STABLY EXOTIC FILLINGS OF 3-MANIFOLDS

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Abstract. We investigate which 3-manifolds bound 4-manifolds that are homeomorphic but not stably diffeomorphic, where stabilising means taking connected sum with copies of $S^2 \times S^2$. We show that every closed, orientable 3-manifold admits such fillings, as do certain families of nonorientable 3-manifolds. In contrast we show that for a 3-manifold containing a 2-sided $\mathbb{RP}^2$, any two smooth, homeomorphic fillings are stably diffeomorphic.

1. Introduction

A *filling* of a closed, smooth 3-manifold Y is a smooth, compact 4-manifold M with $\partial M = Y$. We prove the following.

Theorem 1.1. *Let Y be a closed, smooth 3-manifold. Then Y admits a pair of fillings that are homeomorphic rel. boundary but not stably diffeomorphic rel. boundary if one of the following holds.*

(i) Y is orientable.
(ii) Y is a circle bundle over a closed surface $F \neq \mathbb{RP}^2$.
(iii) Y admits a null-bordant tangential Pin^+ *structure.*

Here two fillings M and M' of Y are *stably diffeomorphic rel. boundary* if there exists $k \geq 0$ such that $M\#k(S^2 \times S^2)$ and $M'\#k(S^2 \times S^2)$ are diffeomorphic via a diffeomorphism restricting to the identity on the boundary. Stable homeomorphism, which is used later, is defined analogously. Note that the fillings in Theorem 1.1 are in particular exotic fillings. By a theorem of Gompf [Gom84], homeomorphism implies stable diffeomorphism for compact, orientable 4-manifolds, so the fillings we produce in Theorem 1.1 are necessarily nonorientable.

Every orientable 3-manifold admits a null-bordant tangential Pin^+ structure, so Theorem 1.1 (i) is implied by Theorem 1.1 (iii). We state (i) separately for emphasis, and because we give the proof for this case separately, in Section 3.

Our examples will be of the form

$$M = Z\#K_3 \text{ and } M' = Z\#11(S^2 \times S^2),$$

for a suitable nonorientable filling Z of Y, where K_3 denotes the Kummer surface. The first examples of this sort, for closed 4-manifolds, are due to Kreck [Kre84]. Using that K_3 splits topologically as $E_8\#E_8\#3(S^2 \times S^2)$, we will show in Lemma 3.1 that M is homeomorphic to M'. But by Rochlin's theorem, there is no such smooth splitting of K_3. This is used in the proof of Theorem 1.1, for Y satisfying (i), (ii), or (iii), and well-chosen fillings Z, to show that the union $M \cup_Y M'$ is nontrivial when considered in the kernel $\ker(\Omega_4(\xi_G) \to \Omega_4(\xi_G^{\mathrm{Top}}))$ of the forgetful map between smooth and topological bordism groups over certain fibrations, defined in Section 2 and dependent on $G = \pi_1(Z)$. Using a result of Kreck [Kre99, Corollary 3] (see Theorem 2.4 for details), we show how unions in this kernel give rise to the eponymous *stably exotic fillings* of Y, namely fillings that are stably homeomorphic but not stably diffeomorphic, both rel. boundary. Working stably is thus the most natural setting in which to isolate this type of exotic behaviour, arising from Rochlin's theorem, since it factors out the effect of gauge theory.

The above strategy does not always produce stably exotic fillings, and indeed not all 3-manifolds admit them.

Theorem 1.2. *Let Y be a closed, smooth 3-manifold that contains a two-sided $\mathbb{RP}^2$. Then Y does not admit stably exotic fillings.*

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Theorems 1.1 and 1.2 partially answer the following question, which remains open in general.

Question 1.3. Which closed, smooth 3-manifolds admit stably exotic fillings?

For example, we do not know whether the 3-manifold constructed in Example 6.8, a T^2-bundle over S^1, admits stably exotic fillings.

We also remark that it is open which 3-manifolds admit exotic fillings, i.e. fillings that are homeomorphic rel. boundary but not diffeomorphic rel. boundary; cf. [EMM22; BKR26, Problem 4.6]. Another related open question asks which orientable 3-manifolds admit exotic orientable fillings.

Proof strategy. We now elaborate further on the proofs of Theorems 1.1 and 1.2. For the argument outlined on the previous page to work, at a minimum the kernel of $\Omega_4(\xi_G) \to \Omega_4(\xi_G^{\mathrm{Top}})$ must have a nontrivial element. Given any group G, together with a choice of abstract *normal* 1*-type data* (see Section 2), we can hope to characterise precisely when this kernel is nontrivial, by considering differentials in the James spectral sequence computing $\Omega_4(\xi_G)$, as in [KP25]. Nontriviality of the kernel turns out to be equivalent to $[K_3] \neq 0$ in $\Omega_4(\xi_G)$. Given a 3-manifold Y, we can then seek to find, as we will do in the proof of Theorem 1.1, a suitable group G with normal 1-type data compatible with Y, and a suitable nonorientable filling Z with $\pi_1(Z) = G$, to realise the nontrivial element of the kernel by the manifold $M \cup_Y M'$ described above. This will prove we have constructed stably exotic fillings. In contrast, in the proof of Theorem 1.2, we show that in the presence of $\mathbb{RP}^2 \times I \subseteq Y$, for any possible ξ_G we will have $[K_3] = 0 \in \Omega_4(\xi_G)$. See Section 2 for more information on ξ_G and ξ_G^{Top}, and Section 4 for our compatibility condition for normal 1-type data with Y.

The cardinality of stably exotic fillings. To supplement the results above, within a fixed (stable) homeomorphism class of fillings, we show in Proposition 8.1 that there are at most two stable diffeomorphism classes rel. boundary. So it is not possible to find any larger classes of stably exotic fillings than we find in Theorem 1.1. On the other hand, in Proposition 8.3 we show that modifying any given stably exotic pair by connected summing both with copies of $S^1 \times S^3$, one can obtain infinitely many stable homeomorphism classes of pairs of stably exotic fillings for any fixed Y that admits them, distinguished by their fundamental groups.

Examples. Now we give some examples designed to illustrate the use of condition (iii) in Theorem 1.1 and to compare the three conditions of Theorem 1.1. As mentioned earlier, Theorem 1.1 (iii) implies Theorem 1.1 (i). There is some overlap between (i) and (ii): for example, (i) implies (ii) when $F = S^2$. But in general both set differences between conditions (i) and (ii) are nonempty. Since Theorem 1.1 (ii) does not imply Theorem 1.1 (i) while Theorem 1.1 (iii) does imply Theorem 1.1 (i), one sees that Theorem 1.1 (ii) does not imply Theorem 1.1 (iii).

Condition (iii) in Theorem 1.1 may seem difficult to check at first glance, in which case the following observations, which we use in the proofs of the upcoming examples, may help. First, Y admits a tangential Pin^+ structure if and only if the orientation character $w_1^Y : \pi_1(Y) \to \mathbb{Z}/2$ factors through $\mathbb{Z}/4$ (Proposition 5.9). Second, a Pin^+ 3-manifold is null-bordant if and only if the spin surface dual to the orientation character is spin null-bordant (Proposition 5.10).

In the following example, we describe a family of Pin^+ nonorientable 3-manifolds admitting stably exotic fillings, by applying Theorem 1.1 (iii). This demonstrates that there exist nonorientable 3-manifolds admitting a null-bordant tangential Pin^+ structure, so Theorem 1.1 (i) does not imply Theorem 1.1 (iii).

Example 1.4. Let F be a closed, orientable surface and let $F \to Y \to S^1$ be a fibre bundle with orientation-reversing monodromy $\varphi\colon F \to F$. Note that Y is nonorientable by construction. Suppose that the fixed subgroup $H^1(F;\mathbb{Z}/2)^{\varphi^*}$ is at least half-rank in $H^1(F;\mathbb{Z}/2)$. We show in Proposition 6.6 that Y admits a null-bordant Pin^+ structure, and hence admits stably exotic fillings by Theorem 1.1 (iii). We show in Example 6.8 that the statement is false in general without the rank assumption on $H^1(F;\mathbb{Z}/2)^{\varphi^*}$.

Next we give an example of a circle bundle over a surface ($\neq \mathbb{RP}^2$) which does not admit a tangential Pin^+ structure. This shows that Theorem 1.1 (iii) does not imply Theorem 1.1 (ii).

Example 1.5. Let $F = \#^3\mathbb{RP}^2$ be the closed surface of nonorientable genus three and $Y := S^1 \times F$. In Proposition 6.9 we show that the orientation character $w_1^Y : \pi_1(Y) \to \mathbb{Z}/2$ does not factor through $\mathbb{Z}/4$. By Proposition 5.9, this shows that Y does not admit a tangential Pin^+ structure.

We have therefore shown that the only logical dependency between the items of Theorem 1.1 is the aforementioned fact that Theorem 1.1 (iii) implies Theorem 1.1 (i).

Organisation. In Section 2 we introduce the key concept of normal 1-type data and explain the significance of this concept to the paper. In Section 3 we prove the orientable case of Theorem 1.1. In Section 4 we relate the existence of stably exotic fillings to the existence of stably exotic closed 4-manifolds, under certain hypotheses. The main result is the partial characterisation in Theorem 4.5. In Section 5.1, we apply Theorem 4.5 to prove Theorem 1.1 (ii), the circle bundle case. In Section 5.2, we prove Theorem 1.1 (iii). In Section 6 we provide the examples promised in Examples 1.4 and 1.5. In Section 7, we prove Theorem 1.2. Finally in Section 8, we consider the number of stably exotic fillings of a given 3-manifold.

Conventions. We assume that all 3-manifolds are closed, connected, and smooth, and that all 4-manifolds are compact, connected, and smooth.

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2. Normal 1-type data

Given a 3-manifold Y, our strategy to prove that it admits stably exotic fillings requires, of course, the construction of stably homeomorphic fillings M and M', together with some obstruction to stable diffeomorphism. The first component will be furnished by a lemma of Kreck, detailed below as Lemma 3.1. For the second component, we exploit the fact that stably homeomorphic fillings M and M' are stably diffeomorphic rel. Y if and only if $M \cup_Y M'$ is null-bordant over the normal 1-type of M (which is the same as the normal 1-type of M' as they are stably homeomorphic); see Theorem 2.4. We will first recall some general facts about normal 1-types of manifolds [Kre99]. We then recall some specific ideas for working with normal 1-types of manifolds with spin universal cover, developed by Teichner [Tei92].

2.1. Normal 1-types.

Definition 2.1. A *normal* 1*-type* for a smooth manifold W is a pair (B, ξ), where B is a space with the homotopy type of a CW complex, $\xi \colon B \to \mathrm{BO}$ is a 2-coconnected fibration with connected fibre, and there exists a 2-connected map $\overline{\nu}_W \colon W \to B$, such that $\xi \circ \overline{\nu}_W$ represents the stable normal bundle $\nu_W \colon W \to \mathrm{BO}$ of W. Such a pair $(W, \overline{\nu}_W)$ is called a *normal* 1*-smoothing of* W.

A *topological normal* 1*-type* for a manifold W is a pair $(B^{\mathrm{Top}}, \xi^{\mathrm{Top}})$, where B^{Top} is a space with the homotopy type of a CW complex, $\xi^{\mathrm{Top}} \colon B^{\mathrm{Top}} \to \mathrm{BTop}$ is a 2-coconnected fibration with connected fibre, and there exists a 2-connected map $\overline{\nu}_W^{\mathrm{Top}} \colon W \to B^{\mathrm{Top}}$, such that $\xi^{\mathrm{Top}} \circ \overline{\nu}_W^{\mathrm{Top}}$ represents the stable normal microbundle $\nu_W^{\mathrm{Top}} \colon W \to \mathrm{BTop}$ of W. Such a pair $(W, \overline{\nu}_W^{\mathrm{Top}})$ is called a *topological normal* 1*-smoothing of* W.

Here BTop is the classifying space for Top, the colimit under inclusion of the groups $\mathrm{Top}(n)$ for $n \geq 0$, consisting of homeomorphisms of $\mathbb{R}^n$ preserving 0.

Remark 2.2. It follows from the Moore–Postnikov factorisation (see e.g. [Bau77]) that every smooth manifold W has a corresponding normal 1-type, and similarly every topological manifold W admits a topological normal 1-type. This theory also guarantees uniqueness, up to fibre homotopy equivalence, of these objects.

Definition 2.3. Let W be a smooth manifold and let (B, ξ) be the normal 1-type of some smooth manifold, not necessarily of W. A lift of the stable normal bundle of W to B is called a ξ*-structure*.

The relevance of the normal 1-type technology to this article comes from the following theorem.

Theorem 2.4 ([Kre99, Corollary 3; CS11, Lemma 2.2])**.**

(i) Let M and M' be compact, smooth 4-manifolds with fixed identifications $\partial M = Y = \partial M'$, and the same normal 1-type $\xi\colon B \to \mathrm{BO}$. Then M and M' are stably diffeomorphic rel. Y if and only if there exist normal 1-smoothings $\overline{\nu}_M$ and $\overline{\nu}_{M'}$, of M and M' respectively, that agree on Y, such that

$$[M \cup_Y M', \overline{\nu}_M \cup -\overline{\nu}_{M'}] = 0 \in \Omega_4(\xi).$$

(ii) Let M and M' be compact, topological 4-manifolds with fixed identifications $\partial M = Y = \partial M'$, and the same topological normal 1-type $\xi^{\mathrm{Top}}\colon B^{\mathrm{Top}} \to \mathrm{BTop}$. Then M and M' are stably homeomorphic rel. Y if and only if there exist topological normal 1-smoothings $\overline{\nu}_M^{\mathrm{Top}}$ and $\overline{\nu}_{M'}^{\mathrm{Top}}$, of M and M' respectively, that agree on Y, such that

$$[M \cup_Y M', \overline{\nu}_M^{\mathrm{Top}} \cup -\overline{\nu}_{M'}^{\mathrm{Top}}] = 0 \in \Omega_4(\xi^{\mathrm{Top}}).$$

2.2. **Normal 1-type data.** When the universal cover of a manifold W is spin, there exists a 2-connected classifying map $c\colon W \to B\pi_1(W)$, and unique cohomology classes $w_i^W \in H^i(\pi_1(W);\mathbb{Z}/2)$, for $i = 1, 2$, such that $c^*(w_i^W) = w_i(\nu_W)$, where ν_W denotes the stable normal bundle of W. In this case, the fibre homotopy type of the normal 1-type is determined by the isomorphism class of the triple $(\pi_1(W), w_1^W, w_2^W)$, in both the smooth and topological categories [Tei92, Theorem 2.2.1 (a)].

Using this, we can describe an algebraic abstraction that will permit us to consider putative 4-manifold fillings of 3-manifolds. The following definitions are due to Teichner [Tei92, Theorem 2.2.1 (b)(IV)].

Definition 2.5. A triple (G, w, v) consisting of a finitely presented group G, an element $w \in H^1(G;\mathbb{Z}/2)$, and an element $v \in H^2(G;\mathbb{Z}/2)$ is called *normal 1-type data*.

Given normal 1-type data, we define an associated pair (B_G, ξ_G), where B_G is a space homotopy equivalent to a CW complex and $\xi_G\colon B_G \to \mathrm{BO}$ is a fibration, as follows. Represent (w, v) via a map $(w, v)\colon K(G, 1) \to K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2)$. Let $P_2(-)\colon \mathrm{BO} \to K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2)$ denote the map from BO to its Postnikov 2-type. We write B_G for the homotopy fibre in

$$B_G \xrightarrow{\zeta_G} \mathrm{BO} \times K(G, 1) \xrightarrow{P_2(-)+(w\times v)} K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2), \tag{2.6}$$

with ζ_G further assumed to be a fibration. Here the operation $+$ uses the H-space structure on $K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2)$. Define the fibration

$$\xi_G := \mathrm{pr}_1 \circ \zeta_G\colon B_G \to \mathrm{BO}\,.$$

We similarly define an associated pair $(B_G^{\mathrm{Top}}, \xi_G^{\mathrm{Top}})$, where B_G^{Top} is a space homotopy equivalent to a CW complex and $\zeta_G^{\mathrm{Top}}\colon B_G^{\mathrm{Top}} \to \mathrm{BTop}$ is a fibration, as follows. Write B_G^{Top} for the homotopy fibre in

$$B_G^{\mathrm{Top}} \xrightarrow{\zeta_G^{\mathrm{Top}}} \mathrm{BTop} \times K(G, 1) \xrightarrow{P_2(-)+(w\times v)} K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2), \tag{2.7}$$

with ζ_G^{Top} a fibration, again using the H-space structure on $K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2)$. Define the fibration

$$\xi_G^{\mathrm{Top}} := \mathrm{pr}_1 \circ \zeta_G^{\mathrm{Top}}\colon B_G \to \mathrm{BTop}\,.$$

Remark 2.8. Using the long exact sequence in homotopy groups of the fibrations in (2.6) and (2.7), it is straightforward to check that $\pi_1(B_G) \cong \pi_1(B_G^{\mathrm{Top}}) \cong G$, $\pi_2(B_G) = 0 = \pi_2(B_G^{\mathrm{Top}})$, and that both ξ_G and ξ_G^{Top} are 2-coconnected. Also, note that the map $\mathrm{BO} \to \mathrm{BTop}$ induces a forgetful map

$$\eta_G\colon B_G \to B_G^{\mathrm{Top}}.$$

For a smooth manifold W whose universal cover is spin, the respective normal 1-type and topological normal 1-type are

$$(B_W, \xi_W) = (B_G, \xi_G) \qquad \text{and} \qquad (B_W^{\mathrm{Top}}, \xi_W^{\mathrm{Top}}) = (B_G^{\mathrm{Top}}, \xi_G^{\mathrm{Top}}),$$

constructed using the normal 1-type data $(G := \pi_1(W), w_1^W, w_2^W)$, as described above [Tei92, Theorem 2.2.1 (b)(IV)]. This explains the following definition.

Definition 2.9. Let W be a smooth manifold with universal cover spin. If the abstract normal 1-type data (G, w, v) is isomorphic to $(\pi_1(W), w_1^W, w_2^W)$, we will say that W *has normal* 1*-type data* (G, w, v).

Given normal 1-type data (G, w, v), the bordism groups $\Omega_4(\xi_G)$ and $\Omega_4(\xi_G^{\mathrm{Top}})$ can in principle be computed using *James spectral sequences*. These have the form

$$E^2_{p,q} = H_p(G; (\Omega_q^{\mathrm{Spin}})^w) \Rightarrow \Omega_*(\xi_G)$$

and

$$(E^2_{p,q})^{\mathrm{Top}} = H_p(G; (\Omega_q^{\mathrm{TopSpin}})^w) \Rightarrow \Omega_*(\xi_G^{\mathrm{Top}}),$$

for $p, q \geq 0$, respectively. Here the coefficients are twisted using the class $w \in H^1(G; \mathbb{Z}/2)$ that pulls back to the orientation character. For ξ_G and ξ_G^{Top} orientable these were constructed by Teichner [Tei93, Section II]. For ξ_G and ξ_G^{Top} nonorientable, if there exists a vector bundle $E \to K(G,1)$ such that $w_1(E) = w \in H^1(G; \mathbb{Z}/2)$ and $w_2(E) = v \in H^2(G; \mathbb{Z}/2)$, then the adaptation of the James spectral sequence was constructed in [Tei97, Before Lemma 2]. The general case was constructed in detail by Galvin–Teichner–Veselá in [GTV26, Appendix B].

Remark 2.10. All 3-manifolds have universal cover spin because all orientable 3-manifolds are spin. The fillings of 3-manifolds we will produce will have normal 1-types corresponding to the, more algebraic, normal 1-type data we defined above. Hence we will not produce any fillings with non-spin universal cover. It is possible that a 3-manifold might have a pair of exotic fillings whose universal covers are non-spin. However such a pair would be stably diffeomorphic [Gom84; FNOP25, Theorem 13.3], so the methods of this paper would not detect them. Hence from now on we will only consider manifolds with spin universal cover.

3. All orientable 3-manifolds admit stably exotic fillings

We begin the proof of Theorem 1.1. We start with a lemma due to Kreck [Kre84], which gives the key construction of potentially exotic nonorientable 4-manifolds, making use of the K_3 surface. This lemma will be used in this section and in subsequent sections.

Lemma 3.1 ([Kre84])**.** *Let X be a compact, nonorientable 4-manifold. Let $M := X \# K_3$ and let $M' := X \# 11(S^2 \times S^2)$. Then M and M' are homeomorphic (and hence stably homeomorphic) rel. boundary.*

Proof. We have

$$M = X \# K_3 \cong X \# E_8 \# E_8 \# 3(S^2 \times S^2) \cong X \# E_8 \# \overline{E}_8 \# 3(S^2 \times S^2) \cong X \# 11(S^2 \times S^2) = M'.$$

Here we use the classification of closed, simply connected 4-manifolds [Fre82], and the fact that indefinite, symmetric, bilinear forms are determined up to isometry by their rank, signature, and parity [MH73], to see that $K_3 \cong E_8 \# E_8 \# 3(S^2 \times S^2)$ and that $E_8 \# \overline{E}_8 \# 3(S^2 \times S^2) \cong \# 11(S^2 \times S^2)$. To replace one copy of E_8 by $\overline{E}_8$ in the central homeomorphism we use that X is nonorientable, so that the orientation of the 4-ball in X used for the connected sum can be reversed by an ambient isotopy. □

Using this lemma and Theorem 2.4, we can prove Theorem 1.1 (i).

Theorem 3.2. *Let Y be a closed, orientable 3-manifold. Then Y admits a pair of stably exotic fillings M and M'.*

Proof. Since Y is orientable, it admits a compact, simply connected, spin filling X. Define

$$M := X \# (S^1 \widetilde{\times} S^3) \# K_3 \quad \text{and} \quad M' := X \# (S^1 \widetilde{\times} S^3) \# 11(S^2 \times S^2),$$

where $S^1 \widetilde{\times} S^3$ is the nonorientable S^3 bundle over S^1. Then M and M' are homeomorphic by Lemma 3.1, and they both have the normal 1-type data $(\mathbb{Z}, w, 0)$ with $0 \neq w \in H^1(\mathbb{Z}; \mathbb{Z}/2) \cong \mathbb{Z}/2$.

If M and M' were stably diffeomorphic relative to Y, then by Theorem 2.4 (ii) there would exist normal 1-smoothings $\overline{\nu}_M$ and $\overline{\nu}_{M'}$ of M and M' that agree on Y and such that $[M \cup_Y M', \overline{\nu}_M \cup -\overline{\nu}_{M'}] = 0 \in \Omega_4(\xi_{\mathbb{Z}})$. We show next that this is not the case.

Let $\overline{\nu}_M$ and $\overline{\nu}_{M'}$ be arbitrary normal 1-smoothings of M and M' that agree on Y. The group $\Omega_4(\xi_{\mathbb{Z}})$ is computed by a James spectral sequence with $E^2_{p,q} \cong H_p(\mathbb{Z}; (\Omega_q^{\mathrm{Spin}})^w)$; see [Tei92] and [GTV26, Appendix B]. Since $H_p(\mathbb{Z}; (\Omega_q^{\mathrm{Spin}})^w) = 0$ for $p > 1$ or $q = 3$, and $w \neq 0$, we have that

$$\Omega_4(\xi_{\mathbb{Z}}) \cong H_0(\mathbb{Z}; (\Omega_4^{\mathrm{Spin}})^w) \cong H_0(\mathbb{Z}; \mathbb{Z}^-) \cong \mathbb{Z}/2,$$

and the map $16\mathbb{Z} \cong \Omega_4^{\mathrm{Spin}} \to \Omega_4(\xi_{\mathbb{Z}}) \cong \mathbb{Z}/2$ is given by $16 \mapsto 1$. In particular, every element $[N, g]$ of $\Omega_4(\xi_{\mathbb{Z}})$ is bordant over $\xi_{\mathbb{Z}}$ to a simply connected, spin 4-manifold N' (with some $\xi_{\mathbb{Z}}$-structure, as in Definition 2.3), and $[N, g]$ is trivial if and only if $\sigma(N')/16 \in \mathbb{Z}/2$ is trivial, where $\sigma(N')$ denotes the signature.

For the pair $(N, g) = (M \cup_Y M', \overline{\nu}_M \cup -\overline{\nu}_{M'})$, such a simply connected $\xi_{\mathbb{Z}}$-bordant spin 4-manifold is provided by $N' = (X \# K_3) \cup_Y (X \# 11(S^2 \times S^2))$ (with some $\xi_{\mathbb{Z}}$-structure). To see this, we claim that, since $S^1 \widetilde{\times} S^3$ bounds $S^1 \widetilde{\times} D^4$, we have $[S^1 \widetilde{\times} S^3] = 0 \in \Omega_4(\xi_{\mathbb{Z}})$ for every choice of $\xi_{\mathbb{Z}}$-structure.

Now we prove the claim. First we note that $B_{\mathbb{Z}}$ is a homotopy fibre, so it suffices to show that the maps to BO and $K(\mathbb{Z}, 1)$ extend and the extensions are compatible in $K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2)$. The map to $K(\mathbb{Z}, 1) \simeq S^1$ extends by basic obstruction theory. Specifically, we note that $S^1 \widetilde{\times} D^4$ is obtained from $S^1 \widetilde{\times} S^3$ by adding a 4-cell and a 5-cell, and $\pi_3(S^1) = 0 = \pi_4(S^1)$. Furthermore, the stable normal bundle $\nu_{S^1 \widetilde{\times} D^4} \colon S^1 \widetilde{\times} D^4 \to \mathrm{BO}$ restricts to $\nu_{S^1 \widetilde{\times} S^3} \colon S^1 \widetilde{\times} S^3 \to \mathrm{BO}$ on the boundary, providing the desired extension. The extensions are compatible in $K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}/2, 2)$ since $w_1(S^1 \widetilde{\times} D^4) \neq 0$ and $w_2(S^1 \widetilde{\times} D^4) = 0$.

The signature of $(X \# K_3) \cup_Y -(X \# 11(S^2 \times S^2))$ is the signature of K_3, which is 16. Thus $[M \cup_Y M', \overline{\nu}_M \cup -\overline{\nu}_{M'}] \neq 0 \in \Omega_4(\xi_{\mathbb{Z}})$ and M is not stably diffeomorphic to M' rel. boundary, as claimed. □

4. A partial characterisation of 3-manifolds admitting stably exotic fillings

In order to prove Theorem 1.1 (ii) and (iii), first we deduce a partial characterisation of 3-manifolds admitting stably exotic fillings. As the universal cover of an arbitrary 3-manifold is spin, the 3-manifold determines normal 1-type data as described in Section 2.2. Now we introduce the notion of abstract normal 1-type data for a putative filling, whose normal 1-type data would interact correctly with the normal 1-type of Y.

Definition 4.1. Let Y be a closed, smooth 3-manifold. The quadruple (G, w, v, φ) is called *Y-filling normal 1-type data* if (G, w, v) comprises normal 1-type data (not necessarily for Y), and $\varphi \colon \pi_1(Y) \to G$ is a homomorphism such that $\varphi^*(w) = w_1^Y \in H^1(Y; \mathbb{Z}/2)$ and $\varphi^*(v) = w_2^Y \in H^2(Y; \mathbb{Z}/2)$.

In the definition above, one should think of G as the fundamental group of the putative filling of Y, the map φ as the putative inclusion-induced map, and w and v as the putative first and second Stiefel–Whitney classes of the filling.

Proposition 4.2. *Let (G, w, v) and (H, x, y) be normal 1-type data. Let $\phi \colon G \to H$ be a homomorphism such that $\phi^*(x) = w$ and $\phi^*(y) = v$. Then there exists a map $\phi_B \colon B_G \to B_H$ covering $\mathrm{Id}_{\mathrm{BO}} \times \phi \colon \mathrm{BO} \times K(G, 1) \to \mathrm{BO} \times K(H, 1)$, up to homotopy. Since ξ_H is a fibration, we can assume that $\xi_H \circ \phi_B = \xi_G$.*

Proof. The assumption that $\phi^*(x) = w$ and $\phi^*(y) = v$ implies that the diagram

$$\begin{tikzcd} \mathrm{BO} \times K(G,1) \arrow[r, "\mathrm{Id}_{\mathrm{BO}} \times \phi_*"] \arrow[d, "P_2(-)+(w \times v)"] & \mathrm{BO} \times K(H,1) \arrow[d, "P_2(-)+(x \times y)"] \\ K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2) \arrow[r, "\mathrm{Id}"] & K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2) \end{tikzcd}$$

commutes up to homotopy. Hence there is a map $\phi_B \colon B_G \to B_H$ of the homotopy fibres covering $\mathrm{Id}_{\mathrm{BO}} \times \phi \colon \mathrm{BO} \times K(G, 1) \to \mathrm{BO} \times K(H, 1)$ up to homotopy, as needed. □

Remark 4.3. When G is the trivial group, $B_G \simeq \mathrm{BSpin}$. Hence by choosing G to be the trivial group in Proposition 4.2, we obtain a map $\mathrm{BSpin} \to B_H$ over BO, for any normal 1-type data (H, x, y). This determines a map $\Omega_4^{\mathrm{Spin}} \to \Omega_4(B_H)$, and so every spin 4-manifold has a ξ_H-structure. In the proof of Theorem 3.2 we produced an instance of this map for $H = \mathbb{Z}$ by analysing the James spectral sequence; this method generalises to arbitrary H.

Corollary 4.4. *Let Y be a closed, smooth 3-manifold and let (H, x, y, φ) be Y-filling normal 1-type data. Then there is a map $\varphi_B \colon B_Y \to B_G$ over BO inducing φ on fundamental groups. Hence, for every normal 1-smoothing $\overline{\nu}_Y$ of Y, we obtain a corresponding ξ_G-structure $\varphi_B \circ \overline{\nu}_Y \colon Y \to B_G$ inducing φ on fundamental groups.*

Proof. Apply Proposition 4.2 with (G, w, v) the normal 1-type data of Y, with φ in place of ϕ. $\square$

Corollary 4.4 produces ξ_G-structures $\overline{\nu}_\varphi$ that might satisfy the hypotheses of (ii) in the following promised partial characterisation of 3-manifolds admitting stably exotic fillings.

Theorem 4.5. *Let Y be a closed, smooth 3-manifold and let (G, w, v, φ) be Y-filling normal 1-type data.*

(i) If Y admits stably exotic fillings with normal 1-type data (G, w, v), then there exist closed stably exotic 4-manifolds with normal 1-type data (G, w, v).
(ii) If φ is surjective and $[Y, \overline{\nu}_\varphi] = 0 \in \Omega_3(\xi_G)$ for some ξ_G-structure $\overline{\nu}_\varphi \colon Y \to B_G$ inducing φ on fundamental groups, and there exist closed stably exotic 4-manifolds with normal 1-type data (G, w, v), then Y admits stably exotic fillings with normal 1-type data (G, w, v).

We will use Theorem 4.5 to deduce the remaining items of Theorem 1.1 in Section 5.1, and to prove Theorem 1.2 in Section 7.

Proof. First we prove (i). Assume that Y admits stably exotic fillings M and M' with normal 1-type data (G, w, v). Since M and M' are stably homeomorphic rel. Y, by Theorem 2.4 (i) there exist normal 1-smoothings $\overline{\nu}_M^{\mathrm{Top}} \colon M \to B_G^{\mathrm{Top}}$ and $\overline{\nu}_{M'}^{\mathrm{Top}} \colon M' \to B_G^{\mathrm{Top}}$, such that $\overline{\nu}_M^{\mathrm{Top}}|_Y = \overline{\nu}_{M'}^{\mathrm{Top}}|_Y$, and such that

$$[M \cup_Y M', \overline{\nu}_M^{\mathrm{Top}} \cup -\overline{\nu}_{M'}^{\mathrm{Top}}] = 0 \in \Omega_4(\xi_G^{\mathrm{Top}}).$$

Recall that the spaces B_G and B_G^{Top} were constructed as certain homotopy fibres in Section 2.2, differing only in that we used BO or BTop in the construction respectively. As such, the obstruction to lifting topological normal 1-smoothings along the forgetful map $\eta_G \colon B_G \to B_G^{\mathrm{Top}}$ is the obstruction to lifting the stable topological normal bundle along the forgetful map BO $\to$ BTop, which is, by definition, the Kirby–Siebenmann invariant of the manifold. Since both M and M' are smooth, this obstruction vanishes for each of $\overline{\nu}_M^{\mathrm{Top}}$ and $\overline{\nu}_{M'}^{\mathrm{Top}}$, and so the lift along η_G exists. Let $\overline{\nu}_M \colon M \to B_G$ and $\overline{\nu}_{M'} \colon M' \to B_G$ be the resulting normal 1-smoothings such that $\overline{\nu}_M^{\mathrm{Top}} = \eta_G \circ \overline{\nu}_M$ and $\overline{\nu}_{M'}^{\mathrm{Top}} = \eta_G \circ \overline{\nu}_{M'}$.

Since M and M' are not stably diffeomorphic rel. Y, Theorem 2.4 (ii) implies that $[M \cup_Y M', \overline{\nu}_M \cup -\overline{\nu}_{M'}] \neq 0 \in \Omega_4(\xi_G)$. Hence there exists a nontrivial element in the kernel of the forgetful map $\Omega_4(\eta_G) \colon \Omega_4(\xi_G) \to \Omega_4(\xi_G^{\mathrm{Top}})$. Each element of $\Omega_4(\xi_G)$ can be represented by a 4-manifold with normal 1-type data (G, w, v), by surgery below the middle dimension. Apply this to surger the trivial element $M \cup_Y M$ and the nontrivial element $M \cup_Y M'$ of $\Omega_4(\xi_G)$ we just constructed to obtain 4-manifolds with normal 1-type data (G, w, v). Automorphisms of the normal 1-type cannot relate a nontrivial element of $\Omega_4(\xi_G)$ with the trivial element. Hence by [Kre99, Theorem C], there exist closed stably exotic 4-manifolds with normal 1-type data (G, w, v). This completes the proof of (i).

Next we prove (ii). If $w = 0$, then closed stably exotic 4-manifolds with normal 1-type data (G, w, v) do not exist by [Gom84]. Hence we assume $w \neq 0$. By our hypothesis that $[Y, \overline{\nu}_\varphi] = 0 \in \Omega_3(\xi_G)$ for some ξ_G-structure $\overline{\nu}_\varphi$ of Y, there exists a filling X of Y with a smoothing $\overline{\nu}_X \colon X \to B_G$ lifting the stable normal bundle along ξ_G and extending $\overline{\nu}_\varphi$ on Y. Using surgery, we can assume that $\overline{\nu}_X$ is a normal 1-smoothing, i.e. is 2-connected (recall from Remark 2.8 that $\pi_2(B_G) = 0$). It follows from the definition of ξ_G and the assumption that $w \neq 0$ that $w_1(\nu X) \neq 0$, and hence X is nonorientable. Define

$$M := X \# K_3 \text{ and } M' := X \# 11(S^2 \times S^2).$$

Then M and M' are homeomorphic by Lemma 3.1.

Extend $\overline{\nu}_X$ to normal 1-smoothings $\overline{\nu}_M$ and $\overline{\nu}_{M'}$ of M and M' respectively. For this, use the unique spin structures on K_3 and $11(S^2 \times S^2)$ together with Remark 4.3 to obtain ξ_G-structures on these manifolds, and take the connected sum. Again both normal 1-smoothings agree with $\overline{\nu}_\varphi$ on Y. For those smoothings,

$$[M \cup_Y M', \overline{\nu}_M \cup -\overline{\nu}_{M'}] = [K_3] \in \Omega_4(\xi_G),$$

because the double $[X \cup_Y X, \overline{\nu}_X \cup -\overline{\nu}_X]$ is null-bordant, as is $\#11(S^2 \times S^2)$.

Now we show that M and M', which both have normal 1-type data (G, w, v), are the desired stably exotic fillings. Suppose then, for a contradiction, that there exists a stable diffeomorphism from M to M' relative to Y, i.e. a diffeomorphism $f\colon M\#\ell(S^2\times S^2)\to M'\#\ell(S^2\times S^2)$ for some $\ell\in\mathbb{N}_0$. We use this to prove that there are no closed, stably exotic 4-manifolds with normal 1-type data (G,w,v), which contradicts the hypotheses of (ii), and thus completes the proof.

By [KP25, Proposition 4.1 (ii)], if $[K_3]=0\in\Omega_4(\xi_G)$, then there are no closed, stably exotic 4-manifolds with normal 1-type data (G,w,v). So we aim to prove the former statement.

We claim that $\overline{\nu}_{M'\#\ell(S^2\times S^2)}\circ f$ and $\overline{\nu}_{M\#\ell(S^2\times S^2)}$ are homotopic rel. Y. For this we use that both restrict to $\overline{\nu}_\varphi$ on the boundary and that $\varphi\colon\pi_1(Y)\to G$ is surjective by assumption, which implies that the inclusion $Y\to M\#\ell(S^2\times S^2)$ is 1-connected. Consider the diagram

$$\begin{tikzcd}
Y \arrow[rr, "\overline{\nu}_\varphi"] \arrow[d] & & B_G \arrow[d, "\xi_G"] \\
M\#\ell(S^2\times S^2) \arrow[rr, "\nu_{M\#\ell(S^2\times S^2)}"'] \arrow[urr, dashed] & & \mathrm{BO}\,.
\end{tikzcd}$$

The maps $\overline{\nu}_{M\#\ell(S^2\times S^2)}$ and $\overline{\nu}_{M'\#\ell(S^2\times S^2)}\circ f$ both give examples of the diagonal dashed arrow that make the diagram commute. Let F be the homotopy fibre of ξ_G. The obstructions to finding a homotopy rel. Y between two given lifts of $\nu_{M\#\ell(S^2\times S^2)}$ along ξ_G lie in

$$H^i(M\#\ell(S^2\times S^2),Y;\pi_i(F)).$$

Since ξ_G is 2-coconnected we have that $\pi_i(F)=0$ for $i\geq 2$. Since $Y\to M\#\ell(S^2\times S^2)$ is 1-connected, the cohomology groups $H^1(M\#\ell(S^2\times S^2),Y;\pi_1(F))$ and $H^0(M\#\ell(S^2\times S^2),Y;\pi_0(F))$ both vanish. Hence any two lifts, in particular $\overline{\nu}_{M\#\ell(S^2\times S^2)}$ and $\overline{\nu}_{M'\#\ell(S^2\times S^2)}\circ f$, are homotopic rel. Y.

It follows that

$$0=[M\#\ell(S^2\times S^2)\cup_Y M'\#\ell(S^2\times S^2),\overline{\nu}_{M\#\ell(S^2\times S^2)}\cup -\overline{\nu}_{M'\#\ell(S^2\times S^2)}]=[K_3]\in\Omega_4(\xi_G),$$

as desired. Hence, by [KP25, Proposition 4.1], there exist no closed stably exotic 4-manifolds with normal 1-type data (G,w,v). $\square$

5. Some nonorientable 3-manifolds that admit stably exotic fillings

In this section we prove Theorem 1.1 (ii) and (iii), using Theorem 4.5 (ii). Recall that the latter result has two main hypotheses. One of these is the existence of closed stably exotic 4-manifolds. To arrange for this, we will leverage the following statements from [Kre84] and [KP25]. Recall that $H^i(\mathbb{Z}/2;\mathbb{Z}/2)=\mathbb{Z}/2$ for $i=1,2$.

Theorem 5.1 ([Kre84])**.** *There exist closed stably exotic 4-manifolds with normal 1-type data $(\mathbb{Z}/2,1,1)$.*

For example, $\mathbb{RP}^4\#K_3$ and $\mathbb{RP}^4\#11(S^2\times S^2)$ are stably exotic.

Theorem 5.2 ([KP25, Theorem A])**.** *Let (G,w,v) be normal 1-type data. If there exist closed stably exotic 4-manifolds with normal 1-type data (G,w,v), then $w\neq 0$ and $w^3=wv\in H^3(G;\mathbb{Z}/2)$.*

If moreover G has cohomological dimension at most 3, the converse also holds, i.e. if $w\neq 0$ and $w^3=wv\in H^3(G;\mathbb{Z}/2)$ then there exist closed stably exotic 4-manifolds with normal 1-type data (G,w,v).

In Section 5.1 we prove Theorem 1.1 (ii). We use Theorem 5.2 to show the existence of closed stably exotic 4-manifolds. The key observation is that circle bundles over aspherical surfaces provide surjective maps $\varphi\colon\pi_1(Y)\to G$, for groups G with cohomological dimension two, that are compatible with the normal 1-type data in the sense of Definition 4.1.

In Section 5.2, we prove Theorem 1.1 (iii) for nonorientable Y, the case of orientable Y having been already dealt with in Section 3. In this case, we apply Theorem 5.1 to provide the closed stably exotic 4-manifolds required for Theorem 4.5 (ii). To satisfy the other hypothesis of the latter theorem, we use the normal 1-type data $(\mathbb{Z}/2,1,1)$, and connect $\Omega_3(\xi_{\mathbb{Z}/2})$ to Pin^+-bordism.

5.1. **Circle bundles admit stably exotic fillings.** The next result proves Theorem 1.1 (ii).

Theorem 5.3. *Let Y be a circle bundle over a surface $\Sigma \neq \mathbb{RP}^2$. Then Y admits stably exotic fillings.*

Proof. If $\Sigma \cong S^2$ then Y is orientable, and this case was covered in Theorem 3.2. Hence we assume for this proof that Σ is aspherical.

Let $p\colon X \to \Sigma$ be the total space of the disc bundle over Σ obtained from Y by filling in each fibre with D^2. This is possible since $\mathrm{Diff}(S^1) \simeq \mathrm{O}(2)$. Then the universal cover $\widetilde{X} \simeq \widetilde{\Sigma}$ is contractible, and in particular is spin. Furthermore, $\pi_1(X) \cong \pi_1(\Sigma)$, and hence $\pi_1(X)$ is cohomologically 2-dimensional. Let $i\colon Y \to X$ be the inclusion map and consider the Y-filling normal 1-type data given by $(G, w, v, \varphi) := (\pi_1(X), w_1^X, w_2^X, \pi_1(i))$. Note that $w^3 = wv \in H^3(G;\mathbb{Z}/2)$ automatically since $H^3(G;\mathbb{Z}/2) = 0$. Hence by Theorem 5.2 there exist closed stably exotic 4-manifolds with normal 1-type data (G, w, v). This gives one of the hypotheses needed to apply Theorem 4.5 (ii).

The map $\varphi\colon \pi_1(Y) \to G = \pi_1(X)$ is surjective by construction. The 4-manifold X shows that Y bounds over the normal 1-type. Thus all the hypotheses Theorem 4.5 (ii) are satisfied, and so Y admits stably exotic fillings. ☐

5.2. **Pin bordism and stably exotic fillings.** We will soon prove Theorem 1.1 (iii). To do so, we will require the following facts about Pin structures and Pin bordism groups from Kirby–Taylor [KT90]. Let Y be a manifold with stable tangent bundle TY and stable normal bundle νY.

Proposition 5.4 ([KT90, Lemma 1.7])**.** *Let Y be a manifold. Let ζ denote a (stable) vector bundle over Y with determinant bundle $\det(\zeta)$.*

(i) A Pin^+ structure on ζ is equivalent to a spin structure on $\zeta \oplus 3\det(\zeta)$. Such a structure exists if and only if $w_2(\zeta) = 0$.

(ii) A Pin^- structure on ζ is equivalent to a spin structure on $\zeta \oplus \det(\zeta)$. Such a structure exists if and only if $w_2(\zeta) + w_1(\zeta)^2 = 0$.

Here is an immediate corollary.

Corollary 5.5. *Let Y be a manifold with stable tangent bundle TY and stable normal bundle νY.*

(i) The manifold Y admits a tangential Pin^+ structure if and only if $w_2(TY) = 0$, if and only if $w_2(\nu Y) + w_1(\nu Y)^2 = 0$, if and only if Y admits an normal Pin^- structure.

(ii) The manifold Y admits a tangential Pin^- structure if and only if $w_2(TY) + w_1(TY)^2 = 0$, if and only if $w_2(\nu Y) = 0$, if and only if Y admits an normal Pin^+ structure.

Remark 5.6. The usual convention is to write $w_i(Y) := w_i(TY)$, but since the normal bundle also plays a large rôle in this article we preserve the T in the notation to avoid potential confusion.

Proposition 5.7 ([KT90, §2])**.** *Every 3-manifold Y has a tangential Pin^- structure, and it has a tangential Pin^+ structure if and only if $w_1^2(TY) = 0$.*

Proof. Details were not given in [KT90], so we provide them here. In this proof, for brevity, denote $v_i := v_i(TY)$ and $w_i := w_i(TY)$ for the Wu and Stiefel–Whitney classes respectively. We similarly omit TY from the notation for the total Wu and Stiefel–Whitney classes.

First, we show that $v_2 = 0$. By definition of the Wu class, we have that $v_2 \cup x = \mathrm{Sq}^2(x)$ for every $x \in H^1(Y;\mathbb{Z}/2)$, and $\mathrm{Sq}^2(x) = 0$ because Sq^2 always vanishes on H^1. By Poincaré duality it follows that $v_2 = 0$ as claimed.

The Wu formula for the total classes is $w = \mathrm{Sq}(v)$. From this we can compute that $v_1 = w_1$ and $w_2 = v_2 + \mathrm{Sq}^1(v_1) = v_2 + w_1^2$, hence $v_2 = w_2 + w_1^2$, as we are working over $\mathbb{Z}/2$. Now since $v_2 = 0$ we must have $w_2 + w_1^2 = 0$, so every 3-manifold admits a tangential Pin^- structure. Also $w_2 = 0$ if and only if $w_1^2 = 0$, so a 3-manifold admits a tangential Pin^+ structure if and only if $w_1^2 = 0$. ☐

Lemma 5.8. *Let Y be a manifold. We have that $w_1^2(TY) = 0$ if and only if $w_1(TY)\colon \pi_1(Y) \to \mathbb{Z}/2$ admits a lift to $\mathbb{Z}/4$ along the surjective map $\mathbb{Z}/4 \twoheadrightarrow \mathbb{Z}/2$.*

Proof. The exact sequence $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$ yields an exact sequence

$$H^1(Y;\mathbb{Z}/4) \to H^1(Y;\mathbb{Z}/2) \xrightarrow{\beta} H^2(Y;\mathbb{Z}/2),$$

where β denotes the Bockstein homomorphism. It is a standard property of Steenrod squares (see e.g. [Hat02, Section 4L]) that $\beta = \mathrm{Sq}^1$. By the universal coefficient theorem and since $\mathbb{Z}/4$ and $\mathbb{Z}/2$ are abelian, we thus have the exact sequence

$$\mathrm{Hom}(\pi_1(Y), \mathbb{Z}/4) \longrightarrow \mathrm{Hom}(\pi_1(Y), \mathbb{Z}/2) \xrightarrow{\beta=\mathrm{Sq}^1} H^2(Y;\mathbb{Z}/2).$$

We see from the sequence that $w_1(TY) \in \mathrm{Hom}(\pi_1(Y), \mathbb{Z}/2)$ lifts to a map to $\mathbb{Z}/4$ if and only if $w_1^2(TY) = \mathrm{Sq}^1(w_1(TY)) = \beta(w_1(TY)) = 0$. □

We deduce the following characterisation for when tangential Pin^+ structures exist on 3-manifolds.

Proposition 5.9. *A closed, nonorientable 3-manifold Y admits a tangential Pin^+ structure if and only if the orientation character $w\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}/2$ factors as $\pi_1(Y) \to \mathbb{Z}/4 \twoheadrightarrow \mathbb{Z}/2$.*

Proof. Combine Proposition 5.7 and Lemma 5.8. □

This will be used in Section 6, when we seek to apply Theorem 5.12 below. The following is also from Kirby–Taylor [KT90].

Proposition 5.10 ([KT90, Theorems 5.1 and 5.2])**.**

(i) There are isomorphisms $\Omega_3^{\mathrm{Pin}^+} \xrightarrow{\cong} \Omega_2^{\mathrm{Spin}} \cong \mathbb{Z}/2$, given by taking a spin surface dual to w_1, and then taking the Arf invariant of its associated quadratic form.

(ii) There is an isomorphism $\Omega_4^{\mathrm{Pin}^+} \cong \mathbb{Z}/16$, under which $[K3]$ maps to $8 \in \mathbb{Z}/16$.

Remark 5.11. It is also true that $\Omega_4^{\mathrm{Pin}^-} = 0$. For the purposes of this paper, this explains why it is not so helpful to look at Pin^- structures, even though every 3-manifold admits a Pin^- structure.

The following theorem proves Theorem 1.1 (iii).

Theorem 5.12. *Let Y be a closed 3-manifold. If Y admits a null-bordant tangential Pin^+ structure, then Y admits stably exotic fillings.*

Proof. We already proved Theorem 5.12 in the case that Y is orientable, in Theorem 3.2. Hence we can restrict here to the case that Y is nonorientable.

Let $\varphi\colon \pi_1(Y) \twoheadrightarrow \mathbb{Z}/2$ be the orientation character of Y. Here φ is surjective since Y is nonorientable. Let $\xi_{\mathbb{Z}/2}$ be the normal 1-type for the normal 1-type data $(\mathbb{Z}/2, w = 1, v = 1)$, constructed as described in Section 2.2. Here $w \in H^1(\mathbb{Z}/2;\mathbb{Z}/2) \cong \mathbb{Z}/2$ and $v \in H^2(\mathbb{Z}/2;\mathbb{Z}/2) \cong \mathbb{Z}/2$ are the generators.

We claim that $[Y, \overline{\nu}_\varphi] = 0 \in \Omega_3(\xi_{\mathbb{Z}/2})$ for some $\xi_{\mathbb{Z}/2}$-structure $\overline{\nu}_\varphi$ inducing $\varphi\colon \pi_1(Y) \to \pi_1(B_{\mathbb{Z}/2}) = \mathbb{Z}/2$. Indeed, let X be a (hypothesised) tangential Pin^+ filling of Y, and let $w_1^X\colon X \to K(\mathbb{Z}/2, 1)$ be the map corresponding to the orientation character of X, which is such that the composition

$$\pi_1(Y) \to \pi_1(X) \xrightarrow{(w_1^X)_*} \mathbb{Z}/2 \tag{5.13}$$

equals φ. Since X has a tangential Pin^+ structure, $\widetilde{X}$ is spin and $w_1(\nu X)^2 + w_2(\nu X) = 0$ by Corollary 5.5 (i). Using the cup product structure on $K(\mathbb{Z}/2, 1) \simeq \mathbb{RP}^\infty$, naturality, the fact that w pulls back to $w_1(\nu X)$, and finally the previous sentence, we therefore have

$$(w_1^X)^*(v) = (w_1^X)^*(w^2) = (w_1^X)^*(w)^2 = w_1(\nu X)^2 = w_2(\nu X) \in H^2(X;\mathbb{Z}/2).$$

By Proposition 4.2 we obtain a map $(w_1^X)_B\colon B_{\pi_1(X)} \to B_{\mathbb{Z}/2}$ over BO that induces $(w_1^X)_*$ on fundamental groups. Thus X admits a $\xi_{\mathbb{Z}/2}$-structure $(w_1^X)_B \circ \overline{\nu}_X$, where $\overline{\nu}_X$ is a normal 1-smoothing of X. This $\xi_{\mathbb{Z}/2}$-structure on X induces a $\xi_{\mathbb{Z}/2}$-structure $\overline{\nu}_\varphi$ on Y that gives φ on fundamental groups by (5.13). We see that X is a null-bordism witnessing that $[Y, \overline{\nu}_\varphi] = 0 \in \Omega_3(\xi_{\mathbb{Z}/2})$. This completes the proof of the claim.

We know from Theorem 5.1 that the normal 1-type data $(\mathbb{Z}/2, 1, 1)$ admits closed stably exotic 4-manifolds. Hence we may apply Theorem 4.5 (ii) to see that Y admits stably exotic fillings. □

Remark 5.14. There is a free transitive action of $H^1(Y;\mathbb{Z}/2)$ on the set of Pin^+ structures, and in applications of Theorem 5.12, once we know that Y admits a Pin^+ structure, we are free to try to modify it by this action to arrange for Y to be Pin^+-null-bordant.

6. Examples

We provide the Pin$^+$ examples promised in the introduction. The first aim is to prove that the examples in Example 1.4 behave as asserted. For this we need a couple of lemmas, before Proposition 6.6, which is the main result of this section.

First, we set up some notation. Let F be a closed, orientable surface of genus g. For a $\mathbb{Z}/2$ vector space P, we write $P^* := \operatorname{Hom}_{\mathbb{Z}/2}(P, \mathbb{Z}/2)$ for the dual vector space. For any $i \geq 0$, let $\operatorname{ev}\colon H^i(F;\mathbb{Z}/2) \xrightarrow{\cong} H_i(F;\mathbb{Z}/2)^*$ denote the evaluation map. We will consider the $\mathbb{Z}/2$-valued *intersection form*

$$\lambda_F^{\mathbb{Z}/2}\colon H_1(F;\mathbb{Z}/2) \times H_1(F;\mathbb{Z}/2) \to \mathbb{Z}/2; \quad (x,y) \mapsto \big(\operatorname{ev} \circ PD^{-1}(y)\big)(x)$$

and the Poincaré dual *cup product form*

$$- \smile -\colon H^1(F;\mathbb{Z}/2) \times H^1(F;\mathbb{Z}/2) \to \mathbb{Z}/2; \quad (f,h) \mapsto \operatorname{ev}(f \smile h)[F]_{\mathbb{Z}/2}.$$

Both can be represented by the diagonal sum of g copies of the standard symplectic matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. A *Lagrangian* for either pairing is a direct summand L with $L = L^\perp$. The proof we give of the next lemma was suggested to us in this formulation by Csaba Nagy; it is closely related to the proof of [Nag24, Theorem 7.6].

Lemma 6.1. *Let (P,ψ) be a hyperbolic form over $\mathbb{Z}/2$ of dimension $2g$ and let V be a subspace of P of dimension at least g. Then there is a Lagrangian $K \subseteq P$ such that $K + V = P$.*

Proof. By passing to a g-dimensional subspace of V, without loss of generality we can assume that V is rank g. Writing $j\colon V \to P$ for the inclusion, we have that $V^\perp$ is the kernel of the surjective map $j^* \circ \operatorname{ad}(\psi)\colon P \to V^*$, so $\dim V^\perp = \dim P - \dim V^* = g = \dim V$. Let $Z := V \cap V^\perp$ and choose direct sum decompositions $V = U \oplus Z$ and $V^\perp = U' \oplus Z$, so that $V + V^\perp = Z \oplus U \oplus U'$. Noting that Z is the radical for the restriction of ψ to both V and to $V^\perp$, we have that ψ restricts to a nonsingular form on both U and U'. As U and U' have the same dimension, we may thus choose an isomorphism of nonsingular alternating forms $f\colon (U, \psi|_U) \to (U\, \psi|_{U'})$. Such an isomorphism determines a Lagrangian $\Delta_f := \{(y, f(y)) \mid y \in U\} \subseteq U \oplus U'$. By the same argument as the second sentence of the proof, $\dim(U \oplus U')^\perp = \dim P - \dim(U \oplus U')$. The restriction of ψ to $U \oplus U'$ is nonsingular, so $(U \oplus U') \cap (U \oplus U')^\perp = \{0\}$. Combining the two previous sentences, we have

$$P = (U \oplus U') \oplus (U \oplus U')^\perp. \tag{6.2}$$

We will now extend Δ_f to a full Lagrangian of (P,ψ). Note that $Z \subseteq (U \oplus U')^\perp$. Since $\dim U + \dim Z = \dim V = g = \dim V^\perp = \dim U' + \dim Z$ we have that $2\dim Z = 2g - \dim U - \dim U'$. Also $2g = \dim P = \dim U + \dim U' + \dim(U \oplus U')^\perp$, and hence $\dim(U \oplus U')^\perp = 2\dim Z$, i.e. $Z \subseteq (U \oplus U')^\perp$ is half-rank. Moreover we already noted that ψ vanishes identically on Z, so Z is isotropic. Since the decomposition in (6.2) is by definition orthogonal with respect to ψ, and ψ is nonsingular, ψ restricts to a nonsingular form on $(U \oplus U')^\perp$. It follows that Z is a Lagrangian for the restriction in $(U \oplus U')^\perp$. Moreover Z admits a complementary Lagrangian L, so that $Z \oplus L = (U \oplus U')^\perp$. To find one, start with a complementary summand L'. Since $\psi|$ is nonsingular, $Z \to (L')^*$ is an isomorphism, and hence by adding elements of Z to the elements of a basis for L', we can change L' to a Lagrangian L that is still complementary to Z.

For any complementary Lagrangian L to Z in $(U \oplus U')^\perp$, the submodule

$$K := \Delta_f \oplus L \subseteq (U \oplus U') \oplus (U \oplus U')^\perp$$

is a Lagrangian, and is such that $K + V = P$. $\square$

For a closed, orientable surface F, let

$$\operatorname{ad}(- \smile -)\colon H^1(F;\mathbb{Z}/2) \to H^1(F;\mathbb{Z}/2)^*; \quad f \mapsto (h \mapsto \operatorname{ev}(f \smile h)[F]_{\mathbb{Z}/2})$$

denote the adjoint of the cup product form. Equivalently, $f \mapsto h \frown (f \frown [F]_{\mathbb{Z}/2})$.

Lemma 6.3. *Let F be a closed, orientable surface of genus g. Let V be a summand of $H^1(F;\mathbb{Z}/2) \cong (\mathbb{Z}/2)^{2g}$ whose rank is at least g. Then there is a Lagrangian $K \subseteq H^1(F;\mathbb{Z}/2)$ of the cup product form such that*

$$V \hookrightarrow H^1(F;\mathbb{Z}/2) \xrightarrow{\operatorname{ad}(-\smile-)} H^1(F;\mathbb{Z}/2)^* \to K^*$$

is surjective.

Proof. By Lemma 6.1, there exists a Lagrangian K for the cup product form such that $K+V=H^1(F;\mathbb{Z}/2)$. Since K is a Lagrangian, $K\subseteq K^\perp$ and hence the composition

$$K \hookrightarrow H^1(F;\mathbb{Z}/2) \xrightarrow{\mathrm{ad}(-\smile-)} H^1(F;\mathbb{Z}/2)^* \to K^* \tag{6.4}$$

is the zero map. We develop the diagram

$$\begin{tikzcd} & K+V \arrow[rr, two heads] \arrow[d, "\cong"] & & (K+V)/K \arrow[d] \\ V \arrow[ur] \arrow[r] & H^1(F;\mathbb{Z}/2) \arrow[r, "\cong"] & H^1(F;\mathbb{Z}/2)^* \arrow[r, two heads] & K^*. \end{tikzcd}$$

The bottom row is the composition in the statement of the lemma. The down-right-right composition $K+V\to K^*$ factors through $(K+V)/K$ by (6.4), which defines the right vertical map. We have labelled several maps as being surjective or isomorphisms. We see that the composition $K+V\to K^*$ is surjective. It follows that the right vertical map $(K+V)/K\to K^*$ is surjective. Moreover, observe that the upper composition $V\to K+V\to (K+V)/K$ is surjective. It follows that the lower composition $V\to K^*$ is surjective, as desired. □

Corollary 6.5. *Let F be a closed, orientable surface of genus g. Let V be a summand of $H^1(F;\mathbb{Z}/2)\cong(\mathbb{Z}/2)^{2g}$ of rank at least g. Then there is a Lagrangian $L\subseteq H_1(F;\mathbb{Z}/2)$ of the intersection form such that*

$$V\hookrightarrow H^1(F;\mathbb{Z}/2)\xrightarrow{\mathrm{ev}} H_1(F;\mathbb{Z}/2)^*\to L^*$$

is surjective.

Proof. Let $K\subseteq H^1(F;\mathbb{Z}/2)$ be the Lagrangian as in Lemma 6.3, and define $L:=PD(K)\subseteq H_1(F;\mathbb{Z}/2)$. Since Poincaré duality determines an isometry from $-\smile-$ to $\lambda_{\mathbb{Z}/2}$, L is a Lagrangian of $\lambda_{\mathbb{Z}/2}$. We have the following diagram. It is straightforward to check that it commutes.

$$\begin{tikzcd} V \arrow[r, hook] & H^1(F;\mathbb{Z}/2) \arrow[r, "\mathrm{ev}"] \arrow[d, "\cong"', "PD"] & H_1(F;\mathbb{Z}/2)^* \arrow[r] \arrow[d, "\cong"', "PD^*"] & L^* \arrow[d, "\cong"', "PD^*|_{L^*}"] \\ & H_1(F;\mathbb{Z}/2) \arrow[r, "\mathrm{ev}"] & H^1(F;\mathbb{Z}/2)^* \arrow[r] & K^*. \end{tikzcd}$$

The down-right composition $H^1(F;\mathbb{Z}/2)\to H^1(F;\mathbb{Z}/2)^*$ sends

$$f\mapsto \big(h\mapsto h(f\frown[F]_{\mathbb{Z}/2})\big)=\big(h\mapsto h\frown(f\frown[F]_{\mathbb{Z}/2})\big)=\big(h\mapsto \mathrm{ev}(h\smile f)[F]_{\mathbb{Z}/2}\big),$$

and so is $\mathrm{ad}(-\smile-)$. Thus V surjects onto K^* in this diagram by Lemma 6.3, and hence by commutativity V also surjects onto L^*. □

We are now ready to provide the examples promised in Example 1.4.

Proposition 6.6. *Let F be a closed, orientable surface of genus g and let $F\to Y\to S^1$ be a fibre bundle with orientation-reversing monodromy $\varphi\colon F\to F$. Suppose that the fixed subgroup $H^1(F;\mathbb{Z}/2)^{\varphi^*}\subseteq H^1(F;\mathbb{Z}/2)$ has rank at least g. Then Y admits a null-bordant* Pin^+ *structure, and hence admits stably exotic fillings by Theorem 1.1 (iii).*

Proof. The orientation character factors through the bundle projection map as

$$\pi_1(Y)\to\pi_1(S^1)\cong\mathbb{Z}\twoheadrightarrow\mathbb{Z}/2.$$

Hence there is a lift to $\mathbb{Z}/4$ and so there is a tangential Pin^+ structure on Y by Proposition 5.9. A copy of the fibre F is dual to w_1. This has an induced spin structure and, by Proposition 5.10 (i), Y is Pin^+ null-bordant if and only if F is spin null-bordant. To arrange the latter, we are free to change the Pin^+ structure on Y, and hence the induced spin structure on F, using the action of $H^1(Y;\mathbb{Z}/2)$.

We have a Wang exact sequence

$$H^0(F;\mathbb{Z}/2)\xrightarrow{0}H^0(F;\mathbb{Z}/2)\to H^1(Y;\mathbb{Z}/2)\to H^1(F;\mathbb{Z}/2)\xrightarrow{\varphi^*-\mathrm{Id}}H^1(F;\mathbb{Z}/2) \tag{6.7}$$

leading to a short exact sequence

$$0\to\mathbb{Z}/2\to H^1(Y;\mathbb{Z}/2)\to H^1(F;\mathbb{Z}/2)^{\varphi^*}\to 0,$$

where $H^1(F;\mathbb{Z}/2)^{\varphi^*}$ denotes the fixed points under φ^*.

Let $\iota\colon F \to Y$ be the inclusion. If $\operatorname{Im}\big(\iota^*\colon H^1(Y;\mathbb{Z}/2) \to H^1(F;\mathbb{Z}/2)\big) = H^1(F;\mathbb{Z}/2)^{\varphi^*} \subseteq H^1(F;\mathbb{Z}/2)$ is at least half-rank, then by Corollary 6.5, there is a Lagrangian L for the $\mathbb{Z}/2$-intersection form such that

$$H^1(F;\mathbb{Z}/2)^{\varphi^*} \to H^1(F;\mathbb{Z}/2) \xrightarrow{\mathrm{ev}} H_1(F;\mathbb{Z}/2)^* \to L^*$$

is surjective. The group $H^1(Y;\mathbb{Z}/2)$ acts transitively on the Pin^+ structures on Y. The effect of the action of $x \in H^1(Y;\mathbb{Z}/2)$ on a spin structure of F is the effect of $\iota^*(x)$. We can therefore make any change of spin structures on F that corresponds to the action of an element of $\operatorname{Im}\iota^* = H^1(F;\mathbb{Z}/2)^{\varphi^*}$. Under the action, $\iota^*(x)$ changes the spin structure on a curve $\gamma \subseteq F$ if and only if $\mathrm{ev}\circ\iota^*(x)[\gamma] = 1 \in \mathbb{Z}/2$. Since $H^1(F;\mathbb{Z}/2)^{\varphi^*}$ maps surjectively to L^*, it follows that by changes of the Pin^+ structure on Y, we can change the spin structure induced on F to realise any desired spin structure on the Lagrangian L. Specifically, we make the spin structure into the bounding one on L. After that, F spin bounds, and hence Y admits a Pin^+ filling by Proposition 5.10 (i). □

Next we give an example, also promised in Example 1.4, demonstrating that Proposition 6.6 is false in general without the hypothesis that $H^1(F;\mathbb{Z}/2)^{\varphi^*}$ is at least half-rank.

Example 6.8. Let $F = T^2$ be a torus. Let $\varphi\colon F \to F$ be a diffeomorphism whose induced map on $H^1(F;\mathbb{Z}/2)$ is represented by $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$, and consider the associated T^2-bundle Y over S^1. The monodromy φ is orientation-reversing, and so Y is nonorientable. As in the proof of Proposition 6.6, the orientation character factors through the bundle projection map as $w^Y : \pi_1(Y) \to \pi_1(S^1) \cong \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2$, so there is a lift to $\mathbb{Z}/4$ and hence Y admits a tangential Pin^+ structure by Proposition 5.9.

One computes that φ^* acts transitively on the nonzero elements of $H^1(F;\mathbb{Z}/2)$, and hence has no nonzero fixed points; in particular the rank of the fixed points is $0 < g/2 = 1/2$. By the Wang sequence (6.7), the map $H^1(Y;\mathbb{Z}/2) \to H^1(F;\mathbb{Z}/2)$ is trivial, and hence every Pin^+ structure on Y induces the same spin structure on F. We need to determine the Arf invariant of this spin structure to identify its spin bordism class. Since the action of φ^* on $H^1(F;\mathbb{Z}/2)\setminus\{0\}$ is transitive, the only spin structure that is invariant under φ^* is the non-bounding structure on curves corresponding to all three nontrivial elements $(1,0)$, $(0,1)$, and $(1,1)$ in $(\mathbb{Z}/2)^2 \cong H^1(F;\mathbb{Z}/2)$. To see this, note that all spin structures restrict to a non-bounding one on at least one of these three curves. This spin structure has Arf invariant nonzero, and hence as a spin surface F is nontrivial in Ω_2^{Spin}. So Y does not Pin^+-bound, by Proposition 5.10 (i). Thus the assumption on the fixed points was necessary in Proposition 6.6. We do not know whether this Y admits stably exotic fillings, for some other normal 1-type data.

Finally we construct a circle bundle over a surface ($\neq \mathbb{RP}^2$) which does not admit a tangential Pin^+ structure, as mentioned in Example 1.5.

Proposition 6.9. *Let $F = \#^3\mathbb{RP}^2$ be the closed surface of nonorientable genus three and let $Y := S^1 \times F$. The orientation character $w_1^Y : \pi_1(Y) \to \mathbb{Z}/2$ does not factor through $\mathbb{Z}/4$, and hence Y does not admit a tangential* Pin^+ *by Proposition 5.9.*

Proof. The orientation character of F factors as

$$w_1^F : \pi_1(F) \to \pi_1(F) \times \pi_1(S^1) \xrightarrow{\cong} \pi_1(Y) \xrightarrow{w_1^Y} \mathbb{Z}/2.$$

Thus if w_1^Y factors through $\mathbb{Z}/4$ then so does w_1^F. Write $\pi_1(F) = \pi_1(\#^3\mathbb{RP}^2) \cong \langle a,b,c \mid a^2b^2c^2\rangle$, where the orientation character sends each of a, b, and c to 1. The orientation character of F factors further through the Hurewicz map as

$$w_1^F : \pi_1(F) \to H_1(F;\mathbb{Z}) \xrightarrow{\overline{w}_1^F} \mathbb{Z}/2.$$

If w_1^F factors through $\mathbb{Z}/4$, so does $\overline{w}_1^F$; since $\mathbb{Z}/4$ is abelian, maps to it factor through the abelianisation. Suppose there is such a factorisation $H_1(F;\mathbb{Z}) \xrightarrow{\varphi} \mathbb{Z}/4 \to \mathbb{Z}/2$. The element $[abc] \in H_1(F;\mathbb{Z})$ has order two, so we must have $\varphi([abc]) = 2 \in \mathbb{Z}/4$, which maps trivially to $\mathbb{Z}/2$. This contradicts the fact that $\overline{w}_1^F([abc]) = 1 \in \mathbb{Z}/2$, which holds since each of a, b, and c are orientation-reversing. Thus $\overline{w}_1^F$ does not factor through $\mathbb{Z}/4$, and hence neither does w_1^F. □

7. No stably exotic fillings in the presence of a two-sided projective plane

We prove Theorem 1.2, whose statement we recall.

Theorem 1.2. *Let Y be a closed 3-manifold that contains a two-sided $\mathbb{RP}^2$. Then Y does not admit stably exotic fillings.*

Proof. Let X be a smooth filling of Y, which exists since the smooth bordism group of unoriented 3-manifolds is trivial. If the universal cover $\widetilde{X}$ is non-spin, then by [Gom84; FNOP25, Theorem 13.3], any 4-manifold stably homeomorphic to X is stably diffeomorphic to X, and we are done. So we may assume that $\widetilde{X}$ is spin.

Let (G, w, v) be the normal 1-type data of X. Then by hypothesis we have maps $\mathbb{RP}^2 \xrightarrow{f} Y \xrightarrow{\iota} X$, where $\iota\colon Y \to X$ is the inclusion map. Since $f(\mathbb{RP}^2) \subseteq Y$ and $\iota(Y) \subseteq X$ have trivial normal bundles, $(\iota \circ f)^* w_i(\nu_X) = w_i(\nu_{\mathbb{RP}^2})$ for $i = 1, 2$. Let $w_i^{\mathbb{RP}^2} \in H^i(\mathbb{Z}/2; \mathbb{Z}/2)$ denote classes pulling back to $w_i(\nu_{\mathbb{RP}^2})$ under the classifying map $c\colon \mathbb{RP}^2 \to K(\mathbb{Z}/2, 1)$, for $i = 1, 2$. Since $w(\nu_{\mathbb{RP}^2}) = 1 + x$, with $x \in H^1(\mathbb{RP}^2; \mathbb{Z}/2)$ the generator, we see that $w_1^{\mathbb{RP}^2} \neq 0$ and $w_2^{\mathbb{RP}^2} = 0$. Consider the composition

$$g\colon \mathbb{Z}/2 \xrightarrow[\cong]{c_*^{-1}} \pi_1(\mathbb{RP}^2) \xrightarrow{(\iota\circ f)_*} \pi_1(X) \xrightarrow{\cong} G.$$

Denoting the induced map $K(\mathbb{Z}/2, 1) \to K(G, 1)$ also by g, we have a commutative diagram

$$\begin{tikzcd} \mathbb{RP}^2 \arrow[r, "\iota\circ f"] \arrow[d, "c"] & X \arrow[d, "c_X"] \\ K(\mathbb{Z}/2,1) \arrow[r, "g"] & K(G,1). \end{tikzcd}$$

The induced square in cohomology for $i = 1, 2$, is:

$$\begin{tikzcd} H^i(\mathbb{RP}^2;\mathbb{Z}/2) & H^i(X;\mathbb{Z}/2) \arrow[l, "(\iota\circ f)^*"'] \\ H^i(\mathbb{Z}/2;\mathbb{Z}/2) \arrow[u, "c^*", "\cong"'] & H^i(G;\mathbb{Z}/2). \arrow[l, "g^*"'] \arrow[u, "c_X^*"] \end{tikzcd}$$

By commutativity of this square, and the fact that the left vertical map is injective, we deduce that $g^*(w) = w_1^{\mathbb{RP}^2} \in H^1(\mathbb{Z}/2, \mathbb{Z}/2)$ and $g^*(v) = w_2^{\mathbb{RP}^2} \in H^2(\mathbb{Z}/2, \mathbb{Z}/2)$. Hence

$$g^*(w^3 - wv) = (w_1^{\mathbb{RP}^2})^3 - w_1^{\mathbb{RP}^2} w_2^{\mathbb{RP}^2} \in H^3(\mathbb{Z}/2; \mathbb{Z}/2).$$

Since $w_1^{\mathbb{RP}^2} \neq 0$, it follows that $(w_1^{\mathbb{RP}^2})^3 \neq 0$, and we saw above that $w_2^{\mathbb{RP}^2} = 0$. It follows that $(w_1^{\mathbb{RP}^2})^3 \neq w_1^{\mathbb{RP}^2} w_2^{\mathbb{RP}^2}$, so $w^3 \neq wv \in H^3(G; \mathbb{Z}/2)$. By Theorem 5.2, there are no closed stably exotic 4-manifolds with normal 1-type data (G, w, v). It now follows from Theorem 4.5 (i) that Y does not admit stably exotic fillings. □

In the notation of the previous proof, since $f^*\big(w_1(\nu_Y)^2 + w_2(\nu_Y)\big) = w_1(\nu_{\mathbb{RP}^2})^2 + w_2(\nu_{\mathbb{RP}^2}) = x^2 \neq 0$, it follows that Y does not admit a normal Pin^- structure, hence Y does not admit a tangential Pin^+ structure by Corollary 5.5. This aligns with Theorem 1.1.

8. The cardinality of stably exotic fillings

We consider the size of the set of stably exotic fillings of a given 3-manifold. First we show there are at most two fillings up to stable diffeomorphism, within a fixed stable homeomorphism class. Then we show that if Y admits stably exotic fillings, then it admits infinitely many stable homeomorphism classes of such.

Proposition 8.1. *Let Y be a closed 3-manifold and let M be a filling of Y. Let $\mathcal{F}_Y(M)$ denote the set of smooth fillings of Y that are stably homeomorphic to M rel. boundary, modulo stable diffeomorphism rel. boundary. Then $|\mathcal{F}_Y(M)| \leq 2$.*

Proof. After possibly stabilising, we can suppose that Y and M are such that there exists M' that is homeomorphic to M rel. boundary but not stably diffeomorphic rel. boundary; if not then $|\mathcal{F}_Y(M)| = 1$ and we are done. Let $\xi_G\colon B_G \to \mathrm{BO}$ be the normal 1-type of M, constructed using the normal 1-type data $(G := \pi_1(M), w_1^M, w_2^M)$, and let $\overline{\nu}_M$ be a normal 1-smoothing. Similarly, let $\xi_G^{\mathrm{Top}} \colon B_G^{\mathrm{Top}} \to B\mathrm{Top}$ be the topological normal 1-type of M, constructed in the same way.

By Theorem 2.4, the forgetful map $\Omega_4(\eta_G)\colon \Omega_4(\xi_G) \to \Omega_4(\xi_G^{\mathrm{Top}})$ is not injective and hence the kernel is isomorphic to $\mathbb{Z}/2$, generated by K_3 (with some normal 1-smoothing) by [KP25, Proposition 4.1].

Now let N be a smooth filling of Y that is stably homeomorphic to M rel. boundary. We will show that N is stably diffeomorphic to either M or M' rel. boundary, which will complete the proof. By Theorem 2.4 (ii), there exists a topological normal 1-smoothing $\overline{\nu}_N^{\mathrm{Top}}$ of N that is bordant to $\overline{\nu}_M^{\mathrm{Top}} := \eta_G \circ \overline{\nu}_M$ rel. boundary. We claim that $\overline{\nu}_N^{\mathrm{Top}}$ can be lifted to a normal 1-smoothing of N which agrees with $\overline{\nu}_M$ on Y. To see this, recall from (2.6) that B_G is the homotopy fibre of the map $p := P_2(-) + (w_1^M \times w_2^M)$. Similarly, from (2.7), we have that B_G^{Top} is the homotopy fibre of the map $p^{\mathrm{Top}} := P_2(-) + (w_1^M \times w_2^M)$. Modelling these homotopy fibres as mapping path spaces, and using superscript I to denote the free path space, we wish to first solve the lifting problem

$$\begin{tikzcd} Y \arrow[r, "(\overline{\nu}_M)|_Y"] \arrow[d, "\mathrm{incl.}"'] & (\mathrm{BO} \times K(G,1)) \times_p (K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2))^I \arrow[d, "\eta_G = (\eta \times \mathrm{Id}) \times \mathrm{Id}"] \\ N \arrow[ur, dashed, "\overline{\nu}_N"] \arrow[r, "\overline{\nu}_N^{\mathrm{Top}}"'] & (\mathrm{BTop} \times K(G,1)) \times_{p^{\mathrm{Top}}} (K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2))^I, \end{tikzcd}$$

where $\eta\colon \mathrm{BO} \to \mathrm{BTop}$ denotes the forgetful map. In this model, $\overline{\nu}_N^{\mathrm{Top}} = (\nu_N^{\mathrm{Top}} \times f) \times \chi$, for some maps $f\colon N \to K(G,1)$ and $\chi\colon N \to (K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2))^I$ such that $p^{\mathrm{Top}}((\nu_N^{\mathrm{Top}} \times f)(x)) = \chi(x)(1)$ for all $x \in N$. Noting that $(w_1 \times w_2) \circ \eta \circ \nu_N = (w_1 \times w_2) \circ \nu_N^{\mathrm{Top}}$, we see that $\overline{\nu}_N := (\nu_N \times f) \times \chi$ is a lift of $\overline{\nu}_N^{\mathrm{Top}}$. To see that $\overline{\nu}_N$ agrees with $\overline{\nu}_M$ on restriction to Y, we write $\overline{\nu}_M = (\nu_M \times f_M) \times \chi_M$ for some f_M and χ_M, and compare to $(\nu_N \times f) \times \chi$. The maps ν_N and ν_M agree on restriction to Y because Y is the common boundary of N and M. We also have $f|_Y = (f_M)|_Y$ and $\chi|_Y = (\chi_M)|_Y$, because $\overline{\nu}_M^{\mathrm{Top}} = (\nu_M^{\mathrm{Top}}, f, \chi)$ and $\overline{\nu}_N^{\mathrm{Top}} = (\nu_N^{\mathrm{Top}}, f_N, \chi_N)$, and these topological normal 1-smoothings agree on restriction to Y, by definition of $\overline{\nu}_N^{\mathrm{Top}}$. Thus $(\overline{\nu}_N)|_Y = (\overline{\nu}_M)|_Y$. Finally, $\overline{\nu}_N$ is a normal 1-smoothing because η_G induces an isomorphism on fundamental groups, and $\pi_2(B_G) = 0$, so $\overline{\nu}_N$ inherits the required connectivity properties from $\overline{\nu}_N^{\mathrm{Top}}$.

By definition of $\overline{\nu}_N^{\mathrm{Top}}$, we have that $[M \cup_Y N, \overline{\nu}_M \cup -\overline{\nu}_N] \in \Omega_4(\xi_G)$ maps trivially to $\Omega_4(\xi_G^{\mathrm{Top}})$. If $[M \cup_Y N, \overline{\nu}_M \cup -\overline{\nu}_N] = 0 \in \Omega_4(\xi_G)$ then N is stably diffeomorphic to M rel. boundary by Theorem 2.4 (i). If not, $[M \cup_Y N, \overline{\nu}_M \cup -\overline{\nu}_N]$ is nonzero in $\ker \Omega_4(\eta_G) \cong \mathbb{Z}/2$. Since M' is not stably diffeomorphic to M rel. boundary, Theorem 2.4 (i) implies that $[M' \cup_Y M, \overline{\nu}_{M'} \cup -\overline{\nu}_M]$ is nontrivial in $\Omega_4(\xi_G)$ for any normal 1-smoothing $\overline{\nu}_{M'}$ that agrees with $\overline{\nu}_M$ on Y. Since M and M' are homeomorphic, by the argument of the previous paragraph, with M' in place of N, there exists such a normal 1-smoothing $\overline{\nu}_{M'}$ such that $[M' \cup_Y M, \overline{\nu}_{M'} \cup -\overline{\nu}_M] \in \ker \Omega_4(\eta_G)$.

Thus we have two representatives of the nontrivial element of $\ker \Omega_4(\eta_G) \cong \mathbb{Z}/2$, and so the sum

$$[M' \cup_Y M, \overline{\nu}_{M'} \cup -\overline{\nu}_M] + [M \cup_Y N, \overline{\nu}_M \cup -\overline{\nu}_N] \in \Omega_4(\xi_G) \tag{8.2}$$

is trivial. Take the product of $(M' \cup_Y M) \sqcup (M \cup_Y N)$ with I, and glue $(M \times I, \overline{\nu}_M \circ \mathrm{pr}_1)$ to

$$\big((M' \cup_Y M) \cup (M \cup_Y N)\big) \times \{1\}$$

along the two copies of M, to obtain a bordism over ξ_G, witnessing the fact that (8.2) is equal to $[M' \cup_Y N, \overline{\nu}_{M'} \cup -\overline{\nu}_N]$. As the bordism class in (8.2) is trivial, we have that N is stably diffeomorphic rel. Y to M' by Theorem 2.4. It follows that $\mathcal{F}_Y(M) = \{M, M'\}$, which completes the proof. □

Now we show, as promised, that whenever we have a pair of stably exotic fillings for a 3-manifold Y, we can find infinitely many stable homeomorphism classes of pairs of stably exotic fillings.

Proposition 8.3. *Let M and M' be stably exotic fillings of Y. Then for all $k \geq 0$,*

$$M_k := M \# k(S^1 \times S^3) \quad \text{and} \quad M'_k := M' \# k(S^1 \times S^3)$$

are also stably exotic fillings of Y. Thus there are infinitely many stable homeomorphism classes of fillings of Y, each of which contains a stably exotic pair.

Proof. Since M and M' are stably homeomorphic rel. Y, so are M_k and M'_k.

Let $H := \pi_1(M)$ and let $\xi_H \colon B_H \to \mathrm{BO}$ be the normal 1-type of M and M', with normal 1-type data (H, x, y). Let F_k denote the free group on k letters and let $G := H * F_k$. Note that $H^1(G;\mathbb{Z}/2) \cong H^1(H;\mathbb{Z}/2) \oplus (\mathbb{Z}/2)^k$ and $H^2(G;\mathbb{Z}/2) \cong H^2(H;\mathbb{Z}/2)$. Under these identifications let $w := (x, 0) \in H^1(G;\mathbb{Z}/2)$ and let $v := y \in H^2(G;\mathbb{Z}/2)$. Since $\#^k(S^1 \times S^3)$ is spin, the normal 1-type data of M_k and M'_k is (G, w, v). Let $\phi \colon G \to H$ be the projection. By Proposition 4.2, there exists a map $\phi_B \colon B_G \to B_H$ covering $\mathrm{Id}_{\mathrm{BO}} \times \phi \colon \mathrm{BO} \times K(G,1) \to \mathrm{BO} \times K(H,1)$ up to homotopy. This map induces

$$(\phi_B)_* \colon \Omega_4(\xi_G) \to \Omega_4(\xi_H).$$

To show M_k and M'_k are not stably diffeomorphic rel. Y, consider the union

$$[M_k \cup_Y M'_k, \overline{\nu}_{M_k} \cup -\overline{\nu}_{M'_k}] \in \Omega_4(\xi_G)$$

for arbitrary normal 1-smoothings $\overline{\nu}_{M_k}$ and $\overline{\nu}_{M'_k}$ that agree on Y. By Theorem 2.4, we need to show that this is nontrivial. It suffices to show that its image

$$[M_k \cup_Y M'_k, \phi_B \circ \overline{\nu}_{M_k} \cup -\phi_B \circ \overline{\nu}_{M'_k}] \in \Omega_4(\xi_H). \tag{8.4}$$

under $(\phi_B)_*$ is nontrivial.

We claim that for each $S^1 \times S^3$ summand, $S^1 \times S^3 = \partial(S^1 \times D^4)$ gives a ξ_H null-bordism of $[S^1 \times S^3, \phi_B \circ \overline{\nu}_{M_k}|_{S^1 \times S^3}]$. To see this, consider the map of fibrations:

$$\begin{tikzcd}
\mathrm{BSpin} \arrow[r] \arrow[d] & \mathrm{BO} \arrow[r] \arrow[d] & K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2) \arrow[d, "="] \\
B_H \arrow[r] & \mathrm{BO} \times K(H,1) \arrow[r] & K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2).
\end{tikzcd}$$

Since the $S^1 \times S^3$ summand maps trivially to $K(H,1)$, and is spin, the restriction of $\phi_B \circ \overline{\nu}_{M_k}$ to $S^1 \times S^3$ lifts to BSpin. Noting that $S^1 \times D^4$ is a spin filling, for either choice of spin structure on $S^1 \times S^3$, it follows that $S^1 \times S^3$ is null-bordant over B_H as claimed.

It follows from the claim that M_k is bordant rel. Y to M over ξ_H, for some normal 1-smoothing $\overline{\nu}_M \colon M \to B_H$. Similarly, M'_k is bordant rel. Y to M' over ξ_H, for some normal 1-smoothing $\overline{\nu}_{M'} \colon M' \to B_H$. Moreover, by construction, (8.4) equals

$$[M \cup_Y M', \overline{\nu}_M \cup -\overline{\nu}_{M'}] \in \Omega_4(\xi_H).$$

Since M and M' are not stably diffeomorphic rel. Y, Theorem 2.4 implies that the latter union is nontrivial in $\Omega_4(\xi_H)$, as desired. □

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