SPANNING 3-DISCS IN THE 4-SPHERE PUSHED INTO THE 5-DISC

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Abstract. I prove that any two smooth spanning discs for the trivial 2-knot in S^4 become isotopic rel. boundary after pushing them into D^5.

1. Introduction

Let $U \subseteq S^4$ be the trivial 2-knot. A smoothly embedded 3-disc $D \subseteq S^4$ with $\partial D = U$ is called a *spanning 3-disc* for U.

Theorem 1.1. *Let D_0 and D_1 be spanning 3-discs for the trivial 2-knot U in S^4. Then including $S^4 \subseteq D^5$, D_0 and D_1 become smoothly isotopic in D^5, relative to U.*

Remark 1.2. This was proven by Daniel Hartman in [H], and after writing this I noticed that an alternative argument was given in the latest version of the introduction to [HKM]. The purpose of this note is to explain an outline of a proof for this fact via the Browder-Novikov-Sullivan-Wall surgery theory [W]. This is invoked in the proof in [HKM], since it cites unknotting theorems, so the mathematical content of the proof I give here is similar to that proof. The difference is that I argue directly using the surgery method, whereas [HKM] cite surgery arguments as a black box and then deduce by a pushing argument that the relative case follows from the closed case.

My proof is analogous to the proof that slice discs for Alexander polynomial one knots are unique up to topological isotopy from Conway-Powell [CP]. But since this is one dimension up, everything also works in the smooth category.

Remark 1.3. Let me mention how this result arose in the modern context. Budney-Gabai [BG] showed that there are spanning 3-discs for U that are not isotopic rel. boundary in S^4. Hughes-Kim-Miller [HKM] showed that there are genus ≥ 2 handlebodies in S^4 with the same boundary that are not isotopic rel. boundary, and remain not isotopic rel. boundary after they are pushed into D^5. The genus one case is open. It might be worthwhile to try to understand the [HKM] examples using the surgery programme.

2. Proof of Theorem 1.1

All manifolds and embeddings are assumed to be smooth. Push D_0 and D_1 into D^5, and by an abuse of notation denote these pushed in copies also by D_0 and D_1 respectively. Write $W_i := D^5 \setminus \nu D_i$ for $i = 0, 1$.

Lemma 2.1. *For $i = 0, 1$, there is a degree one normal map of pairs $(f_i, b_i)\colon (W_i, \partial W_i) \to (S^1 \times D^4, S^1 \times S^3)$ such that f_i is a homotopy equivalence that restricts to a diffeomorphism on ∂W_i.*

Proof. There is a diffeomorphism $\partial W_i \to S^1 \times S^3$, that restricts to diffeomorphisms between the two copies of $S^4 \setminus \nu U$ and between the two copies of

$$D^3 \times S^1 \cong \partial W_i \setminus (S^4 \setminus \overline{\nu} U).$$

Extend this to a diffeomorphism of a collar neighbourhood $\partial W_i \times [0, 1]$ to $S^1 \times S^3 \times [0, 1]$. Compose with the projection to obtain a map $\partial W_i \times [0, 1] \to S^1$. Using obstruction theory, extend this to a map $g\colon W_i \to S^1$. We can do this in such a way that the trace of the push of D_i into D^5, intersected with W_i, coincides with the inverse image of a point in S^1.

Next, the collar neighbourhood $\partial W_i \times [0, 1)$ maps to $(S^1 \times D^4) \setminus (S^1 \times \{0\})$ via a diffeomorphism. Send the rest of W_i to the core $S^1 \times \{0\}$ using g. I shall argue that this is a homotopy equivalence. Note that $\pi_1(W_i) \cong \mathbb{Z}$. The map we have defined induces an isomorphism $\pi_1(W_i) \xrightarrow{\cong} \pi_1(S^1 \times D^4)$.

We can push D_i into D^5 such that the radial function restricted to D_i is a Morse function with a unique critical point, which has index zero. It follows that the exterior has a handle decomposition with a single 0-handle and a single 1-handle, and is therefore diffeomorphic (without any control on the diffeomorphism on the boundary) to $S^1 \times D^4$. In particular W_i is homotopy equivalent to S^1, and so any map $W_i \to S^1$ inducing an isomorphism on π_1 is a homotopy equivalence by Whitehead's theorem.

Since $S^1 \times D^4$ and W_i have trivial tangent bundles, our homotopy equivalence can be augmented with the necessary bundle data to obtain a normal map. □

Lemma 2.2. *The degree one normal maps (f_i, b_i), for $i = 0, 1$, are degree one normally bordant over $S^1 \times D^4 \times [0,1]$, via a cobordism that restricts to a product cobordism between ∂W_0 and ∂W_1.*

Proof. The surgery obstruction map $\mathcal{N}(S^1 \times D^4, S^1 \times S^3) \to L_5(\mathbb{Z}[\mathbb{Z}]) \cong \mathbb{Z}$ is injective. Since the f_i are homotopy equivalences, i.e. (f_i, b_i) lies in the image of the structure set in the surgery sequence, the surgery obstructions of (f_i, b_i) both vanish. Therefore a normal bordism

$$(F, B)\colon \big(Z, W_0 \cup (\partial W_0 \times [0,1]) \cup W_1\big) \to \big(S^1 \times D^4 \times [0,1], S^1 \times D^4 \cup (S^1 \times S^3 \times [0,1]) \cup S^1 \times D^4\big)$$

exists as claimed. □

Lemma 2.3. *The exteriors W_0 and W_1 are s-cobordant rel. boundary.*

Proof. Since the Whitehead group of $\mathbb{Z}$ is trivial, we can ignore decorations on L theory, and every h-cobordism is an s-cobordism. Since $F|_{\partial Z}$ is a homotopy equivalence, there is an obstruction in $L_6(\mathbb{Z}[\mathbb{Z}]) \cong L_6(\mathbb{Z}) \oplus L_5(\mathbb{Z}) \cong \mathbb{Z}/2$ to surgering (F, B) relative to the boundary to an h-cobordism. However the nontrivial element of $L_6(\mathbb{Z})$ is realised by a degree one normal map $S^3 \times S^3 \to S^6$, so we can take connected sum with this to kill the obstruction, if necessary. Once we have arranged for the surgery obstruction in $L_6(\mathbb{Z}[\mathbb{Z}])$ to vanish, one can surger (F, B) until the domain becomes an h-cobordism, which as noted above is necessarily an s-cobordism. □

Remark 2.4. Note that the combination of the proofs of the last two lemmas show that the structure set $\mathcal{S}(S^1 \times D^4, S^1 \times S^3)$ is trivial.

We can now complete the proof of Theorem 1.1. By the s-cobordism theorem, there is a diffeomorphism $W_0 \cong W_1$ that restricts to the composite $\partial W_0 \cong S^1 \times S^3 \cong \partial W_1$, using the original diffeomorphisms from Lemma 2.1. The boundary of $W_i \subseteq D^5$ splits as $\partial W_i = S^4 \setminus \nu U \cup D^3 \times S^1$. Glue in $D^3 \times D^2$ along $D^3 \times S^1 \subseteq \partial W_i$. This recovers D^5, with $D^3 \times \{0\}$ sent to D_i.

Extend the diffeomorphism $W_0 \cong W_1$ across $D^3 \times D^2$, to obtain a diffeomorphism Ψ from D^5 to itself, restricting to Id_{S^4} on the boundary, and mapping D_0 to D_1. Now use that every such diffeomorphism of the 5-disc is isotopic rel. boundary to the identity (this is equivalent to there being no exotic 6-spheres), to isotope the diffeomorphism to the identity. The resulting isotopy $\Phi_t \colon D^5 \to D^5$ satisfies that $\Phi_0 = \Psi$, $\Phi_t|_{S^4} = \mathrm{Id}_{S^4}$, and $\Phi_1 = \mathrm{Id}_{D_5}$. Therefor $\Phi_0(D_0) = \Phi(D_0) = D_1$, while $\Phi_1(D_0) = \mathrm{Id}(D_0) = D_0$. It follows that $D_t := \Phi_t(D_0)$ is a 1-parameter smooth family of smoothly embedded discs interpolating between D_1 and D_0. Thus D_0 and D_1 are isotopic.

Remark 2.5. It was not really necessary for the rel. boundary smooth mapping class group of D^5 to be trivial. We could have avoided applying this theorem, by instead composing the given diffeomorphism with its inverse but with the inverse shrunk down to be supported in a small D^5 away from the D_i. This composite would then be isotopic to the identity.

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