Two new uses for the Alexander Invariant

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Links in S^3, component preserving cobordisms in $S^3 \times I$.

Theorem (J. Pardon)

A smooth cobordism from a Kh*–thin link to a link which is split into m disjoint sublinks must have genus at least* $\lfloor \frac{m}{2} \rfloor$.

$$\left\{ \begin{array}{c} \text{non-split} \\ \text{alternating} \\ \text{links} \end{array} \right\} \subset \left\{ \begin{array}{c} \text{Kh–thin} \\ \text{links} \end{array} \right\}$$

$$\subset \left\{ \begin{array}{c} \text{links with} \\ \det(L) \neq 0 \end{array} \right\} \subset \left\{ \begin{array}{c} \text{links with} \\ \Delta_L(t) \neq 0 \end{array} \right\}.$$

Definition

A link J is weakly m–split if J is the boundary of m disjoint surfaces $\Sigma_1 \sqcup \cdots \sqcup \Sigma_m$ in S^3.

$$\left\{ \begin{array}{c} m-\text{split} \\ \text{links} \end{array} \right\} \subset \left\{ \begin{array}{c} \text{weakly} \\ m-\text{split} \\ \text{links} \end{array} \right\} \supset \left\{ \begin{array}{c} m-\text{comp} \\ \text{boundary} \\ \text{links} \end{array} \right\}.$$

Theorem (Friedl, P.)

A locally flat cobordism between a link L with $\Delta_L(t) \neq 0$ and a weakly m–split link J must have genus at least $\lfloor \frac{m}{2} \rfloor$.

Idea of proof:

Let C a genus g cobordism between L with $\Delta_L(t) \neq 0$ and a weakly m–split link J.

$$X_L := S^3 \setminus \nu L;\ X_J := S^3 \setminus \nu J;\ X_C := S^3 \times I \setminus \nu C.$$

- $\operatorname{rk} H_1(X_L; \mathbb{Q}(t)) = 0$;
- $\operatorname{rk} H_1(X_J; \mathbb{Q}(t)) \geq m - 1$;
- $\chi^{\mathbb{Q}}(X_C) = 2g$;
- $\operatorname{rk} H_2(X_C; \mathbb{Q}(t)) = \chi^{\mathbb{Q}(t)}(X_C) = 2g$;
- By the long exact sequence of the pair, $(X_C, \partial X_C)$, and Poincaré duality:

$$\operatorname{rk} H_2(X_C; \mathbb{Q}(t)) \geq \operatorname{rk} H_1(X_J; \mathbb{Q}(t)) - \operatorname{rk} H_1(X_L; \mathbb{Q}(t))$$

- Therefore $2g \geq m - 1$ as claimed.

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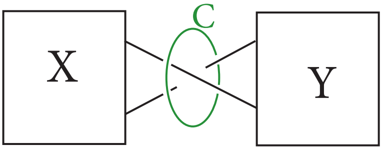
Is it possible to change a crossing on an oriented knot and obtain the same knot?

In general, yes:

Definition

A crossing circle C for a knot $K \subset S^3$ is an unknotted simple closed curve in $S^3 \setminus \nu K$ which bounds a disk whose intersection with K is two points of opposite sign. A crossing change on the knot is obtained by performing ± 1 Dehn surgery along C.

A crossing change is called *nugatory* if C bounds a disk in $S^3 \setminus \nu K$.



Definition

A knot is called *cosmetic* if it admits a crossing change which does not change the knot but which is not nugatory. Such a crossing is called a cosmetic crossing.

Conjecture (Kirby's problem list)

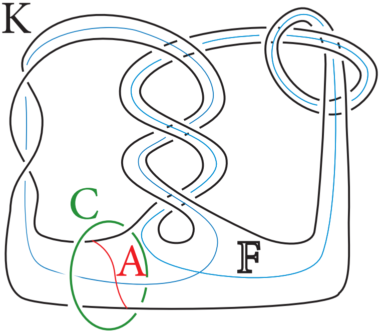
There are no cosmetic knots in S^3.

Known for unknot (Gabai, Scharlemann–Thompson), fibred (Kalfagianni), and two bridge knots (Torisu).

Theorem (Balm, Kalfagianni)

Let K be a genus one cosmetic knot with cosmetic crossing circle $C = \partial D$. There exists a minimal genus Seifert surface F for K such that $F \cap D$ is an essential arc A in F.

The proof uses a deep result of Gabai on taut foliations. Considering the Seifert surface in disk–band form, the crossing change introduces a twist to one of the bands.



Change in Seifert form $\Rightarrow$ potential change in Alexander invariant.

Partial solution for genus one knots:

Theorem (Balm, Friedl, Kalfagianni, P.)

Suppose K is cosmetic, has genus one, and $\Delta_K(t) \neq 1$.

1. *K is algebraically slice;*
2. *K does not admit a unique minimal genus Seifert surface;*
3. *K has more than 12 crossings.*

Idea of proof (1):

S–equivalence of Seifert forms of K and its cosmetic partner K':

$$\begin{bmatrix} a & b \\ b+1 & c \end{bmatrix} \sim \begin{bmatrix} a \pm 1 & b \\ b+1 & c \end{bmatrix}$$

$\Delta_K(t) = \Delta_{K'}(t)$, implies that $c = 0$. Therefore algebraically slice.

Idea of proof (2):

So we have an S–equivalence:

$$\begin{bmatrix} a & b \\ b+1 & 0 \end{bmatrix} \sim \begin{bmatrix} a \pm 1 & b \\ b+1 & 0 \end{bmatrix}.$$

Unique Seifert surface implies Seifert forms are unimodular congruent: this is not possible.

Idea of proof (3):

3 algebraically slice knots of 12 crossings or fewer: 6_1, 9_{46} and $11n_{139}$. Seifert forms:

$$\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}; \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}; \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}.$$

A theorem of Trotter says that S–equivalence implies unimodular congruence for matrices with prime determinant, so none of these matrices are S–equivalent.

On the other hand, there are many S–equivalent Seifert forms of the form

$$\begin{bmatrix} a & b \\ b+1 & 0 \end{bmatrix} \sim \begin{bmatrix} a \pm 1 & b \\ b+1 & 0 \end{bmatrix}.$$

In current work, with E.Kalfagianni, we are trying to use metabelian representations of knot groups, as often used for concordance problems, to solve the conjecture for all genus one knots with non–trivial Alexander polynomial.