

First we prove a lemma that gives a simple way to decide if a stable homotopy class is a fundamental class.

Lemma 7.6. *Let Y be a d -dimensional compact topological manifold with (possibly nonempty) boundary. A class $\alpha \in \pi_d^s(Y/\partial Y)$ is an $\mathbb{S}$ -theory fundamental class if and only if its Hurewicz image $h(\alpha) \in \tilde{H}_d(Y/\partial Y; \mathbb{Z})$ is an integral homology fundamental class.*

Proof. If $q: Y/\partial Y \rightarrow Y/(Y - \text{int}(D^d)) \cong D^d/\partial D^d \cong S^d$ is a coordinate chart collapse map then $q_*(\alpha) \in \pi_d^s(S^d)$. The Hurewicz map $h: \pi_d^s(S^d) \rightarrow \tilde{H}_d(S^d; \mathbb{Z})$ is an isomorphism, so $q_*(\alpha)$ corresponds to $\pm 1 \in \pi_0^s(S^0)$ under the suspension isomorphism if and only if $h(q_*(\alpha)) = q_*(h(\alpha))$ corresponds to $\pm 1 \in \tilde{H}_0(S^0; \mathbb{Z})$ under the suspension isomorphism, i.e. if and only if $h(\alpha)$ is an integral homology fundamental class. $\square$

Let X and A be as in Definition 7.4. For some $N \gg 0$, let $e: X \hookrightarrow \mathbb{R}^{4+N}$ be a smooth embedding, and write ν_e^{Diff} for the normal bundle. Apply the functor $\Sigma^{-N}\Sigma^\infty$ to the corresponding Thom collapse map $S^{4+N} \rightarrow \text{Th}(\nu_e^{\text{Diff}})$ to obtain a Thom collapse map of spectra

$$c: \mathbb{S}^4 \longrightarrow \text{Th}(\nu_X^{\text{Diff}}).$$

The target here is of course the same as $\text{Th}(\nu_X^G)$, for ν_X^G the Spivak fibration of X : the Thom spectrum of a vector bundle only depends on the underlying spherical fibration, and we may freely interchange them. Assume N is large enough that the stable framing on ν_X^{Diff} is represented by a framing of ν_e^{Diff} by $\nu_e^{\text{Diff}} \cong \mathbb{R}^N \times X$. Passing back to the stable version, and its Thom spectrum, we have

$$\text{Th}(\nu_X^{\text{Diff}}) = \Sigma^{-N}\Sigma^\infty \text{Th}(\nu_e^{\text{Diff}}) \cong \Sigma^{-N}\Sigma^\infty \text{Th}(\mathbb{R}^N \times X) \simeq \Sigma^\infty X_+. \quad (16)$$

Under this sequence of equivalences of spectra, the map c corresponds to a class $[X]_\mathbb{S} \in \pi_4^s(X_+)$.

Lemma 7.7. *The class $[X]_\mathbb{S} \in \pi_4^s(X_+)$ is a fundamental class in stable homotopy theory. Furthermore, the image $[A]_\mathbb{S} := \mathbf{t}_*[X]_\mathbb{S} \in \pi_4^s(A_+)$ under the Postnikov truncation map $\mathbf{t}: X \rightarrow A$ is also a fundamental class in stable homotopy theory.*

Proof. The group $H_4(\mathbb{S}^4; \mathbb{Z}) \cong [\mathbb{S}^0, \mathbb{S}^4 \wedge \mathbb{H}\mathbb{Z}] \cong [\mathbb{S}^0, \Sigma^\infty \mathbb{S}_+^4 \wedge \mathbb{H}\mathbb{Z}]$ is isomorphic to $\mathbb{Z}$, generated by the class $[\mathbb{S}^4]$, which is the class that maps to the fundamental class of the 4-sphere under these isomorphisms. It is a property of the Thom collapse map that $c_*[\mathbb{S}^4] \in H_4(\text{Th}(\nu_X^{\text{Diff}}); \mathbb{Z})$ is a generator. So under the Thom isomorphism (with respect to the orientation given by the stable framing) this class corresponds to an integral homology fundamental class $[X] \in H_4(X; \mathbb{Z})$. This is the image of $[X]_\mathbb{S}$ under the Hurewicz map, so $[X]_\mathbb{S}$ is a fundamental class by Lemma 7.6.

For the second statement, we observed already that $\mathbf{t}: X \rightarrow B\pi \simeq A$ has degree ± 1 , meaning that $\mathbf{t}_*[X] \in H_4(A; \mathbb{Z})$ is an integral homology fundamental class. The class $\mathbf{t}_*[X]$ is the Hurewicz image of $[A]_\mathbb{S} := \mathbf{t}_*[X]_\mathbb{S} \in \pi_4^s(A_+)$, so again by Lemma 7.6 it follows that $[A]_\mathbb{S}$ is also a stable homotopy fundamental class. $\square$

Every spectrum is an $\mathbb{S}$ -module, so an $\mathbb{S}$ -orientation on a manifold induces an orientation in any generalised (co)homology theory. This means that for any spectrum F the manifolds X and A have Poincaré duality isomorphisms (see Subsection 4.2). Using these, any map of $\mathbb{S}$ -oriented manifolds $f: X \rightarrow A$ determines an umkehr map

$$f!: F^r(X) \xrightarrow{PD} F_{4-r}(X) \xrightarrow{f_*} F_{4-r}(A) \xrightarrow{PD^{-1}} F^r(A).$$

Lemma 7.8. *If $f_*([X]_{\mathbb{S}}) = [A]_{\mathbb{S}}$ then the umkehr map satisfies the identity $f_! \circ f^* = \text{Id}$.*

Proof. Use respectively the definition of $f_!$, the projection formula, and the given identity $f_*([X]_{\mathbb{S}}) = [A]_{\mathbb{S}}$, to obtain

$$[A]_{\mathbb{S}} \frown (f_! \circ f^*)(-) = f_*([X]_{\mathbb{S}} \frown f^*(-)) = f_*([X]_{\mathbb{S}}) \frown (-) = [A]_{\mathbb{S}} \frown (-).$$

The lemma then follows from the fact that $[A]_{\mathbb{S}} \frown (-)$ is an isomorphism. $\square$

We use the umkehr construction to define the maps $\mathfrak{t}_!$ in diagram (15). Consider the Postnikov truncation map $\mathfrak{t}: X \rightarrow A$. Recall the spaces Top/PL and G/PL were given compatible infinite loop space structures coming from taking fibres of the infinite loop maps $\text{BPL} \rightarrow \text{BTop} \rightarrow \text{BG}$. The umkehr maps $\mathfrak{t}_!$ in diagram (15) are formed using the spectra corresponding to these infinite loop space structures. Thus the central square involving $\mathfrak{t}_!$ in diagram (15), commutes by naturality of the umkehr construction with respect to infinite loop maps.

Lemma 7.9. *The right-hand square involving $\mathfrak{t}_!$ in diagram (15), commutes.*

Proof. The map $\mathfrak{t}_!: [X_+, \Omega(\text{Top/PL})] \rightarrow [A_+, \Omega(\text{Top/PL})]$ is the umkehr map for the cohomology theory represented by the connective spectrum $\Sigma^{-1}(\text{top/pl})$ obtained from the infinite loop space $\Omega(\text{Top/PL})$. This space is a $K(\mathbb{Z}/2, 2)$, so the spectrum $\Sigma^{-1}(\text{top/pl})$ is the Eilenberg–MacLane spectrum $\Sigma^2 H(\mathbb{Z}/2)$. So this umkehr map is identified with the usual one $\mathfrak{t}_!: H^2(X; \mathbb{Z}/2) \rightarrow H^2(A; \mathbb{Z}/2)$, which is indeed Poincaré dual to $\mathfrak{t}_*: H_2(X; \mathbb{Z}/2) \rightarrow H_2(A; \mathbb{Z}/2)$. $\square$

7.4. Mapping into the PL normal invariants. We will consider the following diagram.

$$\begin{array}{ccccc} \pi_1(\widetilde{\text{Homeo}}(X)) & \xrightarrow{\alpha} & Q(X) & \xrightarrow{\text{Sm}} & [\frac{X \times I \times I}{\partial}, \text{B}(\text{Top/O})] & \xrightarrow{\cong} & [X_+, \Omega(\text{Top/O})] \\ \downarrow & & & & & & \downarrow \\ & & & & & & [X_+, \Omega(\text{Top/PL})] \\ \downarrow & & & & & & \downarrow \\ \pi_1(\widetilde{\text{hAut}}(X)) & \xrightarrow{\quad} & \mathcal{S}_{\partial}^{\text{PL}}(X \times I) & \xrightarrow{\eta_{\text{PL}}} & [X_+, \Omega(\text{G/PL})] \\ & \searrow \eta & & & \end{array} \quad (17)$$

The first map in the bottom row involves the map that considers a loop of homotopy automorphisms of X as a PL-manifold structure on $X \times I$. The second map in the bottom row is the PL normal invariant map $\eta_{\text{PL}}: \mathcal{S}_{\partial}^{\text{PL}}(X \times I) \rightarrow [\Sigma X_+, \text{G/PL}]$ followed by the Σ - Ω adjunction. The map η is defined so that the semi-circle commutes.

Lemma 7.10. *Diagram (17) commutes.*

Proof. The commutativity is essentially tautological. Here are the details. Write φ for a loop in $\text{Homeo}(X)$ (based at the identity map). This is represented by a homeomorphism $\psi \in \text{Homeo}_{\partial}(X \times I)$. Write

$$\nu_{X \times I}^{\text{Diff}}: X \times I \longrightarrow \text{BO}, \quad \nu_{X \times I}^{\text{PL}}: X \times I \longrightarrow \text{BPL} \quad \text{and} \quad \nu_{X \times I}^{\text{Top}}: X \times I \longrightarrow \text{BTop}$$

for the stable normal bundle, then its underlying stable PL-bundle, and then stable microbundle. There is a homotopy from $\nu_{X \times I}^{\text{Top}}$ to $(\psi^{-1})^* \nu_{X \times I}^{\text{Top}}$, induced by the stable normal microbundle of the mapping cylinder of ψ . Since we model $\text{BO} \rightarrow \text{BTop}$ by a fibration, this lifts to a homotopy from $\nu_{X \times I}^{\text{Diff}}$ to some lift of the tangent microbundle $\nu_{X \times I}^{\text{Top}}$, and the lift may be identified with $(\psi^{-1})^* \nu_{X \times I}^{\text{Diff}}$.

Consider the clockwise composition. We have $\alpha(\varphi) = [\psi] \in Q(X)$. Under the identifications on the top row, the class $\text{Sm}([\psi])$ is the difference class $d((\psi^{-1})^* \nu_{X \times I}^{\text{PL}}, \nu_{X \times I}^{\text{PL}})$, measuring the difference of these lifts of $\nu_{X \times I}^{\text{Top}}$ from Top to PL , relative to the fixed lift on $\partial(X \times I)$. Under the forgetful map, which sends $\nu_{X \times I}^{\text{Top}}$ to the Spivak normal fibration of $X \times I$, we obtain a PL normal invariant, which is again a difference class, now measuring the difference of the lifts from G to PL

$$d((\psi^{-1})^* \nu_{X \times I}^{\text{PL}}, \nu_{X \times I}^{\text{PL}}) \in \left[\frac{X \times I}{\partial(X \times I)}, \text{G/PL} \right]. \quad (18)$$

Now consider the anticlockwise composition. The first vertical map is forgetful, and the element of the PL structure set we obtain from φ is again just the (class of) ψ . The PL normal invariant obtained from an element of the structure set is just the difference of the lifts to PL of the Spivak normal fibration. Thus it is the element displayed in (18). This shows the diagram commutes, as claimed. $\square$

7.5. Functoriality of PL normal invariants with respect to Postnikov truncation. We will prove that the diagram

$$\begin{array}{ccc} \pi_1(\widetilde{\mathrm{hAut}}(X)) & \xrightarrow{\eta} & [X_+, \Omega(\mathrm{G}/\mathrm{PL})] \\ \downarrow \mathfrak{t}_* & & \downarrow \mathfrak{t}_! \\ \pi_1(\widetilde{\mathrm{hAut}}(A)) & \xrightarrow{\eta} & [A_+, \Omega(\mathrm{G}/\mathrm{PL})] \end{array} \quad (19)$$

commutes. This will complete the proof that diagram (15) commutes and thus the proof of Theorem F. The proof that diagram (19) commutes will crucially use that the manifolds are 4-dimensional: one should not expect it to hold in higher dimensions.

7.5.1. Factoring through the tangential structure set. To prove that diagram (19) commutes we will factor the map η , using that X (and A) are stably framed. The approach is based on the “tangential structure set” idea of Madsen–Taylor–Williams [MTW80], in which those authors consider an enriched structure set $\mathcal{S}_\partial^t(X \times I)$ where objects are not merely homotopy equivalences, but are also covered by maps of their (stable) tangent bundles. The purpose of this subsection is to establish the commutative diagram

$$\begin{array}{ccccc} \pi_1(\widetilde{\mathrm{hAut}}(X)) & \longrightarrow & \mathcal{S}_\partial^t(X \times I) & \longrightarrow & \mathcal{S}_\partial^{\mathrm{PL}}(X \times I) \\ \downarrow \hat{\eta} & & \downarrow \eta_t & & \downarrow \eta_{\mathrm{PL}} \\ [X_+, \Omega^{\infty+1}\mathbb{S}^0] & \xrightarrow[\cong]{(T_{0,1})^*} & [X_+, \Omega\mathrm{G}] & \longrightarrow & [X_+, \Omega(\mathrm{G}/\mathrm{PL})]. \end{array} \quad (20)$$

In this diagram, the homomorphism η , from earlier, is the right-right-down composition. We will see from the development that the right-hand square is defined and commutes without the assumption that X is stably framed, but the top left horizontal arrow and the map $\hat{\eta}$ require the stable framing of X to even be defined.

We begin by defining the bottom left horizontal map.

Definition 7.11. Let u and v be elements of a loop space ΩZ . There is a homotopy equivalence of based spaces

$$T_{u,v}: (\Omega Z, u) \longrightarrow (\Omega Z, v)$$

given by composing the concatenating map $(\Omega Z, u) \longrightarrow (\Omega Z, vu^{-1}u); \gamma \mapsto vu^{-1}\gamma$ with the homotopy inverse of the concatenating map $(\Omega Z, v) \longrightarrow (\Omega Z, vu^{-1}u); \delta \mapsto \delta u^{-1}u$. We call this homotopy equivalence *translating the basepoint* from u to v .

Remark 7.12. The infinite loop space $\Omega^\infty \mathbb{S}^0$ is the colimit $\mathrm{colim}_k \Omega^k S^k$ and therefore can be thought of as a colimit of based maps from a k -sphere to itself. Prominent basepoint choices for this infinite loop space include the identity map 1 and the constant map 0; by default, the basepoint of $\Omega^\infty \mathbb{S}^0$ is 0. Recall that there is an equality of spaces $\mathrm{SG} = \Omega_1^\infty \mathbb{S}^0$, where $\mathrm{SG} \subseteq \mathrm{G}$ and $\Omega_1^\infty \mathbb{S}^0$ denote the components containing the basepoint 1 (see e.g. [MM79, Corollary 3.8]). Thus translating the basepoint $0 \in \Omega^\infty \mathbb{S}^0$ to the basepoint $1 \in \mathrm{SG} \subseteq \mathrm{G}$ determines a based homotopy equivalence $T_{0,1}: (\Omega^\infty \mathbb{S}^0, 0) \simeq (\mathrm{SG}, 1)$. Consequently, there is a based homotopy equivalence of loop spaces $\Omega T_{0,1}: \Omega^{\infty+1} \mathbb{S}^0 \simeq \Omega \mathrm{G}$.

Construction 7.13. We construct a homomorphism of abelian groups

$$\hat{\eta}: \pi_1(\widetilde{\mathrm{hAut}}(X)) \longrightarrow [X_+, \Omega^{\infty+1} \mathbb{S}^0]$$

crucially using that X is stably framed, so that (16) is available. A loop in the Kan semi-simplicial group $\widetilde{\mathrm{hAut}}(X)$ is a homotopy equivalence $\phi: X \times I \rightarrow X \times I$ restricting to the identity on $\partial(X \times I)$. By gluing $X \times \{0\}$ to $X \times \{1\}$ in the domain and projecting to X in the codomain, we obtain a map $\psi: X \times S^1 \rightarrow X$ which restricts to the identity on $X \times \{1\}$. Write $\mathrm{pr}_X: X \times S^1 \rightarrow X$ for

the projection map. Recall that for maps of spectra $f, g: U \rightarrow V$, we write $f - g: U \rightarrow V$ for a representative of the homotopy class $[f] - [g] \in [U, V]$. On suspension spectra the difference

$$\Sigma^\infty \psi - \text{pr}_X: \Sigma^\infty (X \times S^1)_+ \longrightarrow \Sigma^\infty X_+$$

is canonically homotopically trivial when restricted to $\Sigma^\infty (X \times \{1\})_+$, so descends to a map

$$[\Sigma^\infty \psi - \text{pr}_X]: \Sigma^{\infty+1} X_+ = \Sigma^\infty \frac{(X \times S^1)_+}{(X \times \{1\})_+} \longrightarrow \Sigma^\infty X_+. \quad (21)$$

As X is stably framed, it has an $\mathbb{S}$ -theory fundamental class $[X]_{\mathbb{S}} \in [\mathbb{S}^4, \Sigma^\infty X_+]$. We consider the image $[\Sigma^\infty \psi - \text{pr}_X]_*(\Sigma[X]_{\mathbb{S}}) \in [\mathbb{S}^5, \Sigma^\infty X_+]$ of the suspension $\Sigma[X]_{\mathbb{S}} \in [\mathbb{S}^5, \Sigma^{\infty+1} X_+]$, and define $\hat{\eta}(\phi)$ to be element of $[X_+, \Omega^{\infty+1} \mathbb{S}^0]$ corresponding to $[\Sigma^\infty \psi - \text{pr}_X]_*(\Sigma[X]_{\mathbb{S}})$ under the isomorphisms

$$[X_+, \Omega^{\infty+1} \mathbb{S}^0] \cong [\Sigma^{\infty+1} X_+, \mathbb{S}^0] \underset{AD}{\cong} [\mathbb{S}^5, \text{Th}(\nu_X^G)] \underset{(16)}{\cong} [\mathbb{S}^5, \Sigma^\infty X_+]$$

given by adjunction, Atiyah duality and (16). Here the Atiyah duality isomorphism is given by applying the second version of Atiyah duality in Section 4.1 with $E = \mathbb{S}$, $\partial X = \emptyset$, and $r = -1$, together with the fact that $\text{Th}(\nu_X^{\text{Top}}) \simeq \text{Th}(\nu_X^G)$, to obtain

$$[\Sigma^{\infty+1} X_+, \mathbb{S}^0] \cong [\Sigma^\infty X_+, \mathbb{S}^{-1}] \underset{AD}{\cong} [\mathbb{S}^0, \text{Th}(\nu_X^G) \wedge \Sigma^{-1-4} \mathbb{S}^0] \cong [\mathbb{S}^5, \text{Th}(\nu_X^G)].$$

Remark 7.14. We argue that $\hat{\eta}$ is a homomorphism of abelian groups. The domain $\pi_1(\widehat{\text{hAut}}(X))$ of $\hat{\eta}$ is an abelian group because the geometric realisation of $\widehat{\text{hAut}}(X)$ is a group-like topological monoid, so $\pi_1(\widehat{\text{hAut}}(X)) \cong \pi_1(|\widehat{\text{hAut}}(X)|) \cong \pi_2(\text{B}|\widehat{\text{hAut}}(X)|)$. For the codomain, maps into an infinite loop space always form an abelian group. Identifying $\pi_1(\widehat{\text{hAut}}(X))$ with $\pi_0(\widehat{\text{hAut}}_{\partial}(X \times I))$, the group structure on the domain corresponds to stacking, and this corresponds to the group structure on the codomain $[X_+, \Omega^{\infty+1} \mathbb{S}] \cong [\Sigma X_+, \Omega^\infty \mathbb{S}] = [\frac{X \times I}{\partial(X \times I)}, \Omega^\infty \mathbb{S}]$ given by the co- H -space $\frac{X \times I}{\partial(X \times I)} \simeq \Sigma X_+$.

Remark 7.15. Intuitively, we could describe $\hat{\eta}(\phi)$ as measuring how the loop of homotopy equivalences ϕ acts upon the fundamental class $[X]_{\mathbb{S}} \in \pi_4^{\mathbb{S}}(X_+)$.

Next we describe the tangential structure set and tangential normal invariant. The description begins with a recap of the ordinary (PL) normal invariant, which is the right vertical map of diagram (20). We use the somewhat non-standard description in [MTW80, Section 2] (see also [BM76, Section 4]), with the changes necessary to deal with the fact that we work relative to the boundary whereas that paper does not.

7.5.2. PL normal invariants via f -maps. Given a compact space Y , define an f -map (ν^q, t, ξ^q) over Y to consist of two PL microbundles ν^q and ξ^q of the same dimension q over Y together with a fibre homotopy equivalence $t: S(\nu^q) \xrightarrow{\sim} S(\xi^q)$ between their underlying spherical fibrations. The f -maps (ν_i^q, t_i, ξ_i^q) for $i = 1, 2$ are *stably homotopic* if there exist f -maps $(\gamma_i, \text{Id}, \gamma_i)$ for $i = 1, 2$, such that $(\nu_1^q, t_1, \xi_1^q) \oplus (\gamma_1, \text{Id}, \gamma_1)$ and $(\nu_2^q, t_2, \xi_2^q) \oplus (\gamma_2, \text{Id}, \gamma_2)$ are homotopic as f -maps. It is shown in [BM76, Section 4] that the space G/PL classifies f -maps, considered up to stable homotopy.

An element of the structure set $\mathcal{S}_{\partial}^{\text{PL}}(X \times I)$ is represented by a simple homotopy equivalence $(\phi, \partial\phi): (M, \partial M) \xrightarrow{\sim} (X \times I, \partial(X \times I))$ from a PL manifold, with $\partial\phi$ a PL-isomorphism. Next we recall the details of $\eta_{\text{PL}}(\phi)$. There is a canonical map, over ϕ , of PL microbundles $\tilde{\phi}: \nu_M^{\text{PL}} \rightarrow (\phi^{-1})^* \nu_M^{\text{PL}}$. This induces a reduction

$$c_{\phi}: \mathbb{S}^5 \xrightarrow{c_M} \text{Th}(\nu_M^{\text{PL}}) / \text{Th}(\nu_{\partial M}^{\text{PL}}) \xrightarrow{\text{Th}(\tilde{\phi})} \text{Th}((\phi^{-1})^* \nu_M^{\text{PL}}) / \text{Th}(\nu_{\partial(X \times I)}^{\text{PL}})$$

that under the connecting map $\partial: \text{Th}((\phi^{-1})^* \nu_M^{\text{PL}}) / \text{Th}(\nu_{\partial(X \times I)}^{\text{PL}}) \rightarrow \Sigma \text{Th}(\nu_{\partial(X \times I)}^{\text{PL}})$ in the Puppe sequence agrees with the suspension

$$\Sigma c_{\partial(X \times I)}: \mathbb{S}^5 = \Sigma \mathbb{S}^4 \longrightarrow \Sigma \text{Th}(\nu_{\partial(X \times I)}^{\text{PL}}) \quad (22)$$

of the reduction for $\partial(X \times I)$. Further, as ϕ is a homotopy equivalence, the reduction c_{ϕ} is compatible with the $\mathbb{Z}$ -coefficient fundamental class of $X \times I$ in the sense that $h_*[c_{\phi}] \frown U_{H\mathbb{Z}} = [X \times I]_{H\mathbb{Z}} \in H_5(X \times I, \partial(X \times I); \mathbb{Z})$. This compatibility means that the uniqueness theorem for

Spivak fibrations [Bro72, Theorem I.4.19] applies, to give a fibre homotopy equivalence $t_\phi: \nu_{X \times I}^G \xrightarrow{\sim} (\phi^{-1})^* \nu_M^G$ rel. boundary between the underlying spherical fibrations, unique such that

$$\begin{array}{ccc} \mathbb{S}^5 & \xrightarrow{c_M} & \text{Th}(\nu_M^{\text{PL}}) / \text{Th}(\nu_{\partial M}^{\text{PL}}) \\ \downarrow c_{X \times I} & & \downarrow \text{Th}(\tilde{\phi}) \\ \text{Th}(\nu_{X \times I}^{\text{PL}}) / \text{Th}(\nu_{\partial(X \times I)}^{\text{PL}}) & \xrightarrow{\text{Th}(t_\phi)} & \text{Th}((\phi^{-1})^* \nu_M^{\text{PL}}) / \text{Th}(\nu_{\partial(X \times I)}^{\text{PL}}) \end{array} \quad (23)$$

commutes up to homotopy over $\Sigma \text{Th}(\nu_{\partial(X \times I)}^{\text{PL}})$. The f -map data $(\nu_{X \times I}^{\text{PL}}, t_\phi, (\phi^{-1})^* \nu_M^{\text{PL}})$ is classified by the element

$$\eta_{\text{PL}}(\phi) \in [\frac{X \times I}{\partial}, \text{G/PL}] \cong [X_+, \Omega(\text{G/PL})].$$

7.5.3. Tangential PL normal invariants via f -maps. Now we develop the central column of diagram (20) in parallel to the previous discussion, again following [MTW80, Section 2] and [BM76, Section 4]. The space G may be considered as classifying a restricted type of f -map, consisting of the data (ν^q, t) , where ν^q is a single PL microbundle and $t: S(\nu^q) \xrightarrow{\sim} S(\nu^q)$ is a fibre homotopy automorphism of its underlying spherical fibration, up to stable homotopy. For later use, we expand further on this classification. Given (ν^q, t) , we may take a finite dimensional representative ξ of the stable inverse to ν^q and form $(\nu^q, t) \oplus (\xi, \text{Id})$. The resultant bundle is $\nu^q \oplus \xi \simeq \varepsilon^M$ for some integer M , and under this identification $t * \text{Id}$ becomes a fibre homotopy equivalence of the trivial spherical fibration $Y \times S^{M-1} \rightarrow Y \times S^{M-1}$. Such a fibre homotopy equivalence is described by a map $Y \rightarrow \text{G}(M)$. After postcomposing with the inclusion $\text{G}(M) \subseteq \text{G}$, it is the homotopy class of this map which classifies (ν^q, t) up to stable homotopy; see [BM76, Section 4].

The PL *tangential structure set* $\mathcal{S}_\partial^t(X \times I)$ is the (bordism group of) simple homotopy equivalences of pairs $(\phi, \partial\phi): (M, \partial M) \rightarrow (X \times I, \partial(X \times I))$ from a PL manifold, with $\partial\phi$ a PL isomorphism, and equipped with a PL-microbundle map $\hat{\phi}: \nu_M^{\text{PL}} \rightarrow \nu_{X \times I}^{\text{PL}}$ covering ϕ and restricting to the PL derivative of $\partial\phi$ over the boundary. The map $\hat{\phi}$ determines a PL bundle identification $(\phi^{-1})^* \nu_M^{\text{PL}} \cong \nu_{X \times I}^{\text{PL}}$. Under this identification, the PL normal invariant classifies the data $(\nu_{X \times I}^{\text{PL}}, t_{\phi, \hat{\phi}}, \nu_{X \times I}^{\text{PL}})$, where $t_{\phi, \hat{\phi}}: \nu_{X \times I}^G \xrightarrow{\sim} (\phi^{-1})^* \nu_M^G \cong \nu_{X \times I}^G$ is the composition whose first map is t_ϕ and whose second is induced by our identification on the PL level. There is then defined a *tangential normal invariant*

$$\eta_t: \mathcal{S}_\partial^t(X \times I) \longrightarrow [\frac{X \times I}{\partial}, \text{G}] \cong [X_+, \Omega \text{G}],$$

where $\eta_t(\phi, \hat{\phi}) \in [\frac{X \times I}{\partial}, \text{G}] \cong [X_+, \Omega \text{G}]$ is the element classifying the data $(\nu_{X \times I}^{\text{PL}}, t_{\phi, \hat{\phi}})$.

At this point we have enough to verify that the tangential PL normal invariant is a lift of the ordinary PL normal invariant.

Lemma 7.16. *The right-hand square of diagram (20) commutes.*

Proof. Given $(\phi, \hat{\phi})$, the top horizontal arrow is the map that forgets $\hat{\phi}$ and thus the clockwise route sends $(\phi, \hat{\phi})$ to $(\nu_{X \times I}^{\text{PL}}, t_\phi, (\phi^{-1})^* \nu_M^{\text{PL}})$. The bottom horizontal arrow maps $(\nu^q, t) \mapsto (\nu^q, t, \nu^q)$ and hence the anticlockwise route gives $(\nu_{X \times I}^{\text{PL}}, t_{\phi, \hat{\phi}}, \nu_{X \times I}^{\text{PL}})$. Using $\hat{\phi}$, we see these f -maps are isomorphic. $\square$

To prove the left-hand square of diagram (20) commutes, we will first describe another point of view on the element $\eta_t(\phi, \hat{\phi})$. Given $(\phi, \hat{\phi}) \in \mathcal{S}_\partial^t(X \times I)$, take the diagram

$$\begin{array}{ccc} \mathbb{S}^5 & \xrightarrow{c_M} & \text{Th}(\nu_M^{\text{PL}}) / \text{Th}(\nu_{\partial M}^{\text{PL}}) \xrightarrow{\text{Th}(\hat{\phi})} \text{Th}(\nu_{X \times I}^{\text{PL}}) / \text{Th}(\nu_{\partial(X \times I)}^{\text{PL}}) \\ & \searrow \Sigma c_{\partial(X \times I)} & \downarrow \partial \\ & & \Sigma \text{Th}(\nu_{\partial(X \times I)}^{\text{PL}}), \end{array}$$

apply the functor $\text{SW} \circ \Sigma^{-5}$, to obtain

$$\begin{array}{ccc} \mathbb{S} & \xleftarrow{\text{SW} \circ \Sigma^{-5}(\text{Th}(\widehat{\phi}) \circ c_M)} & \Sigma^\infty(X \times I)_+ \\ & \searrow C_{\partial(X \times I)} & \uparrow \\ & & \Sigma^\infty \partial(X \times I)_+. \end{array}$$

To verify this, recall the properties of Spanier–Whitehead duals from Section 4.1, and also apply Lemma 4.2, where the notation $C_{\partial(X \times I)}$ was introduced for the map of spectra induced by the constant map $\partial(X \times I) \rightarrow \text{pt.}$ Applying Σ – Ω adjunction and discarding basepoints, we obtain a diagram of unbased spaces.

$$\begin{array}{ccc} \Omega^\infty \mathbb{S} & \xleftarrow{\text{Ad} \circ \text{SW} \circ \Sigma^{-5}(\text{Th}(\widehat{\phi}) \circ c_M)} & X \times I \\ & \searrow 1 & \uparrow \\ & & \partial(X \times I). \end{array} \quad (24)$$

The adjoint to $C_{\partial(X \times I)}$ is the constant map to the identity basepoint $1 \in \Omega^\infty \mathbb{S}$, by Lemma 4.2. Thus the horizontal map also lands in the path component $\text{SG} \subseteq \Omega^\infty \mathbb{S}$ of the identity map, and there is an induced map $\frac{X \times I}{\partial} \rightarrow \text{SG} \subseteq \text{G}$. The following is asserted in [MTW80, Section 2], without proof. We provide one here.

Lemma 7.17. *The tangential PL normal invariant $\eta_t(\phi, \widehat{\phi}) \in [\frac{X \times I}{\partial}, \text{G}]$ is equal to the element described by (24).*

Proof. There is a square

$$\begin{array}{ccc} \mathbb{S}^5 & \xrightarrow{c_M} & \text{Th}(\nu_M^{\text{PL}}) / \text{Th}(\nu_{\partial M}^{\text{PL}}) \\ \downarrow c_{X \times I} & & \downarrow \text{Th}(\widehat{\phi}) \\ \text{Th}(\nu_{X \times I}^{\text{PL}}) / \text{Th}(\nu_{\partial(X \times I)}^{\text{PL}}) & \xrightarrow{\text{Th}(t_{\phi, \widehat{\phi}})} & \text{Th}(\nu_{X \times I}^{\text{PL}}) / \text{Th}(\nu_{\partial(X \times I)}^{\text{PL}}) \end{array}$$

analogous to (23), which commutes up to homotopy over $\Sigma \text{Th}(\nu_{\partial(X \times I)}^{\text{PL}})$. The element (24) is obtained by Spanier–Whitehead dualising the map $\Sigma^{-5}(\text{Th}(\widehat{\phi}) \circ c_M)$ over $\Sigma \text{Th}(\nu_{\partial(X \times I)}^{\text{PL}})$, hence obtaining a map under $\Sigma^\infty \partial(X \times I)_+$. By the square this map is homotopic under $\Sigma^\infty \partial(X \times I)_+$ to the result of Spanier–Whitehead dualising $\Sigma^{-5}(\text{Th}(t_{\phi, \widehat{\phi}}) \circ c_{X \times I})$. We must therefore show that the Spanier–Whitehead dual of $\Sigma^{-5}(\text{Th}(t_{\phi, \widehat{\phi}}) \circ c_{X \times I})$ over $\Sigma \text{Th}(\nu_{\partial(X \times I)}^{\text{PL}})$ is Σ – Ω –adjoint to the map $\eta_t(\phi, \widehat{\phi}): X \times I \rightarrow \text{G} \subset \Omega^\infty \mathbb{S}$ under $1: \partial(X \times I) \rightarrow \text{G} \subseteq \Omega^\infty \mathbb{S}$.

By Lemma 4.2, the Spanier–Whitehead dual of $\Sigma^{-5}c_{X \times I}$ is the constant map $C_{X \times I}: \Sigma^\infty(X \times I)_+ \rightarrow \mathbb{S}$, so to proceed we must understand the Spanier–Whitehead dual of $\Sigma^{-5}\text{Th}(t_{\phi, \widehat{\phi}})$. To do so, define the map $a: \Sigma^\infty \text{G}_+ \rightarrow \mathbb{S}$ to be the adjoint of the inclusion $\text{G}_+ \rightarrow \Omega^\infty \mathbb{S}$. For brevity, introduce the notation

$$T := \Sigma^{-5} \frac{\text{Th}(\nu_{X \times I}^{\text{PL}})}{\text{Th}(\nu_{\partial(X \times I)}^{\text{PL}})} \quad \text{and} \quad T^\vee := \Sigma^\infty(X \times I)_+,$$

where $T^\vee$ indicates that these are Spanier–Whitehead dual.

Claim. The Spanier–Whitehead dual of

$$\Sigma^{-5} \text{Th}(t_{\phi, \widehat{\phi}}): T \longrightarrow T, \quad \text{over } \Sigma \text{Th}(\nu_{\partial(X \times I)}^{\text{PL}}),$$

is homotopic to the composition

$$T^\vee \xrightarrow{\text{diag}} T^\vee \wedge T^\vee \xrightarrow{\Sigma^\infty \eta_t(\phi, \widehat{\phi}) \wedge \text{Id}} \Sigma^\infty \text{G}_+ \wedge T^\vee \xrightarrow{a \wedge \text{Id}} \mathbb{S} \wedge T^\vee \simeq T^\vee, \quad \text{under } \Sigma^\infty \partial(X \times I)_+. \quad (25)$$

Proof of claim. We first show that the map $\text{Th}(t_{\phi, \widehat{\phi}})$ may be decomposed as

$$\Sigma^5 T \xrightarrow{\Delta} T^\vee \wedge \Sigma^5 T \xrightarrow{\Sigma^\infty \eta_t(\phi, \widehat{\phi}) \wedge \text{Id}} \Sigma^\infty \text{G}_+ \wedge \Sigma^5 T \xrightarrow{a \wedge \text{Id}} \mathbb{S} \wedge \Sigma^5 T \simeq \Sigma^5 T, \quad (26)$$

where Δ denotes the Thom diagonal. To justify this, write $\nu: X \times I \rightarrow \text{BG}(N)$ for the spherical fibration of a finite-dimensional representative of the PL-normal bundle of $X \times I$. Write ξ for the

a finite-dimensional representative of the stable inverse to ν , so that $\nu * \xi \simeq S(\varepsilon^M)$, the trivial spherical fibration for some M . Now consider the self-equivalence

$$t_{\phi, \widehat{\phi}} * \text{Id} * \text{Id} : \nu * \xi * \nu \rightarrow \nu * \xi * \nu.$$

We may identify the first two factors with $S(\varepsilon^M)$. Under this identification, the self-equivalence of spherical fibrations $t_{\phi, \widehat{\phi}} * \text{Id} : \nu * \xi \rightarrow \nu * \xi$ is identified with the automorphism of the trivial spherical fibration $S(\varepsilon^M) = (X \times I) \times S^{M-1}$ described fibre-wise by the map $\eta_t(\phi, \widehat{\phi}) : \frac{X \times I}{\partial} \rightarrow G(M)$. Hence, the map $\text{Th}(t_{\phi, \widehat{\phi}} * \text{Id} * \text{Id})$ can be identified as the M -fold suspension of the composition (26). On the other hand, we can instead identify the latter two factors of $\nu * \xi * \nu$ with $S(\varepsilon^M)$, so that the map $t_{\phi, \widehat{\phi}} * \text{Id} * \text{Id}$ is just the M -fold stabilisation of $t_{\phi, \widehat{\phi}}$, giving $\text{Th}(t_{\phi, \widehat{\phi}} * \text{Id} * \text{Id}) = \Sigma^M \text{Th}(t_{\phi, \widehat{\phi}})$. So the desired factorisation of $\text{Th}(t_{\phi, \widehat{\phi}})$ has been shown.

With this factorisation in hand, we can prove the claim. Write the coevaluation and evaluation maps for the pair T and $T^\vee$ as

$$\text{coev} : \mathbb{S} \longrightarrow T^\vee \wedge T \quad \text{and} \quad \text{ev} : T \wedge T^\vee \longrightarrow \mathbb{S}.$$

Consider the diagram

$$\begin{array}{ccccc} T^\vee & \xrightarrow{\simeq} & \mathbb{S} \wedge T^\vee & \xrightarrow{\text{coev} \wedge \text{Id}} & T^\vee \wedge T \wedge T^\vee \\ \downarrow \text{diag} & & & & \downarrow \text{Id} \wedge \Sigma^{-5} \Delta \wedge \text{Id} \\ & & T^\vee \wedge T^\vee \wedge T \wedge T^\vee & \xrightarrow{\text{Id} \wedge \Sigma^\infty \eta_t(\phi, \widehat{\phi}) \wedge \text{Id} \wedge \text{Id}} & T^\vee \wedge \Sigma^\infty G_+ \wedge T \wedge T^\vee \\ & & \downarrow \text{Id} \wedge \text{Id} \wedge \text{ev} & & \downarrow \text{Id} \wedge a \wedge \text{Id} \wedge \text{Id} \\ T^\vee \wedge T^\vee & \xrightarrow{\simeq} & T^\vee \wedge T^\vee \wedge \mathbb{S} & & T^\vee \wedge \mathbb{S} \wedge T \wedge T^\vee \\ & & \downarrow \Sigma^\infty \eta_t(\phi, \widehat{\phi}) \wedge \text{Id} \wedge \text{Id} & & \downarrow \text{Id} \wedge \text{Id} \wedge \text{ev} \\ & & \Sigma^\infty G_+ \wedge T^\vee \wedge \mathbb{S} & \xrightarrow{a \wedge \text{Id} \wedge \text{Id}} & T^\vee. \end{array}$$

By definition, the Spanier–Whitehead dual of the map $\Sigma^{-5} \text{Th}(t_{\phi, \widehat{\phi}})$ is the composition

$$T^\vee \simeq \mathbb{S} \wedge T^\vee \xrightarrow{\text{coev} \wedge \text{Id}} T^\vee \wedge T \wedge T^\vee \xrightarrow{\text{Id} \wedge \Sigma^{-5} \text{Th}(t_{\phi, \widehat{\phi}}) \wedge \text{Id}} T^\vee \wedge T \wedge T^\vee \xrightarrow{\text{Id} \wedge \text{ev}} T^\vee \wedge \mathbb{S} \simeq T^\vee.$$

Using the factorisation (26), this shows that the Spanier–Whitehead dual of the map $\text{Th}(t_{\phi, \widehat{\phi}})$ is the full clockwise composition in the diagram. The full anticlockwise composition in the diagram is the map (25). Hence the claim is proved if we can show the diagram commutes up to homotopy under $\Sigma^\infty \partial(X \times I)_+$.

To see the right rectangle commutes, simply note that the maps ev and $a \circ \eta_t(\phi, \widehat{\phi})$ operate upon different factors. Now we show that the left rectangle commutes. The diagonal map $T^\vee \rightarrow T^\vee \wedge T^\vee$ is a co-ring structure on $T^\vee$ (true of any suspension spectrum), and the five-fold desuspension of the Thom diagonal $\Sigma^{-5} \Delta : T \rightarrow T^\vee \wedge T$ endows T with the structure of co-module over $T^\vee$. This may be checked from the definition of the Thom diagonal (it corresponds to the well-known fact that the cohomology of the Thom space of a bundle is a module over the cohomology of the base space). In particular, co-associativity $(\text{Id} \wedge \Sigma^{-5} \Delta) \circ \Sigma^{-5} \Delta = (\text{diag} \wedge \text{Id}) \circ \Sigma^{-5} \Delta$ holds. Recall that coev is given by

$$\mathbb{S} \xrightarrow{\Sigma^{-5} c_{X \times I}} T \xrightarrow{\Sigma^{-5} \Delta} T^\vee \wedge T,$$

where the second map is the five-fold desuspension of the Thom diagonal. Hence, the co-associativity statement, above, means composition of the first two maps in the clockwise route around the left rectangle agrees with

$$(\text{diag} \wedge \text{Id} \wedge \text{Id}) \circ (\text{coev} \wedge \text{Id}) : \mathbb{S} \wedge T^\vee \longrightarrow T^\vee \wedge T^\vee \wedge T \wedge T^\vee.$$

Now consider the full composition clockwise around the left rectangle as a map $\mathbb{S} \wedge T^\vee \rightarrow T^\vee \wedge T^\vee \wedge \mathbb{S}$. We may commute the latter two maps in the composition, as follows, because they operate on different factors:

$$(\text{Id} \wedge \text{Id} \wedge \text{ev}) \circ (\text{diag} \wedge \text{Id} \wedge \text{Id}) \circ (\text{coev} \wedge \text{Id}) = (\text{diag} \wedge \text{Id}) \circ (\text{Id} \wedge \text{ev}) \circ (\text{coev} \wedge \text{Id}).$$

But then $(\text{Id} \wedge \text{ev}) \circ (\text{coev} \wedge \text{Id})$ is the identity equivalence $\mathbb{S} \wedge T^\vee \simeq T^\vee \wedge \mathbb{S}$, so this equation is the statement that the left hand square commutes. $\square$

We can now finish the proof of the lemma. By inspection, the following diagram commutes up to homotopy under $\Sigma^\infty \partial(X \times I)_+$.

$$\begin{array}{ccccccc}
T^\vee & \xrightarrow{\Sigma^\infty \eta_t(\phi, \hat{\phi})} & \Sigma^\infty G_+ & \xrightarrow{a} & \mathbb{S} & & \\
\downarrow \text{diag} & \searrow \simeq & \downarrow \simeq & & \nearrow \simeq & & \\
& & T^\vee \wedge \mathbb{S} & \xrightarrow{\Sigma^\infty \eta_t(\phi, \hat{\phi}) \wedge \text{Id}} & \Sigma^\infty G_+ \wedge \mathbb{S} & \xrightarrow{a \wedge \text{Id}} & \mathbb{S} \wedge \mathbb{S} \\
& \nearrow \text{Id} \wedge C_{X \times I} & & \uparrow \text{Id} \wedge C_{X \times I} & & \uparrow \text{Id} \wedge C_{X \times I} & \\
T^\vee \wedge T^\vee & \xrightarrow{\Sigma^\infty \eta_t(\phi, \hat{\phi}) \wedge \text{Id}} & \Sigma^\infty G_+ \wedge T^\vee & \xrightarrow{a \wedge \text{Id}} & \mathbb{S} \wedge T^\vee & \xrightarrow{\simeq} & T^\vee
\end{array}$$

Under Σ - Ω adjunction, the composition along the top row is $\eta_t(\phi, \hat{\phi}) \in [\frac{X \times I}{\partial}, G]$. The anti-clockwise composition is (25) postcomposed with the map $C_{X \times I}$. So by the claim, its Spanier-Whitehead dual is homotopic over $\Sigma \text{Th}(\nu_{\partial(X \times I)}^{\text{PL}})$ to $\text{Th}(t_{\phi, \hat{\phi}}) \circ c_{X \times I}$, as required. $\square$

Remark 7.18. The structure of the proof of Lemma 7.17 can be summarised as follows. We proved the following sequence of homotopies of maps of spectra $T^\vee \rightarrow \mathbb{S}$ under $\Sigma^\infty \partial(X \times I)_+$:

$$\begin{aligned}
\text{Ad}^{-1}(24) &\simeq \text{SW}(\Sigma^{-5}(\text{Th}(\hat{\phi}) \circ c_M)) \simeq \text{SW}(\Sigma^{-5}(\text{Th}(t_{\phi, \hat{\phi}}) \circ c_{X \times I})) \\
&\simeq \text{SW}((\Sigma^{-5} \text{Th}(t_{\phi, \hat{\phi}})) \circ (\Sigma^{-5} c_{X \times I})) \simeq \text{SW}(\Sigma^{-5} c_{X \times I}) \circ \text{SW}(\Sigma^{-5} \text{Th}(t_{\phi, \hat{\phi}})) \\
&\simeq C_{X \times I} \circ (25) \simeq a \circ \Sigma^\infty \eta_t(\phi, \hat{\phi}) \simeq \text{Ad}^{-1} \eta_t(\phi, \hat{\phi}).
\end{aligned}$$

Taking adjoints and forgetting basepoints yields the statement of Lemma 7.17.

Proposition 7.19. *The left-hand square of diagram (20) commutes.*

Proof. Let $\phi: X \times I \rightarrow X \times I$ be a homotopy equivalence which is the identity on the boundary, representing an element of $\pi_1(\widetilde{\text{hAut}}(X))$. As in Construction 7.13, we associate to ϕ the induced map $\psi: X \times S^1 \rightarrow X$ which is the identity on $X \times \{1\}$. Using the framing of ν_X^{PL} we can cover ϕ by the PL-microbundle map $\hat{\phi}: \nu_{X \times I}^{\text{PL}} \rightarrow \nu_{X \times I}^{\text{PL}}$ induced by the framing, giving an element $(\phi, \hat{\phi}) \in \mathcal{S}_\partial^t(X \times I)$. By Lemma 7.17, $\eta_t(\phi, \hat{\phi})$ is described using the diagram

$$\begin{array}{ccc}
\mathbb{S}^5 & \xrightarrow{c_{X \times I}} & \text{Th}(\nu_{X \times I}^{\text{PL}}) / \text{Th}(\nu_{\partial(X \times I)}^{\text{PL}}) \xrightarrow{\text{Th}(\hat{\phi})} \text{Th}(\nu_{X \times I}^{\text{PL}}) / \text{Th}(\nu_{\partial(X \times I)}^{\text{PL}}) \\
& \searrow & \downarrow \partial \\
& & \Sigma \text{Th}(\nu_{\partial(X \times I)}^{\text{PL}}),
\end{array}$$

$\Sigma c_{\partial(X \times I)}$

by applying the functor $\text{SW} \circ \Sigma^{-5}$ and then Σ - Ω adjunction. The basepoint-translated element $(T_{1,0})_*(\eta_t(\phi, \hat{\phi})) \in [\frac{X \times I}{\partial}, \Omega^\infty \mathbb{S}]$ is then described by the same recipe applied to the analogous diagram

$$\begin{array}{ccc}
\mathbb{S}^5 & \xrightarrow{c_{X \times I}} & \text{Th}(\nu_{X \times I}^{\text{PL}}) / \text{Th}(\nu_{\partial(X \times I)}^{\text{PL}}) \xrightarrow{\text{Th}(\hat{\phi}) - \text{Id}} \text{Th}(\nu_{X \times I}^{\text{PL}}) / \text{Th}(\nu_{\partial(X \times I)}^{\text{PL}}) \\
& \searrow & \downarrow \partial \\
& & \Sigma \text{Th}(\nu_{\partial(X \times I)}^{\text{PL}}),
\end{array}$$

0

Using the framing to trivialise these Thom spectra, we may write this diagram as

$$\begin{array}{ccc}
\mathbb{S}^5 & \xrightarrow{[X \times I]_S} & \Sigma^\infty \frac{X \times I}{\partial} \xrightarrow{\Sigma^\infty \phi - \text{Id}} \Sigma^\infty \frac{X \times I}{\partial} \\
& \searrow & \downarrow \partial \\
& & \Sigma^{\infty+1} \partial(X \times I)_+.
\end{array}$$

0

There is a factorisation

$$\Sigma^\infty \phi - \text{Id}: \Sigma^\infty \frac{X \times I}{\partial} \simeq \Sigma^\infty \frac{(X \times S^1)_+}{(X \times \{1\})_+} \xrightarrow{[\Sigma^\infty \psi - \text{pr}_X]} \Sigma^\infty X_+ \simeq \Sigma^\infty (X \times I)_+ \xrightarrow{q} \Sigma^\infty \frac{X \times I}{\partial},$$

where first map is described in (21) and the second is the quotient. The map ∂ in the diagram fits into the cofibre sequence $\Sigma^\infty (X \times I)_+ \xrightarrow{q} \Sigma^\infty \frac{X \times I}{\partial} \xrightarrow{\partial} \Sigma^{\infty+1} \partial(X \times I)_+$, where this cofibre sequence

is obtained from taking the sequence $\partial(X \times I) \rightarrow X \times I \rightarrow \frac{X \times I}{\partial}$, applying $\Sigma^\infty(-)_+$ and extending to the right by one. It follows that the map described by the diagram corresponds, under lifting along q , to the composition

$$\mathbb{S}^5 \xrightarrow{[X \times I]_\mathbb{S}} \Sigma^\infty \frac{X \times I}{\partial} \xrightarrow{[\Sigma^\infty \psi - \text{pr}_X]} \Sigma^\infty X_+. \quad (27)$$

Under the identification $\Sigma^\infty \frac{X \times I}{\partial} \simeq \Sigma^{\infty+1} X_+$, the fundamental class $[X \times I]_\mathbb{S}$ corresponds to $\Sigma[X]_\mathbb{S}$. Using this, the composition (27) corresponds to the element $[\Sigma^\infty \psi - \text{pr}_X]_* (\Sigma[X]_\mathbb{S}) \in [\mathbb{S}^5, \Sigma^\infty X_+] \cong [\mathbb{S}^5, \text{Th}(\nu_X^{\text{PL}})]$ where the isomorphism comes from another use of the framing. Applying the functor $\text{SW} \circ \Sigma^{-5}$ to the composition, we obtain the definition of $\widehat{\eta}(\phi)$. $\square$

7.5.4. Proof that (19) commutes. The equivalence $\Omega G \simeq \Omega^{\infty+1} \mathbb{S}^0$, induced by basepoint translation (Remark 7.12), is *not an equivalence of infinite loop spaces* (for example, their deloopings have different Pontryagin rings [MM79, §3.F]). This will account for a difficulty the following proposition. The diagram in the following proposition is obtained from (19) by applying the alternative factorisation of η that we obtained by proving that diagram (20) commutes. Hence Proposition 7.20 completes the proof that (19) commutes.

Proposition 7.20. *There is a commutative diagram of groups and homomorphisms*

$$\begin{array}{ccccccc} \eta: \pi_1(\widetilde{\text{hAut}}(X)) & \xrightarrow{\widehat{\eta}} & [X_+, \Omega^{\infty+1} \mathbb{S}^0] & \xrightarrow{(T_{0,1})_*} & [X_+, \Omega G] & \longrightarrow & [X_+, \Omega(G/\text{PL})] \\ \downarrow \mathfrak{t}_* & & \downarrow \mathfrak{t}_! & & \downarrow \mathfrak{t}_! & & \downarrow \mathfrak{t}_! \\ \eta: \pi_1(\widetilde{\text{hAut}}(A)) & \xrightarrow{\widehat{\eta}} & [A_+, \Omega^{\infty+1} \mathbb{S}^0] & \xrightarrow{(T_{0,1})_*} & [A_+, \Omega G] & \longrightarrow & [A_+, \Omega(G/\text{PL})], \end{array}$$

where each $\mathfrak{t}_!$ denotes the umkehr map with respect to the indicated infinite loop space structure on the codomain.

Proof. Apart from the left column, the group structures arise from loop space structures indicated by (the first) Ω . These are compatible, so the maps are homomorphisms. The commutativity of the right-hand square is by naturality of the umkehr map with respect to maps of generalised cohomology theories, i.e. maps of infinite loop spaces. The commutativity of the middle square is therefore surprising, as we have pointed out that $\Omega G \simeq \Omega^{\infty+1} \mathbb{S}^0$ is not an equivalence of infinite loop spaces. We will address this below.

Next we prove the commutativity of the left-hand square. Let ϕ be a loop in $\widetilde{\text{hAut}}(X)$ based at the identity map. As in Construction 7.13, we associate to ϕ the induced map $\psi: X \times S^1 \rightarrow X$ which is the identity on $X \times \{1\}$. By functoriality of Postnikov truncation, ψ yields $\mathfrak{t}_*(\psi) \in \pi_1(\widetilde{\text{hAut}}(A))$, and the diagram

$$\begin{array}{ccccc} \mathbb{S}^5 & \xrightarrow{\Sigma[X]_\mathbb{S}} & \Sigma^{\infty+1} X_+ & \xrightarrow{[\Sigma^\infty \psi - \text{pr}_X]} & \Sigma^\infty X_+ \\ & \searrow \Sigma[A]_\mathbb{S} & \downarrow \Sigma \mathfrak{t}_+ & & \downarrow \mathfrak{t}_+ \\ & & \Sigma^{\infty+1} A_+ & \xrightarrow{[\Sigma^\infty \mathfrak{t}_*(\psi) - \text{pr}_A]} & \Sigma^\infty A_+ \end{array}$$

commutes up to homotopy, using that $\mathfrak{t}_*([X]_\mathbb{S}) = [A]_\mathbb{S}$ by Lemma 7.7. The clockwise composition in $[\mathbb{S}^5, \Sigma^\infty A_+] = [\mathbb{S}^5, \text{Th}(\nu_A^{\text{PL}})] \cong_{\text{SW}} [A_+, \Omega^{\infty+1} \mathbb{S}^0]$ is by definition $\mathfrak{t}_! \widehat{\eta}(\phi)$; the anticlockwise composition is $\widehat{\eta}(\mathfrak{t}_!(\phi))$.

To prove commutativity of the middle square, let us write $\mathfrak{t}_!^{\mathbb{S}}$ for the left-hand umkehr map and $\mathfrak{t}_!^G$ for the right-hand one. As X is 4-dimensional we may replace $\Omega^{\infty+1} \mathbb{S}^0$ by its 4-truncation, which we choose to write as $\Omega \tau_{\leq 5} \Omega^\infty \mathbb{S}^0 = \Omega \Omega^\infty \tau_{\leq 5} \mathbb{S}^0$. We then use that $\pi_4^s(\mathbb{S}^0) = 0 = \pi_5^s(\mathbb{S}^0)$ to see that $\tau_{\leq 5} \mathbb{S}^0 = \tau_{\leq 3} \mathbb{S}^0$, and hence we have that $\Omega \tau_{\leq 5} \Omega^\infty \mathbb{S}^0 = \Omega \tau_{\leq 3} \Omega^\infty \mathbb{S}^0$. Now we consider the equivalence of fibre sequences of connected spaces

$$\begin{array}{ccccc} \tau_{[2,3]} \Omega_0^\infty \mathbb{S}^0 & \longrightarrow & \tau_{\leq 3} \Omega_0^\infty \mathbb{S}^0 & \xrightarrow{\rho} & \tau_{\leq 1} \Omega_0^\infty \mathbb{S}^0 \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ \tau_{[2,3]} \text{SG} & \longrightarrow & \tau_{\leq 3} \text{SG} & \longrightarrow & \tau_{\leq 1} \text{SG} \end{array}$$

and make the following observation: the base and fibre each admit a unique structure of an infinite loop space. For the base this is clear as it is a $K(\mathbb{Z}/2, 1)$; for the fibre this follows from Mathew–Stojanoska [MS16, Theorem 5.1.2], though it can also be proven by hand. The citation applied with $n = 2$ says that the functor $\Omega^\infty(-): \mathcal{S}pectra_{[2,3]} \rightarrow \mathcal{S}_*$, from the ∞ -category of spectra with nontrivial homotopy groups only in degrees two and three to the ∞ -category of pointed spaces, is fully faithful, i.e. homotopy equivalence on morphism spaces. If we have two a priori different infinite loop space structures on $\tau_{[2,3]}SG$, they corresponds to two spectra in the domain of $\Omega^\infty(-)$ that are homotopy equivalent in the codomain. The fact that $\Omega^\infty(-)$ is fully faithful leads to an equivalence of spectra, and hence of infinite loop space structures on $\tau_{[2,3]}SG$.

Thus the maps on base and fibre are necessarily equivalences of infinite loop spaces. Looping the top fibre sequence and mapping in X or A , the difference of the umkehr maps $\mathfrak{t}_!^S$ and $\mathfrak{t}_!^G$ fit in to a map of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & [X_+, \Omega\tau_{[2,3]}\Omega_0^\infty\mathbb{S}^0] & \longrightarrow & [X_+, \Omega\tau_{\leq 3}\Omega_0^\infty\mathbb{S}^0] & \xrightarrow{\Omega\rho_*} & [X_+, \Omega\tau_{\leq 1}\Omega_0^\infty\mathbb{S}^0] \longrightarrow 0 \\ & & \downarrow \mathfrak{t}_!^S - \mathfrak{t}_!^G = 0 & & \downarrow \mathfrak{t}_!^S - \mathfrak{t}_!^G & & \downarrow \mathfrak{t}_!^S - \mathfrak{t}_!^G = 0 \\ 0 & \longrightarrow & [A_+, \Omega\tau_{[2,3]}\Omega_0^\infty\mathbb{S}^0] & \longrightarrow & [A_+, \Omega\tau_{\leq 3}\Omega_0^\infty\mathbb{S}^0] & \longrightarrow & [A_+, \Omega\tau_{\leq 1}\Omega_0^\infty\mathbb{S}^0] \longrightarrow 0, \end{array} \quad (28)$$

where the two outer vertical maps are zero because here the two infinite loop space structures used to form the two umkehr maps agree, although we will only make use of the vanishing of the left-hand vertical map.

First we show that the right hand groups are both $\mathbb{Z}/2$. For this we compute the homotopy groups of the codomain $\Omega\tau_{\leq 1}\Omega_0^\infty\mathbb{S}^0$. The space $\tau_{\leq 1}\Omega_0^\infty\mathbb{S}^0$ has non-vanishing homotopy groups only in degrees 0 and 1, where they are the stable 0-stem $\mathbb{Z}$ and the stable 1-stem $\mathbb{Z}/2$. Taking the loop space shifts the homotopy groups down, so that only a $\mathbb{Z}/2$ in degree 0 remains. We see that $\Omega\tau_{\leq 1}\Omega_0^\infty\mathbb{S}^0$ has precisely two path components, each of which is weakly contractible. It follows that $[X_+, \Omega\tau_{\leq 1}\Omega_0^\infty\mathbb{S}^0] \cong \mathbb{Z}/2 \cong [A_+, \Omega\tau_{\leq 1}\Omega_0^\infty\mathbb{S}^0]$, as claimed.

We show that the middle vertical map $\mathfrak{t}_!^S - \mathfrak{t}_!^G$ factors through the map $\Omega\rho_*$ to the quotient, yielding

$$(\mathfrak{t}_!^S - \mathfrak{t}_!^G): [X_+, \Omega\tau_{\leq 3}\Omega_0^\infty\mathbb{S}^0] \xrightarrow{\Omega\rho_*} [X_+, \Omega\tau_{\leq 1}\Omega_0^\infty\mathbb{S}^0] \xrightarrow{T} [A_+, \Omega\tau_{\leq 3}\Omega_0^\infty\mathbb{S}^0].$$

To see this, let $x, x' \in [X_+, \Omega\tau_{\leq 3}\Omega_0^\infty\mathbb{S}^0]$ be such that $\Omega\rho_*(x) = \Omega\rho_*(x')$. Then $\Omega\rho_*(x - x') = 0$, so $x - x'$ lies in the image of $[X_+, \Omega\tau_{[2,3]}\Omega_0^\infty\mathbb{S}^0]$. By commutativity of the left square, and the fact that the left vertical map vanishes, we deduce that $0 = (\mathfrak{t}_!^S - \mathfrak{t}_!^G)(x - x') = (\mathfrak{t}_!^S - \mathfrak{t}_!^G)(x) - (\mathfrak{t}_!^S - \mathfrak{t}_!^G)(x')$. Thus $(\mathfrak{t}_!^S - \mathfrak{t}_!^G)(x) = (\mathfrak{t}_!^S - \mathfrak{t}_!^G)(x')$ and the central $\mathfrak{t}_!^S - \mathfrak{t}_!^G$ factors as $T \circ (\Omega\rho_*)$, as asserted.

Next we argue that the composition

$$[A_+, \Omega\tau_{\leq 3}\Omega_0^\infty\mathbb{S}^0] \xrightarrow{\mathfrak{t}^*} [X_+, \Omega\tau_{\leq 3}\Omega_0^\infty\mathbb{S}^0] \longrightarrow [X_+, \Omega\tau_{\leq 1}\Omega_0^\infty\mathbb{S}^0]$$

is surjective. To see this recall that the last group here is $\mathbb{Z}/2$, and note that by a similar calculation $\Omega\tau_{\leq 3}\Omega_0^\infty\mathbb{S}^0$ has precisely two path components. We can map all of A to either of these components to obtain elements in the domain of the composition that map to each of the elements in the codomain $\mathbb{Z}/2$.

We have developed the following diagram.

$$\begin{array}{ccc} [A_+, \Omega\tau_{\leq 3}\Omega_0^\infty\mathbb{S}^0] & \twoheadrightarrow & [X_+, \Omega\tau_{\leq 1}\Omega_0^\infty\mathbb{S}^0] \\ \downarrow \mathfrak{t}^* & \nearrow \Omega\rho_* & \downarrow T \\ [X_+, \Omega\tau_{\leq 3}\Omega_0^\infty\mathbb{S}^0] & \xrightarrow{\mathfrak{t}_!^S - \mathfrak{t}_!^G} & [A_+, \Omega\tau_{\leq 3}\Omega_0^\infty\mathbb{S}^0]. \end{array}$$

The identity $\mathfrak{t}_! \circ \mathfrak{t}^* = \text{Id}$ from Lemma 7.8 holds for both meanings of $\mathfrak{t}_!$, so that $(\mathfrak{t}_!^S - \mathfrak{t}_!^G) \circ \mathfrak{t}^* = 0$. It now follows by a straightforward diagram chase that $\mathfrak{t}_!^S - \mathfrak{t}_!^G = 0$. Here are the details. Let $x \in [X_+, \Omega\tau_{\leq 3}\Omega_0^\infty\mathbb{S}^0]$. Since the top horizontal map is surjective, there exists $y \in [A_+, \Omega\tau_{\leq 3}\Omega_0^\infty\mathbb{S}^0]$ with $\Omega\rho_* \circ \mathfrak{t}^*(y) = \Omega\rho_*(x)$. Then

$$(\mathfrak{t}_!^S - \mathfrak{t}_!^G)(x) = T \circ (\Omega\rho_*)(x) = T \circ (\Omega\rho_*) \circ \mathfrak{t}^*(y) = (\mathfrak{t}_!^S - \mathfrak{t}_!^G) \circ \mathfrak{t}^*(y) = 0.$$

This shows that the middle square of (28) commutes, completing the proof of the proposition. $\square$

As explained beforehand, the proposition implies that the square (19) commutes. This completes the proof that (15) commutes, and hence completes the proof of Theorem F.

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