SPANNING 3-DISCS IN THE 4-SPHERE PUSHED INTO THE 5-DISC

MARK POWELL

Abstract. I prove that any two smooth collections of spanning 3-discs for the trivial 2-link in S^4 become isotopic rel. boundary after pushing them into D^5.

1. Introduction

Let $U_m \subseteq S^4$ be the trivial m-component 2-link, that is a smooth submanifold of S^4 homeomorphic to a disjoint union $\sqcup^m S^2$. A collection of smoothly embedded 3-discs $D_m \subseteq S^4$ with $D_m \cong \sqcup^m D^3$ and $\partial D_m = U_M$ is called a *spanning 3-disc collection* for U_m.

Theorem 1.1. *Let D^0_m and D^1_m be spanning 3-disc collections for the trivial 2-link U_m in S^4. Then including $S^4 \subseteq D^5$, D^0_m and D^1_m become smoothly isotopic in D^5, relative to U_m.*

Remark 1.2. In the case that $m = 1$, i.e. spanning 3-discs for the trivial 2-knot, this was proven by Daniel Hartman in [H]. An alternative argument was given in the introduction to Hughes-Kim-Miller [HKM]. The case of multiple connected components is new. The proof here is analogous to the proof that slice discs for Alexander polynomial one knots are unique up to topological isotopy from my work with Conway [CP]. But since this is one dimension up, the result holds in the smooth category. In addition, we can work with free groups of arbitrary rank, whereas in the 4-dimensional case we needed to work in a setting where the fundamental group is good.

Remark 1.3. Let me mention how this question arose in the modern context. Budney-Gabai [BG] showed that there are spanning 3-discs for U_1 that are not isotopic rel. boundary in S^4, but which become isotopic in D^5. Hughes-Kim-Miller [HKM] showed that there are genus ≥ 2 handlebodies in S^4 with the same boundary that are not isotopic rel. boundary, and remain not isotopic rel. boundary after they are pushed into D^5. Hartman's result shows that the Budney-Gabai examples are optimal in this sense. The genus one case is open. It might be worthwhile to try to understand the [HKM] examples using the surgery programme.

2. Proof of Theorem 1.1

All manifolds and embeddings are assumed to be smooth. Fix $m \geq 1$. Push D^0_m and D^1_m into D^5, and by an abuse of notation denote these pushed in copies also by D^0_m and D^1_m respectively. Write $W_i := D^5 \setminus \nu D^i_m$ for $i = 0, 1$.

Lemma 2.1. *For $i = 0, 1$, there is a degree one normal map of pairs $(f_i, b_i)\colon (W_i, \partial W_i) \to (\natural^m S^1 \times D^4, \#^m S^1 \times S^3)$ such that f_i is a homotopy equivalence that restricts to a diffeomorphism on ∂W_i.*

Proof. There is a diffeomorphism $\partial W_i \to \#^m S^1 \times S^3$, that restricts to diffeomorphisms between the two copies of $S^4 \setminus \nu U_m$ and between the two copies of

$$\sqcup^m D^3 \times S^1 \cong \partial W_i \setminus (S^4 \setminus \overline{\nu} U_m).$$

Extend this to a diffeomorphism of a collar neighbourhood $\partial W_i \times [0, 1]$ to $(\#^m S^1 \times S^3) \times [0, 1]$. Compose with the projection $(\#^m S^1 \times S^3) \times [0, 1] \to \#^m S^1 \times S^3$ followed by a standard Pontryagin-Thom type map $\#^m S^1 \times S^3 \to \vee^m S^1$ to obtain a map $\partial W_i \times [0, 1] \to \vee^m S^1$. Using obstruction theory, extend this to a map $g\colon W_i \to \vee^m S^1$. Here note that $\pi_i(\vee^m S^1) = 0$ for $i > 1$, so as long as we define the map to $\vee^m S^1$ correctly on the 1-cells, the rest of the obstruction theory proceeds without hindrance. By taking some care, we can do this in such a way that the trace of the push of the kth component of D^i_m into D^5, intersected with W_i, coincides with the inverse image of a point in the kth wedge summand of $\vee^m S^1$.

Next, the open collar neighbourhood $\partial W_i \times [0,1)$ maps to $(\natural^m S^1 \times D^4) \setminus (\vee^m S^1)$ via a diffeomorphism. Send the rest of W_i to the core $\vee^m S^1$ using g. I shall argue that this is a homotopy equivalence. Note that $\pi_1(W_i) \cong F_m$, the free group of rank m. The map we have defined induces an isomorphism $\pi_1(W_i) \xrightarrow{\cong} \pi_1(\natural^m S^1 \times D^4)$.

We can push D^i_m into D^5 such that the radial function restricted to D^i_m is a Morse function with m critical points, each of which has index zero. It follows that the exterior has a handle decomposition with a single 0-handle and m 1-handles, and is therefore diffeomorphic (without any control on the diffeomorphism on the boundary) to $\natural^m S^1 \times D^4$. In particular W_i is homotopy equivalent to $\vee^m S^1$, and so any map $W_i \to \vee^m S^1$ inducing an isomorphism on π_1 is a homotopy equivalence by Whitehead's theorem.

Since $\natural^m S^1 \times D^4$ and W_i have trivial tangent bundles, our homotopy equivalence can be augmented with the necessary bundle data to obtain a normal map. □

Lemma 2.2. *The degree one normal maps (f_i, b_i), for $i = 0, 1$, are degree one normally bordant over $(\natural^m S^1 \times D^4) \times [0,1]$, via a cobordism that restricts to a product cobordism between ∂W_0 and ∂W_1.*

Proof. We claim that the surgery obstruction map $\mathcal{N}(\natural^m S^1 \times D^4, \#^m S^1 \times S^3) \to L_5(\mathbb{Z}[F_m]) \cong \oplus^m \mathbb{Z}$ is injective. To see the claim, recall that the set of normal bordism classes are in one to one correspondence with $[(\natural^m S^1 \times D^4, \#^m S^1 \times S^3), (G/O, *)]$. The potential obstructions to a normal bordism lie in

$$H^2(\natural^m S^1 \times D^4, \#^m S^1 \times S^3; \mathbb{Z}/2) \cong H_3(\natural^m S^1 \times D^4; \mathbb{Z}/2) = 0$$

and

$$H^4(\natural^m S^1 \times D^4, \#^m S^1 \times S^3; \mathbb{Z}) \cong H_1(\natural^m S^1 \times D^4; \mathbb{Z}) \cong \mathbb{Z}^m.$$

The latter obstruction is given as follows. For each i, consider the map $W_i \to \vee^m S^1$ constructed in the previous proof. Take the inverse image of a regular point in the kth S^1 wedge summand in $\vee^m S^1$. This gives a 4-manifold with boundary S^3, X^k_i, say. The boundary is S^3 because of the construction of the map $\partial W_i \to \vee^m S^1$ above. Then consider the difference in signatures $\sigma(X^k_1) - \sigma(X^k_0) \in \mathbb{Z}$. The degree one normal maps (f_1, b_1) and (f_0, b_0) are normally bordant if and only if $\sigma(X^k_1) - \sigma(X^k_0) = 0$ for $k = 1, \dots, m$. On the other hand the surgery obstruction group is $L_5(\mathbb{Z}[F_m]) \cong \oplus^m L_4(\mathbb{Z}[F_m]) \cong \oplus^m \mathbb{Z}$, and the image of a degree one normal map in $\oplus^m \mathbb{Z}$ is detected by $(\sigma(X^k_1) - \sigma(X^k_0))^m_{k=1}$. Thus the surgery obstruction map $\mathcal{N}(\natural^m S^1 \times D^4, \#^m S^1 \times S^3) \to L_5(\mathbb{Z}[F_m])$ is injective as claimed.

Since the f_i are homotopy equivalences, i.e. (f_i, b_i) lies in the image of the structure set in the surgery sequence, the surgery obstructions in $L_5(\mathbb{Z}[F_m])$ of (f_i, b_i) both vanish. By the claim, (f_0, b_0) and (f_1, b_1) are normally bordant. In other words, a normal bordism

$$\begin{aligned}(F,B)\colon \big(Z, W_0 \cup (\partial W_0 \times [0,1]) \cup W_1\big) \to \\ \big((\natural^m S^1 \times D^4) \times [0,1], (\natural^m S^1 \times D^4) \cup (\#^m S^1 \times S^3 \times [0,1]) \cup (\natural^m S^1 \times D^4)\big)\end{aligned}$$

exists as desired. □

Lemma 2.3. *The exteriors W_0 and W_1 are s-cobordant rel. boundary.*

Proof. Since the Whitehead group of F_m is trivial, we can ignore decorations on L theory, and every h-cobordism is an s-cobordism. Since $F|_{\partial Z}$ is a homotopy equivalence, there is an obstruction in $L_6(\mathbb{Z}[F_m]) \cong L_6(\mathbb{Z}) \oplus \bigoplus^m L_5(\mathbb{Z}) \cong \mathbb{Z}/2$ to surgering (F, B) relative to the boundary to an h-cobordism. However the nontrivial element of $L_6(\mathbb{Z})$ is realised by a degree one normal map $S^3 \times S^3 \to S^6$, so we can take connected sum of (F, B) with this to kill the obstruction, if necessary. Once we have arranged for the surgery obstruction in $L_6(\mathbb{Z}[F_m])$ to vanish, one can surger (F, B) until the domain becomes an h-cobordism, which as noted above is necessarily an s-cobordism. □

Remark 2.4. Note that the combination of the proofs of the last two lemmas show that the structure set $\mathcal{S}(\natural^m S^1 \times D^4, \#^m S^1 \times S^3)$ is trivial.

We can now complete the proof of Theorem 1.1. By the s-cobordism theorem, there is a diffeomorphism $W_0 \cong W_1$ that restricts to the composite $\partial W_0 \cong \#^m S^1 \times S^3 \cong \partial W_1$, using the original diffeomorphisms from Lemma 2.1. The boundary of $W_i \subseteq D^5$ splits as

$$\partial W_i = S^4 \setminus \nu U_m \cup \bigsqcup^m D^3 \times S^1.$$

Glue in $\sqcup^m D^3 \times D^2$ along $\sqcup^m D^3 \times S^1 \subseteq \partial W_i$. This recovers D^5, with $\sqcup^m D^3 \times \{0\}$ sent to D^i_m.

Extend the diffeomorphism $W_0 \cong W_1$ across $\sqcup^m D^3 \times D^2$, to obtain a diffeomorphism Ψ from D^5 to itself, restricting to Id_{S^4} on the boundary, and mapping D^0_m to D^1_m. Now use that every such diffeomorphism of the 5-disc is isotopic rel. boundary to the identity (this is equivalent to there being no exotic 6-spheres), to isotope the diffeomorphism to the identity. The resulting isotopy $\Phi_t \colon D^5 \to D^5$ satisfies that $\Phi_0 = \Psi$, $\Phi_t|_{S^4} = \mathrm{Id}_{S^4}$, and $\Phi_1 = \mathrm{Id}_{D_5}$. Therefore $\Phi_0(D^0_m) = \Phi(D^0_m) = D^1_m$, while $\Phi_1(D^0_m) = \mathrm{Id}(D^0_m) = D^0_m$. It follows that $D^t_m := \Phi_t(D^0_m)$ is a 1-parameter smooth family of collections of smoothly embedded 3-discs interpolating between D^1_m and D^0_m. Thus D^0_m and D^1_m are isotopic.

Remark 2.5. It was not really necessary for the rel. boundary smooth mapping class group of D^5 to be trivial. We could have avoided applying this theorem by instead composing the given diffeomorphism with a map isotopic to its inverse, with the inverse shrunk down to be supported in a small D^5 away from the D^i_m. This composite would then be isotopic to the identity.

Acknowledgements. Thanks to Anthony Conway for helpful comments on a first draft, and SeungWon Kim and Maggie Miller for suggesting to extend to the multi-component case. I was partially supported by EPSRC grants EP/T028335/1 and EP/V04821X/1.

References

[BG] Ryan Budney and David Gabai, *Knotted 3-balls in S^4*. Preprint, available at arXiv:1912.09029.

[CP] Anthony Conway and Mark Powell, *Characterisation of homotopy ribbon discs.* Advances in Mathematics, Volume 391, 19 November 2021, 107960.

[H] Daniel Hartman, *Unknotting 3-balls in the 5–ball.* Preprint, available at arXiv:2206.11243.

[HKM] Mark Hughes, SeungWon Kim, and Maggie Miller, *Knotted handlebodies in the 4-sphere and 5-ball.* Preprint, available at arXiv:2111.13255.

[W] C.T.C. Wall, *Surgery on compact manifolds*, 1970.

School of Mathematics and Statistics, University of Glasgow, United Kingdom
Email address: mark.powell@glasgow.ac.uk