Topics in topology || Topological manifolds

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Lectures will be streamed through zoom. Recordings will be made available afterwards. Bonn students: check eCampus for details; others please email us for access.

Zoom link: https://zoom.us/j/96847441282
password: torustrick

Time: 1015 – 1200, German time.
Dates: Wednesdays and Thursdays, from October 28th - February 11th.

We will also hold a weekly online office hour, time to be confirmed.

There is a discussion forum for the course on Piazza. Bonn participants: check eCampus for details; others please email us to be added.

Course website:
maths.dur.ac.uk/users/mark.a.powell/topological-manifolds

Pre-requisites: Basics of algebraic topology (fundamental group, homology groups, Poincaré-Lefschetz duality, fibrations).
Basics of differential topology (tangent and normal bundles, transversality, tubular neighbourhoods) will be useful for context but not strictly required.

Co-requisites: Participants are recommended (but not required) to follow the concurrent course of Professor Lück on Surgery Theory.

Plan for the course:
A manifold is a Hausdorff, paracompact topological space that is locally homeomorphic to $\mathbb{R}^n$. One frequently encounters manifolds with additional structure, such as smooth, piecewise-linear (PL), Riemannian, or symplectic. The aim of this course is to learn about unadulterated topological manifolds. For such manifolds, many essential tools from the start of differential topology are deep theorems. Examples include the existence and uniqueness of collar neighbourhoods for boundaries of manifolds, the existence of tubular neighbourhoods and transversality for submanifolds, and the well-definedness of the connected sum operation on closed, oriented manifolds. We will learn about these results, and what goes into their proofs, for topological manifolds.

Although we will primarily study topological manifolds, smooth and PL manifolds are never far away. We shall need facts from these theories, that we shall recall or prove as needed. Moreover, a natural and fascinating question arises as to whether a given topological manifold admits a smooth/PL structure, and if so how many up to equivalence. An interesting aspect is the strong dependence on the dimension of the manifold in question. The character of the theory is markedly different for each of the dimensions $1, 2, 3, 4$, and 5, whereas for dimensions ≥ 5, there is a more coherent theory with a similar flavour. We shall try to give a feeling for all dimensions, and the connections between them.

Topics from:

(1) Introduction to topological manifolds. Invariance of domain.
(2) Definition of PL and smooth structures. Combinatorial manifolds. Review of basic differential topology and PL topology.
(3) Collars of boundaries, existence and uniqueness.
(4) Decomposition spaces. and the Schoenflies problem.
(5) Local contractibility of homeomorphisms.
(6) A review of surgery and the s-cobordism theorem (3 categories), discuss unknotting in codimension 2.
(7) The torus trick, the stable homeomorphism theorem, and the annulus theorem.
(8) Connected sums, homeomorphisms of S^n.
(9) Microbundles, tangent and normal. The Kister-Mazur theorem.
(10) The product structure theorem, concordance implies isotopy.
(11) Transversality, handle structures, and the isotopy extension theorem.
(12) Smoothing theory.
(13) Normal bundles in codimension one and two.
(14) Homotopy groups of classifying spaces and the Hauptvermutung.
(15) Low dimensions.
(16) Engulfing. Stallings/Zeeman unknotting theorems.
(17) The manifold recognition problem. Disjoint discs, ANR homology manifolds and the Quinn invariant.
(18) The double suspension theorem.
(19) Non-triangulable manifolds, modulo Floer theory.
(20) $\mathbb{R}^n$ has a unique smooth structure except in dim 4.

References:

See the course website for more references and links. Books on topological manifolds include the following.

- Kirby-Siebenmann: Foundational essays on topological manifolds, smoothings, and triangulations.
- Daverman: Decompositions of manifolds.

- Rourke-Sanderson: Introduction to piecewise-linear topology.
- Rushing: Topological embeddings.
- Freedman-Quinn: Topology of 4-manifolds.
- Rudyak: Piecewise linear structures on topological manifolds.