MAT993D: Exercise sheet 2

Exercise 1. Let M be a closed odd dimensional manifold. Show that for any handle decomposition the number of handles of odd index is equal to the number of handles of even index.

Exercise 2. Let M be a closed connected manifold with a handle decomposition without any handles of index 1. Show that M is simply connected.

Exercise 3. Let $\mathbb{K}$ be the Klein bottle. Let $\pi = \pi_1(\mathbb{K})$. Write down a $\mathbb{Z}\pi$-module chain complex for $\mathbb{K}$.

Exercise 4. Let

$$\pi = \langle g_1, \ldots, g_n \,|\, w_1, \ldots, w_m \rangle.$$

Construct a 2-complex with a single 0-cell, one 1-cell per generator, and one 2-cell per relator. Write down the $\mathbb{Z}\pi$-module cellular chain complex for this 2-complex. The *Fox free differential calculus* might help. Let F be the free group on letters $g_1, \ldots, g_n$. Define

$$\frac{\partial}{\partial g_i} \colon F \to \mathbb{Z}F$$

by

$$\frac{\partial g_j}{\partial g_i} = \delta_{ij}, \; \frac{\partial 1}{\partial g_i} = 0, \; \frac{\partial g_j^{-1}}{\partial g_i} = -g_j^{-1}\delta_{ij}$$

and a Leibniz-type (but not exactly the Leibniz rule)

$$\frac{\partial (uv)}{\partial g_j} = ??$$

Find the Leibniz-type rule that allows the inductive free derivation of any word in F, so that the derivative of a word corresponds to the sum of cells in the Cayley graph of F used in the path determined by w. After this, writing down the chain complex of the 2-complex associated to a group presentation should be straightforward. Try out your formula on the fundamental group of the complement in S^3 of a trefoil knot:

$$\pi = \langle g_1, g_2, , g_3 \,|\, g_3 g_2 g_3^{-1} g_1^{-1}, g_2 g_1 g_2^{-1} g_3^{-1} \rangle.$$

Exercise 5. Show that $1-t-t^{-1} \in \mathbb{Z}[\mathbb{Z}/5]$, for $t \in \mathbb{Z}/5$ the generator, is a unit (i.e. find a multiplicative inverse) and hence show that $1-t-t^{-1}$ defines an element η in $\mathrm{Wh}(\mathbb{Z}/5)$. Prove that we obtain a well-defined map

$$\mathrm{Wh}(\mathbb{Z}/5) \to \mathbb{R}$$

by sending the class represented by the $\mathbb{Z}[\mathbb{Z}/5]$-automorphism $f\colon \mathbb{Z}[\mathbb{Z}/5]^n \to \mathbb{Z}[\mathbb{Z}/5]^n$ to $\ln(|\det(\overline{f})|)$, where $\overline{f}\colon \mathbb{C}^n \to \mathbb{C}^n$ is the $\mathbb{C}$-linear map

$$f \otimes_{\mathbb{Z}[\mathbb{Z}/5]} \mathrm{Id}_{\mathbb{C}} \colon \mathbb{Z}[\mathbb{Z}/5]^n \otimes_{\mathbb{Z}[\mathbb{Z}/5]} \mathbb{C} \to \mathbb{Z}[\mathbb{Z}/5]^n \otimes_{\mathbb{Z}[\mathbb{Z}/5]} \mathbb{C}$$

with respect to the $\mathbb{Z}/5$-action on $\mathbb{C}$ given by multiplication with $\exp(2\pi i/5)$. Finally show that η generates an infinite cyclic subgroup in $\mathrm{Wh}(\mathbb{Z}/5)$.

Exercise 6. Consider a 3-component oriented link $L_1 \sqcup L_2 \sqcup L_3$ in S^3 such that for each component L_i bounds an embedded disc D_i in D^4 (here $S^3 = \partial D^4$). Suppose that the D_i intersect each other transversely in algebraically cancelling pairs. Find a collection of Whitney discs for the pairs of intersection points. Let W_{ij} be the union of the Whitney discs pairing up intersections between D_i and D_j, $i \neq j$. For i, j, k distinct, let $\tau(ijk) \in \mathbb{Z}$ be the algebraic count of intersection points in $W_{ij} \cap D_k$. Can you compute τ for the 3-component unlink and for the Borromean rings? (In fact this number is invariant under ambient isotopy of a link, and does not depend on the choice of discs D_i nor on the choice of Whitney discs W_{ij}.) Hint: surfaces in 4-space are best understood in terms of motion pictures i.e. consider the radial direction of D^4 as time, and at each point in time t one sees the intersection of the surfaces with the S^3 of radius $1-t$. Then crossing changes on link diagrams correspond to intersections between surfaces. This will enable you to explicitly construct the discs you need.