# MAT993D: Exercise sheet 1

**Exercise 1.** Show that $\Omega_1(S^1) \neq 0$. In fact $\Omega_1(S^1) \xrightarrow{\cong} \mathbb{Z}$. Use transversality and orientations to describe the map. Can you show that the map is an isomorphism?

**Exercise 2.** Show that every closed $n$-manifold $M$ admits a degree one map $F\colon M \to S^n$.

**Exercise 3.** Show that $U(n)$ is a (compact) manifold. What is its dimension?

**Exercise 4.** Prove that bordism of maps over $X$ is an equivalence relation.

**Exercise 5.** Let $X$ be a 2-complex. Describe a function $f\colon X \to \mathbb{R}$ such that the barycentre of each cell is a critical point, and these are the only critical points. Sketch the gradient vector field. What is the index of each critical point? What is its ascending and descending submanifold?

**Exercise 6.** Let $M$ be a compact manifold of dimension $n$ with $H_*(M) \cong H_*(S^n)$ and $\pi_1(M) = 1$. Show that

$$(M \setminus (D_1^n \sqcup D_2^n); S^{n-1}, S^{n-1})$$

is a simply connected $h$-cobordism. You will probably need the Whitehead and Hurewicz theorems.

**Exercise 7.** Prove that every compact manifold admits a Morse function. At least, read and convince yourself of one of Milnor's proofs; there is one Morse theory and one in the $h$-cobordism book.