# MAT993D: TOPOLOGY OF MANIFOLDS: SURGERY THEORY

MARK POWELL

## 1. Introduction and overview

The course will roughly follow the books of Ranicki [Ran] and Crowley-Lück-Macko [CLM], although some material also comes from all of the books in the bibliography.

### 1.1. **Classification of manifolds.**

**Definition 1.1** (Topological manifold)**.** An $n$-dimensional *manifold* $M$ is a Hausdorff topological space with a countable basis of open sets, such that for all $m \in M$, there is an open set $U \subseteq M$ containing $m$ an open set $V \subset \mathbb{R}^n$ or $V \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$ and a homeomorphism $h|_U : U \to V$.

**Remark 1.2.** Any such space is metrizable and paracompact.

**Definition 1.3.** A *differentiable structure* on an $n$-manifold $M$ is a collection $\{(U_\alpha, h_\alpha)\}$ of *charts* i.e. open sets $U_\alpha \subset M$ with a homeomorphism $h_\alpha \colon U_\alpha \to V_\alpha$ where $V_\alpha$ is an open set in $\mathbb{R}^n$ or $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$, that satisfy the following conditions.

1. $\{U_\alpha\}$ cover $M$, that is $\bigcup_\alpha U_\alpha = M$.
2. $h_\alpha \circ h_\beta^{-1}| \colon h_\beta(U_\alpha \cap U_\beta) \to h_\alpha(U_\alpha \cap U_\beta)$ is a smooth map.
3. The collection $\{(U_\alpha, h_\alpha)\}$ is maximal with respect to (2). That is, adding another coordinate neighbourhood causes (2) to fail.

The collection of charts is called a differentiable atlas. A manifold together with a differentiable structure is called a differentiable or a smooth manifold.

Some famous manifolds include the $n$-sphere $S^n$, the $n$-torus $T^n$, $\mathbb{CP}^{n/2}$, the complex projective space, for $n$ even.

*Can we classify manifolds?* In particular compact, connected manifolds with empty boundary, up to either homeomorphism or diffeomorphism.

To start, we can observe that homeomorphic manifolds have the same dimension, and they are homotopy equivalent so the basic invariants of algebraic topology must coincide. If $M \cong N$, then $H_*(M) \cong H_*(N)$ and $\pi_1(M) \cong \pi_1(N)$. What else? We will see more in due course.

First, what does classification actually mean? Ideally, we would say that we have classified $n$-dimensional manifolds if we can give a set of algebraic objects which are in 1-1 correspondence with diffeomorphism classes of $n$-manifolds. Given an $n$-manifold $M$, we then wish for a procedure that decides to which algebraic object

$M$ corresponds. Perhaps more modestly, one could ask for a procedure that, given two $n$-manifolds, decides whether they are diffeomorphic.

Note that you have already seen this in dimensions 0, 1 and 2. For $n = 0, 1$ there is a unique homeomorphism class of connected, compact manifolds with empty boundary.

For $n = 2$:

$$\{2-\text{manifolds}\}/\sim \leftrightarrow \{2-2g \mid g \in \mathbb{N} \cup \{0\}\} \cup \{2-k \mid k \in \mathbb{N}\}.$$

Here the procedure is well known: decide whether a surface $\Sigma$ is orientable, then choose a triangulation and compute the Euler characteristics. The Euler characteristic is of course the number in the sets above.

We will largely skip dimensions 3 and 4 for now, which are very interesting, but which are not the main aim of this course. Instead we will focus on manifolds of dimension at least 5. We immediately run into a major problem with the question to determine where in a classification a manifold sits: given two finite group presentations, the question of whether they present isomorphic groups is unanswerable in general. Thus the question of whether the fundamental groups $\pi_1(M)$ and $\pi_1(N)$ of two $n$-manifolds $M$ and $N$ are isomorphic is intractable. Instead we try to answer the easier question, where we assume that we already have two manifolds of the right homotopy type:

*Given a homotopy equivalence $f\colon M \to N$ of $n$-dimensional smooth manifolds, is $f$ homotopic to a diffeomorphism?* Next we will describe the surgery programme for answering this question for $n \geq 5$.

**Definition 1.4.** An $(n+1)$-dimensional cobordism $(W; M, N)$ is an $(n+1)$-dimensional manifold with boundary $M \sqcup N$. If $W, M$ and $N$ are oriented then we require that the oriented boundary of $W$ is $\partial W = M \sqcup -N$, where $-N$ denotes $N$ with the opposite orientation.

Note that $\partial(M \times I) = M \sqcup -M$. This can be seen by adopting the outward-normal-first convention when inducing an orientation on a boundary; the outward normal at $M \times \{0\}$ transports to the inward pointing normal at $M \times \{1\}$. Cobordism is an equivalence relation, and with $\sqcup$ the addition, we obtain a group:

$$\Omega_n := \{\text{diffeomorphism classes of smooth, closed, oriented } n\text{-dim manifolds}\}/\text{cobordism}.$$

For example, $\Omega_1 = \Omega_2 = \Omega_3 = 0$ but $\Omega_0 \cong \Omega_4 \cong \mathbb{Z}$.

*Maps are always assumed to be (at least) continuous.* A closed manifold is a compact manifold with empty boundary.

**Definition 1.5.** Given a space $X$, a bordism over $X$ of maps $f_1\colon M_1 \to X$, $f_2\colon M_2 \to X$, with $M_1$, $M_2$ manifolds, is a cobordism $(W; M_1, M_2)$ with a function $F\colon W \to X \times [0,1]$ such that $f_1 = F|\colon M_1 \to X \times \{0\}$ and $f_2 = F|\colon M_2 \to X \times \{1\}$. The equivalence classes of pairs $(M, f)$, with $M$ an oriented, compact, smooth $n$-dimensional manifold form the *bordism group* $\Omega_n(X)$, with addition again by disjoint union.

Note that $\Omega_n(\mathrm{pt}) \cong \Omega_n$. As an example, note that whereas $\Omega_1 = 0$, because a circle bounds a disc, we have that $\Omega_1(S^1) \cong \mathbb{Z}$. Can you prove this?

Homotopic maps are bordant: let $f \sim g$ via a homotopy $h_t \colon M \to X$ with $h_0 = f$ and $h_1 = g$. Then define:

$$\begin{array}{rcl} M \times I & \to & X \times I \\ (m,t) & \mapsto & (h_t(m), t). \end{array}$$

So if a map is homotopic to a diffeomorphism it is also bordant to a diffeomorphism.

**Definition 1.6** (Structure set)**.** Let $X$ be a space. An $n$-dimensional manifold structure on $X$ is a homotopy equivalence $f \colon M \to X$ from an $n$-manifold $M$ to $X$. The *structure set* $\mathcal{S}_n(X)$ is the set of manifold structures up to equivalence, where $(M,f) \sim (N,g)$ if there is an $(n+1)$-dimensional bordism $(W,F)$ between $(M,f)$ and $(N,g)$ with $F \colon W \to X \times I$ a homotopy equivalence.

Given $X$ and $n \geq 5$, surgery theory asks: is $\mathcal{S}_n(X)$ nonempty? If so, how many elements does it have? We will give an overview of surgery in the case that $\pi_1(X) = \{e\}$.

The assumptions on $F$, $f$ and $g$ imply that $W$ is an *$h$-cobordism*.

**Definition 1.7.** A cobordism $(W; M, N)$ is called an $h$-cobordism if the inclusion maps $M \to W$ and $N \to W$ are homotopy equivalences.

The basis of surgery theory is the following theorem, which won Smale the Fields medal. Our first main goal of the course will be to prove this theorem.

**Theorem 1.8** ($h$-cobordism theorem, Smale)**.** *An $(n+1)$-dimensional simply-connected $h$-cobordism $(W; M, N)$ with $n \geq 5$ is trivial i.e. is diffeomorphic to the product cobordism*

$$(M \times I; M \times \{0\}, M \times \{1\}).$$

In particular $M$ and $N$ are diffeomorphic. We close the section by showing how this theorem implies the topological generalised Poincaré conjecture.

**Corollary 1.9.** *Let $M$ be an $n$-dimensional closed manifold with $H_*(M) \cong H_*(S^n)$ and $\pi_1(M) = 1$. Then $M$ is homeomorphic to $S^n$.*

*Proof.* Let $\mathrm{cl}(W := M \smallsetminus (D^n_0 \sqcup D^n_1))$ be obtained by removing two disjoint open balls from $M$. By a homology calculation, the relative Hurewicz theorem and the Whitehead theorem, $W$ is a simply connected $h$-cobordism $(W; S^{n-1}, S^{n-1})$. Therefore there is a diffeomorphism to $F \colon S^{n-1} \times [0,1] \to W$. This can be arranged to be the identity on the first $S^{n-1}$. Now define a homomorphism

$$f \colon D^n_0 \cup S^{n-1} \times I \cup D^n_1 \to M = D^n_0 \cup W \cup D^n_1$$

as follows. On $D^n_0$ use the identity map, and on $S^{n-1} \times I$ use $F$. Let $f_1 \colon S^{n-1} \xrightarrow{\simeq} S^{n-1}$ be the restriction of $F$ to the second $S^{n-1}$ boundary component.

**Claim.** *$f_1$ extends to a homeomorphism $\overline{f_1} \colon D^n_1 \xrightarrow{\simeq} D^n_1$.*

The disc $D^n_1$ is homeomorphic to the quotient pace $S^{n-1} \times [0,1]/S^{n-1} \times \{0\}$. Then the map $f_1 \times \mathrm{Id} \colon S^{n-1} \times [0,1]$ determines a homeomorphism of the quotient space that extends $f_1 \colon S^{n-1} \times \{0\} \xrightarrow{\simeq} S^{n-1} \times \{0\}$. This is known as the Alexander trick. The map is also often written $t \cdot x \mapsto t \cdot f(x)$, with $t \in [0,1]$ and $x \in S^{n-1}$.

This completes the proof of the claim. The map $\overline{f_1}\colon D^n_1 \to D^n_1$ completes the definition of the homeomorphism $f\colon S^n \xrightarrow{\simeq} M$. □

Note that $M$ is not diffeomorphic to $S^n$ in general. $D^n \cup_f D^n$ is always homeomorphic to $S^n$ but may not be diffeomorphic. For example, there are famously 28 diffeomorphism classes of smooth manifolds homeomorphic to $S^7$.

1.2. **The surgery sequence.** In this section we will give a quick survey of the surgery sequence, in the simply connected case. One of the main goals of the course is to present this sequence. Do not worry if everything in this section does not make sense to you yet, we are going to slow down a lot after this section.

**Definition 1.10** (Poincaré complex, simply connected oriented case)**.** A finite CW complex $X$ is called an $n$-dimensional *Poincaré complex* if there exists a homology class $[X] \in H_n(X;\mathbb{Z})$ such that cap product with $[X]$ induces isomorphisms $[X] \cap -\colon H^{n-r}(X;\mathbb{Z}) \xrightarrow{\simeq} H_r(X;\mathbb{Z})$ for all $r = 0, 1, \ldots, n$.

These are Poincaré duality-like isomorphisms between cohomology and homology. Note that every $n$-dimensional manifold is an $n$-dimensional Poincaré complex. We will study cohomology and Poincaré duality in a lot more detail soon, with definitions.

For $X$ to have any chance of $\mathcal{S}_n(X) \neq \emptyset$, we need $X$ to be and $n$-dimensional Poincaré complex. We have:

$$\mathcal{N}_n(X) = \text{Normal bordism classes of degree one normal maps.}$$

We will not define this fully here. It is similar to $\Omega_n(X)$, but with extra bundle data *and* the maps are required to be degree 1.

**Definition 1.11** (Degree 1)**.** A map $f\colon M \to N$ of $n$-dimensional Poincaré complexes is *degree one* if

$$f_*\colon H_n(M;\mathbb{Z}) \to H_n(N;\mathbb{Z})$$

sends $f_*([M]) = [N]$.

The abelian groups

$$L_n(\mathbb{Z}) = L_n(\mathbb{Z}[\{1\}]) = \begin{cases} \mathbb{Z} & n \equiv 0 \mod 4 \\ 0 & n \equiv 1 \mod 4 \\ \mathbb{Z}/2\mathbb{Z} & n \equiv 2 \mod 4 \\ 0 & n \equiv 3 \mod 4 \end{cases}$$

(the simply connected $L$-groups, or simply connected surgery obstruction groups; in general $L_n(\mathbb{Z}[\pi_1(X)])$), complete the surgery sequence of sets:

$$\cdots \to \mathcal{N}_{n+1}(X) \to L_{n+1}(\mathbb{Z}) \xrightarrow{W} \mathcal{S}_n(X) \to \mathcal{N}_n(X) \xrightarrow{\sigma} L_n(\mathbb{Z}) \to \cdots$$

which is exact for $n \geq 5$. One of our main aims, as mentioned above, is to understand what this sequence says. For the existence question, given $X$, try to find a degree one normal map $f\colon M \to X$ from some manifold $M$, i.e. an element of $\mathcal{N}_n(X)$.

**Theorem 1.12** (Browder, Wall surgery obstruction theorem). *$\sigma(f) = 0$ if, and for $n \geq 5$ only if, $f$ is bordant to a homotopy equivalence.*

Thus the surgery obstruction map $\sigma$ measures whether $\mathcal{S}_n(X)$ is nonempty. Once we have found one manifold homotopy equivalent to $X$, the size of the kernel of $\sigma_n$ and the cokernel of $\sigma_{n+1}$ determine the size of the structure set. Beware the maps in the sequence are only maps of sets, making them into group homomorphisms is a major task, and strange group structures are often required. The map labelled $W$ is really an action of $L_{n+1}(\mathbb{Z})$ on the structure set, called Wall realisation. Note that the sequence classifies homotopy equivalences from manifolds to $X$. To obtain the true classification, one also has to mod out by the homotopy self-equivalences of $X$.

To show that two manifolds are diffeomorphic, one typically uses a relative version of the above sequence: find a cobordism between the two manifolds and evaluate the surgery obstruction to change the cobordism into an $h$-cobordism. If the obstruction vanishes, we have an $h$-cobordism and so the manifolds are diffeomorphic.

In principle, if one can compute the groups $L_n(\mathbb{Z}[\pi_1(X)])$, $\mathcal{N}_n(X)$, and *the maps*, as well as the homotopy self-equivalences of $X$, then one can classify manifolds in the homotopy type of $X$! Admittedly, these computations can be rather daunting.

1.3. **What is surgery?** Let $M$ be an $n$-dimensional manifold and let $f\colon S^r \times D^{n-r} \hookrightarrow M$ be a framed embedding of a sphere $S^r$. This is the *data* for an $r$-surgery. Then the $n$-manifold

$$M' := \operatorname{cl}(M \smallsetminus f(S^r \times D^{n-r})) \cup_{S^r \times S^{n-r-1}} D^{r+1} \times S^{n-r-1}$$

is the *output* of the surgery operation.

We say that $M'$ is obtained from $M$ by surgery on $f(S^r)$ or $f(S^r \times D^{n-r})$. Surgery changes $M$ rather drastically. For example, it kills the homotopy class $[f(S^r)] \in \pi_r(M)$ determined by choosing a basing of $f(S^r)$. We always round the corners of an attachment without comment.

**Example 1.13.**

(1) $M = S^1$, $n = 1$. $r = 0$. Choose an embedding of $S^0 \times D^1 \hookrightarrow S^1$. There are two essentially different choices, depending on how the two $D^1$s are oriented. The outcome is either one $S^1$ or $S^1 \sqcup S^1$.

(2) Any $M$, any $n$. If $r = -1$, then since $S^{-1} = \emptyset$ by convention, the output of surgery is
$$M \smallsetminus (\emptyset \times D^{n+1}) \cup D^0 \times S^n = M \sqcup S^n.$$
A $-1$-surgery takes a disjoint union with an $n$-sphere.

(3) $M = T^2$. A 1-surgery on an essential embedded $S^1$ produces $S^2$. On the other hand, with $M = S^2$, and $r = 0$, we obtain $\overline{S^2 \smallsetminus (S^0 \times D^2)} \cup D^1 \times S^1 \cong T^2$.

(4) Start with $M \sqcup N$, and $r = 0$. The outcome of surgery where one $D^n$ is embedded in $M$ and one in $N$, is the connected sum $M \# N$.

(5) Let $M = S^3$ and let $S^1 \times D^2 \hookrightarrow S^3$ as the unknot. Then the *framing* determines the outcome: this is how many times $t$ the $D^2$ component twists

in one complete revolution of $S^1$, with respect to the zero twisting, which is the twisting that extends over a disc. For $t = 0$ we obtain $S^1 \times S^2$. For $t = \pm 1$ we obtain $S^3$ again. For $t = p \notin \{0, 1, -1\}$, we obtain the lens space $L(p, 1)$.

(6) This last example can be extended to produce all 3-manifolds: the Lickorish-Wallace theorem says that every 3-manifold arises as surgery on some link in $S^3$.

1.4. **Handles.** A surgery corresponds to a cobordism, in fact a rather simple type of cobordism called an elementary cobordism, that arises as the *trace of the surgery.*

**Definition 1.14** (Trace of a surgery)**.** Given the data $f\colon S^r \times D^{n-r} \hookrightarrow M^n$ for a surgery, form the $(n+1)$-dimensional cobordism

$$W := M \times [0,1] \cup_{f(S^r \times D^{n-r}) \times \{1\}} D^{r+1} \times D^{n-r}.$$

Note that $\partial W = M \sqcup -M'$, where $M'$ is the output of the surgery, so $W$ is a cobordism from $M$ to the output $M'$. The cobordism $W$ is called the trace of the surgery. A cobordism that arises in this way is called an *elementary cobordism of index $r$.*

Thus there is a relation between surgery and cobordism. To understand this relation better, we will soon learn some Morse theory. First, here is some terminology on handles.

**Definition 1.15.** The manifold $D^r \times D^{n-r}$ is called an $n$-dimensional $r$-handle. The integer $r$, which satisfies $0 \leq r \leq n$, is called the index of the handle.

Note that all $n$-dimensional $r$-handles are just $n$-balls. Especially, $r$-handles and $(n-r)$-handles look very similar, the only difference being the order in which the coordinates appear. Nevertheless, the particular decomposition of $D^n$ as $D^r \times D^{n-r}$ has some significance due to the way in which handles are glued together to create a manifold.

**Definition 1.16.** The following subsets of an $r$-handle $h^r = D^r \times D^{n-r}$ will be of interest.

- The *core* $D^r \times \{0\}$.
- The *cocore* $\{0\} times D^{n-r}$.
- The *attaching sphere* $\partial D^r \times \{0\} \cong S^{r-1}$.
- The *belt sphere* $\{0\} \times \partial D^{n-r} \cong S^{n-r-1}$.

**Example 1.17.** Let $n = 3$.

(i) For $r = 0$, the core is a point, the cocore is the entire 0-handle, the attaching sphere is the empty set, and the belt sphere is an $S^2$.

(ii) For $r = 1$, the core is a $D^1$, the cocore is a disc $D^2$, the attaching sphere is two points $S^0$, and the belt sphere is an $S^1$. The name belt sphere probably arises from this example, although you probably have to imagine a snake who is worried about his trousers falling down in order for it to really make sense.

(iii) For $r = 2$, the core is a disc, the cocore is a $D^1$, the attaching sphere is a circle and the belt sphere is two points.

(iv) For $r = 3$, the core is the whole handle, the cocore is a point, the attaching sphere is $S^2$ and the belt sphere is empty.

Given a cobordism $W$ with $\partial W = \partial_0 W \sqcup \partial_1 W$, and an embedding $f\colon S^{r-1} \times D^{n-r+1} \hookrightarrow \partial_1 W$ of a framed sphere, we say that

$$W' = W \cup_f h^r = W \cup_{S^r \times D^{n-r+1}} D^r \times D^{n-r+1}$$

is obtained from $W$ by attaching an $r$-handle along $f$. Note that $\partial_0 W' = \partial_0 W$ and $\partial_1 W'$ is diffeomorphic to the result of an $(r-1)$-surgery on $\partial_1 W$ with data $f$.

**Theorem 1.18** (Thom-Milnor)**.** *Every differentiable compact $n$-dimensional manifold $M$ can be expressed as a union of handles*

$$M \cong \bigcup^{m_0} h^0 \cup \bigcup^{m_1} h^1 \cup \cdots \cup \bigcup^{m_n} h^n,$$

*with $m_r$ $r$-handles, with the attaching sphere of the $r$-handles in the boundary $\partial M^{(r-1)}$ of the $(r-1)$-skeleton, where*

$$M^{(r-1)} := \bigcup^{m_0} h^0 \cup \bigcup^{m_1} h^1 \cup \cdots \cup \bigcup^{m_{r-1}} h^{r-1}$$

*is the union of the handles of index $0, \ldots, r-1$. Thus $M^{(r)}$ is obtained from $M^{(r-1)}$ by attaching handles of index $r$.*

Such a decomposition of $M$ is called a *handle decomposition.* More generally, we have a corresponding result for cobordisms.

**Theorem 1.19** (Thom-Milnor)**.** *Every $(n+1)$-dimensional cobordism can be expressed as a union of elementary cobordisms:*

$$(W_1; M_0, M_1) \cup_{M_1} (W_2; M_1, M_2) \cup_{M_2} \cdots \cup_{M_{k-1}} (W_k; M_{k-1}, M_k)$$

*with the index of $W_j$ equal to $i_j$ and $i_1 \leq i_2 \leq \cdots \leq i_k$.*

The closed manifolds case above follows when $M_0 = M_k = \emptyset$.

**Corollary 1.20.** *Closed manifolds $M, N$ are cobordant if and only if they can be obtained from one another by a finite sequence of surgeries.*

The proof of Theorem 1.19 is quite long and uses Morse theory.

## 2. Morse theory

The main sources for Morse theory are [M1] and [M2]. For differential topology basics, see also [M3], which everybody should absolutely read (it is very short so no excuse not to), and [Hir]. We will cover a little differential topology but we will also assume some of it without proof. I will recall statements.

Let $M$ be a smooth $n$-manifold. Consider a smooth function $f\colon M \to \mathbb{R}$, i.e. $f \circ h_\alpha^{-1}\colon V_\alpha \to \mathbb{R}$ is a smooth function, where $V_\alpha \subseteq \mathbb{R}^n$ is an open set.

**Definition 2.1** (Critical point)**.** A point $p \in M$ is a *critical point* of $f$ if $df_p = 0\colon T_pM \to T_{f(p)}\mathbb{R}$. That is, if there exist coordinates $x_1, \dots, x_n$ around $p$ such that

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \cdots = \frac{\partial f}{\partial x_n} = 0.$$

**Definition 2.2** (Nondegenerate critical point)**.** A critical point is *nondegenerate* if, in some (and whence any) coordinate system, the Hessian matrix

$$H(p) := \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{ij}$$

is nonsingular, i.e. $\det H(p) \neq 0$.

The matrix $H(p)$ determines a bilinear form $H(p)\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, where the coordinate system determines an identification $T_pM = \mathbb{R}^n$. The *index* of $f$ at $p$ is the maximal dimension of a subspace $V$ of $\mathbb{R}^n$ on which $H(p)$ is negative definite, that is $H_p(v, v) < 0$ for all nonzero $v \in V$.

2.1. **The Morse lemma.** The basis of Morse theory is the famous Morse lemma.

**Lemma 2.3** (The Morse lemma)**.** *Let $p$ be a nondegenerate critical point for a smooth function $f\colon M \to \mathbb{R}$. There is a local coordinate system $(x_1, \dots, x_n)$ in some open set $U \ni p$, with $x_i(p) = 0$ for all $i$, such that*

$$f(x_1, \dots, x_n) = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2$$

*for all of $U$, where $\lambda$ is the index of $f$ at $p$.*

For example, when $n = 2$, nondegenerate critical points are maxima, minima and saddle points. Consider a torus with a height function to $\mathbb{R}$ determined by an embedding into $\mathbb{R}^3$. For one standard embedding, the height function has 4 critical points, one of index 0, one of index 2, and two of index 1.

To prove the Morse lemma, first we need a preliminary lemma.

**Lemma 2.4.** *Let $f\colon V \to \mathbb{R}$ where $V \subseteq \mathbb{R}^n$ is a convex open subset with $0 \in V$, $f$ is a $C^\infty$ function and $f(0) = 0$. Then*

$$f(x_1, \dots, x_n) = \sum_{i=1}^n x_i g_i(x_1, \dots, x_n)$$

*for some $C^\infty$ functions $g_i$ on $V$ with $g_i(0) = \frac{\partial f}{\partial x_1}(0)$.*

*Proof.* We have:

$$\begin{aligned} f(x_1, \dots, x_n) = [f(tx_1, \dots, tx_n)]_{t=0}^1 &= \int_0^1 \frac{\mathrm{d}f(tx_1, \dots, tx_n)}{\mathrm{d}t}\,\mathrm{d}t \\ &= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(tx_1, \dots, tx_n) \cdot x_i \,\mathrm{d}t \end{aligned}$$

Thus we can take

$$g_i(x_1, \dots, x_n) := \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \dots, tx_n)\,\mathrm{d}t.$$

We should check that

$$g_i(0) = \int_0^1 \frac{\partial f}{\partial x_i}(0, \dots, 0)\, \mathrm{d}t = \frac{\partial f}{\partial x_i}(0),$$

as required. □

Now let us prove the Morse lemma.

*Proof of Morse lemma.* First we note that if we can obtain the right form of the function, $\lambda$ will indeed be the index of $f$ at $p$, since the Hessian matrix of the given form is:

$$\begin{pmatrix} -2 & & & & & \\ & \ddots & & & & \\ & & -2 & & & \\ & & & 2 & & \\ & & & & \ddots & \\ & & & & & 2 \end{pmatrix}$$

with $\lambda$ diagonal $-2$s and $n - \lambda$ diagonal 2s. The rest of the entries, which are not shown, are zero. The maximal negative definite subspace of $\mathbb{R}^n$ with respect to the form induced on $\mathbb{R}^n$ by this matrix is $\mathbb{R}^\lambda \times \{0\} \subset \mathbb{R}^\lambda \times \mathbb{R}^{n-\lambda} = \mathbb{R}^n$.

Now we need to show that coordinates can be found around a nondegenerate critical point, with respect to which the function has the desired form. We can assume that $f(p) = f(0) = 0$, that is that we have a chart in which $p$ is sent to $0 \in \mathbb{R}^n$, and that $f(0) = 0$ with respect to these coordinates. Since any translation of a coordinate system must also be in a maximal atlas, this assumption is valid.

Take a possible smaller open set around 0 which is convex, so that we may apply Lemma 2.4 to write

$$f(x_1, \dots, x_n) = \sum_{j=1}^n x_j g_j(x_1, \dots, x_n)$$

in a neighbourhood of 0 for some smooth functions $g_j$. Now, $g_j(0) = \frac{\partial f}{\partial x_j}(0)$ so Lemma 2.4 applied to $g_j$ yields

$$g_j(x_1, \dots, x_n) = \sum_{i=1}^n x_i h_{ij}(x_1, \dots, x_n)$$

in a neighbourhood of 0 for some smooth functions $h_{ij}$ with $h_{ij}(0) = \frac{\partial g_j}{\partial x_i}(0)$. Therefore

$$f(x_1, \dots, x_n) = \sum_{j=1}^n \sum_{i=1}^n x_i x_j h_{ij}(x_1, \dots, x_n)$$

in a neighbourhood $V$ of 0. Differentiating twice and evaluating at 0 (this is a calculation), we find that

$$\frac{\partial^2 f}{\partial x_k \partial x_\ell}(0) = h_{\ell k}(0) + h_{k\ell}(0).$$

Define

$$\overline{h}_{\ell k} = \frac{1}{2}(h_{\ell k} + h_{k\ell})$$

in $V$. Then $\overline{h}_{\ell k}(0) = \frac{1}{2}(h_{\ell k}(0) + h_{k\ell})(0)$ defines a symmetric nonsingular matrix. The fact that the matrix is nonsingular is the hypothesis that $p$ is a nondegenerate critical point. Note that

$$f(x_1, \dots, x_n) = \sum_{j=1}^{n}\sum_{i=1}^{n} x_i x_j \overline{h}_{ij}(x_1, \dots, x_n).$$

Since a symmetric matrix over $\mathbb{R}$ is diagonalisable, if we only needed to alter $f$ at a point we would be done. As we need the form of $f$ in a neighbourhood, we will need to work a little harder. Our aim is to change coordinates so that $\overline{h}_{ij}$ is sent to a matrix of the following form:

$$\begin{pmatrix} -1 & & & & & \\ & \ddots & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}.$$

Proceed by induction: suppose that there exists a neighbourhood $U_1 \ni 0$ with coordinates $(u_1, \dots, u_n)$ such that

$$f(u_1, \dots, u_n) = \pm u_1^2 \pm \dots \pm u_{r-1}^2 + \sum_{i,j \geq r} u_i u_j H_{ij}(u_1, \dots, u_n)$$

on $U_1$ with $H_{ij}(u_1, \dots, u_n)$ symmetric.

We may assume that $H_{rr}(0) \neq 0$ by a linear change of variables. Define:

$$G(u_1, \dots, u_n) := \sqrt{|H_{rr}(u_1, \dots, u_n)|}.$$

There is a smaller neighbourhood $0 \in U_2 \subseteq U_1$ such that $G > 0$ and smooth on $U_2$. Define new coordinates $(v_1, \dots, v_n)$ as follows:

$$v_i = u_i \text{ for } i \neq r$$

and

$$v_r(u_1, \dots, u_n) = G(u_1, \dots, u_n)\Big(u_r + \sum_{i=r+1}^{n} u_i \frac{H_{ir}}{H_{rr}}\Big).$$

**Claim.**

*(i) This change of coordinates has invertible derivative at* $0$.

*(ii) With respect to these new coordinates,*

$$f = \sum_{i \leq r} \pm v_i^2 + \sum_{i,j > r} v_i v_j H'_{ij}(v_1, \dots, v_n).$$

The first part of the claim shows that this is a permissable coordinate change, since by the inverse function theorem there is a smaller neighbourhood $U_3$ containing $0$ in which the proposed coordinate change is a diffeomorphism.

**Theorem 2.5** (Inverse function theorem)**.** *Let $g\colon U \to V$, be a smooth function with $U, V$ open neighbourhoods of $0$ in $\mathbb{R}^n$, and suppose that $g(0) = 0$. Suppose that $Dg(0)$ is invertible. Then there exist neighbourhoods $U_1 \subseteq U$, $V_1 \subseteq V$ of $0$ such that $g|\colon U_1 \to V_1$ is a diffeomorphism.*

The second claim gives the inductive step. Thus it remains to prove the claim. First, if $g$ is the coordinate change function, we can compute

$$Dg(0) = \begin{pmatrix} 1 & & & * & & & \\ & \ddots & & * & & & \\ & & 1 & * & & & \\ & & & G(0) & & & \\ & & & * & 1 & & \\ & & & * & & \ddots & \\ & & & * & & & 1 \end{pmatrix}.$$

Here the stars are unknown terms that do not concern us, and all other entries are zero. We then have

$$\det(Dg(0)) = G(0) = \sqrt{|H_{rr}(0)|} \neq 0.$$

This completes the proof of part (i) of the claim. The second part follows from substitution, and collecting terms in which $v_r$ occurs.

□

**Corollary 2.6.** *Nondegenerate critical points are isolated.*

**Definition 2.7.** A smooth function $f\colon M \to \mathbb{R}$ is called a Morse function if all its critical points are nondegenerate.

2.2. **Morse functions exist.** The aim of this subsection is to explain why the following theorem holds, without giving the full proof.

**Theorem 2.8.**

*(i) Every differentiable manifold admits a Morse function.*

*(ii) For every cobordism $(W; M, N)$ there is a Morse function $f\colon W \to [0, 1]$ with $f(M) = 0$ and $f(N) = 1$ that has no critical points in a neighbourhood of the boundary.*

Part (ii) is proven in [M1, Theorem 2.5]. We will just explain (i) here.

**Definition 2.9.** Let $M^m, N^n$ be smooth manifolds with $m = \dim M \leq \dim N = n$ and let $f\colon M \to N$ be a smooth map. If $\operatorname{rk} Df_p = m$ for all $p \in M$, then $f$ is said to be an *immersion*. If, in addition, $f\colon M \to f(M)$ is a homeomorphism, $f$ is said to be an *embedding*.

The proof of Theorem 2.8 needs for an embedding of a manifold into euclidean space.

**Theorem 2.10** (Weak Whitney embedding theorem)**.** *For every compact differentiable manifold $M$, there exists $K \in \mathbb{N}$ and a smooth embedding $M \hookrightarrow \mathbb{R}^K$.*

The Whitney embedding theory can be improved to $K = 2\dim M + 1$ with a bit more work, and in fact to $K = 2\dim M$ by using the Whitney trick. We will learn about the Whitney trick soon.

*Proof of the weak Whitney embedding theorem.* This proof is from the differential topology book of Munkres. We will assume that $\partial M = \emptyset$ for simplicity. Let $C(r)$ be the $r$-cube

$$\{\underline{x} \in \mathbb{R}^n \mid \max\{|x_i|\, i = 1, \dots n\} < r\}.$$

For each $p \in M$ choose coordinates $(U, h)$ such that $h(U) \supseteq C_1$ and $h(p) = 0$. Define

$$V_p := h^{-1}(\operatorname{Int} C(1))$$

and

$$W_p := h^{-1}(\operatorname{Int} C(1/2)).$$

The sets $W_p$ cover $M$. Let $W_1, \dots W_k$ be a finite subcover. We have corresponding sets $U_1, \dots, U_k$ and $V_1, \dots, V_k$, as well as charts $h_i \colon U_i \to \mathbb{R}^n$.

There exists a smooth function $\varphi \colon \mathbb{R}^m \to \mathbb{R}$ such that

$$\varphi(\mathbf{x}) \begin{cases} = 1 & \mathbf{x} \in C(1/2) \\ > 0 & \mathbf{x} \in \operatorname{Int} C(1) \\ = 0 & \notin C(1). \end{cases}$$

Now, for $i = 1, \dots, k$, define functions $\varphi_i \colon M \to \mathbb{R}$ as follows:

$$\varphi_i(\mathbf{x}) = \begin{cases} \varphi(h_i(\mathbf{x})) & \mathbf{x} \in U_i \\ 0 & x \in M \smallsetminus V_i. \end{cases}$$

The functions $\varphi_i$ are smooth, and well-defined since both options are zero on $M \smallsetminus V_i \cap U_i$. Now let

$$\begin{array}{rcl} \Phi \colon M & \to & \mathbb{R}^k \\ \mathbf{x} & \mapsto & (\varphi_1(\mathbf{x}), \dots, \varphi_k(\mathbf{x})) \end{array}$$

and then define

$$\begin{array}{rcl} f \colon M & \to & \mathbb{R}^k \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{kn} \\ \mathbf{x} & \mapsto & (\Phi(\mathbf{x}), \varphi_1(\mathbf{x}) \cdot h_1(\mathbf{x}), \varphi_2(\mathbf{x}) \cdot h_2(\mathbf{x}), \cdots, \varphi_k(\mathbf{x}) \cdot h_k(\mathbf{x})) \end{array}$$

Here $\varphi_i(\mathbf{x}) \cdot h_i(\mathbf{x})$ is extended to be 0 outside $U_i \subset M$. We note that $f$ is smooth. Next we check that $f$ is injective. Suppose that $f(x) = f(y)$. Then in the first $\mathbb{R}^k$, we have that $\varphi_i(x) = \varphi_i(y)$. So if $x \in V_i$ then $y \in V_i$ too. For such $x, y$, $\varphi_i(x) \cdot h_i(x) = \varphi_i(y) \cdot h_i(y)$. Since $\varphi_i(x) = \varphi_i(y) \neq 0$ we have $h_i(x) = h_i(y)$. Thus $x = y$ since $h_i$ is injective.

Next, $M$ is compact, $f$ is a one-one continuous map to a Hausdorff space, so is a homeomorphism onto its image. It remains to check that $df$ has rank $m$. That is, $f \circ h_i^{-1} \colon h_i(U_i) \to \mathbb{R}^{nk+k}$ has rank $n$ as a map from a subset of $\mathbb{R}^n$ to $\mathbb{R}^{nk+k}$. But every point is inside some $W_i$, and inside $W_i$ we have $\varphi_i(x) = 1$. Thus in $h(W_i)$, $f \circ h_i^{-1}$ followed by projection to the $i$th $\mathbb{R}^n$ component in the definition of $f$, is given by the identity map of $\mathbb{R}^n$. The identity map has rank $n$. Since the $W_i$ cover $M$, $df$ has rank $n$ everywhere. This completes the proof that $M$ is an embedding. $\square$

Now we know that embeddings into Euclidean space always exist, we can use this to find a Morse function.

**Theorem 2.11** (Theorem 6.6 of [M2]). *Let $M \subset \mathbb{R}^K$ be a smooth compact manifold embedded in $\mathbb{R}^K$. For almost all $p \in \mathbb{R}^N$, the function*

$$\begin{array}{rcl} F\colon M & \to & \mathbb{R} \\ x & \mapsto & \|x-p\|^2 \end{array}$$

*has only nondegenerate critical points.*

Thus every compact smooth manifold admits a Morse function.

2.3. **Relating critical points to handles.** Now, let $f\colon M \to \mathbb{R}$ be a Morse function on a smooth compact $n$-manifold $M$. Define

$$L^a := f^{-1}(a) \text{ and } M^a := f^{-1}((-\infty, a]).$$

When $a$ is a regular value, $L^a$ is a smooth $(n-1)$-dimensional submanifold of $M$ and $M^a$ is a manifold with boundary. This last sentence will be explained below.

**Definition 2.12.** Let $f\colon M^m \to N^n$ be a smooth map. A point $p \in M$ is called a critical point if $df_p\colon T_pM \to T_{f(p)}N$ has rank less than $m$. Let $C$ be the critical points of $f$. The points $f(C) \subset N$ are called the critical values of $f$. The points $N \smallsetminus f(C)$ are called the *regular values* of $f$.

**Theorem 2.13** (Sard, Brown). *The set of regular values of a smooth map is everywhere dense in $N$.*

This can be very useful due to the next lemma.

**Lemma 2.14.** *Let $f\colon M \to N$ be smooth, and let $m \geq n$. Suppose that $y \in N$ is a regular value. Then $f^{-1}(y) \subset M$ is a smooth submanifold of dimension $m-n$.*

A nice proof of the previous two statements can be found in [M3].

**Example 2.15.**

(1) Consider

$$\begin{array}{rcl} f\colon \mathbb{R}^n & \to & \mathbb{R} \\ \mathbf{x} & \mapsto & x_1^2 + \cdots + x_n^2 \end{array}$$

Then $f^{-1}(1) = S^{n-1}$ is a smooth manifold.

(2) Identify $\mathbb{R}^{n^2}$ with $n \times n$ matrices over $\mathbb{R}$, and identify $\mathbb{R}^{n(n+1)/2}$ with real symmetric $n \times n$ matrices. Consider the map

$$\begin{array}{rcl} f\colon \mathbb{R}^{n^2} & \to & \mathbb{R}^{n(n+1)/2} \\ P & \mapsto & PP^T \end{array}$$

Then $f^{-1}(\mathrm{Id}) = O(n)$ is a smooth manifold.

Next, we want to relate critical points of a Morse function $f\colon M \to \mathbb{R}$ to handles. In summary, as $a$ increases, if $f^{-1}(a)$ does not contain any critical points, then the diffeomorphism type of $M^a$ does not change. On the other hand passing a critical point of index $r$ corresponds to attaching a handle of index $r$.

**Theorem 2.16.** *Let $M$ be a compact smooth manifold and let $f\colon M \to \mathbb{R}$ be a Morse function.*

(i) *Let $a < b \in \mathbb{R}$ be such that $f^{-1}([a,b])$ does not contain any critical points. Then $M^a \cong M^b$.*
(ii) *Now suppose that $a < b$ and $f^{-1}([a,b])$ contains exactly one critical point $p$ with $a < f(p) < b$, of index $k$. Then $M^b \cong M^a \cup h^k$, the result of attaching an index $k$ handle to $M^a$.*

The next definition is a metric-independent modification of usual the gradient vector field.

**Definition 2.17** (Gradient-like vector field)**.** A vector field $\xi$ on $M$ is gradient-like with respect to a Morse function $f$ if

(i) $\xi(f) > 0$. i.e. $df_p(\xi_p) > 0 \in T_{f(p)}\mathbb{R} = \mathbb{R}$;
(ii) there exist coordinates $(x_1, \dots, x_k, x_{k+1}, \dots, x_n)$ around each critical point $p$, with
$$f = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2$$
and
$$\xi = (-x_1, \dots, -x_k, x_{k+1}, \dots, x_n).$$

**Definition 2.18** (Ascending and descending manifolds)**.** For a critical point $p$ of a Morse function $f$ with a gradient-like vector field $\xi$, define the stable manifold, or the descending manifold of $p$, $W_p^s$, to be all the points that flow to $p$ along the integral curves of $\xi$. That is, the union of all trajectories that limit to $p$ as $t \to +\infty$.

Similarly, define the unstable manifold, or the ascending manifold of $p$, $W_p^u$, to be all the points that flow away from $p$ along the integral curves of $\xi$. That is, the union of all trajectories that limit to $p$ as $t \to -\infty$.

In part (ii) of Theorem 2.16, $W_p^s \cap (M^b \smallsetminus M^a)$ is the core of $h^k$, $W_p^u \cap (M^b \smallsetminus M^a)$ is the cocore, $W_p^s \cap L^a$ is the attaching sphere $S^{k-1} \times \{0\}$ and $W_p^u \cap L^b$ is the belt sphere.

*Idea behind proof of Theorem 2.16.*

(1) Define $X_p := \xi_p / \|\xi_p\|^2$, and let $\eta_x(t)$ be an integral curve to $X_p$ that passes through $x \in f^{-1}(a)$, so that $\eta_x(0) = x$. Then define
$$\begin{aligned} F\colon f^{-1}(a) \times [a,b] &\to f^{-1}([a,b]) \subset M \\ (x,t) &\mapsto \eta_x(t-a). \end{aligned}$$
Then $F$ is the diffeomorphism that we seek.
(2) In a small coordinate neighbourhood of the critical point $p$, in which the origin of the coordinates corresponds to $p$, let $e^k = B(0, \sqrt{\varepsilon}) \times \{0\} \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$, that is
$$\{\mathbf{x} \mid x_1^2 + \cdots + x_k^2 < \varepsilon, x_{k+1} = \cdots = x_n = 0\}.$$
Outside this neighbourhood, $f^{-1}([a,b])$ is a product as in (i). The neighbourhood of $p$ is diffeomorphic to $D^n$, and the Morse function in the neighbourhood determines a decomposition into $D^k \times D^{n-k}$, with the core given by $e^k$. There is a diffeomorphism retract of $M^b$ onto $M^a \cup e^k$.

Full details can be found in [Hir], [M2]. □

Now let $(W, M_0, M_1)$ be a cobordism and let $f: W \to [c, d]$ be a Morse function with $f^{-1}(c) = M_0$ and $f^{-1}(d) = M_1$. We can make a small perturbation of the Morse function if necessary so that each critical point appears at a different critical value. Let $p_1, \dots, p_k$ be the critical points, enumerated so that $f(p_1) < f(p_2) < \dots < f(p_k)$.

Choose $a_i \in [c, d]$ for $i = 1, \dots, k-1$ so that $f(p_i) < a_i < f(p_{i+1}$. Then define

$$W_1 := f^{-1}([c, a_1])$$
$$W_i := f^{-1}([a_{i-1}, a_i]), i = 2, \dots, k-1$$
$$W_k := f^{-1}([a_{k-1}, d]).$$

Then with $M_i := f^{-1}(a_i)$ we have

$$W = W_1 \cup_{M_1} W_2 \cup_{M_2} \cdots \cup_{M_{k-1}} W_k.$$

This proves the Thom-Milnor theorem, apart from the statement that handles/elementary cobordisms can be arranged to be in non-decreasing order.

## 3. Handle moves

3.1. **Transversality, immersions and embeddings.** We need to recall a little more machinery from differential topology in order to perform the handle moves that will be necessary to prove the $h$-cobordism theorem.

**Definition 3.1.** Let $f: P \to M$ and $g: Q \to M$ be two smooth maps of manifolds. Let $x = f(p) = g(q) \in f(P) \cap g(Q)$. We say that $P$ and $Q$ intersection transversely at $x$ if the rank of

$$df_p \oplus dg_q : T_pP \oplus T_qQ \to T_xM$$

is equal to $\dim M$. We write $P \pitchfork Q$ to mean that for all $x \in P \cap Q$, $P$ and $Q$ intersection transversely at $x$. In this case, we also sometimes use $P \pitchfork Q$ to denote the intersection $P \cap Q$.

If $\dim P + \dim Q < \dim M$ then $P \pitchfork Q$ implies that $P \cap Q = \emptyset$. On the other hand, if $\dim P + \dim Q = \dim M$, then the transverse intersection is a union of points. The next theorem essentially says that transverse maps are dense.

**Theorem 3.2.**

(i) *Let $f: M^m \to N^n$ be a smooth map of manifolds with $m \leq n$, and let $A \subset N$ be a submanifold. The map $f$ is arbitrarily close to a map $f'$ with $f'(M) \pitchfork A$.*

(ii) *The transverse intersection $P \pitchfork Q$ between a $k$-dimensional submanifold $P$ and an $\ell$-dimensional submanifold $Q$ is a $(k+\ell-m)$-dimensional submanifold. (Here a manifold of negative dimension is empty)*

In the case that $k + \ell = m$, we will need to be able to assign a sign $\text{sign}_x(P, Q)$ to each transverse intersection point, when $P$, $Q$ and $M$ are oriented. Let $x \in P \cap Q$ and let $\{v_1, \dots, v_k\}$ be a basis for $T_xP$ corresponding to the orientation of $P$. Let $\{v_{k+1}, \dots v_m\}$ be a basis for $T_xQ$ corresponding to the orientation, and let $\{z_1, \dots, z_m\}$ be a basis for $T_xM$ corresponding to the orientation. Let $B \in \mathrm{GL}_m(\mathbb{R})$

be the matrix with columns representing the vectors $v_1, \dots, v_m$ in the basis $\{z_i\}$. Then define

$$\operatorname{sign}_x(P, Q) = \det(B)/|\det(B)|.$$

This sign measures whether the combination of the orientations of $P$ and $Q$ agrees with the orientation of $M$ at the transverse intersection point $x$.

The next two theorems are foundational.

**Theorem 3.3** (Whitney immersion theorem)**.** *For $2n \leq m$ every map $f\colon N^n \to M^m$ is homotopic to an immersion.*

We proved a weak version of the Whitney embedding theorem before, but we will need the strongest version eventually.

**Theorem 3.4** (Strong Whitney embedding theorem)**.**

*(1) For $2n+1 \leq m$ every map $f\colon N^n \to M^m$ is homotopic to an embedding $N \hookrightarrow M$, and for $2n+2 \leq m$ any two homotopic embeddings are isotopic.*

*(2) For $n \geq 3$ and $\pi_1(M) = \{1\}$, every map $f\colon N^n \to M^{2n}$ is homotopic to an embedding.*

Part (ii) needs the Whitney trick, which we will cover in detail soon. We will have to content ourselves here with the sketch proof of the Whitney immersion theorem from [Ran].

**Proposition 3.5.** *Let $k \leq n \leq m$ and let $R_k$ be the subset of the $m \times n$ matrices over $\mathbb{R}$ comprising the matrices of rank exactly $k$. Then $R_k$ is a submanifold of dimension $k(m+n-k)$.*

*Proof.* Define the $m \times n$ matrix

$$J_k := \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$$

where the $I_k$ in the top left corner is the $k \times k$ identity matrix. Observe that $X \in R_k$ if and only if there exist $A \in \mathrm{GL}_m(\mathbb{R})$ and $B \in \mathrm{GL}_n(\mathbb{R})$ such that $X = AJ_kB^{-1}$. Moreover

$$AJ_kB = A'J_k(B')^{-1}$$

if and only if $(A')^{-1}AJ_kB^{-1}B' = J_k$. Thus $R_k = G/H$ where $G := \mathrm{GL}_m(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{R})$ and

$$H := \{(A, B) \in G \mid AJ_kB^{-1} = J_k\}.$$

We have that $(A, B) \in H$ if and only if the matrices are of the form

$$\left( \begin{pmatrix} L & M \\ 0 & N \end{pmatrix}, \begin{pmatrix} L & 0 \\ P & Q \end{pmatrix} \right)$$

for $L \in \mathrm{GL}_k(\mathbb{R})$, $N \in \mathrm{GL}_{m-k}(\mathbb{R})$, $Q \in \mathrm{GL}_{n-k}(\mathbb{R})$, $P \in M_{n-k,k}(\mathbb{R})$ and $M \in M_{k,m-k}(\mathbb{R})$. Then

$$\dim G = m^2 + n^2$$

and

$$\dim H = k^2 + k(m-k) + (m-k)^2 + k(n-k) + (n-k)^2.$$

Thus $\dim R_k = \dim G - \dim H = k(m+n-k)$ as claimed. □

*Sketch of proof of Whitney immersion theorem.* First approximate $f\colon N \to M$ by a $C^\infty$ map. First we work in one chart. In the chart, the derivative of $f$ determines a map

$$df\colon \mathbb{R}^n \to M_{m,n}(\mathbb{R}) = R_0 \cup R_1 \cup \cdots \cup R_n.$$

Make this transverse to each $R_k$ by a small perturbation. Here we want to perturb the derivative, so we alter $f$ in such a way that this gives the desired change on $df$. Now for $0 \leq k \leq n-1$, we compute:

$$\begin{aligned}(df)^{-1}(R_k) &= n - (mn - \dim R_k) = n - mn + \dim R_k \\ &\leq n - mn + \dim R_{n-1} = n - nm + (n-1)(m + n - (n-1)) \\ &= 2n - m - 1 < 0\end{aligned}$$

since $2n \leq m$. Thus generically $df$ missed $R_0 \cup \cdots \cup R_{n-1}$, and so has rank $n$. Now paste together these homotopies in each of the charts to get a map which is globally of rank $n$. □

3.2. **Rearranging handles.** Now let us return to handle decompositions of manifolds and cobordisms. From now on, when we write $W \cup_f h^r$, we mean the cobordism $W$ with an $(n+1)$-dimensional, index $r$ handle $h^r$ attached along $f\colon S^{r-1} \times D^{n-r+1} \to \partial_1 W$.

**Lemma 3.6** (Isotopy lemma)**.** *Suppose that $f, g\colon S^{r-1} \times D^{n-r+1} \to \partial_1 W$ are isotopic embeddings. Then*

$$W \cup_f h^r \cong W \cup_g h^r$$

*relative to $\partial_0 W$.*

*Proof.* Let $k\colon S^{r-1} \times D^{n-r+1} \times [0,1] \to \partial_1 W$ be an isotopy between $f$ and $g$. This extends to an ambient isotopy $K\colon \partial_1 W \times I \to \partial_1 W$, so that $K_t\colon \partial_1 W \times \{t\} \to \partial_1 W$ is a diffeomorphism for all $t$, and $K_0 = \mathrm{Id}$ so $K_0 \circ f = f$, while $K_1 \circ f = g$. Then we can define

$$\begin{array}{rcl}\widetilde{K}\colon \partial_1 W \times I & \to & \partial_1 W \times I \\ (x,t) & \mapsto & (K(x,t), t).\end{array}$$

This is the identity on $\partial_1 W \times \{0\}$ and so extends by the identity on the complement of a collar of $\partial_1 W$ to a diffeomorphism of $F\colon W \to W$ with $F \circ f = K_1 \circ f = g$. □

**Lemma 3.7** (Handle rearrangement)**.** *Suppose that $r \leq s$, let $f\colon S^{s-1} \times D^{n-s+1} \to \partial_1 W$ be an attaching map for an $s$-handle $h^s$, and let $g\colon S^{r-1} \times D^{n-r+1} \to \partial_1(W \cup_f h^s)$ be an attaching map for an $r$-handle $h^r$. There exists another attaching map $\overline{g}\colon S^{r-1} \times D^{n-r+1} \to \partial_1 W$ such that*

$$(W \cup_f h^s) \cup_g h^r \cong (W \cup_{\overline{g}} h^r) \cup_f h^s.$$

*Proof.* Recall that the attaching sphere of $h_r$ is a copy of $S^{r-1} \times \{0\}$, and the belt sphere of $h^s$ is $\{0\} \times S^{n-s}$. We have that $r - 1 + n - s \leq s - 1 + n - s = n - 1 < n$. Thus by transversality, $g$ is isotopic to a map that misses the belt sphere of $h^s$, and therefore can be isotoped until it misses $D^s \times S^{n-s} \subset \partial h^s$ altogether. Call the resulting map $\overline{g}$. We can make the thickening of our attaching maps arbitrarily close to their cores. Thus by the isotopy lemma

$$(W \cup_f h^s) \cup_g h^r \cong (W \cup_f h^s) \cup_{\overline{g}} h^r.$$

But now since $\overline{g}(S^{r-1} \times D^{n-r+1}) \cap h^s = \emptyset$, we can attach the handles in either order. □

At last, this completes the proof of the Thom-Milnor theorem, since we can now arrange for handles to appear in order of increasing index.

### 3.3. **Handle cancellation and sliding.**

**Lemma 3.8** (Handle cancellation lemma)**.** *Consider $W \cup_f h^r \cup_g h^{r+1}$ such that the belt sphere $\{0\} \times S^{n-r}$ of $h^r$ and the attaching sphere $g(S^r \times \{0\}) \subset g(S^r \times D^{n-r})$ of $h^{r+1}$ intersect transversely in exactly one point of $\partial_1(W \cup_f h^r)$. Then*

$$W \cup_f h^r \cup_g h^{r+1} \cong W,$$

*relative to $\partial_0 W$.*

The proof involves checking that an index $r$ and an index $(r+1)$ handle cancel in a standard situation: the effect of adding both is just glueing an $(n+1)$-disc to $W$ along an embedded $n$-disc in $\partial_1 W$. Alternatively there is a Morse theory proof that shows that when there is a unique trajectory between two critical points of a gradient-like vector field, the vector field can be modified in such a way that the two critical points are removed. A new vector field gives rise to a new Morse function by integration.

**Lemma 3.9** (Handle sliding)**.** *Let $h_1^r$ and $h_2^r$ be two $r$-handles attached to $\partial_1 W$ of a cobordism $(W; \partial_0 W, \partial_1 W)$ via $f_i \colon S^{r-1} \times D^{n-r+1} \hookrightarrow \partial_1 W$, $i = 1, 2$. Then $W \cup_{f_1} h_1^r \cup_{f_2} h_2^r$ is diffeomorphic to $W \cup_{f_1} h_1^r \cup_{\overline{f_2}} h_2^r$ where $\overline{f_2}$ is obtained as $f_2 \# f_1'$, where $f_1'$ is a push off of $f_1$. (Some care with framings is required to do the push-off correctly.)*

The handle slide is just an isotopy of the attaching map $f_2$, thus the outcome is diffeomorphic to the starting manifold by the isotopy lemma.

## 4. The handle chain complex

In this section we show how to make our first transition from geometry to algebra. Once we have the handle chain complex, the goal of the proof of the $h$-cobordism theorem is to simplify the handle chain complex. The excitement is to show that the algebraic manipulations can all be realised geometrically.

For the $h$-cobordism theorem, we will be working with simply connected manifolds. However in the sequel we will want to work in more generality. For this reason we will develop the theory in this section for any fundamental group. So let $\pi := \pi_1(W)$.

We define the handle chain groups

$$C_r(W, \partial_0 W; \mathbb{Z}\pi) = C_r(\widetilde{W}, \widetilde{\partial_0 W}) := H_r(\widetilde{W}^{(r)}, \widetilde{W}^{(r-1)}),$$

where the latter is singular homology, and $\widetilde{W}$ is the universal cover of $W$. Recall that for any manifold $M$ with a handle decomposition $M^{(r)}$ is the $r$-skeleton; the union of all the handles of index up to and including $r$. The $r$-handles of $\widetilde{W}$ are in one to one correspondence with the product of $\pi$ and the $r$-handles of $W$. The quotient space $\widetilde{W}^{(r)}/\widetilde{W}^{(r-1)}$ is homotopy equivalent to a wedge of $r$-spheres, and thus $C_r(W, \partial_0 W; \mathbb{Z}\pi) = H_r(\widetilde{W}^{(r)}, \widetilde{W}^{(r-1)})$ is a free $\mathbb{Z}\pi$-module with generator corresponding to the $r$-handles of $W$.

The boundary map in the handle chain complex is defined by the following diagram.

$$\begin{tikzcd}
H_r(\widetilde{W}^{(r)}, \widetilde{W}^{(r-1)}) \arrow[r] & H_{r-1}(\widetilde{W}^{(r-1)}) \arrow[r] & H_{r-1}(\widetilde{W}^{(r-1)}, \widetilde{W}^{(r-2)}) \arrow[d, "="'] \\
C_r(W, \partial_0 W; \mathbb{Z}\pi) \arrow[u, "="'] \arrow[rr, "\partial_r"'] & & C_{r-1}(W, \partial_0 W; \mathbb{Z}\pi).
\end{tikzcd}$$

Next we explain how to compute the boundary map. Choose a basepoint $p$ for $W$ and for each $r$ and for each $r$-handle $h_i^r$, choose a path $\alpha_i^r$ from $p$ to the centre point $q_i^r$ of $h_i^r$. For the $(j, i)$ entry of $\partial_{r+1} \colon C_{r+1}(W, \partial_0 W; \mathbb{Z}\pi) \to C_r(W, \partial_0 W; \mathbb{Z}\pi)$, we count the transverse intersections of the attaching sphere $A(h_i^{r+1}$ with the belt sphere $B(h_j^r)$, with a sign and an element of $\pi$, which we will describe now.

A choice of basis for the handle chain group is equivalent to choosing an orientation of the core of each handle. This induces an orientation on all attaching spheres. Also choose an orientation of the entire $r$-handle. This induces an orientation on the boundary of the handle, the ambient space in which intersection signs are computed. Together with the orientation of the core, the orientation of the $r$-handle induces an orientation of the cocore, which in turn induces an orientation of the belt sphere. Thus intersection points can be counted with sign. A change in choice of orientations of the core corresponds to a change in the choice of basis of the chain complex, replacing that basis element by its negative. Reversing the choice of orientation of the entire $r$-handle changes the orientation of its boundary, and changes the orientation of the belt sphere. Thus this latter choice was immaterial.

Now let $x \in A(h_i^{r+1}) \cap B(h_j^r)$ be an intersection point. Choose paths $\gamma_x^{r+1}$ in $h_i^{r+1}$ from $q_i^{r+1}$ to $x$ and $\gamma_x^r$ in $h_j^r$ from $q_j^r$ to $x$. Let $g$ be the loop

$$\alpha_i^{r+1} \cdot \gamma_x^{r+1} \cdot \overline{\gamma_x^r} \cdot \alpha_j^r.$$

This is the same as measuring which lift of the handle $h_j^r$ to $\widetilde{W}$ has an intersection of its belt sphere with the attaching sphere of a chosen lift of $h_i^{r+1}$. The choice of basing path $\alpha_j^r$ is equivalent to a choice of lift of $h^r$ to $\widetilde{W}$.

**Example 4.1.** A handle chain complex for the torus $S^1 \times S^1$, with

$$\pi = \pi_1(S^1 \times S^1) \cong \mathbb{Z}^2 = \langle \mu, \lambda \,|\, [\mu, \lambda] \rangle$$

is given by

$$\mathbb{Z}\pi \xrightarrow{\begin{pmatrix}\lambda-1\\ 1-\mu\end{pmatrix}} \bigoplus^2 \mathbb{Z}\pi \xrightarrow{\begin{pmatrix}\mu-1 & \lambda-1\end{pmatrix}} \mathbb{Z}\pi.$$

## 5. The $h$-cobordism theorem

For a definitive proof of the $h$-cobordism theorem, that only uses Morse functions and no handles, see Milnor's book [M1]. We will present a less technical treatment using handles following [CLM].

5.1. **Handle trading.** Start with an $r$-handle $h^r_j$. We aim to trade it for an $(r+2)$-handle. Recall that the attaching sphere $A(h^r_j) = S^{r-1}$. Take a push off $e^r_+$ of the core $D^r \times \{0\}$ to $D^r \times \{1\} \subset D^r \times S^{n-r}$. Suppose that

(1) $S^{r-1} \times \{1\} = \partial e^r_+$ is the boundary of a disc $D^r$, called $e^r_-$, embedded in $\partial_1 W^{(r-1)}$.
(2) The sphere $e^r_- \cup e^r_+ = S^r = \partial D^{r+1}$ is the boundary of a disc $e^{r+1} = D^{r+1}$ embedded into $\partial_1 W^{(r+1)}$.

For $2(r+1) < n+1$ all of $e^r_\pm$ and $e^{r+1}$ are embedded by a small homotopy by transversality. Next, create a cancelling pair $h^{r+1}, h^{r+2}$ with attaching data as follows. The attaching sphere of $h^{r+1}$ is $e^r_+ \cup e^r_- \cong S^r$. We have subsets $\partial h^{r+1} \supseteq D^{r+1} \times S^{n-r-1} \supseteq D^{r+1} \times \{pt\}$. The attaching map of $h^{r+2} = e^{r+1} \cup_{e^r_+ \cup e^r_-} (D^{r+1} \times \mathrm{pt})$ Then

$$W^{(r+1)} \cup h^{r+1} \cup h^{r+2} \cong W^{(r+1)}$$

by the cancellation lemma. On the other hand,

$$\begin{aligned} W^{(r+1)} \cup h^{r+1} \cup h^{r+2} &\cong W^{(r-1)} \cup \bigcup_{j\neq i} h^r_j \cup h^r_j \cup \bigcup_j h^{r+1}_j \cup h^{r+1} \cup h^{r+2} \\ &\cong W^{(r-1)} \cup \bigcup_{j\neq i} h^r_j \cup \bigcup_j h^{r+1}_j \cup h^{r+2}, \end{aligned}$$

since $h^r_j$ and $h^{r+1}$ are also in cancelling position. Thus given (1) and (2), for $2r < n-1$ we can exchange an $r$-handle for an $(r+2)$-handle.

5.2. 0-**handles.** For a warm up we remove all the 0-handles of $W$. Since $\partial_0 W \to W$ is a homotopy equivalence, the inclusion induces a bijection on $\pi_0(\partial_0 W) \to \pi_0(W)$. Thus every 0-handle of $W$ is subsequently connected to $\partial_0 W \times I$ by a 1-handle $h^1$. Cancel them.

5.3. 1-**handles.** Assume that all 0-handles have been removed, so the connected components of $W^{(0)}$ are in one-one correspondence with the connected components of $\partial_0 W$. The attaching sphere of a 1-handle $h^1$ is two points, which lie in the same connected component. So the two points are connected by a path $e^1_-$ in $\partial_1 W^{(0)}$.

Now $e^1_+$, which is the core of $h^1$ pushed to its boundary, has endpoints that are joined by a path $e^1_- \subset \partial_1 W^{(0)}$. We have that $\pi_1(\partial_0 W) \to \pi_1(W)$ is surjective. Thus there exists a map of a disc $D^2 \to W^{(2)}$ with boundary $e^1_+ \cup \overline{e}^1_-$, for some path $\overline{e}^1_-$ with the same endpoints as $e^1_-$. This disc can be pushed to the boundary of $W^{(2)}$, and it can be embedded, since $n \geq 5$. Thus we have the data required to trade $h^1$ for a 3-handle.

5.4. **Trading higher handles.** For $r \geq 2$, in order to trade up $r$-handles, we will use the handle chain complex. Suppose that all handles of index less that $r$ have been cancelled already. Start with:

$$\ldots \longrightarrow C_{r+2} \longrightarrow C_{r+1} C'_r \oplus \mathbb{Z}\pi \longrightarrow 0$$

where the $r$-th chain group $C_r \cong \mathbb{Z}\pi \oplus C'_r$ for some $C'_r$. The $\mathbb{Z}\pi$ that we have separated out corresponds to $h^r$. Next, add a cancelling pair in a standard way in a small neighbourhood.

$$\cdots \longrightarrow C_{r+2} \oplus \mathbb{Z}\pi \xrightarrow{\begin{pmatrix} \partial_{r+1} & 0 \\ 0 & 1 \end{pmatrix}} C_{r+1} \oplus \mathbb{Z}\pi \xrightarrow{\begin{pmatrix} \partial_{r+1} & 0 \\ \partial_{r+1} & 0 \end{pmatrix}} C'_r \oplus \mathbb{Z}\pi \longrightarrow 0$$

The hypothesis that the inclusion map is a homotopy equivalence implies that $H_r(W, \partial_0 W; \mathbb{Z}\pi) = 0$. Thus the last map

$$C_{r+1} \oplus \mathbb{Z}\pi \xrightarrow{\begin{pmatrix} \partial_{r+1} & 0 \\ \partial_{r+1} & 0 \end{pmatrix}} C'_r \oplus \mathbb{Z}\pi$$

is surjective. Thus the generator of $\mathbb{Z}\pi$ is hit by the image of $\partial_{r+1}$. Take the new $h^{r+1}$ and slide it over the other $h^{r+1}$s, according to the linear combination of handles that hit the generator of the extra $\mathbb{Z}\pi$ corresponding to $h^r$. We obtain:

$$\cdots \longrightarrow C_{r+2} \oplus \mathbb{Z}\pi \xrightarrow{\begin{pmatrix} \partial_{r+1} & * \\ 0 & 1 \end{pmatrix}} C_{r+1} \oplus \mathbb{Z}\pi \xrightarrow{\begin{pmatrix} \partial_{r+1} & * \\ \partial_{r+1} & 1 \end{pmatrix}} C'_r \oplus \mathbb{Z}\pi \longrightarrow 0$$

Now we are in a position to algebraically cancel a summand of $C_{r+1}$ and $C_r$, to obtain:

$$\cdots \longrightarrow C_{r+2} \oplus \mathbb{Z}\pi \longrightarrow C_{r+1} \longrightarrow C'_r \longrightarrow 0.$$

To replicate this geometrically, we need to know that we can realise a 1 in the boundary map by one geometric intersection in between the relevant belt and attaching spheres of the handles. This will require the Whitney trick. Assume that we can arrange for the belt sphere of the $r$-handle and the attaching sphere of the $(r+1)$-handle to intersect geometrically once. Then we can apply the cancellation lemma, and trade an $r$-handle for an $(r+1)$-handle as required.

5.5. **Proof excluding Whitney trick.** Let us engage in some wishful thinking: assume that we can always realise algebraic intersections by geometric intersections; in particular, that we can always cancel intersections geometrically if they cancel algebraically. We saw in the previous sections that we can trade handles until all handles with $2r < n-1$ are gone. Then, we *turn the handle decomposition upside down.*

An elementary cobordism $(M \times I \cup h^r; M, N)$ is also an elementary cobordism $(N \times I \cup h^{n+1-r}; N, M)$, obtained from turning the first cobordism upside down.

This is equivalent to replacing a Morse function $F$ by $-F$. Handles/critical points of index $r$ become index $n+1-r$. The core and the cocore of $D^r \times D^{n+1-r}$ are swapped, as are the attaching and belt spheres. Perform handle trading again and turn the cobordism back the roight way up. There are now no handles with $2(n+1-r) < n-1$, that is $r > n/2+1$. After this we only have handles left with indices $\lceil \frac{n-1}{2} \rceil$ and $\lfloor n/2+1 \rfloor$. These are two consecutive integers $r, r+1$. Thus the handle chain complex $C_*(W, \partial_0 W; \mathbb{Z}\pi)$ looks like

$$0 \to C_{r+1} \xrightarrow{\partial_{r+1}} C_r \to 0,$$

with $\partial_{r+1} \in \mathrm{GL}(m, \mathbb{Z}\pi)$. Now, we want to prove the $h$-cobordism theorem, so we work with $\pi$ the trivial group. Thus $\partial_{r+1}$ is represented by a matrix over the integers with determinant $\pm 1$. We can stabilise the matrix:

$$\partial_{r+1} \mapsto \begin{pmatrix} \partial_{r+1} & 0 \\ 0 & 1 \end{pmatrix},$$

corresponding to the introduction of a cancelling pair of an $r+1$ and an $r$-handle. We can also change the basis of $C_r$ and $C_{r+1}$, by handle slides. This corresponds to multiplication of $\partial_{r+1}$ by elementary matrices of the form $\mathrm{Id} + mE_{ij}$, where $m \in \mathbb{Z}$ and $E_{ij}$ is the square matrix with a 1 in the $(i,j)$ entry and zeroes elsewhere.

**Lemma 5.1.** *Let $A \in \mathrm{GL}(m, \mathbb{Z})$. There exists a sequence of row and column operations, followed by replacing basis elements $h^r$ with $-h^r$, that transform $A$ into the identity matrix.*

*Proof.* Using the Euclidean algorithm, arrange that the gcd of the first row and the first column appears somewhere in the first row and the first column (there must be a nonzero entry or else $A$ would not be invertible). Then move it to the top left, and use it to clear the rest of the first row and column. Now iterate this procedure on the submatrix of the last $m-k$ rows and the last $m-k$ columns, in the $k-1$th step of the iteration. We obtain a diagonal matrix. But all the entries must be $\pm 1$ since the matrix has determinant 1. One more basis change, replacing certain $h^r$ with $-h^r$, fixes the signs and yields the identity matrix. $\square$

Assuming again that the algebraic intersections can be realised geometrically, we can cancel all the handles, and arrive at a product cobordism. This completes the proof of the $h$-cobordism theorem, once we have stated and proved the Whitney trick, and justified that it can be applied.

An embedding $P \hookrightarrow M$ is said to be $\pi_1$-null if the induced map on fundamental groups $\pi_1(P) \to \pi_1(M)$ is the trivial map.

**Theorem 5.2** (The Whitney trick)**.** *Let $P^p, Q^q \hookrightarrow M^m$ be two $\pi_1$-null embeddings, $p+q=m$, of oriented transversely intersecting submanifolds $P, Q$ of $M$. Suppose in addition that either both $p$ and $q$ are at least 3, or that $p = 1,2$ and $\pi_1(M \smallsetminus Q) \rightarrowtail \pi_1(M)$ is injective. Let $x, y \in P \pitchfork Q$ be intersection points with opposite signs, and suppose that there are paths $\gamma_1 \subset P$ from $x$ to $y$ and $\gamma_2 \subset Q$ from $y$ to $x$ such that $\gamma_1 \cdot \gamma_2 = \{1\} \in \pi_1(M)$. There there is an isotopy of $P$ to an embedding $P'$ such that $P' \pitchfork Q = P \pitchfork Q \smallsetminus \{x, y\}$.*

Here are some remarks on the statement and its use in the proof of the $h$-cobordism theorem. If $p, q \leq 3$ then $m \geq 6$. In the case that $\dim M = 5$, we will always need $\pi_1$-injectivity. An embedding $Q \subset M$ is called $\pi_1$-negligible if $\pi_1(M) \cong \pi_1(M \setminus Q)$.

We will apply the Whitney trick with $M = \partial_1 W^{(r)}$, whose dimension is $n$ for an $(n+1)$-dimensional cobordism $W$, and with $P$ and $Q$ the intersecting attaching and belt spheres of an $r+1$ and an $r$-handle respectively, with a pair of algebraically cancelling intersections. The existence of the paths $\gamma_1$ and $\gamma_2$ with their composite homotopically trivial, is equivalent to the $\pi_1$ elements associated to the intersection points being equal.

Note that a codimension 3 submanifold is always $\pi_1$-negligible by general position. By assumption we have that the concatenation $\gamma_1 \cdot \gamma_2$ bounds an immersed disc in $M$. We need the $\pi_1$ condition to guarantee that this disc can be homotoped to a disc that lies in $M \setminus (P \cup Q)$. An issue is in the case that $Q$ is codimension one or two; then the extra assumption on $\pi_1$ injectivity is required.

For us, there is one potentially problematic case: when we want to cancel intersections between the belt sphere $S^{n-2}$ of a 2-handle and the attaching sphere $S^2$ of a 3-handle. The belt sphere $S^{n-2}$ is codimension 2. However it will not cause any problems, as we argue now. Let $M_1$ be the level set just before a 2-handle attachment, and let $M_2 \cong \partial_1 W^{(2)}$ be the the level set just after it. $M_2$ is obtained from $M_1$ by a surgery on $S^1 \times D^{n-2}$.

Note that
$$M_2 \setminus \{0\} \times S^2 \cong M_1 \setminus S^1 \times \{0\}.$$
But $\pi_1(M_1 \setminus S^1) \cong \pi_1(M_1)$ since $S^1$ is codimension greater than 2. Since there are no 1-handles (they have been traded for 3-handles by this point), $\pi_1(M_1) \cong \pi_1(\partial_0 W)$. Since the fundamental group of $\partial_0 W$ injects into $\pi_1(W)$, we see that the inclusion induced map $\pi_1(M_2 \setminus S^{n-2}) \to \pi_1(M_2)$ must be injective.

5.6. **Vector bundles.** The description and proof of the Whitney trick uses the language of vector bundles, so we introduce it here.

**Definition 5.3.** A $k$-dimensional vector bundle with structure group $\mathrm{GL}_k(\mathbb{R})$ over a space (manifold) $X$ is a space $E$ with a map $\pi \colon E \to X$ such that:

(1) For each $x \in X$, there is a neighbourhood $U_\alpha \ni x$ in $X$ and a homeomorphism (diffeomorphism)
$$h_\alpha \colon \pi^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times \mathbb{R}^k$$
such that
$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[r, "h_\alpha"] \arrow[dr, "\pi"'] & U_\alpha \times \mathbb{R}^k \arrow[d, "\mathrm{pr}_2"] \\ & U_\alpha \end{tikzcd}$$
commutes. The pairs $(U_\alpha, h_\alpha)$ are called *charts*.

(2) For any $\alpha, \beta$, the composition
$$h_\beta \circ h_\alpha^{-1} \colon (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$$

is given by $(x, \mathbf{v}) \mapsto (x, \Phi_{\alpha\beta}(x) \cdot \mathbf{v})$ where $\Phi_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}_k(\mathbb{R})$ is a continuous (smooth) function. The codomain of $\Phi_{\alpha\beta}$, $\mathrm{GL}_k(\mathbb{R})$, is called the *structure group*. (For smooth transition functions, we need that $\mathrm{GL}_k(\mathbb{R})$ is a smooth manifold.)

(3) If $(U_\alpha, h_\alpha)$ is a chart and $V \subset U$, then $(V, h_\alpha|_V)$ is also a chart.
(4) The collection of charts is maximal with respect to the conditions above.

**Example 5.4.**

(i) For a space $X$, $\pi \colon X \times \mathbb{R}^k \to X$ is the *trivial* $k$-plane vector bundle over $X$.
(ii) For a smooth manifold $M$, the tangent bundle $\pi \colon TM \to M$ is a vector bundle, with transition functions determined by the derivatives of the charts of the manifold.
(iii) Let $P^p \subset M^m$ be an embedding of a submanifold $P$ in $M$. The normal bundle of the embedding
$$\mathbb{R}^{m-p} \to \nu_{P \subset M} \to P$$
is subbundle of $TM|_P$ such that
$$\nu_{P \subset M} \oplus TP = TM|_P.$$

In the last example, we used the *Whitney sum of vector bundles*. Given $\pi \colon E \to X$ and $\pi' \colon E' \to X$, the Whitney sum $\pi \oplus \pi' \colon E \oplus E' \to X$ is defined by
$$E \oplus E' = \{(e, e') \in E \times E' \mid \pi(e) = \pi'(e')\}$$
If the fibres are $\pi^{-1}(x) \cong \mathbb{R}^k$ and $(\pi')^{-1}(x) = \mathbb{R}^\ell$ then $(\pi \oplus \pi')^{-1}(x) \cong \mathbb{R}^{k+\ell}$.

**Definition 5.5.** A *framing* of a trivial vector bundle is a choice of isomorphism $E \xrightarrow{\simeq} X \times \mathbb{R}^k$. Equivalently, a framing is a choice of $k$ linearly independent vector fields
$$v_1, \ldots, v_k \colon X \to E$$
with $\pi \circ v_i = \mathrm{Id}_X$.

For example *up to homotopy*, the framings of the trivial vector bundle $S^1 \times \mathbb{R}^2$ are affine correspondence with $\mathbb{Z}$.

Since $O(n) \hookrightarrow \mathrm{GL}_n(\mathbb{R})$ is a deformation retract, given a metric on a smooth manifold $M$ i.e. an inner product on $T_pM$ for all $p \in M$, we can reduce the structure group to $O(n)$ i.e. we can deform the transition functions so that they factor as:
$$\Phi_{\alpha\beta} \colon U_\alpha \cap U_\beta \to O(n) \hookrightarrow \mathrm{GL}_k(\mathbb{R}).$$
Using this we can define the following.

(i) The disc bundle of $E$:
$$D(E) := \{e \in E \mid \| \mathrm{pr}_2 \circ h_\alpha(e) \| \leq 1\}.$$
(ii) The sphere bundle of $E$:
$$S(E) := \{e \in E \mid \| \mathrm{pr}_2 \circ h_\alpha(e) \| = 1\}.$$
(iii) The Thom space of $E$:
$$\mathrm{Th}(E) := D(E)/S(E).$$

We do not need the Thom space right now but we give the definition here anyway.

5.7. **Whitney trick proof.** To prove the Whitney trick, first note that the dimension hypotheses and the fundamental group hypotheses imply that there is an embedded disc $D^2 \hookrightarrow M \setminus P \cup Q$ with boundary $\gamma_1 \cdot \gamma_2$. The use of transversality to embed the disc is the only place that we used the dimension restrictions.

Along $\gamma_1$, we have

$$TM = TP \oplus \nu_{P \subset M} = T\gamma_1 \oplus \nu_{\gamma_1 \subset P} \oplus \nu_{P \subset M},$$

while along $\gamma_2$ we have

$$TM = TQ \oplus \nu_{Q \subset M} = TQ \oplus \nu_{\gamma_2 \subset D} \oplus (\nu_{D^2 \subset M} \cap \nu_{Q \subset M}),$$

The subbundles $\nu_{\gamma_1 \subset P}$ and $(\nu_{D^2 \subset M} \cap \nu_{Q \subset M})$ are $(p-1)$-dimensional.

*Suppose* that we have a framing $v_1, \dots, v_{m-2}$ for $\nu_{D^2 \subset M}$, such that

(1) $v_1, \dots, v_{p-1}$ restricts to a framing for $\nu_{\gamma_1 \subset P}$ along $\gamma_1$.
(2) $v_1, \dots, v_{p-1}$ lie in $\nu_{Q \subset M}$, giving a framing for $(\nu_{D^2 \subset M} \cap \nu_{Q \subset M})$.

Since $\gamma_1$ and $\gamma_2$ are contractible, there is a unique choice of framing of the (sub)bundles over them, up to homotopy. Another way to phrase the condition above is that the framings are essentially given on the boundary of $D^2$. They glue together to give subbundle $E$ of $\nu_{S^1 \subset M}$, where $S^1 = \gamma_1 \cdot \gamma_2$. The question is whether these $(p-1)$ linearly independent vectors extend to a $(p-1)$-frame in the normal bundle of $D^2$. It turns out that this is the case precisely when $E$ is orientable: a $(p-1)$ frame bundle over a circle is trivial if and only if it extend over a disc $D^2$, if and only if it is orientable. In turn, careful consideration of orientations shows that this is the case if and only if the signs of the intersection points $x$ and $y$ (that we are trying to cancel) are opposite i.e. one is $+$ and one is $-$. This should be quite easy to see in the toy case that $p = 2$ and $q = 1$. The 1-plane bundle is an annulus if the intersection signs are opposite and is a Möbius band if the intersection signs of $x$ and $y$ agree. A detailed description can be found on [Sco, Pages 55–7].

Now think of the Whitney disc like the Russian military, and extend $D^2$ very slightly beyond its borders. More precisely, extend $\gamma_1$ slightly beyond $x$ and $y$ and push $\gamma_2$ out along the radial direction of $TD^2|_{\gamma_2}$, that is the direction orthogonal to $T\gamma_2$. Now consider the disc bundle $DE \cong D^2 \times D^{p-1} \cong D^{p+1}$. The boundary

$$\partial(P \times I \cup DE) \cong P \cup P'$$

where $P$ is the original and $P'$ is the outcome of the push of $P$ across the Whitney disc. Note that $DE \cap P = D\nu_{\gamma_1 \subset P} \cong D^p$ by construction of $E$. So there is an isotopy of this $D^p \subset \partial D^{p+1}$ across $D^{p+1}$ to the other hemisphere, producing $P'$. This completes the proof of the Whitney trick.

## 6. Algebraic $K$ theory – $K_0$ and $K_1$

We do not really need to talk about $K_0$ for the course, but since we are going to talk about $K_1$ we really should talk about $K_0$ first briefly, to give some context, and since it is good general culture to be aware of.

### 6.1. Projective modules.

**Definition 6.1.** An $R$-module $P$ is projective if one of the following are true:

(i) There exists a free $R$-module $F$ and an $R$-module $Q$ such that $P \oplus Q \cong F$.

(ii) For any surjective homomorphism $f\colon M \to N$ and for every $g\colon P \to N$ there is a homomorphism $h\colon P \to M$ such that $f \circ h = g$.

$$\begin{tikzcd} & N \arrow[d, "f"] \\ P \arrow[ur, dashed, "h"] \arrow[r, "g"] & M \end{tikzcd}$$

(iii) Every short exact sequence

$$0 \to A \to B \xrightarrow{j} P \to 0$$

splits, that is there is a homomorphism $f\colon P \to B$ such that $j \circ f = \mathrm{Id}$.

You should check that these conditions are equivalent.

### 6.2. The zeroth $K$-group of a ring.

Let $R$ be a ring (with unit). The group $K_0(R)$ is the abelian group arising from the following construction. Consider the isomorphism classes of finitely generated projective modules, and make them into a group by taking the abelian group generated by the isomorphism classes, with the relations given by short exact sequences as follows. If $0 \to A \to B \to C \to 0$ is a short exact sequence of $R$-modules, then we have the relation $[A] - [B] + [C] = 0$. In particular, note that this means that $[A] + [B] = [A \oplus B]$.

A morphism of rings $R \to S$ induces a morphism of $K_0$ groups $K_0(R) to K_0(S)$ by $A \mapsto S \otimes_R A$. We define the reduced $K_0$ group to be

$$\widetilde{K}_0(R) := \operatorname{coker}(K_0(\mathbb{Z}) \to K_0(R)).$$

### 6.3. Colimits.

Let $\{X_i\}_{i \in \mathbb{N} \cup \{0\}}$ be a collection of objects in a category together with maps $f_i\colon X_i \to X_{i+1}$ for each $i$; this is called a directed system. For example, the category could be of groups, $R$-modules, or topological spaces. The colimit $\operatorname{colim} X_i$ is the unique object which has morphisms $\phi_i\colon X_i \to\colon X_i$ for all $i$ such that $\phi_{i+1} \circ f_i = \phi_i$, which satisfies the following universal property. For any object $Y$ with maps $g_i\colon X_i \to Y$ such that $g_{i=1} \circ f_i = g_i$ for all $i$, there is a unique map $\Phi^Y\colon \operatorname{colim} X_i \to Y$ such that $\Phi^Y \circ \phi_i = g_i$. One can take $\operatorname{colim} X_i$ to be the quotient of the disjoint union $\bigsqcup X_i/\sim$ where $x_i \in X_i$ is declared equivalent to $f_i(x_i)$ for all $x_i \in X_i$ and for all $i$. This is sometimes known as a direct limit.

This is a special case of the colimit of a functor $F\colon \mathcal{C} \to \mathcal{D}$ between two categories. The colimit is an object of $\mathcal{D}$, which is the unique object that has maps $\phi(c)\colon F(c) \to \operatorname{colim}_{\mathcal{C}} F$ for each object $c$ of $\mathcal{C}$, such that $\phi(g(c)) \circ F(g) = \phi(c)$ for any morphism $g$ of $\mathcal{C}$, satisfying the following universal property. For any other object $Y$ of $\mathcal{D}$ with the same property, with the maps denoted $\phi^Y(c)\colon F(c) \to Y$, there exists a unique map $\Phi^Y\colon \operatorname{colim}_{\mathcal{C}} F \to Y$ with $\Phi^Y \circ \phi(c) = \phi^Y(c)$.

One can define the notion of a limit, also inverse limit in the case of the direct limit, by reversing all the arrows above.

6.4. **Simple homotopy equivalence.** The notion of simple homotopy equivalence is stronger than having a homotopy equivalence between spaces. An $s$-cobordism is an $h$-cobordism where the inclusion maps of the boundary components are simple homotopy equivalences. The obstruction to being a simple homotopy equivalence, which is called the Whitehead torsion, obstructs an $h$-cobordism from being a product, when the manifolds in question are not simply connected.

Let us now define simple homotopy equivalence.

**Definition 6.2.** Let $(K, L)$ be a CW pair, i.e. a pair of cell complexes. We say that $K$ collapses to $L$ by an elementary collapse $K \searrow L$ if

(i) $K = L \cup e^{n-1} \cup e^n$;
(ii) Let $\partial D^n = S^{n-1} D_+^{n-1} \cup D_-^{n-1}$. We require that there exists a characteristic map $\varphi\colon D^n \to K$ for $e^n$ such that $\varphi|\colon D_+^{n-1}$ is a characteristic map for $e^{n-1}$ and $\varphi(D_-^{n-1}) \subseteq L$.

Under these circumstances there is a elementary collapse map $K \to L$ realising the fact that the inclusion $L \to K$ is a deformation retract. The inclusion is called an elementary expansion.

In general a map $f\colon K \to L$ of CW complexes is a *simple homotopy equivalence* if $f$ is homotopic to a composition

$$K = K_0 \to K_1 \to K_2 \to \cdots \to K_n = L$$

where each map in the sequence is either an elementary expansion, an elementary collapse, or a cellular homeomorphism.

To decide on whether a map is a simple homotopy equivalence we will look at the chain complex $C_*(K, L; \mathbb{Z}[\pi])$ where $\pi$ is the fundamental group of $K$ and $L$ (the map $f$ induces an identification of their fundamental groups.) This will lead to the notion of the Whitehead torsion of a homotopy equivalence. As an easy example, suppose that $C_*(K, L; \mathbb{Z}[\pi])$ is only supported in two degrees $0 \to C_{r+1} \xrightarrow{\partial_{r+1}} C_r \to 0$. This was the situation we found ourselves in after the handle trading in the proof of the $h$-cobordism theorem. In this case the equivalence class of the matrix representing $\partial_{r+1}$ in the group $K_1(R)$ determines the Whitehead torsion. So we need to understand $K_1$ of a ring.

6.5. **The first algebraic $K$-group of a ring.** See chapter 11 of [DK] for more detail on the material of this section. The abelian group $K_1(R)$ measures to what extent an invertible matrix can be simplified using row and column operations, and stabilisation, to become an identity matrix.

Let $R$ be a ring with the property that $R^n \cong R^m$ implies $n = m$. All group rings satisfy this since $\mathbb{Z}$ does and one can apply the augmentation homomorphism $\mathbb{Z}\pi \to \mathbb{Z}$. Let $E_{ij}(r)$ be a matrix with entry $r$ in the $(i, j)$ position and zeroes elsewhere. There is a homomorphism $\mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_{n+1}(R)$ defined by sending a matrix

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

This makes $\{\mathrm{GL}_n(R) \,|\, n \in \mathbb{N}\}$ into a directed system. Define $\mathrm{GL}(R)$ to be the colimit. One can think of infinite matrices that eventually become the identity. Next let $E(R)$ be the normal subgroup of $\mathrm{GL}(R)$ generated by matrices of the form $\mathrm{Id} + E_{ij}(r)$. These are called the *elementary matrices*. Define:

$$K_1(R) := \mathrm{GL}(R)/E(R).$$

The next lemma shows that this is an abelian group.

**Lemma 6.3** (Whitehead lemma)**.** *We have that*

$$E(R) = [\mathrm{GL}(R), \mathrm{GL}(R)],$$

*the commutator subgroup. Thus* $K_1(R) = \mathrm{GL}(R)_{ab}$.

*Proof.* Let $I := \mathrm{Id}$ and $E_{ij} = E_{ij}(1)$, so $E_{ij}(r) = rE_{ij}$. First observe that

$$(I + rE_{ij})^{-1} = I - rE_{ij}$$

and that

$$I + rE_{ik} = (I + rE_{ij})(I + rE_{jk})(I + rE_{ij})^{-1}(I + rE_{jk})^{-1}.$$

Thus for $n \geq 3$, any elementary matrix is a commutator, so $E(R) \subseteq [\mathrm{GL}(R), \mathrm{GL}(R)]$. Next we have the following three identities:

(i)
$$\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} (BA)^{-1} & 0 \\ 0 & BA \end{pmatrix}.$$

(ii)
$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ I - A^{-1} & I \end{pmatrix} \begin{pmatrix} I & -I \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ I - A & I \end{pmatrix}.$$

(iii)
$$\begin{pmatrix} I & X \\ 0 & I \end{pmatrix} = \prod_{i=1}^{m} \prod_{j=1}^{n} I + x_{ij}E_{i,j+m}.$$

The left hand side of (i) represents any commutator. The right hand side of (i) contains three terms of the form of the left hand side of (ii). The right hand side of (ii) contains four terms of the form of the left hand side of (iii). The right hand side of (iii) is contained in $E(R)$. Combining these observations we have that $[\mathrm{GL}(R), \mathrm{GL}(R)] \subseteq E(R)$. This completes the proof of the Whitehead lemma. □

For any commutative ring $R$ there is a homomorphism $\det\colon K_1(R) \to R^\times$ defined by the determinant, where $R^\times$ is the group of units of $R$. In $K_1(R)$, we have that

$$\begin{pmatrix} AB & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

and we saw above that $\begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix}$ is elementary, so we have

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \sim (AB).$$

Thus the multiplication in $\mathrm{GL}(R)$ translated to direct sum in the abelianisation.

**Definition 6.4.** Let $\pi$ be a group. The *Whitehead group* of $\pi$ is

$$\mathrm{Wh}(\pi) := K_1(\mathbb{Z}\pi)/\{(\pm g) | g \in \pi\}.$$

The quotient is generated by all $1 \times 1$ matrices $(\pm g)$ where $g$ is an element of the group $\pi$. There is a short exact sequence

$$0 \to \{\pm 1\} \times \pi_{ab} \to K_1(\mathbb{Z}\pi) \to \mathrm{Wh}(\pi) \to 0.$$

Here are some examples.

(i) The Whitehead group of the trivial group vanishes $\mathrm{Wh}(\{e\}) = 0$. This is essentially the Euclidean algorithm.

(ii) So does the Whitehead group of the infinite cyclic group $\mathrm{Wh}(\mathbb{Z}) = 0$. This is a nontrivial fact.

(iii) The Whitehead group $\mathrm{Wh}(\mathbb{Z}/n)$ of a finite cyclic group of order $n$ is free abelian $\mathrm{Wh}(\mathbb{Z}/n) \cong \mathbb{Z}^k$ where $k = \lfloor n/2 \rfloor + 1 - d(n)$ where $d(n)$ is the number of divisors of $n$.

(iv) Here is an example of two equivalent elements for which stabilisation is necessary to see the equivalence. Let

$$\pi = \langle x, y | y^2 = 1 \rangle = \mathbb{Z} * \mathbb{Z}_2.$$

Define elements of $\mathbb{Z}\pi$, $a := 1 - y$ and $b = x(1 + y)$. Observe that $(1 - ab)(1 + ab) = 1$, so $(1 - ab)$ is a nontrivial unit of the group ring. We will see that this is equivalent to the zero group, but only after stabilisation:

$$\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 - ab & 0 \\ 0 & 1 \end{pmatrix}.$$

So $\big(1 - ab\big) = 0 \in \mathrm{Wh}(\pi)$ but only after stabilisation.

In general the Whitehead group is difficult to compute.

6.6. **The $s$-cobordism theorem and some applications.** We will associate an element $\tau(\widetilde{W}, \widetilde{\partial_0 W}) \in \mathrm{Wh}(\pi)$ to an $h$-cobordism $W$ with $\pi_1(W) = \pi$. This $\tau$ is called the *Whitehead torsion* of the $h$-cobordism.

**Theorem 6.5** ($s$-cobordism theorem)**.** *Let $n \geq 5$, and let $(W; M, N)$ be an $h$-cobordism. Then $W$ is diffeomorphic to $M \times I$ if and only if $\tau(\widetilde{W}, \widetilde{M}) = 0 \in \mathrm{Wh}(\pi)$.*

An $h$-cobordism with vanishing Whitehead torsion is called an $s$-cobordism. The proof of the $s$-cobordism theorem is basically the same as the proof of the $h$-cobordism theorem. The key difference is that diagonalising the matrix of

$$\partial_{r+1} \colon C_{r+1}(\widetilde{W}, \widetilde{\partial_0 W}) \to C_r(\widetilde{W}, \widetilde{\partial_0 W})$$

to be of the form

$$\begin{pmatrix} \pm g_1 & & \\ & \ddots & \\ & & \pm g_n \end{pmatrix}$$

is not always possible. We also have to check that the choices we made in the handle trading process to arrive at this simple form of the chain complex, whereby the chain complex is only supported in two adjacent degrees, do not affect whether the resulting matrix can be diagonalised i.e. whether the matrix vanishes or not in $\mathrm{Wh}(\pi)$. To arrange this we will define the Whitehead torsion on the whole chain complex. For now we just say that in the special case of the simple chain complexes $C_{r+1} \xrightarrow{\partial_{r+1}} C_r$, the Whitehead torsion is just $[\partial_{r+1}] \in \mathrm{Wh}(\pi)$. Before embarking on another algebraic investigation, we give an outline of a cool application of the $h$ and $s$-cobordism theorems to high dimensional knot theory.

**Theorem 6.6** (Unknotting theorem). *Let $n, k \in \mathbb{N}$ be such that $n+k \geq 6$ and let $K\colon S^n \hookrightarrow S^{n+k}$ be a locally flat embedding i.e. each point of the image is locally homeomorphic to a pair $(D^{n+k}, D^n)$. Suppose that either $k \geq 3$ or $k = 2$ and $S^{n+2} \smallsetminus K(S^n) \simeq S^1$. Then $K$ is topologically unknotted, that is there is an orientation preserving homeomorphism $S^{n+k} \to S^{n+k}$ sending $K(S^n)$ to the standard unknot $U(S^n)$.*

We remark that the theorem is true in all dimensions, but we do not prove it. In dimension 3 one needs the loop theorem of Papakriakopolous, while in dimension 4 one needs topological surgery and $s$-cobordism, due to Freedman and Quinn.

*Proof.* Remove a pair $(D^{n+k}, D^n)$ from $(S^{n+k}, S^n)$. We are left with a proper embedding $D^n \hookrightarrow D^{n+k}$. Here is some notation.

(1) Let $N$ be a regular neighbourhood of $K(D^k) \cong D^n \times D^k$.
(2) Let $W = D^{n+k} \smallsetminus N$.
(3) Let $M_0 = W \cap N \cong D^n \times S^{k-1}$.
(4) Let $M_1 = \partial W \smallsetminus M_0$.

We claim that $(W; M_0, M_1)$ is an $h$-cobordism. First, $H_*(W, M_0) \cong H_*(D^{n+k}, N) \cong H_*(D^{n+k}, D^n) = 0$ by excision, then homotopy invariance, then computation.

Now we split into two cases. In the case that $k \geq 3$, we have $\pi_1(W) = 1$ by transversality, and $pi_1(M_0) = \pi_1(D^n \times S^{k-1}) = 1$. So $W$ is (rel. boundary) $h$-cobordism, and is therefore a product.

In the case that $k = 2$ and $W \simeq S^1$, we also have that $M_0, M_1 \cong D^n \times S^1$, and the inclusion map induced an isomorphism on fundamental groups. The Hurewicz and Whitehead theorems therefore imply that the inclusion maps of $M_0$ and $M_1$ are homotopy equivalences. Now $\pi_1(W) \cong \mathbb{Z}$, and $\mathrm{Wh}(\mathbb{Z}) = 0$ (we will just take this as a black box). Thus $(W, M_0, M_1)$ is not just an $h$-cobordism, but rather an $s$-cobordism, and is therefore a product. In both cases we have trivial knot exteriors. Glue back in the removed disc pair $(D^{n+k}, D^n)$, to obtain a topological unknot (the new ambient space may only be topologically $S^{n+k}$, since one needs the Poincaré hypothesis for this. □

6.7. **Whitehead torsion and some homological algebra.** In this section we will define the Whitehead torsion of a chain complex, then of a chain equivalence, then of a homotopy equivalence, and finally of an $h$-cobordism.

Let $(C_*, \partial)$ be a finite, free, based, non-negative $R$-module chain complex. Here finite means that $\bigoplus_r C_r$ is finitely generated, based means that there is a fixed isomorphism $C_r \cong R^{n_r}$, $n_r < \infty$, and non-negative means that $C_r = 0$ for $r < 0$.

A chain complex is called chain contractible if there exists a sequence of homomorphisms $s\colon C_r \to C_{r+1}$ for all $r$ such that $\partial s + s\partial = \mathrm{Id}$. That is, the identity map is chain homotopic to the zero map. Note that a contractible space does not have a contractible chain complex, since a space has $H_0$, whereas for any cycle $x \in C_r$, $x = \mathrm{Id}(x) = \partial s(x) + s\partial(x) = \partial(s(x))$, and is thus a boundary. so $H_*(C_*) = 0$ for $C_*$ contractible. For projective module chain complexes, we have a converse to this statement.

**Lemma 6.7.** *If $H_*(C_*) = 0$ then $C_*$ is chain contractible.*

The details of this proof are an archetype for a common inductive argument in homological algebra.

*Proof.* We want to construct a chain homotopy $s$ fitting into a diagram of maps as follows, with $s\partial + \partial s = \mathrm{Id}$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_2 \arrow[r, "\partial_2"] \arrow[d, "\mathrm{Id}"] \arrow[dl, "s_2"'] & C_1 \arrow[r, "\partial_1"] \arrow[d, "\mathrm{Id}"] \arrow[dl, "s_1"'] & C_0 \arrow[r] \arrow[d, "\mathrm{Id}"] \arrow[dl, "s_0"'] & 0 \\
\cdots \arrow[r] & C_2 \arrow[r, "\partial_2"] & C_1 \arrow[r, "\partial_1"] & C_0 \arrow[r] & 0
\end{tikzcd}$$

First $\partial_1$ is onto since $H_0(C_*) = 0$, so there is a map $s_0 \colon C_0 \to C_1$ such that $\partial_1 \circ s_0 = \mathrm{Id}$. To construct this map, define it on free generators of $C_0$ using a choice of preimages of these generators, then extend by linearity. This uses the fact that $C_0$ is free, and would also work if $C_0$ were projective.

Next, we have a small computation:

$$\partial_1(\mathrm{Id} - s_0\partial_1) = \partial_1 - \partial_1 s_0 \partial_1 = \partial_1 - \mathrm{Id} \circ \partial_1 = 0.$$

Thus $(\mathrm{Id} - s_0\partial_1)(c) \in \ker \partial_1 = \operatorname{im} \partial_2$. since $H_1(C_*) = 0$. Now we have a lifting problem:

$$\begin{tikzcd}
 & C_1 \arrow[dl, dashed, "?\exists"'] \arrow[d, "\mathrm{Id} - s_0\partial_1"] & \\
C_2 \arrow[r, "\partial_2"] & \ker \partial_1 \arrow[r] & 0
\end{tikzcd}$$

We just calculated that the image of the vertical map indeed lies in $\ker \partial_1$, and the horizontal map is surjective by exactness. Then $C_1$ is projective, so the lifting problem can be solved, and the resulting dotted map becomes $s_1$, that satisfies $\partial_2 s_1 + s_0 \partial_1 - \mathrm{Id}$. Now repeat this argument, inducting to the left on the assumption that $s_i$ has been defined for $i < k$. □

The same idea for the proof, working backwards from the map on $C_0$, enables one to prove the *fundamental lemma of homological algebra.* This is an aside in the theory of Whitehead torsion, but is an important fact to know.

**Lemma 6.8.** *Let $P_*$ be a projective $R$-module chain complex, i.e. $P_i$ is a projective module for all $i$, and let $C_*$ be an acyclic $R$-module complex, that is $H_i(C_*) = 0$ for all $i > 0$. Both are assumed to be nonnegative. Let $\varphi \colon H_0(P_*) \to H_0(C_*)$ be a homomorphism. Then:*

*(1) there is a chain map $f_i \colon P_i \to C_i$ ($\partial_C f_{i+1} = f_i \partial_P$) such that $f_0$ induces $\varphi$ on $H_0$;*

*(2) any two such chain maps $f$ and $g$ are chain homotopic $f \sim g$, that is there exists a chain homotopy $h_i \colon P_i \to C_{i+1}$ such that $\partial_C h_i + h_{i-1}\partial_P = f_i - g_i$.*

*Proof.* We give an outline of the proof. Let $M := H_0(P_*)$ and $M' := H_0(C_*)$. First construct, using the idea of the previous lemma, the vertical maps (apart from the far right vertical map, which is given), in the following diagram, using the fact that $P_i$ is projective for all $i$ and that the bottom row is exact, now it has been augmented with $M'$.

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "\partial_P"] \arrow[d, dashed] & P_1 \arrow[r, "\partial_P"] \arrow[d, dashed] & P_0 \arrow[r] \arrow[d, dashed] & M \arrow[r] \arrow[d, "\varphi"'] & 0 \\
\cdots \arrow[r] & C_2 \arrow[r, "\partial_C"] & C_1 \arrow[r, "\partial_C"] & C_0 \arrow[r] & M' \arrow[r] & 0
\end{tikzcd}$$

This shows (i). Now let $f$ and $g$ be two such chain maps as in (ii). Construct a chain homotopy $h$, again using the idea of the proof of the lemma above, fitting into the diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "\partial_P"] \arrow[d, "f_2-g_2"] \arrow[dl, dashed, "h_2"'] & P_1 \arrow[r, "\partial_P"] \arrow[d, "f_1-g_1"] \arrow[dl, dashed, "h_1"'] & P_0 \arrow[r] \arrow[d, "f_0-g_0"] \arrow[dl, dashed, "h_0"'] & M \arrow[r] \arrow[d, "\varphi-\varphi=0"] & 0 \\
\cdots \arrow[r] & C_2 \arrow[r, "\partial_C"] & C_1 \arrow[r, "\partial_C"] & C_0 \arrow[r] & M' \arrow[r] & 0
\end{tikzcd}$$

To do this one needs the following computation:

$$\begin{aligned}
&\partial_C((f_n - g_n) - h_{n-1}\partial_P) = (f_{n-1} - g_{n-1})\partial_P - \partial_C h_{n-1}\partial_P \\
=&f_{n-1}\partial_P - g_{n-1}\partial_P - h_{n-2}\partial_P^2 - f_{n-1}\partial_P + g_{n-1}\partial_P = 0.
\end{aligned}$$

We leave the details to the reader. They can be found, for example, in chapter 2 of [DK]. □

Now we define the Whitehead torsion of a contractible, finite, based free $R$–module chain complex, with a chain contraction $s$. Consider:

$$\partial + s \colon \bigoplus_{r \text{ odd}} C_r = C_1 \oplus C_3 \oplus C_5 \oplus \cdots \to \bigoplus_{r \text{ even}} C_r = C_0 \oplus C_2 \oplus C_4 \oplus \cdots$$

**Lemma 6.9.**

$$\partial + s = \begin{pmatrix} \partial & 0 & 0 & \dots & \\ s & \partial & 0 & & \\ 0 & s & \partial & 0 & \\ 0 & 0 & s & \partial & \ddots \\ & & & & \ddots \end{pmatrix} : C_{\text{odd}}] \to C_{\text{even}}$$

*is an isomorphism of chain complexes.*

*Proof.* The inverse $(\partial + s)^{-1}$ is given by $(1+s^2)^{-1}(s+\partial)$ Concretely, we have matrices:

$$\begin{pmatrix} 1 & 0 & 0 & & & & \\ -s^2 & 1 & 0 & & & & \\ s^4 & -s^2 & 1 & 0 & & & \\ -s^6 & s^4 & -s^2 & 1 & 0 & & \\ s^8 & -s^6 & s^4 & -s^2 & 1 & 0 & \\ \vdots & \vdots & & & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} s & \partial & 0 & & & & \\ 0 & s & \partial & 0 & & & \\ & 0 & s & \partial & 0 & & \\ & & 0 & s & \partial & \ddots & \\ & & & & \ddots & \ddots & \\ & & & & & \ddots & \end{pmatrix} : C_{\text{even}} \to C_{\text{odd}}$$

It can be checked that the given matrix on the left really is the inverse of $\mathrm{Id} + s^2$ and that the two maps together are indeed an inverse to $\partial + s$. $\square$

Since $C_*$ is based, $\partial + s$ determines an element of $\mathrm{GL}(R)$. We want to define

$$\tau(C_*) = [\partial + s] \in \widetilde{K}_1(R) = K_1(R)/[-1] = \mathrm{coker}(K_1(\mathbb{Z}) \to K_1(R)).$$

The next lemma shows that this is independent on the choice of chain contraction. In general the torsion $\tau$ does depend on the choice of basis of $C_*$; for this we will cite a theorem that says this is not the case for CW complexes.

**Lemma 6.10.** *Let $s$ and $s'$ be two chain contractions for $C_*$. Then $\tau(\partial + s) = \tau(\partial + s')$.*

*Proof.* From [CLM]. Consider the two maps

$$(\partial + s)_{\text{odd}} \colon C_{\text{odd}} \to C_{\text{even}}$$

and

$$(\partial + s')_{\text{even}} \colon C_{\text{even}} \to C_{\text{odd}}.$$

Also define

$$\Delta := (s' - s)s; \quad \Delta' := (s - s')s'$$

where

$$\Delta_i, \Delta_i' \colon C_i \to C_{i+2}.$$

Now all the following compositions are represented by upper triangular matrices, with respect to the given basis, thus all represent the zero element of $\widetilde{K}_1(R)$.

$$\begin{aligned} 1 + \Delta'_{\text{odd}} &\colon C_{\text{odd}} \to C_{\text{odd}} \\ 1 + \Delta_{\text{even}} &\colon C_{\text{even}} \to C_{\text{even}} \\ (\partial + s)_{\text{odd}} \circ (1 + \Delta'_{\text{odd}}) \circ (\partial + s')_{\text{even}} &\colon C_{\text{even}} \to C_{\text{odd}} \\ (\partial + s')_{\text{even}} \circ (\mathrm{Id} + \Delta_{\text{even}}) \circ (\partial + s)_{\text{odd}} &\colon C_{\text{odd}} \to C_{\text{odd}} \end{aligned}$$

Let $A = \tau((\partial + s)_{\text{odd}})$ and let $B = (\partial + s')_{\text{even}}$. Then $A + B = 0 \in \widetilde{K}_1(R)$ so $A = -B$. But $A$ is independent of $S$ and $B$ is independent of $s'$. Thus both $A$ and $B$ are independent of both $s$ and $s'$. So $\tau(C_*) = A = \tau((\partial + s)_{\text{odd}})$ is independent of the choice of chain contraction $s$. $\square$

Let $\phi\colon X \to Y$ be a map of topological spaces. The *mapping cylinder* is the identification space

$$\mathcal{M}_\phi\colon X \times I \cup Y/\{(x,1) \sim \phi(x) \,|\, x \in X\}.$$

It is homotopy equivalent to $Y$, and is often used to replace $Y$ by a homotopy equivalent space with $X \to \mathcal{M}(\phi)$ an inclusion. The *mapping cone* is

$$\mathrm{cone}(\phi) = \mathcal{M}_\phi/X \times \{0\}.$$

These topological constructions have algebraic analogues. The *algebraic mapping cylinder* $M(f)$ of a chain map $f\colon C \to D$ has boundary map

$$\partial^{M(f)}_{r+1}\colon M(f)_{r+1} = C_{r+1} \oplus C_r \oplus D_{r+1} \to M(f)_r = C_r \oplus C_{r-1} \oplus D_r$$

given by

$$\begin{pmatrix} \partial^C_{r+1} & (-1)^r \,\mathrm{Id} & 0 \\ 0 & \partial^C_r & 0 \\ 0 & (-1)^r f & \partial^D_{r+1} \end{pmatrix}.$$

The *algebraic mapping cone* $\mathscr{C}$ of a chain map $f\colon C \to D$ has boundary map given by

$$\partial^{\mathscr{C}(f)}_{r+1}\colon \mathscr{C}(f)_{r+1} D_{r+1} \oplus C_r \to \mathscr{C}(f)_r := D_r \oplus C_{r-1}$$

given by

$$\begin{pmatrix} \partial^D_{r+1} & (-1)^r f \\ 0 & \partial^C_r \end{pmatrix}.$$

**Example 6.11.** Let $C_0 = \mathbb{Z}$ and $C_r = 0$ for $r \neq 0$, and let $D_r = C_r$ for all $r$. Let $f = \mathrm{Id}\colon C_0 \to D_0$. Then $\mathscr{C}(f) = (\mathbb{Z} \xrightarrow{\mathrm{Id}} \mathbb{Z})$.

Let $f\colon X \to Y$ be a map of spaces, and let $f\colon C_*(X) \to C_*(Y)$ also denote the induced map on chain complexes. Then

$$C_*(\mathrm{cone}(f), \mathrm{pt}) \simeq \dot{C}_*(\mathrm{cone}(f)) \simeq$$

where $\dot{C}$ is the reduced chain complex of $C$, i.e. $C$ augmented with a $\mathbb{Z}$ in degree $-1$. So the chain complexes of the mapping cone and the algebraic mapping cone differ in degree 0, but are elsewhere identical.

**Definition 6.12.** Let $f\colon C \to D$ be a chain homotopy equivalence of based free finite non-negative chain complexes. Then

$$\tau(f) := \tau(\mathscr{C}(f)) \in \widetilde{K}_1(R).$$

In order for this to make sense we need the following lemma.

**Lemma 6.13.** *If a chain map $f\colon C \to D$ is a chain homotopy equivalence then $\mathscr{C}(f)$ is chain contractible.*

*First proof.* There exist maps $g, h, k$, with $g$ the homotopy inverse to $f$ and $h, k$ homotopies, such that:

$$fg - \mathrm{Id} = \partial h + h\partial$$

and

$$gf - \mathrm{Id} = \partial k + k\partial$$

where $g\colon D_r to C_r$, $h\colon D_r \to D_{r+1}$ and $k\colon C_r \to C_{r+1}$. Then

$$\begin{pmatrix} 1 & (-1)^{r+1}(fk - hf) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h & 0 \\ (-1)^r g & k \end{pmatrix} : D_r \oplus C_{r-1} \to D_{r+1} \oplus C_r$$

is a chain contraction. There is a certain amount of matrix multiplication required of the reader to verify this. □

We give another proof that some will find more satisfying and others might find less so.

*Second proof.* Define the *suspension* $(\Sigma C)_i = C_{i-1}$, with the boundary maps $(\Sigma\partial)_i = \partial_{i-1}$. The short exact sequence of chain complexes

$$0 \to D \to \mathscr{C}(f) \to \Sigma C \to 0$$

induced a long exact sequence in homology (any short exact sequence of chain complexes does, by the snake lemma, that was used in Algebraic Topology I to construct the long exact sequence in homology of a pair, or the Mayer-Vietoris sequence.) Then the fact that $f$ is a homotopy equivalence implies that the connected homomorphism of this sequence $H_{r+1}(\Sigma(C)) \to H_r(D)$ is an isomorphism for all $r \geq 0$, which implies by exactness (and $H_0(\Sigma C) = H_{-1}(C) = 0$) that $H_r(\mathscr{C}(f)) = 0$ for all $r$.

□

Now we have defined the Whitehead torsion of an acyclic chain complex, and of a chain equivalence between such chain complexes, we switch to geometry.

**Definition 6.14.** Let $f\colon X to Y$ be a homotopy equivalence of finite CW complexes. Let

$$\widetilde{f}\colon C_*(X;\mathbb{Z}[\pi_1(X)]) \to C_*(Y;\mathbb{Z}[\pi_1(Y)])$$

be the induced chain map. We have $\pi_1(X) \cong \pi_1(Y) =: \pi$ with an identification given by $f$, so both are $\mathbb{Z}[\pi]$-modules. Choose lifts of each cell to $\widetilde{X}$, or $\widetilde{Y}$ as appropriate. This is equivalent to choosing a basing path for each cell. Then these chosen lifts give rise to a basis for $C_*(X;\mathbb{Z}[\pi])$, $C_*(Y;\mathbb{Z}[\pi])$, which are f.g. free based, nonnegative chain complexes over $\mathbb{Z}[\pi]$. We therefore obtain a basis for $\mathscr{C}(\widetilde{f})$.

The *Whitehead torsion* of the homotopy equivalence $f$ is $\tau(\widetilde{f}) \in \mathrm{Wh}(\pi)$.

We showed that the torsion does not depend on the choice of chain contraction. However right now it can depend on the basis for the chain complex. It does not depend on the choice of lifts since this change is quotiented out in the definition of the Whitehead group. However the next theorem, which we will not prove, says that for $CW$ complexes the torsion is topological invariant i.e. is independent of the choice of cell subdivision.

**Theorem 6.15** (Chapman)**.** *The Whitehead torsion of a homotopy equivalence of CW complexes $\tau(\widetilde{f})$ is well-defined and $\tau(\widetilde{f}) = 0$ if and only if $f$ is a simple homotopy equivalence.*

The torsion of an $h$-cobordism $(W; \partial_0 W, \partial_1 W)$ is now easy to define. The torsion of the $h$-cobordism is the torsion of the homotopy equivalence $\partial_0 W \to W$, or $\partial_1 W \to W$, denoted $\tau(W, \partial_0 W) \in \mathrm{Wh}(\pi_1(W))$. This amounts to the class of the single matrix $\partial_{r+1} \colon C_{r+1}(\widetilde{W}) \to C_r(\widetilde{W})$ obtained after handle trading above and below the middle dimension to get a chain complex supported in only two degrees. But no prior trading is required to define the obstruction, and we see that the obstruction is independent of any choices made in the handle trading process. Eventually, for cobordisms of dimension at least 6, all handles can be cancelled if and only if the torsion of the $h$-cobordism vanishes, in which case it is an $s$-cobordism and therefore is a product.

## 7. Universal coefficient homology and cohomology

In the past you have probably mostly studied homology with ordinary $\mathbb{Z}$ coefficients, or perhaps you have seen deRham cohomology with $\mathbb{R}$ coefficients. We will be particularly concerned with $\mathbb{Z}[\pi_1(X)]$-coefficient homology and cohomology. It is obstructions defined with reference to these groups that precisely determine the answers to the problems of surgery theory; is the manifold structure set of a Poincaré complex $X$ nonempty, and is a given homotopy equivalence of manifolds homotopic to a diffeomorphism?

### 7.1. **Cohomology.**

**Definition 7.1.** An involution on a ring $R$ with unity is a functor $\bar{\cdot} \colon R \to R$ such that $\overline{r+s} = \overline{r} + \overline{s}$, $\overline{1} = 1$, $\overline{rs} = \overline{s} \cdot \overline{r}$ and $\overline{\overline{r}} = r$ for all $r, s \in R$.

Let $(C_*, \partial)$ be a chain complex of left $R$-modules. We have that

$$C^r := (C_r)^* = \mathrm{Hom}_R(C_r, R)$$

is naturally a right $R$-module, by

$$f \cdot r = (c \mapsto f(c)r).$$

There is a coboundary map

$$\begin{array}{rcl} \partial^*_{r+1} \colon C^r & \to & C^{r+1} \\ \partial^*(f) & = & (c \mapsto f \circ \partial(c)). \end{array}$$

This defines a cochain complex; the co- prefix indicates that the degree of the boundary maps is $+1$. The cohomology is then defined to be

$$H^r(C_*) := \frac{\ker(\partial^*_{r+1} \colon C^r \to C^{r+1})}{\mathrm{im}(\partial^*_r \colon C^{r-1} \to C^r)}$$

The ordinary/$\mathbb{Z}$-coefficient cohomology of a space $X$ is $H^r(X) := H^r(C_*(X))$ where $C_*(X)$ is the singular/cellular/handle chain complex with $\mathbb{Z}$-coefficients of $X$. One of the main motivations for introducing cohomology and cochain complexes is to be able to state the Poincaré duality theorem, which is one of the fundamental properties of a manifold, and a key advantage these spaces have over arbitrary CW complexes.

Given a chain complex $C_*$, let $C^{n-*}$ be the chain complex given by

$$\delta_{n-r+1} := (-1)^{n-r+1}\partial^*_{n-r+1} \colon (C^{n-*})_r := C^{n-r} \to (C^{n-*})_{r-1} := C^{n-r+1}$$

Since $\mathbb{Z}$ is abelian, any right module can automatically also be considered as a left module. We do this with $C^{n-*}(M)$, which thus becomes a left $\mathbb{Z}$-module chain complex, instead of a right $\mathbb{Z}$-module cochain complex.

**Theorem 7.2** ($\mathbb{Z}$-coefficient Poincaré duality)**.** *Let $M$ be an $n$-dimensional oriented, compact manifold with empty boundary. Then $C_*(M) \simeq C^{n-*}(M)$*

*Proof.* Let $C_*(M)$ be the handle chain complex. It is quite straightforward to see that this is chain equivalent to a cellular chain complex of a space homotopy equivalent to $M$, obtained by contracting the $D^{n-r}$s of each handle. It is a much more non-trivial fact that the cellular chain complex of a manifold is chain equivalent to the singular complex. So we work with the handle chain complex, which arises from a Morse function $f$ on $M$. Now replace $f$ by $-f$, and note that the handle chain complex associated to $-f$ is $C^{n-*}(M)$. Then we use another non-trivial fact that the chain homotopy type of a cellular chain complex of a space is independent of the choice of CW decomposition, thus the chain complex for both handle decompositions, the original and the upside-down decomposition, are chain equivalent. □

Here are some examples of cohomology of spaces.

(1) Let $\mathbb{T}^n$ be the $n$-torus. Then the cohomology is much the same as the homology:

$$H^r(\mathbb{T}^n;\mathbb{Z}) \cong \bigoplus_{\binom{n}{r}} \mathbb{Z}$$

This uses the Künneth theorem, which I advise you to read about, for example in [Hat], if you have not done so already. There is a homology and a cohomology version.

(2) A chain complex of a lens space is

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{0} \mathbb{Z},$$

and the homology groups are $\mathbb{Z}, \mathbb{Z}_p, 0, \mathbb{Z}$ in dimensions $0, 1, 2, 3$ respectively. The corresponding cochain complex is isomorphic to

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{0} \mathbb{Z},$$

and the cohomology groups are $\mathbb{Z}, 0, \mathbb{Z}_p, \mathbb{Z}$ in dimensions $0, 1, 2, 3$. Note that the torsion shifts from degree one to degree two. On the one hand this is consistent with Poincaré duality. On the other hand, we can also understand this shift in the torsion in terms of the universal coefficient theorem, which will be the subject of the next subsection, after we have discussed more general coefficient systems.

Cohomology satisfies the following properties.

(LES of pair) Let $A \subset X$ be a pair of spaces. Then the following is a long exact sequence in cohomology groups, arising from the short exact sequence in cochain complexes $0 \to C^*(X,A) \to C^*(X) \to C^*(A) \to 0$.

$$\cdots \to H^r(X,A) \to H^r(X) \to H^r(A) \to H^{r+1}(X,A) \to H^{r+1}(X) \to \cdots$$

(Homotopy invariance) Let $f,g\colon (X,A) \to (Y,B)$ be homotopic maps. Then

$$f^* = g^*\colon H^r(Y,B) \to H^r(X,A).$$

(Excision) Let $A \subseteq X$ and let $U \subseteq A$ be such that $\overline{U} \subset \operatorname{Int} A$. Then there is an isomorphism

$$H^r(X,A) \xrightarrow{\simeq} H^r(X \smallsetminus U, A \smallsetminus U).$$

(Mayer-Vietoris) Let $U,V \subseteq X$ be open subsets of $X$ with $U \cup V = X$. Then the following is a long exact sequence in cohomology:

$$\cdots \to H^{r-1}(U \cap V) \to H^r(X) \to H^r(U) \oplus H^r(V) \to H^r(U \cap V) \to H^{r+1}(X).$$

Next we will upgrade this to the $\mathbb{Z}[\pi_1(M)]$-coefficient version, known as universal Poincaré duality. A better version of Poincaré duality states that this chain equivalence is induced by a very specific map, cap product with a fundamental class, that is a generator $[M]$ of $H_n(M;\mathbb{Z})$. (This last statement is true in all coefficients.) We will introduce the cap product, and in particular the Poincaré duality cap product

$$- \cap [M]\colon C^{n-r}(\widetilde{M}) \to C_r(\widetilde{M}),$$

but we will skip the proof that it is this that gives rise to the Poincaré duality chain equivalence. See e.g. [Br],[Hat] for a detailed proof.

**Definition 7.3.** For a ring with involution $R$ we can consider a left $R$ module $C$ as a right $R$ module, denoted $\overline{C}$, via $c \cdot r := \overline{r}c$.

For a space $X$ with $\pi := \pi_1(X)$, define the $\mathbb{Z}\pi$-module cochain complex to be $C^r(X;\mathbb{Z}\pi) := \operatorname{Hom}_{\mathbb{Z}\pi}(\overline{C_r(X;\mathbb{Z}\pi)}, \mathbb{Z}\pi)$. Here we mean homomorphisms of right modules, and then the left $\mathbb{Z}\pi$-module structure of $\mathbb{Z}\pi$ allows us to consider $C^r(X;\mathbb{Z}\pi)$ as a left $\mathbb{Z}\pi$-module.

The $\mathbb{Z}\pi$-module cohomology is then $H^r(X;\mathbb{Z}\pi) := H_r(C^*(X;\mathbb{Z}\pi)$.

Note that the $\mathbb{Z}\pi$-coefficient cohomology is not the same as the ordinary cohomology of the universal cover $\widetilde{X}$ of $X$. The $\mathbb{Z}\pi$-coefficient cohomology is isomorphic, forgetting the module structure, to the cohomology with compact supports of $\widetilde{X}$. For example, any aspherical space has contractible universal cover, so the ordinary cohomology is the same as that of a point. But there is no chance of such cohomology exhibiting Poincaré duality, so we will always work with the $\mathbb{Z}\pi$-module cohomology.

**Definition 7.4** (General coefficients)**.** Let $N$ be an $(R,\mathbb{Z}\pi)$-bimodule, with $X$ a space and $\pi = \pi_1(X)$ as above. The $N$-coefficient chain complex of $X$ is

$$C_*(X;N) := N \otimes_{\mathbb{Z}\pi} C_*(X;\mathbb{Z}\pi).$$

This is a left $R$-module, as is the $N$-coefficient homology

$$H_r(X;N) := H_r(C_*(X;N))$$

The $N$-coefficient cochain complex is

$$C^*(X;N) := \mathrm{Hom}_{\mathrm{right}-\mathbb{Z}\pi}(\overline{C_*(X;\mathbb{Z}\pi)}, N)$$

and the $N$-coefficient cohomology is

$$H^r(X;N) := H_r\big(\mathrm{Hom}_{\mathrm{right}-\mathbb{Z}\pi}(\overline{C_*(X;\mathbb{Z}\pi)}, N)\big).$$

This is again a left $R$-module.

Important examples of $N$ include:

(1) $R = N$ a commutative ring, with a representation $\mathbb{Z}\pi \to \mathrm{Aut}(R)$ factoring as $\mathbb{Z}\pi \to \mathbb{Z} \to \mathrm{Aut}(R)$ via the augmentation map $\mathbb{Z}\pi \to \mathbb{Z}$.
(2) $R = \mathbb{Z}\pi$ with $N = \mathbb{Z}\pi$ a $\mathbb{Z}\pi$-bimodule. This is the universal case already discussed above.
(3) Let $R = \mathbb{F}$ be a field, and consider a representation $\rho\colon \pi \to \mathrm{GL}(d,\mathbb{F})$. Then we obtain homology $H_*^\rho(X;\mathbb{F}^d)$; adding the representation into the notation here is not mandatory, but it might be useful to distinguish in case multiple representations could be used.

**Theorem 7.5** (Universal Poincaré duality)**.** *Let $M$ be an $n$-dimensional oriented, compact manifold with empty boundary, let $\pi = \pi_1(M)$, let $R$ be a ring with involution and let $N$ be an $(R,\mathbb{Z}\pi)$-bimodule. Then $C^{n-*}(M;N) \simeq C_*(M;N)$ is a chain equivalence of left $\mathbb{Z}\pi$-module chain complexes.*

We will return to this later in the context of cap products, but first we want to describe the universal coefficient theorems.

## 7.2. Universal coefficient theorems.

This section did not appear in the lectures, but it is very useful to know, and I prepared something to say about it which I did not get time to say in class, so here it is.

**Definition 7.6** (Left and right exact)**.**

(1) A covariant functor $F\colon R\text{−mod} \to R\text{−mod}$ is called left exact if for any short exact sequence
$$0 \to A \to B \to C \to 0$$
we have that
$$0 \to F(A) \to F(B) \to F(C)$$
is exact.
(2) A contravariant functor $F\colon R\text{−mod} \to R\text{−mod}$ is called left exact if for any short exact sequence
$$0 \to A \to B \to C \to 0$$
we have that
$$0 \to F(C) \to F(B) \to F(A)$$
is exact.

(3) A covariant functor $F\colon R{-}\mathrm{mod} \to R{-}\mathrm{mod}$ is called right exact if for any short exact sequence
$$0 \to A \to B \to C \to 0$$
we have that
$$F(A) \to F(B) \to F(C) \to 0$$
is exact.

(4) A contravariant functor $F\colon R{-}\mathrm{mod} \to R{-}\mathrm{mod}$ is called left exact if for any short exact sequence
$$0 \to A \to B \to C \to 0$$
we have that
$$F(C) \to F(B) \to F(A) \to 0$$
is exact.

A functor that is both left and right exact is called exact.

For example, let $N$ be an $R$-bimodule. Then:

(1) The functor $M \mapsto \mathrm{Hom}_R(M, N)$ is left exact contravariant.
(2) The functor $M \mapsto \mathrm{Hom}_R(N, M)$ is left exact covariant.
(3) The functor $M \mapsto N \otimes_R M$ is right exact covariant.

As an explicit example, consider the chain complex:
$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0.$$
Tensor this with $\mathbb{Z}/2$, to obtain
$$\mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{\cong} \mathbb{Z}/2 \to 0.$$
Tensoring is right exact. On the other hand applying $\mathrm{Hom}(-, \mathbb{Z})$ yields
$$0 \to \mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}) = 0 \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{f \mapsto 2f} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}).$$
This is a left exact functor. We will focus on the case of Hom, since this will be more relevant for the study of intersection forms which play a big role in surgery theory.

**Definition 7.7.** An $R$ module $I$ is said to be injective if the diagram
$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d] & N \arrow[dl, dashed, "\exists"] \\ & I & \end{tikzcd}$$
for any $R$ modules $M$ and $N$.

**Theorem 7.8.** *Any divisible module over a PID is injective.*

A good exercise is to prove this for divisible abelian groups, since $\mathbb{Z}$ is a PID. Here a module $M$ is divisible if for any $m \in M$ and for every $n \in \mathbb{Z} \smallsetminus \{0\}$ there exists an $m' \in M$ such that $nm' = m$.

**Proposition 7.9.** *Let $P$ be a projective module, let $I$ be an injective module, and let $0 \to A \to B \to C \to 0$ be a short exact sequence. Then*

$$0 \to \mathrm{Hom}(C, I) \to \mathrm{Hom}(B, I) \to \mathrm{Hom}(A, I) \to 0$$

*and*

$$0 \to \mathrm{Hom}(P, A) \to \mathrm{Hom}(P, B) \to \mathrm{Hom}(P, C) \to 0$$

*are exact.*

The proposition follows immediately from the definitions. Often in applications the modules in question will not be projective or injective as required, and we want a way to understand the failure of the previous proposition to hold in these cases. For this we use Ext groups, which we will now work towards defining.

**Definition 7.10.** Given an $R$-module $M$, a *projective resolution* is an exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0,$$

where $P_i$ is a projective $R$-module for all $i \in \mathbb{N} \cup \{0\}$.

An *injective resolution of* $M$ is an exact sequence

$$0 \to M \to I_0 \to I_1 \to I_2 \to \cdots$$

where $I_i$ is an injective $R$-module for all $i \geq 0$.

The deleted resolutions are

$$P_* = \cdots \to P_2 \to P_1 \to P_0$$

and

$$I_* = I_0 \to I_1 \to I_2 \to \cdots$$

which are exact apart from at $P_0$ and $I_0$.

We will focus to begin with on projective resolutions and the functor between $R$-modules $M \mapsto \mathrm{Hom}_R(M, N)$. This has what is called a "derived functor" called Ext. We will obtain $R$-modules $\mathrm{Ext}^i_R(M, N)$, with $i \geq 0$.

**Proposition 7.11.** *Let $f \colon M \to N$ be an $R$-module homomorphism. Then there is a unique chain homotopy class of chain maps $P^M_* \to P^N_*$ induces the given map $f \colon M \to N$.*

This follows immediately from the fundamental lemma of homological algebra.

**Definition 7.12** ($\mathrm{Ext}^n_R$)**.** Let $M$ and $N$ be $R$-modules and let $P_* \to M \to 0$ be an $R$-module projective resolution, with $P_*$ the deleted resolution. Form $\mathrm{Hom}_R(P_*, N)$. Then

$$\mathrm{Ext}^n_R(M, N) := H_n(\mathrm{Hom}_R(P_*, N)).$$

Equivalently, let $0 \to N \to I_*$ be an injective resolution of $N$, with $I_*$ the deleted resolution. Then

$$\mathrm{Ext}^n_R(M, N) := H_n(\mathrm{Hom}_R(M, I_*))$$

It turns out that the definitions are equivalent with a projective resolution of the first argument or an injective resolution of the second argument. Here are some straightforward remarks.

(i) $\operatorname{Ext}^0_R(M,N) = \ker(\operatorname{Hom}_R(P_0,N) \to \operatorname{Hom}_R(P_1,N) = \operatorname{Hom}_R(M,N)$.
(ii) If $M$ is projective then $\operatorname{Ext}^i_R(M,N) = 0$ for all $i > 0$.
(iii) If $N$ is injective then $\operatorname{Ext}^i_R(M,N) = 0$ for all $i > 0$.

Now let $0 \to A \to B \to C \to 0$ be a short exact sequence of chain complexes. Each of these modules has a projective resolution, and the short exact sequence lifts to a short exact sequence of chain complexes

$$\begin{tikzcd} 0 \arrow[r] & P^A_* \arrow[r] \arrow[d] & P^B_* \arrow[r] \arrow[d] & P^C_* \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

by the fundamental lemma of homological algebra. Dualising the top row gives rise to

$$0 \to \operatorname{Hom}_R(P^C_*,N) \to Hom_R(P^B_*,N) \to \operatorname{Hom}_R(P^A_*,N) \to 0.$$

This is a short exact sequence of cochain complexes, which induces a long exact sequence in cohomology:

$$\begin{array}{ccccccccc} 0 & \to & \operatorname{Hom}_R(C,N) & \to & \operatorname{Hom}_R(B,N) & \to & \operatorname{Hom}_R(A,N) & \to & \\ & \to & \operatorname{Ext}^1_R(C,N) & \to & \operatorname{Ext}^1_R(B,N) & \to & \operatorname{Ext}^1_R(A,N) & \to & \\ & \to & \operatorname{Ext}^2_R(C,N) & \to & \operatorname{Ext}^2_R(B,N) & \to & \operatorname{Ext}^2_R(A,N) & \to & \cdots \end{array}$$

Here are some examples of the Ext groups. The $\operatorname{Ext}^0$ groups are equal to the corresponding Hom groups, so we omit the discussion of them.

(1) $\operatorname{Ext}^n_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}/p) = 0$ for $n > 0$.
(2) $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n,\mathbb{Z}) = \mathbb{Z}/n$, and they $\operatorname{Ext}^i$ groups vanish for $i > 1$. In general, $\operatorname{Ext}^1_{\mathbb{Z}}(A,\mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}(A,\mathbb{Q}/\mathbb{Z})$, which picks up the torsion subgroup of $A$.
(3) If $R$ is a field, $\operatorname{Ext}^i_R(M,N) = 0$ for $i > 0$.
(4) If $R$ is a PID, then $\operatorname{Ext}^i_R(M,N) = 0$ for $i > 1$.
(5) $\operatorname{Ext}^n_R(\oplus_\alpha A_\alpha, B) = \prod_\alpha \operatorname{Ext}^n_R(A_\alpha, B)$.
(6) $\operatorname{Ext}^n_R(A, \oplus_\alpha B_\alpha) = \prod_\alpha \operatorname{Ext}^n_R(A, B_\alpha)$.

Now we state and prove the universal coefficient theorem for cohomology, in the case that $R$ is a PID. In more generality for rings of homological dimension greater than one, there is a universal coefficient spectral sequence. But we won't cover that here.

**Theorem 7.13** (The universal coefficient theorem)**.** *Let $R$ be a PID, let $M$ be an $R$-module and let $(C_*,\partial)$ be a f.g. free $R$-module chain complex. Then*

$$0 \to \operatorname{Ext}^1_R(H_{r-1}(C_*),M) \xrightarrow{\alpha} H^r(C_*;M) \xrightarrow{\beta} \operatorname{Hom}_R(H_r(C_*),M) \to 0$$

*is an exact sequence of abelian groups, which is natural in chain maps of $C_* \to C'_*$, and which splits, but the splitting is not natural. The map $\beta$ sends $[f] \mapsto ([c] \mapsto f(c))$. If $M$ is an $(R,S)$-bimodule, then this is an exact sequence of $S$-modules.*

*Proof.* This proof is essentially from [Br]. Note that the map $\beta$ is well-defined since $f$ is a cocycle and $c$ is a cycle.

Recall that for $R$ a PID, any submodule of a free module is free. Also recall that for any chain complex $C_*$ we let

$$Z_p := \ker(\partial_p \colon C_p \to C_{p-1})$$

the $p$-cycles, and

$$B_p = \mathrm{im}(\partial_{p+1} \colon C_{p+1} \to C_p),$$

the $p$-boundaries. Of course $H_p(C_*) = Z_p/B_p$. There are two exact sequences of $R$-modules, for each $p$.

(1) $0 \to Z_p \xrightarrow{\chi} C_p \xrightarrow{\theta} B_{p-1} \to 0$. The submodule $B_{p-1}$ is free, whence projective, so the sequence splits. Let $\phi \colon C_p \to Z_p$ be a splitting.

(2) $0 \to B_p \xrightarrow{\gamma} Z_p \xrightarrow{q} H_p(C_*) \to 0$.

The proof will follow from the next diagram, and some fun diagram chasing.

$$\begin{tikzcd}
 & & & 0 & \\
0 \arrow[r] & \mathrm{Hom}(B_p, M) \arrow[r, "\theta^*"] & \mathrm{Hom}(C_{p+1}, M) & \mathrm{Ext}^1(H_{p-1}(C_*), M) \arrow[u] & \\
0 & \mathrm{Hom}(Z_p, M) \arrow[l] \arrow[u, "\gamma^*"'] \arrow[r, shift right, "\phi^*"'] & \mathrm{Hom}(C_p, M) \arrow[u, "\delta"'] \arrow[l, shift right, "\chi^*"'] & \mathrm{Hom}(B_{p-1}, M) \arrow[l, "\theta^*"'] \arrow[u] & 0 \arrow[l] \\
 & \mathrm{Hom}(H_p(C_*), M) \arrow[u, "q^*"'] & \mathrm{Hom}(C_{p-1}, M) \arrow[u, "\delta"] \arrow[r, "\chi^*"] & \mathrm{Hom}(Z_{p-1}, M) \arrow[u, "\gamma^*"'] \arrow[r] & 0 \\
 & 0 \arrow[u] & & &
\end{tikzcd}$$

Here is some explanation of the diagram. The left 3 terms of the top row come from the dual of (1), and $\mathrm{Hom}(-, M)$ is left exact so the part shown is exact. The middle row is also the dual of (1). Here $Z_p$ is a submodule of a free module and hence is free, since $R$ is a PID. Thus $\mathrm{Ext}^1_R(Z_p, M) = 0$, so the middle row is exact. The left column is the dual of (2), and is left exact. The right three terms of the bottom row also come from the dual of (1), which is exact as described above. The middle column is the dual of the chain complex $C_*$. This is not exact, but $\delta^2 = 0$. The right hand column is part of the long exact sequence associated to the dual of (2). The two squares commute.

Now the proof is a diagram chase. Let $f \in \mathrm{Hom}(C_p, M)$, with $f \in \ker \delta$. Go left and up to get an element of $\mathrm{Hom}(B_p, M)$. By commutativity of the top left square, and injectivity of $\theta^*$, this is the zero element. Let $g = \chi^*(f) \in \mathrm{Hom}(Z_p, M)$. Then there is an $h \in \mathrm{Hom}(H_p, M)$ with $g = q^*h$. We define $\beta(f) := h$. To see that this is well defined note that if we replace $f$ with $f + \delta k$, then by commutativity of the bottom right square $\delta k \in \mathrm{im}(\theta^*)$, so maps to zero in $\mathrm{Hom}(Z_p, M)$, and therefore does not change the element of $\mathrm{Hom}(H_p, M)$ by injectivity of $q^*$.

The composition $\phi^* \circ q^*$ induces the splitting. This also shows surjectivity.

The map $\mathrm{Ext}^1(H_{p-1}, M) \to \mathrm{Hom}(C_p, M)$ is defined by lifting $x \in \mathrm{Ext}^1(H_{p-1}, M)$ to an element $y \in \mathrm{Hom}(B_{p-1}, M)$, then taking $\theta^*(y)$. More diagram chases show that this is well defined and injective.

It remains to show exactness at $H^p(C_*)$. This is also a straightforward diagram chase that is left to the reader. □

There is an analogous derived functor Tor for the tensor product. We give a less detailed treatment, but give the main statements here. Given $R$-modules $M$ and $N$ let $P_*^M$ and $P_*^N$ be projective resolutions. Then

$$\mathrm{Tor}_n^R(M, N) := H_n(P_*^M \otimes_R N)$$

or

$$\mathrm{Tor}_n^R(M, N) := H_n(M \otimes_R P_*^N).$$

Note that $\mathrm{Tor}_0^R(M, N) = M \otimes_R N$. Given an exact sequence $0 \to A \to B \to C \to 0$, we have a long exact sequence

$$\begin{array}{ccccccccc} \cdots & \to & \mathrm{Tor}_2^R(A,N) & \to & \mathrm{Tor}_2^R(B,N) & \to & \mathrm{Tor}_2^R(C,N) & \to & \\ & \to & \mathrm{Tor}_1^R(A,N) & \to & \mathrm{Tor}_1^R(B,N) & \to & \mathrm{Tor}_1^R(C,N) & \to & \\ & \to & A \otimes_R N & \to & B \otimes_R N & \to & C \otimes_R N & \to & 0 \end{array}$$

**Theorem 7.14** (Universal coefficient theorem for homology)**.** *Let $R$ be a PID, let $C_*$ be a f.g. free $R$-module chain complex, and let $M$ be an $R$-module. Then there is a split natural short exact sequence of abelian groups*

$$0 \to H_r(C_*) \otimes_R M \to H_r(C_* \otimes_R M) \to \mathrm{Tor}_1^R(H_{r-1}(C_*), M) \to 0.$$

*The splitting is not natural. If $M$ is an $(R, S)$-bimodule, this is a split exact sequence of $S$-modules.*

**Remark 7.15.** Both of the universal coefficient theorems are special cases of corresponding universal coefficient spectral sequences. In fact, there are more general Künneth spectral sequences that imply the universal coefficient spectral sequences and imply the ordinary Künneth theorem.

Here is an application. Let $W$ be a simply connected closed 4-manifold. Then $H_3(M;\mathbb{Z}) = 0 = H_1(M;\mathbb{Z})$ and $H_2(M;\mathbb{Z})$ is torsion-free. (We use Poincaré duality here too. To see this we have $H_3(M) \cong H^1(M) \cong \mathrm{Hom}(H_1(M), \mathbb{Z}) = 0$. The first isomorphism is by Poincaré duality and the second is from the universal coefficient theorem, since $\mathrm{Ext}^1(H_0(M), \mathbb{Z}) = 0$ as $H_0(M)$ is torsion-free. Then $H_1(M) = 0$ since $\pi_1(M) = 0$. Next, $H_3(M) = 0$ implies that $\mathrm{Ext}^1_{\mathbb{Z}}(H_2(M), \mathbb{Z}) \cong H^3(M)$, but $H^3(M) \cong H_1(M) = 0$, so $\mathrm{Ext}^1(H_2(M), \mathbb{Z}) = 0$, and $H_2(M)$ is torsion free as claimed.

## 8. Cap and cup products

Let $X$ be a finite CW complex. Let $D\colon X \to X \times X$ be the diagonal map. On the chain level (singular chains) this corresponds to a map $D\colon C_*(X) \to C_*(X \times X)$.

**Definition 8.1.** The tensor product of two chain complexes $C, D$ is another chain complex $C \otimes D$ with

$$(C \otimes D)_n := \bigoplus_{p+q=n} C_p \otimes D_q$$

and

$$\partial^{\otimes}(c_p \otimes d_q) = \partial c \times d + (-1)^{\deg c = p} c \otimes \partial d.$$

As an exercise, show that $(\partial^{\otimes})^2$.

**Theorem 8.2** (Eilenberg-Zilber theorem)**.** *For any two spaces $X$ and $Y$ there is a chain equivalence*

$$A\colon C_*(X \times Y) \simeq C_*(X) \otimes C_*(Y)$$

*and the chain equivalence is unique up to chain homotopy. The chain equivalence is natural, in the sense that maps on chain complexes induced by continuous maps of spaces commute with the Eilenberg-Zilber maps.*

Define

$$\Delta = A \circ D\colon C_*(X) \to C_*(X) \otimes C_*(X).$$

The *cup product* is a homomorphism

$$\cup\colon H^p(X) \otimes H^q(X) \to H^{p+q}(X)$$

induced by a map $C^p(X) \times C^q(X) \to C^{p+q}(X)$

$$(a, b) \mapsto \Delta^*(a \otimes b).$$

The cup product has the following properties:

(i) Let $1 \in H^0(X)$ be a generator of the zeroth cohomology. Then $1 \cup a = a \cup 1 = a$ for all $a$.
(ii) $(a \cup b) \cup c = a \cup (b \cup c)$.
(iii) $a \cup b = (-1)^{\deg a \deg b} b \cup a$.

Thus the direct sum of all the cohomology groups of a space form a ring, with cup product the multiplication.

The *cap product* is a map

$$\begin{array}{rcl} \cap\colon C^p(X) \times C_{p+q}(X) & \to & C_p(X) \\ (f, x) & \mapsto & E(f \otimes \Delta(x)) = f \cap x \end{array}$$

where $E(f \otimes x \otimes y) = f(y) \cdot x$ evaluated the first coordinate on the last. This has the important property that

$$\partial(f \cap x) = (-1)^p \delta f \cap x + f \cap \partial x.$$

Thus cap product descends to a well-defined map on homology/cohomology

$$\cap\colon H^q(X) \otimes H_{p+q}(X) \to H_q(X).$$

The *Alexander-Whitney diagonal approximation* is defined as follows, on singular chains. A singular simplex $\sigma$ has a front $p$-face

$${}_p\lfloor \sigma(t_1, \ldots, t_p) = \sigma(t_0, \ldots, t_p, 0 \ldots, 0)$$

and a back $q$-face

$$\sigma(t_1, \ldots, t_q)\rfloor_q = \sigma(0, \ldots, 0, t_0, \ldots, t_q).$$

A diagonal approximation is then given by

$$\begin{array}{rcl}\Delta\colon C_*(X) & \to & C_*(X)\otimes C_*(X)\\ \sigma & \mapsto & \sum_{p+q=n} {}_p\lfloor \sigma \otimes \sigma \rfloor_q.\end{array}$$

Using this we can define the cap product, for $f \in C^p(X)$ and $g \in C^q(X)$ by

$$\begin{aligned}(f \cup g)(\sigma) &= \Delta^*(f \otimes g)(\sigma) = (f \otimes g)\Delta(\sigma)\\ &= (f \otimes g)\left(\sum_{p+q=n} {}_p\lfloor \sigma \otimes \sigma \rfloor_q\right)\\ &= f({}_p\lfloor \sigma) \cdot g(\sigma\rfloor_q).\end{aligned}$$

We can also define the cap product, for $\alpha \in C^q(X)$ and $\sigma \in C_{p+q}(X)$ by

$$\alpha \cap \sigma = \alpha(\sigma\rfloor_q) \cdot {}_p\lfloor \sigma.$$

## 9. Universal coefficient Poincaré duality

Let $M$ be an $n$-dimensional manifold, and let $\pi = \pi_1(M)$. We have a diagonal map of the universal cover $\widetilde{M} \to \widetilde{M} \times \widetilde{M}$ and we have a corresponding diagonal approximation map, again by the Eilenberg-Zilber theorem,

$$\widetilde{\Delta}\colon C_*(\widetilde{M}) \to C_*(\widetilde{M}) \to_{\mathbb{Z}} C_*(\widetilde{M}).$$

Tensor both the domain and the codomain with $\mathbb{Z}$ to obtain

$$\operatorname{Id} \otimes \widetilde{\Delta}\colon \mathbb{Z} \otimes_{\mathbb{Z}\pi} C_*(\widetilde{M}) \to \mathbb{Z} \otimes_{\mathbb{Z}\pi} (C_*(\widetilde{M}) \otimes_{\mathbb{Z}} C_*(\widetilde{M})).$$

where on the right the action of $\mathbb{Z}\pi$ on the tensor product $C_*(\widetilde{M}) \otimes_{\mathbb{Z}} C_*(\widetilde{M})$ is via the diagonal action. We obtain

$$\Delta = \operatorname{Id} \otimes \widetilde{\Delta}\colon C(\widetilde{M}) \otimes_{\mathbb{Z}\pi} C_*(\widetilde{M}).$$

The image

$$\Delta([M]) \in (C_*(\widetilde{M}) \otimes_{\mathbb{Z}\pi} C_*(\widetilde{M}))_n$$

determines a chain map

$$- \cap [M]\colon C^{n-*}(\widetilde{M}) \to C_*(\widetilde{M}),$$

using the *slant* chain map, which holds for any f.g. free $R$-module chain complexes:

$$\begin{array}{rl}\backslash\colon C \otimes_R D \to \operatorname{Hom}_R(C^{-*}, D) & \\ x \otimes y & \mapsto \big(f \mapsto \overline{f(x)} \cdot y\big).\end{array}$$

Here note that $C^{-*}$ is a chain complex with $(C^{-*})_r := \operatorname{Hom}_R(C_{-r}, R)$, and not a cochain complex. By definition, given two chain complexs $A_*$ and $B_*$, we have a chain complex $\operatorname{Hom}_R(A_*, B_*)$ with

$$\operatorname{Hom}_R(A_*, B_*)_n = \bigoplus_r \operatorname{Hom}_R(A_r, B_{r+n})$$

and

$$\partial^{\operatorname{Hom}} f = (-1)^r \partial \circ f + f \circ \partial.$$

The chain map $\backslash(\Delta([M])) = - \cap [M]$ is a chain equivalence by Poincaré duality (we are omitting the proof that this is the chain equivalence, (although we saw from

Morse theory that one exists) – to be covered in a student talk). This chain map induces isomorphisms

$$- \cap [M] = PD\colon H^{n-r}(\widetilde{M}) \xrightarrow{\cong} H_r(\widetilde{M}).$$

Recall the following definition.

**Definition 9.1.** An $n$-dimensional Poincaré complex is a connected finite CW complex with an element $[X] \in H_n(X;\mathbb{Z})$ such that $- \cap [X]\colon H^{n-r}(\widetilde{X}) \xrightarrow{\cong} H_r(\widetilde{X})$ is an isomorphism for all $r$.

A Poincaré complex is called *simple* if $- \cap [X]\colon C^{n-r}(\widetilde{X}) \xrightarrow{\cong} C_r(\widetilde{X})$ is a simple chain homotopy equivalence.

Any closed manifold is a simple Poincaré complex. A Poincaré complex has an intersection pairing

$$\begin{array}{rcl} \lambda\colon H_r(\widetilde{X}) \times H_{n-r}(\widetilde{X}) & \to & \mathbb{Z}\pi \\ ([x],[y]) & \mapsto & PD^{-1}(x)(y) \end{array}$$

where $PD^{-1}\colon C_*(\widetilde{X}) \to C^{n-*}(\widetilde{X})$ is a chain homotopy inverse to $PD = - \cap [X]$. This has the following geometric interpretation for manifolds. Let $A, B$ be smooth, compact, oriented, closed manifolds and let $A^a, B^b \subset M^m$ be submanifolds, with $a+b=m$, $A \pitchfork B$ and basing paths $\gamma_A$ and $\gamma_B$ in $M$ from $A, B$ respectively to the basepoint of $M$. Suppose that $A$ and $B$ are $\pi_1$-null, that is the inclusion induced map $\pi_1(A) \to \pi_1(M)$ is the trivial map, and similarly for $B$. (This discussion can be done without the $\pi_1$-null assumption, if we just work over $\mathbb{Z}$-coefficients/ without passing to a covering space.)

The geometric intersection pairing of $A, B$ is defined to be

$$\lambda_{geom}(A,B) = A \cdot B = \sum_{p \in A \pitchfork B} \epsilon_p g(p),$$

where $\epsilon(p)$ is the sign associated to a transverse intersection point and $g(p)$ is the concatenation of paths from the basepoint of $M$, along $\gamma_A$, in $A$ to the intersection point, in $B$ to the end of $\gamma_B$, then back along $B$ to the basepoint of $M$, considered as an element of $\pi_1(M)$.

**Theorem 9.2.** *$\lambda_{geom}(A,B) = \lambda([A],[B])$, where $[A] \in H_a(\widetilde{M})$, $[B] \in H_b(\widetilde{M})$ are the associated homology classes.*

We will not prove this unfortunately, but we will prove the Thom isomorphism theorem, which is the main ingredient in the proof. See Section 10.7 of [DK] for a proof.

## 10. Pontryagin-Thom construction

Recall that for a vector bundle $\eta\colon E \to X$ with an inner product on each fibre we have:

(i) The disc bundle of $E$:

$$D(E) := \{(x, \mathbf{v}) \in E \mid \|\mathbf{v}\| \leq 1\}.$$

(ii) The sphere bundle of $E$:

$$S(E) := \{(x, \mathbf{v}) \in E \mid \|\mathbf{v}\| = 1\}.$$

(iii) The Thom space of $E$:

$$\mathrm{Th}(E) := D(E)/S(E).$$

Note that $\mathrm{Th}(\eta \oplus \epsilon^k) = \Sigma^k \mathrm{Th}(\eta)$, where $\epsilon^k$ is the trivial $k$-plane bundle over $X$. Here $\Sigma^k X$ is the $k$-fold reduced suspension of $X$, $\Sigma^k X := S^k \wedge X$, where $\wedge$ is the *smash product* of two based spaces $X$ and $Y$

$$X \wedge Y = X \times Y/(X \times \{y\} \cup \{x\} \times Y)$$

The space that we quotient out by is homeomorphic to a wedge $X \vee Y \subset X \times Y$. Note as an example, note that $S^k \wedge S^l \cong S^{k+l}$.

Given a manifold $M^n \subset S^{n+k}$ we have a normal bundle of the embedding $\nu(M) \to M$. The total space of the normal bundle can be embedded in $S^{n+k}$, and we always do this from now on.

There is a *collapse map*

$$c \colon S^{n+k} \to \mathrm{Th}(\nu(M))$$

which by definition is the identity on the interior of $D(\nu(M))$ and which maps everything in $S^{n+k}$ that is outside the interior of the normal disc bundle to the basepoint of $\mathrm{Th}(\nu(M))$.

Now let $X$ be a CW complex and fix a $k$-dimensional vector bundle $\xi \colon E \to X$. Define the bordism group

$$\Omega_n(\xi) = \{(M, i, f, \overline{f})\}/\sim$$

where the data is a manifold that maps to $X$ embedded in $S^{n+k}$ with a bundle map of the normal bundle to $\xi$:

$$\begin{tikzcd}
\nu M \arrow[r, "\overline{f}"] \arrow[d] & E \arrow[d, "\xi"] \\
M \arrow[r, "f"] \arrow[d, hook, "i"'] & X \\
S^{n+k} &
\end{tikzcd}$$

The equivalence relation is bordism of this data over $X \times I$.

**Theorem 10.1** (Pontryagin-Thom isomorphism theorem)**.** *The following map is an isomorphism*

$$\begin{array}{rcl}
\Omega_n(\xi) & \xrightarrow{\cong} & \pi_{n+k}(\mathrm{Th}(\xi)) \\
(M, i, f, \overline{f}) & \mapsto & [S^{n+k} \xrightarrow{c} \mathrm{Th}(\nu(M)) \to \mathrm{Th}(\xi)]
\end{array}$$

*For the second map, note that a morphism of vector bundles induces a map of Thom spaces.*

To see that this is a bijection, construct an inverse as follows. Make $f\colon S^{n+k} \to \mathrm{Th}(\xi)$ transverse to the zero section $X \subset \mathrm{Th}(\xi)$ and take the inverse image $M := f^{-1}(X)$. This comes with a map to $X$, an embedding $i$, and a map of its normal bundle to $\xi$. One has to check that homotopic maps induce bordant manifolds, that bordant manifolds induce homotopic maps, and that this really is an isomorphism.

Here is an important special case. Let $X$ be a point. Then the $k$ dimensional vector bundle over $X$ is of course trivial. A square

$$\begin{tikzcd} \nu M \arrow[r, "\overline{f}"] \arrow[d] & \mathbb{R}^k \arrow[d, "\xi"] \\ M \arrow[r, "f"] & \mathrm{pt} \end{tikzcd}$$

is the same as a framing of the normal bundle of $M$. Then $\Omega_n(\xi_k) = \Omega_n^{fr(k)}$, so $\operatorname{colim} \Omega_n(\xi_k) = \Omega_n^{fr}$, the framed bordism group. The Thom space of $\xi$ is just $S^k$ in this case, so the Pontryagin -Thom construction together with the colimit (by definition $\operatorname{colim} \pi_{n+k}(S^k) = \pi_n^S$, the stable $n$-stem), yields an isomorphism

$$\Omega_n^{fr} \cong \pi_n^S.$$

In particular, by investigating framed bordism of oriented 1- and 2-manifolds, one can show that $\pi_1^S$ and $\pi_2^S$ are isomorphic to $\mathbb{Z}/2$. Note that stable framings on a circle are in one to one correspondence with $\pi_1(SO) = \mathbb{Z}/2$. To see the framed bordism of surfaces requires more work, in particular the Arf invariant that we will return to later.

10.1. **Pontryagin Thom for $\Omega_n(X)$.** In this section we give a construction that converts the bordism groups $\Omega_n(X)$ into a colimit of homotopy groups of certain Thom spaces. Let $\xi_k\colon E_k \to BSO(k)$ be the universal $k$-plane vector bundle. That is,

$$BSO(k) = \widetilde{Gr}_k(\mathbb{R}^\infty),$$

the Grassmannian of oriented $k$-dimensional subspaces of $\mathbb{R}^\infty$. Then

$$E_k = \{(V, x) | V \in \widetilde{Gr}_k(\mathbb{R}^\infty), x \in V\}.$$

which maps to $BSO(k)$ via $(V, x) \mapsto V$.

**Theorem 10.2.** *Every oriented $k$-plane vector bundle over $X$ is the pull back $f^*\xi_k$, for some function $f\colon X \to BSO(k)$ of the universal vector bundle. There is a one to one correspondence between isomorphism classes of vector bundles and homotopy classes of maps $X \to BSO(k)$.*

The trivial bundle $X \times \mathbb{R}^k$ corresponds to a null homotopic map.

Consider the map of bundles

$$\begin{tikzcd} E_k \oplus \epsilon \arrow[r, "\overline{j_k}"] \arrow[d, "\xi_k \oplus \epsilon"'] & E_{k+1} \arrow[d, "\xi_{k+1}"'] \\ BSO(k) \arrow[r, "j_k"] & BSO(k+1) \end{tikzcd}$$

Here $j_k$ shifts all coordinates one to the right and adds the basis vector $(1, 0, 0, \dots)$ to each plane. This is covered by a unique map $\overline{j_k}$. This yields

$$\begin{tikzcd} X \otimes (E_k \oplus \epsilon) \arrow[r, "\overline{i_k}:=\mathrm{Id}\times\overline{j_k}"] \arrow[d, "\gamma_k\oplus\epsilon=(\mathrm{Id}\times\xi_k)\oplus\epsilon"'] & X \times E_{k+1} \arrow[d, "\gamma_{k+1}=\mathrm{Id}\times\xi_{k+1}"] \\ X \times BSO(k) \arrow[r, "i_k:=\mathrm{Id}\otimes j_k"] & X \times BSO(k+1) \end{tikzcd}$$

This square defines a map of bordism groups

$$\Omega_n(\overline{i_k}) \colon \Omega_n(\gamma_k) \to \Omega_n(\gamma_{k+1}).$$

Let $\nu_k \colon \Omega_n(\gamma_k) \to \Omega_n(X)$ be given by $(M, i, f\overline{f}) \mapsto (M, \pi \circ f)$ where $\pi \colon X \times BSO(k) \to X$ is the projection. Note that $\nu_k = \nu_{k+1} \circ \Omega_n(\overline{i_k})$. One can show that $\operatorname{colim} \Omega_n(\gamma_k) = \Omega_n(X)$. On the other hand, the suspension homomorphism gives rise to a map

$$\pi_{n+k}(\mathrm{Th}(\gamma_k)) \to \pi_{n+k+1}(\Sigma\,\mathrm{Th}(\gamma_k)) = \pi_{n+k+1}(\mathrm{Th}(\gamma_k \oplus \epsilon)) \xrightarrow{\overline{i_k}} \pi_{n+k+1}(\mathrm{Th}(\gamma_{k+1})).$$

We can therefore consider

$$\operatorname{colim} \pi_{n+k}(\mathrm{Th}(\gamma_k)).$$

Each of the individual terms $\Omega_n(\gamma_k) \cong \pi_{n+k}(\mathrm{Th}(\gamma_k))$ are isomorphic by the Pontryagin Thom construction, and the maps are induced by the same map, therefore the colimits are equal, and we have

$$\Omega_n(X) \cong \operatorname{colim} \pi_{n+k}(\mathrm{Th}(\gamma_k)).$$

This translates bordism groups into homotopy theory, and is the beginning of the subject of spectra. We will need the Pontryagin Thom construction again soon.

Our aim is to understand degree one normal maps with target $X$. The first question of surgery theory will be, does a degree one normal map to $X$ exist, and if so how many are there up to normal bordism. The existence of such a map is a necessary condition for there to be a manifold homotopy equivalent to $X$. (If there is such a manifold $M$, then pull back its normal bundle using the homotopy inverse $g \colon X \to M$ of $f$, to get a bundle on $X$ as required).

**Definition 10.3.** Let $X$ be a Poincaré complex. An $n$-dimensional normal map over $X$ consists of a stable vector bundle $\xi \colon E \to X$, an $n$-dimensional manifold $M$ embedded in $S^{n+k}$ for $k$ large, and a bundle map

$$\begin{tikzcd} \nu(M) \arrow[r, "b"] \arrow[d] & E \arrow[d, "\xi"] \\ M \arrow[r, "f"] & X \end{tikzcd}$$

where $\nu(M)$ is the stable normal bundle. The set $\mathcal{N}_n(X)$ is the set of normal bordism classes of *degree one* normal maps over $X$. Note that here the bundle $\xi$ is not fixed, and we must have that $f$ is of degree one.

As a warm up, suppose that we fix a $k$-dimensional vector bundle $\xi\colon E \to X$ and forget about the degree one condition for now. Then the normal bordism classes of normal maps $(f, b)\colon (M, \nu(M)) \to (X, \xi)$ are in one to one correspondence with the stable homotopy group $\pi_n^S(\mathrm{Th}(\xi))$, again by the Pontryagin-Thom construction.

However, this is not enough, since we will want to understand the normal bordism classes where the vector bundle $\xi$ is not fixed.

## 11. Fibrations

We need the notion of a fibration, so we briefly introduce this notion here.

**Definition 11.1.** A fibration is a map $f\colon X \to Y$ with the *homotopy lifting property.* This is the property that for any space $A$ with homotopy between two maps of $A$ to $Y$, and such that one of these maps lifts to $X$, we can also lift the homotopy to $X$. That is, for any $A$ and any maps, a dotted map exists as shown in the next diagram:

$$\begin{tikzcd} A \times \{0\} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ A \times [0,1] \arrow[r] \arrow[ur, dashed, "\exists"] & Y. \end{tikzcd}$$

For example, any fibre bundle is a fibration (but not vice versa). The fibres of a fibration are in general not homeomorphic to each other, but they are all homotopy equivalent, if $Y$ is bath connected.

To see this, let $f\colon E \to B$ be a fibration and consider a path $\alpha\colon I \to B$ with $\alpha(i) = b_i$ for $i = 0, 1$. Let $E_{b_i} := f^{-1}(b_i)$. Let $H\colon E_{b_0} \times I \to B$ be given by $(e, t) \mapsto \alpha(t)$. We have a diagram:

$$\begin{tikzcd} E_{b_0} \times \{0\} \arrow[r] \arrow[d] & E \arrow[d] \\ E_{b_0} \times I \arrow[r, "H"] \arrow[ur, dashed, "\exists", "\widetilde{H}"'] & B \end{tikzcd}$$

Then we obtain a map $\widetilde{H}_1\colon E_{b_0} \to E_{b_1}$. One has to show that for any choice of path $\alpha'$ with the same endpoints, and for any choice of lift $\widetilde{H}'$ that fits into the diagram associated to $\alpha'$ (or the same $\alpha$), we have that

$$\widetilde{H}_1 \sim \widetilde{H}'_1\colon E_{b_0} \to E_{b_1}.$$

Also note that $\alpha^{-1}$ gives rise to a map $E_{b_1} \to E_{b_0}$. $\alpha \circ \alpha^{-1}$ is homotopic to the constant path, the associated map of fibres is homotopic to the identity. Therefore $\widetilde{H}_1(\alpha)$ and $\widetilde{H}_1(\alpha^{-1})$ are homotopy inverses.

**Definition 11.2.** A map of fibrations is a commuting square

$$\begin{array}{ccc} E & \xrightarrow{\overline{f}} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{f} & B' \end{array}$$

A fibre homotopy between two maps of fibrations from $p$ to $p'$ is a commuting square

$$\begin{array}{ccc} E \times I & \xrightarrow{\overline{H}} & E' \\ {\scriptstyle p\times \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle p'} \\ B \times I & \xrightarrow{H} & B' \end{array}$$

A fibre homotopy equivalence between fibrations $p\colon E \to B$ and $p'\colon E' \to B$ is a pair of fibration maps $(\mathrm{Id}, \overline{f})$ and $(\mathrm{Id}, \overline{g})$ with $\overline{f}\colon E \to E'$ and $\overline{g}\colon E' \to E$, both covering the identity on $B$, such that

$$(\mathrm{Id}, \overline{f} \circ \overline{g})\colon (B, E') \to (B, E')$$

and

$$(\mathrm{Id}, \overline{g} \circ \overline{f})\colon (B, E) \to (B, E)$$

are both fibre homotopic to the identity, via a fibre homotopy that is the identity on $B$ throughout.

We will be interested mainly in fibrations whose fibres are homotopy spheres.

**Definition 11.3.** A $(k-1)$-spherical fibration over $X$ is a fibration with homotopy fibre $S^{k-1}$, that is all fibres are homotopy equivalent to $S^{k-1}$.

$$S^{k-1} \to E \to X.$$

The sphere bundle of a vector bundle is such a spherical fibration. For any spherical fibration there is an associated disc fibration

$$D^k \to D(E) \to X$$

where

$$D(E) = \mathcal{M}(E \to X)$$

is the mapping cylinder. A spherical fibration also has a Thom space

$$\mathrm{Th}(E) = D(E)/E = \mathrm{cone}(E \to X).$$

The Thom space of a vector bundle is the same as the Thom space of the sphere bundle of the vector bundle, so they deserve the same name.

Let $G(k)$ be the monoid of self homotopy equivalences of $S^{k-1}$. There is a classifying space $BG(k)$ for fibrations with $(k-1)$-sphere fibre and a universal $(k-1)$-spherical fibration over this classifying space.

**Theorem 11.4.** *The fibre homotopy equivalence classes of $(k-1)$-spherical fibrations over $X$ are in one to one correspondence with homotopy classes of maps $X \to BG(k)$.*

We will tend to think of a vector bundle as a map $X \to BO(k)$ and a spherical fibration as a map $X \to BG(k)$.

## 12. Thom isomorphism theorem

In this section we present the Thom isomorphism theorem. We will only prove it in the case that the base space $X$ is a Poincaré complex, but it is true in more generality. See Chapter 10 of Milnor and Stasheff [MS] for the proof when $X$ is any CW complex.

Let $\eta\colon E \to X$ be an oriented $k$-dimensional vector bundle over an $n$-dimensional Poincaré complex $X$. (We could also replace $\eta$ by a $(k-1)$-spherical fibration and $SE$ by $E$ in what follows, and get a Thom isomorphism theorem for spherical fibrations.) Let $i\colon X \to DE$ be the 0-section. We can consider the Poincaré pair $(DE, SE)$.

The definition of cup and cap products descend to relative cup and cap products, as follows. Let $A \subseteq X$. The relative cup product is:

$$\cup\colon H^p(X) \times H^q(X,A) \to H^{p+q}(X,A).$$

or

$$\cup\colon H^p(X,A) \times H^q(X) \to H^{p+q}(X,A).$$

The relative cap product is

$$\cap\colon H^p(X,A) \times H_{p+q}(X,Y) \to H_q(X)$$

or

$$\cap\colon H^p(X) \times H_{p+q}(X,Y) \to H_q(X,A).$$

We will need two properties of cup and cap products in the upcoming proof, so we record them here.

(i) For any $x, y, z$ in the appropriate groups, $x \cap (y \cap z) = (x \cup y) \cap z$.
(ii) For a map $f\colon X \to Y$ and any $x, y$, we have that $y \cap f_*(x) = f_*(f^*(y) \cap x)$.

Given a map $f\colon X^n \to Y^m$ of Poincaré complexes, or more generally $f\colon (X, \partial X) \to (Y, \partial Y)$ of Poincaré pairs, there are Umkehr or shriek maps:

$$f_! := (- \cap [X]) \circ f^* \circ (- \cap [Y])^{-1}\colon H_{m-p}(Y) \to H_{n-p}(X)$$

$$f_! := (- \cap [X]) \circ f^* \circ (- \cap [Y])^{-1}\colon H_{m-p}(Y, \partial Y) \to H_{n-p}(X, \partial X)$$

for the homology; note they go in the opposite direction to the usual functoriality, because we have used duality. Then for cohomology we have:

$$f^! := (- \cap [Y])^{-1} \circ f_* \circ (- \cap [X])\colon H^{n-p}(X) \to H^{m-p}(Y)$$

$$f^! := (- \cap [Y])^{-1} \circ f_* \circ (- \cap [X])\colon H^{n-p}(X, \partial X) \to H^{m-p}(Y, \partial Y).$$

The next theorem is an important one, the Thom isomorphism theorem. It posits the existence of a cohomology class of the Thom space of a vector bundle, called the Thom class, which can be combines with cup and cap products to link the homology and cohomology of the Thom space with the homology and homology of the base space, with a shift in degree by $k$. (Note that $H^{r+k}(DE, SE)$ is the reduced $(r+k)$th cohomology of the Thom space $\mathrm{Th}(\eta)$.)

Below, we only give the $\mathbb{Z}$-coefficient version for vector bundles. As remarked above, it can be easily translated to hold for spherical fibrations. Also, there is a $\mathbb{Z}\pi$-equivariant version.

**Theorem 12.1** (Thom isomorphism theorem)**.** *There is a cohomology class $\tau_E \in H^k(DE, SE)$, called the* Thom class, *such that*

$$H^r(X) \xrightarrow{\eta^*} H^r(DE) \xrightarrow{\cup\tau_E} H^{r+k}(DE, SE)$$

*and*

$$H_{r+k}(DE, SE) \xrightarrow{\tau_E\cap -} H_r(DE) \xrightarrow{\eta_*} H_r(X).$$

*are isomorphisms, that coincide with $i^!$ and $i_!$ respectively.*

*Proof.* We will prove the cohomology version only. The homology version is similar and is an exercise. Recall that $i\colon X \to DE$ is the zero section. This is a homotopy equivalence. Thus $i_*\colon H_r(X) \to H_r(DE)$ and $\eta_*\colon H_r(DE) \to H_r(X)$ are inverse isomorphisms. Let

$$\tau_E := (- \cap [DE])^{-1}(i_*([X])) \in H^k(DE, SE)$$

be the Thom class. We know that $i^!$ is an isomorphism, since $i_*$ is an isomorphism. Therefore we need to show that $i^! := (- \cap [DE])^{-1} \circ i_* \circ (- \cap [X])$ coincides with $(- \cup \tau_E) \circ \eta^*$. Let $\beta \in H^r(X)$, and let $\alpha \in H^r(DE)$ be such that $i^*(\alpha) = \beta$. So $\eta^*(\beta) = \alpha$. Then we have:

$$\begin{aligned}
i^!(\beta) &= (- \cap [DE])^{-1} \circ i_*(\beta \cap [X]) \\
&= (- \cap [DE])^{-1} \circ i_*(i^*(\alpha) \cap [X])) \\
&= (- \cap [DE])^{-1}(\alpha \cap i_*[X]) \\
&= (- \cap [DE])^{-1}(\alpha \cap (\tau_E \cap [DE])) \\
&= (- \cap [DE])^{-1}((\alpha \cup \tau_E) \cap [DE]) \\
&= \alpha \cup \tau_E \\
&= \eta^*(\beta) \cup \tau_E
\end{aligned}$$

as required. As remarked above the homology version is similar, but left to the reader. □

## 13. The Spivak normal fibration and normal invariants

Let $X$ be an $n$-dimensional connected finite Poincaré complex.

**Definition 13.1** (Spivak normal structure)**.** A *Spivak normal structure* on $X$ is a $(k-1)$-spherical fibration $\rho_X\colon E \to X$ with a pointed map $c_X\colon S^{n+k} \to \mathrm{Th}(\rho_X)$ such that there is a choice of Thom class $\tau_E \in H^k(DE, E)$ and a fundamental class $H_n(X)$ such that

$$[X] = (\rho_X)_*(\tau_E \cap h(c_X)).$$

Here $h\colon \pi_{n+k}(\mathrm{Th}(\rho_X)) \to H_{n+p}(\mathrm{Th}(\rho_X))$ is the Hurewicz homomorphism, $\tau_E \cap h(c_X) \in H_n(DE)$, and then $(\rho_X)_*\colon H_n(DE) \to H_n(X)$ sends it to $H_n(X)$, where it is required to coincide with the fundamental class $[X]$ of $X$.

For example, an $n$-dimensional manifold $M$ embedded in $S^{n_k}$ has Spivak normal structure with $\rho_M$ the sphere bundle of the normal bundle of the embedding, and $c_M\colon S^{n+k} \to \mathrm{Th}(\rho_M)$ the Pontryagin-Thom collapse map.

We need to know how to stabilise spherical fibrations.

**Definition 13.2** (Join of spaces)**.** The *join* of two spaces $X$ and $Y$ is

$$X * Y = X \times Y \times [0,1]/\sim$$

where the equivalence relation identifies the following sets to points. For each $y \in Y$, the set $\{(x,y,0) \,|\, x \in X\}$ becomes a point. For each $x \in X$, the set $\{(x,y,1) \,|\, y \in Y\}$ becomes a point. For example, $S^{k-1} * S^{l-1} = S^{k+l-1}$.

The sphere bundle of the sum $\xi_0 \oplus \xi_1$ of two vector bundles $\xi_i\colon E_i \to X$, $i = 0,1$, is the *fibrewise join*:

$$\{(e_0,e_1) \in E_0 \times E_1 \times [0,1] \,|\, \xi_0(e_0) = \xi_1(e_1)\}/\sim$$

with the identifications for each fixed $e_0 \in E_0$, of $(e_0,e_1,0) \sim (e_0,e_1',0)$ for all $e_1, e_1' \in E_1$, and for each fixed $e_1 \in E_1$, of $(e_0,e_1,1) \sim (e_0',e_1,0)$ for all $e_0, e_0' \in E_0$.

Fibrewise join with the trivial spherical fibration $S^{l-1} \times X \to X$ induced stabilisation, that corresponds to a composition

$$X \xrightarrow{\eta} BG(k) \to BG(k+l).$$

We write the new spherical fibration as $\eta \oplus \epsilon^l$ to mirror the vector bundle notation.

A stable spherical fibration over $X$ is an equivalence class of spherical fibrations with $\xi \sim \eta$ if there are natural numbers $k,l$ with $\xi \oplus \epsilon^k$ and $\eta \oplus \epsilon^l$ fibre homotopy equivalent.

Stable spherical fibrations over $X$ are in one to one correspondence with homotopy classes of maps $X \to BG := \mathrm{colim}_{k\to\infty} BG(k)$. The next theorem is fundamental, but we will not have time to prove it. The rest of this section will unfortunately just be an outline of the theory of Spivak normal fibrations and normal invariants: the theory to decide whether there is a degree one normal map over $X$.

**Theorem 13.3** (Existence and uniqueness of stable Spivak normal structure)**.**

*(i) A finite CW complex $X$ is an $m$-dimensional Poincaré complex if and only if there is an embedding $X \hookrightarrow S^{m+k}$ with a thickening $(Y, \partial Y)$ such that $\rho\colon \partial Y \to Y \simeq X$ is a spherical fibration with fibre $S^{k-1}$.*

*(ii) The map $c\colon S^{m+k} \to Y/\partial Y = \mathrm{Th}(\rho)$ makes $(X,\rho,c)$ a Spivak normal structure.*

*(iii) For an $m$-dimensional Poincaré complex $X$, all Spivak normal structures on $X$ are stably fibre homotopy equivalent. Any Spivak normal structure induced from (ii) via an embedding $X \hookrightarrow S^{m+k}$, determines a unique fibre homotopy equivalence class.*

The proof of this theorem uses more homotopy theory than we have access to at present.

**Definition 13.4.** The *Spivak normal fibration* of a Poincaré complex $X$ is the unique *stable* spherical fibration over $X$ arising from a *stable* Spivak normal structure on $X$.

Let $J\colon BO(k) \to BG(k)$ be the forgetful map sending a vector bundle to its underlying sphere bundle. The stable version of $J$ fits into a fibration sequence

$$G/O \to BO \xrightarrow{J} BG \to B(G/O).$$

Note that the extension of this sequence to the right to $B(G/O)$ is nontrivial. The fibration sequence induces a long exact sequence in homotopy groups

$$\pi_n(G/O) \to \pi_n(BO) \to \pi_n(BG) \to \pi_{n-1}(G/O).$$

The homotopy groups of $G/O$ are quite well understood, and are given by

$$\pi_n(G/O) \cong 0, \mathbb{Z}/2, 0, \mathbb{Z}, 0, \mathbb{Z}/2, 0, \mathbb{Z} \oplus \mathbb{Z}/2, (\mathbb{Z}/2)^2, \mathbb{Z}/2 \oplus \mathbb{Z}/3$$

for $n$ from 1 to 10. The space $G/O_z$, as the fibre, up to homotopy equivalence, of a point $z \in BG$, can be described as

$$\{(y, w) \in BO \times [[0,1], BG] \mid J(y) = w(0), w(1) = z\}$$

We will soon be interested in the homotopy classes of maps $[X, G/O]$, so it is good to know that the space $G/O$ can be understood. We will not go into any details of this during this course.

**Proposition 13.5.** *A spherical fibration $\eta\colon X \to BG$ admits a vector bundle reduction if and only if there exists a lift*

$$\begin{tikzcd} & BO \arrow[d, "J"] \\ X \arrow[ur, dashed] \arrow[r, "\eta"] & BG \end{tikzcd}$$

*which holds if and only if the composition $X \to BG \to B(G/O)$ is null homotopic.*

If such a lift exists, the homotopy classes of maps $X \to G/O$ correspond to the different choices of lift. A map $X \to G/O$ is the same as a map $\eta\colon X \to BO$, that is a stable vector bundle, and a homotopy $h$ of $J\eta$ to the constant map to $z \in BG$.

We want to work from the existence of a vector bundle reduction of the Spivak normal fibration to the existence of a degree one normal map over $X$, and an understanding of the number of different choices of such normal maps. This existence question is the primary surgery obstruction, homotopy theoretic in nature. Degree one normal maps will then be the starting point for the definition of the secondary surgery obstruction for there to be a manifold homotopy equivalent to a Poincaré complex $X$. The secondary obstruction is the obstruction to finding a homotopy equivalence in a given normal bordism class of degree one normal maps, that is an element of $\mathcal{N}_n(X)$. With this small description of our goal in mind, let us continue with the promised passage from vector bundle lift to a normal map.

**Definition 13.6.** A *normal $k$-invariant* for an $n$-dimensional connected finite Poincaré complex $X$ is a $k$-dimensional vector bundle $\xi\colon X \to BO(k)$ and a map $\rho\colon S^{n+k} \to \mathrm{Th}(\xi)$ such that there is a Thom class $U_\xi$ and a fundamental class $[X]$ for which $\xi(h_*(\rho) \cap U_\xi) = [X]$.

Normal $k$-invariants $(\xi, \rho)$, $(\xi', \rho')$ are equivalent if there exists a bundle map $(\mathrm{Id}, f)\colon \xi \xrightarrow{\simeq} \xi'$ covering the identity with

$$\rho' = f_* \circ \rho\colon S^{n+k} \to \mathrm{Th}(\xi) \to \mathrm{Th}(\xi').$$

Let $\mathcal{T}_n(X,k)$ be the equivalence classes of normal $k$-invariants on $X$, and define

$$\mathcal{T}_n(X) := \mathrm{colim}_{k\to\infty} \mathcal{T}_n(X,k).$$

By the same proof as the Pontryagin-Thom construction, we have that

$$\mathcal{T}_n(X) \cong \mathcal{N}_n(X).$$

That is, by transversality we pass from a normal invariant to a normal map, and the inverse uses the embedding in $S^{n+k}$ and the collapse map followed by the map on Thom spaces induced by the normal map: $S^{n+k} \to \mathrm{Th}(\nu_M) \xrightarrow{b} \mathrm{Th}(\xi)$. The cool thing is that the Thom isomorphism theorem, together with the Spivak condition $\xi(h_*(\rho) \cap U_\xi) = [X]$ guarantees that the transversality condition yields a *degree one* map.

In summary, we have the following theorem.

**Theorem 13.7** (Browder-Novikov normal invariant theorem)**.** *Let $X$ be a connected finite $n$-dimensional Poincaré complex. The following are equivalent.*

*(i)* $\mathcal{T}_n(X) \neq \emptyset$.
*(ii) There exists a degree one normal map $(f,b)\colon M \to X$.*
*(iii) The Spivak normal fibration $\nu_X\colon X \to BG$ admits a vector bundle reduction $\eta\colon X \to BO$ $(J \circ \eta = \nu_X)$.*
*(iv) The composition $X \to BG \to B(G/O)$ is null homotopic.*

We have the following consequence.

**Proposition 13.8.** *An $n$-dimensional simple Poincaré complex $X$ is homotopy equivalent to an $n$-dimensional manifold if and only if there exists a lift of the Spivak normal fibration to a map $X \to BO$, such that the resulting normal bordism class of degree one normal maps contains a homotopy equivalence.*

Thus we need a procedure for deciding if a degree one normal map is normal bordant to a homotopy equivalence. This will be the subject of most of the remainder of the course.

First let us look at the different available choices of a degree one normal map, once we know that one exists. In order to find a homotopy equivalence in the normal bordism class, we are at liberty to pick any choice of normal map over the given $X$.

**Theorem 13.9.** *There is a bijection of the normal invariant set $\mathcal{T}_n(X)$ with $[X, G/O]$. The bijection is unnatural, that is it depends on a choice.*

*Proof.* We know that the different choices of lift of the Spivak normal fibration are in one to one correspondence with $[X, G/O]$, by the fibration sequence $G/O \to BO \to BG$ discussed above. Here we will describe a free transitive action of $[X, G/O]$ on the normal invariants explicitly.

Recall that a map $X \to G/O$ is the same as a map $\alpha\colon X \to BO$, that is a stable vector bundle, and a homotopy $\beta$ of $J\eta$ to the constant map to $z \in BG$. Represent this by a map $\alpha\colon X \to BO(j)$, for suitably large $j$.

Fix $\xi\colon BO(k)$, for large $k$, and a map $\rho\colon S^{n+k} \to \mathrm{Th}(\xi)$. Define a new normal invariant $(\eta, \sigma)$, as follows. Let

$$\eta = \xi\colon \alpha\colon X \to BO(k+j)$$

be the Whitney sum. Then let $\sigma$ be the composition

$$\sigma\colon S^{n+k+j} \xrightarrow{\Sigma^j \rho} \Sigma^j \,\mathrm{Th}(\xi) = \mathrm{Th}(\xi \oplus \epsilon^j) \xrightarrow{\mathrm{Id} \oplus \mathrm{Th}(\beta)^{-1}} \mathrm{Th}(\xi \oplus \alpha) = \mathrm{Th}(\eta).$$

This is a free and transitive action of $[X, G/O]$ on the normal invariant set, and it defines a bijection as required. □

Here is a summary of where we are so far. Given a finite connected CW complex $X$, we want to know whether there is an $n$-dimensional manifold $M$ and a homotopy equivalence $M \to X$. First, we need a fundamental class $[X] \in H_n(X;\mathbb{Z})$ such that cap product with this class determines a simple chain homotopy equivalence $C^{n-*}(\widetilde{X}) \to C_*(\widetilde{X})$. Next we need a null homotopy of the composition $X \to BG \to B(G/O)$ determined by the Spivak normal fibration. Finally, we need for the resulting normal bordism class of degree one normal maps to contain a homotopy equivalence.

It is this last step that will occupy us in the ensuing sections.

## 14. Surgery below the middle dimension

Recall that a degree one normal map, or DONM, to a finite, connected, $n$-dimensional Poincaré complex $X$ consists of a manifold $M$ with a degree one map $f\colon M \to X$, a stable vector bundle $\xi\colon E \to X$, and a bundle map from the stable normal bundle

$$\begin{tikzcd} \nu(M) \arrow[r, "b"] \arrow[d] & E \arrow[d, "\xi"] \\ M \arrow[r, "f"] & X \end{tikzcd}$$

covering $f$. Recall that $\mathcal{N}_n(X)$ denotes the normal bordism classes of degree one normal maps.

Our aim is to define, for a group $\pi$ and an integer $n$, a group $L_n(\mathbb{Z}\pi)$ and a function

$$\sigma\colon \mathcal{N}_n(X) \to L_n(\mathbb{Z}\pi)$$

such that

$$\mathcal{S}_n(X) \to \mathcal{N}_n(X) \xrightarrow{\sigma} L_n(\mathbb{Z}\pi)$$

is an exact sequence of pointed sets. (Even though both of the second two sets can be given a group structure, the map $\sigma$ is not a homomorphism.) The exactness corresponds to the statement that for $n \geq 5$ a degree one normal map $f\colon M \to X$ is normal bordant to a homotopy equivalence if and only if the surgery obstruction $\sigma(M \xrightarrow{f} X)$ vanishes.

We will make use of the Whitehead and Hurewicz theorems, so we give their statements here.

**Theorem 14.1** (J.H.C. Whitehead)**.** *Let $f\colon X \to Y$ be a map between CW complexes with $f_*\colon \pi_i(X) \xrightarrow{\simeq} \pi_i(Y)$ for all $i$. Then $f$ is a homotopy equivalence.*

Given a pair of spaces $(X, A)$, recall that $\pi_n(X, A)$ is homotopy classes of squares

$$\begin{array}{ccc} S^{n-2} & \longrightarrow & A \\ \downarrow & & \downarrow \\ D^n & \longrightarrow & X. \end{array}$$

The Hurewicz map is

$$\begin{array}{rcl} \rho\colon \pi_n(X, A) & \to & H_n(X, A) \\ (f, g) & \mapsto & (f, g)_*([D^n, S^{n-1}]). \end{array}$$

Let $\pi_n(X, A)^\dagger = \pi_n(X, A)/x \sim \alpha(x)$ for all $\alpha \in \pi_1(A)$; recall that $\pi_1(A)$ acts on $\pi_n(X, A)$.

**Theorem 14.2** (Relative Hurewicz theorem)**.** *Suppose that $X$ and $A$ are path connected, $n > 1$ and that $\pi_k(X, A) = 0$ for all $k < n$. Then $H_k(X, A) = 0$ for all $k < n$ and $\rho\colon \pi_n(X, A)^\dagger \to H_n(X, A)$ is an isomorphism.*

Here is a corollary that spells out the version that we will use. We will be applying the Hurewicz theorem to universal covers, so we can assume that $X$ and $A$ are simply connected.

**Theorem 14.3** (Relative Hurewicz theorem II)**.** *Let $(X, A)$ be a pair of spaces with $\pi_1(A) = \pi_0(A) = \{1\}$. Suppose that $n \geq 2$ and $\pi_0(X, A) = \pi_1(X, A) = \{1\}$. Then:*

*(1) If in addition $\pi_k(X, A) = 0$ for all $k < n$ then $H_k(X, A) = 0$ for all $k < n$ and the Hurewicz map $\rho\colon \pi_n(X, A) \xrightarrow{\simeq} H_n(X, A)$ is an isomorphism.*
*(2) If in addition $H_k(X, A) = 0$ for all $k < n$ then $\pi_k(X, A) = 0$ for all $k < n$ and the Hurewicz map $\rho\colon \pi_n(X, A) \xrightarrow{\simeq} H_n(X, A)$ is an isomorphism.*

For a map of spaces $f\colon X \to Y$, define $\pi_n(f) := \pi_n(\mathcal{M}_f, X)$, with $\mathcal{M}_f$ the mapping cylinder. There is a long exact sequence in homotopy groups

$$\pi_n(X) \to \pi_n(Y) \to \pi_n(f) \to \pi_{n-1}(f) \to \pi_{n-1}(X) \to \pi_{n-1}(Y)$$

The next corollary follows from the combination of the Hurewicz and Whitehead theorems.

**Corollary 14.4.** *Suppose that $\pi_1(X) \cong \pi_1(Y)$ and there exists a map $f\colon X \to Y$ that induces this isomorphism. Denote the induced map on the homology of universal covers by $\widetilde{f}\colon H_*(\widetilde{X}) \to H_*(\widetilde{Y})$. Suppose that $H_k(\widetilde{f}) = 0$ for all $k$. Then $f$ is a homotopy equivalence.*

Our plan for improving a degree one normal map is to inductively kill $\pi_k(f)$ by changing $M$, until $\pi_k(f) = 0$ for all $k$.

**Proposition 14.5.** *Suppose that $Y$ and $X$ are finite CW complexes, and that $f\colon Y \to X$ is a $(k-1)$-connected map, with $k \geq 2$. Then $\pi_k(f)$ is a finitely generated $\mathbb{Z}[\pi_1(Y)]$-module.*

*Proof.* Denote $\pi = \pi_1(X) = \pi_1(Y)$, let $\widetilde{f}\colon \widetilde{Y} \to \widetilde{X}$ be the induced map on universal covers. Then $\pi_i(f) = H_i(\widetilde{f}) = H_i(\mathscr{C}(\widetilde{f}))$ for $i \leq k$, where here we abuse notation and write $\widetilde{f}\colon C_*(\widetilde{Y}) \to C_*(\widetilde{X})$ for the induced map on cellular chain complexes.

Define $D_* := \mathscr{C}(\widetilde{f})$. The fact that $f$ is $(k-1)$-connected implies that

$$0 \to \ker \partial_k \to D_k \xrightarrow{\partial_k} D_{k-1} \xrightarrow{\partial_{k-1}} \cdots \to D_0 \to 0$$

is exact. Each $D_i$ is finitely generated and free since the CW complexes are finite.

**Claim.** $\ker \partial_k$ *is stably free and finitely generated.*

Note that the claim implies the proposition, since the homology is a quotient of $\ker \partial_k$ and is therefore also finitely generated. To prove the claim, begin by noting that we have an exact sequence

$$0 \to \ker \partial_1 \to D_1 \to \operatorname{im} \partial_1 = D_0 \to 0.$$

Then $D_0$ is projective so this sequence splits, and we have $D_1 = \ker \partial_1 \oplus D_0$. So $\ker \partial_1$ is stably free and finitely generated. Next, we have a short exact sequence

$$0 \to \ker \partial_2 \to D_2 \to \ker \partial_1 \to 0$$

which gives rise to a short exact sequence

$$0 \to \ker \partial_2 \to D_2 \oplus D_0 \to \ker \partial_1 \oplus D_0 = D_1 \to 0$$

Thus $\ker \partial_2 \oplus D_1 \cong D_2 \oplus D_0$ and so $\ker \partial_2$ is stable free and finitely generated. Proceed by induction to see that $\ker \partial_k \oplus D_{\text{odd}} \cong D_{\text{even}}$, or the same with even and odd switched. This completes the proof of the claim and therefore of the proposition. □

Now that we know $\pi_k(f)$ is finitely generated, we could do a finite number of cell attachments to $M$ to kill $\pi_k(f)$. But of course we want to do a sequence of surgeries instead, so that we always have a manifold. Recall that $\pi_{k+1}(f)$ consists of squares of the form:

$$\begin{tikzcd} S^k \arrow[r, "g"] \arrow[d] & M \arrow[d, "f"] \\ D^{k+1} \arrow[r, "h"] & X. \end{tikzcd}$$

Suppose that $f$ is degree one, and that we can represent $g$ by a framed embedding

$$\overline{g}\colon S^k \times D^{n-k} \hookrightarrow M.$$

Then

$$M' = M \smallsetminus \overline{g}(S^k \times D^{n-k}) \cup D^{k+1} \times S^{n-k-1}$$

maps to $X$, with $h$ being used to extend $f|_{M \smallsetminus (S^k \times D^{n-k})}$. If $2k < n$, we have that $g$ is homotopic to an embedding by the Whitney embedding theorem. If $2k = n$, then we can only arrange an immersion in general. The middle dimensional surgery obstruction will be based around this problem. For now, assume that $n > 2k$. Then we can arrange to have an embedding, but why should it be framed? For this, we need to use the bundle data. More details on this will be coming soon, but first let us investigate the effect of a surgery on the relative homotopy groups.

**Theorem 14.6** (Homotopy effect of surgery)**.** *Let $f\colon M \to X$ be an $n$-dimensional degree one map. Let $k$ be a nonnegative integer such that $2(k+1) \leq n$. Let $g\colon S^k \times D^{n-k} \hookrightarrow M$ be an embedding with an extension to a map $h\colon D^{k+1} \times D^{n-k} \to X$, such that $x := (g|_{S^k\times\{0\}}, h|_{D^{k+1}\times\{0\}})$ represents an element of $\pi_{k+1}(f)$. Let $f'\colon M' \to X$ be the result of surgery on $g$. Then*

$$\pi_{k+1}(f') \cong \pi_{k+1}(f)/\langle x\rangle$$

*and*

$$\pi_{j+1}(f') = \pi_{j+1}(f)$$

*for $j < k$.*

*Proof.* We look at the effect of the two steps of surgery on the relative homotopy groups independently. First, the map

$$\pi_{j+1}(M \smallsetminus S^k \to X) \to \pi_{j+1}(M \to X)$$

is surjective for $j + k < n$ and if injective for $j + k + 1 < n$. This follows from general position: a sphere can be made disjoint from $S^k$ by general position in a certain range, while a homotopy can be made to miss $S^k$ in a slightly smaller range. Next,

$$\pi_{j+1}(M \smallsetminus S^k \to X) \to \pi_{j+1}(f')$$

is injective for $j < k$ and surjective for $j \leq k$. For an isomorphism $\pi_{j+1}(f) \cong \pi_{j+1}(f')$ we therefore need $j < k$ and $j < n - k - 1$. By assumption, $2(k+1) \leq n$, which implies the first and last inequalities of $n - k - 1 \geq n/2 > n/2 - 1 \geq k$. Thus when $j < k$ we have $j < n - k - 1$ automatically, and therefore we have an isomorphism of relative homotopy groups as required for $j < k$. For surjectivity, we again need for $j < n-k-1$, since we need for $\pi_{j+1}(M \smallsetminus S^k \to X) \to \pi_{j+1}(M \to X)$ to be an isomorphism, and we need that$j \leq k$. But $j = k$ indeed satisfies both. The kernel of the map $\pi_{j+1}(M \smallsetminus S^k \to X) \to \pi_{j+1}(f')$ is the submodule generated by $x$. □

Now we return to bundle data. We need to argue why we can do surgery to kill a given element of $\pi_{r+1}(f)$.

Here is Wall's [Wal] version of a normal map. We have an $n$-dimensional degree one map $f\colon M \to X$, the tangent bundle $TM \to M$ of the manifold $M$, and an $m$-dimensional vector bundle $\xi \to X$. The normal data is a stable trivialisation of $TM \oplus f^*\xi$, that is an isomorphism of bundles:

$$b\colon TM \oplus f^*\xi \oplus \epsilon^a \cong \epsilon^{n+m+a}.$$

This is equivalent to requiring that $f^*\xi$ be the stable normal bundle of some embedding of $M$, and one can show that the resulting normal bordism group is isomorphic to $\mathcal{N}_n(X)$.

Bordism of normal maps requires a cobordism $F\colon (W; M, M') \to (X, \times I; X \times \{0\}, X \times \{1\})$ with the vector bundle $\xi \to X \times I$ extended trivially, and a stable trivialisation of $TW \oplus F^*\xi$ extending

$$b \text{ on } TM \oplus f^*\xi \oplus \text{inward normal}$$

and

$$b' \text{ on } TM' \oplus (f')^*\xi \oplus \text{outward normal.}$$

To perform surgery, choose a representation

$$\begin{tikzcd} S^r \arrow[r, "q"] \arrow[d, "i"] & M \arrow[d, "f"] \\ D^{r+1} \arrow[r, "Q"] & X. \end{tikzcd}$$

for an element in $\pi_{r+1}(f)$. The trivialisation $b$ induces a stable trivialisation of

$$q^*TM \oplus q^*f^*\xi \cong q^*TM \oplus i^*Q^*\xi$$

on $S^r$. But $D^{r+1}$ is contractible so $i^*Q^*\xi$ is canonically framed. Thus we obtain a framing of $q^*TM \oplus \epsilon^m \oplus \epsilon^a$.

On the other hand $q^*TM \cong \nu_{S^r \subset M} \oplus TS^r$, and $TS^r \oplus \epsilon \cong \epsilon^{r+1}$. Thus $\nu_{S^r \subset M} \oplus \epsilon^{r+m+a-1}$ is a trivial vector bundle. For $r < n/2$, the next lemma then guarantees that the normal bundle is trivial.

**Lemma 14.7.** *If $k > m$ then any two $k$-plane bundles on an $m$-dimensional CW complex $X$ are isomorphic if and only if they are stably isomorphic.*

*Proof.* We begin the proof by claiming that there is a fibration

$$S^k \to BO(k) \to BO(k+1).$$

To see this, let $V_n(\mathbb{R}^m)$ denote the Stiefel manifold of orthonormal $n$ frames in $\mathbb{R}^m$. In particular, $V_1(\mathbb{R}^{k+1})$ is the space of length one vectors in $\mathbb{R}^{k+1}$, so is diffeomorphic to $S^k$. Recall that there is a universal $k+1$-plane vector bundle $\gamma_{k+1} \to BO(k+1)$, where $BO(k+1)$ is the Grassmannian of $k+1$-planes in $\mathbb{R}^\infty$ and $\gamma_{k+1}$ is the set of pairs $(W, v)$ with $W \in BO(k+1)$ and $v \in W$. The projection map of the universal bundle sends $(W, v) \mapsto W$. By considering the 1-frame bundle of this vector bundle, we obtain a fibre bundle

$$S^k \to V_1(\gamma_{k+1}) \to BO(k+1).$$

Next, there is a homotopy equivalence $V_1(\gamma_{k+1}) \to BO(k)$, which sends

$$(W, b) \mapsto (\operatorname{span} b)^\perp$$

The reader should find a homotopy inverse. We therefore replace $V(\gamma_{k+1})$ by $BO(k)$ to get the fibration claimed.

The long exact sequence of the fibration implies, since $\pi_j(S^k = 0$ for $j < k$, that $(BO(k+1), BO(k))$ is a $k$-connected pair. It follows that there is a bijection

$$[X, BO(k)] \leftrightarrow [X, BO(k+1)]$$

for any CW complex $X$ of dimension less than $k$. Thus for such $X$, $[X, BO(k)] = [X, BO]$, and two $k$-plane bundles are isomorphic if and only they are stably isomorphic. $\square$

Now we know that $\nu_{S^r \subset M}$ is trivial for $r < k$ (recall $n = 2k$). Thus we can thicken our embedding of $S^r$ to an embedding of $S^r \times D^{n-r}$, and perform surgery.

Next, although we now know that we can perform surgery on one element, we need to be able to repeat the process. Surgery on one element required bundle data. Thus in order to have an iterative procedure, we need to be able to extend bundle data across the trace of a surgery to the outcome $M'$ of a surgery. It turns out that this can always be done. Achieving it requires an investigation of immersions. The ability to repeat surgeries for $2k < n$, and the fact that each successive $\pi_j(f)$ is finitely generated as a $\mathbb{Z}[\pi_1(X)]$-module, will enable us to prove the following theorem, which is the main result of this section.

**Theorem 14.8** (Surgery below the middle dimension)**.** *Let $X$ be a connected, finite, $n$-dimensional Poincaré complex, and let $(f, b)\colon M \to X$ be a degree one normal map. There exists a finite sequence of surgeries to a normal bordant degree one normal map $(f', b')\colon M' \to X$ with $f'$ a $k$-connected map.*

As mentioned above, we need a further investigation of the bundle data to ensure that it extends across the trace of a surgery. This needs some important facts about immersions, as mentioned above.

**Definition 14.9.**

(i) Recall that an *immersion* of $M$ into $N$ is a smooth map $f\colon M \looparrowright N$ such that $df_p\colon T_pM \to T_{f(p)}N$ is an injection for all $p \in M$.

(ii) The *normal bundle of an immersion* $f\colon M \looparrowright N$ is

$$\nu(q) = f^*(TN)/TM.$$

(iii) A *regular homotopy* between immersions $f_0$ and $f_1$ is a homotopy $h\colon M \times [0,1] \to N$ with $h(-,i) = f_i$ for $i = 0, 1$, such that $h_t\colon M \looparrowright N$ is an immersion for all $t \in [0,1]$, with

$$\begin{array}{rcl} dh\colon TM \times [0,1] & \to & TN \\ ((p,x),t) & \mapsto & (h_t(p), dh_t(x)) \end{array}$$

a smooth bundle map that restricts to a bundle monomorphism for each $t$.

The next theorem gives a classification of immersions, and will be crucial for doing surgery. The idea is that we will use our bundle data to fix a regular homtopy class of immersions, and then we will look for an embedding in that regular homotopy class of immersions only. This will guarantee that the bundle data extends across the trace of a surgery.

In the next theorem let $\pi_0(\mathrm{Imm}(M, N))$ be the regular homotopy classes of immersions, and let $\pi_0(\mathrm{Mono}(TM, TN))$ be the isotopy classes of bundle monomorphisms from $TM$ to $TN$.

**Theorem 14.10** (Smale-Hirsch classification of immersions)**.** *Let $M^m$ and $N^n$ be smooth manifolds.*

*(i) Suppose that $1 \leq m < n$. Then the differential induces a bijection*

$$\pi_0(\mathrm{Imm}(M,N)) \xrightarrow{\simeq} \pi_0(\mathrm{Mono}(TM,TN)).$$

*(ii) Suppose that $1 \leq m \leq n$ and that $M$ has a handle decomposition comprising $r$-handles for $r \leq n-2$ only. Then the differential induces an isomorphism*

$$\pi_0(\mathrm{Imm}(M,N)) \xrightarrow{\simeq} \mathrm{colim}_{a\to\infty}\, \pi_0(\mathrm{Mono}(TM \oplus \epsilon^a, TN \oplus \epsilon^a)).$$

One striking consequence of this theorem is that all immersions of $S^2$ into $\mathbb{R}^3$ are regularly homotopic to each other, and can therefore the 2-sphere can be turned inside out by regular homotopies. This is called the *eversion* of the sphere.

Here is an application of the theorem to the special case of immersions of spheres $S^k$ in the disc $D^n$. Let $f\colon S^k \looparrowright D^n$, for $2 \leq k \leq n-2$. There is a stable bundle monomorphism uniquely associated to this immersion $ST(f)\colon S^k \times \mathbb{R}^{k+1} \to \mathbb{R}^{n+1}$, defined as the composition

$$TS^k \oplus \epsilon \xrightarrow{df \oplus \mathrm{Id}} TD^n \oplus \epsilon \xrightarrow{\simeq} D^n \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$$

where the last map is the projection. For each $x \in S^k$, we get a $(k+1)$-frame $\{v_i(x) := ST(f)(x, v_i) \mid i = 1, \ldots, k+1\}$ in $\mathbb{R}^{n+1}$. Since $(k+1)$-frames in $\mathbb{R}^{n+1}$ correspond to elements of $\mathrm{GL}_{n+1}(\mathbb{R})/\mathrm{GL}_{n-k}(\mathbb{R})$, and regularly homotopic immersions correspond to homotopic maps $S^k \to \mathrm{GL}_{n+1}(\mathbb{R})/\mathrm{GL}_{n-k}(\mathbb{R})$, we see that there is a bijection

$$\pi_0(\mathrm{Imm}(S^k, D^n)) \cong [S^k, \mathrm{GL}_{n+1}(\mathbb{R})/\mathrm{GL}_{n-k}(\mathbb{R})].$$

For framed immersions there is a bijection to $[S^k, \mathrm{GL}_{n+1}(\mathbb{R})]$. Next, $O(j)$ is homotopy equivalent to $\mathrm{GL}_j(\mathbb{R})$ by the Gramm-Schmidt process, and we know that

$$[S^k, O(j)] = [S^k, O]$$

for $j \geq k+2$. Thus we can replace GL with $O$ and we can stabilise. Thus

$$\pi_0(\mathrm{Imm}^{fr}(S^k, D^n)) \cong [S^k, O]$$

and

$$\pi_0(\mathrm{Imm}(S^k, D^n)) \cong [S^k, O/O(n-k)].$$

The fundamental group $\pi_1(O/O(n-k)) = \{e\}$, so $[S^k, O/O(n-k)] \cong \pi_k(O/O(n-k))$. When $n = 2k$ and $k \geq 3$ is equal to $2j, 2j+1$, these homotopy groups are known to be equal to:

$$\pi_{2j}(O/O(2j)) \cong \mathbb{Z}$$

and

$$\pi_{2j+1}(O/O(2j+1)) \cong \mathbb{Z}/2.$$

*Proof of surgery below the middle dimension.* Recall from our discussion above that a class in $\pi_{r+1}(f)$ can be represented by an immersion with trivial normal bundle, so an immersion $\overline{q}\colon S^r \times D^{n-r} \to M$ with a stable trivialisation of $\overline{q}^*TM$, that is a framing of $\overline{q}^*TM \oplus \epsilon^{a'}$, for some integer $a'$. Since $\overline{q}^*TM \cong T(S^r \times D^{n-r})$, this is equivalent to a stable bundle monomorphism $T(S^r \times D^{n-r}) \to TM$, and thus there is determined a regular homotopy class of immersions $S^r \times D^{n-r} \to M$. Note that $S^r \times D^{n-r}$ has a handle decomposition with handles of dimension 0 and $r$ only, and $r \leq n-2$, so (ii) of the Smale-Hirsch theorem applies. Since $r < n/2$, this immersion can be represented by an embedding in the regular homotopy class.

Note that the stable trivialisation was determined by the canonical trivialisation of $i^*Q^*\xi$ on $D^{r+1} \times D^{n-r}$. Thus the trivialisation extends across $W = M \times I \cup D^{r+1} \times D^{n-r}$, the trace of the surgery. Thus we can perform a surgery to kill a homotopy class in $\pi_{r+1}(f)$. As long as we proceed inductively on $r$, we saw above that $\pi_{r+1}(f)$ is finitely generated over $\mathbb{Z}[\pi_1(X)]$ and that surgery to kill a generator of $\pi_{r+1}(f)$ does not change the lower relative homotopy groups. Once they are killed they stay dead. The argument above works until just below the middle dimension. Thus we can find a normal bordant $k$-connected map, as claimed. $\square$

Now we can arrange up to normal bordism that any degree one normal map in dimension $n = 2k$ is $k$-connected. We still need to try to kill $\pi_{k+1}(f)$. We will show that by Poincaé duality, it is enough to do this, and then we will have $\pi_j(f) = 0$ for all $j$. Then $f$ will be a homotopy equivalence, by the Whitehead theorem. In general., this last step cannot be done. We will define the *surgery obstruction group* $L_{2k}(\mathbb{Z}[\pi_1(X)])$ and the *surgery obstruction map*

$$\sigma \colon \mathcal{N}_n(X) \to L_{2k}(\mathbb{Z}[\pi_1(X)]),$$

for which $\sigma(f,b) = 0$ if and only if $(f,b)$ is normal bordant to a homtopy equivalence. The main idea, as mentioned before, is that in the middle dimension the intersection form of $M$ gives an obstruction to embedding spheres, and thus it may not be possible to find enough embedded spheres on which to perform surgery in order to simplify $M$.

## 15. Surgery in the middle dimension

Let $(f,b) \colon M \to X$ be a degree one $k$-connected normal map (with vector bundle $\xi \to X$) where $M$ is a $2k = n$ dimensional manifold and $X$ is an $n$-dimensional connected finite Poincaré complex. We want to decide whether it is possible to do further surgeries on $M$ to convert $f$ into a homotopy equivalence.

Let $p \in M$ and $s \in S^k$ be basepoints. Let $I_k(M)$ be set of immersions $q \colon S^k \looparrowright M$ together with a path $w$ in $M$ from $p$ to $q(s)$, up to regular homotopy of $q$ and homotopy of $w$. We call this pointed regular homotopy. With the basing information $I_k(M)$ is a $\mathbb{Z}[\pi_1(M)]$-module: just precompose $w$ with a loop in $pi_1$ to define the action. The addition is defined by connected sum along the given paths.

**Lemma 15.1.** *There is a natural homomorphism $\pi_{k+1}(f) \to I_k(M)$.*

To define the homomorphism, represent the map by a regular homotopy class of framed immersions as determined by the stable framing $b$ of $TM \oplus f^*\xi$. This proceeds exactly as in the previous section. There is a little extra work to see that the normal bundle is always framed in this setting. The bundle data gives rise to a stable trivialisation of $q^*TM$, which induces a stable trivialisation of $TS^k$. This can be destablised by Lemma 14.7 to give a trivialisation of $TS^k \oplus \epsilon^{n-k}$, and an identification of this stabilisation with $q^*TM$, which determines a stable bundle monomorphism $T(S^k \times D^{n-k}) \to TM$, and thus a regular homotopy class of immersions $S^k \times D^{n-k}$ is determined.

15.1. **Intersection and self intersection forms.** We will often write $\pi$ for $\pi_1(M) = \pi_1(X)$.

There is an intersection form

$$\begin{array}{rcl} \lambda\colon I_k(M) \times I_k(M) & \to & \mathbb{Z}\pi \\ ((q_1, w_1), (q_2, w_2)) & \mapsto & \sum_{p \in q_1(S^k) \pitchfork q_2(S^k)} \epsilon_p g_p. \end{array}$$

Here $\epsilon_p$ arises from transversality (we have to choose transverse representatives for $q_1$ and $q_2$) and orientations. Let $\gamma_i^p$ be a path in $q_i(S^k)$ from $q(s)$ to $p$. Then $g_p$ is defined by

$$w_1 \cdot \gamma_1^p \cdot \overline{\gamma_2^p} \cdot \overline{w_2}.$$

This defines the ordinary intersection of two immersions. This is a homotopy invariant.

The *self-intersection number* is a group homomorphism

$$\mu\colon I_k(M) \to Q_{(-1)^k}(\mathbb{Z}\pi)$$

where

$$Q_{(-1)^k}(\mathbb{Z}\pi) := \frac{\mathbb{Z}\pi}{x \sim (-1)^k \overline{x}}.$$

This is defined by

$$(q, w) \mapsto \sum_{p \in q(S^k) \text{ with } |q^{-1}(p)|=2} \epsilon_p g_p.$$

Here $\epsilon_p$ is defined by the orientations as usual. To define $g_p$ choose two paths $\gamma_1^p$ and $\gamma_2^p$ from $q(s)$ to $p$, which arrive at $p$ on opposite sheets. Then

$$g_p := w \cdot \gamma_1^p \cdot \overline{\gamma_2^p} \cdot \overline{w}.$$

The fact that we have no preferred order for the sheets at each intersection point gives rise to the indeterminacy i.e. this is only well defined in $Q_{(-1)^k}(\mathbb{Z}\pi)$. $\mu((q, w))$ is invariant under pointed regular homotopy.

$\lambda$ and $\mu$ have the following properties. For all $q, q_1, q_2 \in I_k(M)$ and for all $a, b \in \mathbb{Z}\pi$:

(i) $\lambda(aq_1, bq_2) = a\lambda(q_1, q_2)\overline{b}$.
(ii) $\mu(aq) = a\mu(q)\overline{a}$ for $a \in \pi$.
(iii) $(-1)^k \lambda(q_2, q_1) = \overline{\lambda(q_1, q_2)}$.
(iv) $\lambda(q, q) = \mu(q) + (-1)^k \overline{\mu(q)} + \chi(\nu_q)$ where $\chi(\nu_q) \in \mathbb{Z} \subset \mathbb{Z}\pi$ is the Euler number of the normal bundle of $q$.
(v) $\mu(q_1 + q_2) - \mu(q_1) - \mu(q_2) = \mathrm{pr}(\lambda(q_1, q_2))$. Here $\mathrm{pr}\colon \mathbb{Z}\pi \to Q_{(-1)^k}(\mathbb{Z}\pi)$ is the projection.

We invite the reader to check these properties.

**Theorem 15.2** (Wall embedding theorem)**.** *Let $M$ be a compact oriented manifold of dimension $n = 2k \geq 6$. An immersion $(q, w) \in I_k(M)$ is pointed regular homotopic to an embedding if and only if $\mu(q) = 0$.*

*Proof.* One direction is trivial. The other direction uses the Whitney trick. □

Now we give a formal algebraic setting, of which the intersection form we just described is a special case.

**Definition 15.3** (Symmetric form)**.** A $(-1)^k$-symmetric form consists of a f.g. projective $\mathbb{Z}\pi$-module $P$ and a $\mathbb{Z}\pi$-module morphism $\lambda\colon P \to P^* = \overline{\mathrm{Hom}_{\mathbb{Z}\pi}(P, \mathbb{Z}\pi)}$ with $\lambda = T\lambda := (-1)^k\lambda^*$. (These properties correspond to (i) and (iii) above) The form is nonsingular if $\lambda$ is an isomorphism.

The form is nondegenerate if $\lambda$ is an injection.

**Definition 15.4** (Quadratic form)**.** A $(-1)^k$-quadratic form is a $(-1)^k$ symmetric form together with a *quadratic enhancement* $\mu\colon P \to Q_{(-1)^k}(\mathbb{Z}\pi)$ such that for all $q, q_1, q_2 \in P$ and for all $a \in \mathbb{Z}\pi$ we have:

(i) $\mu(aq) = a\mu(q)\overline{a}$.
(ii) $\lambda(q,q) = \mu(q) + (-1)^k\overline{\mu(q)}$.
(iii) $\mu(q_1 + q_2) - \mu(q_1) - \mu(q_2) = \mathrm{pr}(\lambda(q_1, q_2))$. Here $\mathrm{pr}\colon \mathbb{Z}\pi \to Q_{(-1)^k}(\mathbb{Z}\pi)$ is the projection.

Note that in the surgery step we will always consider spheres with trivial normal bundle, so we do not need to consider the $\chi$ term.

A quadratic form is equivalent to a $\mathbb{Z}\pi$-module homomorphism $\psi \in \mathrm{Hom}_{\mathbb{Z}\pi}(P, P^*)/(1-T)$. Note that $(1+T)\psi$ is symmetric and does not depend on the choice of representative, since $(1+T)(1-T) = 0$, as $T^2 = 1$. To see the equivalence, define $\lambda(p,q) = ((1+T)\psi)(p)(q)$ and $\mu(p) = \psi(p)(p)$.

Next we want to answer the following question: why is $\lambda$ of a $k$-connected degree one normal map a nonsingular form? This will be the subject of the next two lemmas.

**Definition 15.5** (Surgery homology and cohomology kernels)**.** Let $f\colon M \to X$ be 2-connected map between Poincaré complexes and let $\widetilde{f}\colon \widetilde{M} \to \widetilde{X}$ be the induced map on universal covers. Define the homology kernel

$$K_r(M) := H_{r+1}(\widetilde{f}\colon \widetilde{M} \to \widetilde{X})$$

and the cohomology kernel

$$K^r(M) := H^{r+1}(\widetilde{f}\colon \widetilde{M} \to \widetilde{X}).$$

**Lemma 15.6.** *For a degree one 2-connected map $f\colon M \to X$, for all $r$ we have splittings of kernels, and of duality maps:*

$$\begin{tikzcd} H^r(\widetilde{M}) \arrow[r, "\cong"] \arrow[d, "-\cap[M]"'] & K^r(M) \oplus H^r(\widetilde{X}) \arrow[d, "(-\cap[M])|\oplus(-\cap[X])"] \\ H_{n-r}(\widetilde{M}) \arrow[r, "\cong"] & K_{n-r}(M) \oplus H_{n-r}(\widetilde{X}). \end{tikzcd}$$

*Proof.* Consider the following square of chain complexes.

$$\begin{tikzcd} C_*(\widetilde{X}) \arrow[r, "(-\cap[X])^{-1}"] & C^{n-*}(\widetilde{X}) \arrow[d, "\widetilde{f}^*"'] \\ C_*(\widetilde{M}) \arrow[u, "\widetilde{f}_*"'] & C^{n-*}(\widetilde{M}) \arrow[l, "(-\cap[M])^{-1}"] \end{tikzcd}$$

Define $f_!\colon C_*(\widetilde{X}) \to C_*(\widetilde{M})$ to be the right-down-left composition, and it defines a splitting of $\widetilde{f}_*$ up to homotopy, that is $\widetilde{f}_* \circ f_! \sim \mathrm{Id}$. To see this note that since $f_*([M]) = [X]$, we have:

$$\widetilde{f}_*(\widetilde{f}^*(x) \cap [M]) = x \cap f_*([M]) = x \cap [X]$$

for any $x \in C^{n-*}(\widetilde{X})$. Let $y = x \cap [X]$. Then $(- \cap [X])^{-1}(y) = x$ up to chain homotopy so

$$\widetilde{f}_* \circ f_!(y) = \widetilde{f}_*(\widetilde{f}^*((- \cap [X])^{-1}(y)) \cap [M]) = y,$$

the diagram above commutes up to homotopy, and $f_!$ is a homotopy splitting as claimed.

On the level of homology, the long exact sequence therefore breaks up into short exact sequences:

$$\xrightarrow{0} H_{k+1}(\widetilde{f}) \to H_k(\widetilde{M}) \to H_k(\widetilde{X}) \xrightarrow{0} H_k(\widetilde{f}).$$

Since $H_{k+1}(\widetilde{f}) = K_k(M)$, and $f_!$ gives a splitting of this short exact sequence, we have a direct sum decomposition $H_k(\widetilde{M}) \cong K_k(M) \oplus H_k(\widetilde{X})$ as desired.

Next we want to see that the duality maps also split as claimed. Consider the diagram:

$$\begin{tikzcd}
0 & K^{n-k}(M) \arrow[l] \arrow[r, shift left] \arrow[d, dotted] & H^{n-k}(\widetilde{M}) \arrow[l, shift left] \arrow[r, shift left, "f^!"] \arrow[d, "\cong"', "-\cap[M]"] & H^{n-k}(\widetilde{X}) \arrow[l, shift left, "f^*"] \arrow[d, "\cong"', "-\cap[X]"] & 0 \arrow[l] \\
0 \arrow[r] & K_k(M) \arrow[r, shift right] & H_k(\widetilde{M}) \arrow[l, shift right] \arrow[r, shift right, "f_*"'] & H_k(\widetilde{X}) \arrow[l, shift right, "f_!"'] \arrow[r] & 0
\end{tikzcd}$$

The splitting $f_!$ of the bottom row induces a splitting $s\colon H_k(\widetilde{M}) \to K_m(M)$. Identify

$$K_k(M) \cong \ker(f_*\colon H_k(\widetilde{M}) \to H_k(\widetilde{X})$$

and

$$K^{n-k}(M) \cong \operatorname{coker}(f^*\colon Hn - k(\widetilde{X}) \to H^{n-k}(\widetilde{M}).$$

For $[x] \in K^{n-k}(M) = H^{n-k}(\widetilde{M})/H^{n-k}(\widetilde{X})$, we have that $s(x \cap [M])$ is well defined by commutativity of the diagram. This produces the dotted arrow in the diagram. The right-up-left composition of the quotient, $(- \cap [M])^{-1}$ and the inclusion give an inverse to the dotted arrow. We also denote the dotted arrow by $- \cap [M]$, and use it to define the intersection pairing next. $\square$

The *surgery pairing* $\lambda\colon K_k(M)\times K_k(M)\to\mathbb{Z}\pi$ of a $k$-connected map $f\colon M\to X$ is given by $(x,y)\mapsto\langle(-\cap[M])^{-1}x,y\rangle$. It fits into the diagram

$$\begin{tikzcd} K_k(M)\times K_k(M) \arrow[r, "\lambda"] \arrow[d, "\cong"] & Z\pi \arrow[dd, equal] \\ \pi_{k+1}(\widetilde{f})\times\pi_{k+1}(\widetilde{f}) \arrow[d, "\cong"] & \\ I_k(\widetilde{f})\times I_k(\widetilde{f}) \arrow[r, "\lambda"] & \mathbb{Z}\pi \end{tikzcd}$$

Here $I_k(\widetilde{f})$ denotes the pointed immersions of $S^k$ into $M$ that are null homotopic in $X$. Both the geometric and algebraic pairings are denoted by $\lambda$, since they are equal (we mentioned this fact above, but we will not prove it unfortunately.). The algebraic pairing is necessary to prove nonsingularity.

**Lemma 15.7.** *The Kronecker map $K^{n-k}(M)\to K_{n-k}(M)^*$ is an isomorphism.*

It follows from the lemma that $\lambda$ is nonsingular, since the adjoint of $\lambda$ is the composition of the inverse of Poincaré duality and the kronecker evaluation map in the lemma.

*Proof.* Define $D_* := \mathscr{C}(\widetilde{f})$. This is a projective $R=\mathbb{Z}\pi$-module chain complex with $H_i(D_*)=0$ for $i\le k$, since $H_i(D_*)=K_{i-1}(M)$. Define a chain complex $E_*$ as follows:

$$\cdots\to E_{k+3}=D_{k+3}\xrightarrow{d_{k+3}}E_{k+2}=D_{k+2}\xrightarrow{d_{k+2}}E_{k+1}=\ker(d_{k+1})\xrightarrow{d_{k+1}}0$$

Now, we have that

$$0\to\operatorname{im}d_{k+1}\to D_k\to D_{k-1}\to D_{k-2}\to\cdots\to D_0\to 0$$

is exact. It follows by an argument given in a previous lemma that $\operatorname{im}d_{k+1}$ is projective. Then the short exact sequence

$$0\to\ker d_{k+1}\to D_{k+1}\to\operatorname{im}d_{k+1}\to 0$$

splits and so $\ker d_{k+1}$ is projective. This together with the fact that the inclusion $E\to D$ induces isomorphisms on homology implies, by the fundamental lemma of homological algebra, that this inclusion is a homtopy equivalence.

Next we have an exact sequence

$$\cdots\to E_{k+2}\to E_{k+1}\to H_{k+1}(E_*)\to 0.$$

Applying the functor $\operatorname{Hom}(-,\mathbb{Z}\pi)$ to this yields an exact sequence

$$0\to H_{k+1}(E_*)^*\to E^*_{k+1}\to E^*_{k+2}\to\cdots$$

Thus

$$\begin{aligned} K^k(M) &= H^{k+1}(D_*)=H^{k+1}(E_*)=\ker(E^*_{k+1}\to E^*_{k+2})\\ &= H_{k+1}(E)^*=H_{k+1}(D)^*=K_k(M)^*. \end{aligned}$$

□

**Corollary 15.8.** *Suppose that $K_r(M) = 0$ for $r < k$. Then $K_{n-r}(M) \cong K^r(M) \cong K_r(M)^* = 0$ for $r < k$.*

Therefore it suffices to kill the kernel homology groups up to and including the middle dimension. Then they will all vanish, and thus the relative homotopy groups $\pi_r(f)$ will all vanish, and so $f$ will be a homotopy equivalence. Above the middle dimension comes for free by duality. We have also now shown that $\lambda$ is nonsingular.

15.2. **The surgery obstruction $L$-groups.** In this section let $R$ be a ring with involution, such as $\mathbb{Z}\pi$, and we will consider f.g. projective $R$ modules, usually denoted by $P$.

Two $(-1)^k \left\{ \begin{array}{c} \text{symmetric} \\ \text{quadratic} \end{array} \right\}$ forms, $\left\{ \begin{array}{c} (P_1, \lambda_1) \text{ and } (P_2, \lambda_2) \\ (P_1, \psi_1) \text{ and } (P_2, \psi_2) \end{array} \right\}$ can be added, to form

$$\left( P_1 \oplus P_2, \left\{ \begin{array}{c} \lambda_1 \oplus \lambda_2 \\ \psi_1 \oplus \psi_2 \end{array} \right\} \right)$$

where for example $\lambda_1 \oplus \lambda_2((p_1, p_2), (q_1, q_2)) = \lambda_1(p_1, q_1) + \lambda_2(p_2, q_2)$.

For $\epsilon \in \{\pm 1\}$ and for a f.g. $R$-module $P$, the hyperbolic $\left\{ \begin{array}{c} \text{symmetric} \\ \text{quadratic} \end{array} \right\}$ form $\left\{ \begin{array}{c} H^\epsilon(P) \\ H_\epsilon(P) \end{array} \right\}$ is given by

$$\left\{ \begin{array}{c} (P \oplus P^*, \lambda_H) \\ (P \oplus P^*, \psi_H) \end{array} \right\}$$

where

$$\lambda_H \colon P \oplus P^* \xrightarrow{\begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}} P^* \oplus P \xrightarrow{\mathrm{Id} \oplus \mathrm{Canon}} P^* \oplus P^{**} \cong (P \oplus P^*)^*$$

and

$$\psi_H \colon P \oplus P^* \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} P^* \oplus P \xrightarrow{\mathrm{Id} \oplus \mathrm{Canon}} P^* \oplus P^{**} \cong (P \oplus P^*)^*.$$

Note that $\psi_H + T\psi_H = \lambda_H$. Given a $2k$-dimensional $\left\{ \begin{array}{c} \text{manifold } M \\ \text{DONM } f \colon M \to X \end{array} \right\}$, the $\left\{ \begin{array}{c} \text{symmetric} \\ \text{quadratic} \end{array} \right\}$ intersection form associated to $M \# S^k \times S^k$ is $\left\{ \begin{array}{c} \lambda_M \oplus H^{(-1)^k}(P) \\ \psi_f \oplus H_{(-1)^k}(P) \end{array} \right\}$.

**Definition 15.9** ($L$-groups)**.** Given a ring with involution $R$, the $\left\{ \begin{array}{c} \text{symmetric} \\ \text{quadratic} \end{array} \right\}$ $L$-group $\left\{ \begin{array}{c} L^{2k}(R) \\ L_{2k}(R) \end{array} \right\}$ is the group of *nonsingular* $(-1)^k$- $\left\{ \begin{array}{c} \text{symmetric} \\ \text{quadratic} \end{array} \right\}$ forms $\left\{ \begin{array}{c} (P, \lambda) \\ (P, \psi) \end{array} \right\}$ with $P$ a finitely generated projective $R$-module and

$$\left\{ \begin{array}{c} \lambda \in \ker(1 - T \colon \operatorname{Hom}_R(P, P^*) \to \operatorname{Hom}_R(P, P^*)) \\ \psi \in \operatorname{coker}(1 - T \colon \operatorname{Hom}_R(P, P^*) \to \operatorname{Hom}_R(P, P^*)) \end{array} \right\}$$

with addition $\oplus$, zero element the trivial form on the zero module, modulo the equivalence relation $\left\{\begin{array}{c}(P_1,\lambda_1)\sim(P_2,\lambda_2)\\(P_1,\psi_1)\sim(P_2,\psi_2)\end{array}\right\}$ if there exist $m,n\in\mathbb{N}\cup\{0\}$ such that

$$\left\{\begin{array}{c}(P_1,\lambda_1)\oplus\bigoplus^m H^\epsilon(R)\cong(P_2,\lambda_2)\oplus\bigoplus^n H^\epsilon(R)\\(P_1,\psi_1)\oplus\bigoplus^m H_\epsilon(R)\cong(P_2,\psi_2)\oplus\bigoplus^n H_\epsilon(R)\end{array}\right\}.$$

An isomorphism of $\left\{\begin{array}{c}\text{symmetric}\\\text{quadratic}\end{array}\right\}$ forms $\left\{\begin{array}{c}(P,\lambda)\cong(P',\lambda')\\(P,\psi)\cong(P',\psi')\end{array}\right\}$ is an isomorphism of $R$-modules $\phi\colon P\xrightarrow{\cong}P'$ such that $\left\{\begin{array}{c}\lambda=\phi^*\lambda'\phi\\\psi=\phi^*\psi'\phi\end{array}\right\}$.

The equivalence relation in the definition of $L$-groups is sometimes called Witt equivalence.

Note that when $R=\mathbb{Z}\pi$, you may see $L_{2k}(\mathbb{Z}\pi)$ written as $L_{2k}(\pi)$ in the literature. When $\pi=\{e\}$, the trivial group, we have

$$L_{2k}(\mathbb{Z})\cong\begin{cases}\mathbb{Z} & k\text{ even}\\\mathbb{Z}/2 & k\text{ odd.}\end{cases}$$

In the case of $k$ even, the isomorphism is given by the signature divided by 8: note that the existence of a quadratic refinement means that the intersection form $\lambda$ is even, and it turns out that the signature of nonsingular even forms is always divisible by 8. The quadratic refinement is determined to be half the diagonal values $\lambda(x,x)$, it is just its existence that guarantees divisibility by 8.

In the case of $k$ odd, the isomorphism to $\mathbb{Z}/2$ is given by the Arf invariant. This depends on just on the existence of a quadratic refinement, but also on the particular quadratic refinement. The signature and the Arf invariant were treated in student talks.

15.3. **The surgery obstruction theorem.** Let $n=2k>5$, and let $f\colon M\to X$ be an $n$-dimensional DONM. Define a map

$$\sigma\colon\mathcal{N}_n(X)\to L_n(\mathbb{Z}[\pi_1(X)])$$

as follows. Given $(f,b)\colon M\to X$, perform surgery to make $f$ be $k$-connected. So $\pi_1(M)\cong\pi_1(X)=\pi$. Then $(K_k(M),\lambda,\mu)$ determines an element $\sigma(f,b)$ of $L_n(\mathbb{Z}[\pi_1(X)])$. (Recall that the pair $(\lambda,\mu)$ are equivalent to a single map $\psi\in\operatorname{coker}(1-T)$.)

**Proposition 15.10.** *The surgery obstruction map $\sigma$ is well-defined.*

**Theorem 15.11** (Surgery obstruction theorem)**.** *The surgery obstruction $\sigma(f,b)=0$ if and only if $(f,b)$ is normally bordant to a homotopy equivalence ($n\geq 5$).*

This corresponds to "exactness at the normal invariants" of the surgery sequence.

The majority of the rest of the course is devoted to proving this theorem. To begin, we need some more homological algebra, to show that $K_k(M)$ is a stably f.g. free module. Actually this is needed to see that the surgery obstruction map $\sigma$ is well-defined, since the modules in $L$-theory are supposed to be projective. But beyond this, the surgery kernel being stably free will be useful later in the proof of the surgery obstruction theorem.

Some more remarks on being well-defined: we already saw that the form on the middle dimensional surgery kernel of a highly-connected degree one map is nonsingular. One could worry whether the choices in the way in which we made the map highly-connected affect the Witt equivalence class of the resulting element of the $L$-group. However all different choices are normal bordant to one another, and we will show that normal bordant highly connected maps have Witt equivalent intersection forms on their surgery kernels.

**Lemma 15.12.** *Suppose that a finite chain complex $D_*$ has chain groups $D_i$ that are stably f.g. free for $i \geq 0$, and suppose the following:*
*(a) $H_i(D_*) = 0$ for $i \neq k$.*
*(b) $H^{k+1}(\mathrm{Hom}_R(D_*, V)) = 0$ for all $R$-modules $V$.*
*Then $H_k(D_*)$ is stably f.g. free.*

We will show why this lemma gives us the conclusion we need, in the next corollary, then we will prove the lemma.

**Corollary 15.13.** *Let $f\colon M \to X$ be a $k$-connected degree one map. Then the surgery kernel $K_k(M)$ is stably f.g. free.*

*Proof.* Let $D_* := \mathscr{C}(\widetilde{f}_!)$ (recall that $\widetilde{f}_!\colon C_*(\widetilde{X}) \to C_*(\widetilde{M})$ is the Umkehr map). Note that $H_i(D_*) = K_i(M)$. Then we have shown that $K_i(M) = 0$ for $i \neq k$, so (a) of Lemma 15.12 holds for $D_*$. We therefore just have to show that (b) holds.

Similarly to in the proof of Lemma 15.7, define a chain complex $E$:

$$\cdots \to E_{k+2} = D_{k+2} \xrightarrow{d_{k+2}} E_{k+1} = D_{k+1} \xrightarrow{d_{k+1}} E_k = \ker(d_k) \xrightarrow{d_k} 0$$

Note that there is a degree shift here relative to the previous incarnation of $E$, because in this lemma we are using $D_* = \mathscr{C}(\widetilde{f}_!)$ whereas in the previous lemma we had $\mathscr{C}(\widetilde{f})$. Nevertheless the same proof as used in that lemma shows that $E \simeq D$. Thus $E^{n-*} \simeq D^{n-*}$. Now $H_i(D_*) = 0$ for $i \neq k$. Thus $H_i(D^{n-*}) = 0$ for $i \neq k$ by Poincaré duality $D_* \simeq D^{n-*}$ for the surgery kernel.

Let $V$ be any $R$-module. Then

$$H^j(\mathrm{Hom}_R(D_*, V)) \cong H^j(\mathrm{Hom}_R(D^{n-*}, V)) \cong H^j(\mathrm{Hom}_R(E^{n-*}, V)).$$

But $E^{n-i} = 0$ for $i > k$ so $H^j(\mathrm{Hom}_R(D_*, V)) = 0$ for $j > k$. In particular $H^{k+1}(\mathrm{Hom}_R(D_*, V)) = 0$.

We can now apply Lemma 15.12 to see that $H_k(D_*) \cong K_k(M)$ is stably f.g. free. □

*Proof of Lemma 15.12.* We begin the proof of this purely algebraic lemma with a claim.

**Claim.** $\mathrm{im}\, d_{k+1}$ *is a direct summand of $D_k$ and $H_k(D_*)$ is f.g. projective.*

Here is the proof of the claim. We will just say projective, but each time we really mean finitely generated projective. We apply $H^{k+1}(\mathrm{Hom}(D_*, V)) = 0$ with

$V = \operatorname{im} d_{k+1}$. This implies that

$$\operatorname{Hom}_R(D_k, \operatorname{im} d_{k+1}) \to \operatorname{Hom}_R(D_{k+1}, \operatorname{im} d_{k+1}) \to \operatorname{Hom}_R(D_{k+2}, \operatorname{im} d_{k+1})$$

is exact. Now the map $d_{k+1} \in \operatorname{Hom}_R(D_{k+1}, \operatorname{im} d_{k+1})$ maps to zero in $\operatorname{Hom}_R(D_{k+2}, \operatorname{im} d_{k+1})$, so lies in the image of $\operatorname{Hom}_R(D_k, \operatorname{im} d_{k+1})$. That is, there exists a map $\alpha\colon D_k \to \operatorname{im} d_{k+1}$ with $\alpha \circ d_{k+1} = d_{k+1}$. This gives rise to a splitting map of the short exact sequence

$$0 \to \operatorname{im} d_{k+1} \to D_k \to D_k / \operatorname{im} d_{k+1} \to 0.$$

Thus $\operatorname{im} d_{k+1}$ is a direct summand of $D_k$ as claimed, and in particular we note that $\operatorname{im} d_{k+1}$ is projective. We can replace $D_k$ by $\ker d_k$ to obtain a split exact sequence

$$0 \to \operatorname{im} d_{k+1} \to \ker d_k \to H_k(D_*) \to 0.$$

Thus $\operatorname{im} d_{k+1} \oplus H_k(D_*) \cong \ker d_k$. So we need to see that $\ker d_k$ is projective. But we saw the argument before for this: since the homology of $D_*$ vanishes in degrees less than $k$, we have an exact sequence

$$0 \to \operatorname{im} d_k \to D_{k-1} \to D_{k-2} \to \cdots \to D_0 \to 0$$

which implies that $\operatorname{im} d_k$ is projective. Then

$$0 \to \ker d_k \to D_k \to \operatorname{im} d_k \to 0$$

splits so that $\ker d_k$ is projective. Then the fact from above that $\operatorname{im} d_{k+1} \oplus H_k(D_*) \cong \ker d_k$, together with the facts that $\operatorname{im} d_{k+1}$ and $\ker d_k$ are projective, implies that $H_k(D_*)$ is projective. This completes the proof of the claim.

Now let us proof the lemma. The fact that $H_i(D_*) = 0$ for $i > k$ implies that

$$\cdots \to D_{k+3} \to D_{k+2} \to D_{k+1} \to \operatorname{im} d_{k+1} \to 0$$

is exact. By the claim, $\operatorname{im} d_{k+1}$ is projective. Thus

$$\operatorname{im} d_{k+1} \oplus \bigoplus_{j>0 \text{ even}} D_{k+j} \cong \bigoplus_{j>0 \text{ odd}} D_{k+j}$$

Since $D_*$ is a finite complex, $\operatorname{im} d_{k+1}$ is stably f.g. free. Similarly, the exact sequence

$$0 \to \ker d_k \to D_k \to D_{k-1} \to \cdots \to D_0 \to 0$$

implies that $\ker d_k$ is stably f.g. free. Then the sequence

$$0 \to \operatorname{im} d_k \to \ker d_k \to H_k(D_*) \to 0$$

splits since $H_k(D_*)$ is projective by the claim. But then $H_k(D_*) \oplus \operatorname{im} d_k \cong \ker d_k$. Both $\operatorname{im} d_k$ and $\ker d_k$ are stably free, so $H_k(D_*)$ must be too. This completes the proof of the lemma. □

**Theorem 15.14.** *Let $f\colon M \to X$ be a $k$-connected $2k$-dimensional degree one normal map with $\sigma(f, b) = 0 \in L_{2k}(\mathbb{Z}\pi)$, where $\pi = \pi_1(X)$. Then $(f, b)$ is normal bordant to a homotopy equivalence.*

*Proof.* First, note that surgery on trivial elements of $\pi_{k+1}(f)$, that is on null homotopic embeddings $S^k \subset M$, results in converting $M$ to $M \# S^k \times S^k$. Thus the surgery kernel form becomes $(K_k(M), \lambda, \mu) \oplus H_{(-1)^k}(\mathbb{Z}\pi)$. Since $K_k(M)$ is stably free, and by assumption the intersection form is stably hyperbolic, after enough surgeries on trivial elements we can arrange that

$$(K_k(M), \lambda, \mu) \cong \bigoplus^{v} H_{(-1)^k}(\mathbb{Z}\pi) \cong H_{(-1)^k}(\mathbb{Z}\pi^v)$$

for some $v \in \mathbb{N} \cup \{0\}$. If we had $v = 0$, then we would have $K_i(M) = 0$ for all $i$ and $f$ would be a homotopy equivalence. Thus we need a mechanism for reducing $v$ by one. Then we will obtain a proof of the theorem by induction.

Let $\{b_1, \dots, b_v, c_1, \dots, c_v\}$ be a $\mathbb{Z}\pi$-basis for $K_k(M)$, with respect to which

$$\lambda = \bigoplus^{v} \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix}$$

and $\mu(b_i) = 0$, for $i = 1, \dots, v$. Perform surgery on $b_v$. That is, $\mu(b_v) = 0$, so using the Whitney trick we can represent the homotopy class by a framed embedding. Surgery gives rise to a normal bordism

$$F\colon (W; M, M') \to (X \times I; X \times \{0\}, X \times \{1\})$$

where

$$W := M \times I \cup D^{k+1} \times D^{n-k} = M \times I \cup D^{k+1} \times D^k.$$

A degree one bordism has surgery kernel $\mathbb{Z}\pi$-modules

$$K_k(W) := H_{k+1}(C_*(\widetilde{W}) \to C_*(\widetilde{X \times I}))$$

$$K_k(W, M) := H_{k+1}(C_*(\widetilde{W}, \widetilde{M}) \to C_*(\widetilde{X \times I}, \widetilde{X} \times \{0\}))$$

$$K_k(W, M') := H_{k+1}(C_*(\widetilde{W}, \widetilde{M'}) \to C_*(\widetilde{X \times I}, \widetilde{X} \times \{1\}))$$

and

$$K_k(W, \partial W) := H_{k+1}(C_*(\widetilde{W}, \widetilde{\partial W}) \to C_*(\widetilde{X \times I}, \widetilde{X} \times \{0, 1\})).$$

These fit into a commutative braid of exact sequences, that we will use to determine $K_k(M')$.

$$\begin{tikzcd}
0 \arrow[dr] \arrow[rr, bend left] & & K_{k+1}(W,M) \arrow[dr, "\theta"'] \arrow[rr, bend left] & & K_k(M) \arrow[dr] \arrow[rr, bend left, "\chi"] & & K_k(W,M') \arrow[dr] \arrow[rr, bend left] & & 0 \\
& 0 \arrow[ur] \arrow[dr] & & K_{k+1}(W,\partial W) \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & K_k(W) \arrow[ur] \arrow[dr] & & 0 \arrow[ur] & \\
& & 0 \arrow[ur] \arrow[rr, bend right] & & K_k(M') \arrow[ur] \arrow[rr, bend right] & & 0 \arrow[ur] & &
\end{tikzcd}$$

Each of the four interlocking exact sequences arises from the long exact sequence of a pair. We have that

$$K_{k+1}(W, M) \cong \mathbb{Z}\pi,$$

generated by the core of $D^{k+1} \times D^k$, while

$$K_k(W, M') \cong \mathbb{Z}\pi,$$

generated by the cocore of $D^{k+1} \times D^k$.

**Claim.** *There is a basis $(b'_1, \ldots, b'_v, c'_1, \ldots, c'_{v-1})$ of $K_{k+1}(W, \partial W)$ such that $\alpha(b'_i) = b_i$ and $\alpha(c'_j) = \alpha(c_j)$.*

The map $\chi$ is given by

$$\begin{array}{rcl} K_k(M) & \to & K_k(W, M') \\ x & \mapsto & \lambda(x, b_v). \end{array}$$

Thus $\chi(c_v) = 1$, $\chi(c_i) = 0$ for $i \neq v$, and $\chi(b_i) = 0$ for all $i$. The kernel of this is a free module isomorphic to $\mathbb{Z}\pi^{2v-1}$ generated by $b_1, \ldots, b_v, c_1, \ldots, c_{v-1}$. Since $\alpha$ is an injection, the claim follows.

The map $\theta \colon K_{k+1}(W, M) \to K_{k+1}(W, \partial W)$ is determined by $1 \mapsto b_v$. Therefore we see that $K_k(M') \cong \mathbb{Z}\pi^{2v-2}$ generated by elements $(b''_1, \ldots, b''_{v-1}, c''_1, \ldots, c''_{v-1})$, where $b''_i = \beta(b'_i)$ and $c''_i = \beta(c_i)$ for $i = 1, \ldots, v-1$. With respect to this basis, we have

$$(K_k(M'), \lambda, \mu) = \bigoplus^{v-1} H_{(-1)^k}(\mathbb{Z}\pi),$$

since the remaining summands can be assumed, by the Whitney trick, to be disjoint from the sphere $b_v$ on which surgery was performed, and therefore their intersection numbers are unaltered. This completes proof of the inductive step, and therefore completes the proof of the theorem. □

This gives one direction in the proof of the surgery obstruction theorem. Now for the converse.

**Theorem 15.15.** *Suppose that $2k$-dimensional degree one normal maps $(f, b) \colon M \to X$ and $(f', b') \colon M' \to X$ are normally bordant via a degree one normal bordism*

$$((F, B); (f, b), (f', b') \colon (W; M, M') \to (X \times I; X \times \{0\}, X \times \{1\}).$$

*Then $\sigma(f, b) = \sigma(f', b') \in L_{2k}(\mathbb{Z}[\pi_1(X)])$.*

Note that we do not need to assume that $f$ and $f'$ are $k$-connected; we will fix this at the beginning of the proof. It is important for the proof that the surgery obstruction is well-defined that we do not need to assume this in the statement of the theorem, as remarked above. In particular, we see that the choice of how to make a map highly connected does not affect the Witt class of the resulting element in $L_{2k}(\mathbb{Z}\pi)$.

*Proof.* First, by surgery below the middle dimension on $M$ and $M'$, we can assume that both $f$ and $f'$ are $k$-connected. By surgery on trivial elements we can assume that $K_k(\partial W)$ is a free $\mathbb{Z}\pi$-module. By surgery below the middle dimension on $W$ we can assume that $F$ is also $k$-connected.

Next, we can also arrange that $K_k(W, \partial W) = 0$ by the following argument. Since $K_{k-1}(\partial W) = 0$ the map $K_k(W) \to K_k(W, \partial W)$ is onto. Thus each generator of $K_k(W, \partial W)$ can be represented by an embedding $S^k \to \operatorname{Int} W$, which can be thickened to an embedding of $S^k \times D^{k+1}$ by the bundle data argument used in previous sections. Consider an embedded path from $M'$ to a point on the boundary of $S^k \times D^{k+1}$, and delete the interior of $S^k \times D^{k+1}$ together with a neighbourhood of the path. The effect is to quotient $K_k(W, \partial W)$ by this generator, and to connect sum $M'$ with $S^k \times S^k$. But this adds a hyperbolic summand to $\sigma(f', b')$, which does not change the Witt class of the intersection form.

We now have the following commutative diagram.

$$\begin{tikzcd}
K_{k+1}(W,\partial W) \arrow[r] \arrow[d, "\cong"', "(-\cap[W,\partial W])^{-1}"] & K_k(M) \oplus K_k(M') \arrow[r, "i_*"] \arrow[d, "\cong"', "(-\cap[M])^{-1} \oplus (-\cap -[M'])^{-1}"] & K_k(W) \arrow[r] \arrow[d, "\cong"', "(-\cap[W,\partial W])^{-1}"] & 0 \\
K^k(W) \arrow[r] & K^k(M) \oplus K^k(M') \arrow[r] & K^{k+1}(W,\partial W) \arrow[r] & 0
\end{tikzcd}$$

As in our talk on signatures, this diagram can be used to show that $\ker i_*$ is a lagrangian for the intersection form; that is, $\lambda(\ker i_*, \ker i_*) = 0$ and $\ker i_*$ is a submodule of half rank. Also $\mu(\ker i_*) = 0$. To see this represent an element of the kernel by an immersion from a disjoint union of spheres $\coprod S^k$ into $M \sqcup M'$, bounding an immersion of a punctured $S^{k+1} \looparrowright W$. The double points of this latter immersion are a collection of arcs and circles. The arcs pair up double points of the original immersions $\coprod S^k \looparrowright M$.

A nonsingular quadratic form on a free $\mathbb{Z}\pi$-module, that admits a lagrangian, is isomorphic to the hyperbolic form (exercise to check this, or see Lemma 4.80 of [CLM]). Thus $\sigma(f, b) - \sigma(f', b') = 0 \in L_{2k}(\mathbb{Z}\pi)$ (there is a minus sign coming from orientations of fundamental classes) and so $\sigma(f, b) = \sigma(f', b')$ as claimed. $\square$

## 16. Solution to the manifold existence question in the simply connected case

Our results now allow us to give a fairly satisfactory description of a solution to the question of whether a manifold exists that is homotopy equivalent to a given CW complex.

Let $\xi \colon E \to X$ be a stable vector bundle. There are special cohomology classes $p_j(\xi) \in H^{4j}(X, \mathbb{Z})$ called the Pontryagin classes. These can be defined as the generators of $H^*(BO; \mathbb{Q})$, lifted to the integers, and pulled back along the classifying map of $\xi$. The $L_j$-polynomial of the vector bundle $\xi$ is a homogeneous polynomial of degree $j$ in the $p_i$ of $\xi$,

$$\mathcal{L}_k(\xi) = \mathcal{L}_j(p_1, \dots, p_j) \in H^{4j}(X; \mathbb{Q}).$$

The coefficients of these polynomials are defined in terms of the Bernoulli numbers. We will just give the first two:

$$\mathcal{L}_1(\xi) = \frac{1}{3} p_1(\xi);$$

$$\mathcal{L}_2(\xi) = \frac{1}{45}(7p_2 - p_1^2).$$

These polynomials are crucial for the statement of the *Hirzebruch signature theorem.*

**Theorem 16.1** (Hirzebruch signature theorem)**.** *Let $M$ be a closed smooth oriented $4k$-dimensional manifold, let $\xi := TM$ and define $p_j(M) := p_j(\xi)$. Then*

$$\operatorname{sign}(M) = \langle\, \mathcal{L}_k(p_1(M), \dots, p_k(M)), [M]\,\rangle$$

For example, for a closed 4-manifold, the Hirzebruch signature theorem gives rise to the neat formula that the signature and the first Pontryagin class are related by $\operatorname{sign}(M) = p_1/3$. For example, when $M = \mathbb{CP}^2$, $\operatorname{sign}(M) = 1$ and $p_1(M) = 3[M]^*$.

**Theorem 16.2** (Simply connected surgery theorem)**.** *Let $n \leq 5$ and let $X$ be an $n$-dimensional simply connected finite, connected Poincaré complex. Then $X$ is homotopy equivalent to a closed manifold if and only if the Spivak normal fibration has a vector bundle reduction, i.e. $X \to BG \to B(G/O)$ is null homotopic, and if there is an associated normal invariant with:*

*(i) if $n = 4k$, $\langle \mathcal{L}(\xi)^{-1}, [X]\rangle = \operatorname{sign} X$;*
*(ii) if $n = 4k + 2$, then the Arf invariant of the associated quadratic form in $L_{4k+2}(\mathbb{Z})$ vanishes;*
*(iii) if $n$ is odd, there is no further obstruction.*

**Remark 16.3.**

(1) For the first two conditions, if the condition is not satisfied for an initial particular choice of normal invariant, perhaps the action of $[X, G/O]$ on the normal invariants can remedy this.
(2) We have not had time to define the surgery obstruction for $n$ odd, however there are surgery obstruction $L$-groups and it is true that $L_{2k+1}(\mathbb{Z}) = 0$ for all $k$, so the homotopy theoretic construction is the only obstruction in odd dimensions.

*Proof.* Only item (i) needs a proof, and we will just sketch it. Recall that the signature gives an isomorphism $\operatorname{sign}\colon L_{4k}(\mathbb{Z}) \xrightarrow{\simeq} 8 \cdot \mathbb{Z}$. For a choice of degree one normal map, we need for

$$0 = \operatorname{sign}(K_k(M) \otimes \mathbb{R}, \lambda \otimes \mathrm{Id}) = \operatorname{sign}(M) - \operatorname{sign}(X).$$

But:

$$\operatorname{sign}(M) = \langle \mathcal{L}(TM), [M]\rangle = \langle \mathcal{L}(\nu(M))^{-1}, [M]\rangle = \langle \mathcal{L}(\xi)^{-1}, [X]\rangle$$

The first equality follows from the Hirzebruch signature theorem. The second equality uses that $TM$ and $\nu(M)$ are stable inverses. The third equality uses that we have a degree one normal map of $\nu(M) \to M$ to $\xi \to X$. □

## 17. The surgery exact sequence

We have seen the exactness of $\mathcal{S}_n(X) \to \mathcal{N}_n(X) \to L_n(\mathbb{Z}[\pi_1(X)])$. We want to see how this sequence can be extended to the left. Recall that $\mathcal{S}_n(X)$ is the set of manifolds homotopy equivalent to $X$ up to $h$-cobordism over $X$.

There is an action of $L_{n+1}(\mathbb{Z}\pi)$ on $\mathcal{S}_n(X)$. When $n + 1 = 2k$ is even, this is defined roughly for a manifold $M$ with a homotopy equivalence to $X$, by taking $M \times I$ and attaching handles $D^k \times D^k$, then plumbing these handles together. That is, identify to each other small neighbourhoods $D^k \times D^k$ of a point on each of the handles, but switch the coordinates $(x, y) \sim (y', x')$. So a piece of the core of one handle becomes a parallel of the cocore of the other, and vice versa. This induces an intersection point between the corresponding elements of $K_k(W)$, where $W$ is the $n+1$-dimension bordism we are building. Repeatedly perform this plumbing to create a desired intersection form (one has to be more careful when $\pi \neq \{e\}$, but we will not go into detail on this here). This produces a cobordism $W$ from $M$ to another manifold $M'$, which it turns out is also homotopy equivalent to $X$.

There is a resulting short exact sequence of sets, for $n \neq 5$:

$$L_{n+1}(\mathbb{Z}[\pi_1(X)]) \to \mathcal{S}_n(X) \to \mathcal{N}_n(X) \to L_n(\mathbb{Z}[\pi_1(X)]).$$

The first map is the action just described. There is an analogous construction when $n+1$ is odd. The second map uses the homotopy inverse $X \to M$ to pull back the normal bundle of $M$ to get a vector bundle $\xi$ on $X$ as required for an element of $\mathcal{N}_n(X)$.

Why is the surgery sequence exact at the structure set? The action just described produces a bordism, which can be made into a normal bordism. Thus the images of $M$ and $M'$ in $\mathcal{N}_n(X)$ coincide. On the other hand, given any two normally bordant homotopy equivalences $f\colon M \to X$ and $f'\colon M' \to X'$, the surgery obstruction of the bordism gives the lift to an element $x \in L_{n+1}(\mathbb{Z}[\pi_1(X)])$. The intersection form of the surgery kernel of the bordism is non-singular because the degree one map to $X \times I$ is a homotopy equivalence on the boundary. We can also use $x$ to construct a bordism from $M$ to $M''$, using the plumbing construction above. Then glue the two bordisms together to get a bordism from $M'$ to $M''$ with vanishing surgery obstruction. Then surgery on this relative to the boundary produces a bordism homotopy equivalent to $X \times I$, hence by the $h$-cobordism theorem $M' \cong M''$, and so we see that normally bordant elements of the structure set are indeed related by the action of $L_{n+1}(\mathbb{Z}[\pi_1(X)])$.

There is also a *simple* version of the surgery sequence, obtained by replacing $\mathcal{S}_n(X)$ with $\mathcal{S}^s_n(X)$; here $h$-cobordisms are required to be simple in the equivalence relation. One also replaces $L_m(\mathbb{Z}[\pi_1(X)])$, for $m = n, n+1$, with $L^s_m(\mathbb{Z}[\pi_1(X)])$, the simple $L$-group. Here the intersection form $\lambda$ is required to vanish in the Whitehead group $\mathrm{Wh}(\pi_1(X))$. Isomorphisms of forms in the equivalence relation are also required to be simple.

## 18. The uniqueness question

We have been mostly focusing on the existence question so as not to over complicate matters. Here are some brief remarks on one common strategy for the application of surgery theory to the uniqueness question.

There exists a version for everything we have done for manifolds with boundary, where the surgeries happen away from the boundary. There is a corresponding sequence

$$L_{n+1}(\mathbb{Z}[\pi_1(X)]) \to \mathcal{S}_n(X \times I, \partial) \to \mathcal{N}_n(X \times I, \partial) \to L_n(\mathbb{Z}[\pi_1(X)]).$$

To decide if there is a unique manifold in a given homotopy type, let $M$ be an $n$-dimensional manifold and apply the above sequence with $X = M$. Suppose there is another manifold $N$ with a homotopy equivalence $N \to M$. There is a two-step process. First, does there exist a normal bordism $W$ over $M \times I$ between $N$ and $M$. Suppose the answer is yes. Next, is there a choice of $W$ that is normal bordant relative to the boundary to a homotopy equivalence? If the answer to this question is also yes (as decided by surgery), then we have an $h$-cobordism! Thus for $n \geq 5$ the $h$ cobordism is diffeomorphic to a product, $N$ and $M$ are diffeomorphic, and the original homotopy equivalence $N \to M$ is homotopic to a diffeomorphism.

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Département de Mathématiques, Université du Québec à Montréal, Canada
*E-mail address*: mark@cirget.ca