MAT993D: Exercise sheet 3

Exercise 1. Compute $K_0(\mathbb{F})$ for a field $\mathbb{F}$ and $K_0(R)$ for a PID R. What is $K_0(\mathbb{C}[S_3])$, where S_3 is the symmetric group?

Exercise 2. Prove that the different definitions of projective coincide.

Exercise 3.

(i) Express the rational numbers as a colimit of the directed system

$$\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 3} \mathbb{Z} \xrightarrow{\cdot 4} \mathbb{Z} \xrightarrow{\cdot 5} \mathbb{Z} \cdots$$

(ii) Express the direct limit as the colimit of a category.

(iii) Express the Seifert Van Kampen theorem as the statement that a certain colimit commutes with the functor from spaces to groupoids sending a space to its fundamental groupoid.

Exercise 4. Show that $K_1(\mathbb{Z}) \cong \mathbb{Z}/2$ and that $K_1(\mathbb{F}) \cong \mathbb{F}^\times$, where $\mathbb{F}$ is any field.

Exercise 5. Show that an abelian group is divisible if and only if it is injective.

Exercise 6. What are $\mathrm{Ext}^n_{\mathbb{Z}[\mathbb{Z}]}(\mathbb{Z}, \mathbb{Z}[\mathbb{Z}])$ and $\mathrm{Ext}^n_{\mathbb{Z}[\mathbb{Z}]}(\mathbb{Z}, \mathbb{Z})$? What is $\mathrm{Ext}^n_A(A, \mathbb{Q})$, for any ring with unity A?

Exercise 7. Prove the universal coefficient theorems for homology and cohomology. In the lecture notes, some diagram chases are left to the reader, so do these diagram chases to make sure the maps are well-defined and that the sequence is exact. Then see if you can do the whole thing for the homology, Tor version on your own.

Exercise 8. Prove the homology version of the Thom isomorphism theorem.