TOPOLOGICAL MANIFOLDS

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Preface

These are notes from a lecture course on topological manifolds held at the University of Bonn in the winter semester 2020-21. The lecturers were Mark Powell and Arunima Ray. The notes were typed up in a collaborative effort of the lecturers and many participants of the course, as listed. Almost all of the pictures were drawn by Danica Kosanović. The lectures were live streamed, and recordings are available upon request. Please contact Mark or Aru for access.

1 Overview

We begin with an overview of the field of topological manifold theory in general and a preview of what we will discuss. First, we define topological manifolds.

Definition 1.1 (Topological manifold)**.** A topological space M is said to be an n-dimensional *topological manifold* if it is

(i) Hausdorff, i.e. any two points may be separated by open neighbourhoods;
(ii) locally Euclidean, i.e. for every $x \in M$ there is an open neighbourhood $U \ni x$ that is homeomorphic to either $\mathbb{R}^n$ or $\mathbb{R}^n_+ := \{\vec{y} \in \mathbb{R}^n \mid y_1 \geq 0\}$; and
(iii) paracompact, i.e. any open cover has a locally finite refinement.

Note that by allowing the possibility of $\mathbb{R}^n_+$ we are defining what some authors call a "manifold with boundary". With our definition we avoid having to specify that a boundary is permitted, however this means we must then stipulate when one is expressly forbidden.

You may also have seen other definitions of manifolds, e.g. requiring second countability or metrisability. We will see presently that some other definitions are equivalent to the one above, and in the exercises you will explore examples of spaces lacking one or other of the above properties.

The word "manifold" comes from German. Specifically, Riemann used the term *Mannigfaltigkeit* in his PhD thesis to describe a certain generalisation of surfaces. This was translated to "manifoldness" by Clifford. Prior to Riemann, mathematicians had classically studied geometry, first Euclidean, then spherical and hyperbolic. Surfaces were studied in depth, including by Riemann. As you probably know the first systematic account of the field of topology was in *Analysis situs* by Poincaré, and the first definition he wrote down was of what he called a manifold. In modern terms, he defined a *smooth* manifold. Here is a quick reminder of the definition (the modern one, not Poincaré's).

Definition 1.2 (Smooth manifold)**.** Let M^n be a topological manifold. A *chart* on M is a pair (U, φ) where $U \subseteq M$ is open and $\varphi\colon U \xrightarrow{\cong} \mathbb{R}^n$ is a homeomorphism. If (U, φ) and (V, ψ) are two charts on M such that $U \cap V \neq \emptyset$ then the map $\psi \circ \varphi^{-1}$ is said to be a *transition map* (this is a homeomorphisms, as a composite of homeomorphisms). If $\psi \circ \varphi^{-1}$ is further a diffeomorphism then the charts (U, φ) and (V, ψ) are said to be *smoothly compatible.*

A *smooth atlas* for M is a collection of smoothly compatible charts for M whose domains cover M. A *smooth structure* on M is a maximal smooth atlas, where *maximal* means that any chart smoothly compatible with the atlas is already contained in the atlas.

A couple of remarks are in order. First, Poincaré's original definition of a (smooth) manifold had been as a subset of Euclidean space satisfying a given collection of smooth functions. The Whitney embedding theorem from the 1930s showed that every smooth n-manifold (satisfying the definition above) embeds in $\mathbb{R}^{2n+1}$, and so the two notions coincide.

Second, the definition above indicates a recipe for imposing more structure on topological manifolds. By requiring the transition maps to be smooth, we obtain smooth manifolds. Similarly, by imposing further (or fewer) conditions, e.g. symplectic, complex, C^1, etc., we may produce more categories of manifolds. In this course, we will focus on unadulterated topological manifolds, with occasional cameos by smooth manifolds and *piecewise-linear* manifolds. We define the latter next.

As you probably noticed in your algebraic topology courses, it is often convenient to work with simplicial complexes rather than purely abstract spaces, e.g. when computing homology groups. This was especially true in the early days of topology.

Definition 1.3. A manifold is said to be *triangulated* if it is homeomorphic to the geometric realisation of a (locally finite) simplicial complex.

A *piecewise-linear* manifold, often called a PL manifold is a manifold with a particularly nice triangulation.

Definition 1.4 (PL manifold (preliminary))**.** An n-manifold is *piecewise linear* (PL) if it has a triangulation such that the link of every vertex is a PL $(n-1)$-sphere or PL $(n-1)$-ball.

Rest assured, we will carefully define what a PL sphere is later in the course. An intuitive way to think about the definition is that it is a strengthening of the "locally Euclidean" condition in the definition of a manifold, specifically that not only does each point have a neighbourhood homeomorphic to Euclidean space, but that such neighbourhoods may further be taken to be PL equivalent to Euclidean space. An alternative definition of PL manifolds requires that the transition maps be piecewise-linear maps on Euclidean space (also to be defined carefully in the future). In other words, a PL manifold is a topological manifold with a maximal PL atlas. A result of Dedecker [Ded62] shows that the two definitions coincide.

By definition, both smooth and PL manifolds are topological manifolds, by forgetting the extra structure. By results of Cairns (1934) and Whitehead (1940) every smooth manifold is PL. Since the very inception of manifold theory, e.g. in *Analysis situs*, there has been much interest in the relationship between these three categories. Here are some other fundamental questions.

(1) Is a given CW complex homotopy equivalent to a TOP manifold? PL? DIFF?
(2) Given two manifolds, are they homotopy equivalent? Are they homeomorphic? If they are PL or smooth, are they PL homeomorphic or diffeomorphic respectively?
(3) When do manifolds embed in one another?
(4) For a given topological manifold M, what is the space of self-homeomorphisms $\mathrm{Homeo}(M)$? Given a pair of manifolds M and N, what is the space of embeddings $\mathrm{Emb}(M,N)$?

These are huge, very general questions. Too general, to expect to be able to have answers of a manageable level of complexity. We will make some initial steps on the long quest to answering interesting special cases of them in this course.

To guide our investigations, it might help to focus on some more specific questions, that we shall aim to discuss in the course, and which represent some highlights of the theory.

- The (generalised) **Poincaré conjecture**: if a closed n-dimensional manifold M is homotopy equivalent to the n-sphere S^n, is M homeomorphic to S^n?
  - Yes, classical for $n \leq 2$, Perelman for $n = 3$ (2003), Freedman for $n = 4$ (1982), Smale, Stallings, Newman for $n \geq 4$ (1960s). True in PL category for $n \neq 5$, for $n = 4$ the PL question is equivalent to the DIFF question. In DIFF it has been reduced to problems in homotopy theory for $n > 4$, while it is wide open for $n = 4$.
- The **Schoenflies problem**: is every embedding of S^{n-1} in S^n equivalent to the standard (equatorial) embedding?
  - Solved in the topological category by M. Brown (1960), assuming bicollared, false otherwise. It is true in smooth category for $n \geq 5$, open for $n = 4$.
- Can topological manifolds manifolds be **triangulated**?
  - Not always, for $n = 4$ Casson (1980s), for $n \geq 5$ Manolescu (2013).
- **Double suspension problem**: Let M be a (homology) manifold with $H_*(M;\mathbb{Z}) \cong H_*(S^n;\mathbb{Z})$. Is the double suspension $\Sigma^2 M$ a TOP manifold? If yes, then $\Sigma^2 M \cong S^{n+2}$.
  - Yes, Cannon, Edwards.

We will substantially address some of the above questions in this course. Some basic tools in smooth and PL topology include:

- tangent bundles;
- tubular neighbourhoods;
- handle decompositions; and
- transversality.

In stark contrast, these are difficult theorems in the topological category; we will see how to prove them.

Along with the tools listed above, there are also certain standard "tool theorems", such as:

- the h- and s-cobordism theorems and
- the surgery exact sequence.

Indeed, one of the key consequences of transversality and the existence of handle decompositions in the topological category is making the s-cobordism theorem and surgery available.

An inane comment is that working purely in the topological category makes some things easier and some things harder. More specifically, major theorems like the Poincaré conjecture and the Schoenflies theorem are now known in the topological category, since it is comparatively easier to detect a topological ball or sphere, compared to a smooth one. The other side of the coin is that basic tools such as transversality and handlebody decompositions are harder to achieve in the topological category, since we do not impose so much structure to get these via the usual methods. Consequently, "standard" facts like the well-definedness of the connected sum operation become highly nontrivial to prove.

Along with the drastic contrast between categories, there is also a sharp distinction in the behaviour of low- and high-dimensional manifolds. A slogan here is that dimension 4 is a sort of phase transition. This is exemplified by the following facts:

- the topological manifold $\mathbb{R}^n$ has a unique smooth structure if $n \neq 4$ and uncountably many smooth structures if $n = 4$; and
- a topological manifold M admits a topological handlebody decomposition precisely if M is not a non-smoothable 4-manifold.

1.1 Conventions

We will use the following notation for equivalence relations:

- $\simeq$ for homotopy equivalences;
- $\cong$ for homeomorphism;
- $\cong_{C^\infty}$ for diffeomorphisms.
- $\cong_{\mathrm{PL}}$ for PL homeomorphisms.

2 Definitions of topological manifolds

Let us discuss the definition of a topological manifold in more detail. We will present a series of alternative definitions. First we recall some terms.

Definition 2.1. A topological space X is *Hausdorff* if for every $x, y \in X$, with $x \neq y$, there exist disjoint open sets $U \ni x$ and $V \ni y$.

Definition 2.2. A subset $U \subseteq X$ of a topological space X is a *neighbourhood* of $x \in U$ if there is an open set $V \subseteq U$ with $x \in V$ and $\overline{V} \subseteq U$.

Definition 2.3. A collection of subsets $\{V_\alpha\}$ of X is *locally finite* if for every $x \in X$ there is a neighbourhood $U \ni x$ with $U \cap V_\alpha \neq \emptyset$ for finitely many α.

Definition 2.4. A topological space X is *paracompact* if every open cover $\{U_\alpha\}$ of X has a locally finite refinement. Here a refinement is another cover $\{V_\beta\}$ such that for each β, $V_\beta \subseteq U_\alpha$ for some α.

Now we recall the definition of a topological manifold from above.

Definition 2.5 (Topological manifold)**.** A topological space M is an n-dimensional *topological manifold* (often from now on, a *manifold*) if it is

(i) Hausdorff;
(ii) locally n-Euclidean; and
(iii) paracompact

Here a space M is said to be *locally n-Euclidean* if for every $x \in M$ there is an open neighbourhood $U \ni x$ that is homeomorphic to either $\mathbb{R}^n$ or

$$\mathbb{R}^n_+ := \{\vec{y} \in \mathbb{R}^n \mid y_1 \geq 0\}.$$

We refer to such a U as a *coordinate neighbourhood.*

The *interior* $\operatorname{Int} M$ of the manifold M is the union of all the points that have an open neighbourhood homeomorphic to $\mathbb{R}^n$. The *boundary* of M is defined as the complement of the interior

$$\partial M := M \setminus \operatorname{Int} M.$$

A manifold is *closed* if it is compact and $\partial M = \emptyset$. A manifold is *open* if it is noncompact and $\partial M = \emptyset$.

Example 2.6. The line $\mathbb{R}$ is a topological manifold. It is straightforward to see that $\mathbb{R}$ is locally Euclidean and Hausdorff. It is second countable because open intervals with rational centre and rational length form a countable basis for the topology. We will show below that connected, second countable, locally Euclidean, Hausdorff spaces are paracompact.

Example 2.7. The line with two origins, namely the quotient space of $\mathbb{R} \sqcup \mathbb{R}$ where x in the first $\mathbb{R}$ is identified with x in the second $\mathbb{R}$ for all $x \neq 0$, is locally Euclidean and paracompact, but is not Hausdorff.

Example 2.8. Let Ω be the first uncountable ordinal. Take one copy of $[0,1)$ for each ordinal less than Ω. The *long line* is formed from stacking these half open intervals: define an order on the union of all $[0,1)_\omega$, $\omega < \Omega$, as follows. If $x, y \in [0,1)_\omega$ then define $x \leq y$ if and only if $x \leq y$. If $x \in [0,1)_\omega$ and $y \in [0,1)_{\omega'}$, with $\omega \neq \omega'$ then define $x < y$ if and only if $\omega < \omega'$.

Theorem 2.9. *Let M be a Hausdorff, locally n-Euclidean topological space. Then the following are equivalent.*

(i) M is paracompact;
(ii) Every component M_α of M admits an exhaustion by compact sets. That is there is a countable sequence $\{C_i\}_{i=1}^\infty$ of compact sets C_i with $C_i \subseteq \operatorname{Int} C_{i+1}$ and $\bigcup_{i=1}^\infty C_i = M_\alpha$.

(iii) Every component of M is second countable.
(iv) M is metrisable.

Proof. We will only give the argument for the case of empty boundary.

((iv))⇒((i)) Here we quote a result that every metric space is paracompact [Mun00, Theorem 41.4].

((iii)) ⇒ ((iv)). We start by showing that every Hausdorff and locally Euclidean space M is regular.

To do this, first we claim that for every $x \in M$ and open $U \ni x$, there exists $W \ni x$ open with $x \in W \subseteq \overline{W} \subseteq U$. To prove the claim, let V be an open set containing x from the locally Euclidean hypothesis, and let $\varphi\colon V \to \mathbb{R}^n$ be a homeomorphism. Then $\varphi(U \cap V) \subseteq \mathbb{R}^n$ is open. It follows that there exists $\varepsilon > 0$ such that

$$Z := \overline{B_{\varepsilon/2}(\varphi(x))} \subseteq B_\varepsilon(\varphi(x)) \subseteq \varphi(U \cap V),$$

where $B_\delta(y)$ is the ball of radius δ and centre y. Now Z is closed and bounded and therefore is compact in $\mathbb{R}^n$ by the Heine-Borel theorem. It follows that $\varphi^{-1}(Z)$ is compact, and then since M is Hausdorff, $\varphi^{-1}(Z)$ is closed. Now take W to be the point-set interior of Z, $\mathring{Z}$. Then $x \in W \subseteq \overline{W} \subseteq U \cap V \subseteq U$, as desired.

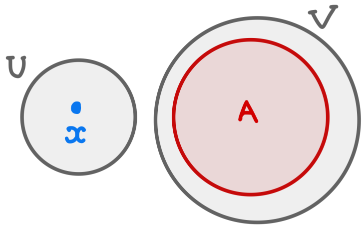


FIGURE 1. A regular topological space is the one in which for every point x and a closed set A there exist open sets U and V separating them.

Using the claim, we show that M is *regular*, meaning that for any closed set C and point x not in C, there exist open sets V, W with $x \in W$, $C \subseteq V$, and $V \cap W = \emptyset$. So fix C and x as above, and let $U := M \setminus C$, which is open. Then by the previous claim there exists and open set W with $x \in W \subseteq \overline{W} \subseteq U$. Define $V := M \setminus \overline{W}$, which contains C. Indeed $V \cap W = \emptyset$, so M is regular as asserted.

Now, the Urysohn metrisation theorem says that every Hausdorff, regular, second countable space is metrisable. This gives a metric on each connected component of M. Make each component diameter at most 1 by replacing the metric d with d', where $d'(x,y) := \min\{d(x,y), 1\}$. Then set the distance between any two points in distinct connected components to be 2. This gives a metric on all of M, which completes the proof that ((iii)) ⇒ ((iv)).

((ii)) ⇒ ((iii)) Cover each C_i by finitely many coordinate neighbourhoods. It follows that each component of M has a countable cover by coordinate neighbourhoods. Each of these is open and second countable, so the entire component of M is also second countable.

((i))⇒((ii)) Let C denote a component of M. Since C is locally Euclidean, there exists an open cover $\{U_\alpha\}$ where each $\overline{U}_\alpha$ is compact. Let $\{V_\beta\}$ be a locally finite refinement. Then $\overline{V}_\beta$ is a closed subset of a compact set so is compact.

We claim that each V_β intersects finitely many other sets $V_{\beta'}$, since V_β is compact. To see this, suppose it is false and choose $x_\alpha \in V_\alpha \cap V_\beta$ for infinitely many α. Since the V_β came from coordinate neighbourhoods, they are also sequentially compact (since $\mathbb{R}^n$ is a metric space). Therefore the set $\{x_\alpha\}$ has a limit point y in $\overline{V_\beta}$. Any neighbourhood of y intersects infinitely many of the x_α, and therefore intersects infinitely many of the subsets V_α. This contradicts local finiteness, so completes the proof of the claim that V_β intersects finitely many other $V_{\beta'}$.

Now define Γ to be a graph with a vertex for each set V_β and an edge whenever $V_\beta \cap V_{\beta'} \neq \emptyset$. The graph Γ is connected since C is, and it is locally finite, meaning that each vertex is connected to finitely many edges.

We claim that a locally finite connected graph Γ is countable i.e. has countably many vertices. To see this, fix a vertex γ and let $\Gamma_n \subseteq \Gamma$ be the full subgraph consisting of all the vertices that can be reached from γ by a path intersecting at most n edges. Local finiteness implies that Γ_n is finite. Since Γ is locally connected it is path connected, and since a path intersects finitely many edges by compactness, every vertex is contained in Γ_n for some n. Therefore $\Gamma = \bigcup_{i=0}^{\infty} \Gamma_n$ is countable as claimed.

We deduce that $\{V_\beta\}$ is countable, so equals $\{V_1, V_2, \dots\}$ after relabelling. Define $C_1 := \overline{V}_1$. Note that C_1 is contained in a union of finitely many V_i. Call them $V_{i_1}, \dots, V_{i_k}$. Then define

$$C_2 := \overline{V}_2 \cup \bigcup_{j=1}^{k} \overline{V}_{i_j}.$$

Iterate this idea to define C_3, C_4, and so on. This completes the proof of ((i))⇒((ii)). □

Sol. on p.139. **Exercise 2.1**. (PS1.1) Give an example of a locally n-Euclidean, paracompact space that is not Hausdorff.

Sol. on p.139. **Exercise 2.2**. (PS1.2) Give an example of a locally n-Euclidean, Hausdorff space that is not paracompact.

See 2.2.

Sol. on p.139. **Exercise 2.3**. (PS1.3) Every compact topological manifold embeds in $\mathbb{R}^N$ for some N.

3 Invariance of domain

We study the invariance of domain theorem, following Casson's notes [Cas71]. This theorem has a simple and innocuous looking statement, but it is foundational to the theory of manifolds. The word domain is an old-fashioned word for an open set in $\mathbb{R}^n$, used frequently in complex analysis. The next theorem was one of the early triumphs of homology theory.

Theorem 3.1 (Brouwer, 1910)**.** *Let $U \subseteq \mathbb{R}^n$ be open and let $f\colon U \to \mathbb{R}^n$ be continuous and injective. Then $f(U) \subseteq \mathbb{R}^n$ is open and $f\colon U \to f(U)$ is a homeomorphism, i.e. f is an embedding.*

Definition 3.2. A map $f\colon X \to Y$ is called an *embedding* if f is injective and is a homeomorphism onto its image.

Note that in the smooth category, an embedding is also required to be an immersion, meaning that at each point the derivative is an injective linear map on tangent spaces. By the inverse function theorem this implies that an immersion is a local diffeomorphism. The condition for an embedding to be an immersion is equivalent to an embedding being a diffeomorphism onto its image.

Corollary 3.3. *Let $V \subseteq \mathbb{R}^n$ such that $V \cong U$ with $U \subseteq \mathbb{R}^n$ open. Then V is open in $\mathbb{R}^n$.*

Proof. Let $f\colon U \to V$ be the homeomorphism given in the statement. Apply invariance of domain to deduce that $f(U) = V$ is open. $\square$

An important consequence of invariance of domain is that the notion of dimension is well-defined for manifolds. Note that a topological manifold is locally path-connected, so it is connected if and only if it is path connected.

Proposition 3.4. *There is a well-defined dimension for nonempty connected topological manifolds. That is, a nonempty Hausdorff, paracompact topological space that is locally n-Euclidean cannot be locally m-Euclidean for $m \neq n$.*

As a consequence, we will drop the n prefix from n-Euclidean from now on.

Proof. We will only argue for the case of empty boundary. Let M be an n-dimensional manifold and let A and B neighbourhoods of a point p in M together with homeomorphisms $\varphi\colon A \xrightarrow{\cong} \mathbb{R}^n$ and $\psi\colon B \xrightarrow{\cong} \mathbb{R}^m$. Suppose without loss of generality that $m < n$. Then we have a homeomorphism

$$\psi \circ \varphi^{-1}\colon U := \varphi(A \cap B) \to V := \psi(A \cap B) \subseteq \mathbb{R}^m \subseteq \mathbb{R}^n.$$

Here we include $\mathbb{R}^m \subseteq \mathbb{R}^n$ using the standard inclusion. Then $U \subseteq \mathbb{R}^n$ is open and $U \cong V$. So by Corollary 3.3 we see that V is open in $\mathbb{R}^n$. But any open ball around a point in V is not contained in $\mathbb{R}^m$, so is certainly not contained in V. It follows that V cannot be open in $\mathbb{R}^n$. This contradiction implies that the initial set up cannot exist, which proves the proposition. $\square$

Corollary 3.5. *Let M be an n-manifold. Then $\partial M = M \setminus \operatorname{Int} M$ is an $(n-1)$-manifold without boundary.*

Proof. Let $x \in M$ and let $f\colon \mathbb{R}^n_+ \to M$ be a map that is a homeomorphism onto its image, which is an open neighbourhood of x.

Claim. *We have that $x \in \partial M$ if and only if $x \in f(\mathbb{R}^{n-1})$, where we consider $\mathbb{R}^{n-1} \subseteq \mathbb{R}^n$ via $\vec{x} \mapsto (0, \vec{x})$.*

Note that the claim in particular says that the boundary can potentially be nonempty. It could have been, a priori, that every point with an $\mathbb{R}^n_+$ neighbourhood also secretly lives in the interior by virtue of a different $\mathbb{R}^n$ neighbourhood. This is not the case.

Let us prove the claim. We will prove the contrapositive of each inclusion. So suppose that $x \notin f(\mathbb{R}^{n-1})$. Then $x \in f(\mathbb{R}^n_+ \setminus \mathbb{R}^{n-1}) \cong \mathbb{R}^n$, so $x \in \operatorname{Int} M$. Therefore $x \notin \partial M$.

Now suppose that $x \notin \partial M$. Then $x \in \operatorname{Int} M$. So there exists $U \ni x$ open in M with $U \cong \mathbb{R}^n$. Therefore there is a neighbourhood V of x with $V \subseteq U$ and $V \subseteq f(\mathbb{R}^{n-1}) \subseteq M$, with V homeomorphic to an open subset of $\mathbb{R}^n$. Therefore

$$f^{-1}(V) \subseteq \mathbb{R}^n_+ \subseteq \mathbb{R}^n.$$

By invariance of domain, $f^{-1}(V)$ is open in $\mathbb{R}^n$.

Now suppose for a contradiction that $x \in f(\mathbb{R}^{n-1})$ then $f^{-1}(x) \in \mathbb{R}^{n-1}$. But $f^{-1}(V)$ cannot simultaneously be open in $\mathbb{R}^n$ and be an open neighbourhood of $f^{-1}(x) \in \mathbb{R}^{n-1}$. Therefore $x \notin f(\mathbb{R}^{n-1})$. This completes the proof of the claim that $x \in \partial M$ if and only if $x \in f(\mathbb{R}^{n-1})$.

Now we prove the corollary. Let $y \in \partial M$. Let $g\colon \mathbb{R}^n_+ \xrightarrow{\cong} M$ be a coordinate neighbourhood. Then $g(\mathbb{R}^n_+) \cap \partial M = g(\mathbb{R}^n_+) \cap g(\mathbb{R}^{n-1}) = g(\mathbb{R}^{n-1})$ is an open set in ∂M homeomorphic to $\mathbb{R}^{n-1}$, so ∂M is locally $(n-1)$-Euclidean, as required. Note that ∂M is certainly Hausdorff and paracompact. □

Corollary 3.6. *Let M^m, N^n be manifolds. Then $M \times N$ is an $(m+n)$-manifold with*

$$\partial(M \times N) = M \times \partial N \cup_{\partial M \times \partial N} \partial M \times N$$

Proof. Each point in $M \times N$ has an open neighbourhood homeomorphic to one of $\mathbb{R}^m \times \mathbb{R}^m$, $\mathbb{R}^m \times \mathbb{R}^n_+$, $\mathbb{R}^m_+ \times \mathbb{R}^n_+$, or $\mathbb{R}^m \times \mathbb{R}^n_+$. Apart from the first one, the other three are all homeomorphic to $\mathbb{R}^{m+n}_+$. As we showed in the proof of the previous corollary, the boundary $\partial(M \times N)$ is precisely the points which have one of the neighbourhoods of the latter three types. □

The boundary of a smooth product has corners, but we do not have to worry about corner points in the topological category. A helpful example to consider is that the disc and the square are homeomorphic. What is the smooth structure on a square? Is it equivalent to the smooth structure on a disc?

Having explained some important consequences of invariance of domain, now we begin to prove it. We will need the following two manifolds:

$$S^n = \{\vec{x} \in \mathbb{R}^{n+1} \mid \|\vec{x}\| = 1\}$$
$$D^n = \{\vec{x} \in \mathbb{R}^{n+1} \mid \|\vec{x}\| \leq 1\}.$$

Lemma 3.7. *Let $X \subseteq S^n$ be a subset of the n-sphere which is homeomorphic to a disc, $X \cong D^k$. Then for all degrees $r \in \mathbb{N}_0$, the reduced homology groups of the complement vanish, $\widetilde{H}_r(S^n \setminus X) = 0$.*

Proof. The proof is by induction on k. For $k = 0$, $S^n \setminus \{\mathrm{pt}\} \cong \mathbb{R}^n$ so is contractible.

Now assume that the lemma holds for k. Choose a homeomorphism

$$f\colon D^k \times I \cong D^{k+1} \cong X$$

Let $t \in I = [0,1]$. Note that for every $t \in I$ we have

$$\widetilde{H}_*(S^n \setminus f(D^k \times \{t\})) = 0$$

by the inductive hypothesis. Let $[\alpha] \in \widetilde{H}_r(S^n \setminus X)$ be a class in reduced homology for some $r \geq 0$; we want to show that α is the trivial class. We can write $\alpha = \partial c_t$ for some chain c_t in $C_{r+1}(S^n \setminus D^k \times \{t\})$. Since c_t is a sum of finitely many singular simplices, its image is compact. Therefore there exists an open interval J_t of t in I such that c_t lies in $S^n \setminus f(D^k \times J_t)$. Since I is compact, we can find a finite partition

$$0 = t_0 < t_1 < t_2 < \cdots < t_\ell = 1$$

such that $[t_i, t_{i+1}] \subseteq J_\tau$ for some $\tau \in J_\tau$. For $0 \le p \le q \le \ell$, we consider the inclusion induced homomorphisms

$$\phi_{p,q} \colon \widetilde{H}_r(S^n \setminus X) \to \widetilde{H}_r(S^n \setminus f(D^k \times [t_p, t_q])).$$

We know that $\phi_{p-1,p}(\alpha) = 0$ for every p because α bounds c_τ in $C_{r+1}(S^n \setminus f(D^k \times [t_{p-1}, t_p]))$ for some $\tau \in I$.

We want to show that $\phi_{0,\ell}(\alpha) = 0$. Then since $\phi_{0,\ell} = \mathrm{Id} \colon \widetilde{H}_r(S^n \setminus X) \to \widetilde{H}_r(S^n \setminus X)$, it will follow that $\alpha = 0$ as desired. We show by induction that $\phi_{0,i}(\alpha) = 0$. For $i = 1$, this holds as the case $p = 1$ of $\phi_{p-1,p}(\alpha) = 0$.

The sets $S^n \setminus f(D^k \times [t_p, t_{p+1}])$ are open. We apply Mayer-Vietoris for

$$S^n \setminus f(D^k \times \{t_i\}) = S^n \setminus f(D^k \times [0, t_i]) \cup_{S^n \setminus f(D^k \times [0, t_{i+1}])} S^n \setminus f(D^k \times [t_i, t_{i+1}]).$$

Since we know that $\widetilde{H}_s(S^n \setminus f(D^k \times \{t_i\})) = 0$, we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{H}_r(S^n \setminus f(D^k \times [0, t_{i+1}])) \arrow[r, "\cong"] & \widetilde{H}_r(S^n \setminus f(D^k \times [0, t_i])) \oplus \widetilde{H}_r(S^n \setminus f(D^k \times [t_i, t_{i+1}])) \arrow[r] & 0 \\
& \widetilde{H}_r(S^n \setminus X) \arrow[u, "\phi_{0,i+1}"] \arrow[ur, "\phi_{0,i} \oplus \phi_{i,i+1}"'] & &
\end{tikzcd}$$

The diagonal map sends α to 0, so we deduce that $\phi_{0,i+1}(\alpha) = 0$. Then by induction $\phi_{0,\ell}(\alpha) = 0$, so $\alpha = 0$ as desired. $\square$

Lemma 3.8. *If $X \subseteq S^n$ is homeomorphic to S^k, then*

$$\widetilde{H}_r(S^n \setminus X) \cong \widetilde{H}_r(S^{n-k-1}) \cong \begin{cases} \mathbb{Z} & r = n - k - 1 \\ 0 & \text{else.} \end{cases}$$

Proof. The proof is by induction on k. For $k = 0$, for any two points p, q in S^n, we have that $S^n \setminus \{p, q\}$ is homeomorphic to $\mathbb{R}^n \setminus \{0\}$, which is homotopy equivalent to S^{n-1}. Now assume the lemma holds for $k - 1$. Let $f \colon S^k \xrightarrow{\cong} X$ be a homeomorphism. Let D_+ and D_- be hemispheres of S^k, with $S^k = D_+ \cup D_-$ and $D_+ \cap D_- \cong S^{k-1}$. Write $X_\pm := f(D_\pm$ and $X_e = X_+ \cap X_-$. Then note that $S^n \setminus X = (S^n \setminus X_+) \cap (S^n \setminus X_-)$. Furthermore $S^n \setminus X_+ \cup S^n \setminus X_- = S^n \setminus X_e$. In addition $S^n \setminus X_\pm$ and $S^n \setminus X_e$ are open. Now Lemma 3.7 yields $\widetilde{H}_r(S^n \setminus X_\pm)$ so that the Mayer-Vietoris sequence yields:

$$0 \to \widetilde{H}_{r+1}(S^n \setminus X_e) \xrightarrow{\cong} \widetilde{H}_r(S^n \setminus X) \to 0.$$

By the inductive hypothesis

$$\widetilde{H}_{r+1}(S^n \setminus X_e) \cong \begin{cases} \mathbb{Z} & r + 1 = n - (k-1) - 1 \\ 0 & \text{else} \end{cases} \cong \begin{cases} \mathbb{Z} & r = n - k - 1 \\ 0 & \text{else.} \end{cases}$$

$\square$

Corollary 3.9 (Jordan-Brouwer separation)**.** *Let $f \colon S^{n-1} \to S^n$ be an injective, continuous map. Then f is an embedding and $S^n \setminus f(S^{n-1})$ has two connected components, both of which are open in S^n.*

We will use the following closed map lemma, also sometimes known as the compact-Hausdorff lemma.

Lemma 3.10. *Let $f \colon X \to Y$ be a continuous injective map from a compact space X to a Hausdorff space Y. Then f is a homeomorphism onto its image and a closed map.*

Proof. Let U be a closed set in X. Then U is compact since X is compact. Therefore $f(U)$ is compact. So $f(U)$ is closed because Y is Hausdorff. It follows that $f^{-1} \colon f(X) \to X$ is continuous, so that $f \colon X \to f(X)$ is a homeomorphism. $\square$

Proof of Corollary 3.9. By the closed map lemma $f\colon S^{n-1} \to S^n$ is an embedding and has closed image, so in particular $S^n \setminus f(S^{n-1})$ is open. Since $S^n \setminus f(S^{n-1})$ is locally path-connected, and since S^n is a manifold, the number of components equals the number of path components. By Lemma 3.8, $\widetilde{H}_0(S^n \setminus f(S^{n-1})) \cong \mathbb{Z}$, which shows that there are two path components. Components are always closed, and since there are finitely many components, both are open as well. They are open in $S^n \setminus f(S^{n-1})$, and therefore they are also open in S^n since $S^n \setminus f(S^{n-1}$ is open. □

Corollary 3.11. *Let $f\colon D^n \to S^n$ be injective and continuous. Then $f(\operatorname{Int} D^n)$ is open in S^n.*

Proof. By Lemma 3.7, $\widetilde{H}_0(S^n \setminus f(D^n)) = 0$, so $S^n \setminus f(D^n)$ is connected. Now

$$S^n \setminus f(S^{n-1}) = f(\operatorname{Int} D^n) \cup S^n \setminus f(D^n).$$

The left hand space is not connected by Corollary 3.9: it has exactly two connected components. The two spaces on the right hand side are connected.

We deduce that $f(\operatorname{Int} D^n)$ is precisely one of the two open components in $S^n \setminus f(S^{n-1})$, so is open by Corollary 3.9. □

Now we have finally assembled the ingredients necessary to prove invariance of domain.

Proof of Invariance of Domain Theorem 3.1. Postcompose $f\colon U \to \mathbb{R}^n$ with the inclusion into S^n. For a point $x \in U$, there exists a small closed metric ball B still contained in U. The map $f|_B\colon B \to S^n$ fulfills the conditions of Corollary 3.11 so that $f(\operatorname{Int} B)$ is open in S^n, hence an open neighbourhood of $f(x)$. We have shown that every point in the image of f has a neighbourhood inside the image of f, hence f has open image. Furthermore, since interiors of closed balls constitute a basis for the topology of U, this argument also shows that f is an open map. □

4 More foundational properties of topological manifolds

Here are some further properties of topological manifolds, whose proofs are omitted, for now.

Theorem 4.1. *Every m-dimensional topological manifold has covering dimension m.*

That is, every open cover has an order m refinement, so there are at most $m+1$ sets in the refinement in any nonempty intersection. That is, $\bigcap_{i=1}^{k} V_{\beta_i} \neq \emptyset$ implies that $k \leq m+1$.

Theorem 4.2. *Every topological manifold admits a partition of unity.*

That is, there exist functions $\{\phi_\alpha \colon X \to I\}$ such that (i) $\{\phi_\alpha^{-1}((0,1])\}$ is locally finite, and (ii) $\sum_\alpha \phi_\alpha(x) = 1$ for all $x \in X$.

Theorem 4.3. *Every component of an m-dimensional topological manifold embeds in $\mathbb{R}^N$ for some N. In fact $N = (m+1)(m+2)$ suffices.*

Theorem 4.4. *Every topological manifold is homotopy equivalent to a cell complex.*

Theorem 4.5. *Every topological manifold is an ANR (Absolute Neighbourhood Retract) and an ENR (Euclidean Neighbourhood Retract).*

Sol. on p.140. **Exercise 4.1.** (PS6.1) Every connected topological manifold with empty boundary is *homogeneous.* That is, for any two points $a, b \in M$, there exists a homeomorphism $h \colon M \to M$ with $h(a) = b$.

Hint: show that for any two points a, b in $\operatorname{Int} D^n$, there is a homeomorphism of D^n mapping a to b and fixed on the boundary. Next show that the orbit of any given point in M under the action of $\operatorname{Homeo}(M)$ is both open and closed in M.

5 Wild embeddings

One of our goals is to give an answer to the following problem.

Question 5.1 (Schoenflies problem)**.** Is every embedding $f\colon S^{n-1} \hookrightarrow S^n$ *equivalent* to the equator $S^{n-1} \subseteq S^n$? That is, is there a homeomorphism of pairs $H\colon (S^n, f(S^{n-1})) \to (S^n, S^{n-1})$?

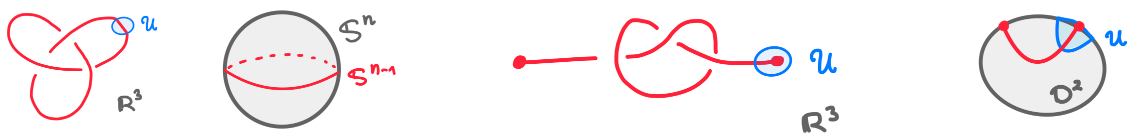
In order to get a feeling for this problem, we study some wild embeddings. We will see that some fascinating pathologies can occur.

Recall from Definition 3.2 that an embedding is a continuous injective map which is a homeomorphism onto its image. Also recall that we denote $\mathbb{R}^n_+ := \mathbb{R}^1_+ \times \mathbb{R}^{n-1}$. For $m \leq n$ let $\mathbb{R}^m_+ \subseteq \mathbb{R}^n_+$ be the product of $\mathbb{R}^1_+$ with the inclusion $\mathbb{R}^{m-1} \subseteq \mathbb{R}^{n-1}$.

Definition 5.2. Let $e\colon M^m \hookrightarrow N^n$ be an embedding. We say that e is *locally flat at* $x \in M$ (or at $e(x) \in N$) if there exists a neighbourhood U of $e(x)$ in N and a homeomorphism:

$$\begin{cases} h\colon U \to \mathbb{R}^n \text{ such that } h(U \cap e(M)) = \mathbb{R}^m \subseteq \mathbb{R}^n, & \text{if } x \in \operatorname{Int} M,\ e(x) \in \operatorname{Int} N, \\ h\colon U \to \mathbb{R}^n \text{ such that } h(U \cap e(M)) = \mathbb{R}^m_+ \subseteq \mathbb{R}^n, & \text{if } x \in \partial M,\ e(x) \in \operatorname{Int} N, \\ h\colon U \to \mathbb{R}^n_+ \text{ such that } h(U \cap e(M)) = \mathbb{R}^m_+ \subseteq \mathbb{R}^n_+, & \text{if } x \in \partial M,\ e(x) \in \partial N. \end{cases}$$

We say that e is *locally flat* if it is locally flat at each point; it is *wild at* $x \in M$ if it is not locally flat at x. We say e is *proper* if for each $x \in \operatorname{Int} M$ the first condition holds and for each $x \in \partial M$ the last condition holds.



(A) (B) (C)

Figure 2. Examples of locally flat embeddings (in red).

Remark 5.3. With our definition, the boundary of a manifold M is not locally flat in M. More on boundaries in the next section. We will see that the boundary is collared. There is an opposing school of thought that holds that the definition of locally flat ought to be such that a boundary is locally flat, but this is an inconvenient choice for a number of reasons. For example, we will want to understand when locally flat embeddings have normal bundles, or at least well-behaved regular neighbourhoods.

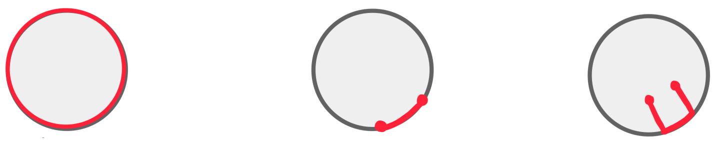


Figure 3. Examples of non locally flat embeddings (in red) in D^2.

Note that local flatness is preserved under homeomorphism of pairs. We will identify some nice properties of locally flat embeddings, giving us a tool to detect those which are wild.

Definition 5.4. Let $A \subseteq X$ be a closed subset of a topological space.

(1) We say that A is *k-locally co-connected at $a \in A$*, written k-LCC at a, if for every neighbourhood U of a there exists an open neighbourhood V with $a \in V \subseteq U$ such that any $S^k \to V \setminus A$ extends as

$$\begin{tikzcd} S^k \arrow[r] \arrow[d, hook] & V \setminus A \arrow[d, hook] \\ D^{k+1} \arrow[r] & U \setminus A \end{tikzcd}$$

In other words, $\pi_k(V \setminus A) \to \pi_k(U \setminus A)$ is trivial for every choice of basepoints for which this makes sense.

(2) We say that A has a *1-abelian local group at $a \in A$*, written 1-alg, if for every neighbourhood U of a there exists an open neighbourhood V with $a \in V \subseteq U$ such that the inclusion induced homomorphism $\pi_1(V \setminus A) \to \pi_1(U \setminus A)$ has abelian image.

(3) We say that A is *locally homotopically unknotted in X at $a \in A$* if A is both 1-alg and k-LCC at a for every $k \neq 1$.

Remark 5.5. The notion of 1-alg above has some equivalent formulations. We may instead ask that each loop which is null-homologous in $V \setminus A$ is null-homotopic in $U \setminus A$. Alternatively, we may require that the image of $\pi_1(V \setminus A)$ in $\pi_1(U \setminus A)$ is isomorphic to $\mathbb{Z}$. The interested reader should check that these are indeed equivalent.

Remark 5.6. The use of co-connected should not be confused with the use of this word, in other contexts, to describe vanishing of relative homotopy groups in a range.

Example 5.7. Suppose $M^m \subseteq N^n$ is locally flat. If U is as in the first case of Definition 5.2 we have $(U, M \cap U) \cong (\mathbb{R}^n, \mathbb{R}^m)$, so $U \setminus M \cap U \cong \mathbb{R}^n \setminus \mathbb{R}^m \cong \mathbb{R}^m \times (\mathbb{R}^{n-m} \setminus \{0\}) \simeq S^{n-m-1}$. Therefore,

- If $n - m = 1$, then $\operatorname{Int} M$ is k-LCC for all $k \geq 1$ except $k = 0$.
- If $n - m = 2$, then $\operatorname{Int} M$ is locally homotopically unknotted in N at every point.
- If $n - m > 2$, then $\operatorname{Int} M$ is k-LCC for all $k \leq n - m - 2$.

If U is as in the second case of Definition 5.2, we have

$$U \setminus M \cap U \cong \mathbb{R}^n \setminus \mathbb{R}^m_+ \cong (\mathbb{R}^{n-m+1} \setminus \mathbb{R}^1_+) \times \mathbb{R}^{m-1},$$

which is contractible, so ∂M is k-LCC in $\operatorname{Int} M$ for all k in this case. If U is as in the third case of Definition 5.2, we have

$$\begin{aligned} U \setminus M \cap U \cong (\mathbb{R}^1_+ \times \mathbb{R}^{n-1}) \setminus (\mathbb{R}^1_+ \times \mathbb{R}^{m-1}) &\cong \mathbb{R}^1_+ \times (\mathbb{R}^{n-1} \setminus \mathbb{R}^{m-1}) \\ &\cong \mathbb{R}^1_+ \times \mathbb{R}^{m-1} \times (\mathbb{R}^{n-m} \setminus \{0\}) \simeq S^{n-m-1}, \end{aligned}$$

so ∂M is k-LCC in ∂N for all $k \leq n - m - 2$ in this case.

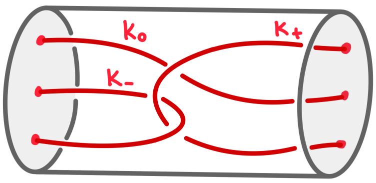
Remark 5.8. The converse in the second case is also true: if $e \colon M \hookrightarrow N$ is an embedding, $n - m = 2$, and $\operatorname{Int} M$ is locally homotopically unknotted in N at every point, then e is locally flat. This is due to Chapman for dimension ≥ 5 [Cha79] and Quinn for dimension 4 [Qui82] (see also [FQ90]).

There are converses in the other codimensions as well, such as in [Č73]. Indeed, these may be applied to certain generalisations of manifolds. See [FQ90, Sec. 9.3] and [DV09, Chap. 7, Chap. 8] for further details.

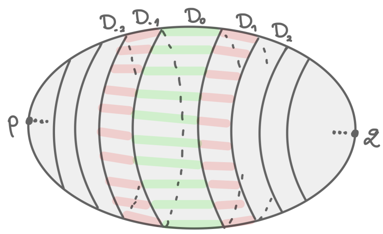
Remark 5.9. In the topological literature "flat" sometimes means equivalent to the standard embedding, i.e. the only 'flat' knot in S^3 is the unknot. In low-dimensional topology 'flat' usually means 'has a trivial normal bundle', so any smooth knot in S^3 is flat. In the topological terminology, the Schoenflies problem is asking whether any codimension one embedding of a sphere is 'flat'. We will try to avoid this controversy by just specifying what we mean.

Question 5.10. Are all embeddings locally flat?

The answer is no. Let us give an example, due to Artin and Fox [FA48]. We will embed the building block C from Fig. 4a into each of the balls D_n for $n \in \mathbb{Z}$, which are the slices of D^3 depicted in Fig. 4b.



(A) Our building block is the ball $C := D^2 \times [0,1]$ containing properly embedded arcs $K = K_0 \cup K_- \cup K_+$.



(B) The slices of D^3.

FIGURE 4. Construction of Fox-Artin examples.

The (double) Fox-Artin arc is the image of all arcs K, together with the limiting points:

$$\alpha := \{p\} \cup \bigcup_{n=-\infty}^{n=\infty} f_n(K) \cup \{q\}$$

FIGURE 5. Fox-Artin arc α

Proposition 5.11. *The fundamental group $\pi_1(\mathbb{R}^3 \setminus \alpha)$ is non-trivial. Thus, $(\mathbb{R}^3, \alpha)$ is not equivalent to $(\mathbb{R}^3, [0,1])$.*

Proof. Let us consider the nested sequence subspaces of $\mathbb{R}^3$ given by

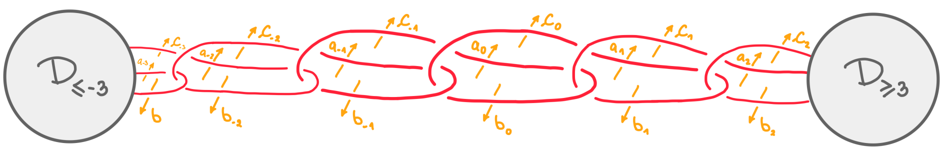
$$X_m := \mathbb{R}^3 \setminus \left(\overline{\bigcup_{|n| \geq m} D_n} \cup \alpha \right)$$

In other words, as m increases we are "carving out" more and more material from $\mathbb{R}^3$. Thus, we want to compute the fundamental group of $X := \mathbb{R}^3 \setminus \alpha = \bigcup_{m \geq 1} X_m$.

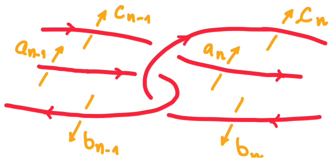
The hypothesis of Seifert-van Kampen theorem are satisfied (check!), so

$$\pi_1 X \cong \varinjlim \pi_1(X_m),$$

the direct limit of the sequence of homomorphisms $\pi_1 X_m \to \pi_1 X_{m+1}$ induced by inclusions. To compute $\pi_1 X_m$ we see X_m as the complement of (here with $m = 3$):



where we use the standard method to compute the fundamental group of a complement of a graph (cf. Wirtinger presentation of the knot group), using the convention



Thus, we obtain

$$\pi_1 X_m = \left\langle \{a_n, b_n, c_n\}_{-m \le n < m-1} \;:\; \begin{cases} \left.\begin{array}{l} a_n c_n = c_n c_{n-1}, \\ b_{n-1} c_n = c_n a_{n-1}, \\ b_{n-1} b_n = c_n b_{n-1} \end{array}\right\} & -m \le n < m-1, \\ b_{-m}^{-1} a_{-m} c_{-m} = 1, \\ b_{m-1}^{-1} a_{m-1} c_{m-1} = 1 \end{cases} \right\rangle$$

Note that under inclusion map $a_n, b_n, c_n \in \pi_1 X_m$ each map to $a_i, b_i, c_i \in \pi_1 X_{m+1}$.

Therefore, by the definition of direct limit we have

$$\pi_1 X = \left\langle \{a_n, b_n, c_n\}_{n \in \mathbb{Z}} \;:\; \left.\begin{array}{l} a_n c_n = c_n c_{n-1}, \\ b_{n-1} c_n = c_n a_{n-1}, \\ b_{n-1} b_n = c_n b_{n-1}, \\ b_n^{-1} a_n c_n = 1 \end{array}\right\} \text{ for all } n \right\rangle$$

Now eliminating $a_n = c_n c_{n-1} c_n^{-1}$ by the first relation, and $b_n = a_n c_n = c_n c_{n-1} c_n^{-1} c_n = c_n c_{n-1}$ by the last, the two remaining relations both reduce to one:

$$\pi_1 X = \left\langle \{c_n\}_{n \in \mathbb{Z}} \;:\; c_{n-1} c_{n-2} c_n c_{n-1} = c_n c_{n-1} c_{n-2} \text{ for all } n \right\rangle$$

We claim that this is a nontrivial group. Indeed, there is a homomorphism $\pi_1 X \to S_5$ to the symmetric group on five letters, given by $c_n \mapsto \begin{cases} (12345), & n \text{ odd} \\ (14235), & n \text{ even} \end{cases}$ (check relation satisfied!). $\square$

Slightly modifying this example gives another wild arc but for which the argument using the fundamental group will not work. Namely, we use the same building block from Fig. 4a but now put it into the slices only of one half of the ball, see Fig. 6.

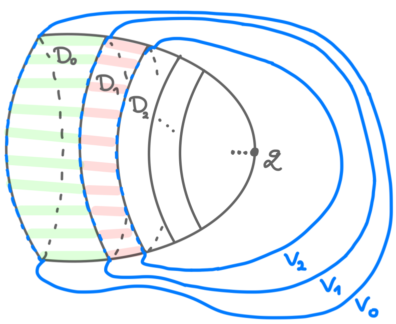


FIGURE 6. The model half-ball for the arc β together with the open sets V_n (in blue) from the proof of Proposition 5.12.

The resulting Fox-Artin arc $\beta := \bigcup_{n \ge 0} f_n(K) \cup \{q\}$ is shown in Fig. 7. This has a simply connected complement $\pi_1(\mathbb{R}^3 \setminus \beta) \cong 1$. Indeed, the computation is similar as in the previous proof but now the loop c_n is trivial. Actually, $\mathbb{R}^3 \setminus \beta \cong \mathbb{R}^3 \setminus \{pt\}$ (see [FA48]). However, we show that β is nevertheless wild.

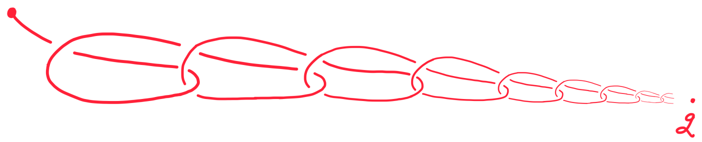


FIGURE 7. Fox-Artin arc β.

Proposition 5.12. *β is not 1-LCC at q. Therefore, β is a wild embedding.*

Proof. Let V_n be open sets as in Fig. 6. If β was 1-LCC at q, then there would exist $N \geq 0$ such that $\pi_1(V_N \setminus \beta) \to \pi_1(V_0 \setminus \beta)$ is trivial (since by definition of 1-LCC can find $V \subseteq V_0$, but then can find $V_N \subseteq V$ for some N).

Now $\pi_1(V_N \setminus \beta)$ is generated by $c_N, c_{N-1}, \dots$ subject to $c_{n-1}c_{n-2} = c_n c_{n-1}$ for $n \geq N+1$ and $c_n c_{n-1} c_{n-2} = 1$, and each c_n maps to $c_n \in \pi_1(V_0 \setminus \beta)$ under the homomorphism induced by the inclusion. However, each c_n is nontrivial in $\pi_1(V_0 \setminus \beta)$, which we can see using the same homomorphism to S_5 as in previous proof. □

Sol. on p.140. **Exercise 5.1**. (PS2.1) Prove that the arc γ in Figure 8 is locally flat, and indeed there is a homeomorphism f of pairs mapping $(\mathbb{R}^3, \gamma)$ to $(\mathbb{R}^3, [0,1])$.

Hint: Find a nested sequence of balls $\{B_i\}$ so that $\cap B_i$ is the compactification point and each B_i intersects γ at a single point. For each i there is an isotopy that is the identity on $(S^3 \setminus \operatorname{Int} B_i) \cup B_{i+1}$ and that straightens out $\gamma \cap (B_i \setminus \operatorname{Int} B_{i+1})$. The desired homeomorphism f is a limit of a composition of such homeomorphisms.

Sol. on p.141. **Exercise 5.2**. (PS2.2) The arc δ is the union of γ and a standard interval $[0,1]$ (see Figure 8). Prove that δ is not locally flat. The arc δ is an example of a 'mildly wild' arc, i.e. it is a union of two locally flat arcs.

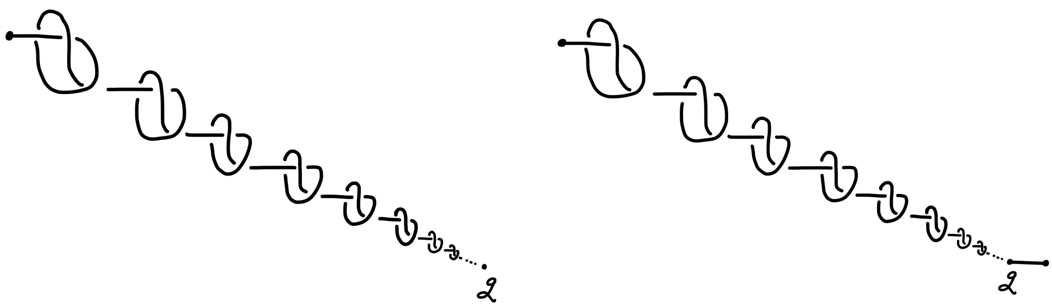


FIGURE 8. Arcs γ and δ respectively.

Hint: use the Seifert-van Kampen theorem to prove that δ is not 1-alg at the "union point".

The examples so far have concerned arcs in $\mathbb{R}^3$, or equivalently in S^3 if we pass to the 1-point compactification. Of course our original question had been in terms of embedded spheres. Such examples can also be generated from our arcs. For example, by taking two parallel copies of

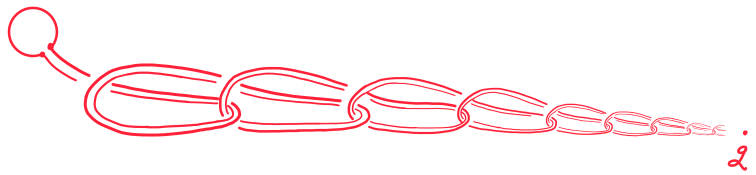


Figure 9. Fox-Artin 1-sphere.

each strand in the building block for the Fox-Artin arc, we can produce (double and single) Fox-Artin 1-spheres, see Fig. 9.

Alternatively, by replacing each strand in the building block by a tube, we produce (double and single) Fox-Artin 2-spheres, see Fig. 10. Similar proofs as above show that these are not locally flat. A more well-known example of a non-locally flat S^2 in S^3 is the *Alexander horned sphere* (see. e.g. [Hat02, Ex. 2B.2]). In this section we have only just begun scratching the surface of the world of wild embeddings. In the rest of this course we will focus on locally flat embeddings. For more on wild embeddings, see [FA48] and [DV09].

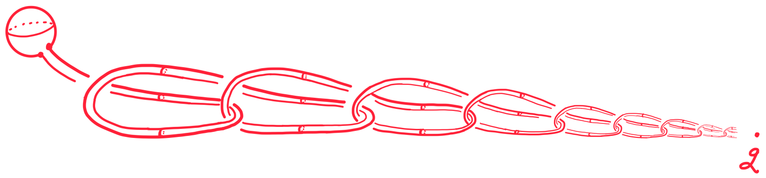


Figure 10. Fox-Artin 2-sphere.

In terms of our original Question 5.1, whether every embedding $S^{n-1} \hookrightarrow S^n$ is equivalent to the standard one (the equator), we see now that we must restrict to locally flat embeddings. We will see later what the exact conditions will be, see Section 7.

Example 5.13 (Alexander horned sphere and Alexander gored ball)**.** Alexander horned sphere, depicted in Figure 11.

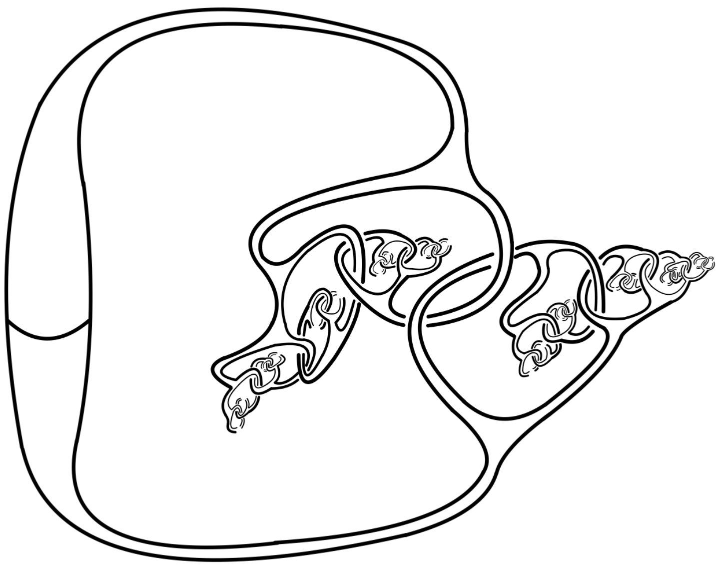


Figure 11. The Alexander horned sphere

The Alexander gored ball is the complement of the Alexander horned ball. This space has nontrivial perfect fundamental group. It is therefore not homeomorphic to a ball. It shows that the Schoenflies theorem does not hold without the locally flat hypothesis. See also Remark 6.7.

6 Collars and bicollars

The goal of this section is to show that the boundary of every manifold admits a collar.

Definition 6.1. A manifold M is said to have a *collared boundary* if there exists a closed embedding $C\colon \partial M \times [0,1] \hookrightarrow M$ such that $(x,0) \mapsto x$.

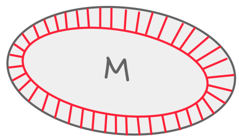


FIGURE 12. The red region indicates the collar.

Definition 6.2. A *submanifold* of X is a subset that is the image of a locally flat embedding.

Definition 6.3. A submanifold Y of X is said to be *two-sided* if there exists a connected neighbourhood N of X that is separated by Y. i.e. $N \setminus Y$ has two components (see Fig. 13a.

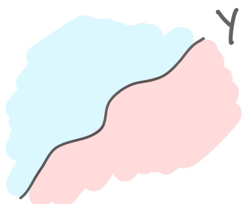
Definition 6.4. A submanifold Y of X is said to be *bicollared* if $f\colon Y \hookrightarrow X$ can be extended to an embedding $f\colon Y \times [-1,1] \hookrightarrow X$ with $(y,0) \mapsto f(y)$.

Now that we know what a collar is, we look at a result by Morton Brown [Bro62] which shows that boundaries of manifolds admit collars. We will present the proof of this result by Robert Connelly [Con71], which is simpler than Brown's proof.

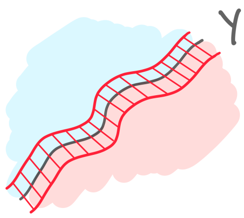
Theorem 6.5 (The Collaring Theorem)**.** *Every manifold has a collared boundary.*

If $\partial M = \emptyset$, then this theorem is vacuous but still true.

Corollary 6.6. *Let Y be a locally flat, two-sided, without boundary, codimension one connected submanifold of X^m. Then Y is bicollared.*



(A) Submanifold with two sides, blue and red.



(B) Bicollaring a two-sided submanifold.

Proof. Consider a connected neighbourhood N of X cut along Y. Let $N \setminus Y = L \cup R$. Now because Y is locally flat, $L \cup Y$ and $R \cup Y$ are manifolds with boundary and thus have collars.

$$Y \times [0,1] \hookrightarrow R \cup Y$$
$$Y \times [-1,0] \hookrightarrow L \cup Y$$

Hence, we can glue these collars to get a topological bicollar. □

Note that the argument given does not work in the smooth category: more work would be required to glue together two smooth collars and obtain a smooth bicollar.

Now that we have seen the collaring theorem and one of its corollaries, it is time to prove the collaring theorem. First we look at an outline of the proof in the smooth category.

Outline of proof in the smooth case.

– Consider an inward pointing nonvanishing vector field on ∂M, and extend it to a vector field on M that is nonvanishing on a neighbourhood of ∂M.
– Integrate the vector field to obtain a flow. By considering a suitably small time period, the flow is defined.
– Propagating the boundary along the flow gives rise to a collar. □

Now we prove the collaring theorem. We will consider the compact case only. The idea of the proof extends to the non-compact case, but we will not give the details here to avoid a too-lengthy side discussion of open covers.

Proof of Theorem 6.5 in the compact case. Let M be an n-dimensional compact manifold. We outline the proof.

– Add an exterior collar $\partial M \times [-1,0]$ to M to obtain
$$M^+ := M \cup \partial M \times [-1,0]$$
by gluing along the boundary, i.e. $x \in \partial M$ $(x,0) \in \partial M \times \{0\}$.
– Construct a homeomorphism $G\colon M \to M^+$, by an induction over charts covering ∂M, gradually stretching more of a neighbourhood of ∂M in M over the exterior collar.
– The inverse image $G^{-1}(\partial M \times [-1,0])$ gives us the desired collar.

Since ∂M is compact, there is a finite collection $U_1, \dots, U_m \subseteq \partial M$ forming an open cover of ∂M by coordinate neighbourhoods, such that for each $i = 1, \dots, m$ we can find local collars for the closures of the U_i, $\overline{U}_i$. Let us call these local embeddings
$$h_i\colon \overline{U_i} \times [0,1] \hookrightarrow M.$$
We may suppose in addition that they satisfy
– $h_i^{-1}(\partial M) = \overline{U_i} \times \{0\}$;
– $h_i(x,0) = x$;
– $h_i(\overline{U}_i \times [0,1))$ is open in M.

Let $\{V_i\}_{i=1}^n$ be another cover with
$$V_i \subseteq \overline{V}_i \subseteq U_i.$$
To find such a collection of U_i, V_i and h_i, take an arbitrary collection of pairs (U_i, V_i) with the V_i covering ∂M, and with the U_i subsets of coordinate neighbourhoods so that they give local collars on $\overline{U}_i$. Then apply compactness to find a finite subcollection.

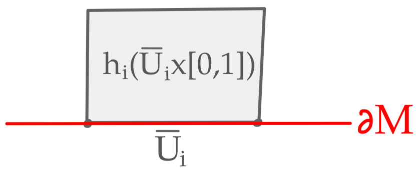


FIGURE 14. A local collar.

We will use the embeddings H_i defined as follows:
$$H_i\colon \overline{U_i} \times [-1,1] \to M^+$$
$$(x,t) \mapsto \begin{cases} h_i(x,t) & t \geq 0 \\ (x,t) & t < 0. \end{cases}$$
This is well-defined and continuous since $h_i(x,0) = x$. We will build a homeomorphism $M \to M^+$ mapping ∂M to $\partial M \times \{-1\}$. Our goal is to inductively define maps $f_i\colon \partial M \to [-1,0]$ and embeddings $g_i\colon M \to M^+$ for $i = 0, 1 \dots, m$ satisfying:

(1) $f_i(x) = -1$ for all $x \in \bigcup_{j \leq i} \overline{V}_j$

(2) $g_i(x) = (x, f_i(x))$ for all $x \in \partial M$
(3) $g_i(M) = M \cup \{(x,t) \mid t \geq f_i(x)\}$.

Once this is completed, since $\bigcup_i V_j = \partial M$, we will have that $f_m(x) = -1$ for all $x \in \partial M$. Therefore $g_m(M) = M^+$, so $G := g_m$ will be our desired homeomorphism, and $g_m^{-1}(\partial M \times [-1,0]) \subseteq M$ will be a collar. Here note that g_m^{-1} being a homeomorphism implies it is a closed map, so $g_m^{-1}(\partial M \times [-1,0])$ will be closed. Also $g_m^{-1}(x,-1) = x$ by (2) for all $x \in \partial M$.

In the case $i = 0$ define $f_0 \equiv 0$ and define $g_0 \colon M \to M^+$ to be the inclusion map. Now suppose for the inductive step that f_{i-1} and g_{i-1} have been defined. We will construct

$$\phi_i \colon H_i^{-1} g_{i-1}(M) \to \overline{U}_i \times [-1,1],$$

embeddings that "push V_i down," taking $H_i^{-1} g_{i-1}(M) \subseteq \overline{U}_i \times [-1,1]$ and reimbedding it in $\overline{U}_i \times [-1,1]$ in such a way that V_i is also pushed down into the exterior collar $\partial M \times [-1,0]$. We will require that:

$$\phi_i H_i^{-1} g_{i-1}(\overline{V}_i) = \overline{V}_i \times \{-1\}$$
$$\phi_i|_{\overline{U}_i \setminus U_i \times [-1,1] \cup \overline{U}_i \times \{1\}} = \mathrm{Id}\,.$$

Find a Urysohn function $\lambda_i \colon \overline{U}_i \to [0,1]$ such that λ_i is 0 on $\overline{U}_i \setminus U_i$ and is 1 on $\overline{V}_i$. Since ∂M is paracompact and Hausdorff it is normal [Mun00, Theorem 41.1], so the Urysohn lemma applies to find such a continuous function.

Write

$$b(x) := (1 - \lambda_i(x)) f_{i-1}(x) - \lambda_i(x).$$

For each x let $S_x \colon [f_{i-1}(x), 1] \to [b(x), 1]$ be the linear map sending $f_{i-1}(x) \mapsto b(x)$ and $1 \mapsto 1$. Define $\phi_i \colon H_i^{-1} g_{i-1}(M) \to \overline{U}_i \times [-1,1]$ to be the map sending $(x,t) \mapsto (x, S_x(t))$. Then using

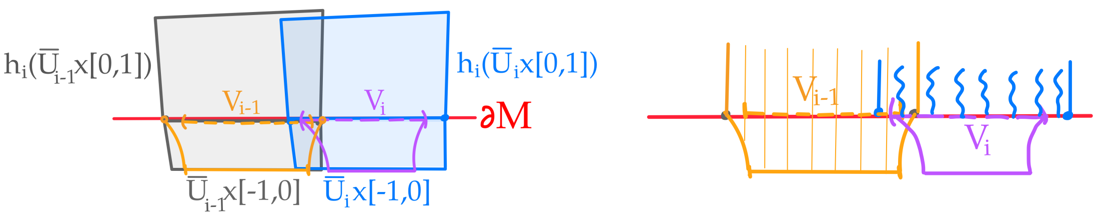


FIGURE 15. The local collars and pushing down into the exterior collar.

ϕ_i we can define the map

$$\Phi_i(x) \colon g_{i-1}(M) \to M^+$$
$$x \mapsto \begin{cases} H_i \phi_i H_i^{-1}(x) & x \in H_i(\overline{U}_i \times [-1,1]) \\ x & \text{else.} \end{cases}$$

The function H_i^{-1} pulls back into the local collar union the local exterior collar $\overline{U}_i \times [-1,1]$, then ϕ_i stretches the local collar in $\overline{V}_i$ over all of $\overline{V}_i \times [-1,0]$, before H_i pushes everything forward into M^+ again. This conjugation method will be used again in the proof of the Schoenflies theorem, and is a powerful way to define global functions that have a desired effect or can be easily defined only in local coordinates.

Then the map

$$g_i := \Phi_i \circ g_{i-1} \colon M \to M^+$$

is the required map for the inductive step. One must check that conditions (1) and (3) are satisfied by the above construction. Use (2) to define f_i from g_i. This completes the induction step. Hence $g_m^{-1}(\partial M \times [-1,0])$ gives us the required collar. $\square$

In fact we have a relative version of collaring: if one already has a collar on an open subset of ∂M, then the given collar can be extended to a collar on all of ∂M, restricting to the given collar on a specified closed subset of that open set. Collars are also essentially unique, due to Armstrong [Arm70], in the following sense. Given two collars $C_1, C_2 \colon \partial M \times [0,2] \to M$, there is an ambient isotopy taking $C_1|_{[0,1]}$ to $C_2|_{[0,1]}$. We will not prove this here.

Remark 6.7. Uniqueness would not hold if we asked for an isotopy between the entire collars. To see this, one needs to know that the Alexander gored ball AGB (the closure of the complement of the Alexander horned sphere embedded in S^3) is (a) not homeomorphic to D^3, and (b) becomes homeomorphic to D^3 after adding an exterior collar $S^2 \times [0,1]$ to its boundary the Alexander horned sphere (which is homeomorphic to S^2). So there is a homeomorphism $f \colon AGB \cup S^2 \times [0,1] \to D^3$. If collars were unique without passing first to a subcollar, then there would be an isotopy from $f(S^2 \times [0,1])$ to the standard collar, which would imply that the complement $D^3 \setminus f(S^2 \times (0,1])$ is again homeomorphic to D^3. But this complement is also homeomorphic to the AGB, so we obtain a contradiction.

Sol. on p.141. **Exercise 6.1**. (PS2.3) Let M be an n-dimensional manifold with nonempty boundary. Let U be an open subset of ∂M that is collared, that is there exists an embedding $U \times [0,1] \hookrightarrow M$ with $(u,0) \mapsto u$ for all $u \in U$. Let $C \subseteq U$ be a closed subset. Then there exists a collaring of ∂M extending the given collaring on C.

7 The Schoenflies theorem

7.1 Overview of proof strategy

The goal is to prove that every locally flat embedding $i \colon S^{n-1} \hookrightarrow S^n$ bounds a ball on both sides. By Corollary 6.6, we may assume that the embedding is bicollared.

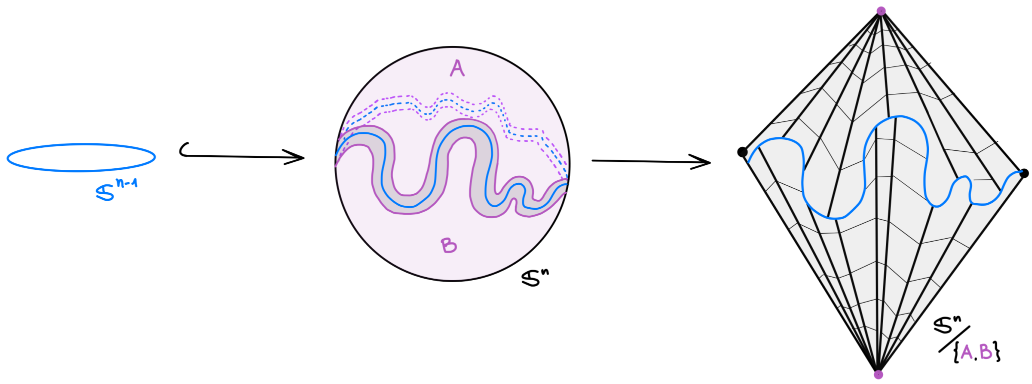


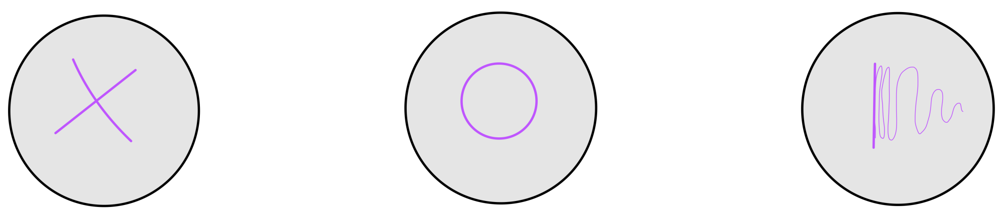
FIGURE 16. Idea of the proof of Schoenflies theorem: collapse the two components of the complement of a bicollar. Between the second and third pictures, the collar has been stretched out to cover most of the sphere, and the complementary regions have been shrunk to the two poles.

The key point is that the result of crushing each boundary component of an annulus $S^{n-1} \times [-1,1]$ to a (distinct) point is the sphere S^n, in other words, the sphere S^n is identified with the unreduced suspension of the sphere S^{n-1}. By the Jordan-Brouwer separation theorem (Corollary 3.9), we know that $S^n \setminus (i(S^{n-1}) \times [-1,1])$ has two components, call them A and B. (In fact, since we have a bicollared embedding, we can prove this much faster using the Mayer-Vietoris sequence.) Our goal will be to crush each of A and B to a point. The result is then seen to be the sphere S^n. This shows that there is a homeomorphism ${}^{(i(S^{n-1}) \times [0,1])}\!/\!_{A} \to D^n$ where the latter is a hemisphere of S^n. This does not seem like progress unless we know something about the quotient space ${}^{(i(S^{n-1}) \times [0,1])}\!/\!_{A}$. In fact, we will show that there is a homeomorphism

$$A \cup (i(S^{n-1}) \times [0,1]) \xrightarrow{\cong} {}^{(i(S^{n-1}) \times [0,1])}\!/\!_{A} \cong D^n.$$

This leads us to the following abstraction.

Question 7.1. Given $X \subseteq M^n$, when is ${}^{M}\!/\!_{X} \cong M$?



(A) ${}^{D^2}\!/\!_{X} \cong D^2$ (B) ${}^{D^2}\!/\!_{O} \cong D^2 \vee S^2$ (C) ${}^{D^2}\!/\!_{S} \cong D^2$

FIGURE 17. Some quotients of D^2.

Consider the three examples in Fig. 17. The first two are not hard to see, but how would you prove the last one, that for the topologist's (closed) sine curve $X = S$ we have ${}^{D^2}\!/\!_{S} \cong D^2$? We explore answers to these questions in the following two subsections.

7.2 Whitehead manifold

In this short interlude, we describe a famous example of a subset $X \subseteq S^3$ where ${S^3}/{X}$ is not homeomorphic to S^3.

Let $V_0 \subseteq \mathbb{R}^3$ be the unknotted solid torus $S^1 \times D^2$. Let V_1 be the embedded solid torus in V_0 shown in Fig. 18. In other words, we have a homeomorphism $h\colon V_0 \to S^1 \times D^2 = V_1$.

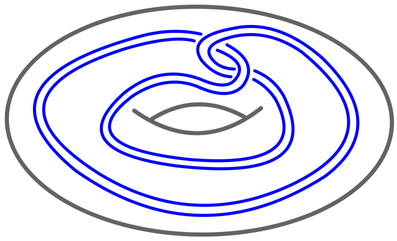


FIGURE 18. The building block for the Whitehead manifold.

Recursively define solid tori $V_i := h(V_{i-1})$. The infinite intersection $X := \bigcap V_i$ is called the *Whitehead continuum*, and its complement $\mathscr{W} := S^3 \setminus X$ is called the *Whitehead manifold* .

Sol. on p.141. **Exercise 7.1**. (Non-HW) The Whitehead manifold is contractible. *Hint:* Use Theorem 4.4.

Sol. on p.141. **Exercise 7.2**. (Non-HW) The quotient ${S^3}/{X}$ is not a manifold. *Hint:* show that the quotient is not 1-LCC at the image of X. Prior knowledge of 3-manifold topology and knots and links will be useful.

Definition 7.2. A noncompact space M is *simply connected at infinity* if for every compact set $C_1 \subseteq W$ there exists a compact set $C_2 \supseteq C_1$ so that $\pi_1(W \setminus C_2) \to \pi_1(W \setminus C_1)$ is trivial.

The solution of the previous exercise in fact shows that the Whitehead manifold $\mathscr{W}$ is not simply connected at infinity. Since $\mathbb{R}^3$ is simply connected at infinity, this shows that the Whitehead manifold is not homeomorphic to $\mathbb{R}^3$.

The Whitehead manifold is historically significant. Whitehead wanted to prove the Poincaré conjecture by showing that any punctured homotopy 3-sphere is homeomorphic to $\mathbb{R}^3$ (this suffices by passing to 1-point compactifications). In this vein, he conjectured that any contractible, open 3-manifold is homeomorphic to $\mathbb{R}^3$, but soon found the Whitehead manifold as a counterexample.

Remark 7.3. While the quotient ${S^3}/{X}$ is not a manifold, it is known that ${S^3}/{X} \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^4$! This can be proven using the techniques of the next section.

Remark 7.4. We have seen that simple connectivity at infinity is an obstruction to being homeomorphic to Euclidean space. It turns out that it is the only siginificant one. In other words, an open, contractible n-manifold is homeomorphic to $\mathbb{R}^n$ if and only if it is simply connected at infinity [Edw63, Wal65, Fre82, Sta62]. In dimension three this requires the Poincaré conjecture [Per02, Per03b, Per03a] (see also [MT07, KL08]).

7.3 Shrinking cellular sets

We are working towards the proof of the Schoenflies theorem by Brown [Bro60]. This material can also be found in [DV09, Dav07, Bin83].

Definition 7.5. Let M^n be a manifold. A subset $X \subseteq \operatorname{Int} M$ is *cellular* if there exist embedded, closed subsets $B_i \subseteq M$, $i \geq 1$, such that

- $B_i \cong D^n$ for all i;
- $B_{i+1} \subseteq \mathring{B}_i$ for all i; and
- $X = \bigcap_{i=1}^{\infty} B_i$.

Equivalently, X is closed and for every open $U \supseteq X$ there exists $B \cong D^n$ and $X \subseteq \mathring{B} \subseteq B \subseteq U$.

The name cellular is because D^nin an n-cell. It does not mean the space is a CW complexes.

Example 7.6. The first and last example from Fig. 17 are cellular, as can be seen in Fig. 19.

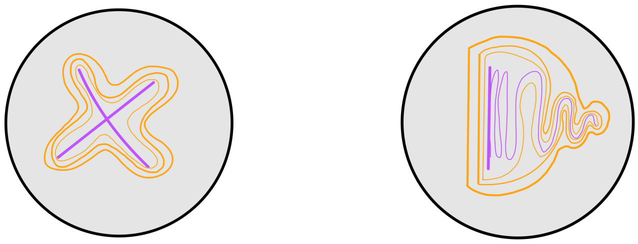


FIGURE 19. Examples of cellular sets, with $\{B_i\}$ in orange.

Proof of equivalence of definitions. Suppose that the first definition holds. Then $X = \bigcap_{i=1}^{\infty} B_i$, which is closed since each B_i is closed. Thus, $X \subseteq B_1$ is a closed subset of a compact set so is compact.

We claim that given $U \supseteq X$ open, there is a natural number n such that $B_n \subseteq U$. Suppose this is false. Then there exists a sequence $\{x_i\}$ with $x_i \in B_i \cap (M \setminus U) \neq \emptyset$ for each i. Since $\{x_i\} \subseteq B_1$, which is sequentially compact, after passing to a convergent subsequence, we have a limit point x. Then $x \in \bigcap_{i=1}^{\infty} B_i = X \subseteq U$, since each B_i is sequentially compact. But $M \setminus U$ is closed, so contains all its limit points, so $x \in M \setminus U$, which is a contradiction.

Suppose now the second definition holds. Since $\operatorname{Int} M$ is open, by hypothesis there exists $B_1 \cong D^n$ such that $X \subseteq \mathring{B}_1 \subseteq B_1 \subseteq \operatorname{Int} M$. Since X is closed and B_1 is compact, we have that X is compact. Let us now recursively define B_i so that $X \subseteq \mathring{B}_i \subseteq B_i \subseteq \mathring{B}_{i-1}$. Fix some metric d on M (recall that manifolds are metrisable, see Theorem 2.9)

Firstly, for $i \in \mathbb{Z}$ define the open set

$$U_i := \Big\{ y \in M \mid d(X, y) < \frac{1}{i} \Big\}.$$

Since $\mathring{B}_{i-1} \cap U_i$ is open, by hypothesis we can pick $B_i \subseteq \mathring{B}_{i-1} \cap U_i$ such that $B_i \cong D^n$ and $X \subseteq \mathring{B}_i$. By construction $X = \bigcap B_i$, since the elements in the intersection must have zero distance from X. This completes the proof. $\square$

The following proposition is why we are interested in cellular sets.

Proposition 7.7. *Let M be a manifold. If $X \subseteq \operatorname{Int} M$ is cellular, then $M \cong {}^{M}\!/_{X}$.*

Remark 7.8. In fact, the above proposition can be strengthened to say that the quotient map $\pi\colon M \to {}^{M}\!/_{X}$ can be "approximated by homeomorphisms", that is we say that X *shrinks*. In order to understand how to approximate functions we need to have a metric on the collection of functions, which is what we recall in the next remark.

Remark 7.9 (Function spaces.). Let X and Y be compact metric spaces. The *uniform metric* is defined as

$$d(f, g) := \sup_{x \in X} d_Y(f(x), g(x)),$$

for continuous maps $f, g\colon X \to Y$. We write $\mathcal{C}(X,Y)$ for the metric space of continuous functions from X to Y equipped with this metric. By [Mun00, Thm. 43.6 and 45.1], $\mathcal{C}(X,Y)$ is complete. For $A \subseteq X$, let

$$\mathcal{C}_A(X,X) := \{f \in \mathcal{C}(X,X) \mid f|_A = \mathrm{Id}_A\}.$$

Note that $\mathcal{C}_A(X,X) \subseteq \mathcal{C}(X,X)$ is closed, so $\mathcal{C}_A(X,X)$ is complete with respect to the induced metric.

Remark 7.10. Let M be a compact manifold and $X \subseteq M$. For X closed, the quotient space M/X is metrisable. Then the quotient map $\pi\colon M \to M/X$ is said to be *approximable by homeomorphisms*, that is the set X is said to *shrink*, if there is a sequence $\{h_i\colon M \xrightarrow{\cong} M/X\}$ of homeomorphisms converging to π.

Remark 7.11. While we will not prove it in this course, a closed subset $X \subseteq \operatorname{Int} M$ of a manifold shrinks if and only if X is cellular (see e.g. [Dav07]). Observe that asking for the quotient map to be approximable by homeomorphisms is stronger than merely requiring the quotient space to be homeomorphic to the original manifold M. A natural question then is whether whenever we have $M/X \cong M$ the set X must be cellular. We will see presently that $\mathbb{R}^n/X \cong \mathbb{R}^n$ implies that X is cellular. However this is not true in general.

We will use the following notion in the proof.

Definition 7.12. For a continuous map $f\colon X \to Y$ and $y \in Y$ we say $f^{-1}(y)$ is an *inverse set* if $|f^{-1}(y)| > 1$.

Proof of Proposition 7.7. We restrict to the case M is compact. The aim is to describe a surjective continuous map $f\colon M \to M$ with $f|_{\partial M} = \mathrm{Id}$, whose only inverse set is X. Then we will obtain a well-defined continuous map $\overline{f}\colon M/X \to M$ completing the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\pi"] & M \\ M/X \arrow[ur, "\exists \overline{f}"'] & \end{tikzcd}$$

since f is constant on the fibres of π. Since $\overline{f}$ is bijective, and is a closed map by the closed map lemma (Lemma 3.10) it will follow that it is a homeomorphism. Note that on ∂M we will have $\overline{f} \circ \pi = \mathrm{Id}$.

Since X is cellular, there exist embedded, closed subsets $B_i \subseteq M$, $i \geq 1$, such that $B_i \cong D^n$ for all i, $B_{i+1} \subseteq \mathring{B}_i$ for all i, and $X = \bigcap_{i=1}^{\infty} B_i$. Fix a metric on M (which is metrisable by Theorem 2.9). We will define f as a limit of a sequence of homeomorphisms. First we define maps $f_i\colon M \to M$ recursively. Set $f_0 = \mathrm{Id}_M$ and assume for the inductive step that $f_i\colon M \xrightarrow{\cong} M$ has been defined. Let $g_i'\colon D^n \to B_i \subseteq M$ be a homeomorphism.

Claim. *For each $i \geq 1$ there exists a homeomorphism $h_i\colon M \to M$ shrinking $f_i(B_{i+1})$ in $f_i(B_i)$ to diameter less than $\frac{1}{i+1}$ and $h_i|_{M \setminus \operatorname{Int}(f_i(B_i))} = \mathrm{Id}$.*

Proof of the claim. Consider the homeomorphism $g := f_i \circ g_i'\colon D^n \to f_i(B_i)$. Let $\mathfrak{s}$ be the shrinking map as in Figure 20. More precisely, we choose a closed collar neighbourhood C of ∂D^n which is disjoint from $g^{-1}f_i(B_{i+1})$. This exists since the latter set is closed. Choose a round ball $U \subseteq D^n$ of diameter r so that $g(U) \subseteq f_i(B_{i+1})$ and $g(U)$ has diameter $< \frac{1}{i+1}$. The map $\mathfrak{s}$ shrinks $\overline{D^n \setminus C}$ until it lies within U, while acting by the identity on ∂D^n, stretching out the collar C to interpolate.. It is instructive to think of this as a radial shrink.

Then define $h_i\colon M \to M$ by setting $h_i|_{M \setminus f_i(\mathring{B}_i)} = \mathrm{Id}$ and $h_i|_{f_i(B_i)} = g \circ \mathfrak{s} \circ g^{-1}$. $\square$

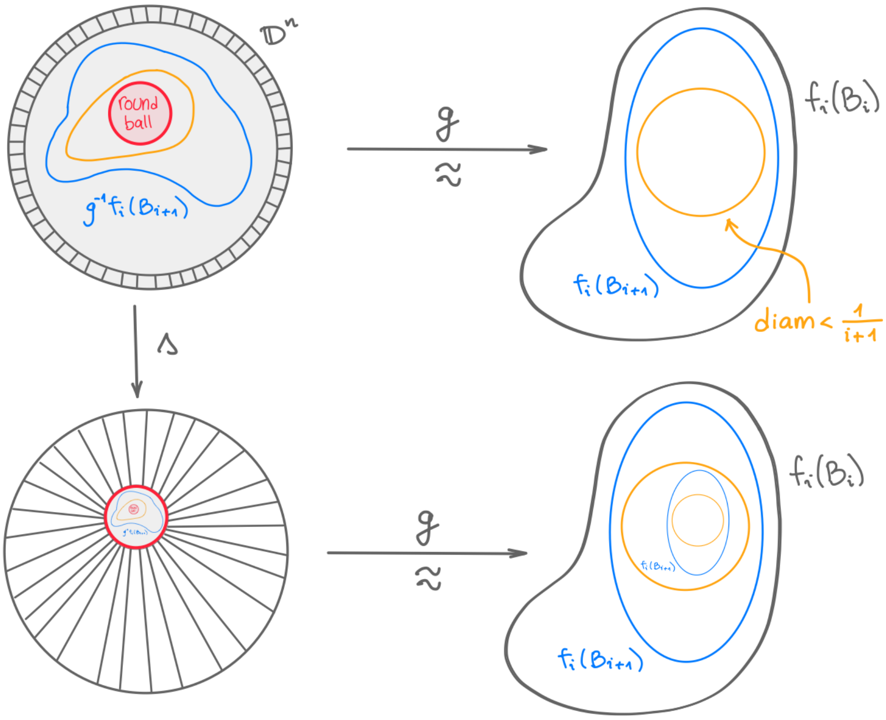


FIGURE 20. Proof of Proposition 7.7: the map $\mathfrak{s}$ shrinks the complement of the collar to the round ball.

Now we finish the inductive step by defining

$$f_{i+1} := h_i \circ f_i \colon M \xrightarrow{\cong} M$$

Note that $\operatorname{diam}(f_i(B_i)) < \frac{1}{i}$ for all i and $f_{i+1} = f_i$ on $M \setminus f_i(\mathring{B}_i)$. We also have that $f_i|_{\partial M} = \mathrm{Id}$.

We assert that $\{f_i\}$ is a Cauchy sequence in the complete metric space of functions $C_{\partial M}(M, M)$ (Remark 7.9), by showing that $d(f_m, f_n) < 1/n$ for $m > n$. Indeed, these maps agree on $M \setminus f_n(\mathring{B}_n)$, while for $x \in B_n$, both $f_m(x)$ and $f_n(x)$ lie in $f_n(B_n)$ (since $f_m(x) = h_{m-1} \circ h_{m-2} \circ \cdots \circ f_n(x)$), and $\operatorname{diam} f_n(B_n) < 1/n$.

Finally, define $f := \lim\{f_i\}$. We have that $f|_{\partial M} = \mathrm{Id}$ since each f_i restricts to the identity on ∂M. It remains only to show that f has the desired inverse sets.

Firstly, we claim that $f(X) = \{y\}$. Otherwise, for $x, x' \in X$ with $f(x) = y \neq y' = f(x')$, we can choose $i \geq 1$ such that $d(f, f_i) < \frac{d(y,y')}{3}$ and $\frac{1}{i} < \frac{d(y,y')}{3}$, so

$$\begin{aligned} d(y, y') = d(f(x), f(x')) &\leq d(f(x), f_i(x)) + d(f_i(x), f_i(x')) + d(f_i(x'), f(x')) \\ &< \frac{d(y,y')}{3} + \frac{1}{i} + \frac{d(y,y')}{3} < d(y, y') \end{aligned}$$

which is a contradiction. Here we used the fact that $\operatorname{diam} f_i(X) < 1/i$.

Secondly, observe that $f|_{M \setminus X}$ is injective. Namely, for any $p, q \in M \setminus X$ there is $i \geq 1$ such that $p, q \notin B_i$, so $f(p) = f_i(p)$ and $f(q) = f_i(q)$. If $f(p) = f(q)$ then $f_i(p) = f_i(q)$, but f_i is a homeomorphism, so $p = q$.

Finally, we claim that $f(X) \cap f(M \setminus X) = \emptyset$. Let $x \in X$. Again, for $p \in M \setminus X$ we have $f(p) = f_i(p)$ for some $i \geq 1$ such that $p \notin B_i$, so $f(p) = f_i(p)$ if $x \in X \subseteq B_{i+1}$ then

$$d(f(p), f(x)) = d(f_i(p), f(x)) \geq d(f_i(p), f_i(B_{i+1})) > 0.$$

For the final inequality we used that $d(p, B_{i+1}) > 0$ (since $B_{i+1} \subseteq \mathring{B}_i$ and $p \notin B_i$) and f_i is a homeomorphism. Therefore, $f(p) \neq f(x)$, finishing the proof that the only inverse set of f is X, as desired. $\square$

From the sketch of the proof of the Schoenflies theorem at the beginning of this section, we should remember that the goal was to quotient out the sphere by the complementary components of the given embedding and conclude that the result is still a sphere. The above result indicates that we will be able to do so if these complementary regions are cellular, and indeed they are, as we will see in the following two propositions.

Proposition 7.13. *Let $f\colon D^n \to S^n$ be a continuous map with $X \subseteq \operatorname{Int} D^n$ the only inverse set. Assume that $f(\operatorname{Int} D^n)$ is open in S^n. Then X is cellular in D^n.*

Remark 7.14. The assumption that $f(\operatorname{Int} D^n)$ is open in S^n is in fact redundant, as we will see in Lemma 7.17. However, the proof of Lemma 7.17 is somewhat involved so the reader may prefer to skip it.

Proof. By invariance of domain we know that $f(D^n) \neq S^n$. Specifically, if the map were surjective, then for a boundary point of D^n we would get a neighbourhood homeomorphic to $\mathbb{R}^n$ which is impossible. Choose a point $z \in S^n \setminus f(D^n)$ and identify $S^n \setminus \{z\}$ with $\mathbb{R}^n$. Let $f(X) =: y \in S^n$ and let U be some open set with $X \subseteq U \subseteq D^n$. Then $f(U)$

We again have the diagram

$$\begin{tikzcd} D^n \arrow[r, "f"] \arrow[d, "\pi"] & S^n \\ D^n/X \arrow[ur, "\exists \overline{f}"'] & \end{tikzcd}$$

where $\overline{f}$ is an embedding. Then since $U \subseteq \operatorname{Int} D^n$ is a saturated open set, we know that $\pi(U)$ is open in D^n/X by the definition of the quotient topology, and then since $\overline{f}$ is an embedding $f(U) = \pi \circ \overline{f}(U)$ is open in $f(\operatorname{Int} D^n)$. Since $f(\operatorname{Int} D^n)$ is open in S^n by hypothesis, we have that $f(U)$ is open in S^n.

We want to find an open ball in U containing X, implying that X is cellular (by the second definition). Using that $f(U)$ is an open neighbourhood of y and $f(D^n)$ is compact, we choose

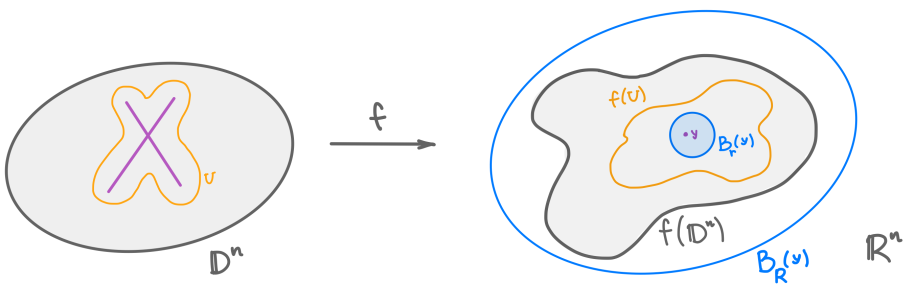


FIGURE 21. Proof of Proposition 7.13

$r, R > 0$ such that $B_r(y) \subseteq f(U) \subseteq f(D^n) \subseteq B_R(y)$ as in Fig. 21. Let $\mathscr{s}\colon S^n \to S^n$ be a 'shrinking' map similar to before (cf. Fig. 20), such that $\mathscr{s}|_{B_{r/2}(y)} = \mathrm{Id}$ and

$$\mathscr{s}(f(D^n) \subseteq \mathscr{s}(B_R(y)) \subseteq B_r(y) \subseteq f(U).$$

Then define a map $\sigma\colon D^n \to D^n$ by

$$x \mapsto \begin{cases} x, & x \in X \\ f^{-1}\mathscr{s}f(x), & x \notin X. \end{cases}$$

Note that in the second case $sf(x) \neq y$, so there is a unique preimage under f, implying that σ is a well-defined map. It is continuous (since f is a closed map restricted to $M \setminus X$) and a homeomorphism onto its image (by the closed map lemma (Lemma 3.10). Therefore, $\sigma(D^n)$ is the desired ball, since $X \subseteq \sigma(\mathring{D}^n) \subseteq \sigma(D^n) \subseteq f^{-1}sf(D) \subseteq f^{-1}f(U) = U$. ☐

Getting even closer to the situation of the Schoenflies theorem, we now generalise to a map from a sphere with two inverse sets.

Proposition 7.15. *Suppose $f\colon S^n \to S^n$ is surjective, continuous, and has precisely two inverse sets A and B. Then each of A and B are cellular.*

Proof. In an effort to prevent confusion, let S and T denote the domain and codomain respectively, so $f\colon S \to T$. Let $a := f(A)$ and $b := f(B)$. Since A and B are closed and disjoint, we can pick a standard open disc $U \subseteq S$, disjoint from A and B. Then $D := S \setminus U \cong D^n$ is a disc and $A \cup B \subseteq \mathring{D}$.

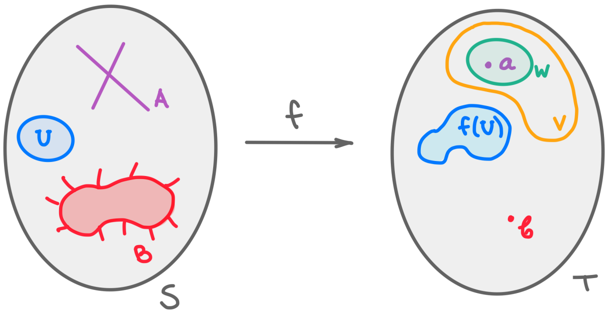


FIGURE 22. Proof of Proposition 7.15

Choose an open set $V \subseteq f(\mathring{D})$ with $a \in V$ and $b \notin V$. We can find such a V because $f(\mathring{D}) = T \setminus f(\overline{U})$ is open, since $f(\overline{U})$ is closed (as f is a closed map by the closed map lemma (Lemma 3.10)).

Now choose a homeomorphism $h\colon T \to T$ such that $f(D)$ is mapped to V, fixing some set W with $a \in W \subsetneq V$. Similarly as in the proof of the previous proposition we have a well-defined map $\psi\colon D \to S$ mapping

$$x \mapsto \begin{cases} x & x \in f^{-1}(W) \\ f^{-1}hf(x) & x \in D \setminus A. \end{cases}$$

since f^{-1} is one-to-one on $V \setminus \{a\}$.

As before we see that ψ is continuous using the pasting lemma and the fact that f is a closed map on $S \setminus (A \cup B)$.

Our goal is to apply Proposition 7.15 to ψ. We check that $B \subseteq \mathring{D}$ is the only inverse set. This is the case since A is in the 'identity' part of the definition of ψ, and h is a homeomorphism so $f^{-1}hf$ has same inverse set as $f|_{D \setminus A}$ (since h maps $f(D)$ into V), which is just B. Moreover, we check that $\psi(\mathring{D}) = f^{-1}hf(\mathring{D})$ is open since f is continuous, h is a homeomorphism, and we saw earlier that $f(\mathring{D})$ is open.

Therefore, by Proposition 7.13 applied to ψ the set B is cellular. By a similar argument A is cellular as well. ☐

Remark 7.16. You might be wondering why we did not directly find a ball neighbourhood D of A disjoint from B and apply Proposition 7.13 to it. A priori all we know about A and B is that they are closed sets, as preimages of closed sets. Then finding such a ball neighbourhood of A amounts precisely to showing that A has such a ball neighbourhood in the open set $S \setminus B$. With little information about B, i.e. when this open neighbourhood is arbitrary, this is the same as showing that A is cellular.

The proof of the following lemma can be safely skipped.

Lemma 7.17. *Let $f\colon D^n \to S^n$ be a continuous map with finitely many inverse sets, all lying within $\operatorname{Int} D^n$. Then $f(\operatorname{Int} D^n)$ is open in S^n.*

Proof. As before by invariance of domain we know that $f(D^n) \neq S^n$. Choose a point $P \in S^n \setminus f(D^n)$. Let X denote the union of all the inverse sets. As a finite union of closed sets it is closed. Define $U := (\operatorname{Int} D^n) \setminus X$ and observe it is open in D^n. It is nonempty since otherwise $X \cap \partial D^n \neq \emptyset$. Then $f|_U$ is an injective continuous map from an open subset of $\mathbb{R}^n$ to $\mathbb{R}^n = S^n \setminus \{P\}$, so by invariance of domain, $f(U)$ is open in S^n and $f|_U$ is a homeomorphism. Then for all $z \in U$, $f(z)$ lies in the interior of $f(\operatorname{Int} D^n)$. It remains only to show that the *singular points* of f, that is, the images of the inverse sets, lie in the topological interior of $f(\operatorname{Int} D^n)$. Let Y denote the collection of singular points of f. By hypothesis Y is a finite collection of isolated points.

Let $y \in Y$ be a singular point of f. Choose a sequence $\{a_i\} \subseteq U$ such that $\{a_i\} \to x$ for some $x \in X$ with $f(x) = y$. Then we know that the sequence $\{f(a_i)\} \subseteq f(U)$ converges to y by the continuity of f. Suppose y is not in the topological interior of $f(\operatorname{Int} D^n)$. Then, perhaps after passing to a subsequence, choose open coordinate ball neighbourhoods $\{B_i\}$ centred at y and with strictly decreasing radii converging to 0, such that $f(a_i) \in B_i$ for all i and choose $w_i \in B_i \cap (S^n \setminus f(\operatorname{Int} D^n)) \neq \emptyset$ so that $\{w_i\} \to y$. Within each B_i choose a path γ_i joining w_i and $f(a_i)$ with $\gamma_i \cap Y = \emptyset$ and a parametrisation $\alpha_i\colon [0,1] \to S^n$ with $\alpha_i(0) = f(a_i)$ and $\alpha_i([0,1]) = \gamma_i$. Such as path can be found since Y is a finite set. Then $\alpha_i^{-1}(f(U))$ is open for each i since $f(U)$ is open in S^n. For each i let $[0,\tau_i)$ denote the component of $\alpha_i^{-1}(f(U))$ containing 0. Then we have $f^{-1}\alpha_i([0,\tau_i)) \subseteq U$.

Fix i. Note that $\alpha_i(\tau_i) \notin Y$ by construction and thus $f^{-1}\alpha_i(\tau_i)$ is a single point v_i. We claim that $v_i \in \partial D^n$. We know that $v_i \notin U$ since then $\alpha_i(\tau_i) \in f(U)$ which is a contradiction. If $v_i \in X$, then $f(v_i) = \alpha_i(\tau_i) \in Y$ which is a contradiction. Thus, $v_i \in \partial D^n$.

We have seen that $\{v_i\} \subseteq \partial D^n$ where the latter is a compact space. Thus, we assume that $\{v_i\}$ converges after passing to a convergent subsequence. Let $u \in \partial D^n$ denote the limit of $\{v_i\}$. By continuity of f, $\{f(v_i)\} \to f(u) \in f(\partial D^n)$. On the other hand, by construction, $\{f(v_i)\} \to y$, since each $f(v_i) \in B_i$ and $\{B_i\}$ are centred at y with radii decreasing to 0. Since limits of sequences are unique in Hausdorff spaces, we have that $y = f(u)$ for $u \in \partial D^n$, which implies that $X \cap \partial D^n \supseteq f^{-1}(y) \cap \partial D^n \neq \emptyset$, which is a contradiction. □

We are now in shape to prove the Schoenflies theorem, following Brown [Bro60].

Remark 7.18. Prior to Brown's work, an alternative proof was given by Mazur in [Maz59] in the case that the embedding has a "flat spot". This hypothesis was removed by Morse in [Mor60], just a few pages after Brown's proof [Bro60] in the same journal. Mazur's argument uses an infinite stacking procedure, and the cancellation procedure known as the *Mazur swindle*. Both approaches are worth knowing, in particular since the smooth Schoenflies conjecture for $S^3 \subseteq S^4$ remains open. Nonetheless we cite Brown for the theorem since he provided the first complete argument.

Theorem 7.19 (Generalised Schoenflies theorem)**.** *Let $n \geq 1$ and let $i\colon S^{n-1} \hookrightarrow S^n$ be a locally flat embedding. Then there is a homeomorphism of pairs $(S^n, i(S^{n-1})) \cong (S^n, S^{n-1})$. where the latter is the equatorial sphere S^{n-1} in S^n.*

In particular, the closure of each component of $S^n \setminus i(S^{n-1})$ is homeomorphic to D^n.

Proof. By Corollary 6.6 we know i is bicollared: there is an embedding $I\colon S^{n-1} \times [-1,1] \to S^n$ such that $I|_{S^{n-1}\times\{0\}} = i$. Moreover, by Jordan-Brouwer Separation (Corollary 3.9), the complement has two components; see Fig. 16. Observe that we could also have applied the Mayer-Vietoris sequence directly, since we have a bicollar.

Now consider the composite

$$f\colon S^n \xrightarrow{\pi} {S^n}\big/{\{A,B\}} \xrightarrow{\cong} S^n,$$

where the quotient map collapses each of A and B to a (distinct) point and the second map is the homeomorphism identifying the unreduced suspension of S^{n-1} with S^n. Note that f maps $i(S^{n-1})$ to the equatorial sphere $S^{n-1} \subseteq S^n$. Since f has precisely two inverse sets A and B, by Proposition 7.15 we have that A and B are both cellular. Let

$$U := A \cup I(S^{n-1} \times (0,1]),$$

namely the component of $S^n \setminus i(S^{n-1})$ containing A. Then $f|_{\overline{U}}\colon \overline{U} \to D \cong D^n$, the upper hemisphere of S^n. We check that $\overline{U}$ is a manifold. As a subspace of S^n it is Hausdorff and second countable. The interior U is an open set of S^n so the only potential problem is at the boundary. But since we have the bicollar, the boundary points are also well behaved.

In the diagram below we have the function $\overline{f}$ as before, using the fact that $f|_{\overline{U}}$ is constant on the fibres of π. Then $\overline{f} \circ \pi = f|_{\overline{U}}$. Since A is cellular, by Proposition 7.7 there exists a homeomorphism h with $h|_{\partial\overline{U}} = \pi|_{\partial\overline{U}}$.

$$\begin{tikzcd}
\overline{U} \arrow[r, "f|_{\overline{U}}"] \arrow[d, "\pi"] \arrow[d, bend right=60, "\exists h\ \cong"'] & D \cong D^n \\
{\overline{U}}\big/{A} \arrow[ur, "\exists \overline{f}"', "\cong"] &
\end{tikzcd}$$

Then we have the homeomorphism $\overline{f} \circ h\colon \overline{U} \to D^n$, and moreover, $\overline{f} \circ h|_{\partial\overline{U}} = \overline{f} \circ \pi|_{\partial\overline{U}} = f|_{\partial\overline{U}}$. A similar argument for B and $V := B \cup I(S^{n-1} \times (0,1])$ shows that $\overline{V}$ is homeomorphic to the lower hemisphere of S^n. Moreover, since the induced maps on the boundary coincide, we can glue the maps together to get a homeomorphism $H\colon S^n \to S^n$ mapping $i(S^{n-1})$ to the equatorial $S^{n-1} \subseteq S^n$ as desired. □

7.4 Schoenflies in the smooth category

The proof given above only applies in the topological category. In particular, there is no analogue of Proposition 7.7 in the smooth category. Nonetheless, the smooth Schoenflies theorem in known in almost all dimensions, as we now sketch. See [Mil65, Sec. 9] for further details.

Theorem 7.20 (Schoenflies theorem, smooth version)**.** *Let Σ be a smooth embedded S^{n-1} in S^n. If $n \geq 5$, then there exists a diffeomorphism of pairs $(S^n, \Sigma) \to (S^n, S^{n-1})$.*

Proof. The complement $S^n \setminus \Sigma$ has two components, by Corollary 3.9 or directly applying the Mayer-Vietoris sequence. It is easy to check that the closure of each component of $S^n \setminus \Sigma$ is a smooth simply connected n-manifold, with boundary diffeomorphic to S^{n-1}, and which has integral homology of the n-ball (i.e. it is a $\mathbb{Z}$-homology ball).

Claim. *For $n \geq 5$, every smooth, compact, simply connected integer homology balls with boundary diffeomorphic to S^{n-1} is diffeomorphic to D^n.*

We will also need the following theorem.

Theorem 7.21 (Palais [Pal60])**.** *Any two smooth orientation-preserving smooth embeddings of D^n in a connected oriented smooth n-manifold are smoothly equivalent (that is, there is an orientation-preserving diffeomorphism of the ambient manifold taking one to the other).*

Given the above two ingredients, we prove the theorem as follows. By Palais's theorem we can obtain a diffeomorphism of S^n taking one component of $S^n \setminus \Sigma$ to the standard hemisphere of S^n. This diffeomorphism must then take the other component to the other hemisphere, and their shared boundary Σ to the equator, giving the desired diffeomorphism of pairs. □

Proof of the claim. Let W^n be a smooth compact simply connected $\mathbb{Z}$-homology n-ball with $\partial W \cong_{C^\infty} S^{n-1}$.

Remove a small ball $D_0 \subseteq \operatorname{Int} W$. Then $W \setminus \mathring{D}_0$ is an h-cobordism, i.e. a smooth manifold with precisely two boundary components, such that the inclusion of each boundary component is a homotopy equivalence.

By the h-cobordism theorem of Smale [Sma62] (see also [Mil65]), since $n \geq 6$ and $W \setminus \mathring{D}_0$ is simply connected, we have that $W \setminus \operatorname{Int} D_0 \cong_{C^\infty} S^{n-1} \times [0,1]$. Therefore, by putting the disc D_0 back in we have $W \cong D^n$.

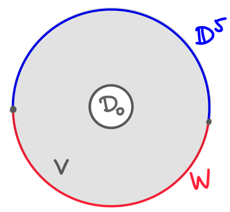


FIGURE 23. The proof of the Schoenflies theorem in the smooth category for $n = 5$.

For $n = 5$ we need a bespoke argument. Let $M = W \cup_f D^5$ where $f \colon \partial W \xrightarrow{\cong_{C^\infty}} S^4 = \partial D^5$. Then M is a $\mathbb{Z}HS^5$ (i.e. has the integral homology of S^5). Then by [Ker69, KM63, Wal62] we know that $M = \partial V$ for some smooth compact contractible 6-manifold V, see Fig. 23.

Now run the same argument as above: remove a small disc D_0 from V to get $V \setminus \operatorname{Int} D_0 \cong_{C^\infty} S^5 \times [0,1]$. Therefore, $M \cong_{C^\infty} S^5$ and we can again use Palais's theorem to conclude that $W \cong_{C^\infty} D^5$. □

Remark 7.22. The smooth Schoenflies theorem also holds in dimensions less than or equal to 3. In dimension one it only requires that S^1 is path connected, in dimension two the Riemann mapping theorem gives a proof. In dimension 3, the result is known as Alexander's theorem [Ale24] (see Hatcher's 3-manifolds notes for a more modern exposition.) In dimension four, the Schoenflies problem remains open (and is equivalent to the version in the PL category).

Remark 7.23. We may wonder to what extent the techniques in this section apply to the topological category. The h-cobordism theorem is an extremely powerful tool, but the proof fundamentally uses handle decompositions. Handle decompositions exist in the smooth category. Work of Kirby-Siebenmann can be used to find *topological* handle decompositions – explaining this is one of the goals of our course. Characteristically, this will be harder than in the smooth category. Every topological manifold, other than non-smoothable 4-manifolds, admit topological handle decompositions. A fun fact: smooth, compact, simply connected 5-dimensional h-cobordisms are not in general smoothly products (as shown by Donaldson [Don87]) but they are topologically products, i.e. homeomorphic to products (as shown by Freedman [Fre82]).

Remark 7.24. Palais' theorem (Theorem 7.21) implies that connected sum of smooth manifolds is well-defined. To show that connected sum of topological manifolds is well-defined, we will need the topological Annulus Theorem. This is due to Kirby for $n \geq 5$, and we will study its proof later (Quinn proved it for $n = 4$).

Remark 7.25. Brown's proof of the Schoenflies theorem belongs in the beautiful field of *decomposition space theory*. Other notable applications include Freedman's proof of the 4-dimensional Poincaré conjecture [Fre82] and Cannon's proof of the double suspension theorem [Can79].

Sol. on p.141. **Exercise 7.3.**Show that the following weak version of the Schoenflies theorem holds for all n in the smooth category: Let Σ denote a smoothly embedded S^{n-1} in S^n. If one of the two components of $S^n \setminus \Sigma$ is a smooth ball, then so is the other.

Sol. on p.141. **Exercise 7.4.**Show that the smooth Poincaré conjecture implies the smooth Schoenflies conjecture, in any dimension. Does the converse hold? (Why not?)

Sol. on p.141. **Exercise 7.5.** (PS3.1) Is the double Fox-Artin arc in the interior of D^3 cellular?

Note that the above exercise shows that cellularity is not a property of a space, but rather of an embedding. That is, we have found a non-cellular embedding of an arc in D^3. Of course there also exist cellular embeddings of arcs in D^3.

Sol. on p.142. **Exercise 7.6.** (PS3.2) Let M be a compact n-manifold so that $M = U_1 \cup U_2$ where each U_i is homeomorphic to $\mathbb{R}^n$.

(a) Prove that M is homeomorphic to S^n. You may use the Schoenflies theorem.

(b) Conclude that if a closed n-manifold M is an (unreduced) suspension SX for some space X, then M is homeomorphic to the sphere S^n.

Note: (b) reduces the double suspension problem to showing that the double suspension is a manifold, not specifically a sphere.

Remark 7.26. The above can be shown independently of the Schoenflies theorem, using a result of Brown characterising Euclidean space [Bro61].

Sol. on p.142. **Exercise 7.7.** (PS3.3) Let $\Sigma \subseteq S^n$ be an embedded copy of S^{n-1} and let U be one of the two path components of $S^n \setminus \Sigma$. If the closure $\overline{U}$ is a manifold, then $\overline{U}$ is homeomorphic to D^n.

Sol. on p.142. **Exercise 7.8.** (PS3.4) Let $f\colon D^n \to D^n$ be a locally collared embedding of a disc into the interior of a disc. Prove that $D^n \setminus f(D^n)$ is homeomorphic to $S^{n-1} \times (0,1]$. *Hint:* show that $f(D^n)$ is cellular.

Note, the result that $D^n \setminus \operatorname{Int}(f(D^n))$ is homeomorphic to $S^{n-1} \times [0,1]$, for $n \geq 4$, is the famous annulus theorem due to Kirby and Quinn. Why doesn't the annulus theorem follow easily from this exercise?

8 Spaces of embeddings and homeomorphisms

In the upcoming chapters, we will develop a topological analogue of the tangent bundle of a smooth manifold. Essentially, this will lead to an $\mathbb{R}^n$-fibre bundle over our manifold with structure group $\mathrm{Homeo}_0(\mathbb{R}^n)$. Hence we will need some terminology and facts about fibre bundles and groups of homeomorphisms, which is the content of this chapter. We also introduce spaces of embeddings, which generalise homeomorphisms, since a surjective embedding is a homeomorphism. Embedding spaces will be used often; the first instance is in the study of topological tangent bundles.

Definition 8.1. A *topological group* is a group G that is also a topological space, such that the group operation is a continuous map $G \times G \to G$ and such that the inverse map $g \mapsto g^{-1}$ is also a continuous map from G to itself.

Definition 8.2. A *fibre bundle* consists of a base space B, total space E and fibre F, together with a map $p\colon E \to B$, a topological group G with a continuous group action $G \times F \to F$ on F, a maximal collection $\{U_\alpha\}$ of open subsets of B and homeomorphisms $\varphi_\alpha\colon U_\alpha \times F \to p^{-1}(U_\alpha)$ called *charts*, such that

(1) $\{U_\alpha\}$ covers B;
(2) for any $V \subseteq U_\alpha$ open, $\varphi_\alpha|_V$ is a chart;
(3) the following diagrams commute:

$$\begin{tikzcd}
p^{-1}(U_\alpha) \arrow[dr, "p|"] & \\
& U_\alpha \\
U_\alpha \times F \arrow[uu, "\varphi_\alpha", "\cong"'] \arrow[ur, "pr"'] &
\end{tikzcd}$$

(4) if φ, φ' are charts over $U \subseteq B$, then there exists a continuous *transition function* $\theta_{\varphi,\varphi'}\colon U \to G$ such that for all $u \in U$ and $f \in F$ we have
$$\varphi'(u, f) = \varphi(u, \theta_{\varphi,\varphi'}(u) \cdot f).$$

Vector bundles are the special case, with $F = \mathbb{R}^n$ and $G = GL_n(\mathbb{R})$ or $O(n)$. There is a chain of inclusions of topological groups
$$O(n) \subseteq \mathrm{GL}_n(\mathbb{R}) \subseteq \mathrm{Diff}(\mathbb{R}^n) \subseteq \mathrm{Homeo}(\mathbb{R}^n)$$
where $\mathrm{Diff}(\mathbb{R}^n)$ is the topological group of diffeomorphisms of $\mathbb{R}^n$ and $\mathrm{Homeo}(\mathbb{R}^n)$ is the topological group of homeomorphisms of $\mathbb{R}^n$; we discuss these spaces (and their topologies) in this section. The first inclusion is a homotopy equivalence, which can be seen by performing the Gram-Schmidt process in a parametrised fashion. The second is also homotopy equivalence. We will not give details of these facts here as they belong in the world of differential topology.

8.1 The compact-open topology on function spaces

Let $\mathrm{Top}(n) := \mathrm{Homeo}_0(\mathbb{R}^n)$ be the group of homeomorphisms of $\mathbb{R}^n$ that fix the origin. This can be made into a topological group with the compact-open topology. We will be talking about such spaces frequently, so let us briefly explain the compact-open topology.

The compact-open topology is defined more generally, for $\mathcal{C}(X, Y)$, the set of all continuous functions from a space X to a space Y. By definition, it has a subbasis of open sets for $\mathcal{C}(X, Y)$ of the form
$$V(K, U) = \{f \in \mathcal{C}(X, Y) | f(K) \subseteq U\}$$
with $K \subseteq X$ compact and $U \subseteq Y$ open.

If a sequence of functions $\{f_i\}$ converges to $f\colon X \to Y$ in this topology, then the functions get closer to f (corresponding to smaller U) on progressively larger compact sets (corresponding to larger K). For details on the compact-open topology, we refer to [Hat02, Appendix], for example. A standard exercise is the following.

Proposition 8.3. *If X is compact, Y a metric space, then compact-open topology coincides with the uniform topology arising from $d_x(f,g) \coloneqq \sup_{x\in X} d_y(f(x),g(x))$.*

Here are the key facts about the compact-open topology on continuous functions. Sometimes $\mathcal{C}(X,Y)$ is denoted Y^X.

Proposition 8.4. *Let X,Y,Z be locally compact, Hausdorff spaces (for example topological manifolds).*

(1) Composition
$$\circ\colon \mathcal{C}(X,Y)\times\mathcal{C}(Y,Z)\to\mathcal{C}(X,Z)$$
is a continuous map.

(2) $f\colon X\times Y\to Z$ is continuous if and only if its adjoint
$$\begin{aligned}\widehat{f}\colon Y &\to \mathcal{C}(X,Z)\\ y &\mapsto (x\mapsto f(x,y))\end{aligned}$$
is continuous.

(3) The adjoint map from the previous item gives rise to a homeomorphism
$$\mathcal{C}(Y,\mathcal{C}(X,Z))\xrightarrow{\cong}\mathcal{C}(X\times Y,Z).$$
This is sometimes called the exponential rule because it can be rephrased as $Z^{X\times Y}\cong(Z^X)^Y$.

(4) The map
$$\begin{aligned}\mathcal{C}(X,Y)\times\mathcal{C}(X,Z) &\to \mathcal{C}(X,Y\times Z)\\ (f,g) &\mapsto (x\mapsto (f(x),g(x)))\end{aligned}$$
is a homeomorphism. (This also has a nice exponential mnemonic $Y^X\times Z^X=(Y\times Z)^X$.)

(5) If M is a manifold, $\mathrm{Homeo}(M)\to\mathrm{Homeo}(M)$ with $h\mapsto h^{-1}$ is continuous.

Corollary 8.5. *For X,Y as above, the map $\mathrm{ev}\colon X\times\mathcal{C}(X,Y)\to Y$ given by $(x,f)\mapsto f(x)$ is continuous.*

Proof. This follows from the exponential rule (3). Namely, it says that the adjoint map is surjective, so in particular $\mathrm{Id}\in\mathcal{C}(\mathcal{C}(X,Y),\mathcal{C}(X,Y))$ is an adjoint of some continuous map $\theta\in\mathcal{C}(X\times\mathcal{C}(X,Y),Y)$. By definition, this means that $\widehat{\theta}(f)=(x\mapsto\theta(x,f))$ is equal to $\mathrm{Id}(f)=(x\mapsto f(x))$, so $\theta(x,f)=f(x)$ and $\theta=\mathrm{ev}$. □

Remark 8.6. Versions of these facts hold with the hypotheses on X, Y, and Z relaxed a little. Since we mostly care about topological manifolds, we restrict to all spaces locally compact and Hausdorff. References for the convenient category of topological spaces and k-ification include Steenrod's paper and May's concise course.

The key consequence of Proposition 8.4 is that with the compact-open topology $\mathrm{Homeo}(M)$ is a topological group.

Remark 8.7. Why do we define the compact-open topology by only controlling functions on compact sets? Consider the space of homeomorphisms of $\mathbb{R}^n$ fixing the origin, $\mathrm{Homeo}_0(\mathbb{R}^n)$. Suppose we required that for f to lie in an open set around g, it must satisfy $|f(x)-g(x)|<\varepsilon$ for all $x\in\mathbb{R}^n$. Then for $g=\mathrm{Id}$, no rotation f about 0 would satisfy this for any ε. So even the simple operation of rotating around the origin would not constitute an isotopy.

On the other hand with the compact-open topology, rotating does give rise to an isotopy of homeomorphisms.

8.2 Spaces of embeddings and isotopy

We can consider embeddings from X to Y, $\mathrm{Emb}(X, Y)$ with the compact-open topology. This is the subspace of the continuous maps $C(X, Y)$ consisting of all the injective continuous maps that are homeomorphisms onto their images.

An isotopy of embeddings $X \hookrightarrow Y$ is a continuous map $[0, 1] \to \mathrm{Emb}(X, Y)$ or equivalently, a continuous map $\psi\colon X \times [0, 1] \to Y$ with $\psi(-, t)$ an embedding for each t (the equivalence follows from the exponential rule). We call this an isotopy between the embeddings $\psi(-, 0)$ and $\psi(-, 1)$.

Remark 8.8. Beware: under this definition, the trefoil and the unknot are isotopic, because we can pull the trefoil arbitrarily tight, until in the limit as $t \to 1$, it becomes the unknot. In knot theory, when people say colloquially that two knots are isotopic, they usually mean some other equivalence relation, such as "ambiently isotopic" as in the next definition, or smoothly isotopic.

Definition 8.9. Two embeddings $f, g\colon X \to Y$ are said to be *ambiently isotopic* if there exists

$$\Phi\colon [0, 1] \to \mathrm{Homeo}(Y)$$

with $\Phi(0) = \mathrm{Id}$ and $\Phi(1) \circ f = g$.

FIGURE 24. The unknot and the trefoil.

As expected, the trefoil and the unknot (see Fig. 24) are not ambiently isotopic. The following flowchart summarizes the relations between different notions of isotopy.

$$\begin{array}{ccccc} \text{isotopic} & \Longleftarrow & \text{locally flat isotopic} & \Longleftarrow & \text{smoothly isotopic} \\ & & \Updownarrow & & \Updownarrow \\ & & \text{ambiently isotopic} & \Longleftarrow & \text{smoothly ambiently isotopic} \end{array}$$

The upwards and the left implications are straightforward. The downwards implications use the isotopy extension theorems. The smooth isotopy extension theorem says that smooth isotopy and smooth ambient isotopy are equivalent. We used knots in S^3 as a convenient example here, but these theorems apply to submanifolds more generally, although of course one needs smooth submanifolds to make sense of the implications involving smoothly isotopic or smoothly ambient isotopic. We will talk later about the topological isotopy extension theorem due to Edwards and Kirby, and the definition of a locally flat isotopy in detail later. The proof uses the torus trick.

The trefoil and the unknot are isotopic only in a rather weak sense, since they are neither locally flat isotopic, ambiently isotopic, smoothly isotopic, nor smoothly ambiently isotopic.

In the upcoming proof of Kister's theorem, we will construct an isotopy of embeddings which is not an ambient isotopy.

8.3 Immersions

Definition 8.10. A smooth map $f\colon M \to N$ between smooth manifolds is a *smooth immersion* if for every $x \in M$ the derivative $df|_x\colon T_xM \to T_{f(x)}N$ of f at x is injective. Equivalently, f is locally a (smooth) embedding.

We can use the second description to define an analogous notion in the topological category.

Definition 8.11. A continuous map $f\colon M^m \to N^n$ between topological manifolds is an *immersion*, denoted $f\colon M \looparrowright N$, if for every $x \in M$ there is an open neighbourhood $U \ni x$ such that $f|_U$ is an embedding.

Note that an injective immersion is not necessarily an embedding, since an embedding is required to be a homeomorphism onto its image. For example, the exponential map induces an injective and surjective immersion $[0,1) \to S^1$ which is not an embedding, since it is not a homeomorphism.

In the smooth category, spaces of immersions are well-understood thanks to the work of Smale and Hirsch [Sma59, Hir59]. Their theory can be used to describe the homotopy type of those spaces in terms of mapping spaces of vector bundles, which are more accessible to the tools of algebraic topology. The following theorem is one consequence of that theory. Let $V_k(\mathbb{R}^n)$ denote the Stiefel space of k-frames in $\mathbb{R}^n$.

Theorem 8.12 (Hirsch [Hir59, Thm. 6.1])**.** *A smooth k-manifold M smoothly immerses into $\mathbb{R}^n$ with $k < n$ if and only if there is a section of the bundle $V_k(\mathbb{R}^n) \to E \to M$ associated to the Stiefel bundle $V_k(\mathbb{R}^k) \to V_k(M) \to M$ of k-frames in M.*

Let us derive a consequence, which will be used in Section 10.2, and several times thereafter. Recall that a manifold is said to be parallelisable if it has trivial tangent bundle, $TM \cong M \times \mathbb{R}^n$.

Corollary 8.13. *Every smooth parallelisable n-manifold admits a smooth immersion into $\mathbb{R}^{n+1}$.*

Proof. Since $T_k(M) \cong M \times V_k(\mathbb{R}^k)$ is the trivial bundle, any associated bundle has a section. The minimal n allowed by the theorem is $n = k + 1$. □

This can be improved in case the manifold is open (i.e. each component is noncompact and with empty boundary), or just requiring that each component is not closed (so either open or with non-empty boundary). The crucial property of such manifolds is that they can be deformed into a neighbourhood of their $(n-1)$-skeleton.

Theorem 8.14 (Hirsch [Hir61])**.** *Every smooth open parallelisable n-manifold admits a smooth immersion into $\mathbb{R}^n$.*

9 Microbundles and topological tangent bundles

We shall provide an answer to the question: what sort of tangent bundles do topological manifolds have? The short answer is: a topological manifold M has a tangent microbundle. Moreover, within the total space of a rank n microbundle, so in particular within the total space of the tangent microbundle of M, there is an $\mathbb{R}^n$-fibre bundle over M. The main references for this material are Milnor's paper on microbundles [Mil64] and Kister's paper "Microbundles are fibre bundles" [Kis64].

Every smooth manifold has a tangent vector bundle $p\colon TM \to M$. However, the linear vector bundle transition functions arise from the derivatives of the transition functions between the charts in a smooth atlas, and this a priori does not work for topological manifolds. Our aim is to get an analogue of tangent bundles for topological manifolds: a fibre bundle with structure group $G = \mathrm{Homeo}(\mathbb{R}^n)$, or in fact slightly better, the subgroup of those homeomorphisms of $\mathbb{R}^n$ that fix the origin:

$$G = \mathrm{Homeo}_0(\mathbb{R}^n) := \Big\{ f\colon \mathbb{R}^n \to \mathbb{R}^n \mid f \text{ is a homeomorphism}, f(\vec{0}) = \vec{0} \Big\}.$$

9.1 Microbundles

To find topological tangent bundles, we need to use the notion of microbundles, which is due to Milnor. Microbundles will also be a useful tool in other contexts.

Remark 9.1. For a vector bundle $p\colon E \to B$ there is a zero section $z\colon B \to E$ with $p \circ z = id$. Moreover, for every α we have a commutative diagram:

$$\begin{tikzcd} & p^{-1}(U_\alpha) \arrow[dr, "p|"] & \\ U_\alpha \arrow[ur, "z"] \arrow[dr, "\times 0"'] & & U \\ & U_\alpha \times F \arrow[uu, "\varphi_\alpha"', "\cong"] \arrow[ur, "\mathrm{pr}"'] & \end{tikzcd}$$

Such a zero section is also defined also for any fibre bundle with fibre $\mathbb{R}^n$ and group $G = \mathrm{Homeo}_0(\mathbb{R}^n)$. On the other hand for such fibre bundles the concept of a sphere or disc bundle is not well-defined.

The idea behind microbundles is to see the fibre at $x \in B$ as the germ of charts over Euclidean neighbourhoods of that point.

Definition 9.2. A *microbundle* $\mathfrak{X}$ of fibre dimension n consists of

(1) a base space B
(2) a total space E
(3) a pair of continuous maps
$$B \xrightarrow{i} E \xrightarrow{j} B$$
such that $j \circ i = \mathrm{Id}_B$ (we call i "the injection" and j "the projection"), and which satisfy *local triviality*: for all $b \in B$, there exist open sets $U \ni b$ and $V \ni i(b)$ and a homeomorphism $V \xrightarrow{\cong} U \times \mathbb{R}^n$, so that $i(U) \subseteq V$, $j(V) \subseteq U$ and the following diagram commutes
$$\begin{tikzcd} & V \arrow[dr, "j|_V"] \arrow[dd, "\cong"] & \\ U \arrow[ur, "i|_U"] \arrow[dr, "\mathrm{Id} \times 0"'] & & U \\ & U \times \mathbb{R}^n \arrow[ur, "\mathrm{pr}_1"'] & \end{tikzcd}$$

Example 9.3. The *standard trivial microbundle* $\mathfrak{e}^n_B$ of fibre dimension n over a space B. This is given by

$$B \xrightarrow{\times 0} B \times \mathbb{R}^n \xrightarrow{\mathrm{pr}_1} B$$

Taking $U := B$ and $V := B \times \mathbb{R}^n$, we will satisfy the local triviality condition at any $x \in B$.

Example 9.4. We introduce the *underlying microbundle* of an $\mathbb{R}^n$ fibre bundle. If ξ is a fibre bundle $p\colon E \to B$ with fibre $F = \mathbb{R}^n$, group $G = \mathrm{Homeo}_0(\mathbb{R}^n)$, and zero section $i\colon B \to E$, then $B \xrightarrow{i} E \xrightarrow{j=p} B$ is a microbundle, denoted by $|\xi|$. Indeed, the local triviality is satisfied by charts $V := p^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times \mathbb{R}^n$.

Remark 9.5. There exist non-isomorphic vector bundles with isomorphic underlying microbundles (see Definition 9.7), see [Mil64, Lemma 9.1].

Example 9.6. The key example is the *tangent microbundle* of a topological manifold. If M be a topological manifold, then

$$M \xrightarrow{\Delta} M \times M \xrightarrow{\mathrm{pr}_1} M$$

is a microbundle, called the *tangent microbundle* of M and denoted by $\mathfrak{t}_M$ of M. Here Δ is the diagonal map $m \mapsto (m,m)$, so $\mathrm{pr}_1 \circ \Delta = \mathrm{Id}_M$ is immediate.

To check the local triviality at $x \in M$, let $U \ni x$ and $f\colon U \xrightarrow{\cong} \mathbb{R}^n$ a chart of M. We define

$$\begin{aligned} h\colon U \times U &\to U \times \mathbb{R}^n \\ (u,v) &\mapsto (u, f(v) - f(u)) \end{aligned}$$

Then h is a homeomorphism with inverse $(a,b) \mapsto (a, f^{-1}(b + f(a)))$, and taking $V := U \times U$, gives the desired commutative diagram

$$\begin{tikzcd} & V \arrow[dr, "\mathrm{pr}_1"] \arrow[dd, "h", "\cong"'] & \\ U \arrow[ur, "\Delta|_U"] \arrow[dr, "\times 0"'] & & U \\ & U \times \mathbb{R}^n \arrow[ur, "\mathrm{pr}"'] & \end{tikzcd}$$

This is a bit surprising, as the total space does not seem like a total space of a tangent bundle, since it has too much topology (see Fig. 25 for an example). The idea is that we really only have to look at small neighbourhoods of the zero section, not all of the total space $M \times M$.

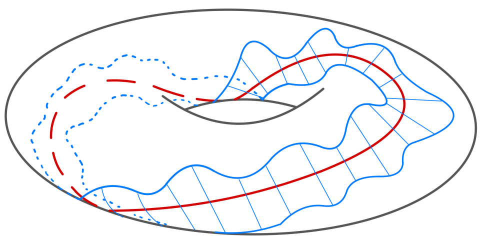


FIGURE 25. Tangent microbundle for $M = S^1$.

For a smooth manifold M it is natural to ask about the relationship between the smooth tangent bundle and the tangent microbundle. In order to address this, we introduce the notion of equivalence for microbundles.

Definition 9.7. Two microbundles $\mathfrak{X}_1$ and $\mathfrak{X}_2$ over the same base space B are said to be *isomorphic*, written $\mathfrak{X}_1 \cong \mathfrak{X}_2$, if there exist neighbourhoods $E_n \supseteq V_n \supseteq i_n(B)$ for $n = 1, 2$ and a homeomorphism $V_1 \xrightarrow{\cong} V_2$ such that the following diagram commutes

$$\begin{tikzcd} & V_1 \arrow[dd, "\cong"'] \arrow[dr, "j_1|_{V_1}"] & \\ B \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & B \\ & V_2 \arrow[ur, "j_2|_{V_2}"'] & \end{tikzcd}$$

Definition 9.8. A microbundle over B will be called *trivial* if it is isomorphic to the standard trivial microbundle $\mathfrak{e}^n_B$ (see Example 9.3).

In other words, the total space of a microbundle is not relevant up to isomorphism, only neighbourhoods of $i(B)$ in it. For example, in Fig. 25 the blue neighbourhood of $\Delta(S^1) \subseteq S^1 \times S^1$ forms a microbundle over S^1 which is isomorphic to the tangent microbundle $\mathfrak{t}_{S^1}$. More generally, we have the following theorem.

Theorem 9.9. *Let M be a smooth manifold with tangent bundle τ_M. Then the underlying microbundle $|\tau_M|$ is isomorphic to the tangent microbundle $\mathfrak{t}_M$.*

Proof. Choose a Riemannian metric on M. The underlying microbundle $|\tau_M|$ of the tangent bundle is by definition $M \xrightarrow{i} TM \xrightarrow{p} M$, where TM is the total space.

Recall that the exponential map sends $(p, \vec{v}) \in TM$ to $\exp(p, \vec{v}) \in M$ defined as the endpoint of the unique geodesic $\gamma \colon [0, 1] \to M$ with $\gamma(0) = p$ and $\gamma'(0) = \vec{v}$. This map is defined in a neighbourhood $E' \supseteq i(M)$ in TM. Then the map

$$h \colon E' \to M \times M, \quad (p, \vec{v}) \mapsto (p, \exp(p, \vec{v}))$$

is a local diffeomorphism from a neighbourhood of $(p, 0)$ in TM to neighbourhood of $(p, p) \in M \times M$, thanks to the inverse function theorem.

We claim that the restriction of h on a perhaps smaller neighbourhood $i(M) \subseteq E'' \subseteq E'$ is a homeomorphism onto some neighbourhood $\Delta(M) \subseteq V \subseteq M \times M$. This follows from a point-set topology argument, inductively covering $i(M)$ by open sets on which h is injective. We skip the argument and refer to [Whi61, Lemma 4.1].

Finally, the following diagram commutes by definition

$$\begin{tikzcd} & E'' \arrow[dd, "h"] \arrow[dr, "p"] & \\ M \arrow[ur, "i"] \arrow[dr, "\Delta"'] & & M \\ & V \arrow[ur, "\mathrm{pr}_1"] & \end{tikzcd}$$

so we have $|\tau_M| = \mathfrak{t}_M$. □

9.2 Kister's theorem

In this section we prove Kister's theorem [Kis64], which shows that every microbundle on a manifold is isomorphic to the underlying microbundle of an $\mathbb{R}^n$-fibre bundle (with structure group $\mathrm{Homeo}_0(\mathbb{R}^n)$, see Definition 8.2). In particular, a topological manifold M has the best type of tangent bundle one could hope for: its tangent microbundle $\mathfrak{t}_M$ can be replaced with the so-called *topological tangent bundle*. Throughout this section we fix an integer $n \geq 1$.

Theorem 9.10 (Kister's Theorem [Kis64])**.** *Let B be a topological manifold or a locally finite simplicial complex and let $\mathfrak{X} = (B \overset{i}{\hookrightarrow} E \overset{j}{\hookrightarrow} B)$ be a microbundle of rank n. Then there exists*

$F \subseteq E$ with $i(B) \subseteq F$ such that $F \xrightarrow{j|_F} B$ is an $\mathbb{R}^n$ fibre bundle with $i\colon B \to F$ a 0-section and underlying microbundle $\mathfrak{X}$. Moreover, any two such $\mathbb{R}^n$-bundles are isomorphic.

The main ingredient in the proof of this theorem is the following result. Let us denote $\mathrm{Emb}_0^n := \mathrm{Emb}_0(\mathbb{R}^n, \mathbb{R}^n)$ for short and let $i\colon \mathrm{Homeo}_0(\mathbb{R}^n) \hookrightarrow \mathrm{Emb}_0^n$ be the natural inclusion. Note that a point $g \in \mathrm{Emb}_0^n$ is in the subspace $\mathrm{Homeo}_0(\mathbb{R}^n)$ if and only if g is surjective.

Theorem 9.11 ([Kis64])**.** *There is a continuous map $F\colon \mathrm{Emb}_0^n \times [0,1] \to \mathrm{Emb}_0^n$, $F(g,t) = F_t(g)$ such that*

(1) $F_0 = \mathrm{Id}_{\mathrm{Emb}_0^n}$,
(2) $\mathrm{Im}(F_1) \subseteq \mathrm{Homeo}_0(\mathbb{R}^n)$,
(3) $\mathrm{Im}(F_t \circ i) \subseteq \mathrm{Homeo}_0(\mathbb{R}^n)$ *for all* $t \in [0,1]$.

Since $F_1 \circ i$ is not required to equal $\mathrm{Id}_{\mathrm{Homeo}_0(\mathbb{R}^n)}$, this is not a deformation retraction. However, the map F does show that the inclusion i is a homotopy equivalence: F_t is a homotopy between $\mathrm{Id}_{\mathrm{Emb}_0^n}$ and $i \circ F_1$, while the map $F_t \circ i$ is a homotopy between $\mathrm{Id}_{\mathrm{Homeo}_0(\mathbb{R}^n)}$ and $F_1 \circ i$.

Here is a warm up lemma before we start the proof of Theorem 9.11, demonstrating how embeddings or homeomorphisms can be deformed in a canonical way, that is continuously.

Lemma 9.12. *The inclusion $i_0\colon \mathrm{Homeo}_0(\mathbb{R}^n) \hookrightarrow \mathrm{Homeo}(\mathbb{R}^n)$ is a homotopy equivalence.*

Proof. For $x \in \mathbb{R}^n$ let $t_x\colon \mathbb{R}^n \to \mathbb{R}^n$ be the translation $t_x(y) := y + x$. Define the map

$$\Theta\colon \mathrm{Homeo}(\mathbb{R}^n) \times [0,1] \to \mathrm{Homeo}(\mathbb{R}^n), \quad \Theta(g,s) := t_{-sg(0)} \circ g.$$

It is continuous in both variables g and s (see Proposition 8.4). We have $\Theta_0 = \mathrm{Id}$, $\mathrm{Im}(\Theta_1) \subseteq \mathrm{Homeo}_0(\mathbb{R}^n)$, and $\Theta_s \circ i_0 = \mathrm{Id}$ for all $s \in [0,1]$, so Θ is a (strong) deformation retraction. □

The proof of the Theorem 9.11 will be significantly harder, but the principle is the same; the key will be the following lemma. Let $D_r \subseteq \mathbb{R}^n$ be the disc of radius r and centre 0.

Lemma 9.13 (Stretching lemma)**.** *Let $0 \leq a < b$ and $0 < c < d$ and let $g, h \in \mathrm{Emb}_0^n$ be such that $h(\mathbb{R}^n) \subseteq g(\mathbb{R}^n)$ and $h(D_b) \subseteq g(D_c)$. Then there is an isotopy of homeomorphisms $\varphi_t(g,h,a,b,c,d)\colon \mathbb{R}^n \to \mathbb{R}^n$ for $t \in [0,1]$, such that*

(1) $\varphi_0 = \mathrm{Id}_{\mathbb{R}^n}$,
(2) $\varphi_1(h(D_b)) \supseteq g(D_c)$,
(3) φ_t *fixes* $\mathbb{R}^n \setminus g(D_d)$ *and* $h(D_a)$ *pointwise; and*
(4) $\varphi\colon \mathrm{Emb}_0^n \times \mathrm{Emb}_0^n \times \mathbb{R}^5 \to \mathrm{Homeo}_0(\mathbb{R}^n)$ *with* $(g,h,a,b,c,d,t) \mapsto \varphi_t$ *is continuous.*

Proof. The idea is to expand $h(D_b)$ so it covers $g(D_c)$ in a "canonical way". The naive stretching will be identity on $\mathbb{R}^n \setminus g(D_d)$ but not on $h(D_a)$, so we will first "push" $h(D_a)$ into a "safe region", then do the stretching, and then pull it back out – this is an instance of what is known as a *push-pull* argument.

We work in g-coordinates, which is possible since $h(\mathbb{R}^n)$ is contained in $g(\mathbb{R}^n)$. We will draw g-balls as round and h-balls as crooked, see Fig. 26. Moreover, we define:

– $b' :=$ the radius of $g^{-1}h(D_b)$ (in g-coordinates: the radius of the largest disc in $h(D_b)$),
– $a' :=$ the radius of $g^{-1}h(D_a)$ (in g-coordinates: the radius of the largest disc in $h(D_a)$),
– $a'' :=$ the radius of $h^{-1}g(D_{a'})$ (in h-coordinates: the radius of the largest disc contained in $g(D_{a'})$).

Thus, we have $0 \leq a' \leq b' < c < d$ and $0 \leq a'' \leq a < b$. Note that these numbers are defined canonically in terms of g, h and a, b, c, d.

First let $\Theta_t(a,b,c,d)\colon \mathbb{R}^n \to \mathbb{R}^n$ be a stretching isotopy of homeomorphisms of $\mathbb{R}^n$, defined on all rays from 0 as the piecewise linear function from Fig. 27. More precisely, Θ_t is the identity on $[0,a]$ and $[d,\infty)$, sends b to $(1-t)b + tc$, and is extended linearly on $[a,b]$ and $[b,d]$. In particular, $\Theta_0 = \mathrm{Id}$ and Θ_1 stretches D_b over D_c and is fixed on D_a and outside of D_d.

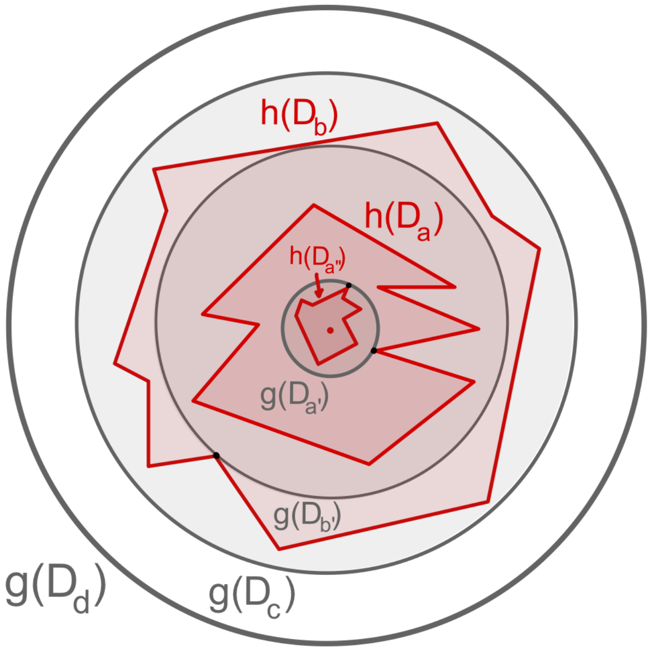


FIGURE 26. Nested balls in the stretching lemma, shown in g-coordinates.

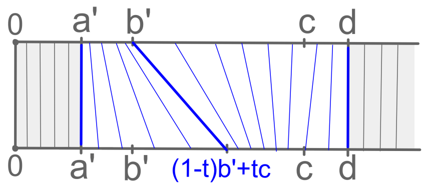


FIGURE 27. The stretching function on the positive real line $[0, \infty)$.

To transfer Θ_t to g coordinates we define $\psi_t \colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$\psi_t := \begin{cases} g \circ \Theta_t(a', b', c, d) \circ g^{-1}, & \text{on } g(D_d), \\ \mathrm{Id}, & \text{elsewhere.} \end{cases}$$

Thus, ψ_t stretches $g(D_{b'})$ over $g(D_c)$ and $h(D_b)$ over $g(D_c)$. However, ψ_t moves $h(D_a)$, so we now modify it using the push-pull argument as mentioned above. Namely, consider the streching homeomorphism $\Theta_1(0, a, a'', b)$, which actually look like a contraction since $a'' \leq a$, see Fig. 28. Then let

$$\sigma := \begin{cases} h \circ \Theta_1(0, a, a'', b) \circ h^{-1}, & \text{on } h(D_b), \\ \mathrm{Id}, & \text{elsewhere.} \end{cases}$$

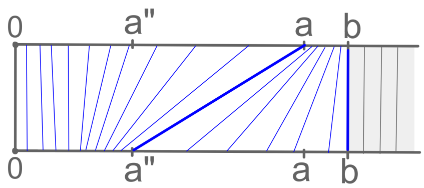


FIGURE 28. The map σ applies the depicted map $\Theta_1(0, a, a'', b)$ in h-coordinates.

Finally, for $t \in [0,1]$ define the desired map by
$$\varphi_t \colon = \sigma^{-1} \circ \psi_t \circ \sigma.$$
This first pushes using σ, then stretches using ψ_t, then pulls back using σ^{-1}. The first three properties in the statement of the lemma are straightforward to check.

It remains to check continuity of φ_t, which, although quite reasonable, requires some work. We will state the following three key propositions, whose proofs can be found in [Kis64].

Proposition 9.14. *Let $g \in \mathrm{Emb}_0^n$ and $r, \varepsilon > 0$. Then there is a $\delta > 0$ so that if $g_1 \in \mathrm{Emb}_0^n$ satisfies $d(g_1|_{D_{r+\varepsilon}}, g|_{D_{r+\varepsilon}}) < \delta$, then*
- *(i) $g_1(D_{r+\varepsilon}) \supseteq g(D_r)$,*
- *(ii) $d(g_1^{-1}|_{g(D_r)}, g^{-1}|_{g(D_r)}) \leq \varepsilon$.*

Proposition 9.15. *Let C a compact set, $h \colon C \to \mathbb{R}^n$ an embedding, $D \subseteq \mathbb{R}^n$ a compact set containing $h(C)$ in its interior, and $g \colon D \to \mathbb{R}^n$ another embedding. For every $\varepsilon > 0$ there is a $\delta > 0$ such that if $g_1 \colon D \to \mathbb{R}^n$ and $h_1 \colon C \to \mathbb{R}^n$ are embeddings whose distance from g and h respectively is bounded above by δ, then $g_1 \circ h_1$ is defined and at distance at most ε from $g \circ h$.*

Proposition 9.16. *Let $g, h \in \mathrm{Emb}_0^n$ and $a \geq 0$ such that $h(D_a) \subseteq g(\mathbb{R}^n)$. Let r be the radius of $g^{-1}h(D_a)$. Then $r = r(g, h, a)$ is continuous in the variables g, h and a.*

Now we come back to the proof of continuity of φ_t. We first show that σ is continuous: by Proposition 9.16 a' depends continuously on g, h, a, and a'' depends continuously on h, g, a', so $\Theta(0, a, a'', b)$ depends continuously on g, h, a, b.

Now σ would be the same function if we slightly modify the domain on which it is possibly not trivial, that is if we set $\sigma = h\Theta_1(0, a, a'', b)h^{-1}$ on $h(D_{b+2})$. Since $h(D_{b+1}) \subseteq \operatorname{Int} h(D_{b+2})$ there is a neighbourhood N of h in Emb_0^n such that $h_1 \in N$ implies $h_1(D_{b+1}) \subseteq h(D_{b+2})$.

Hence, if $h_1 \in N$, $b_1 \in (0, b+1)$ and g_1, a_1 satisfy the hypotheses of the Lemma 9.13, then $\sigma_1 = \sigma(g_1, h_1, a_1, b_1)$ can be defined as $h_1\Theta(0, a_1, a_1'', b_1)h_1^{-1}$ on $h(D_{b+2})$ and 1 everywhere else, where $a_1'' = a''a_1$.

We may assume, using Proposition 9.14, that N has been chosen such that $h_1(D_{b+3}) \supseteq h(D_{b+2})$ for $h_1 \in N$. Hence $h_1^{-1}|_{h(D_{b+2})}$ is defined. Proposition 9.14 also shows that this function varies continuously with h_1. Using Proposition 9.15, we conclude that $\theta(0, a_1, a_1'', b_1)h_1^{-1}|_{h(D_{b+2})}$ varies continuously with g_1, h_1, a_1 and b_1. Applying Proposition 9.15 one last time we see that $\sigma_1|_{h(D_b+2)} = h_1\theta(0, a_1, a_1'', b_1)h_1^{-1}|_{h(D_{b+2})}$ varies continuously with g_1, h_1, a_1, b_1. Hence, $\sigma(g, h, a, b)$ is continuous.

The proof that ψ_t is continuous is analogous. Since composing embeddings is continuous by Proposition 8.4, we have that ϕ_t is continuous in g, h, a, b, c, d, and t. $\square$

With Stretching Lemma 9.13 in our pocket, we are ready to prove Theorem 9.11. This will then imply Kister's Main Theorem 9.10.

Proof of Theorem 9.11. For $g \in \mathrm{Emb}_0^n$ we want to define an isotopy $F_t(g) \in \mathrm{Emb}_0^n$ from $g = \mathrm{Id}_{\mathrm{Emb}_0^n}(g)$ and $F_1(g) \in \mathrm{Homeo}_0(\mathbb{R}^n)$. Let $R_g \colon [0, \infty) \to [0, \infty)$ be the piecewise linear function such that $R_g(0) = 0$ and $R_g(i)$ for $i \in \mathbb{N}$ is the radius of the largest disc inside $g(D_i)$. We apply R_g on rays from the origin in $\mathbb{R}^n$, that is:
$$h_g \colon \mathbb{R}^n \to \mathbb{R}^n, \quad h_g(r, \theta) := (R_g(r), \theta).$$
Note that $h_g(\mathbb{R}^n) \subseteq g(\mathbb{R}^n)$ is a round open disc, and $h_g(D_i) \subseteq g(D_i)$ for all $i \in \mathbb{N}_0$. Moreover, h_g is continuous in g, since it depends only on the radius function R_g, and this depends continuously on g by Proposition 9.16.

The idea of the proof is to first isotope g to an embedding $F_{1/2}(g)$ whose image is $h_g(\mathbb{R}^n)$, and then expand this open disc in a uniform way to an embedding $F_1(g)$ whose image is all of $\mathbb{R}^n$.

Step 1. Perform an isotopy from g to an embedding whose image is the open disc $h_g(\mathbb{R}^n)$. To achieve this, we will define an isotopy $\alpha_t^g \colon \mathbb{R}^n \to g(\mathbb{R}^n)$ such that

(1) $\alpha_0^g = h_g$;
(2) $\alpha_1^g(\mathbb{R}^n) = g(\mathbb{R}^n)$;
(3) α_t^g is continuous in g and t.

We apply Lemma 9.13 for g, $h = h_g$ and $a = 0$, $b = c = 1$, $d = 2$, to obtain the stretching isotopy φ_t. Then for $t \in [0, 1/2]$ define

$$\alpha_t^g := \varphi_{2t} \circ h_g.$$

We see that $\alpha_0^g = h_g$, $g(D_1) \subseteq \alpha_{1/2}^g(D_1)$, and $\alpha_{1/2}^g(D_2) \subseteq g(D_2)$.

Now we consider the interval $[1/2, 3/4]$. Again by Lemma 9.13 applied to g, $h = \alpha_{1/2}^g$, and $a = 1$, $b = c = 2$, $d = 3$, we obtain a new isotopy φ_t. Then for $t \in [1/2, 3/4]$ define

$$\alpha_t^g := \varphi_{4t-2} \circ \alpha_{1/2}^g.$$

We have $\alpha_{1/2}^g$ same as above, $g(D_2) \subseteq \alpha_{3/4}^g(D_2)$, and $\alpha_{3/4}^g(D_3) \subseteq g(D_3)$. Moreover, $\alpha_t^g|_{D_1} = \alpha_{1/2}^g|_{D_1}$ for all $t \in [1/2, 3/4]$.

Now continue this procedure, considering for each $n \in \mathbb{N}$ the interval $[1 - 1/2^n, 1 - 1/2^{n-1}]$. To make sure that the limit function α_1 is defined, we need the following proposition; again, the proof can be found in [Kis64].

Proposition 9.17. *If $\alpha\colon \mathrm{Emb}_0^n \times [0,1) \to \mathrm{Emb}_0^n$ is continuous and for all $t \in [1 - (1/2)^n, 1)$ and $n \geq 1$ satisfies $\alpha_t(g)|_{D_n} = \alpha_{1-(1/2)^n}(g)|_{D_n}$, then α can be extended to $\mathrm{Emb}_0^n \times I$.*

Applying Proposition 9.17 to our α_t^g gives α_1^g such that $\alpha_1^g(\mathbb{R}^n) = g(\mathbb{R}^n)$. Then for $t \in [0, 1/2]$ we define

$$F_t(g) := \alpha_{1-2t}^g \circ (\alpha_1^g)^{-1} \circ g$$

Note that at $F_0(g) = g$ and $F_{1/2}(g) = h \circ (\alpha_1^g)^{-1} \circ g$ has image $F_{1/2}(g)(\mathbb{R}^n) = h_g(\mathbb{R}^n)$. We now expand this open disc to the whole of $\mathbb{R}^n$.

Step 2: Perform a concatenation of piecewise linear isotopies moving h_g to $\mathrm{Id}_{\mathbb{R}^n}$. To do this, we define an isotopy $\beta_t^g\colon \mathbb{R}^n \to \mathbb{R}^n$ such that

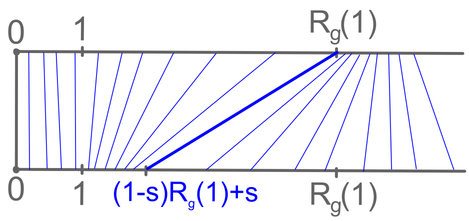
(1) $\beta_0^g = h_g$;
(2) $\beta_1^g = \mathrm{Id}$; and
(3) β_t^g is continuous in g and t.

This is quite similar to what we have done before, but easier since Lemma 9.13 is now not needed. Define h_g on rays from the origin as before. For time $[0, 1/2]$ move $R_g(1)$ to 1 by an isotopy of piecewise linear functions, as in FIiure 29a.

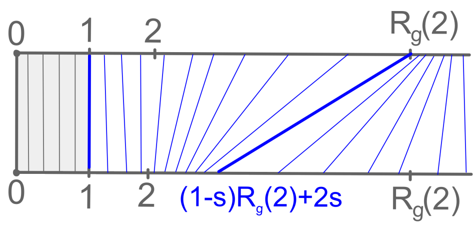
That is, for $s \in [0, 1]$ let $\theta_s(R_g(1)) = (1-s)R_g(1) + s$, and extend linearly in $[0, R_g(1)]$ and $[R_g(1), \infty)$. Then for $t \in [0, 1/2]$ define

$$\beta_t^g = \theta_{2t} \circ h.$$

In $[1/2, 3/4]$ move $R_g(2)$ to 2 in a similar fashion, while fixing $[0, 1]$, as in Fig. 29b. Then continue in the same way for all positive integers, defining an isotopy β_t^g for all $t \in [0, 1]$ analogously to the definition of α_t^g above, so that $\beta_1^g = \mathrm{Id}$ (again one must check that the isotopy is continuous at $t = 1$).



(A) The first isotopy θ_s.



(B) The second isotopy θ_s.

Now we can define the second half of F by

$$F_t(g) := \begin{cases} \alpha^g_{1-2t} \circ (\alpha^g_1)^{-1} \circ g & t \in [0, 1/2] \\ \beta^g_{2t-1} \circ (\alpha^g_1)^{-1} \circ g & t \in [1/2, 1] \end{cases}$$

At $t = 1/2$, we have $\beta^g_0 = h$ so that $h_g \circ (\alpha^g_1)^{-1} \circ g = \alpha^g_0 \circ (\alpha^g_1)^{-1} \circ g$, so the composite function is continuous at $1/2$. We also know that $\beta^g_1 = \mathrm{Id}$ so that at $t = 1$ we have $(\alpha^g_1)^{-1} \circ g$. Since $\alpha^g_1(\mathbb{R}^n) = g(\mathbb{R}^n)$, $(\alpha^g_1)^{-1} \circ g$ is a homeomorphism.

One needs to check that F is indeed continuous in g and t. We also note that if g is a homeomorphism, then $F_t(g)$ is a homeomorphism for every t, by inspecting the proof. □

Now we can use this result to prove Kister's Theorem 9.10: microbundles contain $\mathbb{R}^n$-fibre bundles. More precisely, if B a locally finite simplicial complex or a topological manifold and $\mathfrak{X} = B \xrightarrow{i} E \xrightarrow{j} B$ is a microbundle, we want to prove there is an open set $E_1 \subseteq E$ containing $i(B)$ such that $j|_{E_1} : E_1 \to B$ is a fibre bundle with $\mathrm{Homeo}_0(\mathbb{R}^n)$ as structure group. We call such a bundle an *admissible bundle* for $\mathfrak{X}$.

Proof of Theorem 9.10. The strategy of the proof is as follows.

(i) Prove the theorem for a locally finite simplicial complex B by induction on simplices.
(ii) Deduce for $M = B$ a topological manifold.

For the second item, although M is in general not a simplicial complex, it is an Euclidean neighbourhood retracts, i.e. there is an open neighbourhood $M \subseteq V \subseteq \mathbb{R}^N$ with a retraction $r : V \to M$ see Theorem 4.5. Then $r^*\mathfrak{X}$ is a microbundle on V of the same rank, and since V is an open subset of $\mathbb{R}^N$, it admits a smooth structure. In particular, V admits a locally finite triangulation, so we can apply (i) to obtain an admissible fibre bundle ξ inside $E(r^*\mathfrak{X})$. The restriction of ξ along the inclusion $i : M \hookrightarrow V$ gives the desired $\mathbb{R}^n$-fibre bundle $i^*\xi$ over M with

$$E(i^*\xi) \subseteq E(i^*r^*(\mathfrak{X})) = E((r \circ i)^*\mathfrak{X}) = E(\mathrm{Id}^*\,\mathfrak{X}) = E(\mathfrak{X}).$$

Now, to prove (i) we induct both on simplices and on the dimension m of the simplicial complex. For each m we consider the following two statements, for microbundles of a fixed rank n.

$X_m :=$ "Every microbundle over a locally finite m-dim. simplicial complex admits a bundle."

$U_m :=$ "Any two such admissible bundles for such a microbundle are isomorphic."

Both X_0 and U_0 hold since every microbundle over a point is trivial, and therefore the same holds over a collection of 0-simplices with the discrete topology. For the induction step we prove that X_{m-1} and U_{m-1} together imply X_m, and that X_m implies U_m.

Let us show the first claim. Let K be a locally finite simplicial complex with a microbundle

$$\mathfrak{X} = K \xrightarrow{i} E \xrightarrow{j} K$$

Let K' denote the $(m-1)$-skeleton of K and pick an m-simplex σ of K not in K'.

Since σ is contractible, it admits a trivial admissible bundle ξ_σ, and homeomorphism h_σ fitting into the diagram:

$$\begin{tikzcd} & \sigma \arrow[dl, "\times 0"'] \arrow[dr, "i"] & \\ \sigma \times \mathbb{R}^n \arrow[rr, "h_\sigma", "\cong"'] \arrow[dr, "\mathrm{pr}_1"'] & & E(\xi_\sigma) \arrow[dl, "j|"] \\ & \sigma & \end{tikzcd}$$

Let D be an open set in E such that $i(K) \subseteq D$ and

$$j^{-1}(\sigma) \cap D \subseteq E(\xi_\sigma).$$

Then consider the following restriction of $\mathfrak{X}$ to K':

$$\mathfrak{X}' = \{K' \xrightarrow{i'} j^{-1}(\sigma) \cap D \xrightarrow{j'} K'\}$$

By X_{m-1} we know that $\mathfrak{X}'$ admits an $\mathbb{R}^n$-bundle η over K'. In order to now glue η and ξ_σ we have to make them compatible along the collar of the boundary $\partial\sigma$. Note that since ξ_σ is trivial, $\xi|_{\partial\sigma}$ is a trivial fibre bundle. But now $\eta|_{\partial\sigma}$ and $\xi|_{\partial\sigma}$ are admissible bundles for the same microbundle, so by U_{m-1} they are isomorphic. In particular, $\eta|_{\partial\sigma}$ is also trivial and we have a homeomorphism h_η fitting into the diagram:

$$\begin{tikzcd}
 & \partial\sigma \arrow[dl, "\times 0"'] \arrow[dr, "i"] & \\
\partial\sigma \times \mathbb{R}^n \arrow[rr, "h_\eta", "\cong"'] \arrow[dr, "\mathrm{pr}_1"'] & & E(\eta|_{\partial\sigma}) \arrow[dl, "j|"] \\
 & \partial\sigma &
\end{tikzcd}$$

Thus, over $\partial\sigma$ we have two trivialisastions h_σ and h_η, and we can consider $h_\sigma^{-1}h_\eta$, which us a fibrewise embedding of a fibre of η into a fibre of ξ over $\partial\sigma$. For every $p \in \partial\sigma$ we thus define $g^p \colon \mathbb{R}^n \to \mathbb{R}^n$ with $g^p \in \mathrm{Emb}_0^n$ by the formula

$$h_\sigma^{-1} \circ h_\eta(p, q) = (p, g^p(q)).$$

Now, let σ_1 be a smaller m-simplex in σ, and as in Fig. 30 identify $\sigma \setminus \mathrm{Int}\,\sigma_1 \cong \partial\sigma \times [0,1]$ so that $\partial\sigma = \partial\sigma \times \{0\}$ and $\partial\sigma_1 = \partial\sigma \times \{1\}$.

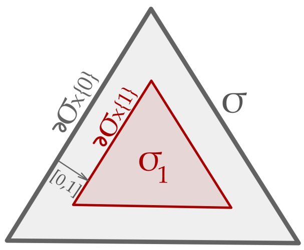


FIGURE 30. A parametrisation $\partial\sigma \times [0,1]$ of the grey region $\sigma \setminus \sigma_1$ between the two simplices.

We now use the map $F\colon \mathrm{Emb}_0^n \times I \to \mathrm{Emb}_0^n$ constructed in Theorem 9.11. For brevity, for each $(p,t) \in \partial\sigma \times I$, we write $g_t^p := F(g^p, t)\colon \mathbb{R}^n \to \mathbb{R}^n$. At $t = 0$ this is the embedding $g_0^p = g^p$, while at $t = 1$ it is a homeomorphism.

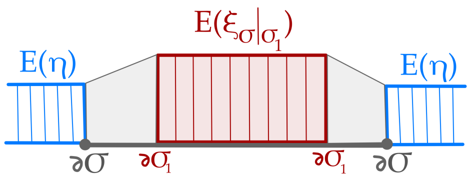


FIGURE 31. We define the bundle E_1 by gluing the two bundles over simplices σ and σ_1, together with a transition in the region $\sigma \setminus \sigma_1$ given by Kister's isotopy.

Consider the space

$$E_1 := E(\eta) \cup \left\{h_\sigma\big((p,t), g_t^p(q)\big) \mid (p,t) \in \partial\sigma \times I \cong \sigma \setminus \mathrm{Int}\,\sigma_1,\ q \in \mathbb{R}^n\right\} \cup E(\xi_\sigma|_{\sigma_1})$$

To complete the proof of X_m it remains to show that the projection

$$j|_{E_1} \colon E_1 \to K' \cup \sigma$$

is indeed a fibre bundle (so that it is an admissible bundle for $\mathfrak{X}|_{K' \cup \sigma}$). The idea is that as $t \in [0,1]$ increases, the image of $h_\sigma((p,t), g_t^p(q))$ expands. Since g_1^p is a homeomorphism, for each $p \in \partial\sigma$ the image fills up the entire fibre $\xi_\sigma|_{\partial\sigma \times \{1\}} = \xi_\sigma|_{\partial\sigma_1}$.

We define a trivialisation over $\sigma \setminus \operatorname{Int}\sigma_1$ by

$$\begin{aligned} f \colon (\sigma \setminus \operatorname{Int}\sigma_1) \times \mathbb{R}^n &\to E_1 \\ ((p,t),q) &\mapsto h_\sigma((p,t), g_t^p(q)). \end{aligned}$$

On the other hand, for $(p,1) \in \partial\sigma \times \{1\} = \partial\sigma_1$ we have $e^p \in \operatorname{Homeo}_0(\mathbb{R}^n)$ given by

$$e^p(q) = \operatorname{pr}_2 \circ f^{-1} h_\sigma((p,1),q).$$

Then we can let

$$\begin{aligned} e \colon \sigma \times \mathbb{R}^n &\to j^{-1}(\sigma) \cap E_1 \\ e((p,t),q) &= \begin{cases} h_\sigma((p,t),q), & (p,t) \in \sigma_1, \\ f((p,t), e^p(q)), & (p,t) \in \sigma \setminus \operatorname{Int}\sigma_1. \end{cases} \end{aligned}$$

Since $f^{-1}h_\sigma((p,1),q) = ((p,1), e^p(q))$, we have for all $p \in \partial\sigma$ and $q \in \mathbb{R}^n$ that

$$h_\sigma((p,1),q) = f((p,1), e^p(q)).$$

Therefore, e is a well-defined homeomorphism, and a local trivialisation of $j|_{E_1}$ over $\operatorname{Int}\sigma$.

We also need to show that $j|_{E_1}$ is locally trivial over $\partial\sigma$, and also that X_m implies U_m. These are rather similar in spirit to the proofs we have just done, so we omit them, referring to [Kis64] for details. □

Sol. on p.142. **Exercise 9.1**. (PS4.1) Every microbundle over a paracompact contractible space B is isomorphic to the trivial microbundle over B.

Sol. on p.143. **Exercise 9.2**. (PS4.3) For X compact and Y a metric space, the compact-open topology on $\mathcal{C}(X,Y) := \{f \colon X \to Y \mid f \text{ continuous}\}$ coincides with the uniform topology coming from

$$d(f,g) := \sup_{x \in X} d_Y(f(x), g(x)).$$

Sol. on p.144. **Exercise 9.3**. (PS5.1) Let X and Y be compact metric spaces with $X \times \mathbb{R}$ homeomorphic to $Y \times \mathbb{R}$. Then $X \times S^1$ is homeomorphic to $Y \times S^1$.

Hint: let $h \colon X \times \mathbb{R} \to Y \times \mathbb{R}$ be a homeomorphism, and consider the two product structures on $Y \times \mathbb{R}$, the intrinsic one and the one coming from $h(X \times \mathbb{R})$. Use a push-pull construction (repeated infinitely many times) to create a periodic homeomorphism $H \colon X \times \mathbb{R} \to Y \times \mathbb{R}$, i.e. for some $p \in \mathbb{R}$, $H(x,t) = H(x, t+p)$ for all $t \in \mathbb{R}$, $x \in X$.

10 Homeomorphisms of $\mathbb{R}^n$ and the torus trick

The torus trick was invented by Kirby [Kir69] to prove the annulus theorem in dimension ≥ 5. Since that proof uses some nontrivial input from PL topology, we prefer to introduce it using another application, namely to show the local contractibility of $\mathrm{Homeo}(\mathbb{R}^n)$, which is one of the main results of this section (see Corollary 10.13) and was first proved by Černavskiĭ [Č73]. This use of the torus trick requires much less input from outside the topological category.

The torus trick turned out to be a very useful method of proof, in many different contexts. Its key applications include topological transversality, isotopy extension, existence of topological handle decompositions, topological invariance of simple homotopy type, and smoothing theory – these are all major advances in the understanding of topological manifolds. We will discuss some of these applications later. The torus trick can also be applied in low dimensional manifolds of dimension 2 and 3 to show that they admit unique smooth structures [Ham76, Hat13].

10.1 Homeomorphisms bounded distance from the identity and Alexander isotopies

We begin our study of $\mathrm{Homeo}(\mathbb{R}^n)$ with an elementary but extremely useful observation.

Definition 10.1. A homeomorphism $h \in \mathrm{Homeo}(\mathbb{R}^n)$ is *bounded distance from* Id if there exists $K > 0$ such that $|h(x) - x| < K$ for all $x \in \mathbb{R}^n$.

In the literature such homeomorphisms are often called 'bounded'. We prefer the longer descriptor to avoid a non-traditional and potentially confusing use of that term.

Proposition 10.2. *If $h \in \mathrm{Homeo}(\mathbb{R}^n)$ is bounded distance from* Id, *then h is isotopic to* $\mathrm{Id}_{\mathbb{R}^n}$.

Proof. Define the map

$$H(x,t) := \begin{cases} t \cdot h(\frac{x}{t}), & t \neq 0, \\ x, & t = 0. \end{cases}$$

Note that $H(-,0) = \mathrm{Id}_{\mathbb{R}^n}$, $H(-,1) = h$ and each $H(-,t)$ is a homeomorphism of $\mathbb{R}^n$. Moreover, H is clearly continuous on $\mathbb{R}^n \times (0,1]$. For $x_0 \in \mathbb{R}^n$, we check continuity at $(x_0, 0)$ directly next.

Given $\varepsilon > 0$ we choose $\delta = \min\{\frac{\varepsilon}{2}, \frac{\varepsilon}{2K}\}$. Then for any (x,t) with $|(x,t) - (x_0,0)| < \delta$, we in particular have $|x - x_0| < \delta$ and $t < \delta$, so

$$|H(x,t) - H(x_0,0)| = \begin{cases} |t \cdot h(\frac{x}{t}) - x_0| \leq |t \cdot h(\frac{x}{t}) - x| + |x - x_0| < tK + \delta < \delta K + \delta < \varepsilon & t \neq 0 \\ |x - x_0| < \delta < \varepsilon & t = 0 \end{cases}$$

where for the second inequality in the first case we used the fact that h is bounded distance from Id, namely

$$|t \cdot h\left(\frac{x}{t}\right) - x| = t|h\left(\frac{x}{t}\right) - \frac{x}{t}| < tK. \qquad \square$$

We next derive a few consequences, all of which go under the name "Alexander trick" or "Alexander isotopy". The first of the cases below does not use Proposition 10.2 and has a somewhat easier proof.

Proposition 10.3 (Alexander isotopies)**.** *Let n be a positive integer.*

(0) Every $f \in \mathrm{Homeo}(S^{n-1})$ extends to $F \in \mathrm{Homeo}(D^n)$.

(i) For any $h \in \mathrm{Homeo}(D^n)$, if $h|_{\partial D^n} = \mathrm{Id}$ then h is isotopic to Id_{D^n}.

(ii) For any $f, g \in \mathrm{Homeo}(D^n)$ if $f|_{\partial D^n} = g|_{\partial D^n}$ then f and g are isotopic.

(iii) For any $h \in \mathrm{Homeo}(\mathbb{R}^n)$ if $h|_{D^n} = \mathrm{Id}$ then h is isotopic to $\mathrm{Id}_{\mathbb{R}^n}$.

(iv) For any $f, g \in \mathrm{Homeo}(\mathbb{R}^n)$, if $f|_{D^n} = g|_{D^n}$ then f and g are isotopic.

(v) For any $f, g \in \mathrm{Homeo}(\mathbb{R}^n)$, if $f|_U = g|_U$ for an open $U \subseteq \mathbb{R}^n$ then f and g are isotopic.

Proof. For (10.3.0), we use that the disc D^n is homeomorphic to the cone on the sphere S^{n-1}, so we can define the extension F the cone of the map f, by setting $F(t,z) = t \cdot f(z)$ for $(t,z) \in D^n$ corresponding to $z \in S^{n-1}$.

For (10.3.i) we extend h by the identity map to a homeomorphism of $\mathbb{R}^n$, which is clearly bounded distance from Id. By Proposition 10.2 this extension is isotopic to $\mathrm{Id}_{\mathbb{R}^n}$ via a 1-parameter family of maps $H(-,t)$, each of which restricts to the identity on the complement of the open unit disc, so their restrictions give the desired isotopy from h to Id_{D^n}. This isotopy rescales a given point "outwards", applies h and then pulls it back in. For each x there is a small enough t so that $\frac{x}{t}$ is outside the unit disc, where we apply the identity. In other words, the region where the identity is applied expands inwards as t decreases.

Now (10.3.ii) follows directly from (10.3.i): we apply it to $h := g^{-1}f$ to get an isotopy H, and then observe that gH is an isotopy from f to g.

To prove (10.3.iii) we define an isotopy $H\colon \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ from $\mathrm{Id}_{\mathbb{R}^n}$ to h given by

$$H(x,t) := \begin{cases} \frac{1}{t}h(tx) & t \neq 0 \\ x & t = 0. \end{cases}$$

(Here the identity expands "outwards" as t decreases.) We may check continuity as in the proof of Proposition 10.2. Namely, continuity away from $t = 0$ is again immediate, and given $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$, we choose $\delta = \min\{\frac{1}{1+|x_0|}, 1, \varepsilon\}$. Then if $|(x,t) - (x_0, 0)| < \delta$, we have $t < \delta$ and $|x|-|x_0| \leq |x-x_0| < \delta \leq 1$. In particular, $|x| < 1+|x_0|$ and $|tx| = t|x| < \delta(1+|x_0|) \leq \frac{1+|x_0|}{1+|x_0|} = 1$, so tx is contained in D^n. Therefore, $h(tx) = tx$ by hypothesis, and we have

$$|H(x,t) - H(x_0,0)| = \begin{cases} \left|\frac{1}{t}h(tx) - x_0\right| = \left|\frac{1}{t}(tx) - x_0\right| = |x - x_0| < \delta < \varepsilon, & t \neq 0, \\ |x - x_0| < \delta < \varepsilon, & t = 0. \end{cases}$$

For (10.3.iv), apply (10.3.iii) to $g^{-1}f$.

For (10.3.v) choose a disc within U and rescale it to a unit disc D^n, then apply (10.3.iv). □

The local contractibility of $\mathrm{Homeo}(\mathbb{R}^n)$ will be a consequence of the following theorem.

Theorem 10.4 (Černavskiĭ [Č73], Kirby [Kir69])**.** *For any $n \geq 0$ there exists $\varepsilon > 0$ such that every homeomorphism $h\colon \mathbb{R}^n \to \mathbb{R}^n$ satisfying $|h(x) - x| < \varepsilon$ for all $x \in D^n$ is isotopic to $\mathrm{Id}_{\mathbb{R}^n}$.*

In other words, if $h \in \mathrm{Homeo}(\mathbb{R}^n)$ and $\mathrm{Id}_{\mathbb{R}^n}$ are close on the unit disc D^n, then they are isotopic. Contrast this with Proposition 10.2, where we require that they are close everywhere to reach the same conclusion. Observe also that ε does not depend on h, but only on n.

10.2 Torus trick – the proof of the Černavskiĭ-Kirby theorem

The proof of Theorem 10.4 given by Černavskiĭ [Č73] is explicit and similar in spirit to the proof of Kister's theorem (Theorem 9.10). We will instead present Kirby's proof [Kir69] using the torus trick. Given $h \in \mathrm{Homeo}(\mathbb{R}^n)$ the strategy is to construct a homeomorphism $\widetilde{h} \in \mathrm{Homeo}(\mathbb{R}^n)$ with the following key properties:

(1) $\widetilde{h}$ and h agree on an open set, and are therefore isotopic (Proposition 10.3.v);
(2) $\widetilde{h}$ is bounded distance from Id, and therefore isotopic to the identity (Proposition 10.2).

How can we build the map $\widetilde{h}$? The next lemma shows that a homeomorphism of the n-torus

$$T^n := S^1 \times \cdots \times S^1,$$

that is homotopic to the identity, induces a homeomorphism of $\mathbb{R}^n$ which is bounded distance from Id. This will be a key step in the proof and indicates why the n-torus is such a key player.

Lemma 10.5. *Given $f \in \mathrm{Homeo}(T^n)$ there exists $\widetilde{f} \in \mathrm{Homeo}(\mathbb{R}^n)$ so that*

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\widetilde{f}", "\cong"'] \arrow[d, "e"] & \mathbb{R}^n \arrow[d, "e"] \\ T^n \arrow[r, "f", "\cong"'] & T^n \end{tikzcd}$$

commutes, where $e\colon \mathbb{R}^n \to T^n$ is the universal covering map. Moreover, if f is homotopic to Id_{T^n}, then $\widetilde{f}$ is bounded distance from $\mathrm{Id}_{\mathbb{R}^n}$.

Proof. Fix $x_0 \in T^n$ and $y_0 \in e^{-1}(x_0)$. There exists a lift $\widetilde{f}$ of fe since $\{0\} = (fe)_*(\pi_1(\mathbb{R}^n)) \leq e_*(\pi_1(\mathbb{R}^n)) = \{0\}$. Similarly, there exists a $\widetilde{g}$ lifting ge for $g := f^{-1}$ so that $\widetilde{g}\widetilde{f}(y_0) = y_0$, and the diagram below commutes.

$$\begin{tikzcd} (\mathbb{R}^n, y_0) \arrow[r, "\widetilde{f}"] \arrow[d, "e"] \arrow[rr, bend left, "\mathrm{Id}_{\mathbb{R}^n}"] & (\mathbb{R}^n, \widetilde{f}(y_0)) \arrow[r, "\widetilde{g}"] \arrow[d, "e"] & (\mathbb{R}^n, \widetilde{g}\widetilde{f}(y_0) = y_0) \arrow[d, "e"] \\ (T^n, x_0) \arrow[r, "f", "\cong"'] \arrow[rr, bend right, "\mathrm{Id}_{T^n}"'] & (T^n, f(x_0)) \arrow[r, "g", "\cong"'] & (T^n, x_0) \end{tikzcd}$$

Note that both $\widetilde{g} \circ \widetilde{f}$ and $\mathrm{Id}\colon \mathbb{R}^n \to \mathbb{R}^n$ are lifts of $g \circ f \circ e = \mathrm{Id} \circ e = e$, and they agree on y_0 so by the uniqueness of lifting, we have that $\widetilde{g} \circ \widetilde{f} = \mathrm{Id}$. The same argument with the rôles of f and g switched shows that $\widetilde{f} \circ \widetilde{g} = \mathrm{Id}$, so $\widetilde{f}$ is the desired homeomorphism.

To prove the last statement, we use the following claim.

Claim. *If f is homotopic to Id_{T^n}, then $\widetilde{f}$ commutes with the deck transformations.*

Recall that the deck transformations of the cover $e\colon \mathbb{R}^n \to T^n$ are translations $\tau_a\colon \mathbb{R}^n \to \mathbb{R}^n$, given by $x \mapsto x + a$ for $a \in \mathbb{Z}^n$.

Proof of the claim. Fix some $a \in \mathbb{Z}^n$. We will prove that $\widetilde{f} \circ \tau_a = \tau_a \circ \widetilde{f}$. Observe that we have $fe = e\widetilde{f}$ since $\widetilde{f}$ is a lift of fe. For any $m \in Z^n$, the deck transformation τ_m is by definition a lift of e so we have $e = e\tau_m$.

By assumption, there is a homotopy $F\colon T^n \times [0,1] \to T^n$ from $F_0 = f$ to $F_1 = \mathrm{Id}_{T^n}$, and we consider the map

$$F \circ (e \times \mathrm{Id})\colon \mathbb{R}^n \times [0,1] \xrightarrow{e \times \mathrm{Id}} T^n \times [0,1] \xrightarrow{F} T^n$$

where

$$F \circ (e \times \mathrm{Id})|_{\mathbb{R}^n \times 0} = f \circ e = e \circ \widetilde{f}$$

and

$$F \circ (e \times \mathrm{Id})|_{\mathbb{R}^n \times 1} = e.$$

The map F is in particular a homotopy, so by the homotopy lifting property we can find $\widetilde{F}\colon \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ with

- $\widetilde{F_0} = \widetilde{f}$, namely a lift of $F \circ (e \times \mathrm{Id})|_{\mathbb{R}^n \times 0}$;
- $e\widetilde{F} = F \circ (e \times \mathrm{Id})$; and
- $\widetilde{F_1}$ a lift of $F \circ (e \times \mathrm{Id})|_{\mathbb{R}^n \times 1} = e$.

Since $\widetilde{F}_1$ is a lift of e, there exists $c \in \mathbb{Z}^n$ such that $\widetilde{F}_1 = \tau_c$.

Define $G := \tau_{-a} \circ \widetilde{F} \circ (\tau_a \times \mathrm{Id})\colon \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ and note that for $t \in [0,1]$ we have

$$\begin{aligned} e \circ G(x,t) &= e \circ \tau_{-a} \circ \widetilde{F}(\tau_a(x), t) && \text{by definition of } G, \\ &= e \circ \widetilde{F}(\tau_a(x), t) && \text{since } e = e \circ \tau_m \text{ for any } m \in Z^n, \end{aligned}$$

$$
\begin{aligned}
&= F(e \circ \tau_a(x), t) && \text{by definition of } \widetilde{F}, \\
&= F(e(x), t) && \text{since } e = e \circ \tau_m \text{ for any } m \in Z^n.
\end{aligned}
$$

Therefore, both G and $\widetilde{F}$ are lifts of $F \circ (e \times \mathrm{Id})$ ending in $G_1 = \widetilde{F}_1 = \tau_c$. By the uniqueness of lifting $G := \tau_{-a}\widetilde{F}(\tau_a \times \mathrm{Id}) = \widetilde{F}$. In particular, $\tau_{-a}\widetilde{f}\tau_a(x) = \tau_{-a}\widetilde{F}(\tau_a(x), 0) = \widetilde{F}(x, 0) = \widetilde{f}(x)$, finishing the proof of the claim. □

Let us now complete the proof of the lemma. Given $x \in \mathbb{R}^n$, we can write $x = x_0 + a = \tau_a(x_0)$, for some $x_0 \in I^n$ the unit cube and $a \in \mathbb{Z}^n$. Then

$$|\widetilde{f}(x) - x| = |\widetilde{f}(\tau_a(x_0)) - \tau_a(x_0)| = |\tau_a(\widetilde{f}(x_0)) - \tau_a(x_0)| = |\widetilde{f}(x_0) - x_0|$$

Therefore, $\widetilde{f}$ is indeed bounded distance from $\mathrm{Id}_{\mathbb{R}^n}$ since

$$\sup_{x \in \mathbb{R}^n} |\widetilde{f}(x) - x| = \sup_{x_0 \in I^n} |\widetilde{f}(x_0) - x_0| < \infty. \qquad \square$$

Returning to Theorem 10.4, we would like to leverage the above fact about homeomorphisms of tori, and the induced maps on $\mathbb{R}^n$. To do so, we need to first find a torus – and for this we will use smooth manifold topology, namely Smale-Hirsch theory. Recall the notions of smooth and topological immersions from Section 8.3.

Corollary 10.6 (of Theorem 8.14)**.** *For all n there is a smooth immersion $\alpha \colon T^n \setminus \{pt\} \looparrowright \mathbb{R}^n$.*

Proof. The circle S^1 is parallelisable, and the product of parallelisable manifolds is parallelisable. An open subset of a parallelisable manifold is parallelisable, so Theorem 8.14 gives the result. □

Let us point out that one need not rely on this machinery – there are explicit constructions of immersed punctured n-tori in $\mathbb{R}^n$, for example by Ferry [Fer74], Milnor [KS77, p. 43], and Barden [Rus73, p. 290].

As a final ingredient in the upcoming proof of Theorem 10.4 we will need the following application of the Schoenflies theorem.

Proposition 10.7. *Let Σ be a bicollared S^{n-1} in T^n for $n \geq 3$. Then Σ bounds a ball in T^n.*

Proof. First we prove that Σ is separating. This can be seen using the following portion of the Mayer-Vietoris sequence for $T^n = T^n \setminus \Sigma \cup \nu\Sigma$, where $\nu\Sigma$ is the image of the bicollar of Σ.

$$H_1(T^n) \xrightarrow{0} H_0(\Sigma \sqcup \Sigma) \longrightarrow H_0(T^n \setminus \Sigma) \oplus H_0(\Sigma) \longrightarrow H_0(T^n) \longrightarrow 0$$

where the first map is trivial since Σ is null-homologous in T^n for $n \geq 3$ (recall that T^n is an Eilenberg-Maclane space).

Let A and B denote the closures of the two components of $T^n \setminus \Sigma$. Then

$$\mathbb{Z}^n \cong \pi_1(T^n) \cong \pi_1(A) * \pi_1(B).$$

Since an abelian group cannot be represented as a nontrivial free product, one of the two pieces, say A, has trivial fundamental group. Then A lifts to the universal cover $\mathbb{R}^n$. In other words, the restriction of e to each component of the preimage of A is a homeomorphism. On the other hand, the boundary of each such component is a bicollared sphere in $\mathbb{R}^n$ and by the Schoenflies theorem each component is a ball. Therefore A is a ball, completing the proof. □

Remark 10.8. An alternative proof of this would be to notice that if Σ were non-separating, there would be an arc connecting one side of Σ to the other. Taking a tubular neighbourhood of this arc along with the bicollar of Σ we see that T^n is represented as a connected sum $M \# S^1 \times S^{n-1}$, indicating that $\pi_1(T^n) \cong \pi_1(M) * \mathbb{Z}$ for some closed n-manifold M. Since $n \geq 3$, and an abelian group cannot be represented as a nontrivial free product, we have a contradiction.

We are now ready to see our first application of the torus trick. We begin with a sketch, and encourage the reader to consult Fig. 33, which summarises all the steps.

Sketch of the proof of Theorem 10.4. We are given a homeomorphism $h\colon \mathbb{R}^n \to \mathbb{R}^n$ which is ε-close to $\mathrm{Id}_{\mathbb{R}^n}$ on the unit disc, for some ε we will need to choose with care.

We will first use Corollary 10.6 to define an immersion $\alpha\colon T^n \setminus 2\mathring{D}^n \subseteq T^n \setminus \mathring{D}^n \looparrowright \mathbb{R}^n$, where $D^n \subseteq 2D^n \subseteq T^n$ are some carefully chosen discs. Next we will define another embedding $\widehat{h}\colon T^n \setminus 2\mathring{D}^n \hookrightarrow T^n \setminus \mathring{D}^n$ such that the lower square of the following diagram commutes.

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, hook, two heads, "\widetilde{h}"] \arrow[d, "e"'] & \mathbb{R}^n \arrow[d, "e"] \\
T^n \arrow[r, hook, two heads, "\overline{h}", "\cong"'] & T^n \\
T^n \setminus 2\mathring{D}^n \arrow[u, hook] \arrow[r, hook, "\widehat{h}", "\text{emb.}"'] \arrow[d, "\alpha"] & T^n \setminus \mathring{D}^n \arrow[u, hook] \arrow[d, "\alpha"'] \\
\mathbb{R}^n \arrow[r, hook, two heads, "h", "\cong"'] & \mathbb{R}^n
\end{tikzcd} \tag{10.9}$$

We will then use the Schoenflies theorem on the torus (Proposition 10.7) to lift $\widehat{h}$ to a homeomorphism $\overline{h}\colon T^n \to T^n$. Let us warn the reader that the middle square in the diagram does not quite commute – see the full proof for details. In order to have that $\widetilde{h}$ *is isotopic to* h, we will ensure in each of the these steps that $\widetilde{h}$ and h agree on an open set, and then use Proposition (10.3.v).

The final step consists of showing that $\overline{h}$ *is isotopic to* $\mathrm{Id}_{\mathbb{R}^n}$. As before, the choice of ε will be important here. Since $\overline{h}$ is homotopic to the identity, the induced homeomorphism $\widetilde{h}\colon \mathbb{R}^n \to \mathbb{R}^n$ is bounded distance from Id by Lemma 10.5, and is consequently isotopic to the identity (Proposition 10.2). Therefore, h is also isotopic to the identity, as desired. $\square$

With the sketch out of the way, here are the details in the proof of Theorem 10.4.

Proof of Theorem 10.4. Let us identify S^1 with ${}^{[0,1]}\!\big/_{0\sim 1}$, so that $[0, \frac{1}{2}]$ is viewed as a subset of S^1, and we have the closed ball

$$B := [0, \frac{1}{2}]^n \subseteq T^n \setminus \{\mathrm{pt}\}$$

for a suitably chosen point $\mathrm{pt} \in T^n$. Moreover, we choose closed concentric balls $A \subsetneq 2A \subsetneq 3A$ centred at $\mathrm{pt} \in T^n$ and disjoint from B. Abusing notation we also write $B := [0, \frac{1}{2}]^n \subseteq \mathbb{R}^n$. The proof consists of building the maps in the following diagram.

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "\widetilde{h}"] \arrow[d, "e"] & \mathbb{R}^n \arrow[d, "e"] \\
T^n \arrow[r, "\overline{h}", "\cong"'] & T^n \\
T^n \setminus 3\mathring{A} \arrow[u, hook] \arrow[r, "\widehat{h}|"] \arrow[d, hook] & \widehat{h}(T^n \setminus 3\mathring{A}) \arrow[u, hook] \arrow[d, hook] \\
T^n \setminus 2\mathring{A} \arrow[r, "\widehat{h}"] \arrow[d, "\alpha"] & T^n \setminus \mathring{A} \arrow[d, "\alpha"] \\
\mathbb{R}^n \arrow[r, "h", "\cong"'] & \mathbb{R}^n
\end{tikzcd} \tag{10.10}$$

Our original homeomorphism h appears in the bottom row, and we start building the diagram from there upwards. We present the proof in a collection of steps and lemmas, so that the many details do not obscure the bigger picture and the structure of the proof is clear.

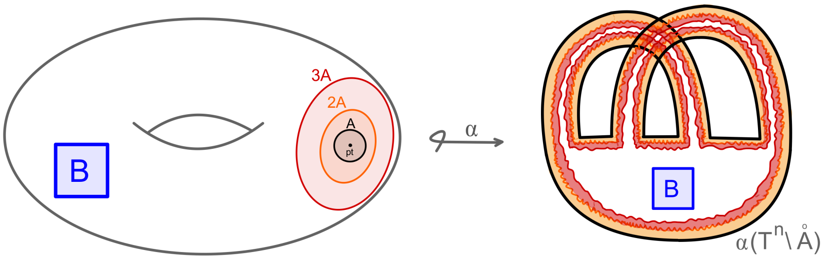


FIGURE 32. The key players in the proof of Theorem 10.4.

Step 1. *Construct an immersion* $\alpha\colon T^n \setminus \{\mathrm{pt}\} \looparrowright \mathbb{R}^n$ *such that* $\alpha(T^n \setminus 2\mathring{A}) \subseteq D^n$ *and* $\alpha|_B = \mathrm{Id}$.

We will obtain α by modifying an immersion $\beta\colon T^n \setminus \{\mathrm{pt}\} \looparrowright \mathbb{R}^n$ from Corollary 10.6. We begin with a smooth immersion, but the smoothness will not be important for the proof. Recall that an immersion is by definition a local embedding, and that β is an open map by invariance of domain. Therefore, we can find a closed ball $B' \subseteq B$ such that $\beta|_{B'}$ is a homeomorphism and $\beta(\partial B')$ is a bicollared $(n-1)$-sphere in $\mathbb{R}^n$.

We then choose homeomorphisms $j\colon T^n \setminus \{\mathrm{pt}\} \to T^n \setminus \{\mathrm{pt}\}$ and $k\colon \mathbb{R}^n \to \mathbb{R}^n$ such that j takes B' to B, and k takes $B'' := \beta(B')$ to B. In more detail, to construct j, we may choose B' to be an n-cube within B, so that j consists of a (cubical) contraction within B and the identity elsewhere. To construct k, we observe that B and B'' are homeomorphic, and such a homeomorphism may be extended to all of $\mathbb{R}^n$ by extending over the complements, which are punctured discs by the Schoenflies theorem.

Then the composition $k \circ \beta \circ j^{-1}\colon T^n \setminus \{\mathrm{pt}\} \looparrowright \mathbb{R}^n$ is still an immersion, which now takes B to itself. By modifying the construction above, we further assume that $\alpha|_B = \mathrm{Id}$. Compose this immersion with a radial squeeze R fixing B so that the resulting immersion

$$\alpha := R \circ k \circ \beta \circ j^{-1}\colon T^n \setminus \{\mathrm{pt}\} \looparrowright \mathbb{R}^n$$

has $\alpha(T^n \setminus 2\mathring{A}) \subseteq D^n$. Here we are using that $T^n \setminus 2\mathring{A}$ is compact and therefore has bounded image under $k \circ \beta \circ j^{-1}$.

Step 2. *Choose* $\varepsilon > 0$ *as required in the statement of the theorem.*

Let us denote $D_\theta(x) := \{y \mid d(x,y) < \theta\}$. We will choose ε in several steps.

(1) Choose $\varepsilon_1 > 0$ such that $\alpha|_{D_{2\varepsilon_1}(x)}$ is a homeomorphism for every $x \in T^n \setminus \mathring{A}$.

Namely, choose for every $x \in T^n \setminus \mathring{A}$ an open neighborhood $U_x \cong \mathbb{R}^n$ such that $\alpha|_{U_x}$ is a homeomorphism; this is an open cover of the compact space $T^n \setminus \mathring{A}$, so has a finite Lebesgue number $2\varepsilon_1$ (meaning that any $D_{2\varepsilon_1}(x)$ is contained in a member of the cover).

(2) Choose $\varepsilon_2 > 0$ so that $D_{\varepsilon_2}(\alpha(x)) \subseteq \alpha(D_{\varepsilon_1}(x))$ for every $x \in T^n \setminus \mathring{A}$.

Namely, consider the map

$$\begin{aligned} T^n \setminus \mathring{A} &\to \mathbb{R}_{>0} \\ x &\mapsto \varepsilon_x := d\big(\alpha(x), \mathbb{R}^n \setminus \alpha(D_{\varepsilon_1}(x))\big). \end{aligned}$$

Above ε_x is positive for each x since $D_{\varepsilon_1}(x)$ and hence $\alpha(D_{\varepsilon_1}(x))$ is open, so $\mathbb{R}^n \setminus \alpha(D_{\varepsilon_1}(x))$ is closed.

Lemma 10.11. *The above map is continuous.*

We defer the proof of the lemma to the end of this step. Since $T^n \setminus \mathring{A}$ is compact, we may choose $z \in T^n \setminus \mathring{A}$ that realises the minimum of the above function. In particular, this minimum is nonzero and we define $\varepsilon_2 := \varepsilon_z > 0$.

(3) Choose $\varepsilon_3 > 0$ with $\varepsilon_3 < \varepsilon_2$ and so that if $y \in \mathbb{R}^n$ satisfies $|y - \alpha(x)| < \varepsilon_3$ for some $x \in T^n \setminus 2\mathring{A}$, then $y \in \alpha(T^n \setminus \mathring{A})$.
This is achieved by taking
$$\varepsilon_3 < \min\left\{\varepsilon_2,\, d(\alpha(T^n \setminus 2\mathring{A}), \mathbb{R}^n \setminus \alpha(T^n \setminus \mathring{A}))\right\}.$$
Since $\alpha(T^n \setminus 2\mathring{A})$ is compact and $\mathbb{R}^n \setminus \alpha(T^n \setminus \mathring{A})$ is closed, their mutual distance is positive so $\varepsilon_3 > 0$.
(4) Finally, define the required $\varepsilon > 0$ by setting $\varepsilon := \frac{\varepsilon_3}{2}$. Observe that the only input in the definition of ε is the map α.

Proof of Lemma 10.11. Fix $\eta > 0$ and $x \in T^n \setminus \mathring{A}$. The map $\alpha|_{\overline{D_{2\varepsilon_1}(x)}}$ is uniformly continuous by the Heine-Cantor theorem since $\overline{D_{2\varepsilon_1}(x)}$ is compact. So there exists $\delta > 0$ such that $d(p,q) < \delta$ for $p, q \in D_{2\varepsilon_1}(x)$ implies that $d(\alpha(p), \alpha(q)) < \frac{\eta}{2}$. Assume that $0 < \delta < \varepsilon_1$.

Claim. *If $d(p,q) < \delta < \varepsilon_1$ for $p, q \in T^n \setminus \mathring{A}$ then for all $z \in \partial D_{\varepsilon_1}(p)$ there exists $z' \in \partial D_{\varepsilon_1}(q)$ so that $d(z, z') < \delta$.*

We defer the proof of the claim to the end of this step. Given $y \in T^n \setminus \mathring{A}$ with $d(x,y) < \delta$, we want to show $|\varepsilon_x - \varepsilon_y| < \eta$. Since $\varepsilon_x := d(\alpha(x), \mathbb{R}^n \setminus \alpha D_{\varepsilon_1}(x))$, there exists $z \in \partial D_{\varepsilon_1}(x)$ with $\varepsilon_x = d(\alpha(x), \alpha(z))$. Choose, using the subclaim, some $z' \in \partial D_{\varepsilon_1}(y)$ with $d(z, z') < \delta$. Then
$$\begin{aligned}\varepsilon_y := d(\alpha(y), \mathbb{R}^n \setminus \alpha D_{\varepsilon_1}(y)) &\leq d(\alpha(y), \alpha(z')) \\ &\leq d(\alpha(y), \alpha(x)) + d(\alpha(x), \alpha(z)) + d(\alpha(z), \alpha(z')) \\ &< \frac{\eta}{2} + \varepsilon_x + \frac{\eta}{2} = \eta + \varepsilon_x\end{aligned}$$
Here we have used the fact that $z, z' \in \overline{D_{2\varepsilon_1}(x)}$, since $z \in \partial D_{\varepsilon_1}(x)$ and $z' \in \partial D_{\varepsilon_1}(y)$, along with $d(x, z') \leq d(x,y) + d(y, z') < \delta + \varepsilon_1 < 2\varepsilon_1$.

A similar proof shows that $\varepsilon_x < \varepsilon_y + \eta$. This completes the proof of the lemma. □

Proof of the claim. Since $\delta < \varepsilon$, by the definition of ε_1 we know that $\alpha|_{D_{\varepsilon_1}(p) \cup D_{\varepsilon_1}(q)}$ is a homeomorphism onto its image, since $D_{\varepsilon_1}(p) \cup D_{\varepsilon_1}(q) \subseteq D_{2\varepsilon_1}(p)$. In the upcoming proof, we will therefore assume that we are working in $\mathbb{R}^n$.

In case $z \in \partial D_{\varepsilon_1}(q)$, we just choose $z' = z$.

The next possibility is that $z \in \mathring{D}_{\varepsilon_1}(q)$. Let z' be the intersection point of the ray starting at q and passing through z, with $\partial D_{\varepsilon_1}(q)$ (so $q < z < z'$). Then
$$d(z, p) = \varepsilon_1 \leq d(p,q) + d(q,z) < \delta + d(q,z)$$
so $\varepsilon_1 - \delta < d(q,z)$. Then
$$\varepsilon_1 - \delta + d(z, z') < d(q,z) + d(z,z') = d(q, z') = \varepsilon_1$$
so $d(z, z') < \delta$.

The final possibility is that $z \in \mathbb{R}^n \setminus \overline{D_{\varepsilon_1}(q)}$. Then let z' be the point of intersection of the ray from p to z, with $\partial D_{\varepsilon_1}(q)$. Then
$$\varepsilon_1 = d(z', q) \leq d(z', p) + d(p,q) < d(z', p) + \delta$$
so $\varepsilon_1 - \delta < d(z', p)$. So
$$\varepsilon_1 - \delta + d(z, z') < d(z', p) + d(z, z') = d(p, z) = \varepsilon_1$$
so $d(z, z') < \delta$, as needed. □

Step 3. *Define the embedding $\widehat{h}\colon T^n \setminus 2\mathring{A} \hookrightarrow T^n \setminus \mathring{A}$ that fits into the bottom square of the diagram in* (10.10).

Recall from the hypothesis that we are given a homeomorphism $h\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $|h(x) - x| < \varepsilon$ for every $x \in D^n$. Define

$$\begin{aligned} \widehat{h}\colon T^n \setminus 2\mathring{A} &\hookrightarrow T^n \setminus \mathring{A} \\ x &\mapsto \alpha|^{-1}_{B_{\varepsilon_1}(x)} \circ h \circ \alpha|_{B_{\varepsilon_1}(x)}(x). \end{aligned}$$

Claim. *The function $\widehat{h}$ is well-defined.*

Proof of claim. By definition of the immersion α, we know that $\alpha(T^n \setminus 2\mathring{A}) \subseteq D^n$. As a consequence, by our assumption on h, we know that for every $x \in T^n \setminus 2\mathring{A}$, we have $|h(\alpha(x)) - \alpha(x)| < \varepsilon < \varepsilon_3$. By definition of ε_3, this implies that $h(\alpha(x)) \in \alpha(T^n \setminus \mathring{A})$. But now since, $|h(\alpha(x)) - \alpha(x)| < \varepsilon_3 < \varepsilon_2$ and using the definition of ε_2, we deduce that $h(\alpha(x)) \in B_{\varepsilon_2}(\alpha(x)) \subseteq \alpha(B_{\varepsilon_1}(x))$. As, by construction, α is a homeomorphism on $B_{\varepsilon_1}(x)$, it makes sense to write $\alpha|^{-1}_{B_{\varepsilon_1}(x)}(h\alpha(x))$. □

Claim. *The function $\widehat{h}$ is continuous.*

Proof of claim. It suffices to prove that for all $x \in T^n \setminus 2\mathring{A}$, there exists an open neighborhood U of x such that $\widehat{h}|_U$ is continuous.

Fix $x \in T^n \setminus 2\mathring{A}$. Since α is continuous, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $d(\alpha(x), \alpha(y)) < \frac{\varepsilon_3}{2}$. Let $y \in U := B_{\varepsilon_1}(x) \cap B_\delta(x)$. Note that $B_{\varepsilon_1}(x) \cup B_{\varepsilon_1}(y) \subseteq B_{2\varepsilon_1}(x)$ and so $\alpha|_{B_{\varepsilon_1}(x) \cup B_{\varepsilon_1}(y)}$ is a homeomorphism.

We have that $h\alpha|_{B_{\varepsilon_1}(y)}(y) = h\alpha|_{B_{\varepsilon_1}(x)}(y) = h\alpha(y)$. Further

$$|h\alpha(y) - \alpha(x)| \leq |h\alpha(y) - \alpha(y)| + |\alpha(y) - \alpha(x)| \quad \leq \frac{\varepsilon_3}{2} + \frac{\varepsilon_3}{2} = \varepsilon_3 < \varepsilon_2,$$

so $h\alpha(y) \in B_{\varepsilon_2}(\alpha(x)) \subseteq \alpha B_{\varepsilon_1}(x)$. As before, we know that $h\alpha(y) \in \alpha B_{\varepsilon_1}(y)$.

By definition, $\widehat{h}(y) = \alpha|^{-1}_{B_{\varepsilon_1}(y)} h\alpha(y)$. Consider $y' := \alpha|^{-1}_{B_{\varepsilon_1}(x)} h\alpha(y)$. We assert that $y' = \widehat{h}(y)$ since $y' \in B_{\varepsilon_1}(x)$ with $\alpha(y') = h\alpha(y)$, and $\widehat{h}(y) \in B_{\varepsilon_1}(y)$ with $\alpha(\widehat{h}(y)) = h\alpha(y)$, where $\alpha|_{B_{\varepsilon_1}(x) \cup B_{\varepsilon_1}(y)}$ is a homeomorphism.

In other words, $\widehat{h}|_U = \alpha|^{-1}_{B_{\varepsilon_1}(x)} \circ h \circ \alpha|_U$ where the latter is continuous as a restriction of a continuous map. This completes the proof of the claim. □

Claim. *The function $\widehat{h}$ is an embedding.*

Proof of claim. First we prove that $\widehat{h}$ is injective. Assume by way of contradiction that $\widehat{h}(x) = \widehat{h}(y)$ for some $x \neq y$. Note that for every $z \in T^n \setminus 2\mathring{A}$, we have $\widehat{h}(z) \in B_{\varepsilon_1}(z)$. In particular, we have $d(\widehat{h}(x), x) < \varepsilon_1$ and $d(\widehat{h}(y), y) < \varepsilon_1$, or put differently, $x, y \in B_{\varepsilon_1}(\widehat{h}(x))$ because we assumed that $\widehat{h}(x) = \widehat{h}(y)$. Since $\alpha|_{B_{\varepsilon_1}(\widehat{h}(x))}$ is a homeomorphism (by definition of ε_1) and $x \neq y$, we deduce that $\alpha(x) \neq \alpha(y)$. Since h is a homeomorphism, this implies that $h(\alpha(x)) \neq h(\alpha(y))$. Using the definition of $\widehat{h}$, this can be written as $\alpha(\widehat{h}(x)) \neq \alpha(\widehat{h}(y))$. This contradicts the fact that $\widehat{h}(x) = \widehat{h}(y)$, and therefore shows that $\widehat{h}$ is injective.

As a continuous, injective map from a compact space to a Hausdorff space $\widehat{h}$ is further a closed map, and therefore by the closed map lemma it is an embedding. □

Finally, we note that $\widehat{h}$ and h agree on $\widetilde{B} := [\varepsilon_3, \frac{1}{2} - \varepsilon_3]^n \subseteq B \subseteq T^n \setminus 2\mathring{A}$. To see this observe that α is fixed on B, and thus for $x \in \widetilde{B}$, we have that $h(x) \in B$ since $|h\alpha(x) - \alpha(x)| = |h(x) - x| < \frac{\varepsilon_3}{2}$. Since α is fixed on B, $\alpha|_B(h(x)) = h(x)$ so $\widehat{h}(x) = \alpha|^{-1}_B(h(x)) = h(x)$ for $x \in \widetilde{B}$.

Step 4. *Extend the embedding $\widehat{h}\colon T^n \setminus 2\mathring{A} \hookrightarrow T^n \setminus \mathring{A}$ to a homeomorphism $\overline{h}\colon T^n \xrightarrow{\cong} T^n$, as in the middle two squares of the diagram in* (10.10)*.*

Note that $\widehat{h}(\partial 3A)$ is a bicollared $(n-1)$-sphere in T^n. By Schoenflies theorem for the n-torus for $n \geq 3$ (Proposition 10.7), this sphere bounds an embedded ball in T^n; since $\widehat{h}(T^n \setminus 3\mathring{A})$ is clearly not a ball, the other component of $T^n \setminus \widehat{h}(\partial 3A)$, call it C, must be homeomorphic to a ball. We can now use the Alexander coning trick (Proposition 10.3.0) to extend the homeomorphism $\widehat{h}|_{T^n\setminus 3\mathring{A}}$ of S^{n-1} to a homeomorphism $\overline{h}\colon T^n \xrightarrow{\cong} T^n$ of D^n, as required (that is, over $3A$ in the domain and C in the codomain). We leave it to the reader to consider the cases $n \leq 2$.

Step 5. *Show that $\overline{h}$ is isotopic to the identity Id_{T^n}.*

Since the universal cover of T^n is contractible, $\pi_i(T^n) = 0$ for $i > 1$ and thus T^n is a $K(\mathbb{Z}^n, 1)$. Now, homotopy classes of maps between Eilenberg-MacLane spaces correspond to the induced maps on the homotopy groups. Since $\overline{h}$ may not preserve basepoints, we must consider the induced map on the outer automorphism group of the fundamental group (since changing the basepoint corresponds to an inner automorphism). Now, as $\pi_1(T^n)$ is abelian, it suffices to show that $\overline{h}$ is homotopic to Id_{T^n} it suffices to prove that $\overline{h}$ preserves free homotopy classes of loops.

To this end, consider a copy γ of $S^1 \times \{*\} \times \dots \{*\} \subseteq T^n \setminus 3\mathring{A}$ and let us show that $\overline{h}(\gamma)$ is freely homotopic to γ. Since we have $\alpha(\overline{h}(\gamma)) = h(\alpha(\gamma))$ it will suffice to check the following.

Claim. *There is a homotopy $\Gamma\colon S^1 \times [0,1] \to \mathbb{R}^n$ from $\Gamma_0 = h(\alpha(\gamma))$ to $\Gamma_1 = \alpha(\gamma)$ such that Γ_t is at most distance ε for all $t \in [0,1]$.*

Indeed, such a homotopy can be lifted to a free homotopy from $\widehat{h}(\gamma)$ (and thus also from $\overline{h}(\gamma)$) to γ, as desired.

Proof of claim. For all $y \in S^1$ we have $d(h\alpha(y), \alpha(\gamma)) \leq d(h\alpha(y), \alpha(y)) < \varepsilon$, for our chosen constant $\varepsilon := \frac{\varepsilon_3}{2}$ from Step 2. Therefore, $h(\alpha(\gamma)) \subseteq N_\varepsilon(\alpha(\gamma)) \subseteq \alpha(T^n \setminus \mathring{A})$. We define Γ as the straight line homotopy

$$(y,t) \;\mapsto\; th(\alpha(y)) + (1-t)\alpha(y),$$

and observe that $\Gamma_t \subseteq N_\varepsilon(\alpha(\gamma)) \subseteq \alpha(T^n \setminus \mathring{A})$ for all $t \in [0,1]$. Indeed, for all $y \in \gamma$ we have

$$d(F_t(y), \alpha(\gamma)) \leq d(F_t(y), \alpha(y)) = t|h\alpha(y) - \alpha(y)| < \varepsilon. \qquad \square$$

Step 6. *Conclude the proof.*

Define $\widetilde{h}\colon \mathbb{R}^n \to \mathbb{R}^n$ to be the map on universal covering spaces induced by $\overline{h}\colon T^n \to T^n$, also ensuring that $B \subseteq \mathbb{R}^n$ is mapped onto $B \subseteq T^n$ by the "identity". Indeed, recall that the universal covering map $e\colon \mathbb{R}^n \to T^n$ denotes the exponential map, so this is in a way automatic by our identification of S^1 with ${[0,1]}/{0 \sim 1}$.

Since $\overline{h}$ is isotopic to the identity, by Lemma 10.5 the induced homeomorphism $\widetilde{h}\colon \mathbb{R}^n \to \mathbb{R}^n$ on the universal covers is bounded distance from the identity. By Proposition 10.2 we deduce that $\widetilde{h}$ is isotopic to $\mathrm{Id}_{\mathbb{R}^n}$.

On the other hand, we claim that h and $\widetilde{h}$ agree on the ball

$$\widetilde{B} := [2\varepsilon, \frac{1}{2} - 2\varepsilon] \;\subseteq\; B := [0, \frac{1}{2}]^n \subseteq T^n \setminus 3\mathring{A}.$$

Indeed, let $x \in \widetilde{B}$. Then $h(x) \in B$ because $|h(x) - x| < \varepsilon$ and $\alpha(x) = x$ and $\alpha|_B(h(x)) = h(x)$, as α fixes B. Now the definition of $\widehat{h}$ now gives

$$\widehat{h}(x) = \alpha|_B^{-1} h\alpha(x) = \alpha|_B^{-1} h(x) = h(x).$$

implying also $\widetilde{h}(x) = h(x)$. Consequently, $\widetilde{h}$ and h are isotopic by Proposition 10.3.v, and so h is also isotopic to the identity. This concludes the proof of Černavskiĭ-Kirby Theorem 10.4. $\square$

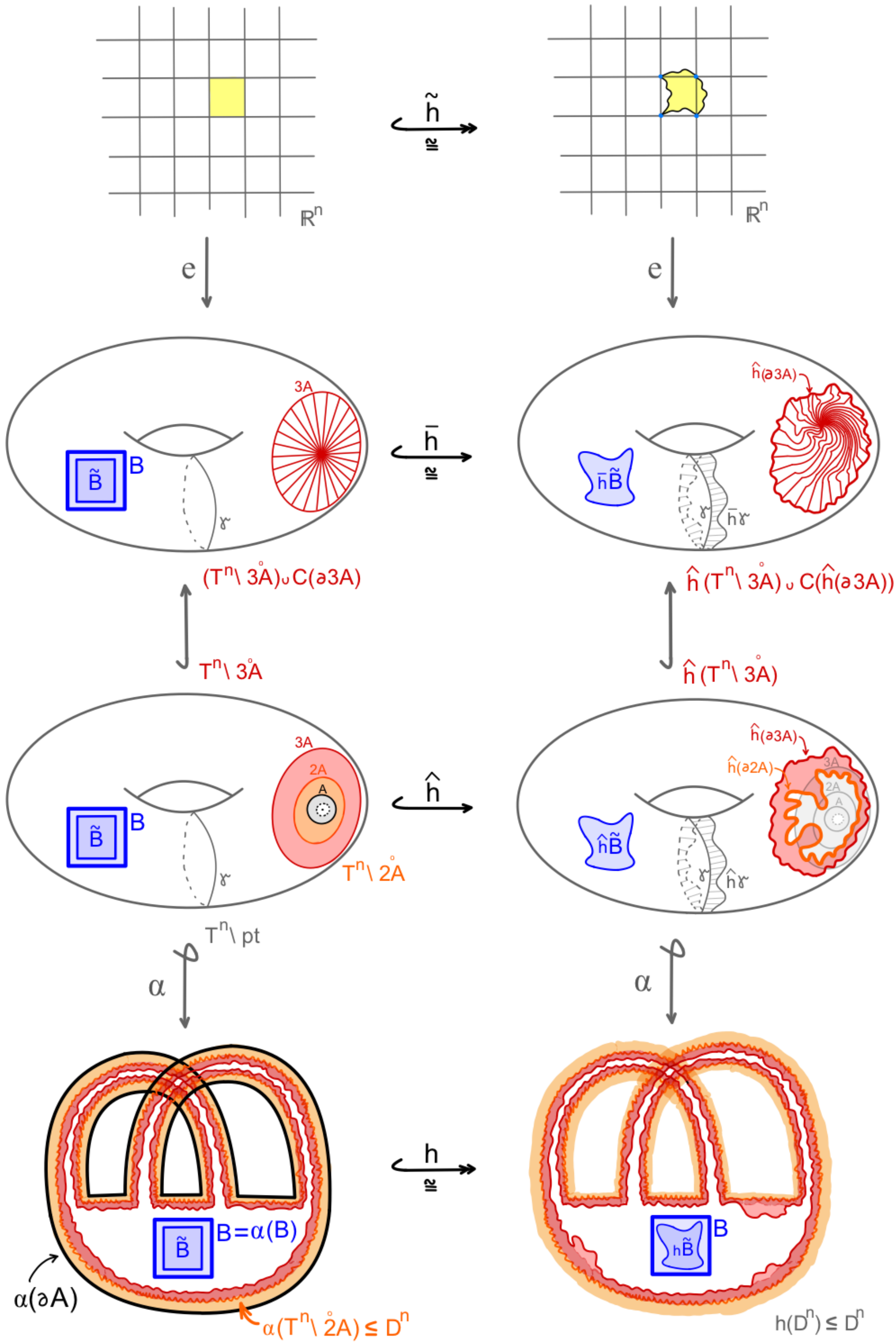


FIGURE 33. Recap of the proof of Theorem 10.4.

10.2.1 Recap of the torus trick. Since the proof in the previous section contained many details, we recap its salient features. See Fig. 33.

We began with an immersion α into $\mathbb{R}^n$ of the punctured torus $T^n \setminus \mathrm{pt}$, which has specified regions B and $A \subseteq 2A \subseteq 3A$. We chose ε so that the image of $T^n \setminus 2\mathring{A}$ under h lies within the unit disc D^n, for any $h \in \mathrm{Homeo}(\mathbb{R}^n)$ satisfying $|h(x) - x| < \varepsilon$ for all $x \in D^n$. This enabled us to define the lift $\widehat{h} \colon T^n \setminus 2\mathring{A} \to T^n \setminus \mathring{A}$, see the middle row in Fig. 33.

Then by the Schoenflies theorem and the Alexander trick we extended $\widehat{h}|_{T^n \setminus 3\mathring{A}}$ to a homeomorphism of the whole torus, $\overline{h} \colon T^n \to T^n$. We checked that $\overline{h}$ is homotopic to Id_{T^n}, as T^n is a $K(\mathbb{Z}^n, 1)$ and, roughly speaking, h does not move generators of $\pi_1(T^n)$ too much.

Finally, $\overline{h}$ induces a homeomorphism $\widetilde{h}$ of $\mathbb{R}^n$ which only moves fundamental domains by a small amount, so it is bounded distance from the identity, and therefore is isotopic to the $\mathrm{Id}_{\mathbb{R}^n}$. On the other hand, we arranged that $\widetilde{h}$ and h agree on an open subset of B, so by an Alexander isotopy h and $\widetilde{h}$ are isotopic. Therefore, we conclude that h is isotopic to $\mathrm{Id}_{\mathbb{R}^n}$, as desired.

10.3 Local contractibility

As mentioned at the beginning of this section, the key use of Theorem 10.4 is in proving that $\mathrm{Homeo}(\mathbb{R}^n)$ is locally contractible.

Definition 10.12. A space X is *locally contractible* at $x \in X$ if for every neighbourhood $U \ni x$ there is a neighbourhood $x \in V \subseteq U$ and a map $H \colon V \times [0,1] \to U$ such that $H(y, 0) = y$ and $H(y, 1) = x$ for all $y \in V$. We say X is *locally contractible* if the previous is true at every $x \in X$.

Corollary 10.13 ([Č73],[Kir69])**.** $\mathrm{Homeo}(\mathbb{R}^n)$ *is locally contractible.*

Proof. For $\varepsilon, \delta > 0$, let D^n_δ be the closed disc of radius δ at the origin and define

$$V(D^n_\delta, \varepsilon) := \{f \in \mathrm{Homeo}(\mathbb{R}^n) \mid d(f(x), x) < \varepsilon \text{ for every } x \in D^n_\delta\}.$$

This is a neighbourhood of $\mathrm{Id}_{\mathbb{R}^n} \in \mathrm{Homeo}(\mathbb{R}^n)$ under compact open topology – actually, such sets comprise a basis for the compact open topology on $\mathcal{C}(M, N)$, see Exercise 10.2 below.

Given $U \ni x$ choose ε, δ such that $V(D^n_\delta, \varepsilon) \subseteq U$. Our goal is to produce a homotopy

$$H \colon V(D^n_\delta, \varepsilon) \times [0,1] \to V(D^n_\delta, \varepsilon) \subseteq U$$

such that $H_0 = \mathrm{Id}$ and $H_1 = \{\mathrm{Id}_{\mathbb{R}^n}\}$. In other words, for $h \in V(D^n_\delta, \varepsilon)$ the path $H_t(h)$ is an isotopy from h to $\mathrm{Id}_{\mathbb{R}^n}$. Note that Theorem 10.4 provides such paths, but it remains to see that they glue together into a continuous map H, i.e. we need to make sure that all constructions in the proof of Theorem 10.4 were *canonical* in terms of h.

This can be done, see ? for details. In particular, the application of Schoenflies theorem in Step 4 is also canonical, meaning that the map

$$\mathrm{Emb}^{bicoll}(S^{n-1}, S^n) \to \mathrm{Emb}(D^{n-1}, S^n)$$

given by the Schoenflies theorem is continuous. For Brown's proof of that theorem, this was shown carefully by Gauld [Gau71].

Therefore, $\mathrm{Homeo}(\mathbb{R}^n)$ is locally contractible at $\mathrm{Id}_{\mathbb{R}^n}$. The rest of the proof is completed by the next exercise. ☐

Sol. on p.145. **Exercise 10.1**. (PS7.2) Let M be a manifold. Show that $\mathrm{Homeo}(M)$ is locally contractible at each $f \in \mathrm{Homeo}(M)$ if and only if it is locally contractible at Id_M.

Sol. on p.146. **Exercise 10.2**. (PS7.1) Let M and N be manifolds. Let d be a metric on N. Show that the collection of sets of the form

$$W(f, K, \varepsilon) := \{f \in \mathcal{C}(M, N) \mid d(f(x), g(x)) < \varepsilon \text{ for all } x \in K\}$$

where $K \subseteq M$ is compact and $\varepsilon > 0$ is a basis for the compact open topology on $\mathcal{C}(M,N)$.

Let us point out that $\mathrm{Homeo}(\mathbb{R}^n)$ is not globally contractible.

Sol. on p.146. **Exercise 10.3**. (PS6.2)

(i) The space of homeomorphisms of $\mathbb{R}^2$ is not contractible.
(ii) The space of orientation preserving homeomorphisms of $\mathbb{R}^2$ is not contractible.

Hint: recall from Corollary 8.5 that $\mathrm{ev}\colon M \times \mathrm{Homeo}(M) \to M$, $\mathrm{ev}(x,f) = f(x)$ is continuous, and from Lemma 9.12 that $\mathrm{Homeo}_0(\mathbb{R}^2) \hookrightarrow \mathrm{Homeo}(\mathbb{R}^2)$ is a homotopy equivalence; then construct a loop of homeomorphisms that does not contract to a point.

Remark 10.14. Kneser [Kne26] (see also [Fri73]) showed that $\mathrm{Homeo}(\mathbb{R}^2) \simeq O(2)$. Further, we know that $O(2) \cong S^1 \sqcup S^1$.

Later in the course we will sketch the proof that $\mathrm{Homeo}(M)$ for every compact manifold M is locally contractible [Č73, EK71]. However, the corresponding fact for noncompact manifolds is not true, as demonstrated by the following exercise.

Sol. on p.146. **Exercise 10.4**. (PS7.3) For $i \in \mathbb{N}$, let B_i denote the ball of radius $\frac{1}{3}$ centred at $(i,0) \in \mathbb{R}^2$. Define $M := \mathbb{R}^2 \setminus \bigcup_i B_i$.

Let $h_i \in \mathrm{Homeo}(M)$ be a homeomorphism which is the identity outside the disc of radius 1 centred at $(i+\frac{1}{2},0)$, and which maps B_i to B_{i+1} and vice versa. Why does such a homeomorphism exist?

Show that h_i is not homotopic to the identity for any i, but $\{h_i\}$ converges to the identity in the compact open topology on $\mathrm{Homeo}(M)$.

Conclude that $\mathrm{Homeo}(M)$ is not locally contractible, nor locally path connected.

11 Piecewise linear manifolds

In the next section we will state and prove the stable homeomorphism theorem and the annulus theorem. One remarkable aspect of these proofs is that they require the use of piecewise-linear (PL) structures, as well as some deep theorems from the theory of PL manifolds.

11.1 Definitions

In this section we introduce PL manifolds. Similar to how we define smooth structures on manifolds, we first establish a notion of piecewise-linear maps between subsets of Euclidean space (with its standard structure).

Definition 11.1. An r-*simplex* in $\mathbb{R}^n$ is the convex hull of $r+1$ linearly independent points. Let $K \subseteq \mathbb{R}^n$ be a compact subset. An injective map $f\colon K \hookrightarrow \mathbb{R}^n$ is said to be *piecewise-linear* if K can be written as a finite union of simplices with each mapped affinely by f.

Next we apply Definition 11.1 to define piecewise-linear structures on general topological manifolds.

Definition 11.2. Let M be an n-manifold. A *piecewise-linear (PL) structure* on M is a family $\mathcal{F} = \{\phi\colon \Delta^n \hookrightarrow M \mid \Delta^n \subseteq \mathbb{R}^n \text{ a standard simplex}\}$ such that

(1) every point $p \in M$ has a neighbourhood of the form $\phi(\Delta^n)$ for some $\phi \in \mathcal{F}$, called a PL chart;
(2) for $\phi, \psi \in \mathcal{F}$, the composition $\psi^{-1}\phi|\colon \phi^{-1}\psi(\Delta^n) \to \mathbb{R}^n$ is piecewise-linear;
(3) $\mathcal{F}$ is maximal with respect to the above two properties.

In the first item, by invariance of domain, if p is in the interior of M, then $p \in \phi(\mathring{\Delta}^n)$, while if $p \in \partial M$, then $p \in \phi(\partial\Delta^n)$.

Definition 11.3. For $m \leq n$, let M^m and N^n be topological manifolds with PL structures $\mathcal{F}$ and $\mathcal{G}$ respectively. An embedding $h\colon M \hookrightarrow N$ is said to be *piecewise-linear* if for all $p \in M$, there exists a PL chart $\phi\colon \Delta^m \hookrightarrow M$ with $p \in \phi(\Delta^m)$ and $\psi\colon \Delta^n \hookrightarrow N$ with $h(p) \in \psi(\Delta^n)$ such that

$$\psi^{-1}h\phi|\colon \phi^{-1}(h^{-1}(\psi(\Delta^n))) \to \mathbb{R}^m$$

is PL in the sense of Definition 11.1. Here note that $\phi^{-1}(h^{-1}(\psi(\Delta^n))) \subseteq \mathbb{R}^m$ is compact.

For $m = n$, the above definition says that $h\colon M \hookrightarrow N$ is a *piecewise-linear embedding* if whenever $\phi \in \mathcal{F}$, we have that $h\phi \in \mathcal{G}$.

A homeomorphism $h\colon M \to N$ is said to be a *PL-homeomorphism* if h is a PL embedding. This implies that h^{-1} is a PL embedding. For a proof of this, see Hudson [Hud69].

Here are some properties of PL manifolds. For a vertex v we define the *star* $\mathrm{St}(v)$ as the union of all simplices which have v as a vertex, and the *link* $\mathrm{Lk}(v)$ as all the faces of $\mathrm{St}(v)$ not containing v.

(1) A compact n-manifold M has a PL-structure if and only if M has a triangulation such that the link of every vertex v is equivalent to a PL sphere S^{n-1} (if $v \in \operatorname{Int} M$) or a PL disc D^{n-1} (if $v \in \partial M$). Here equivalent means that there exists a subdivision such that the result is simplicially homeomorphic. This is due to Dedecker [Ded62] and also appears in Hudson's book [Hud69].
(2) The Cairns-Whitehead theorem says that every smooth manifold has a PL structure, unique up to PL homeomorphism. Further, every diffeomorphism of smooth manifolds determines a PL homeomorphism of the corresponding PL manifolds.
(3) The compositions of PL embeddings are PL. This implies that PL-homeomorphism is an equivalence relation.
(4) A PL structure $\mathcal{F}$ on M induces a PL structure $\partial\mathcal{F}$ on ∂M.
(5) Two PL manifolds with PL homeomorphic boundaries glue together to give a PL manifold.

11.2 Theorems from PL topology

We will need to make use of the following deep theorems on PL manifolds. We will not be going into the proofs at this stage.

Theorem 11.4 (PL Poincaré conjecture)**.** *Let $n \geq 5$. If M^n is a closed PL manifold homotopy equivalent to S^n, then M is PL-homeomorphic to S^n.*

The PL Poincaré conjecture for dimensions at least 5 is due to Smale. Initially there was a category losing version, i.e. PL input, topological output, due to Stallings. Stallings also excluded dimensions 5 and 6. But these defects were soon rectified. Zeeman extended Stallings' techniques to dimension 6, but dimension 5 came from Smale, at the same time as he proved the stronger PL input, PL output version in all dimensions at least five. Smale also proved the smooth input, PL output version.

The purely topological Poincaré conjecture, with topological input and output, in all dimensions at least five, is due to Newman. His proof used engulfing, as did Stallings and Zeeman's initial PL proofs. Kirby-Siebenmann's technology gave an alternative proof of the purely topological version in dimension at least 6.

Theorem 11.5 (Structures on $S^n \times \mathbb{R}$)**.** *Let $n \geq 4$. There is a unique PL structure on $S^n \times \mathbb{R}$. That is, if M is a PL manifold homeomorphic to $S^n \times \mathbb{R}$, then M is PL homeomorphic to $S^n \times \mathbb{R}$.*

This is due to Browder [Bro65] for $n \geq 5$ and to Wall [Wal67] for $n = 4$. The proofs use Siebenmann's thesis [Sie65], results of Wall [Wal64], and Stallings [Sta62], and notably the *PL* Poincaré conjecture mentioned above.

The last deep theorem we will need from PL topology is due to Hsiang-Shaneson [HS69] and Wall [Wal69].

Theorem 11.6 (Homotopy tori)**.** *Let $n \geq 5$. Let M^n be a closed PL manifold, and let $f \colon T^n \to M$ be a homotopy equivalence. Then there is a finite cover of both such that the lift $\widetilde{f} \colon \widetilde{T}^n \to \widetilde{M}$ is homotopic to a PL homeomorphism.*

The finite cover of the torus in the domain is also PL-homeomorphic to the torus. We will explain this theorem in a later chapter.

11.3 Handle decompositions

We will soon need the notion of handle decompositions of manifolds. An n-dimensional index k handle is a copy of $B^k \times B^{m-k}$, and its *attaching region* is the part $\partial B^k \times B^{m-k}$ of its boundary, see Fig. 34. The *core* of a handle is $B^k \times \{0\}$. Given a manifold M^m and a topological embedding $\psi \colon \partial B^k \times B^{m-k} \hookrightarrow \partial M$ we consider $M \cup_\psi (B^k \times B^{m-k})$, the manifold obtained by attaching a k-handle to M along ψ.

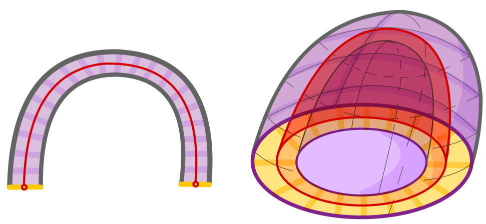


FIGURE 34. The 2-dimensional 1-handle $B^1 \times B^1$ attaches along the yellow $S^0 \times B^1$, and the 3-dimensional 2-handle $B^2 \times B^1$ along the yellow $S^1 \times B^1$. Their cores are shown in red.

A (topological) handle decomposition of a manifold M is a decomposition $M = \bigcup h_i^{k_i}$ into union of handles attached along their attaching regions via topological embeddings as described above. It is said to be *piecewise linear* or *smooth* if the attaching maps are PL or smooth embeddings respectively. In the latter case, we must also smooth the corners to obtain a smooth manifold after attaching a handle, but this can be done in an essentially unique way.

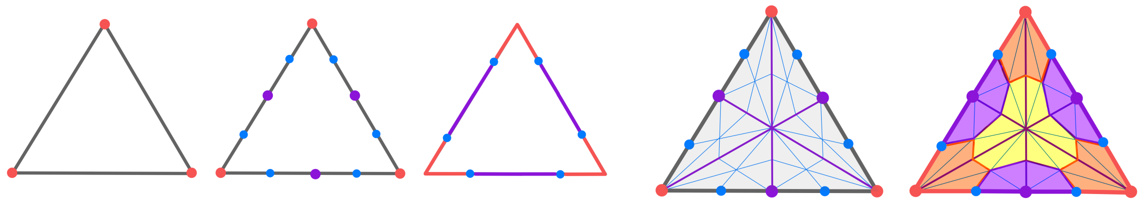
Remark 11.7. Every closed topological manifold has a topological handle decomposition, unless it is non-smoothable and has dimension $m = 4$. For $m > 6$ we will show how to do this later, following [KS77, Essay III]. For $m = 5$, this is due to Quinn [Qui82]. Smooth manifolds have smooth handle decompositions. This suffices for the existence of handle decompositions in dimension ≤ 3 and for smooth 4-manifolds. To see that nonsmoothable 4-manifolds do not have topological handle decompositions, observe that the handle attaching maps are all 3-dimensional and can be isotoped to be smooth embeddings. Consequently a topological handle decomposition would yield a smooth handle decomposition, and thereby a contradiction.

There are also relative handle decompositions, but we will not go into this for the moment.

Triangulations yield handle decompositions. Explicitly, for a k-simplex σ in a triangulation T of a manifold M, we obtain a handle of index k given by

$$\mathrm{St}(\widehat{\sigma}) \subseteq T''$$

where T'' is the second barycentric subdivision of T, $\widehat{\sigma}$ is the barycentre of σ, and St denotes the star. See Fig. 35 for an example and [Hud69, p. 233] for further details.



(A) Circle

(B) 2-simplex

FIGURE 35. Construction of a handlebody decomposition from a triangulation. 0-handles are coloured orange, 1-handles are purple, and the 2-handle is yellow.

12 Stable homeomorphisms and the Annulus Theorem

We now turn our attention to the following fundamental result.

Theorem 12.1 (Annulus Theorem (AC_n))**.** *If $h\colon D^n \hookrightarrow \operatorname{Int} D^n \subseteq D^n$ is a locally collared embedding, then*

$$D^n \setminus h(\operatorname{Int} D^n) \cong S^{n-1} \times [0,1].$$

As before, by Brown's theorem (Corollary 6.6), locally bicollared codimension one embeddings are globally bicollared, so nothing is lost by considering collared embeddings of D^n in $\operatorname{Int} D^n$, see Fig. 36. Note that this is not true if we omit locally bicollared condition – a counterexample is the Alexander gored ball mentioned in Remark 6.7.

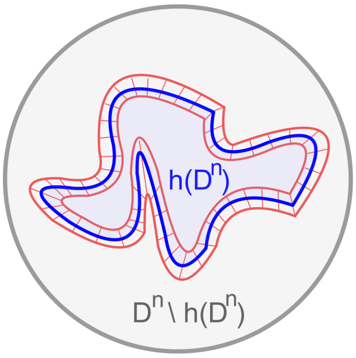


FIGURE 36. The Annulus Theorem asserts that $D^n \setminus h(\operatorname{Int} D^n)$, the closed grey region in the picture, is homeomorphic to an annulus.

For the smooth and PL versions of this theorem see ??. For $n = 2, 3$ the above result follows from the classical fact that surfaces and 3-manifolds have canonical triangulations/smoothings, as shown by Radó [Rad24] and Moise [Moi52] respectively. Kirby [Kir69] proved the case $n \geq 5$ using the torus trick, and we will explain this proof shortly. The case $n = 4$ is due to Quinn [Qui82], and uses very different techniques.

After the Schoenflies problem, which shows that a locally bicollared codimension one sphere Σ in S^n separates S^n into two balls, the following problem is a natural extension.

Question 12.2. Let Σ, Σ' be locally bicollared disjoint codimension one spheres in S^n. By the Jordan Brouwer separation theorem (Corollary 3.9), the space $S^n \setminus (\Sigma \cup \Sigma')$ has three components, two of which are homeomorphic to an open ball by the Schoenflies theorem. Is the third region, i.e. the the region "between" Σ and Σ', homeomorphic to an annulus?

Using the Schoenflies theorem (twice), we can see that this question is indeed equivalent to the annulus problem.

Let us extend the given bicollared embedding $h\colon D^n \hookrightarrow \operatorname{Int} D^n \subseteq D^n$ to a homeomorphism $h\colon \mathbb{R}^n \to \mathbb{R}^n$ which agrees with it on $D^n \subseteq \mathbb{R}^n$. Namely, we may include the codomain in $\mathbb{R}^n$ to get $h\colon D^n \hookrightarrow \mathbb{R}^n$, and then extend to a homeomorphism $h\colon \mathbb{R}^n \to \mathbb{R}^n$ using the Schoenflies theorem and the Alexander trick (Proposition 10.3.0). More specifically, the Schoenflies theorem implies that the complement of $h(D^n)$ in $\mathbb{R}^n$ is a punctured disc, so we extend h over $\mathbb{R}^n \setminus \operatorname{Int} D^n$, seen as the unit disc minus the center, by coning off $h|_{\partial D^n}$ and forgetting the cone points.

Lemma 12.3. *For $h \in \operatorname{Homeo}(\mathbb{R}^n)$ with $h(D^n) \subseteq \operatorname{Int} D^n$ we have that $D^n \setminus h(\operatorname{Int} D^n)$ is homeomorphic to an annulus if and only if for some $K \geq 1$ we have $KD^n \setminus h(\operatorname{Int} D^n)$ is homeomorphic to an annulus, where KD^n is the closed disc of radius K.*

Proof. Adding $KD^n \setminus D^n$ to $D^n \setminus h(\operatorname{Int} D^n)$ just adds a collar to the boundary of the latter manifold. The following remark shows that this cannot change its homeomorphism type. □

Remark 12.4. Adding or subtracting a boundary collar does not change the homeomorphism type of manifolds with boundary. More precisely, if M is a manifold with boundary and

$$M' := M \underset{\substack{m \mapsto (m,0) \\ m \in \partial_1 M}}{\cup} (\partial_1 M) \times [0,1].$$

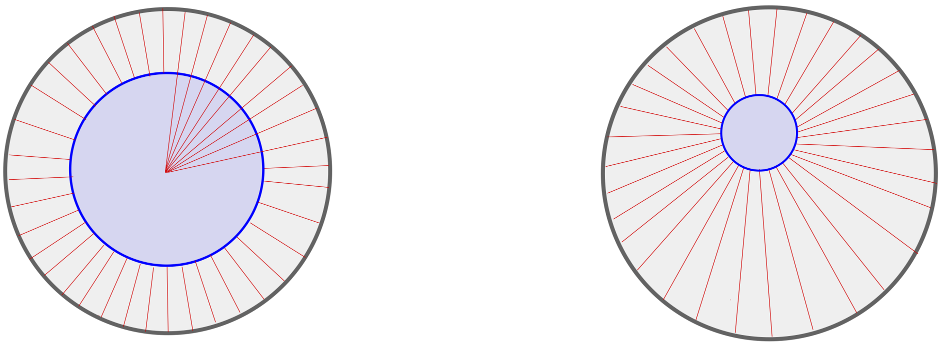
where $\partial_1 M \subseteq \partial M$ is a component of the boundary of M, then $M' \cong M$. This follows from the fact that manifold boundaries have collars (Theorem 6.5).

Conversely, if M' is a manifold with boundary with a collar $\phi\colon \partial_1 M' \times [0,1] \hookrightarrow M'$ and $M := M' \setminus \phi(\partial_1 M' \times [0,1))$, where $\partial_1 M' \subseteq \partial M'$ is a component of the boundary of M', then assuming that M is a manifold with boundary we have $M' \cong M$. This can be seen similarly to the previous paragraph, since M' is the result of adding a collar to M. It is imperative that M be a manifold for this assertion to be true. For a counterexample, see the discussion of the Alexander gored ball from Remark 6.7.

Definition 12.5. Given $h \in \mathrm{Homeo}(\mathbb{R}^n)$ we say that AC_n *holds for* h if $KD^n \setminus h(\mathrm{Int}\, D^n)$ is an annulus for some $K > 0$.

From the preceding discussion we see that AC_n would be true if AC_n holds for each $h\colon \mathbb{R}^n \to \mathbb{R}^n$ satisfying $h(D^n) \subseteq \mathrm{Int}\, D^n$.

Before describing our proof strategy, we discuss some situations where we may directly spot an annulus in $\mathbb{R}^n$. Firstly, for $0 < r < R \in \mathbb{R}$, the region $\overline{B_R(0)} \setminus B_r(0) = \{(\theta, t) \mid \theta \in [0, 2\pi), t \in [r, R]\}$ is explicitly homeomorphic to an annulus using polar coordinates, see Fig. 37a. By translation the same is true for concentric round spheres centred at any point in $\mathbb{R}^n$. Similarly, the region between any two nested round spheres as in Fig. 37b is an annulus. The subtlety in the annulus problem is that the 'inner' sphere is not necessarily round. Since topological embeddings, even bicollared ones, can be quite complicated, it is no longer obvious how to find the coordinates to see that the region between the two spheres is an annulus.



(A) Concentric disks.

(B) Nested disks.

FIGURE 37. Examples of annuli in $\mathbb{R}^n$ (for $n = 2$).

It is instructive to see how far the Schoenflies theorem can take us. Given a homeomorphim $h\colon \mathbb{R}^n \to \mathbb{R}^n$ we saw earlier that the complementary region $\mathbb{R}^n \setminus h(\mathrm{Int}\, D^n)$ is a punctured disc, namely an annulus (open at one end). By truncating, we find many many closed annuli with one of the two desired boundary components. But the second boundary component is not 'round'. Indeed our goal is to see that some $KD^n \setminus h(\mathrm{Int}\, D^n)$ is an annulus, so we precisely require the second boundary component to be round.

12.1 Stable homeomorphisms

Both Kirby's proof of AC_n for $n \geq 5$ and Quinn's for $n = 4$ proceed via proving the stable homeomorphism theorem and then using results of Brown and Gluck, as we now explain.

Definition 12.6. A homeomorphism h of $\mathbb{R}^n$ is *stable* if it can be written as a composition $h = h_k \circ \cdots \circ h_1$ of homeomorphisms $h_i \in \operatorname{Homeo}(\mathbb{R}^n)$ such that for all $1 \leq i \leq k$ there exists an open set $U_i \neq \emptyset$ with $h_i|_{U_i} = \operatorname{Id}_{U_i}$.

Remark 12.7. We do not need to restrict ourselves to $\mathbb{R}^n$ here. Given any homeomorphism $h\colon M \to M$ of a manifold M, we say h is stable if it can be written as a composition $h = h_k \circ \cdots \circ h_1$ of homeomorphisms $h_i \in \operatorname{Homeo}(M)$ such that for all $1 \leq i \leq k$ there exists an open set $U_i \neq \emptyset$ with $h_i|_{U_i} = \operatorname{Id}_{U_i}$. See [BG64b, Section 4] for more details. For now we focus on the case of homeomorphisms of $\mathbb{R}^n$, since those are most relevant to us.

It is a standard result that any orientation preserving diffeomorphism of $\mathbb{R}^n$ is stable, as well as any PL-homeomorphism (see Proposition 12.17). In contrast, the following is harder to prove.

Theorem 12.8 (Stable homeomorphism theorem (SH_n))**.** *Every orientation preserving homeomorphism of $\mathbb{R}^n$ is stable.*

As mentioned, this was proven by Kirby [Kir69] for $n \geq 5$, and by Quinn [Qui82] for $n = 4$. Stable homeomorphisms were defined and systematically studied by Brown and Gluck in a sequence of papers in 1964 [BG63, BG64c, BG64b, BG64a], explicitly as a means of attacking the Annulus Theorem 12.1 (then conjecture, AC_n). In particular, they establish the following key relationship.

Theorem 12.9. *For any $n \geq 1$ the following implications hold.*

$$SH_n \implies AC_n \tag{12.10}$$

$$\bigcup_{k \leq n} AC_k \implies SH_n \tag{12.11}$$

Proof of $SH_n \implies AC_n$. It will suffice to show that AC_n holds for every stable homeomorphism $h \in \operatorname{Homeo}(\mathbb{R}^n)$ (we are using the reformulation of AC_n from Lemma 12.3). First we consider the case when $h|_U = \operatorname{Id}$ for some open set U. The goal is to find $L > 0$ so that $LD^n \setminus h(\operatorname{Int} D^n)$ is (homemorphic to) an annulus.

We will choose L large enough so that $\operatorname{Int}(LD^n) \supseteq h(D^n)$ and $LD^n \cap U \neq \emptyset$. In order to do this, note that $h(D^n)$ is bounded, so it is contained in some large enough round ball centred at the origin. If U is also bounded, choose L large enough so that LD^n contains both $h(D^n)$ and U. Otherwise, if U is unbounded, choose a bounded subset of U and apply the same reasoning.

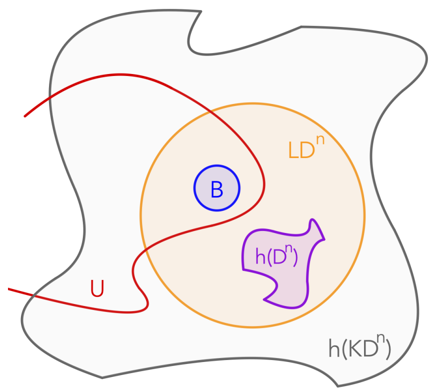


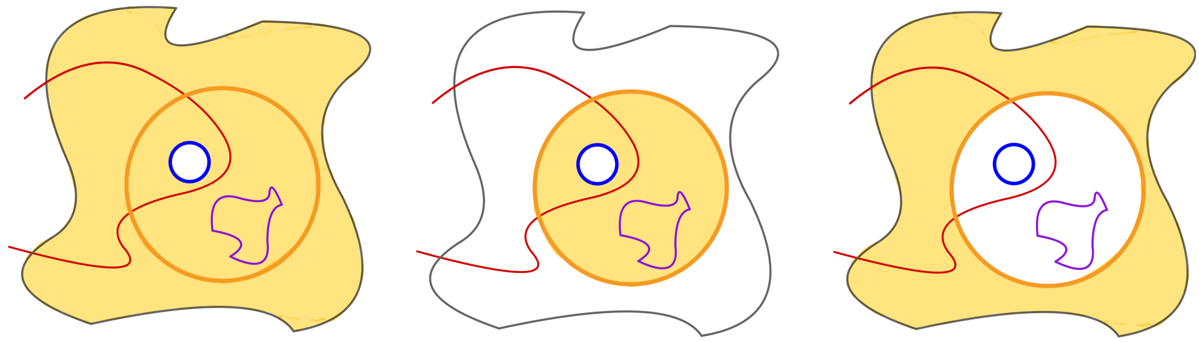
FIGURE 38. The setup in the proof of $SH_n \implies AC_n$ from Theorem 12.9.

Let us show that for this choice of L the space $LD^n \setminus h(\operatorname{Int} D^n)$ is an annulus. We will use to auxiliary disks B and $h(KD^n)$, see Figure 38. Namely, since $\operatorname{Int}(LD^n) \cap U$ is open we can pick $B \subseteq \operatorname{Int}(LD^n) \cap U$ a standard round closed ball in $\mathbb{R}^n$. Moreover, choose $K > 0$ large enough such that $\mathrm{Int}(h(KD^n)) \supseteq LD^n$. This is possible since LD^n is bounded and the sequence $\{h(iD^n)\}_{i\geq 1}$ is a compact exhaustion of $\mathbb{R}^n$.

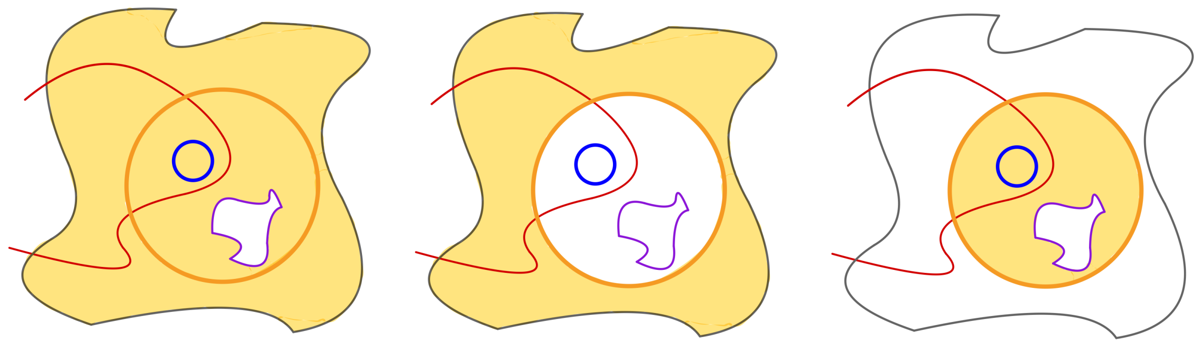
First let us show that $h(KD^n) \setminus \mathrm{Int}\, B$ is an annulus (yellow region in the first picture in Fig. 39a). We have $h(KD^n) \setminus \mathrm{Int}\, B = h(KD^n \setminus \mathrm{Int}\, B)$, since h is the identity on $U \supseteq B$ by hypothesis. Moreover, $KD^n \setminus \mathrm{Int}\, B$ is an annulus, being the region between two nested round spheres, and h is a homeomorphism, so $h(KD^n \setminus \mathrm{Int}\, B)$ is also an annulus.

Secondly, $LD^n \setminus \mathrm{Int}\, B$ is also an annulus, again as the region between two nested round spheres (yellow region in the second picture in Fig. 39a).

From this and Remark 12.4 it follows that $h(KD^n) \setminus \mathrm{Int}(LD^n)$ is an annulus, since it is a manifold with boundary obtained by subtracting a boundary collar, namely $LD^n \setminus \mathrm{Int}\, B$, from $h(KD^n) \setminus B$ (the first row of Fig. 39a). That $h(KD^n) \setminus \mathrm{Int}(LD^n)$ is a manifold with boundary follows from the fact that ∂LD^n is bicollared in $\mathrm{Int}(h(KD^n))$.



(A) As $h(KD^n) \setminus \mathrm{Int}\, B$ and $LD^n \setminus \mathrm{Int}\, B$ are annuli, their difference $h(KD^n) \setminus \mathrm{Int}(LD^n)$ is as well.



(B) As $h(KD^n) \setminus h(\mathrm{Int}\, D^n)$ and $h(KD^n) \setminus \mathrm{Int}(LD^n)$ are annuli, their difference $LD^n \setminus h(\mathrm{Int}\, D^n)$ is as well.

FIGURE 39. Arguments in the proof of $SH_n \implies AC_n$.

Next, $h(KD^n) \setminus \mathrm{Int}(h(D^n)) = h(KD^n \setminus \mathrm{Int}\, D^n)$ is the homeomorphic image of an annulus, thus an annulus itself. Now another application of Remark 12.4 shows that $LD^n \setminus h(\mathrm{Int}\, D^n)$ is an annulus, see Fig. 39b. Namely, it is the manifold with boundary obtained by subtraction of a boundary collar, namely $h(KD^n) \setminus \mathrm{Int}(LD^n)$, from the annulus $h(KD^n) \setminus \mathrm{Int}\, h(D^n)$.

We have thus shown that $LD^n \setminus h(\mathrm{Int}\, D^n)$ is an annulus, proving AC_n for h. The case of a general stable homeomorphism follow immediately from the following claim. $\square$

Claim. *If AC_n holds for homeomorphisms $h, k\colon \mathbb{R}^n \to \mathbb{R}^n$, then it holds for $h \circ k$.*

Proof. By hypothesis, there exists $K > 0$ large enough so that $KD^n \setminus k(\mathrm{Int}\, D^n)$ is an annulus. Then

$$Y := h(KD^n) \setminus h \circ k(\mathrm{Int}\, D^n) = h(KD^n \setminus k(\mathrm{Int}\, D^n))$$

is also an annulus since h is a homeomorphism.

Again by hypothesis, there exists $L > 0$ large enough so that $LD^n \setminus h(\operatorname{Int} D^n)$ is an annulus. By choosing a larger L if necessary, we assume further that $\operatorname{Int}(LD^n)$ contains $h(KD^n)$. Then we claim that

$$Z := LD^n \setminus h(\operatorname{Int} KD^n)$$

is also an annulus by Remark 12.4. To see this, observe that

$$Z \cup \Big(h(KD^n) \setminus h(\operatorname{Int} D^n)\Big) = LD^n \setminus h(\operatorname{Int} D^n)$$

is an annulus, so Z is a manifold with boundary obtained by removing a boundary collar from an annulus. Here we used the fact that $h(KD^n \setminus \operatorname{Int} D^n)$ is an annulus, since it is the homeomorphic image of the region between concentric round spheres, see Fig. 37a.

Now

$$LD^n \setminus h \circ k(\operatorname{Int} D^n) = LD^n \setminus h(\operatorname{Int} KD^n) \cup h(KD^n) \setminus h \circ k(\operatorname{Int} D^n) = Z \cup Y$$

is obtained by gluing two annuli together along a common boundary component, so is also an annulus, showing that AC_n holds for $h \circ k$. □

12.1.1 Properties of stable homeomorphisms. Before giving Kirby's proof of SH_n for $n \geq 5$ we gather together the relevant facts about stable homeomorphisms, starting with the following pleasant property of stable homeomorphisms.

Proposition 12.12. *Every stable $h \in \operatorname{Homeo}(\mathbb{R}^n)$ is isotopic to* Id.

Proof. Write $h = h_k \circ \cdots \circ h_1 \colon \mathbb{R}^n \to \mathbb{R}^n$ as in the definition. Since each h_i agrees with $\operatorname{Id}_{\mathbb{R}^n}$ on some open set, it is isotopic to it by Proposition (10.3.v). Therefore, the composite map $h = h_k \circ \cdots \circ h_1$ is isotopic to $\operatorname{Id}_{\mathbb{R}^n}$ as well. □

We now show that stability is a 'local' property of homeomorphisms, namely, that if a homeomorphism agrees with a stable homeomorphism on an open set, it must itself be stable.

Lemma 12.13. *Let $h, k \in \operatorname{Homeo}(\mathbb{R}^n)$ be such that there exists a nonempty open set U with $h|_U = k|_U$. Then h and k are either both stable or both unstable.*

Proof. We can write $k = h \circ (h^{-1} \circ k)$, where $(h^{-1} \circ k)|_U = \operatorname{Id}$, so $h^{-1} \circ k$ is stable. Then h stable implies that k is stable, since the composition of stable maps is stable. A similar argument shows that k stable implies h stable. □

Since stability is a 'local' property, the following is a natural notion of stability for maps between subsets of $\mathbb{R}^n$.

Definition 12.14. Let $U, V \subseteq \mathbb{R}^n$ be open. A homeomorphism $h \colon U \to V$ is *stable* if every $x \in U$ has a neighbourhood $W_x \subseteq U$ such that $h|_{W_x}$ extends to a stable homeomorphism of $\mathbb{R}^n$.

In particular, the restriction of a stable homeomorphism of $\mathbb{R}^n$ is stable in the above sense. Since we will use the torus trick in the upcoming proof of SH_n the following result is reassuring.

Proposition 12.15 ([Con63, Lem. 5, p. 335])**.** *If $h \in \operatorname{Homeo}(\mathbb{R}^n)$ is bounded distance from the identity, then h is stable.*

Proof. We begin with a helpful lemma.

Lemma 12.16. *Translations of $\mathbb{R}^n$ are stable.*

Proof. Consider a translation $t \colon \mathbb{R}^n \to \mathbb{R}^n$ and a strip $S_1 := \mathbb{R} \times [-1, 1]^{n-1}$ aligned with the direction of the translation, and another strip $S_2 := \mathbb{R} \times [-2, 2]^{n-1}$ containing it. Construct two homeomorphisms, the first that fixes S_1 and moves $\mathbb{R}^n \setminus S_2$ by the translation. The second homeomorphism fixes $\mathbb{R}^n \setminus S_2$ and applies the translation to S_1. In the difference $S_2 \setminus S_1$, we interpolate, so that the composition of the two homeomorphisms is the given translation. □

Returning to the proof of the proposition, by the above lemma, since compositions of stable maps are stable, we may assume, without loss of generality, that $h(0) = 0$. Let $\rho\colon [0,\infty) \to [0,2)$ be a homeomorphism with $\rho|_{[0,1]} = \mathrm{Id}$. Then we define the homeomorphism

$$\begin{aligned}\gamma\colon \mathbb{R}^n &\xrightarrow{\cong} \mathrm{Int}(2D^n)\\ \vec{x} &\mapsto \rho(|\vec{x}|)\frac{\vec{x}}{|\vec{x}|}\end{aligned}$$

Observe that by construction, $\gamma|_{D^n} = \mathrm{Id}$. Next we define a homeomorphism

$$\begin{aligned}H\colon \mathbb{R}^n &\xrightarrow{\cong} \mathbb{R}^n\\ x &\mapsto \begin{cases}\gamma h\gamma^{-1}(x) & x \in \mathrm{Int}\,2D^n\\ x & x \in \mathbb{R}^n \setminus \mathrm{Int}\,2D^n.\end{cases}\end{aligned}$$

We leave it to the reader to verify that H is continuous and a homeomorphism. The continuity uses that h is bounded distance from the identity.

We assert that h and H agree in a neighbourhood of 0. Specifically, h and H agree on the nonempty open set $U := h^{-1}(\mathrm{Int}\,D^n) \cap \mathrm{Int}\,D^n$, as we now show. First we know that $0 \in U$ since $h(0) = 0$, so $U \neq \emptyset$. Let $x \in U$. Then $\gamma^{-1}(x) = x$ since $\gamma|_{D^n} = \mathrm{Id}$. Next we know that $h\gamma^{-1}(x) = h(x) \in \mathrm{Int}\,D^n$ since $U \subseteq h^{-1}(\mathrm{Int}\,D^n)$. Finally we use again that $\gamma|_{D^n} = \mathrm{Id}$ to we see that $H(x) := \gamma h\gamma^{-1}(x) = \gamma h(x) = h(x)$.

By definition, we have that $H|_{\mathbb{R}^n\setminus 2D^n} = \mathrm{Id}$, so H is stable. Then by Lemma 12.13, the homeomorphism h must also be stable. □

12.2 Stable homeomorphism in the smooth and PL categories

Recall that our present goal is to prove that every homeomorphism of $\mathbb{R}^n$ is stable. The next proposition shows this is only interesting in the topological category.

Proposition 12.17. *Every orientation preserving diffeomorphism of $\mathbb{R}^n$ is stable. Every orientation preserving PL homeomorphism is stable.*

We will use the *smooth isotopy extension theorem*, see e.g. [Hir94, Chap. 8] or [Lee13].

Theorem 12.18 ((Smooth) isotopy extension theorem)**.** *Let $U \subseteq M$ be an open subset of a smooth manifold, and let $A \subseteq U$ compact. Let $F\colon U \times [0,1] \to M$ be a smooth isotopy such that the track of the isotopy*

$$\begin{aligned}\widehat{F}\colon U \times [0,1] &\to M \times [0,1]\\ (x,t) &\mapsto (F(x,t),t)\end{aligned}$$

has open image. Then there is an isotopy $H\colon M \times [0,1] \xrightarrow{\widehat{H}} M \times [0,1] \xrightarrow{proj} M$ with H_t a diffeomorphism for all t, H has compact support (i.e. $H_t = \mathrm{Id}$ outside some compact set for each t) and there exists a neighbourhood $V \supseteq A \times [0,1]$ such that $\widehat{H}|_V = \widehat{F}|_V$.

Sketch proof. Use tangent vectors to the curves $\widehat{F}(x \times [0,1]) \subseteq M \times [0,1]$ to get a vector field on $\widehat{F}(U \times [0,1])$. Extend the latter to all of $M \times [0,1]$, with compact support, and then integrate. □

There is also a PL isotopy extension theorem (see, e.g. [RS82]).

Proof of Proposition 12.17. First we address the smooth case. Recalling that translations are stable (Lemma 12.16), it suffices to consider $h\colon \mathbb{R}^n \to \mathbb{R}^n$ a diffeomorphism with $h(0) = 0$. Define a smooth isotopy

$$\begin{cases}\frac{1}{t}h(tx) & 0 < t \leq 1\\ dh|_{x=0} & t = 0\end{cases}$$

from h to a linear map. Recall that $\mathrm{GL}_n(\mathbb{R})$ has two path components detected by $\det > 0$ (if orientation preserving) or $\det < 0$ (if orientation reversing). Choose a smooth path in $\mathrm{GL}_n(\mathbb{R})$ from the linear map to Id. Putting the last two steps together, we have produced a smooth isotopy $H\colon \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$, satisfying $H_0 = \mathrm{Id}$ and $H_1 = h$, where H_t diffeomorphism for all t.

Let $U \ni 0$ be open. Apply the smooth isotopy extension theorem to $H|_{U\times[0,1]}$ to get $\widehat{H}$. Here $H|_{U\times[0,1]}$ has open image since it is the restriction of an ambient isotopy,

$$\begin{tikzcd}
\mathbb{R}^n \times [0,1] \arrow[drr, "H"] & & \\
U \times [0,1] \arrow[u, hook, "\text{incl.}"] \arrow[d, hook, "\text{incl.}"'] \arrow[rr, "H|_{U\times[0,1]}"] & & \mathbb{R}^n \\
\mathbb{R}^n \times [0,1] \arrow[urr, "\widehat{H}"'] & &
\end{tikzcd}$$

In the above diagram, the top triangle commutes, and there exists a neighbourhood V of $0 \times [0,1]$, such that the bottom triangle commutes on V, i.e. $\widehat{H}|_V = (H|_{U\times[0,1]})|_V = H|_V$. Since the isotopy extension theorem provides an isotopy with compact support, we know in particular that $\widehat{H}_1$ restricts to the identity outside some compact set.

Since $\widehat{H}_1$ agrees with Id on some nonempty open set, we see that $\widehat{H}_1$ is stable by definition. Moreover, $h = H_1$ agrees with $\widehat{H}_1$ on $proj(V)$, so h is also stable by Lemma 12.13. This completes the proof of the first statement.

For the PL statement, we will use a similar argument. First we know that every orientation preserving PL embedding of D^n in $\mathbb{R}^n$ is isotopic to the identity [RS82]. The PL isotopy extension theorem then shows that a germ near 0 can be extended to a homeomorphism which is the identity outside some compact set, as in the previous argument. $\square$

We need a definition of stability for PL homeomorphisms.

Definition 12.19. A homeomorphism $h\colon M \to N$ between oriented, PL manifolds is *stable* at $x \in \operatorname{Int} M$ if there are PL coordinate charts $\phi\colon \Delta^n \to M$, with $x \in \phi(\mathring{\Delta}^n)$ and $\psi\colon \Delta^n \to N$, with $h(x) \in \psi(\mathring{\Delta}^n)$, with $h(\phi(\mathring{\Delta}^n)) \cap \psi(\mathring{\Delta}^n) \neq \emptyset$ such that the composition

$$\psi^{-1} h \phi|\colon \phi^{-1} h^{-1} \psi(\Delta^n) \to \mathbb{R}^n$$

extends to a stable homeomorphism of $\mathbb{R}^n$.

Observe that h is only assumed to be a homeomorphism in the above definition. We know already from Proposition 12.17 that orientation preserving PL homeomorphisms are stable. Similar to above, we may define a notion of stability for diffeomorphisms of connected, oriented, smooth manifolds, but we omit this, since we will not need it.

Remark 12.20. We have restricted ourselves to defining stability of PL homeomorphisms. However, we may also define a notion of stable manifolds. Similar to how PL and smooth manifolds are defined by describing the allowed transition maps, a stable manifold is one where the transition maps are stable, in the sense of Definition 12.14. See [BG63, BG64c, BG64b, BG64a] for further details. In particular, every orientable smooth or PL manifold admits a stable structure [BG64b, Theorem 10.4]. The above definition indicates the correct notion of stability for a homeomorphism of a manifold with a stable structure.

Next we show that whether a given homeomorphism is stable is a local property, namely we need only check for stability at a single (arbitrary) point. For this we first need a lemma.

Lemma 12.21. *Let M be a connected PL manifold. For any given pair $x, y \in \operatorname{Int} M$ there exists a PL coordinate chart with image containing both x and y.*

More precisely, for $x, y \in \operatorname{Int} M$ and a PL coordinate chart $\phi\colon \Delta^n \to M$ giving a neighbourhood of x, there exists an orientation preserving PL homeomorphism $f\colon M \to M$ such that $f^{-1}\phi$ is a PL coordinate chart giving a neighbourhood of both x and y.

Proof. Let $\phi\colon \Delta^n \to M$ be a PL coordinate chart with $x \in \phi(\mathring{\Delta}^n)$. Choose $b \in \phi(\mathring{\Delta}^n)$ with $b \neq x$. Choose an open set $W \ni x, y$ with $x \notin W$.

There exists an orientation preserving PL homeomorphism $f\colon M \to M$ with $y \mapsto b$ and $f|_{M\setminus W} = \mathrm{Id}$. Since $x \notin W$, we know that $f(x) = x$. Then $f^{-1}(\phi(\Delta^n)) \ni y, x$ and $f^{-1} \circ \phi\colon \Delta^n \to M$ is a PL coordinate chart with $x, y \in f^{-1} \circ \phi(\mathring{\Delta}^n)$, as claimed. □

Proposition 12.22 ([BG64b, Theorem 7.1])**.** *Let M and N be connected, oriented, PL manifolds. A homeomorphism $h\colon M \to N$ is stable at some $x \in \operatorname{Int} M$ if and only if it is stable at every $x \in \operatorname{Int} M$.*

Proof. Assume that h is stable at $x \in \operatorname{Int} M$ with respect to PL coordinate charts ϕ at x and ψ at $h(x)$. In other words, the composition

$$\psi^{-1}h\phi|\colon \phi^{-1}h^{-1}\psi(\Delta^n) \to \mathbb{R}^n$$

extends to a PL homeomorphism of $\mathbb{R}^n$. Choose $y \in \operatorname{Int} M$ with $y \neq x$. We will show that h is stable at y, which will complete the proof.

We claim that h is stable at x with respect to $f^{-1}\phi$ and ψ at $h(x)$, for f as in the lemma. To see this, we must consider the composition $\psi^{-1}hf^{-1}\phi = \psi^{-1}h\phi \circ \phi^{-1}f\phi$, when both functions are defined. Here we know by hypothesis that $\psi^{-1}h\phi$ is stable, and also that $\phi^{-1}f\phi$ is since f is an orientation preserving PL homeomorphism. The composition of stable homeomorphisms is stable, and therefore, h is stable at x with respect to $f^{-1}\phi$ and ψ at $h(x)$.

But then h is stable at every point in $f^{-1}\phi(\mathring{\Delta}^n)$, and so h is also stable at $y \in M$. □

Lest the reader be concerned that we have two distinct notions of stability for a homeomorphism, the following proposition should lay the mind at ease.

Proposition 12.23 ([BG64b, Theorem 13.1])**.** *For homeomorphisms of $\mathbb{R}^n$, Definition 12.6 and Definition 12.19 agree. For the second definition, we fix some PL structure on $\mathbb{R}^n$. In particular, the statement shows that the choice is irrelevant, assuming one exists.*

Indeed, the two definitions agree in general (see Remark 12.7), assuming a PL structure exists on the given manifold. This shows that for a given manifold M, a given homeomorphism is stable regardless of the PL structure on M. However, we must still choose the same PL structure on both domain and codomain (in this case, both are M). Specifically, given two distinct PL structures Σ and Σ' on a manifold M, denoting the corresponding PL manifolds as M_Σ and $M_{\Sigma'}$ respectively, even the identity map $\mathrm{Id}\colon M_\Sigma \to M_{\Sigma'}$ need not be stable.

Proof of Proposition 12.23. It is clear that Definition 12.6 implies Definition 12.19 by the definition of a PL structure.

Suppose a homeomorphism $h\colon \mathbb{R}^n \to \mathbb{R}^n$ is stable under Definition 12.19 at some $x \in \mathbb{R}^n$ with respect to some PL structure on $\mathbb{R}^n$. Let $\phi\colon \Delta^n \to \mathbb{R}^n$ be a PL coordinate chart with $x, h(x) \in \phi(\mathring{\Delta}^n)$. Such a chart exists by Lemma 12.21. By hypothesis, $\phi^{-1}h\phi$ is stable at $\phi^{-1}(x) \in \mathbb{R}^n$. It is shown in [BG64b] that $\phi^{-1}h\phi|$ restricted to some neighbourhood of $\phi^{-1}(x)$ extends to a homeomorphism h' of $\mathbb{R}^n$ such that $h'|_{\partial\Delta^n} = \mathrm{Id}$. Then $\phi h'\phi^{-1}\colon \phi(\Delta^n) \xrightarrow{\cong} \phi(\Delta^n)$ agrees with h on a neighbourhood of x, since on such a neighbourhood, $h' = \phi^{-1}h\phi$ and so $\phi h'\phi^{-1} = \phi\phi^{-1}h\phi\phi^{-1} = h$. Moreover, on $\phi(\partial\Delta^n)$, we have that $\phi h'\phi^{-1} = \phi\phi^{-1} = \mathrm{Id}$. Extend by the identity to get a homeomorphism $h_1\colon \mathbb{R}^n \to \mathbb{R}^n$.

Then observe that h_1 agrees with the identity on an open set and thus h_1 is stable in the sense of Definition 12.6. We also know that h_1 agrees with h on a neighbourhood of x, and so h_1 is stable in the sense of Definition 12.6 by Lemma 12.13. □

We need one final property of stable homeomorphisms for use in the proof of the stable homeomorphism theorem.

Proposition 12.24. *Let $M, N, \widetilde{M}, \widetilde{N}$ be connected oriented PL manifolds. If in a commutative diagram*

$$\begin{tikzcd} \widetilde{M} \arrow[r, "\widetilde{f}", "\cong"'] \arrow[d, "\alpha"'] & \widetilde{N} \arrow[d, "\beta"] \\ M \arrow[r, "f", "\cong"'] & N \end{tikzcd}$$

the vertical arrows α and β are local PL homeomorphisms, then the homeomorphism $\widetilde{f}$ is stable if and only if the homeomorphism f is stable.

Note that codimension zero PL immersions and PL covering maps are local PL homeomorphisms.

Proof. Suppose that $\widetilde{f}$ is stable. Let ϕ and ψ be coordinate charts for $\widetilde{M}$ and $\widetilde{N}$ respectively, so that the composition $\psi^{-1} \circ \widetilde{f} \circ \phi|$ extends to a stable homeomorphism of $\mathbb{R}^n$. Observe that suitable small restrictions of $\alpha\phi$ and $\beta\psi$ are PL coordinate charts for M and N respectively. Then a suitably small restriction of $\psi^{-1}\beta^{-1}f\alpha\phi$ extends to a stable homeomorphism of $\mathbb{R}^n$, since $\psi^{-1}\beta^{-1}f\alpha\phi$ agrees with $\psi^{-1} \circ \widetilde{f} \circ \phi|$ on a small enough neighbourhood. In light of Proposition 12.22, this finishes the proof of one direction. The other direction is similar. □

12.3 Proof of the stable homeomorphism theorem

We now have the ingredients to prove the stable homeomorphism theorem for $n \geq 5$, that every orientation preserving homeomorphism of $\mathbb{R}^n$ is stable [Kir69].

Proof. We begin with an orientation preserving homeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$. As before, the proof consists of building from the bottom up the maps in the following diagram, where all manifolds are endowed with PL structures – those without subscripts have their standard PL structure, while nonstandard PL structures are denoted by subscripts and will be defined shortly.

We begin with a PL immersion $\alpha\colon T^n \setminus x \to \mathbb{R}^n$ for some $x \in T^n$, as provided by Corollary 10.6. Here we are using the fact that a smooth map induces a PL map as described in Section 11.

Since f is a homeomorphism, the composition $f \circ \alpha$ is a topological immersion.

$$\begin{tikzcd}
\mathbb{R}^n \arrow[rr, "\widetilde{h}"] \arrow[d, "e"] & & \mathbb{R}^n \arrow[d, "e"] \\
T^n \arrow[r, "\overline{h}", "\cong"'] \arrow[d] & \widetilde{T^n_\Sigma} \arrow[r, "g^{-1}", "\cong_{PL}"'] \arrow[d] & T^n \\
T^n \arrow[r, "\widehat{h}", "\cong"'] & T^n_\Sigma & \\
T^n \setminus \mathring{B} \arrow[r, "h|"] \arrow[u, hook] & (T^n \setminus \mathring{B})_\Sigma \arrow[u, hook] & \\
T^n \setminus A \arrow[r, "h|"] \arrow[u, hook] \arrow[d, hook] & (T^n \setminus A)_\Sigma \arrow[u, hook] \arrow[d, hook] & \\
T^n \setminus x \arrow[r, "h"] \arrow[d, looseness, "\alpha"] & (T^n \setminus x)_\Sigma \arrow[d, looseness, "f\alpha"] & \\
\mathbb{R}^n \arrow[r, "f", "\cong"'] & \mathbb{R}^n &
\end{tikzcd} \tag{12.25}$$

Let $(T^n \setminus x)_\Sigma$ denote the topological manifold $T^n \setminus x$ endowed with a PL structure Σ induced by the immersion $f \circ \alpha$. In other words, with respect to this induced PL structure, the map $f \circ \alpha \colon (T^n \setminus x)_\Sigma \looparrowright \mathbb{R}^n$ is a PL immersion. The map h completes the square. On the level of topological manifolds h is the identity map. We use a different symbol here in an attempt to avoid confusion – Since Σ is not equivalent to the standard PL structure on $T^n \setminus x$, the map h is not *a priori* a stable map. Observe that by Proposition 12.24 the map h is stable if and only if f is stable.

Let A be an open ball around $x \in T^n$. Then $A \setminus x$ is an open submanifold of $(T^n \setminus x)_\Sigma$ and therefore inherits a PL structure; we denote the corresponding manifold by $(A \setminus x)_\Sigma$. Observe that $A \setminus x$ is homeomorphic to $S^{n-1} \times \mathbb{R}$. By Theorem 11.5, we know that $(A \setminus x)_\Sigma$ is PL homeomorphic to $S^{n-1} \times \mathbb{R}$, the latter with its standard PL structure. Choose one of those radial copies of S^{n-1} in $(A \setminus x)_\Sigma$ and call it S. There sphere S is bicollared in T^n and therefore by the Schoenflies theorem on the n-torus (Proposition 10.7), since $n \geq 3$, bounds a closed ball B in T^n. The sphere $S = \partial B \subseteq (T^n \setminus x)_\Sigma$ carries the standard PL structure on S^{n-1} and therefore we can glue together $(T^n \setminus \mathring{B})_\Sigma$ and D^n carrying its standard PL structure (inducing the standard PL structure on its boundary, to produce a PL structure on T^n. We still call this Σ. The torus T^n endowed with this PL structure, that is the PL manifold T^n_Σ occurs in the second and third line of the diagram. He we used the Alexander trick (Proposition Proposition 10.3) to extend the map $h| \colon T^n \setminus \mathring{B} \to (T^n \setminus \mathring{B})_\Sigma$ to a homeomorphism $\widehat{h} \colon T^n \to T^n_\Sigma$. In particular, while the map h was only the identity map under an alias, the map $\widehat{h}$ may not be the identity everywhere (of course it agrees with h on $T^n \setminus \mathring{B}$).

Next we need another tool from PL topology. Specifically, we know from Theorem 11.6 that we can lift both T^n and T^n_Σ along finite-sheeted PL covering maps so that the induced map $\overline{h} \colon T^n \to \widetilde{T^n_\Sigma}$ is homotopic to a PL homeomorphism $g \colon T^n \to \widetilde{T^n_\Sigma}$. Here we have used the fact that every finite sheeted cover of T^n is also T^n. The inverse of the PL homeomorphism g appears in the second line of the diagram.

Since $\overline{h}$ and g are homotopic, we know that $g^{-1} \circ \overline{h}$ is homotopic to the identity. By Lemma 10.5 the map $\widetilde{h} \colon \mathbb{R}^n \to \mathbb{R}^n$, induced by the universal covering map $e \colon \mathbb{R}^n \to T^n$, is bounded distance from Id. Then by Proposition 12.15 it follows that $\widetilde{h}$ is stable.

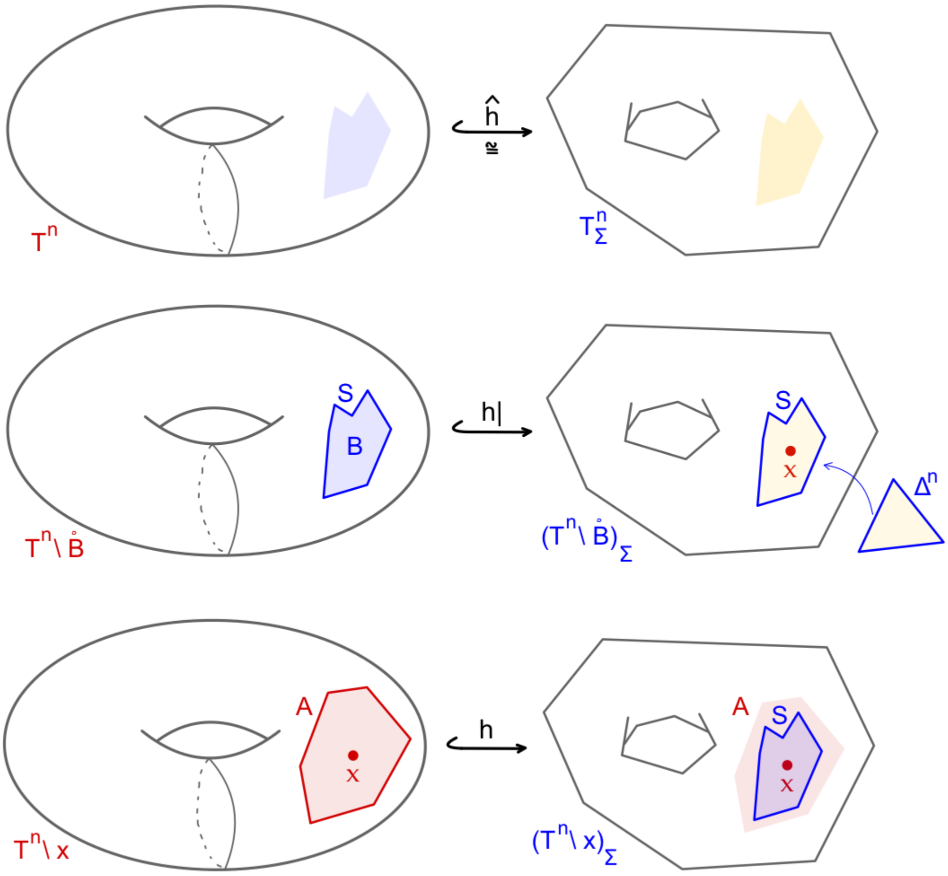


FIGURE 40. Caption

Having reached the top of the diagram, now we climb back down. Since $\widetilde{h}$ is stable, we know that $g^{-1} \circ \overline{h}$ is stable by Proposition 12.24. Next, we know that the map g is a PL homeomorphism, which we may further assume to be orientation preserving by . The composition of stable maps is stable so $\overline{h} = g \circ (g^{-1}\overline{h})$ is stable. Then $\widehat{h}$ is stable by Proposition 12.24. A restriction of a stable map is stable, so h is stable, and then finally f is stable. This completes the proof. □

Remark 12.26. In Kirby's paper proving the stable homeomorphism theorem [Kir69], he initially only reduced it to the Hauptvermutung for tori, that is to a conjecture regarding the number of PL structures on the n-torus. The key insight that one could pass to finite sheeted covers is credited to Siebenmann. Indeed, as we will soon see there do exist nonstandard PL structures on the n-torus for $n \geq 5$, so the step cannot be bypassed. Therefore perhaps Siebenmann deserves some nontrivial credit for the result.

Remark 12.27. Why can we not use the proof above in dimension four? For one thing, the input from PL manifold theory depended on the powerful machinery of surgery theory, which does not work in dimension four. However, as mentioned before, AC_n as well as SH_n is indeed true in dimension four, as proved by Quinn [Qui82].

Remark 12.28. Why do we need to resort to PL technology in the above proof? Is it possible to use just smooth technology? The key difference between the smooth and PL categories that we exploit in the proof is that the PL Poincaré conjecture is true in all dimensions (recall this was used in the proofs of the results of Wall [Wal67] and Browder [Bro65]), but in many dimensions is known to be false in the smooth category.

12.4 Consequences of SH_n and AC_n

We present a couple of important consequences of these theorems, namely that orientation preserving homeomorphisms of both $\mathbb{R}^n$ and S^n are isotopic to the identity, for $n \geq 5$, and that for dimension at least six, connected sum of manifolds is well-defined in the same sense as this holds in the PL and smooth categories.

Theorem 12.29. *For $n \geq 5$ every orientation preserving homeomorphism of $\mathbb{R}^n$ is isotopic to the identity.*

Proof. Every orientation preserving homeomorphism is stable and stable homeomorphisms are isotopic to the identity. Use the Alexander trick for each homeomorphism in the composite, each of which is the identity on an open subset. □

Theorem 12.30. *For $n \geq 5$ every orientation preserving homeomorphism of S^n is isotopic to the identity.*

Proof. Consider an orientation preserving homeomorphism $f\colon \mathbb{R}^n \cup \{\infty\} = S^n \xrightarrow{\cong} S^n = \mathbb{R}^n \cup \{\infty\}$. Isotope f so that $f(\infty) = \infty$ (for example, via a rotation). The restriction $f|_{\mathbb{R}^n}$ is an orientation preserving homeomorphism, so $f|_{\mathbb{R}^n}$ is stable and thus $f\colon S^n \to S^n$ is stable. So $f = f_1 \circ \cdot \circ f_k$ with $f_i|_{U_i} = \mathrm{Id}$, where $U_i \subseteq S^n$ open. Now use Alexander trick to isotope f_i to Id and conclude that f isotopic to Id. □

Remark 12.31. There exist orientation preserving diffeomorphisms of S^n that are not smoothly isotopic to the identity. For example, Milnor's exotic spheres can be built by gluing together two copies of D^7 along an orientation preserving diffeomorphism of S^6. It is an open question whether every orientation preserving diffeomorphism of S^4 is smoothly isotopic to the identity.

Theorem 12.32. *Let $n \geq 6$.*

(1) Connected sum of a pair of oriented, connected topological n-manifolds is well-defined.
(2) Connected sum of connected topological n-manifolds is well-defined provided at least one of the two manifolds is nonorientable.

Example 12.33. The choices of orientation are important, since $\mathbb{CP}^{2n}\#\mathbb{CP}^{2n}$ and $\mathbb{CP}^{2n}\#\overline{\mathbb{CP}^{2n}}$ are not even homotopy equivalent.

We restrict to $n \geq 6$ in Theorem 12.32 because we will use Theorem 12.30 for S^{n-1} in the proof. In fact Theorem 12.30 holds for all n, but since we are focusing on the high dimensional development here, we only state and prove the theorem in dimension at least six.

To make sense of Theorem 12.32, we need to define connected sum. Since the most subtleties occur in the oriented case, we work in that case from now on.

Definition 12.34. Let M_1 and M_2 be connected, oriented n-manifolds. Let $\phi\colon D^n \to M_1$ be a orientation preserving locally collared embedding, and let $\psi\colon D^n \to M_2$ be an orientation reversing locally collared embedding. Then we define

$$M_1\#M_2 := \frac{M_1 \setminus \operatorname{Int}\phi(D^n) \sqcup M_2 \setminus \operatorname{Int}\psi(D^n)}{\phi(\theta) \sim \psi(\theta), \quad \theta \in S^{n-1}}.$$

So the content of Theorem 12.32 is the following proposition.

Proposition 12.35. *For $n \geq 6$ the manifold $M_1\#M_2$ is independent of the choice of ϕ and ψ.*

Proof. It suffices to prove that the connected sum is independent of the choice of ϕ. So let $\phi'\colon D^n \to M_1$ be another orientation preserving locally collared embedding. We aim first to construct a homeomorphism $h\colon M_1 \to M_1$ such that $h \circ \phi'$ and ϕ have the same image.

Step 1: There is an orientation preserving homeomorphism $h_1\colon M_1 \to M_1$ sending $\phi'(0)$ to $\phi(0)$. Namely, manifolds are homogeneous: for any two points in the interior of a manifold, there is an orientation preserving homeomorphism sending one point to the other. See exercise.

Step 2: There is an orientation preserving homeomorphism $h_2 \colon M_1 \to M_1$ such that

$$h_2 \circ h_1(\phi'(D^n)) \subseteq \operatorname{Int} \phi(D^n).$$

To see this, use that $h_1\phi'(D^n)$ is locally collared, hence globally collared since the boundary is codimension one. Then one can stretch the collar out while radially shrinking $h_1\phi'(D^n)$ until it lies within the desired interior, see Fig. 41.

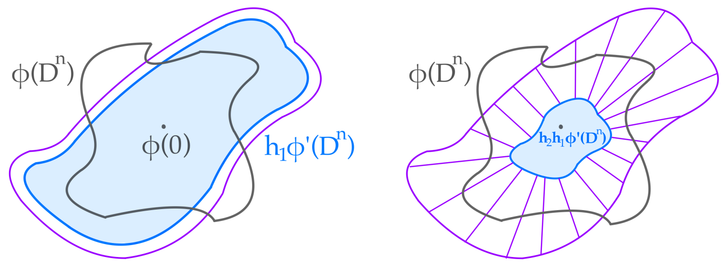


FIGURE 41. Step 2 of the proof: shrink $h_1\phi'(D^n)$ into $\operatorname{Int}\phi(D^n)$.

Step 3: There is an orientation preserving homeomorphism $h_3 \colon M_1 \to M_1$ such that

$$h_3 \circ h_2 \circ h_1(\phi'(D^n)) = \phi(D^n).$$

To see this we apply the Annulus Theorem 12.1. The region $\phi(D^n) \setminus \operatorname{Int}(h_2 \circ h_1 \circ \phi'(D^n)$ is homeomorphic to $S^{n-1} \times [0,1]$, and a choice of such a homeomorphism may be used, together with the outside collar on $\phi(D^n)$, to stretch out $h_2 \circ h_1 \circ \phi'(D^n)$ until it covers all of $\phi(D^n)$.

We write $h = h_3 \circ h_2 \circ h_1$ and

$$\phi'' := h_3 \circ h_2 \circ h_1 \circ \phi'.$$

Our aim is now to show that ϕ'' and ϕ determine homeomorphic connected sums. Since h is a homeomorphism, ϕ' and ϕ'' certainly produce homeomorphic connected sums $M_1 \#_{\phi,\psi} M_2$ and $M_1 \#_{\phi'',\psi} M_2$. So it suffices to show that ϕ and ϕ'' produce homeomorphic connected sums. Although $\phi(D^n) = \phi''(D^n) \subseteq M_1$ coincide, there is still the problem that the gluing maps that they determine, of $\phi(\partial D^n)$ and $\phi''(\partial D^n)$ respectively with $\psi(\partial D^n) \subseteq M_2 \setminus \operatorname{Int}\psi(D^n)$ differ.

However, we observe that the map

$$\phi^{-1} \circ \phi'' \colon S^{n-1} = \partial D^n \to \partial D^n = S^{n-1}$$

is an orientation preserving homeomorphism , so it is isotopic to the identity by Theorem 12.30, i.e. there is a family of homeomorphisms $F_t \colon \partial D^n \to \partial D^n$ with $F_0 = \phi^{-1} \circ \phi''$ and $F_1 = \mathrm{Id}$. Now consider a homeomorphism

$$\begin{aligned} H \colon \phi(\partial D^n) \times I &\to \phi(\partial D^n) \times I \\ (\phi(x), t) &\mapsto (\phi \circ F_t(x), t). \end{aligned}$$

Note that $H(\phi(x), 0) = (\phi''(x), 0)$ and $H(\phi(x), 1) = (\phi(x), 1)$. We will use H to define a homeomorphism of a collar of $\phi(\partial D^n) = \phi''(D^n)$, which exists by Brown's collaring Theorem 6.5 since $\phi(D^n)$ is locally collared by assumption. Fix a choice of such a collar

$$G \colon \phi(\partial D^n) \times I \to M_1 \setminus \operatorname{Int}\phi(D^n),$$

with $G(\phi(\partial D^n) \times \{0\}) = \phi(\partial D^n)$. As $\phi(D^n) = \phi''(D^n)$ we have $M_1 \setminus \operatorname{Int}\phi''(D^n) = M_1 \setminus \operatorname{Int}\phi(D^n)$, and we can view G also as a collar for $\phi''(\partial D^n)$ in $M_1 \setminus \operatorname{Int}\phi''(D^n)$. We define a homeomorphism $K \colon M_1 \#_{\phi,\psi} M_2 \to M_1 \#_{\phi'',\psi} M_2$ as in Fig. 42, namely

$$K(x) := \begin{cases} \mathrm{Id}, & x \in M_2 \setminus \operatorname{Int}\psi(D^n) \cup M_1 \setminus \Big(\phi(D^n) \cup G(\phi(\partial D^n) \times I)\Big), \\ G(H(\phi(x), t)), & x = G(\phi(y), t) \text{ for } y \in \partial D^n. \end{cases}$$

Since $H(\phi(x), 1) = (\phi(x), 1)$ the map K is continuous at $G(\phi(x), 1)$ for all $x \in \partial D^n$. Since

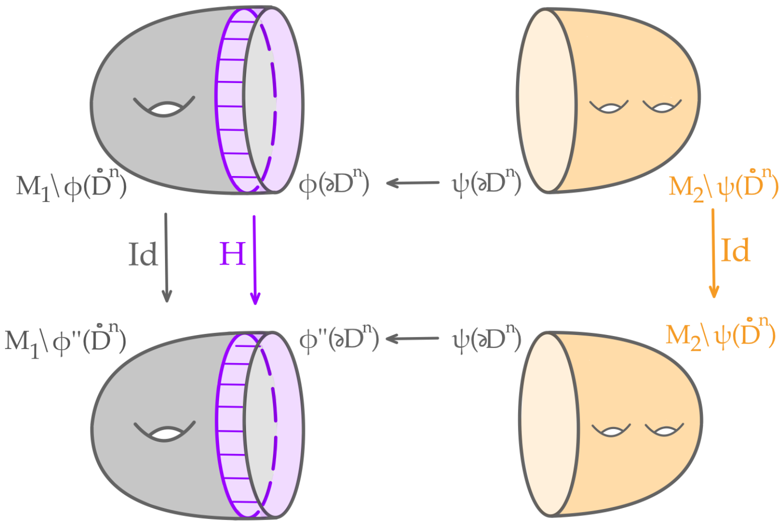


FIGURE 42. The final homeomorphism K.

$H(\phi(x),0) = (\phi''(x),1)$, and $(\phi(x),0) \sim \psi(x)$ in the domain of K, whereas $(\phi''(x),0) \sim \psi(x)$ in the codomain, the map is well-defined and continuous at $\phi(\partial D^n) = \psi(\partial D^n)$. This completes the proof that connected sum is well-defined for manifolds of dimension at least 6. $\square$

Sol. on p.147. **Exercise 12.1**. (PS8.1) Prove that every homeomorphism $h\colon T^n \to T^n$ is stable, where T^n denotes the n-torus $S^1 \times \cdots \times S^1$. *Hints:*

- Easy mode: Apply SH_n.
- Expert mode: The result can be proved independently of SH_n, and was the key step in Kirby's proof of SH_n. (We sidestepped it by using a slightly stronger result about PL homotopy tori.) First prove the case where the induced map on fundamental groups is the identity. Then show that for any $n \times n$ matrix A with integer entries and determinant one, there exists a diffeomorphism $h : T^n \to T^n$ such that $h_* = A$ where $h_*\colon \pi_1(T^n, x) \to \pi_1(T^n, x)$. Prove that diffeomorphisms of T^n are stable.

Sol. on p.148. **Exercise 12.2**. (PS8.2) Use the torus trick to show that a homeomorphism of $\mathbb{R}^n$ is stable if and only if it is isotopic to the identity. *Hints:*

(1) It suffices to show that the space of stable homeomorphisms of $\mathbb{R}^n$, denoted $\mathrm{SHomeo}(\mathbb{R}^n)$, is both open and closed in $\mathrm{Homeo}(\mathbb{R}^n)$.
(2) Use the torus trick from our proof of local contractibility of $\mathrm{Homeo}(\mathbb{R}^n)$ to show that an open neighbourhood of the identity in $\mathrm{Homeo}(\mathbb{R}^n)$ consists of stable homeomorphisms. Conclude that every stable homeomorphism of $\mathbb{R}^n$ has an open neighbourhood consisting of stable homeomorphisms.
(3) Every coset of $\mathrm{SHomeo}(\mathbb{R}^n)$ in $\mathrm{Homeo}(\mathbb{R}^n)$ is open since $\mathrm{Homeo}(\mathbb{R}^n)$ is a topological group. Conclude that $\mathrm{SHomeo}(\mathbb{R}^n)$ is closed in $\mathrm{Homeo}(\mathbb{R}^n)$.

Sol. on p.148. **Exercise 12.3**.(PS9.1) Prove the "topological weak Palais theorem". That is, let $n \geq 6$, let M be a connected n-manifold, and let $\phi, \psi\colon D^n \to \operatorname{Int} M$ be locally collared embeddings. Then there exists a homeomorphism $h\colon M \to M$ with $h \circ \phi = \psi\colon D^n \to M$.

13 PL homotopy tori

We give an outline of the surgery theoretic classification of closed n-manifolds homotopy equivalent to the torus T^n, for $n \geq 5$. This classification played a key rôle in the proof of the stable homeomorphism theorem.

This section will not contain proofs. It is intended to be understandable to those who do not have a background in surgery theory. Along the way we will try to point out where some key tools of PL manifold theory are being used, in the hope that this acts as motivation for our attempt to establish the same tools for topological manifolds. That is, given transversality, handle structures, and immersion theory, we will be able to apply surgery theory in the topological category to obtain similarly strong results on classification of topological manifolds within a homotopy type.

13.1 Classification theorems

The aim is to prove the following two theorems, due to Hsiang-Shaneson [HS69] and Wall [Wal69].

Remark 13.1. The most complete proof was given by Hsiang and Shaneson, although it seems that Wall knew the same result, and was in the middle of writing his extensive book on non-simply connected surgery theory when Kirby announced his proof of SH_n modulo the homotopy tori question. Kirby's proof still needed input from surgery theory, but the theory was so well developed by that point that this was a problem the experts could quickly solve. Wall produced a short announcement of the answer, promising details in his book. Hsiang-Shaneson announced the result at the same time, and using ideas of Farrell, were able to give their own account prior to Wall's book being completed. Perhaps due to this, Wall's book contains fewer details, so the more comprehensive account seems to be Hsiang-Shaneson [HS69].

Theorem 13.2. *Let $n \geq 5$. There is a bijection between the set of closed PL n-manifolds $M \simeq T^n$, up to PL homeomorphism, and*

$$\frac{(\wedge^{n-3}\mathbb{Z}^n) \otimes \mathbb{Z}/2}{\mathrm{GL}_n(\mathbb{Z})}.$$

Here $\wedge^{n-3}\mathbb{Z}^n$ denotes the exterior algebra. The 0 element corresponds to T^n.

Example 13.3. For $n = 5$ we have that $(\wedge^2\mathbb{Z}^5) \otimes \mathbb{Z}/2 \cong (\mathbb{Z}/2)^{10}$, and the quotient by $\mathrm{GL}_5(\mathbb{Z})$ contains 3 elements, represented by 0, $e_1 \wedge e_2$ and $e_1 \wedge e_2 + e_3 \wedge e_4$. The key to checking this is to note that by change of bases

$$e_1 \wedge e_2 + e_2 \wedge e_3 \sim e_1 \wedge e_2 + e_2 \wedge (e_3 + e_1) = e_2 \wedge e_3 \sim e_1 \wedge e_2.$$

Thus even in dimension 5, where there are no exotic spheres, there are two fake PL-tori. That is, they are homotopy equivalent but not PL homeomorphic.

The proof of SH_n used the following result, which is stronger than just enumerating the homotopy tori.

Theorem 13.4. *Let $n \geq 5$. Every closed PL n-manifold $M \simeq T^n$ has a finite cover PL-homeomorphic to T^n.*

Actually, the proof used a further refinement of this, namely that a lift of any homotopy equivalence is homotopic to a homeomorphism. This will be immediate from the fact that we work with the structure set.

Remark 13.5. The analogue of Theorem 13.2 in the smooth category does not hold, since one may connect sum on an exotic sphere, to produce new fake tori. On the other hand, this phenomenon disappears when we pass to finite covers, and the analogue of Theorem 13.4 is also true in the smooth category. In the topological category, there are no fake homotopy tori, but we will need to develop tools such as topological transversality in order to see this.

We will give an introduction to surgery theory in the specific case of the torus T^n. Perhaps this will help readers understand the general theory.

13.2 The structure set

Our primary aim will be to compute the *structure set* of T^n, the set of pairs:

$$\mathcal{S}_{PL}(T^n) := \left\{(M^n \text{ closed PL manifold}, f \colon M \xrightarrow{\simeq} T^n)\right\} / s\text{-cobordism over } T^n.$$

Here, for the equivalence relation, (M, f) and (N, g) are s-cobordant over T^n if there is an $(n+1)$-dimensional cobordism W with $\partial W = M \sqcup -N$ with a map $F \colon W \to T^n$ extending f and g, such that the inclusion maps $M \to W$ and $N \to W$ are simple homotopy equivalences. This means that W can be obtained from either M or N by a sequence of elementary expansions and collapses. See e.g. [Coh73], [DK01, final chapter], or Crowley-Lueck-Macko for more on simple homotopy type. Recall that if the same holds without the simple requirement, then W is called an h-cobordism over T^n.

Here are two simplifications of the structure set. First, it turns out that whether a homotopy equivalence is simple can be decided by an algebraic obstruction in the *Whitehead group*. For a group π, let $\mathbb{Z}[\pi]$ be the group ring, that is sums $\sum_{g\in\pi} n_g g$, with $n_g \in \mathbb{Z}$, and finitely many of the n_g nonzero.

Theorem 13.6 (Bass-Heller-Swan [BHS64])**.** *For $n \geq 0$, the Whitehead group* $\mathrm{Wh}(\mathbb{Z}[\mathbb{Z}^n]) = 0$.

This means that every matrix in $\mathrm{GL}_k(\mathbb{Z}[\mathbb{Z}^n])$ can be converted into a diagonal matrix with entries $\pm g$ by a sequence of operations: taking a block sum with an identity matrix, reversing this operation, or elementary row and column operations. That $\mathrm{Wh}(\mathbb{Z}) = 0$ is a straightforward consequence of the Euclidean algorithm. That $\mathrm{Wh}(\mathbb{Z}[\mathbb{Z}^n]) = 0$ is a much harder theorem.

The algebraic moves in the Whitehead group mirror geometric handle moves that can be performed to a handle decomposition of an h-cobordism. In fact the vanishing of the Whitehead group implies that these moves can be done in order to cancel all handles.

Theorem 13.7 (The s-cobordism theorem; Smale [Sma62], Barden-Mazur-Stallings [Bar63, Sta67, Maz63])**.** *For $n \geq 5$, let $(W^{n+1}; M^n, N^n)$ be a PL s-cobordism. Then*

$$W \cong_{PL} M \times I \cong_{PL} N \times I.$$

In particular $M \cong_{PL} N$.

Remark 13.8. This is also true in the smooth category [Mil65]. It also holds in the topological category, although that needs the results of Kirby-Siebenmann [KS77] that we are currently learning. In the topological category it also holds for $n = 4$, by work of Freedman and Quinn [FQ90] that we will not cover.

Remark 13.9. The proof of the s-cobordism theorem uses handle structures and transversality, so being able to establish versions of these tools for topological manifolds is a prerequisite for proving the topological s-cobordism theorem.

The outcome of these two theorems is that:

$$\mathcal{S}_{PL}(T^n) = \frac{\{f \colon M \xrightarrow{\simeq} T^n\}}{h\text{-cobordism over } T^n} \cong \frac{\{f \colon M \xrightarrow{\simeq} T^n\}}{s\text{-cobordism over } T^n} \cong \frac{\{f \colon M \xrightarrow{\simeq} T^n\}}{\text{PL homeomorphism over } T^n}$$

for $n \geq 5$. So we see that computing the structure set is extremely relevant for the aim of classifying manifolds homotopy equivalent to T^n.

13.3 Normal bordism and the surgery obstruction

The idea of manifold classification via surgery theory is to invoke the power of bordism theory, and to introduce auxiliary stable normal bundle data. This is hard to motivate at first, but it turns out that introducing this extra data is what enables the whole machine to run. Here is an attempt at motivation. Homotopy equivalences are in particular degree one normal maps. Also h-cobordisms are in particular normal bordisms. The powerful machinery of bordism theory allows us to compute the set of degree normal maps up to normal bordism. In addition the normal bundle data provides just the right amount of extra control to enable the definition of an algebraic obstruction to a normal bordism class containing a homotopy equivalence.

The initial goal is to compute *normal bordism classes of degree one normal maps*. Here a degree one normal map is a bundle map

$$\begin{tikzcd} \nu_M \arrow[r, "F"] \arrow[d] & \xi \arrow[d] \\ M \arrow[r, "f"] & T^n. \end{tikzcd}$$

Here we assume that $M \subseteq \mathbb{R}^q$ for some large q and ν_M is the stable normal bundle, while ξ is some stable bundle. We will not discuss the correct notion of a PL bundle theory here. We require that f has degree one, that is both M and T^n are equipped with fundamental classes and $f_*\colon H_n(M) \to H_n(T^n)$ sends $[M]$ to $[T^n]$.

We consider degree one normal maps up to degree one normal bordism. That is a cobordism $(W^{n+1}; M, N)$ with data

$$\begin{tikzcd} \nu_W \arrow[r, "G"] \arrow[d] & \Xi \arrow[d] \\ W \arrow[r, "g"] & T^n \times I \end{tikzcd}$$

restricting to the given degree one normal maps $M \to T^n \times \{0\}$ and $N \to T^n \times \{1\}$, and such that $g_*\colon H_n(W, \partial W) \to H_n(T^n \times I, T^n \times \{0,1\})$ preserves the relative fundamental classes.

Let $\mathcal{N}_{PL}(T^n)$ be the set of normal bordism classes of normal maps with target the PL manifold T^n.

Theorem 13.10. *Let $n \geq 5$. A normal bordism class $[(M, f, F, \xi)]$ contains a homotopy equivalence $M \to T^n$ if and only if the* surgery obstruction $\sigma(M, f, F, \xi) = 0 \in L_n(\mathbb{Z}[\mathbb{Z}^n])$.

Let us explain this theorem. The idea is to try to perform surgery (to be defined presently) on M to convert f into a homotopy equivalence. There is an algebraic obstruction to this in the L-group, which we will define. If the algebraic obstruction vanishes, then the sequence of surgeries exists as desired.

A surgery on an n-manifold consists of cutting out an embedding of $S^r \times D^{n-r}$, for some r, and gluing in $D^{r+1} \times S^{n-r-1}$ instead:

$$M' := M \setminus S^r \times \mathring{D}^{n-r} \cup_{S^r \times S^{n-r-1}} D^{r+1} \times S^{n-r-1}.$$

Associated with a surgery is a cobordism, called the trace of the surgery, given by

$$M \times I \cup D^{r+1} \times D^{n-r},$$

where $D^{r+1} \times D^{n-r}$ is attached along the given embedding $S^r \times D^{n-r}$ in $M \times \{1\}$.

Using Smale-Hirsch immersion theory (due to Haefliger-Poenaru [HP64] in the PL category), one can perform surgeries "below the middle dimension" to obtain $f'\colon M' \to T^n$ with f' $\lfloor n/2 \rfloor$-connected. That is, f' is an isomorphism on π_i for $0 \leq i < \lfloor n/2 \rfloor$ and is a surjection on $\pi_{\lfloor n/2 \rfloor}(M') \to \pi_{\lfloor n/2 \rfloor}(T^n)$. (In our case, the latter is automatic since $\pi_{\lfloor n/2 \rfloor}(T^n) = 0$.)

We want to kill $\ker(\pi_{\lfloor n/2 \rfloor}(M') \to \pi_{\lfloor n/2 \rfloor}(T^n)) = \pi_{\lfloor n/2 \rfloor}(M')$. We can do this by surgery if and only if $f'\colon M' \to T^n$ is normally bordant to a homotopy equivalence, in which case we have a candidate for a fake torus. The fact that making a map an isomorphism on homotopy groups only up to the middle dimension suffices to achieve a homotopy equivalence follows from Poincaré duality, universal coefficients, and the Hurewicz and the Whitehead theorems. These last set of surgeries are possible if and only if an algebraic obstruction in the L-group $L_n(\mathbb{Z}[\mathbb{Z}^n])$, which we will soon define, vanishes. This obstruction is well-defined, meaning that it only depends on the original normal bordism class. In particular it is independent of the choices we made in the initial surgeries below the middle dimension, although this is not at all obvious. The L-groups are the obstructions to finding a collection of disjoint embeddings of $S^{\lfloor n/2\rfloor} \times D^{n-\lfloor n/2\rfloor}$, framed embedded spheres, such that surgery on them gives a homotopy equivalence $f''\colon M'' \to T^n$. We next define the L groups. Note that a group ring $\mathbb{Z}[\pi]$ has an involution defined by sending $g \mapsto g^{-1}$ and extending linearly.

Definition 13.11. In even degrees, $L_{2k}(\mathbb{Z}[\mathbb{Z}^n])$ is the group of nonsingular, $(-1)^k$-Hermitian, sesquilinear forms on finitely generated, free $\mathbb{Z}[\mathbb{Z}^n]$-modules, given by some $\varphi\colon P \to P^* = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}^n]}(P, \mathbb{Z}[\mathbb{Z}^n])$, and further equipped with a quadratic enhancement. We will not define quadratic enhancements in detail, but in particular note that a form with a quadratic enhancement is even. We impose the equivalence relation of *stable isometry*, where by definition φ and φ' are *Witt equivalent* if

$$\varphi \oplus \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix}^a \cong \varphi' \oplus \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix}^b.$$

The form $\begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix}$ on $\mathbb{Z}\pi \oplus \mathbb{Z}\pi$ is called the standard $(-1)^k$-hyperbolic form.

In odd degrees, $L_{2k+1}(\mathbb{Z}[\mathbb{Z}^n])$ is the group of nonsingular *formations*. These are $(-1)^k$ hyperbolic forms with two lagrangians, that is half-rank summands on which the form vanishes. We shall not describe the equivalence relation on formations.

The data of a formation is rather like the algebraic data one can obtain from a Heegaard splitting of a 3-manifold.

We have now seen that the following is an exact sequence of sets:

$$\mathcal{S}_{PL}(T^n) \to \mathcal{N}_{PL}(T^n) \xrightarrow{\sigma} L_n(\mathbb{Z}[\mathbb{Z}^n]).$$

Here the first map is to consider normal bordism classes, and the second is the surgery obstruction map. Exactness encodes the theorem above that the surgery obstruction of a degree one normal map vanishes if and only if that normal bordism class contains a homotopy equivalence.

Proposition 13.12. *For $[(M, f, F, \xi)] \in \mathcal{N}_{PL}(T^n)$, $\sigma(M, f, F, \xi) = 0$ if and only if (M, f, F, ξ) is normally bordant to $(T^n, \mathrm{Id}, Id, \nu_{T^n})$. That is, there is a unique normal bordism class containing a homotopy equivalence.*

We will explain more about the computation of σ later, but first more on the overall strategy.

13.4 Wall realisation and the size of each normal bordism class

Once we know which normal bordism classes contain at least one homotopy equivalence, we can ask how many are there in each normal bordism class, and how many distinct PL manifolds does this give rise to. The first question amounts to completing the computation of the structure set. We saw that every manifold homotopy equivalent to T^n is normally bordant to T^n. It helps to ask the following question.

Question 13.13. Given a normal bordism from M to T^n, is that normal bordism itself bordant (via a bordism of bordisms) to a homotopy equivalence, and hence to an h-cobordism?

If the answer is yes, then $(M, f) = (T^n, \mathrm{Id})$ in $\mathcal{S}_{PL}(T^n)$ and $M \cong_{PL} T^n$. What about if we are allowed to first change the given normal bordism, and then ask this question? If the answer is no for all choices of initial normal bordism, then indeed the pairs (M, f) and (T^n, Id) must be distinct.

Proposition 13.14 (Browder [Bro72], Novikov [Nov64], Wall)**.** *A normal bordism (W, g, G, Ξ) over $T^n \times I$ is normally bordant to an h-cobordism if and only if its surgery obstruction $\sigma(W, g, G, \Xi) = 0 \in L_{n+1}(\mathbb{Z}[Z^n])$.*

In fact, all possible surgery obstructions can be realised for normal bordisms, fixing one end of the normal bordism but not the other.

Theorem 13.15 (Wall)**.** *The group $L_{n+1}(\mathbb{Z}[\mathbb{Z}^n])$ acts on $\mathcal{S}_{PL}(T^n)$ with stabiliser*

$$\mathrm{Im}(\sigma\colon \mathcal{N}_{PL}(T^n \times I, T^n \times \{0,1\}) \to L_{n+1}(\mathbb{Z}[\mathbb{Z}^n]).$$

The action produces a normal bordism starting with (T^n, Id) with any given surgery obstruction. The output of the action is the homotopy equivalence obtained by restricting to the other end of the constructed normal bordism.

We deduce that

$$\mathcal{S}_{PL}(T^n) \leftrightarrow \frac{L_{n+1}(\mathbb{Z}[\mathbb{Z}^n])}{\mathrm{Im}\,\sigma}.$$

Wall realisation extends the sequence above to the surgery exact sequence:

$$\mathcal{N}_{PL}(T^n \times I, T^n \times \{0,1\}) \xrightarrow{\sigma} L_{n+1}(\mathbb{Z}[\mathbb{Z}^n]) \to \mathcal{S}_{PL}(T^n) \to \mathcal{N}_{PL}(T^n) \xrightarrow{\sigma} L_n(\mathbb{Z}[\mathbb{Z}^n]).$$

Proposition 13.16. *We have*

$$\frac{L_{n+1}(\mathbb{Z}[\mathbb{Z}^n])}{\mathrm{Im}\,\sigma} \cong (\wedge^{n-3}\mathbb{Z}^n) \otimes \mathbb{Z}/2.$$

Thus $|\mathcal{S}_{PL}(T^n)| = 2^{\binom{n}{3}}$, all in the normal bordism class of the identity.

Now, how many distinct manifolds does this entail? We have to factor out by the choice of homotopy equivalence to T^n. Note that $T^n \simeq K(\mathbb{Z}^n, 1)$, since the universal cover is $\mathbb{R}^n$, which is contractible. Thus homotopy self-equivalences of T^n up to homotopy are in bijection with isomorphisms of $\pi_1(T^n) \cong \mathbb{Z}^n$, in other words with $\mathrm{GL}_n(\mathbb{Z})$. Therefore the manifold set is given by:

$$\mathcal{M}_{PL}(T^n) = \frac{\{M^n \mid M \simeq T^n\}}{\text{PL-homeomorphism}} \cong \frac{\mathcal{S}_{PL}(T^n)}{\text{self-homotopy equivalences}} \cong \frac{(\wedge^{n-3}\mathbb{Z}^n) \otimes \mathbb{Z}/2}{\mathrm{GL}_n(\mathbb{Z})}.$$

This completes our sketch of the proof of Theorem 13.2. We could leave Proposition 13.12 and Proposition 13.16 as black boxes. But we also want to understand Theorem 13.4, and for that we will need to understand the proofs of these propositions.

13.5 Computations of the surgery obstruction maps

We want to know that for any homotopy torus $M \simeq T^n$, the 2^n-fold cover corresponding to the kernel of $\mathbb{Z}^n \to (\mathbb{Z}/2)^n$, sending $e_i \mapsto e_i$, satisfies $\widetilde{M} \cong_{PL} T^n$.

13.5.1 The L-groups. First, the L-groups of $\mathbb{Z}[\mathbb{Z}^n]$ are known.

Theorem 13.17 (Shaneson)**.** *Let G be a finitely presented group and suppose that* $\mathrm{Wh}(\mathbb{Z}[G]) = 0$*. Then*

$$L_m(\mathbb{Z}[\mathbb{Z} \times G]) \cong L_m(\mathbb{Z}[G]) \oplus L_{m-1}(\mathbb{Z}[G]).$$

This proof is a geometric proof of an algebraic fact, and uses transversality. It is the algebraic analogue of the geometric splitting in bordism groups

$$\Omega_m(X \times S^1) \cong \Omega_m(X) \oplus \Omega_{m-1}(X).$$

Corollary 13.18.

$$L_m(\mathbb{Z}[\mathbb{Z}^n]) \cong \bigoplus_{0\leq i\leq n} \bigoplus \binom{n}{i} L_{m-i}(\mathbb{Z}).$$

The L-groups of $\mathbb{Z}$ are given as follows. They are 4-periodic for $j \geq 0$.

$$L_j(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & j \equiv 0 \mod 4 \\ 0 & j \equiv 1 \mod 4 \\ \mathbb{Z}/2 & j \equiv 2 \mod 4 \\ 0 & j \equiv 3 \mod 4. \end{cases}$$

For $j \equiv 2$, the nontrivial element is detected by an Arf invariant, which depends on the quadratic enhancement. For $j \equiv 0$, the isomorphism is given by taking the signature of the form, and dividing by 8. It is an algebraic fact that every symmetric, even, nonsingular form has signature divisible by 8.

13.5.2 Normal invariants. Next we bring in Sullivan's work, to compute $\mathcal{N}_{PL}(T^n)$. The general fact, for a manifold or more generally for a Poincaré complex X with $\mathcal{N}_{PL}(X) \neq \emptyset$ is that

$$\mathcal{N}_{PL}(X) \cong [X, G/PL].$$

Here square brackets indicate homotopy classes of maps. This translates a bordism question into a homotopy theory question. It is particularly useful because, as we shall see, the homotopy groups of G/PL can be determined, as a consequence of the PL Poincaré conjecture. Let us introduce the notation.

- G_n is the monoid of homotopy self-equivalences of S^{n-1}.
- PL_n is the PL-homeomorphisms of $\mathbb{R}^n$ fixing 0. (In fact to define this space carefully uses semi-simplicial spaces, which will be too much of a distraction for now. So we shall conveniently lie about it, and we will return to the proper definition later when we study smoothing theory.)
- $G := \underrightarrow{\operatorname{colim}}_n G_n$ is the colimit. Here given $f\colon S^{n-1} \to S^{n-1}$ we can take its reduced suspension $\Sigma f\colon \Sigma S^{n-1} \cong S^n \to \Sigma S^{n-1} \cong S^n$, which gives the maps in the directed system needed for the colimit.
- $PL = \underrightarrow{\operatorname{colim}}_n PL_n$. Here a PL-homeomorphism of $\mathbb{R}^n$ induces one of $\mathbb{R}^{n+1}$ by taking the product with $\mathrm{Id}_{\mathbb{R}}$.

Using these, BG and BPL are the associated classifying spaces. Similarly BG_n and BPL_n are the versions prior to taking colimits. In particular BG_n is the classifying space for fibrations with fibre S^{n-1}, BPL_n is the classifying space for $\mathbb{R}^n$ fibre bundles with PL_n structure group, BG is the classifying space for stable spherical fibrations, and BPL is the classifying space for stable classes of PL bundles. A classifying space can be constructed using semi-simplicial techniques. Again we will postpone the precise definitions. At this point, what we need to know is that for a CW complex X, homotopy classes of maps, for examples $[X, BG_n]$, are in bijective correspondence with fibre homotopy equivalence classes of fibrations with fibre homotopy equivalent to S^{n-1}, and $[X, BPL_n]$ is in bijective correspondence with isomorphism classes of $\mathbb{R}^n$ fibre bundles with PL_n structure group. Similarly $[X, BG]$ and $[X, BPL]$ correspond to equivalence classes of stable fibrations and fibre bundles respectively.

The forgetful map $BPL \to BG$ has *homotopy fibre* G/PL, so there is a fibration sequence

$$G/PL \to BPL \xrightarrow{\psi} BG$$

with

$$G/PL = \{(x,\gamma) \mid x \in BPL, \gamma\colon [0,1] \to BG,\ \gamma(0) = \psi(x),\ \gamma(1) = \text{basepoint of } BG\}.$$

The bijection $\mathcal{N}_{PL}(X) \cong [X, G/PL]$ works as follows. Let X be a compact n-manifold for simplicity. Then X has a spherical normal fibration coming from embedding X in Euclidean space. Fixing one PL normal bundle, the different lifts of the spherical normal fibration are in bijective correspondence with $[X, G/PL]$. Each such lift corresponds to a PL bundle over X embedding in S^N for some N. There is an associated collapse map from S^N to the Thom space of the PL normal bundle. Make this map transverse to the zero section and take the inverse image. This yields a manifold $M \subseteq S^N$ with a degree one map to the zero section X. Pulling back the bundle which equals the normal bundle of X in the Thom space gives a bundle over M, with a bundle map. So we obtain a degree one normal map. It turns out that this method gives rise to the claimed bijection.

One key fact about G/PL is that it can be delooped. That is, for some space Y we have $G/PL \simeq \Omega Y$. This is due to Boardman-Vogt [BV68]. Using this we can specialise to $X = T^n$ and compute:

$$[T^n, G/PL] = [T^n, \Omega Y] = [\Sigma T^n, Y] = [\bigvee S^{k+1}, Y] = [\bigvee S^k, \Omega Y] = [\bigvee S^k, G/PL].$$

Here we use that in a CW decomposition of T^n, all the attaching maps become null-homotopic after suspension. This reduces ΣT^n to a wedge of spheres. We have been imprecise with which spheres are involved. There is one wedge summand S^{k+1} for each k-cell of T^n, with $k \geq 1$.

Let us consider the surgery exact sequence for S^n. We have

$$[\Sigma S^n, G/PL] \to L_{n+1}(\mathbb{Z}) \to \mathcal{S}_{PL}(S^n) \to [S^n, G/PL] \to L_n(\mathbb{Z}) \to \cdots$$

By the PL Poincaré conjecture, $\mathcal{S}_{PL}(S^n) \cong \{[S^n]\}$ for $n \geq 5$. Therefore for $n \geq 6$ we have:

$$\pi_n(G/PL) \cong L_n(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & j \equiv 0 \mod 4 \\ 0 & j \equiv 1 \mod 4 \\ \mathbb{Z}/2 & j \equiv 2 \mod 4 \\ 0 & j \equiv 3 \mod 4. \end{cases}$$

In particular $\pi_n(G/PL)$ is 4-periodic. We can compute what happens in the low dimensions using knowledge of the homotopy groups of G and $O = \varinjlim_n O(n)$. Here is a summary, which relies on a certain amount of background knowledge. We will quote the relevant facts, to at least give some indication of what is needed. It is all independent of the theory of topological manifolds. There is a fibration

$$PL/O \to BO \to BPL$$

where PL/O is by definition the homotopy fibre.

Theorem 13.19. *The space PL/O is 6-connected.*

This follows from classical, deep theorems on smoothing PL manifolds in low dimensions. The long exact sequence in homotopy groups

$$\pi_n(PL/O) \to \pi_n(G/O) \to \pi_n(G/PL) \to \pi_{n-1}(PL/O)$$

for $n \leq 6$ implies that $\pi_n(G/PL) \cong \pi_n(G/O)$ for $n \leq 6$. The homotopy groups of BG are related to the stable homotopy groups of spheres by a shift. The homotopy groups of BO are known by Bott periodicity. The homotopy groups are connected by the J homomorphism. We have a long exact sequence

$$\cdots \to \pi_2(G/O) \to \pi_2(BO) \xrightarrow{J} \pi_2(BG) \to \pi_1(G/O) \to \pi_1(BO) \xrightarrow{J} \pi_1(BG)$$

We also know the following information on the groups and the maps in this sequence

$$\begin{tikzcd} \pi_1(BO) \arrow[r, "J"] \arrow[d, "\cong"] & \pi_1(BG) \arrow[d, "\cong"] \\ \mathbb{Z}/2 \arrow[r, "\cong"] & \mathbb{Z}/2 \end{tikzcd}$$

$$\begin{tikzcd} \pi_2(BO) \arrow[r, "J"] \arrow[d, "\cong"] & \pi_2(BG) \arrow[d, "\cong"] \\ \mathbb{Z}/2 \arrow[r, "\cong"] & \mathbb{Z}/2 \end{tikzcd}$$

$$\begin{tikzcd} \pi_3(BO) \arrow[r, "J"] \arrow[d, "\cong"] & \pi_3(BG) \arrow[d, "\cong"] \\ 0 \arrow[r] & \mathbb{Z}/2 \end{tikzcd}$$

$$\begin{tikzcd} \pi_4(BO) \arrow[r, "J"] \arrow[d, "\cong"] & \pi_4(BG) \arrow[d, "\cong"] \\ \mathbb{Z} \arrow[r, two heads] & \mathbb{Z}/24. \end{tikzcd}$$

In addition $\pi_5(BG) = \pi_5(BO) = \pi_6(BG) = 0$. It is then straightforward to compute that the 4-periodicity persists into the low dimensions, namely for $n \in \{1,2,3,4,5\}$ we have:

$$\pi_n(G/PL) \cong \begin{cases} 0 & n = 1,3,5 \\ \mathbb{Z}/2 & n = 2 \\ \mathbb{Z} & n = 0,4. \end{cases}$$

So in fact $\pi_n(G/PL) \cong L_n(\mathbb{Z})$ for all $n \geq 0$. Moreover,

$$[T^n, G/PL] \cong \mathcal{N}_{PL}(T^n) \cong\cong \bigoplus_{0 \leq i < n} \bigoplus \binom{n}{i} L_{n-i}(\mathbb{Z})$$

and

$$L_n(\mathbb{Z}[\mathbb{Z}^n]) \cong \bigoplus_{0 \leq i \leq n} \bigoplus \binom{n}{i} L_{n-i}(\mathbb{Z})$$

are almost isomorphic, the only difference being the extra copy of $L_0(\mathbb{Z}) \cong \mathbb{Z}$ when $i = n$ that appears in $L_n(\mathbb{Z}[\mathbb{Z}^n])$.

13.5.3 The surgery obstruction map is injective.

Proposition 13.20. *The surgery obstruction map* $\sigma\colon [T^n, G/PL] \to L_n(\mathbb{Z}[\mathbb{Z}^n])$ *is injective.*

Proof. Here is a sketch of the proof. Suppose that $\xi \in [T^n, G/PL]$ (we use the notation for a bundle since the set $[T^n, G/PL]$ indexes PL fibre bundles lifting the normal spherical fibration). Suppose that $\sigma(\xi) = 0$. We induct on n. Since $\pi_1(G/PL) = 0$, the base case holds.

We are going to ignore issues with low dimensions for this sketch. Really at the start of the induction we should cross with $\mathbb{CP}^2$ to get into sufficiently high dimensions, and use that crossing with $\mathbb{CP}^2$ realises the 4-periodicity of the surgery obstruction. To avoid the details of this, let us assume we have already done the induction as far as $n = 5$. Recall the computation above that gives the first equality:

$$[T^n, G/PL] = [\bigvee S^{k+1}, Y] = \prod [S^{k+1}, Y] = \prod [S^k, G/PL].$$

The maps in the product are sent under σ to the surgery obstructions of sub-tori $T^k \subseteq T^n$. They are null-homotopic by the inductive hypothesis, except for on the top cell. To understand the obstruction on the top cell we have the following diagram.

$$\begin{tikzcd} {[T^n, G/PL]} \arrow[r] & L_n(\mathbb{Z}[\mathbb{Z}^n]) \\ {[S^n, G/PL]} \arrow[u] \arrow[r, "\cong"] & L_n(\mathbb{Z}) \arrow[u, hook] \end{tikzcd}$$

Here the left vertical arrow is given by collapsing the $(n-1)$-skeleton. That the right vertical arrow is injective follows easily from the definitions: a stable isometry over $\mathbb{Z}[\mathbb{Z}^n]$ augments to one over $\mathbb{Z}$. Since the right-then-up route is an injection, it follows that $\xi = 0$ as desired. $\square$

This shows that indeed there is a unique normal bordism class in $\mathcal{N}_{PL}(T^n)$ that contains a homotopy equivalence.

13.5.4 Constructing normal maps producing given elements of $L_{n+1}(\mathbb{Z}[\mathbb{Z}^n])$. We are left with the question: what is the image of σ? On the left of the surgery exact sequence, this image in $L_{n+1}(\mathbb{Z}[\mathbb{Z}^n])$ equals the stabiliser of $\mathrm{Id}_{T^n} \in \mathcal{S}_{PL}(T^n)$, and the orbit of this element is what we want to compute.

We construct the degree one normal maps that give elements of $L_{n+1}(\mathbb{Z}[\mathbb{Z}^n])$, as suggested by the title of this subsection. Let $J \subseteq \{1,\ldots,n\}$ and write

$$H := \{1,\ldots,n\} \setminus J.$$

These subsets correspond to sub-tori $T_J, T_H \subseteq T^n$. For example if $J = \{1,2,4\} \subseteq \{1,\ldots,5\}$ then $T_J = S^1 \times S^1 \times \{*\} \times S^1 \times \{*\}$. Write

$$m = |J|.$$

Let

$$\begin{array}{ccc} \nu_M & \xrightarrow{F} & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & D^{m+1} \end{array}$$

be a degree one normal map, restricting to a PL homeomorphism on the boundary $\partial M \to S^m$, realising the generator of

$$L_{m+1}(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & m+1 \equiv 0 \mod 4 \\ 0 & m+1 \equiv 1 \mod 4 \\ \mathbb{Z}/2 & m+1 \equiv 2 \mod 4 \\ 0 & m+1 \equiv 3 \mod 4. \end{cases}$$

if $m+1 \neq 4$. If $m+1 = 4$, then we instead realise twice the generator of $L_4(\mathbb{Z}) \cong \mathbb{Z}$. Such a degree one normal map exists by Kervaire and Milnor's plumbing construction, which is a special case of Wall realisation. This gives such an element for $n \neq m+1$. Part of this construction is the fact that an m-dimensional homology sphere bounds a contractible $(m+1)$-dimensional PL manifold. This is true by surgery methods for $m+1 \geq 5$, but it is not true for $m = 3$ in general. For example the Poincaré homology sphere does not bound a contractible PL 4-manifold. More generally, we have Rochlin's important theorem. This theorem will be the underlying source of the main differences between the PL and topological categories in high dimensions.

Theorem 13.21 (Rochlin [Roc52])**.** *Let X be a smooth or PL, closed, spin 4-manifold. Then 16 divides the signature of X.*

Therefore it is not possible to realise the generator of $L_4(\mathbb{Z})$ by a degree one normal map $M \to D^4$. Here the signature of X is the signature of the middle dimensional intersection form on $H_2(X;\mathbb{R})$, which is nonsingular.

Spin 4-manifolds have even intersection forms, by the Wu formula $w_2(X) \cap x = x \cdot x \in \mathbb{Z}/2$ for all $x \in H_2(X;\mathbb{Z})$. Then it is an algebraic fact that 8 divides the signature. The converse, that even intersection form implies spin, is also true if $H_1(X;\mathbb{Z})$ has no 2-torsion. That 16 divides the signature uses the existence of a smooth or PL structure. In fact Freedman showed that there is a simply-connected topological 4-manifold with even intersection form and signature 8, so Rochlin's theorem does not hold for topological 4-manifolds.

It is perhaps rather remarkable that this theorem on 4-manifolds will have so many consequences for high dimensional manifolds.

Now we construct the normal maps desired. We use the boundary connected sum $\natural$ in the construction, which means choosing a copy of D^m in $T_J \times \{1\}$ and in ∂M, and identifying them. Take $N \to T^m \times I$ to be the normal bordism over $T^m \times I$ given by:

$$\begin{array}{ccc} N & \xrightarrow{=} & ((T_J \times I)\natural M) \times T_H \\ \downarrow & & \downarrow \\ T^n \times I & \xrightarrow{=} & ((T_J \times I)\natural D^{m+1}) \times T_H \end{array}$$

These can be concatenated, and sums of them realised every element of $L_{n+1}(\mathbb{Z}[\mathbb{Z}^n])$ *except* for the summand

$$\bigoplus^{\binom{n}{n-3}} L_4(\mathbb{Z}) \cong \bigoplus^{\binom{n}{n-3}} \mathbb{Z} \cong \wedge^{n-3}\mathbb{Z}^n.$$

Note that $(n+1)-(n-3)=4$. In this summand, only the even elements are realised.

So to get nontrivial manifolds τ^n homotopy equivalent to T^n, apply Wall realisation to $\mathrm{Id}\colon T^n \to T^n$ with an element of $\bigoplus^{\binom{n}{n-3}} L_4(\mathbb{Z})$ with a nonzero number of odd entries. The manifold on the far end of the resulting normal bordism will be a homotopy torus that is not PL homeomorphic to T^n.

13.5.5 Detecting homotopy tori. Suppose that we have an n-manifold $N \simeq T^n$ that we wish to show is not homeomorphic to T^n. We describe an obstruction for doing this. We will see that the obstruction vanishes in the 2^n-fold cover, which will complete our sketch of the proof of Theorem 13.4.

Let N be a closed PL n-manifold, $n \geq 5$, and let $f\colon N \xrightarrow{\simeq} T^n$ be a homotopy equivalence. Let $((W; N, T^n)$ be a normal bordism over $T^n \times I$ with $F\colon W \to T^n \times I$, and $F|_{T^n} = \mathrm{Id}\colon T^n \to T^n \times \{1\}$ and $F|_N = f\colon N \to T^n \times \{0\}$. Let $J \subseteq \{1,\dots,n\}$ be a subset with $|J|=3$, and consider the corresponding subtorus $T_J \subseteq T^n$. Also let $H := \{1,\dots,n\} \setminus J$. Consider

$$F \times \mathrm{Id}\colon W \times \mathbb{CP}^2 \to T^n \times I \times \mathbb{CP}^2.$$

This raises the dimensions sufficiently to be able to apply high dimensional surgery theory and the Whitney trick when we need it. Make $F \times \mathrm{Id}$ transverse, using PL-transversality, to $T_J \times I \times \mathbb{CP}^2$. This is codimension $n-3$ in $T^n \times I \times \mathbb{CP}^2$ and therefore the inverse image of $T_J \times I \times \mathbb{CP}^2$ is dimension $n+1+4-(n-3)=8$. By a result called the Farrell-Hsiang splitting theorem, and the fact that $\mathrm{Wh}(\mathbb{Z}[\mathbb{Z}^n])=0$, we can assume that the inverse image is a homotopy equivalence on the boundary. We take the surgery obstruction of

$$(F \times \mathrm{Id})^{-1}(T_J \times I \times \mathbb{CP}^2) \to T_J \times I \times \mathbb{CP}^2$$

in

$$\begin{aligned} &L_8(\mathbb{Z}) \cong \mathbb{Z} \\ &(P,\varphi) \mapsto \mathrm{sign}(\varphi \otimes \mathbb{R})/8, \end{aligned}$$

that is we take the signature of the intersection form and divide it by 8. Then we consider this modulo 2 in $\mathbb{Z}/2$. It turns out that this is independent of the choice of bordism W. This procedure gives a function

$$\Upsilon\colon \{J \subseteq \{1,\dots,n\} \mid |J|=3\} \to \mathbb{Z}/2.$$

This can be translated to an element of $(\wedge^{n-3}\mathbb{Z}^n) \otimes \mathbb{Z}/2$. This gives the bijection we claimed

$$\mathcal{S}_{PL}(T^n) \cong (\wedge^{n-3}\mathbb{Z}^n) \otimes \mathbb{Z}/2.$$

Hsiang-Shaneson also show that the action of $\mathrm{GL}_n(\mathbb{Z})$ is equivariant with respect to this bijection, so that the classification of PL homotopy tori is as claimed.

Finally, we see from the description of the obstruction that passing to the 2^n fold cover $\widetilde{N}$ of N, and therefore to the corresponding cover of W, will have the effect of replacing each inverse image of $T_J \times I \times \mathbb{CP}^2$ by an even number of copies of itself. Therefore the associated map Υ will be identically zero, so that $\widetilde{N} \cong_{PL} T^n$, as desired for Theorem 13.4.

Remark 13.22. Throughout the chapter, we have used simple homotopy type for PL manifolds, PL transversality, PL immersion theory, and we have mentioned smooth handlebody theory. These tools are essential for developing and using surgery theory. Having seen these tools be so important in the remarkable classification theorem for homotopy tori that we have just discussed, the reader of this chapter will now hopefully be motivated to learn these methods in the topological category. With their help, we will be able to apply similar methods to classify topological manifolds. These will be conseqeunces of the Product Structure Theorem, which we will study soon.

We also remark that in the calculations, we used a number of deep results from algebraic topology, in particular on the J homomorphism, on stable homotopy groups of spheres, and on the homotopy groups of BO, as well as Rochlin's theorem.

Sol. on p.150. **Exercise 13.1**. (PS9.2) Up to PL-homeomorphism, how many closed PL manifolds homotopy equivalent to T^6 are there?

14 Local contractibility for manifolds and isotopy extension

The goal of this section is to review the main results of Edwards and Kirby [EK71]. This paper builds on the ideas of Kirby from [Kir69], and in particular, we will see another torus trick. This will be similar in flavour to the proof of Theorem 10.4, and we will work purely in the topological category (other than the initial input of an immersed torus) – no further input from PL topology will be necessary. In particular, there are no dimension restrictions in this section.

We will highlight two results. The following was first proved by Černavskiĭ using push-pull methods. We will give the torus trick proof from [EK71].

Theorem 14.1 ([Č73, EK71])**.** *If M is a compact manifold, then* $\mathrm{Homeo}(M)$ *is locally contractible.*

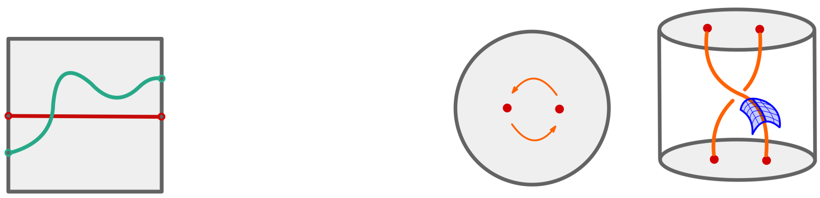
For the next result, we need some preliminary definitions, see Fig. 43.

Definition 14.2. Let M be a manifold and $U \subseteq M$ a subset, with the inclusion denoted by $g\colon U \hookrightarrow M$. An embedding $h\colon U \hookrightarrow M$ is *proper* if $h^{-1}(\partial M) = g^{-1}(U)$. An isotopy $h_t\colon U \to M$ is *proper* if each h_t is proper.

Definition 14.3. A proper isotopy $h_t\colon N \to M$ is *locally flat* if for each $(x,t) \in N \times [0,1]$ there exists a neighbourhood $[t_0,t_1]$ of $t \in [0,1]$ and level preserving embeddings $\alpha\colon D^n \times [t_0,t_1] \to N \times [0,1]$ and $\beta\colon D^n \times D^{m-n} \times [t_0,t_1] \to M \times [0,1]$ onto neighbourhoods of (x,t) such that the following diagram commutes:

$$\begin{tikzcd} D^n \times 0 \times [t_0,t_1] \arrow[r] \arrow[d, "\alpha"] & D^n \times D^{m-n} \times [t_0,t_1] \arrow[d, "\beta"] \\ N \times [0,1] \arrow[r, "(x,t)\mapsto(h_t(x),t)"] & M \times [0,1] \end{tikzcd}$$

Recall that the bottom map is called the *track* of the isotopy.



(A) Red inclusion $B^1 \subseteq B^1 \times B^1$ and a green proper embedding.

(B) Locally flat isotopy and its track.

Figure 43

The definition of a locally flat isotopy says that the track is a locally flat submanifold in a level preserving way. Since the track is in particular locally flat, we infer that, for example, the naïve isotopy taking the trefoil to the unknot is not locally flat. This can be seen using local fundamental groups.

Theorem 14.4 (Isotopy extension theorem [EK71, Corollary 1.2, Corollary 1.4], [Lee69])**.**

1. *Let $h_t\colon C \to M$, $t \in [0,1]$ be a proper isotopy of a compact set $C \subseteq M$, such that h_t extends to a proper isotopy of a neighborhood $U \supseteq C$. Then h_t can be* covered by an (ambient) isotopy, *that is, there exists $H_t\colon M \to M$, satisfying $H_0 = \mathrm{Id}_M$ and $h_t = H_t \circ h_0$ for all $t \in [0,1]$.*
2. *For manifolds M and N with N compact, any locally flat proper isotopy $h_t\colon N \to M$ is covered by an ambient isotopy. If $h_t = h_0$ for all t on a neighbourhood of ∂N, then we may assume that $H_t|_{\partial M} = \mathrm{Id}_{\partial M}$.*

In both cases, we may assume that H has compact support, that is, $H_t = \mathrm{Id}_M$ outside some compact set, for each t.

Remark 14.5. Part (a) of the theorem above was proved independently in both [EK71] and [Lee69]. Both papers use techniques of Kirby from [Kir69].

14.1 Handle straightening

We will consider the following spaces of embeddings.

Definition 14.6. For a manifold M and subsets $C \subseteq U \subseteq M$ we define

$$\mathrm{Emb}_C(U, M) := \{f\colon U \hookrightarrow M \mid f \text{ is proper}, f|_C = \mathrm{incl}\},$$

equipped with the compact open topology. If $C = \emptyset$ we write $\mathrm{Emb}(U, M)$.

The following lemma is the key ingredient in [EK71]. The proof will use the torus trick. Throughout this section, the notation rB^i refers to the i-dimensional closed ball of radius r centred at the origin in $\mathbb{R}^i$.

Lemma 14.7 (Handle straightening)**.** *There exists a neighbourhood $Q \subseteq \mathrm{Emb}_{\partial B^k \times 4B^n}(B^k \times 4B^n, B^k \times \mathbb{R}^n)$ of the inclusion $\eta\colon B^k \times 4B^n \hookrightarrow B^k \times \mathbb{R}^n$, and a* deformation of Q into the subspace $\mathrm{Emb}_{\partial B^k \times 4B^n \cup B^k \times B^n}(B^k \times 4B^n, B^k \times \mathbb{R}^n)$, modulo $\partial(B^k \times 4B^n)$, *and fixing η.*

In more detail, such a deformation of Q is a map

$$\Psi\colon Q \times [0, 1] \to \mathrm{Emb}_{\partial B^k \times 4B^n}(B^k \times 4B^n, B^k \times \mathbb{R}^n)$$

for which

(1) $\Psi(Q \times 1) \subseteq \mathrm{Emb}_{\partial B^k \times 4B^n \cup B^k \times B^n}(B^k \times 4B^n, B^k \times \mathbb{R}^n)$.
(2) $\Psi(h, t)|_{\partial(B^k \times 4B^n)} = h|_{\partial(B^k \times 4B^n)}$ for all $h \in Q$ and $t \in [0, 1]$, and
(3) $\Psi(\eta, t) = \eta$ for all $t \in [0, 1]$.

The proof of this lemma is analogous to the proof of the Černavskiĭ-Kirby theorem we saw in Section 10.2. The goal will be to construct $\widetilde{h} \in \mathrm{Homeo}(B^k \times \mathbb{R}^n)$ for an h suitably close to η, such that

$$\widetilde{h}|_{B^k \times B^n} = h|_{B^k \times B^n} \quad \text{and} \quad \widetilde{h}|_{\partial B^k \times B^n \cup B^k \times (\mathbb{R}^n \setminus \mathrm{Int}\, 3B^n)} = \mathrm{Id}\,.$$

We will then use an Alexander isotopy $\widetilde{H}_t$ of the target space $B^k \times \mathbb{R}^n$ from Id to $\widetilde{h}$ to define the desired deformation:

$$\Psi(h, t) := \widetilde{H}_t^{-1} \circ h.$$

Indeed, $\Psi(h, 0) = h$ and the restriction of $\Psi(h, 1) = \widetilde{h}^{-1} \circ h$ to the core region is the standard inclusion, since $\widetilde{h}$ and h agree there. We will arrange to have Ψ constant on $\partial(B^k \times 4B^n)$. Our construction will be "canonical", so the different isotopies can be sewn together to produce the desired map $\Psi\colon Q \times [0, 1] \to \mathrm{Emb}_{\partial B^k \times 4B^n}(B^k \times 4B^n, B^k \times \mathbb{R}^n)$.

Proof. Let C_1 denote a collar of ∂B^k in B^k and let C denote $C_1 \times 3B^n$. It suffices to consider $h \in \mathrm{Emb}_{\partial B^k \times 4B^n \cup C}(B^k \times 4B^n, B^k \times \mathbb{R}^n)$ by [EK71, Proposition 3.2]. Roughly speaking, by using the collar C, the proposition gives an explicit deformation from a neighbourhood of η in $\mathrm{Emb}_{\partial B^k \times 4B^n}(B^k \times 4B^n, B^k \times \mathbb{R}^n)$ to $\mathrm{Emb}_{\partial B^k \times 4B^n \cup C}(B^k \times 4B^n, B^k \times \mathbb{R}^n)$.

Thus, we begin with a setup as in Fig. 44. Our goal is to build $\widetilde{h} \in \mathrm{Homeo}(B^k \times \mathbb{R}^n)$ such that

$$\widetilde{h}|_{\partial B^k \times \mathbb{R}^n \cup B^k \times (\mathbb{R}^n \setminus \mathrm{Int}(3B^n))} = \mathrm{Id} \quad \text{ and } \widetilde{h}|_{B^k \times B^n} = h|_{B^k \times B^n}.$$

Let $S^1 := {[-4, 4]}/{\sim}$, so that $T^n \supseteq aB^n$ for $a < 4$. Define $B^n := [-1, 1]^n$. Choose closed, nested balls $D_1^k \subseteq D_2^k \subseteq D_3^k \subseteq \mathring{B}^k$, such that $D_i^k \subseteq \mathring{D}_{i+1}^k$ for each i and $B^k \setminus D_1^k \subseteq C_1$. Choose closed, nested balls $D_1^n \subseteq D_2^n \subseteq D_3^n \subseteq T^n \setminus 2B^n$,

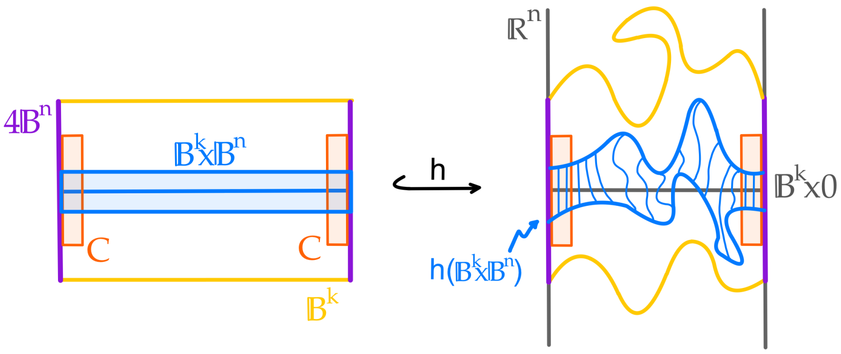


FIGURE 44. The setup of the Handle Straightening Lemma 14.7.

As in the proof of Theorem 10.4 in Section 10.2 we will construct the following tower of maps.

$$\begin{tikzcd}
B^k \times \mathbb{R}^n \arrow[r, hook, two heads, "\widetilde{h}", "\cong"'] & B^k \times \mathbb{R}^n \\
3B^k \times 3B^n \arrow[r, "\widetilde{h}"] \arrow[u, hook] \arrow[d, "\gamma"'] & \mathbb{R}^k \times \mathbb{R}^n \arrow[u, hook] \arrow[d, "\gamma"] \\
B^k \times \mathbb{R}^n \arrow[r, hook, two heads, "\check{h}", "\cong"'] \arrow[d, "\mathrm{Id} \times e"'] & B^k \times \mathbb{R}^n \arrow[d, "\mathrm{Id} \times e"] \\
B^k \times T^n \arrow[r, hook, two heads, "\overline{h}", "\cong"'] & B^k \times T^n \\
(B^k \times T^n) \setminus (D^k_3 \times D^n_3) \arrow[r, hook, "\widehat{h}|"] \arrow[u, hook] \arrow[d, hook] & (B^k \times T^n) \setminus (D^k_1 \times D^n_1) \arrow[u, hook] \arrow[d, hook] \\
(B^k \times T^n) \setminus (D^k_2 \times D^n_2) \arrow[r, hook, "\widehat{h}"] & (B^k \times T^n) \setminus (D^k_1 \times D^n_1) \\
B^k \times (T^n \setminus D^n_2) \arrow[r, hook, "\widehat{h}"] \arrow[u, hook] \arrow[d, looparrowright, "\mathrm{Id} \times \alpha_0"'] & B^k \times (T^n \setminus D^n_1) \arrow[u, hook] \arrow[d, looparrowright, "\mathrm{Id} \times \alpha_0"] \\
B^k \times 4B^n \arrow[r, hook, two heads, "h"] & B^k \times \mathbb{R}^n
\end{tikzcd}$$

See Fig. 45 for a schematic version.

We start with an immersed torus $\alpha_0 \colon T^n \setminus D^n_1 \looparrowright \operatorname{Int}(3B^n)$ with $\alpha_0|_{2B^n} = \mathrm{Id}$. Then define the map $\alpha := \mathrm{Id} \times \alpha_0 \colon B^k \times (T^n \setminus D^n_2) \to B^k \times 4B^n$. We choose Q so that it is possible to construct the lift $\widehat{h}$. This is quite similar to the proof of Theorem 10.4 so we skip the details. Briefly, the map $\widehat{h}$ is defined to agree with $\alpha^{-1} \circ h \circ \alpha$ on small neighbourhoods. The set Q is chosen small enough so that the image of $B^k \times (T^n \setminus D^n_2)$ under $\widehat{h}$ lies within $B^k \times (T^n \setminus D^n_1)$.

Observe that $\widehat{h}|_{(B^k \setminus D^k) \times (T^n \setminus D^n_2)} = \mathrm{Id}$, since h agrees with the inclusion map on $C := C_1 \times 3B^n$, and by construction we have $\alpha(T^n \setminus D^n_2) \subseteq \operatorname{Int} 3B^n$ and $B^k \setminus D^k \subseteq C_1$. Thus we can extend $\widehat{h}$ by the identity to obtain the map in the third row of the diagram from the bottom. In Proposition 10.7, we showed a version of the Schoenflies theorem for the torus. There is also a version for $B^k \times T^n$, which can be made canonical. Applying this, followed by the Alexander coning trick, we obtain the homeomorphism $\overline{h}$ in the next row in the diagram. More precisely,

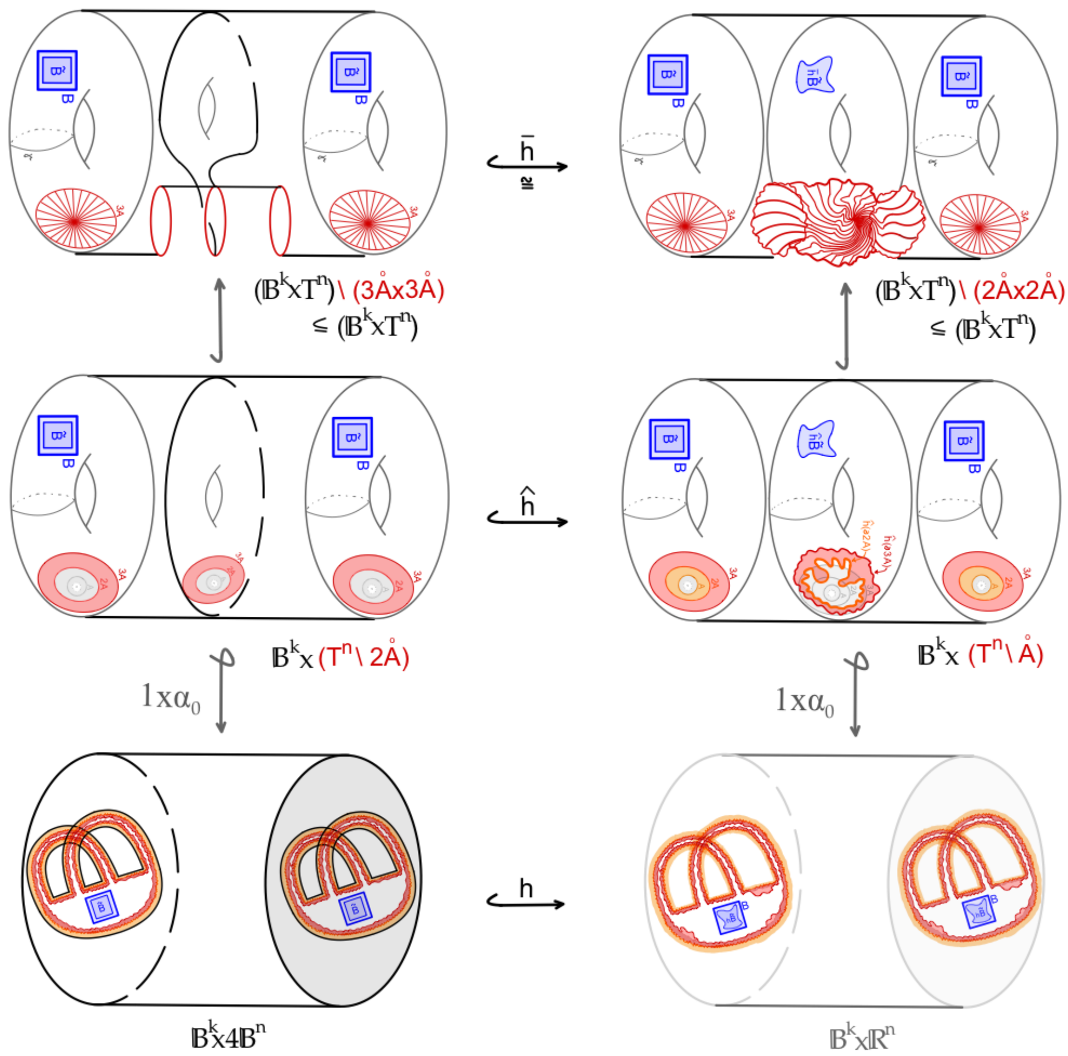


FIGURE 45. The torus trick for handle straightening.

we first consider the restriction of $\widehat{h}$ to $(B^k \times T^n) \setminus (D^k_3 \times D^n_3)$, and observe that the image of $\partial(D^k_3 \times D^n_3)$ is a bicollared sphere in $B^k \times T^n$, and therefore bounds a ball in $B^k \times T^n$. Extending the map over this ball in the codomain and the ball $D^k_3 \times D^n_3$ in the domain produces the homeomorphism

$$\overline{h}\colon B^k \times T^n \xrightarrow{\cong} B^k \times T^n.$$

By choosing Q to be small enough, we may assume that $\overline{h}$ is homotopic to $\mathrm{Id}_{B^k \times T^n}$ (see Step 5 of Section 10.2), so that its lift $\breve{h}$ to universal covers is bounded distance from Id (see Proposition 10.2). As in the proof of Theorem 10.4, we choose the covering map e so that $B^k \times 2B^n$ is mapped by the identity.

Recall that our goal is to define a homeomorphism $\widetilde{h}\colon B^k \times \mathbb{R}^n \to B^k \times \mathbb{R}^n$ which restricts to the identity outside a compact set. While $\breve{h}$ is a homeomorphism of $B^k \times \mathbb{R}^n$, it cannot be our desired map, since being obtained as a lift to a covering space, if it were to restrict to the identity outside a compact set, it would equal the identity map everywhere. In the next (and final) step of the construction of $\widetilde{h}$ we will modify $\breve{h}$ to arrange for the desired behaviour. Roughly speaking, we will rescale so that all the nontrivial behaviour of $\breve{h}$ is concentrated in a compact region in such a way that we can extend by the identity everywhere else. The strategy is similar to the proof of Proposition 12.15.

Observe that $\breve{h}|_{\partial B^k \times \mathbb{R}^n} = \mathrm{Id}$ since $h|_C$ coincides with the inclusion map and $\alpha_0(T^n \setminus D^n_1) \subseteq 3B^n$. So we can extend $\breve{h}$ by the identity to get a map $\breve{h}\colon \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^k \times \mathbb{R}^n$.

Define $\gamma\colon \operatorname{Int}(3B^k \times 3B^n) \xrightarrow{\cong} \mathbb{R}^n$ as a radial expansion fixed on $2B^k \times 2B^n$. Then define

$$\widetilde{h}\colon B^k \times \mathbb{R}^n \to B^k \times \mathbb{R}^n \text{ as } \begin{cases} \gamma^{-1}\breve{h}\gamma, & \text{on } B^k \times 3B^n, \\ \mathrm{Id}, & \text{on } B^k \times (\mathbb{R}^n \setminus \operatorname{Int}(3B^n)). \end{cases}$$

The above map is continuous since $\breve{h}$ is bounded distance from the identity. The homeomorphism $\widetilde{h}$ agrees with h on $B^k \times B^n$, by our definition of γ and α. It also satisfies $\widetilde{h}|_{\partial B^k \times \mathbb{R}^n \cup B^k \times (\mathbb{R}^n \setminus \operatorname{Int}(3B^n))} = \mathrm{Id}$. To see this, first we note that $\widetilde{h}|_{B^k \times (\mathbb{R}^n \setminus \operatorname{Int}(3B^n))}$ by explicit construction. We know that $\widetilde{h}|_{\partial B^k \times \mathbb{R}^n} = \mathrm{Id}$ since $\breve{h}|_{\partial B^k \times \mathbb{R}^n} = \mathrm{Id}$. Each step in the construction has been canonical, so $\widetilde{h}$ depends continuously on h, and from our construction we note that for $h = \eta$ we have $\widetilde{h} = \mathrm{Id}$. This finishes the construction of $\widetilde{h}$.

To finish off the proof of the lemma, extend $\widetilde{h}$ by the identity map to get $\widetilde{h}\colon B^k \times \mathbb{R}^n \to B^k \times \mathbb{R}^n$, which depends continuously on h (see Step 3 of Section 10.2).

Define the isotopy

$$\widetilde{H}_t\colon B^k \times \mathbb{R}^n \to B^k \times \mathbb{R}^n \text{ by } \begin{cases} t\widetilde{h}(\frac{1}{t}x) & t > 0 \\ x & t = 0 \end{cases}$$

(compare Proposition 10.3) taking the identity to $\widetilde{h}$. Define

$$\Psi(h,t) := \widetilde{H}_t^{-1} \circ h\colon B^k \times 4B^n \to B^k \times \mathbb{R}^n$$

so that $\Psi(h,0) = \widetilde{H}_0^{-1} h = \mathrm{Id}^{-1} h = h$ and $\Psi(h,1) = \widetilde{H}_1^{-1} h = \widetilde{h}^{-1} h$, as desired.

We finally check that this isotopy has all the desired properties. By choosing Q small enough we arrange that $h(B^k \times \partial 4B^n) \cap B^k \times 3B^n = \emptyset$. Then since H_t restricts to the identity on $B^k \times (\mathbb{R}^n \setminus \operatorname{Int}(3B^n))$, we see that Ψ is modulo $\partial(B^k \times 4B^n)$. Since h and $\widetilde{h}$ agree on $B^k \times B^n$, we see that $\Psi(h,1)|_{B^k \times B^n} = \widetilde{h}^{-1} h|_{B^k \times B^n}$ is the inclusion, so indeed Ψ gives a deformation of Q into $\mathrm{Emb}_{\partial B^k \times 4B^n \cup B^k \times B^n}(B^k \times 4B^n, B^k \times \mathbb{R}^n)$. The map $\Psi\colon Q \times [0,1] \to \mathrm{Emb}_{\partial B^k \times 4B^n}(B^k \times 4B^n, B^k \times \mathbb{R}^n)$ is continuous since our construction has been canonical throughout. Finally, one should check that $\Psi(\eta, t) - \eta$ for all t. $\square$

14.2 Applying handle straightening

The following theorem generalises the last lemma to "straightening a compact set" in a manifold.

Theorem 14.8 ([EK71, Theorem 5.1])**.** *Let M be a manifold and $C \subseteq U \subseteq M$ where U is an open neighbourhood of the compact set C. Then there exists a neighbourhood P of the inclusion $\eta\colon U \hookrightarrow M$ and a deformation*

$$\phi\colon P \times [0,1] \to \mathrm{Emb}(U, M)$$

into $\mathrm{Emb}_C(U, M)$ modulo the complement of a compact neighbourhood of C in U, and fixing η.

Using this we easily prove Theorem 14.1, i.e. that $\mathrm{Homeo}(M)$ for a compact manifold M is locally contractible.

Proof of Theorem 14.1. Set $C := U := M$, and note $\mathrm{Emb}(M, M) = \mathrm{Homeo}(M)$ (as embeddings are always proper in this section) and $\mathrm{Emb}_M(M, M) = \{\mathrm{Id}_M\}$. Then apply Theorem 14.8. $\square$

Sketch proof of Theorem 14.8. Assume $\partial M = \emptyset$ for simplicity. (The case of nonempty boundary can be reduced to this case by using a boundary collar. See [EK71] for more details.) Let

$\{h_i\colon W_i \xrightarrow{\cong} \mathbb{R}^n\}_{1\leq i\leq r}$ be a finite cover of C by Euclidean neighbourhoods, with $W_i \subseteq U$ for each i. Such a cover exists since C is compact and M is a manifold. Write $C = \bigcup_{i=1}^r C_i$ where each $C_i \subseteq W_i$ is compact, and define $D_i := \bigcup_{j\leq i} C_i$ for $1 \leq i \leq r$ (see Fig. 46).

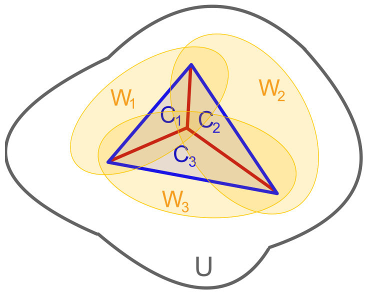


FIGURE 46. Proof of Theorem 14.8

We will induct on $i \geq 0$ and prove that for every $i \geq 0$, there exists a neighbourhood P_i of $\eta\colon U \hookrightarrow M$ in $\mathrm{Emb}(U, M)$ and a deformation $\phi_i\colon P_i \times [0,1] \to \mathrm{Emb}(U, M)$ into $\mathrm{Emb}_{U\cap V_i}(U, M)$ where V_i is some neighbourhood of D_i. (We are focussing on building the deformation rather than the "modulo" or "fixing" portions of the conclusion.)

For the base case $i = 0$, we just take $P_0 = \mathrm{Emb}(U, M)$ and $\phi_0 = \mathrm{Id}$. Now assume the inductive hypothesis for some $i \geq 0$. To prove the $i + 1$ case, identity W_{i+1} with $\mathbb{R}^m$ (using the map h_{i+1}) for convenience. That is, we have $C_{i+1} \subseteq \mathbb{R}^m$ compact, and $V_i \cap \mathbb{R}^m$ is a neighbourhood in $\mathbb{R}^m$ of the closed set $D_i \cap \mathbb{R}^m$.

Let N be a compact neighbourhood of $C_{i+1}\cap D_i$ in $\mathrm{Int}(V_i\cap\mathbb{R}^m)$. Choose a (small) triangulation of $\mathbb{R}^m(= W_{i+1})$. Define K to be the subcomplex of this triangulation consisting of all the simplices that intersect $C_{i+1} \cup N$. Let L be the subcomplex consisting of all the simplices that intersect N. Then we obtain a handle decomposition of K relative to L as explained in Section 11.3. Observe that we have the following properties:

1. $D_i \cap C_{i+1} \subseteq L \subseteq \mathrm{Int}(V_i \cap \mathbb{R}^m)$
2. $C_{i+1} \subseteq K$
3. $\overline{K \setminus L} \cap D_i = \emptyset$
4. If A is a handle of $K\setminus L$ with index k, there exists an embedding $\mu\colon B^k\times\mathbb{R}^n \hookrightarrow \mathbb{R}^m$, where $m = k+n$, such that $\mu(B^k\times B^n) = A$ and $\mu(B^k\times\mathbb{R}^n)\cap(D_i\cup L\cup\overline{K^k \setminus A}) = \mu(\partial B^k \times B^n)$ where K^k is the k-skeleton of K.

Let $A_1, \ldots, A_j, \ldots, A_s$ be the handles of $K \setminus L$ of non-decreasing index. Now we will induct on j. This will finally enable us to apply handle straightening (Lemma 14.7) to each A_j. Specifically, for each $j \geq 0$, define $D'_j := D_i \cup L \cup \bigcup_{\ell\leq j} A_j$. We will prove that for each $j \geq 1$ there exists a neighbourhood P'_j of the inclusion $\eta\colon U \hookrightarrow M$ in $\mathrm{Emb}(U, M)$ and a deformation $\phi'_j\colon P'_j \times [0,1] \to \mathrm{Emb}(U, M)$ into $\mathrm{Emb}_{U\cap V'_j}(U, M)$ where V'_J is some neighbourhood of D'_j in M. The base case $j = 0$ is satisfied the hypothesis in the bigger induction proof. Now assuming the case for some j, we prove the $j + 1$ case.

We know that for A_{j+1} there is a corresponding map $\mu\colon B^k \times \mathbb{R}^n \hookrightarrow \mathbb{R}^m$. Reparametrise in the $\mathbb{R}^n$ coordinate, fixing B^n so that $\mu(B^k \times 4B^n) \subseteq \mathrm{Int}(V'_j)$.

Now by handle straightening (Lemma 14.7), we know that there is some neighbourhood Q of the inclusion $\eta_0\colon B^k \times 4B^n \hookrightarrow B^k \times \mathbb{R}^n$ in $\mathrm{Emb}_{\partial B^k\times 4B^n}(B^k \times 4B^n, B^k \times \mathbb{R}^n)$ and a deformation ψ of Q into $\mathrm{Emb}_{\partial B^k\times 4B^n\cup B^k\times 2B^n}(B^k \times 4B^n, B^k \times \mathbb{R}^n)$, modulo $\partial(B^k \times 4B^n)$ and fixing η_0. Let Q' be a neighbourhood of the inclusion $\eta\colon U \hookrightarrow M$ in $\mathrm{Emb}_{U\cap V'_j}(U, M)$ such that if $h \in Q'$ then $h \circ \mu(B^k \times 4B^n) \subseteq \mu(B^k \times \mathbb{R}^n)$ and $\mu^{-1}h\mu|_{B^k\times 4B^n} \in Q$.

Next we will use ψ to define V'_{j+1} and deform Q'. For $h \in Q'$, define

$$h_t := \begin{cases} h & \text{on } U \setminus \mu(B^k \times 4B^n) \\ \mu\psi(\mu^{-1}h\mu, t)\mu^{-1}) & \text{on } \mu(B^k \times 4B^n) \end{cases}$$

Define $V'_{j+1} := (V'_j \cup \mu(B^k \times 2B^n)) \setminus \mu(B^k \times [2,4]B^n)$. Then $h_0 = h$ and $h_1 \in \mathrm{Emb}_{U \cap V'_{j+1}}(U, M)$. Define $\psi'(h,t) := h_t$, a deformation a of Q'. By the continuity of ϕ'_j, there exists a neighbourhood P'_{j+1} of η in $\mathrm{Emb}(U,M)$ so that $P'_{j+1} \subseteq P'_j$ and $\phi'_j(P'_{j+1} \times 1) \subseteq Q'$. Define the deformation ϕ'_{j+1} to be the result of performing the deformations ψ' and $\phi'_j|_{P'_{j+1} \times [0,1]}$ in order. This completes the induction on j, which completes in turn the induction on i. This completes the proof (sketch). □

14.3 Proof of the isotopy extension theorem

We recall the statement of the isotopy extension theorem.

Theorem 14.9 (Isotopy extension theorem [EK71, Corollary 1.2, Corollary 1.4])**.**

(1) Let $h_t\colon C \to M$, $t \in [0,1]$ be a proper isotopy of a compact set $C \subseteq M$, such that h_t extends to a proper isotopy of a neighborhood $U \supseteq C$. Then h_t can be covered by an (ambient) isotopy, *that is, there exists $H_t\colon M \to M$, satisfying $H_0 = \mathrm{Id}_M$ and $h_t = H_t \circ h_0$ for all $t \in [0,1]$.*

(2) For manifolds M and N with N compact, any locally flat proper isotopy $h_t\colon N \to M$ is covered by an ambient isotopy. If $h_t = h_0$ for all t on a neighbourhood of ∂N, then we may assume that $H_t|_{\partial M} = \mathrm{Id}_{\partial M}$.

In both cases, we may assume that H has compact support, that is, $H_t = \mathrm{Id}_M$ outside some compact set, for each t.

Proof. We prove the first part of the theorem. The plan is to construct H_t in small steps. Choose a compact neighbourhood V of C satisfying $C \subseteq V \subsetneq U$. Let h_t denote the extended isotopy $h_t\colon U \hookrightarrow M$. Such an extension exists by hypothesis.

Fix $T \in [0,1]$. By Theorem 14.8 we know there exists a neighbourhood P of the inclusion $\eta\colon h_T(U) \hookrightarrow M$ and a deformation $\phi\colon P \times [0,1] \to \mathrm{Emb}(h_T(U), M$ into $\mathrm{Emb}_{h_T(C)}(h_T(U,M)$ modulo $h_T(U \setminus V)$.

Let $N(T) \subseteq [0,1]$ denote a neighbourhood of $T \in [0,1]$ such that the composite

$$h_T(U) \xrightarrow{h_T^{-1}} U \xrightarrow{h_t} M$$

is in P for all $t \in N(T)$. Observe that $h_t \circ h_T^{-1} \in \mathrm{Emb}(h_T(U), M)$. Define

$$(H_T)_t\colon M \xrightarrow{\cong} M$$
$$x \mapsto \begin{cases} h_t \circ h_T^{-1} \left(\Phi(h_t \circ h_T^{-1}, 1)\right)^{-1}(x), & x \in h_T(U), \\ x, & x \in M \setminus h_T(U). \end{cases}$$

We need to check that the above is a continuous function. Recall that ϕ is modulo $h_T(U \setminus V)$, so for $x \in h_T(U \setminus V)$ we have $\phi(h_t \circ h_T^{-1}, 1)(x) = h_t \circ h_T^{-1}(x)$, so $(H_T)_t(x) = x$, showing that the two definitions match up. For continuity we also need to observe that Φ is continuous with respect to t, since the argument changes as t changes.

Next we check that H_T covers h_T locally. Since $\phi(h_t \circ h_T^{-1}, 1) \in \mathrm{Emb}_{h_T(C)}(h_T(U), M)$, we know that for $x \in h_T(C)$ we get $\phi(h_t \circ h_T^{-1}, 1)^{-1}(x) = x$. Thus, $(H_T)_t \circ h_T(y) = h_t \circ h_T^{-1} \circ h_T(y) = h_t(y)$, that is, $(H_T)_t \circ h_T|_C = h_t|_C$ for $t \in N(T)$.

We now use compactness of the interval $[0,1]$ to choose a finite partition $0 = t_0 < t_1 < \cdots < t_n = 1$. By the above argument, we have ambient isotopies $H_{i,t}\colon M \to M$ with $t \in [t_i, t_{i+1}]$ such that $h_t|_C = H_{i,t} \circ h_{t_i}|_C$ for all $t \in [t_i, t_{i+1}]$.

In order to build the desired ambient isotopy H_t out of these local isotopies, we induct on $i \geq 0$. For $i = 0$ we have $H_0 = \mathrm{Id}_M$. Assume inductively that we have constructed $H_t \colon M \xrightarrow{\cong} M$ with $t \in [0, t_i]$ such that $H_t \circ h_0|_C = h_t|_C$ for all $t \in [0, t_i]$ and $H_0 = \mathrm{Id}_M$.

We define H_t on $[t_i, t_{i+1}]$ by setting

$$H_t(x) = H_{i,t} \circ H_{i,t_i}^{-1} \circ H_{t_i} \text{ for } [t_i, t_{i+1}]$$

At $t = t_i$, we see that $H_{i,t_i} \circ H_{i,t_i}^{-1} \circ H_{t_i} = H_{t_i}$, so we have a well-defined map on $M \times [0, t_{i+1}]$. Additionally, we see that for $x \in C$ and $t \in [t_i, t_{i+1}]$,

$$\begin{aligned} H_t \circ h_0(x) &= H_{i,t} \circ H_{i,t_i}^{-1} \circ H_{t_i} \circ h_0(x) \\ &= H_{i,t} \circ H_{i,t_i}^{-1} \circ h_{t_i}(x) \\ &= H_{i,t} \circ h_{t_i}(x) \\ &= h_t(x) \end{aligned}$$

where we have used that $h_t|_C = H_{i,t} \circ h_{t_i}|_C$ for $t \in [t_i, t_{i+1}]$ and $H_{t_i} \circ h_0|_C = h_{t_i}|_C$.

For part b) of Theorem 14.4, we only have locally flat neighbourhoods (instead of a global neighbourhood U from part a)). The proof consists of applying part(a) in each local neighbourhood, and then gluing together these local isotopies, as above, to produce the desired ambient isotopy. For more details, we refer the reader to [EK71, Proof of Corollary 1.4]. □

Sol. on p.150. **Exercise 14.1.**(PS10.1) Prove the "strong Palais theorem". That is, let $n \geq 6$, let M be a connected oriented n-manifold and let $\phi, \psi \colon D^n \to M$ be locally collared embeddings with the same orientation-behaviour. Then there exists an isotopy $H_t \colon M \to M$ satisfying $H_0 = \mathrm{Id}$, and $H_1 \circ \phi = \psi$.

Sol. on p.152. **Exercise 14.2.**(PS10.2) Let M be a compact manifold. Prove that $\mathrm{Homeo}(\mathrm{Int}(M))$ is locally contractible. Recall that we saw earlier that the homeomorphism group of a noncompact manifold need not be locally contractible. The above gives an alternative proof that $\mathrm{Homeo}(\mathbb{R}^n)$ is locally contractible.

Hint: Let C be the compact manifold formed by removing an open collar of the boundary of M. Argue that a neighbourhood of the identity map in $\mathrm{Homeo}(\mathrm{Int}(M))$ can be deformed into $\mathrm{Homeo}_C(M)$, consisting of the homeomorphisms of M which restrict to the identity on C. Now deform $\mathrm{Homeo}_C(M)$ to $\{\mathrm{Id}\}$ using the collar.

Fix an orientation on S^m for every m. Let $f \colon S^n \to S^{n+2}$ be a locally flat embedding. We call $K := f(S^n)$ an n-knot. If S^n and S^{n+2} have their standard smooth structures, and if f is a smooth embedding, then we call K a smooth n-knot.

Sol. on p.152. **Exercise 14.3.**(PS11.1) For $n \geq 5$, show that the embeddings f and g defining two n-knots $K = f(S^n)$ and $J = g(S^n)$ are locally-flat isotopic if and only if there is an orientation preserving homeomorphism $F \colon S^{n+2} \to S^{n+2}$ such that $F(K) = J$, and $F|_K \colon K \to J$ is orientation preserving, with respect to the orientations induced by f and g.

Hint: You may use the isotopy extension theorem, as well as SH_m and its consequences. (The same holds for all $n \geq 1$, but we do not have the tools to prove it from the course.)

Sol. on p.153. **Exercise 14.4.**(PS11.2) For $n \geq 1$, show that the embeddings f and g defining two smooth n-knots $K = f(S^n)$ and $J = g(S^n)$ are smoothly isotopic if and only if there is an orientation preserving diffeomorphism $F\colon S^{n+2} \to S^{n+2}$ such that $F(K) = J$ and $g^{-1} \circ F \circ f\colon S^n \to S^n$ is smoothly isotopic to the identity.

You may use the smooth version of the isotopy extension theorem. The theorems from the course may not be very helpful.

15 Normal microbundles and smoothing of a manifold crossed with Euclidean space

We want to prove the following theorem, which is the start of smoothing theory. It gives a criterion in terms of microbundles under which, for a topological manifold M, there is a smooth structure on $M \times \mathbb{R}^q$ for some $q \geq 0$.

Theorem 15.1. *Let M be a topological manifold. Then $M \times \mathbb{R}^q$ admits a smooth structure for some q if and only if $\mathfrak{t}_M$ is stably isomorphic to $|\xi|$ for some vector bundle ξ over M.*

In order to prove this, we will need some more of the theory of microbundles, especially the notion of a normal microbundle to a locally flat embedding. Note that a locally flat embedding need not admit a normal microbundle.

15.1 Constructions of microbundles

Let $\mathfrak{X} = \{B \xrightarrow{i} E \xrightarrow{j} B\}$ be a microbundle.

Definition 15.2 (Restriction)**.** Define the *restricted microbundle* for a subset $A \subseteq B$, by

$$\mathfrak{X}|_A = \{A \xrightarrow{i|_A} j^{-1}(A) \xrightarrow{j|_{j^{-1}(A)}} A\}.$$

Restricted microbundle is a special case of the following construction (when f is an inclusion).

Definition 15.3 (Pullback)**.** Given a map $f\colon A \to B$ we define the *pullback microbundle*

$$f^*\mathfrak{X} \coloneqq \{A \xrightarrow{i'} E' \xrightarrow{pr_1} A\}$$

where $E' = \{(a, e) \in A \times E \mid f(a) = j(e)\}$ is the pullback and the map $i'\colon A \to E'$ is given by $i'(a) = (a, i \circ f(a))$.

In other words, the following diagram commutes

$$\begin{tikzcd}
A \arrow[dr, dotted, "i'"] \arrow[drr, bend left, "i \circ f"] \arrow[ddr, bend right, "\mathrm{Id}"'] & & \\
& E' \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & E \arrow[d, "j"] \\
& A \arrow[r, "f"] & B.
\end{tikzcd}$$

Theorem 15.4. *If A is paracompact, $\mathfrak{X} = \{B \to E \to B\}$ a microbundle, and $f, g\colon A \to B$ are homotopic, $f \simeq g$, then the two pullbacks $f^*\mathfrak{X} \cong g^*\mathfrak{X}$ are isomorphic.*

We refer the reader to Milnor's paper [Mil64, Theorem 3.1 & Section 6] for the proof. This theorem is important, as we shall use it several times, in particular to see (via an exercise on the problem sheets) that a microbundle over a contractible space is trivial.

Definition 15.5 (Whitney sums)**.** Given two microbundles $\mathfrak{X}_1, \mathfrak{X}_2$ over the same base B, their *Whitney sum* is the microbundle $\mathfrak{X}_1 \oplus \mathfrak{X}_2 \coloneqq \{B \xrightarrow{i_1 \times i_2} E(\mathfrak{X}_1 \oplus \mathfrak{X}_2) \xrightarrow{p} B\}$, using the pullback

$$\begin{tikzcd}
B \arrow[dr, dotted, "i_1 \times i_2"] \arrow[drr, bend left, "i_2"] \arrow[ddr, bend right, "i_1"'] & & \\
& E(\mathfrak{X}_1 \oplus \mathfrak{X}_2) \arrow[r] \arrow[d] \arrow[dr, dotted] & E(\mathfrak{X}_2) \arrow[d, "j_2"] \\
& E(\mathfrak{X}_1) \arrow[r, "j_1"] & B
\end{tikzcd}$$

where the two dotted maps are canonical maps $i_1 \times i_2 = (i_1(b), i_2(b))$ and $p(e_1, e_2) = j_1(e_1) = j_2(e_2)$.

Definition 15.6 (Cartesian product)**.** Given two microbundles $\mathfrak{X}_1, \mathfrak{X}_2$ over possibly distinct base spaces $B(\mathfrak{X}_1)$ and $B(\mathfrak{X}_2)$, we define the *product microbundle* $\mathfrak{X}_1 \times \mathfrak{X}_2$ by

$$B(\mathfrak{X}_1) \times B(\mathfrak{X}_2) \xrightarrow{i_1 \times i_2} E(\mathfrak{X}_1) \times E(\mathfrak{X}_2) \xrightarrow{j_1 \times j_2} B(\mathfrak{X}_1) \times B(\mathfrak{X}_2)$$

Remark 15.7. With these definitions, the Whitney sum of two microbundles over the same base is the same as the pullback $\Delta^*(\mathfrak{X}_1 \times \mathfrak{X}_2)$ of the product, along the diagonal $\Delta\colon B \to B \times B$.

Lemma 15.8. *The tangent microbundle of a product $\mathfrak{t}_{M \times N}$ is isomorphic to the product of tangent microbundles $\mathfrak{t}_M \times \mathfrak{t}_N$.*

Proof. In the following diagram, the outside vertical maps are the identity maps, and the middle vertical map permutes the coordinates as appropriate to make the diagram commute.

$$\begin{tikzcd}
M \times N \arrow[r, "\Delta_{M\times N}"] & M \times N \times M \times N \arrow[r, "\mathrm{pr}_{1,2}"] & M \times N \\
M \times N \arrow[r, "\Delta_M \times \Delta_N"] \arrow[u, "\mathrm{Id}"] & M \times M \times N \times N \arrow[r, "\mathrm{pr}_{1,3}"] \arrow[u] & M \times N \arrow[u, "\mathrm{Id}"]
\end{tikzcd}$$

The top row describes $\mathfrak{t}_{M \times N}$, while the bottom row describes $\mathfrak{t}_M \times \mathfrak{t}_N$. Since the middle map is a homeomorphism, the two microbundles are isomorphic. □

Recall that $\mathfrak{e}^n_B$ denotes the standard trivial microbundle of fibre dimension n over B.

Definition 15.9. Two microbundles $\mathfrak{X}, \mathfrak{X}'$ over B are *stably isomorphic* if

$$\mathfrak{X} \oplus \mathfrak{e}^q_B \cong \mathfrak{X}' \oplus \mathfrak{e}^r_B$$

for some $q, r \geq 0$. We denote the stable isomorphism class of $\mathfrak{X}$ by $[\mathfrak{X}]$ and define the operation

$$[\mathfrak{X}] + [\mathfrak{X}'] \coloneqq [\mathfrak{X} \oplus \mathfrak{X}'].$$

Since Whitney sum is commutative and associative, this operation makes the set of stable isomorphism classes of microbundles over B into a commutative monoid, with $[\mathfrak{e}^n_B]$ as the unit. Thanks to the following theorem, if B is a manifold then all elements have inverses.

Theorem 15.10. *Let B be a manifold or finite CW complex. Let $\mathfrak{X}$ be a microbundle over B. Then there exists a microbundle η over B such that $\mathfrak{X} \oplus \eta$ is trivial.*

For the proof see [Mil64, Theorem 4.1].

Definition 15.11. Denote the abelian group of microbundles with base B a manifold or finite CW complex, up to stable isomorphism, with Whitney sum as the group operation, by $k_{\mathrm{TOP}}(B)$.

15.2 Normal microbundles

Definition 15.12 (Normal microbundle)**.** Let $M^m \subseteq N^n$ be a submanifold. We say that M has a *microbundle neighbourhood* in N if there exists a neighbourhood $U \supseteq M$ and a retraction $j\colon U \to M$ such that

$$M \xrightarrow{\mathrm{incl}} U \xrightarrow{j} M$$

is a microbundle. We call it a *normal microbundle* $\mathfrak{n}_{M \hookrightarrow N}$ of M in N.

Remark 15.13. If M has a normal microbundle, then M is locally flat. This is superfluous, since we actually defined a submanifold as being locally flat. However, it is worth emphasising, since the converse is false in general, i.e. locally flat submanifolds need not have normal microbundles. This is somewhat unfortunate, but will turn out to be manageable.

Note that the situation is special in codimensions 1 and 2, where it is known that locally flat embeddings admit normal microbundles. In fact they admit normal bundles in these codimensions.

Milnor [Mil64, Theorem 5.8] proved that for every embedding $M \subseteq N$, there is an integer q such that the composition $M \to N \to N \times \mathbb{R}^q$ admits a normal microbundle. Stern improved this later with quantitative bounds as follows. Intermediate results were also proven by Hirsch, but Stern's bounds seem to be the best known.

Theorem 15.14 (Stern, [Ste75, Theorem 4.5])**.** *Let $M^m \subseteq N^n$ be a submanifold of codimension $q = n - m$ and pick $j \in \{0, 1, 2\}$.*

(1) If $m \leq q + 1 + j$ and $q \geq 5 + j$, then there exists a normal microbundle.
(2) Any two normal microbundles $\mathfrak{n}$ and $\mathfrak{n}'$ for M are isomorphic if $m \leq q + j$.

In particular, for all submanifolds $M \subseteq N$, $M \subseteq N \times \{0\} \subseteq N \times \mathbb{R}^q$ admits an essentially unique normal microbundle for some $q \gg 0$.

Our short term goal is to use microbundles to give a description of when for a given manifold M, the product $M \times \mathbb{R}^q$ admits a smooth structure for some q. For this we need to develop more theory of normal microbundles.

Lemma 15.15. *Every trivial microbundle is isomorphic to the trivial $\mathbb{R}^n$-fibre bundle. More precisely, if $\mathfrak{X} = \{B \to E \to B\}$ is isomorphic to the trivial microbundle over B of rank n, over a paracompact space B, then there exists $U \subseteq E$ with $U \cong B \times \mathbb{R}^n$ such that the following diagram commutes*

$$\begin{tikzcd} & B \times \mathbb{R}^n \cong U \arrow[d] \arrow[dr, "pr_1"] & \\ B \arrow[ur, "\times 0"] \arrow[r] & E \arrow[r] & B. \end{tikzcd}$$

To prove this one observes that E can be assumed to be an open subset of $B \times \mathbb{R}^n$ and then rescales this, see [Mil64, Lemma 2.3]. We apply this to the case when a normal microbundle is trivial, obtaining a criterion under which we can find an actual product neighbourhood.

Corollary 15.16. *Suppose $M^m \subseteq N^n$ admits a trivial normal microbundle. Then M is* flat*, that is there exists an embedding $M \times \mathbb{R}^{n-m} \hookrightarrow N$ with $(x, 0) \mapsto x$ for all $x \in M$.*

One can ask to what extent is a normal microbundle unique.

Theorem 15.17. *Assume $M^m \subseteq N^n$ is a submanifold which has a normal microbundle. Then*

$$\mathfrak{t}_M \oplus \mathfrak{n}_{M \hookrightarrow N} \cong \mathfrak{t}_N|_M$$

Recall that Theorem 15.10 states that $k_{TOP}(M)$ is a group, namely that any microbundle over a finite CW complex B has a stable inverse. We can now show this for manifolds.

Proof of Theorem 15.10 for B a manifold. Consider an embedding $M \subseteq \mathbb{R}^d$ for some d. By Theorem 15.14, by possibly increasing d, M has a normal microbundle $\mathfrak{n}_{M \hookrightarrow \mathbb{R}^d}$. By Theorem 15.17 we have

$$\mathfrak{t}_M \oplus \mathfrak{n}_{M \hookrightarrow \mathbb{R}^d} \cong \mathfrak{t}_{\mathbb{R}^d}|_M \cong \mathfrak{e}^d_M,$$

so $[\mathfrak{t}_M]$ has a stable inverse. □

Corollary 15.18. *Let $M \subseteq N$ be a submanifold. Then $[\mathfrak{t}_M] = [i^*\mathfrak{t}_N]$ if and only if there exists $q > 0$ such that $M = M \times \{0\} \subseteq N \times \mathbb{R}^q$ has a product neighbourhood $M \times \mathbb{R}^q$.*

Proof. By Theorem 15.17, we have $[\mathfrak{t}_M] + [\mathfrak{n}_{M \hookrightarrow N}] \cong [\mathfrak{t}_N|_M] \cong [i^*\mathfrak{t}_N] \cong [\mathfrak{t}_M]$. Now we can subtract these classes to obtain $[\mathfrak{n}_{M \hookrightarrow N}] \cong [\mathfrak{e}_M]$. Hence, by Corollary 15.16 the submanifold $M \times \{0\} \subseteq N \times \mathbb{R}^q$ has a product neighbourhood for some large q. □

Normal microbundles will also be useful in connection with topological transversality for submanifolds.

15.3 Precursor to smoothing theory

The following theorem is a preliminary step towards answering the question of when topological manifolds admit smooth structures.

Theorem 15.19. *Let M be a topological manifold. Then $M \times \mathbb{R}^q$ admits a smooth structure for some q if and only if $\mathfrak{t}_M$ is stably isomorphic to $|\xi|$ for some vector bundle ξ over M.*

Proof. Suppose that $M \times \mathbb{R}^q$ admits a smooth structure for some q. We have the following sequence of isomorphisms of microbundles

$$|\tau_{M\times\mathbb{R}^q}| \cong \mathfrak{t}_{M\times\mathbb{R}^q} \cong \mathfrak{t}_M \times \mathfrak{t}_{\mathbb{R}^q} \cong \mathfrak{t}_M \times \mathfrak{e}^q_{\mathbb{R}^q}.$$

For the first isomorphism we used Theorem 9.9, while the second is by Lemma 15.8. The third holds because the tangent microbundle of $\mathbb{R}^n$ is trivial. Restricting to $M \times \{0\}$ we have

$$|\tau_{M\times\mathbb{R}^q}||_{M\times\{0\}} \cong (\mathfrak{t}_M \times \mathfrak{e}^q_{\mathbb{R}^q})|_{M\times\{0\}} \cong \mathfrak{t}_M \oplus \mathfrak{e}^q_M,$$

where the final isomorphism follows from the commutative diagram

$$\begin{tikzcd}
M \times \mathbb{R}^q \arrow[r, "\Delta_M \times \Delta_{\mathbb{R}^d}"] & M \times M \times \mathbb{R}^q \times \mathbb{R}^q \arrow[r, "pr_{1,3}"] & M \times \mathbb{R}^q \\
M \arrow[u, "\mathrm{Id}\times 0"] \arrow[r, "\Delta_M \times 0"] & M \times M \times \mathbb{R}^q \arrow[u, "\mathrm{Id}\times\mathrm{Id}\times 0\times\mathrm{Id}"] \arrow[r, "pr_1"] & M \arrow[u, "\mathrm{Id}\times 0"]
\end{tikzcd}$$

Namely, the top row describes the product $\mathfrak{t}_M \times \mathfrak{e}^q_{\mathbb{R}^q}$, and by definition its restriction to $M \times \{0\}$ is obtained by precomposing with $\mathrm{Id} \times 0$ and restricting $p_{1,3}$ to the image of $\mathrm{Id} \times 0$. But this agrees with the bottom row, which is precisely the microbundle $\mathfrak{t}_M \oplus \mathfrak{e}^q_M$ over M.

Therefore, $\mathfrak{t}_M$ is stably isomorphic to the underlying microbundle of the smooth bundle $\tau_{M\times\mathbb{R}^q}|_{M\times\{0\}}$ (since restriction commutes with taking underlying microbundles). This completes the proof of the forwards direction.

Now for the converse, assume that $[\mathfrak{t}_M] = [|\xi|]$ for some smooth vector bundle ξ over M. Since topological manifolds are Euclidean Neighbourhood Retracts, there is an embedding $M \subseteq V \subseteq \mathbb{R}^k$ and a retraction $r\colon V \to M$ where V is open.

Therefore, ξ extends to a vector bundle $\xi' = r^*\xi$ over V. Since V is a smooth manifold and $BO(k)$ is an infinite union of finite dimensional smooth manifolds given by Grassmannians $\mathrm{Gr}_k(\mathbb{R}^q)$, by finite dimensionality of V we can approximate the classifying map $V \to BO(k)$ of ξ' by a map into a smooth manifold $\mathrm{Gr}_k(\mathbb{R}^q)$ for some q. In other words, we can assume ξ' is a smooth vector bundle, so that the total space $E(\xi')$ is a smooth manifold. Now $V \hookrightarrow E(\xi')$ and

$$\tau_V \oplus \xi' \cong \tau_E|_V.$$

Since $V \subseteq \mathbb{R}^k$ is open and a restriction of a trivial bundle $\tau_{\mathbb{R}^k} \cong \mathcal{E}^k$ is also trivial, we have that $\tau_V \cong \mathcal{E}^k$. Hence, restricting to M gives

$$\mathcal{E}^k \oplus \xi \cong \tau_E|_M$$

and therefore for the underlying microbundles

$$|\mathcal{E}^k| \oplus |\xi| \cong \mathfrak{t}_E|_M.$$

By assumption $|\xi|$ is stably isomorphic to $\mathfrak{t}_M$, so $\mathfrak{t}_E|_M$ is also stably isomorphic to $\mathfrak{t}_M$.

From Corollary 15.18 it follows that $M \times \{0\} \subseteq E \times \mathbb{R}^s$ has a product neighbourhood, that is $M \times \mathbb{R}^q \subseteq E \times \mathbb{R}^s$ is an open subset of a smooth manifold. Therefore it has a smooth structure obtained from pulling back the smooth structure on $E \times \mathbb{R}^s$ as in Proposition 15.20 below. □

Proposition 15.20. *Let $U \subseteq M$ be a topological manifold embedded as an open subset of a smooth manifold. Then U admits a smooth structure.*

Proof. Choose a collection of charts for M that cover U, $\{V_\alpha\}$. Refine the cover so that all the intersections $\{V_\alpha \cap U\}$ are again charts, homeomorphic to $\mathbb{R}^n$. This is possible as we can choose small open balls around every point contained in U, and restrictions of homeomorphisms are homeomorphisms. Note that since U is open this is a collection of open subsets.

Then the transition functions of $\{V_\alpha \cap U\}$ are restrictions of the transition functions for the V_α, so they are again smooth. The maximal smooth atlas containing $\{V_\alpha \cap U\}$ is a smooth structure on U. □

Remark 15.21. If we can show that $\mathfrak{t}_M$ is stably isomorphic to $|\xi|$ for some smooth vector bundle ξ over M, can we get a smooth structure on M? We could ask a similar question in the PL category, assuming we had a good definition of a PL bundle. There is such a definition, but we will not introduce it here. We now know from Theorem 15.19 that one can find a smooth structure for $M \times \mathbb{R}^q$ for some $q \geq 0$.

The work of Kirby and Siebenmann, which we will study soon, shows that, for manifolds of dimension at least 5, one can improve a smooth or PL structure on $M \times \mathbb{R}$ to a smooth structure on M. So in fact the result we have just proven will be extremely useful, since it is the starting point for actually finding a smooth or PL structure on M itself.

Kirby and Siebenmann's results, when combined with the results of surgery theory, will also allow us to compute the number of distinct smooth or PL structures on a given underlying topological manifold of dimension at least 5. The theorem just proven gives the first hint that such a procedure might be possible.

Sol. on p.153. **Exercise 15.1**. (PS4.2) Let $M^m \subseteq N^n$ be a submanifold with a normal microbundle $\mathfrak{n}_M$. Then

$$\mathfrak{t}_M \oplus \mathfrak{n}_M \cong \mathfrak{t}_N|_M.$$

Look in Milnor [Mil64] for the idea, but fill in the details.

16 Classifying spaces

We will now study classifying spaces $B\,\mathrm{TOP}(n)$, $B\,\mathrm{PL}(n)$ and $BO(n)$ for the corresponding three types of $\mathbb{R}^n$ fibre bundles, and their stable analogues $B\,\mathrm{TOP}$, $B\,\mathrm{PL}$ and BO.

The relationship between these objects is that we will define the limiting classifying spaces, for $\mathrm{CAT} = \mathrm{TOP}, \mathrm{PL}$, or O, as

$$B\mathrm{CAT} := \bigcup_n B\mathrm{CAT}(n)$$

using the inclusions $B\mathrm{CAT}(n) \hookrightarrow B\mathrm{CAT}(n+1)$ induced by crossing with the identity map on $\mathbb{R}$.

The stable classifying spaces in particular will play a key rôle in smoothing and PL-ing theory, which we are going to discuss in the next section. This theory enables us to decide whether one can put the extra corresponding extra structure, smooth or PL respectively, on a topological manifold, and can decide how many such structures exist.

The theory of classifying spaces gives rise to universal spaces whose homotopy types measure the difference between the categories. These spaces, quite amazingly, allow us to convert geometric computations for specific manifolds into global statements for all manifolds.

The key property of classifying spaces that we will use is that homotopy classes of maps to them correspond to isomorphism classes of the related bundles. For example, $[X, BO(n)]$ is in bijection with the isomorphism classes of n-dimensional vector bundles over the CW complex X, and $[X, BO]$ is in bijection with the collection of stable isomorphism classes of vector bundles over X.

To connect with the previous section, there is a forgetful map $f\colon BO \to B\,\mathrm{TOP}$ and for a topological manifold M there is a classifying map $\mathfrak{t}_M\colon M \to B\,\mathrm{TOP}$ of the stable tangent microbundle. The tangent microbundle is, stably, the underlying microbundle of a smooth vector bundle if and only if there is a lift $\tau_M\colon M \to BO$ with $\mathfrak{t}_M = f \circ \tau_M\colon M \to BO \to B\,\mathrm{TOP}$. The analogous statement holds for PL instead of O. So in particular classifying spaces can decide whether there exists a smooth or PL structure on $M \times \mathbb{R}^q$ for some q. We will also see that they can quantify these structures.

The classifying spaces $BO(n)$ may be already familiar to you; they are given by the Grassmannian of n-planes in $\mathbb{R}^\infty$. See e.g. [MS74]. The others take a bit more work to describe. To do so we briefly recall the notion of a semi-simplicial set.

16.1 Semi-simplicial sets

Definition 16.1. Define the category $\Delta_\bullet$ to have objects

$$\{\{0,1,\dots,n\} \mid n \in \mathbb{N}_0\}$$

and morphisms the injective order-preserving maps

$$\{0,1,\dots,m\} \to \{0,1,\dots,n\}$$

for $m \le n$.

We write $[n] := \{0,1,\dots,n\}$. There are n injective order preserving maps

$$[n-1] := \{0,1,\dots,n-1\} \to [n] := \{0,1,\dots,n\}.$$

Definition 16.2. A *semi-simplicial set/group/space* is a functor

$$S_\bullet\colon \Delta_\bullet^{\mathrm{op}} \to \begin{cases} \text{Set} \\ \text{Group} \\ \text{Space} \end{cases}$$

from the opposite category of $\Delta_\bullet$ to the appropriate category of sets, groups, or spaces.

For a comprehensive source on semi-simplicial sets and spaces, we refer to [ER19]. This definition is quick to make, but not so easy to parse, so we unwind it a little. A semi-simplicial set consists of the following.

(i) For each $p \in \mathbb{N}_0$, a set, the *set of p-simplices*, $X_p := X_\bullet([p])$.
(ii) A collection of maps $\partial_i^p \colon X_p \to X_{p-1}$, for $i = 0, \dots, p$, such that
$$\partial_i^{p-1} \circ \partial_j^p = \partial_{j-1}^{p-1} \circ \partial_i^p \colon X_p \to X_{p-2} \quad i < j.$$
These are called the *face maps*.

Example 16.3. As an example, a simplicial complex determines a semi-simplicial set, where the p simplices are the p-simplices, and the face maps give rise to the ∂_i^p.

Here is another famous example. For any space Y, the *singular semi-simplicial set $Y_\bullet$* of Y is the semi-simplicial set with p-simplices Y_p given by the singular p-simplices, that is the continuous maps of the geometric simplex Δ_p to Y. Precomposing with the inclusion of the ith face $\Delta^{p-1} \to \Delta^p$ gives the map $\partial_i^p \colon Y_p \to Y_{p-1}$.

Remark 16.4. You may have heard of the notion of simplicial sets. These have extra structure, the so-called *degeneracy maps*. In some contexts, having this extra structure is very important. Our aim is to pass to geometric realisations, and the geometric realisation of a simplicial set and its underlying semi-simplicial set are homotopy equivalent, so there is no need to

Here is our main example.

Example 16.5. Let Γ be a monoid, e.g. $\Gamma = \mathrm{TOP}(n)$. We define the semi-simplicial set $B\Gamma_\bullet$ as the following collection of data. $B\Gamma_0$ is a singleton, and for each $p > 0$ we have the set of p-simplices:
$$B\Gamma_p := \Big\{(g_1, \dots, g_p) \mid g_i \in \Gamma\Big\}$$
and for every $0 \leq i \leq p$ a boundary map
$$\partial_i^p \colon B\Gamma_p \to B\Gamma_{p-1} \quad (g_1, \dots, g_p) \mapsto \begin{cases} (g_2, \dots, g_p), & i = 0 \\ (g_1, \dots, g_i \cdot g_{i+1}, g_{i+2}, \dots, g_p), & 1 \leq i \leq p-1 \\ (g_1, \dots, g_{p-1}), & i = p. \end{cases}$$
Then we have that $\partial_i^{p-1} \circ \partial_j^p = \partial_{j-1}^{p-1} \circ \partial_i^p$ for $i < j$, fulfilling the definition of a semi-simplicial set.

This is easy to generalise to any small category, with the 0-simplices the objects, and with p-tuples of composable morphisms as the p-simplices.

In another direction, if Γ is a topological monoid or group then $B\Gamma_p$ is a *space*, and $B\Gamma_\bullet$ is a semi-simplicial *space*.

Note that this definition of a semi-simpicial space also encapsulates any monoid Γ, since we can give a monoid Γ the discrete topology, in order to make it into a topological monoid, albeit in a somewhat uninteresting way. When we apply this machinery with $\Gamma = O(n)$ or $\mathrm{TOP}(n)$, the topology on these spaces is not the discrete topology, it will be the usual topology on $O(n)$ as a subset of $\mathbb{R}^{n^2}$, and the compact-open topology on $\mathrm{TOP}(n)$.

Definition 16.6 (Geometric realisation)**.** Let $X_\bullet$ be a semi-simplicial set/space. The *geometric realisation* $\|X_\bullet\|$ of $X_\bullet$ is defined as the quotient
$$\|X_\bullet\| := \bigsqcup_{p \geq 0} X_p \times \Delta^p \big/ \sim.$$
Here we consider X_p as a space using the discrete topology, in the case that $X.$ is a semi-simplicial set, and we use the given topology on X_p in the case that $X_\bullet$ is a semi-simplicial space. Δ_p is a

space, with the subspace topology from $\mathbb{R}^{p+1}$:

$$\Delta^p := \{(x_0, \ldots, x_p) \in \mathbb{R}^{p+1} \mid \sum_{j=0}^{p} x_j = 1,\, x_j \geq 0 \text{ for all } j\}.$$

Let $\iota_i^{p-1} \colon \Delta^{p-1} \hookrightarrow \Delta^p$ be the inclusion of ith face, for $i = 0, \ldots, p$. The equivalence relation is given by:

$$(x, \iota_i^{p-1}(y)) \sim (\partial_i^p x, y)$$

for $x \in X_p$, $y \in \Delta^{p-1}$ and $0 \leq i \leq p$.

Definition 16.7. Given a (topological) monoid Γ, for the semi-simplicial set (space) from Example 16.5, define $B\Gamma := \|B\Gamma_\bullet\|$, the geometric realisation of this semi-simplicial set (space).

This concludes our short introduction to semi-simplicial sets. We have just included enough information in order to be able to describe the constructions of the classifying spaces we will need. The properties of classifying spaces will be assumed without proof, since this theory is not special to the world of topological manifolds. To understand these properties in more detail, we would need to expand on the theory of semi-simplicial sets and spaces as well.

16.2 Defining classifying spaces

Definition 16.8. Apply Example 16.5 and Definition 16.7 to the topological groups (and therefore monoids) $\mathrm{TOP}(n)$ and $O(n)$ to obtain semi-simplicial spaces $B\,\mathrm{TOP}(n)_\bullet$ and $BO(n)_\bullet$. Similarly this construction applied to the topological monoid $G(n)$ of homotopy self-equivalences of S^{n-1} yields the semi-simplicial space $BG(n)_\bullet$. Then we have:

$$\begin{aligned} B\,\mathrm{TOP}(n) &:= \|B\,\mathrm{TOP}(n)_\bullet\| \\ BO(n)' &:= \|BO(n)_\bullet\|. \end{aligned}$$

Theorem 16.9. *We have a homotopy equivalence $BO(n)' \simeq BO(n) := \mathrm{Gr}_n(\mathbb{R}^\infty)$.*

It can be useful to have both models for the same homotopy type. The former can be more easily compared with $B\,\mathrm{TOP}(n)$, while the latter can be useful for computations, and the fact that it is a limit of smooth manifolds was used when we found a smooth structure on $M \times \mathbb{R}^q$ in Section 15.3.

For the piecewise linear case we need a slightly more involved construction.

Example 16.10. We define a semi-simplicial set $\mathrm{PL}(n)_\bullet$. Define $\mathrm{PL}(n)_p$ to be the set of PL homeomorphisms $\Delta^p \times \mathbb{R}^n \to \Delta^p \times \mathbb{R}^n$ such that

$$\begin{tikzcd} \Delta^p \times \mathbb{R}^n \arrow[rr] \arrow[dr] & & \Delta^p \times \mathbb{R}^n \arrow[dl] \\ & \Delta^p & \end{tikzcd}$$

commutes, where the downwards arrows are projection onto the first factor. Define a map $\mathrm{PL}(n)_p \to \mathrm{PL}(n)_{p-1}$ by sending $f \colon \Delta^p \times \mathbb{R}^n \to \Delta^p \times \mathbb{R}^n$ to the restriction

$$f|_{\iota_i^{p-1}(\Delta^{p-1})\times\mathbb{R}^n} \colon \Delta^{p-1}) \times \mathbb{R}^n \to \Delta^{p-1}) \times \mathbb{R}^n.$$

This determines a semi-simplicial set $\mathrm{PL}(n)_\bullet$. Note that we do not have a natural topology on the sets here. We will use the geometric realisation to obtain a topology. That is, we define:

$$\mathrm{PL}(n) := \|\,\mathrm{PL}(n)_\bullet\|.$$

Example 16.11. Now we define $B\,\mathrm{PL}(n)$. For each $p \geq 0$, note that $\mathrm{PL}(n)_p$, the PL-homeomorphisms of $\Delta^p \times \mathbb{R}^n$ over Δ^p, is a group (not a topological group). Form the semi-simplicial set $B(PL(n)_p)_\bullet$ via the procedure in Example 16.5.

Then we define a semi-simplicial space $Y_\bullet$ by

$$Y_p := \|B(\mathrm{PL}(n)_p)_\bullet\|$$

Now each Y_p is a space. The face maps $\partial_i^q(Y)\colon Y_q \to Y_{q-1}$ of $Y_\bullet$ are induced by the face maps of $\mathrm{PL}(n)_\bullet$:

$$\mathrm{Id} \times (\partial_i^p)^q \colon \Delta^q \times (\mathrm{PL}(n)_p)^q \to \Delta^q \times (\mathrm{PL}(n)_{p-1})^q.$$

Finally we define

$$B\,\mathrm{PL}(n) := \|Y_\bullet\|$$

as the geometric realisation of the semi-simplicial space $Y_\bullet$. We performed a level-wise B construction, and then we combined the levels into a semi-simplicial space $Y_\bullet$, and then realising that gave the classifying space $B\,\mathrm{PL}(n)$.

Ultimately, for our intended applications to smoothing and PLing of topological manifolds, we will need the stable classifying spaces. For each of $\mathrm{CAT} = \mathrm{TOP}, \mathrm{PL}, O$, define

$$\mathrm{CAT}(n) \hookrightarrow \mathrm{CAT}(n+1)$$

inclusions induced by crossing with the identity map on $\mathbb{R}$. These in turn induce maps $B\mathrm{CAT}(n) \hookrightarrow B\mathrm{CAT}(n+1)$. If necessary replace these by cofibrations using mapping cylinders, and define

$$B\mathrm{CAT} := \bigcup_n B\mathrm{CAT}(n)$$

to be the infinite union. This defines stable classifying spaces

$$BO,\ B\,\mathrm{PL},\ \text{and}\ B\,\mathrm{TOP}\,.$$

The key fact about all of these classifying space is the following theorem.

Theorem 16.12. *If X be a paracompact space. For $n \in \mathbb{N}$ there is a universal* CAT *bundle $\gamma^n_{\mathrm{CAT}} \to B\mathrm{CAT}(n)$ such that the correspondence*

$$[f\colon X \to B\mathrm{CAT}(n)] \mapsto f^*(\gamma^n_{\mathrm{CAT}})$$

induces a $1-1$ correspondence between homotopy classes of maps $[X, B\mathrm{CAT}(n)]$ and isomorphism classes of CAT *$\mathbb{R}^n$-bundles.*

Similarly stable isomorphism classes of such bundles are in $1-1$ correspondence with homotopy classes of maps $[X, B\mathrm{CAT}]$.

Taking the classifying map of the underlying $\mathrm{TOP}(n)$ bundle of the universal CAT $\mathbb{R}^n$ bundle γ^n_{CAT} induces a homotopy class of maps

$$p_{\mathrm{CAT}}(n)\colon B\mathrm{CAT}(n) \to B\,\mathrm{TOP}(n)$$

. This respects the stabilisations, so that

$$\begin{tikzcd}
\cdots \arrow[r] & B\mathrm{CAT}(n) \arrow[r] \arrow[d, "p_{\mathrm{CAT}}(n)"] & B\mathrm{CAT}(n+1) \arrow[r] \arrow[d, "p_{\mathrm{CAT}}(n+1)"] & \cdots \\
\cdots \arrow[r] & B\,\mathrm{TOP}(n) \arrow[r] & B\,\mathrm{TOP}(n+1) \arrow[r] & \cdots
\end{tikzcd}$$

commutes. In the limit we obtain a map

$$p_{\mathrm{CAT}}\colon B\mathrm{CAT} \to B\,\mathrm{TOP}\,.$$

Studying the failure of this map to be a homotopy equivalence measures the difference between topological and CAT manifolds. We will start this process in the next section.

16.3 Comparing stable classifying spaces.

Let CAT stand for PL or DIFF. Since many facts and proofs will work equally well for both PL and DIFF categories, it will be convenient to have the notation CAT that refers to either of them.

We define the spaces

$$\begin{aligned}{}^{\mathrm{TOP}}\!/_{\mathrm{CAT}} &:= \operatorname{hofib}\Big(B\mathrm{CAT} \xrightarrow{p_{\mathrm{CAT}}} B\,\mathrm{TOP}\Big)\\ &:= \{(x,\gamma) \mid x \in B\mathrm{CAT},\ \gamma\colon [0,1] \to B\,\mathrm{TOP},\ \gamma(0) = x, \gamma(1) = p_{\mathrm{CAT}}(x)\}\end{aligned}$$

as the homotopy fibre of the map $p_{\mathrm{CAT}}\colon B\mathrm{CAT} \to B\,\mathrm{TOP}$. This is the same as replacing this map by a fibration, changing $B\mathrm{CAT}$ by a homotopy equivalence to a path space, and then taking the fibre at the basepoint. So there is a homotopy fibre sequence

$$^{\mathrm{TOP}}\!/_{\mathrm{CAT}} \xrightarrow{j} B\mathrm{CAT} \xrightarrow{p_{\mathrm{CAT}}} B\,\mathrm{TOP}\,.$$

Similarly there are fibre sequences

$$^{\mathrm{TOP}(n)}\!/_{\mathrm{CAT}(n)} \longrightarrow B\mathrm{CAT}(n) \longrightarrow B\,\mathrm{TOP}(n)$$

$$^{\mathrm{PL}}\!/_{O} \longrightarrow BO \longrightarrow B\,\mathrm{PL}$$

$$^{\mathrm{PL}(n)}\!/_{O(n)} \longrightarrow BO(n) \longrightarrow B\,\mathrm{PL}(n).$$

In each case the left-most space is by definition the homotopy fibre of the right hand map. We will restrict attention to the stable versions from now on, since it is these that are relevant for smoothing and PL-ing theory in the next section.

Theorem 16.13 (Boardman-Vogt [BV68])**.** *The space* $^{\mathrm{TOP}}\!/_{\mathrm{CAT}}$ *has the homotopy type of a loop space, that is there exists a space* $B^{\mathrm{TOP}}\!/_{\mathrm{CAT}}$ *such that*

$$^{\mathrm{TOP}}\!/_{\mathrm{CAT}} \simeq \Omega B^{\mathrm{TOP}}\!/_{\mathrm{CAT}}.$$

In fact, the stable classifying spaces and their homotopy fibres $^{\mathrm{TOP}}\!/_{\mathrm{CAT}}$*,* $B\mathrm{CAT}$*,* $B\,\mathrm{TOP}$*,* $B\,\mathrm{PL}$*,* BO*, and* $^{\mathrm{PL}}\!/_{O}$ *are infinite loop spaces.*

An elementary consequence is that

$$\pi_i\left(^{\mathrm{TOP}}\!/_{\mathrm{CAT}}\right) \cong \pi_i\left(\Omega B^{\mathrm{TOP}}\!/_{\mathrm{CAT}}\right) \cong \pi_{i+1}\left(B^{\mathrm{TOP}}\!/_{\mathrm{CAT}}\right).$$

Now we relate CAT bundle structures to lifts of classifying maps. A CAT structure Σ on a manifold M determines a CAT tangent bundle, and therefore a lift of the stable classifying map $\mathfrak{t}_M\colon M \to B\,\mathrm{TOP}$ to $B\mathrm{CAT}$.

Let M be a topological manifold with ∂M equipped with a CAT structure. Then we have the diagram:

(16.14)
$$\begin{tikzcd} & \partial M \arrow[r, hook, "incl"] \arrow[d, "\rho"'] & M \arrow[d, "\mathfrak{t}_M"] \arrow[dl, dotted, "\theta"'] & \\ ^{\mathrm{TOP}}\!/_{\mathrm{CAT}} \arrow[r, "j"] & B\mathrm{CAT} \arrow[r, "F"] & B\,\mathrm{TOP} \arrow[r, "G"] & B^{\mathrm{TOP}}\!/_{\mathrm{CAT}}. \end{tikzcd}$$

The bottom row is a fibration sequence, and the first three entries form a principal fibration. Therefore, questions about existence and uniqueness of CAT structures on $\mathfrak{t}_M$, extending the given CAT structure on the topological tangent bundle of ∂M, are equivalent to the existence and uniqueness respectively of a lift of the map θ. This leads to the following theorem. To state it we need the notion of concordant bundle structures.

Definition 16.15. Let ξ be a TOP $\mathbb{R}^n$ bundle over a CAT manifold X. Consider stable CAT-bundles ξ_0 and ξ_1 over X with $|\xi_0| \cong_s \xi \cong_s |\xi_1|$, i.e. ξ_0 and ξ_1 are stable CAT bundles lifting ξ. A (stable) *concordance* between ξ_0 and ξ_1 is a CAT bundle γ over $X \times I$ extending ξ_i on $X \times \{i\}$, for $i = 0, 1$, and with a stable isomorphism $|\gamma| \cong_s \xi \times \mathfrak{t}_I$ to the product of ξ with the topological tangent bundle of the interval I.

A concordance between CAT lifts of a topological $\mathbb{R}^n$ fibre bundle over X is a CAT lift of the product bundle on $X \times I$.

Theorem 16.16. *Let M be a topological manifold with dimension at least 5 and ∂M given a fixed* CAT *structure.*

(i) The stable tangent microbundle $\mathfrak{t}_M$ is stably isomorphic to $|\xi|$ for some CAT *bundle ξ if and only if there exists a lift $\theta\colon M \to B\mathrm{CAT}$ with $\mathfrak{t}_M \simeq F \circ \theta$, if and only if the map $G \circ \mathfrak{t}_M \colon M \to B^{\mathrm{TOP}}\!/_{\mathrm{CAT}}$ is null homotopic.*

(ii) Moreover, the set $[(M, \partial M), (^{\mathrm{TOP}}\!/_{\mathrm{CAT}}, *)]$ acts freely and transitively on the concordance classes of stable CAT *bundles $\theta\colon M \to B\mathrm{CAT}$ lifting $\mathfrak{t}_M$ and extending ρ. So after fixing one such lift θ, assuming one exists, there is a one to one correspondence between concordance classes of stable* CAT *bundles lifting $\mathfrak{t}_M$ and $[(M, \partial M), (^{\mathrm{TOP}}\!/_{\mathrm{CAT}}, *)]$.*

So we have two tasks. On the one hand, we need to understand the homotopy type of the spaces TOP /CAT, so we can understand when a bundle's classifying map lifts from $B\,\mathrm{TOP}$ to $B\mathrm{CAT}$. On the other hand, given such a suitable lift, we need to make use of it to actually produce a CAT structure, or an equivalence of CAT structures. This is the topic of smoothing/PLing theory, which we discuss next.

17 Introduction to the product structure theorem, and smoothing and PL-ing theory

As before, we let CAT stand for PL or DIFF.

Definition 17.1. A CAT structure Γ on a topological manifold M is a maximal CAT atlas, that is the transition functions are CAT; we shall write M_Γ to indicate M with the CAT structure Γ.

We use the term CAT isomorphism for a PL homeomorphism or a diffeomorphism, as appropriate.

Definition 17.2. Two CAT structures Γ, Γ' on M are CAT-isomorphic if there is a homeomorphism $h\colon M \to M$ with $h^{-1}(\Gamma') = \Gamma$.

If h is homotopic to Id_M rel. ∂M, then (M, h) and (M, Id) represent the same element of the structure set $\mathcal{S}_{\mathrm{CAT}}(M, \partial M)$.

Definition 17.3. An *isotopy* between CAT structures Σ and Σ' on a manifold M is a path of (TOP) homeomorphisms $h_t\colon M \to M$ from $h_0 = \mathrm{Id}_M$ to a CAT isomorphism $h_1\colon M_\Sigma \to M_{\Sigma'}$.

Each h_t can be used to pullback Σ', so an isotopy gives a continuous family of CAT structures on M, starting with Σ' and ending with Σ.

Definition 17.4. A *concordance* of CAT structures on M is a CAT structure Γ on $M \times I$, where $I = [0,1]$. We say that Γ is a concordance from Σ_0 to Σ_1, with $\Sigma_i \coloneqq \Gamma|_{M\times\{i\}}$.

Note that isotopic CAT structures are concordant and CAT-isomorphic. We will discuss the other possible implications below.

For both of CAT equals PL or DIFF, we shall discuss the following two important theorems, and their consequences.

(1) Concordance implies Isotopy.
(2) The Product Structure Theorem.

We will start with statements of these results and their applications to the questions of whether a topological manifold admits a smooth or PL structure. We will give the proofs later on. We are going to start with the simplest statements, and gradually introduce more complications in relative versions as we go on.

The results will again rely on PL or smooth results associated with the s-cobordism theorem and surgery theory. In particular the proof of the product structure theorem relies on the stable homeomorphism theorem. As such dimension restrictions will again appear, and in fact the results will in general be false if one tries to extend them to include 4-manifolds. We will state the precise dimension restrictions at each stage. Dimension at least six is always safe. For some results about 5-manifolds, such as in the well-definedness of connected sum (Theorem 12.32), results about codimension one 4-manifolds will appear, meaning that we have to be careful.

We should note that many of the problems with dimension 4 were fixed by Quinn, but at the moment we are presenting the state of topological manifolds in 1978, that is after Kirby-Siebenmann's book appeared but before Quinn's work.

Theorem 17.5 (Concordance implies Isotopy for CAT structures)**.** *Assume $\partial M = \emptyset$ and $\dim M \geq 5$ and let Γ be a* CAT *structure on $M \times I$, that is, a concordance from Σ_0 to Σ_1.*

Then there exists an isotopy $h_t\colon M \times I \to M \times I$ with

(1) $h_0 = \mathrm{Id}_{M\times I}$*,*
(2) $h_1\colon (M \times I)_{\Sigma\times I} \to (M \times I)_\Gamma$ *is a* CAT *isomorphism,*
(3) $h_t|_{M\times\{0\}} = \mathrm{Id}_{M\times\{0\}}$ *for all* $t \in [0,1]$*.*

We say that $h_t|_{M\times\{1\}}$ is an isotopy of CAT structures from Σ_1 to Σ_0.

Example 17.6. The cardinality of the set of smooth structures on S^7 up to diffeomorphism is 15. Up to concordance, or up to orientation preserving diffeomorphism, there are 28 of them, and indeed the smooth structures on S^n considered up to concordance forms an abelian group θ_n, the group of homotopy spheres, with addition by connected sum and the standard smooth S^7 as the identity element. This group was computed in many cases by Kervaire and Milnor [KM63].

Remark 17.7. Note that for CAT structures isotopy implies both concordance and diffeomorphism. The above result shows that concordance implies isotopy for closed manifolds of dimension ≥ 5. However, diffeomorphism does not imply concordance (nor isotopy) as the above example shows.

Theorem 17.8 (Product Structure Theorem)**.** *Assume $\partial M = \emptyset$ and $\dim M \geq 5$, and let Θ be a* CAT *structure on $M \times \mathbb{R}^q$ for some $q \geq 1$. Then there is a concordance $(M \times \mathbb{R}^q \times I)_\Gamma$ from Θ to $(M \times \mathbb{R}^q)_{\Sigma \times \mathbb{R}^q}$, where the latter* CAT *structure is the product of a structure Σ on M and the standard structure on $\mathbb{R}^q$.*

In particular, M admits a CAT *structure, which is moreover unique up to concordance.*

Corollary 17.9. *Assume $\partial M = \emptyset$ and $\dim M \geq 5$. Suppose that the stable tangent microbundle satisfies $\mathfrak{t}_M \cong |\xi|$, where $|\xi|$ is the underlying microbundle of a* CAT *bundle $\xi \to M$. Then M admits a* CAT *structure.*

Proof. By the 'precursor to smoothing' Theorem 15.19 we know that there exists a CAT structure on $M \times \mathbb{R}^q$ for some $q \geq 1$. By Theorem 17.8 we obtain a CAT structure on M. □

Here if CAT = DIFF bundle then a CAT bundle is a vector bundle. A CAT = PL bundle is an $\mathbb{R}^n$ fibre bundle, where $n = \dim M$, with structure group PL(n), the PL homeomorphisms of $\mathbb{R}^n$ that fix the origin. There is a theory of PL-microbundles, and there is an analogue to Kister's theorem, due to Kuiper-Lashof [KL66], which says that every PL microbundle contains a PL fibre bundle. We are unfortunately omitting to develop the theory of PL bundles, with the assurance that it is analogous to the smooth theory of vector bundles in so far as we will need it.

Remark 17.10. Strictly speaking, we only considered smooth structures in Section 15.3, but the same proofs work in PL category. Namely, we obtained the smooth structure on $M \times \mathbb{R}^q$ by realising $M \times \mathbb{R}^q$ as an open subset in a large dimensional Euclidean space, and we then pulled back the smooth structure from the Euclidean space. We can do the same with the PL structure. For now we will take it on faith, and refer to [KS77, Essay IV, Theorem 3.1 and Proposition 5.1].

Recall that Theorem 15.19 relied on Theorem 15.14 on the stable existence of normal microbundles for topological submanifolds. There are different proofs for this by Milnor, Hirsch, and Stern.

Now we upgrade our statement of the product structure theorem to a relative form, that will be useful for questions about the uniqueness of structures.

Theorem 17.11 (Relative Product Structure Theorem)**.** *Let M be a manifold and fix an open subset $U \subseteq M$. Assume $\dim M \geq 6$, or $\dim M = 5$ with $\partial M \subseteq U$. Let Θ be a* CAT *structure on $M \times \mathbb{R}^q$ for some $q \geq 1$, and suppose there exists a* CAT *structure ρ on U with $\Theta|_{U \times \mathbb{R}^q} = \rho \times \mathbb{R}^q$.*

Then ρ extends to a CAT *structure Σ, and there is a concordance $(M \times \mathbb{R}^q \times I)_\Gamma$ from Θ to $(M \times \mathbb{R}^q)_{\Sigma \times \mathbb{R}^q}$ relative to $U \times \mathbb{R}^q$.*

Corollary 17.12. *Let Σ_0, Σ_1 be* CAT *structures on M with $\dim M \geq 4$ and $\partial M = \emptyset$. The* CAT *structures induce* CAT *tangent bundles with microbundle isomorphisms $|TM_{\Sigma_i}| \to \mathfrak{t}_M$ for $i = 0, 1$. Suppose that there exists a concordance between the* CAT *bundle structures TM_{Σ_0} and TM_{Σ_1}. That is, there is a* CAT *bundle $\xi \to M \times I$ restricting to TM_{Σ_i} on $M \times \{i\}$, and a stable microbundle isomorphism $|\xi| \xrightarrow{\cong_s} \mathfrak{t}_{M \times I}$.*

Then Σ_0 and Σ_1 are concordant. Moreover if $\dim M \geq 5$ then they are isotopic.

Proof. Let U be the union of open collars on $M \times \{i\}$ for $i = 0, 1$, and put product structures $\Sigma_i \times [0, \varepsilon)$ on each of these collars. By a relative version of Milnor's Theorem 15.19, there exists $q \geq 1$ and a CAT structure on $M \times I \times \mathbb{R}^q$ restricting to $\Sigma_i \times \mathbb{R}^q$ on $U \times \mathbb{R}^q$. Then by Theorem 17.11 we obtain a CAT structure on $M \times I$. That is, Σ_0 and Σ_1 are concordant.

If $\dim M \geq 5$ then we have that they are also isotopic by Theorem 17.5. □

Remark 17.13. The requirement that $\partial M = \emptyset$ is not necessary, but was added to make the notation in the statement and the proof easier. More care is needed to state a relative version, in which one assumes that in a closed set $C \subseteq M$ containing the boundary we already have a fixed concordance.

Remark 17.14. We also did not prove a relative version of Theorem 15.19. The proof proceeds analogously, but with more care required.

Remark 17.15. Note that this implies that each of the uncountably many exotic structures on $\mathbb{R}^4$ are concordant to one another, while they are not diffeomorphic to each other and therefore are not isotopic. So concordance implies isotopy is false for 4-manifolds.

To compare to [KS77], we have shown that the *smoothing rule* σ, which in [KS77, Essay IV, Proposition 3.4] is defined by exactly the procedure we have used to obtain CAT structures, is a well-defined map from stable concordance classes of stable CAT bundle structures on $\mathfrak{t}_M$ to concordance classes of CAT structures on M.

That is, the smoothing rule is to apply the method of Theorem 15.19 to obtain a CAT structure on $M \times \mathbb{R}^q$, for some q, from a CAT bundle whose underlying microbundle is the tangent microbundle of M, and then apply the product structure theorem to obtain a CAT structure on M. We have shown that concordant stable CAT structures on $\mathfrak{t}_M$ give rise to concordant CAT structures on $M \times \mathbb{R}^q$, and the product structure theorem gives uniqueness of the resulting smooth structure on M up to concordance.

In fact, [KS77, Essay IV, Theorem 4.1] shows that σ is a bijection from stable concordance classes of stable CAT bundle structures on $\mathfrak{t}_M$ to concordance classes of CAT structures on M. We also now include the possibility that the boundary is nonempty, but we assume that the structures are already equal on the boundary.

Theorem 17.16 ([KS77, Essay IV, Theorem 4.1])**.** *Let M be a topological manifold with $\dim M \geq 5$ and ∂M given a fixed* CAT *structure. Then the smoothing rule gives rise to a bijection between stable concordance classes of stable* CAT *bundle structures on $\mathfrak{t}_M$ and the set of concordance classes of* CAT *structures on M.*

Next we will refine the smoothing rule using classifying spaces. By combining Theorem 17.16 with Theorem 16.16, we obtain the following theorem. We use the fact from Theorem 16.16 that the CAT bundle structures on the topological tangent bundle of M are controlled by lifts of the classifying map, and therefore are controlled by maps to $B^{\mathrm{TOP}}\!/_{\mathrm{CAT}}$ and ${}^{\mathrm{TOP}}\!/_{\mathrm{CAT}}$.

Theorem 17.17. *Let M be a topological manifold with dimension at least 5 and ∂M given a fixed* CAT *structure.*

The map $G \circ \mathfrak{t}_M : M \to B^{\mathrm{TOP}}\!/_{\mathrm{CAT}}$ is null homotopic if and only if M admits a CAT *structure extending the structure on ∂M.*

Moreover, the set $[(M, \partial M), ({}^{\mathrm{TOP}}\!/_{\mathrm{CAT}}, *)]$ acts freely and transitively on the concordance classes of CAT *structures fixing the structure on ∂M. So after fixing one such* CAT *structure, assuming one exists, there is a one to one correspondence between concordance classes of* CAT *structures extending the structure on ∂M and $[(M, \partial M), ({}^{\mathrm{TOP}}\!/_{\mathrm{CAT}}, *)]$.*

Corollary 17.18 ([Sta62])**.** *For $n \geq 5$, $\mathbb{R}^n$ has a unique* CAT *structure.*

Proof. Assuming Theorem 17.17, since $\mathbb{R}^n$ is contractible we have that $[\mathbb{R}^n, {}^{\mathrm{TOP}}\!/_{\mathrm{CAT}}] = \{*\}$. Therefore there is a unique CAT structure on $\mathbb{R}^n$, $n \geq 5$, as claimed. □

This corollary was first proved by Stallings in [Sta62]. Actually we will need this statement for $n \geq 6$ in the proof of the product structure theorem. So we had better give an independent argument, and indeed we shall do so later (our argument will be different from Stallings' argument). Nevertheless for a user of the theory, it is often easier to remember the one central theorem, and deduce everything else from it, which is why we have also pointed it out as a corollary.

For Theorem 17.17 to be useful for non-contractible spaces we need to understand something about the homotopy type of the spaces ${}^{\mathrm{TOP}}\!/\!{}_{\mathrm{CAT}}$ and $B{}^{\mathrm{TOP}}\!/\!{}_{\mathrm{CAT}}$. This is the topic of the next section, but for the piecewise-linear case, the homotopy type is easy to describe.

Theorem 17.19 (Kirby-Siebenmann)**.** *We have a homotopy equivalence* ${}^{\mathrm{TOP}}\!/\!{}_{\mathrm{PL}} \simeq K(\mathbb{Z}/2, 3)$*.*

We will prove this soon. Let us observe some consequences now, however. It follows that the obstruction for existence of a lift θ as in (16.14) lies in the group

$$[(M, \partial M), (B{}^{\mathrm{TOP}}\!/\!{}_{\mathrm{PL}}, *)] \cong H^4(M, \partial M; \mathbb{Z}/2),$$

This obstruction is called the *Kirby-Siebenmann* invariant of $(M, \partial M)$. If this obstruction vanishes, all such lifts are classified by the group

$$[(M, \partial M), ({}^{\mathrm{TOP}}\!/\!{}_{\mathrm{PL}}, *)] \cong H^3(M, \partial M; \mathbb{Z}/2).$$

In particular Theorem 17.17 and Theorem 17.19 imply the following remarkable theorem.

Theorem 17.20. *Let M be a topological manifold with dimension at least* 5 *and ∂M given a fixed* PL *structure.*

(1) Suppose $H^4(M, \partial M; \mathbb{Z}/2) = 0$. Then the PL structure on ∂M extends to M.
(2) Suppose that $H^3(M, \partial M; \mathbb{Z}/2) = 0$. Then any two PL *structures Σ_0 and Σ_1 on M satisfying $\Sigma_0|_{\partial M} = \Sigma_1|_{\partial M}$ are isotopic.*

Remark 17.21. Note the corollary that a compact topological manifold with dimension at least 5 has finitely many PL structures rel. boundary, up to isotopy (it may of course have zero such structures).

18 The homotopy groups of TOP/PL and TOP/O

We have seen that it would be extremely useful to know about the homotopy groups of TOP/PL and TOP/O. This section explains how to compute them. To start, the homotopy groups $\pi_k(\mathrm{TOP}/\mathrm{PL})$ and $\pi_k(\mathrm{TOP}/O)$ for $k \geq 5$ are easy to compute.

Lemma 18.1. *For $k \geq 5$ we have*

$$\pi_k\left(\mathrm{TOP}/\mathrm{PL}\right) = 0, \quad \textit{and} \quad \pi_k\left(\mathrm{TOP}/O\right) \cong \Theta_k$$

where Θ_k is the group of homotopy spheres, that is h-cobordism classes of smooth, closed, oriented k-manifolds homotopy equivalent to S^k.

Proof. For $k \geq 5$, the set $[S^k, \mathrm{TOP}/\mathrm{CAT}]$ is in one-to-one correspondence with concordance classes of CAT structures on S^k, which, via the CAT h-cobordism theorem, equals $\{*\}$ for CAT = PL and equals Θ_k for CAT = DIFF. $\square$

Recall, for example, that famously $\Theta_7 \cong \mathbb{Z}/28$. Unlike the PL Poincaré conjecture, the smooth Poincaré conjecture is not true in many dimensions. It is true in dimensions 5, 6, 12, 56, and 61. It is open in infinitely many dimensions. The groups of homotopy spheres are related to the homotopy groups of spheres, more precisely to the cokernel of the J-homomorphism. So difficulties computing the latter translate into difficulties computing the former. It is known that Θ_k is finite for all $k \geq 5$.

18.1 Smoothing of piecewise-linear manifolds and the homotopy groups of PL/O

There is an analogous theory for the smoothing of piecewise-linear manifolds. There is a fibration sequence

$$\mathrm{PL}/O \to BO \to B\,\mathrm{PL}$$

with PL/O by definition the homotopy fibre. Also PL/O is an infinite loop space, so admits a delooping B^{PL}/O.

Theorem 18.2 (Cairns-Hirsch, Hirsch-Mazur)**.** *Given a closed PL manifold M, the map*

$$M \xrightarrow{\mathrm{t}_M} B\,\mathrm{PL} \longrightarrow B^{\mathrm{PL}}/O$$

is null homotopic if and only if M is smoothable. Moreover concordance classes of smooth structures on M are in 1-1 correspondence with $[M, \mathrm{PL}/O]$

We can describe the homotopy groups $\pi_k(\mathrm{PL}/O)$. By the Poincaré conjecture, and its smooth failure (Smale, Stallings, Zeeman, Kervaire-Milnor),

$$\pi_k(\mathrm{PL}/O) \cong \Theta_k$$

for $k \geq 5$. Note that Kervaire-Milnor computed that $\Theta_5 = \Theta_6 = 0$ and $\Theta_7 \cong \mathbb{Z}/28$.

In addition, $\pi_k(PL/O) = 0$ for $k \leq 4$. This follows from direct geometric proofs that PL manifolds of dimension $k \leq 4$ admit smooth structures, due to Munkres, Smale, and Cerf. We therefore have the following fact.

Theorem 18.3. *The space PL/O is 6-connected.*

To summarise, in general a PL manifold may admit no smooth structures, or multiple smooth structures. Since Θ_k is finite, a given compact PL manifold admits finitely many smooth structures, up to concordance. These are detected via maps to PL/O. In dimensions at most 5, every PL manifold admits a unique smooth structure. This is not to be confused with the fact that a given topological 4-manifold may admit infinitely many PL (and therefore smooth) structures.

18.2 Homotopy groups of $^{\mathrm{TOP}}/_{O}$

We will focus on the question of putting a PL structure on a topological manifold, since this has a particularly clean answer. This can be seen, e.g. from Lemma 18.1. In contrast, for CAT = DIFF, we have to account for nontrivial smooth homotopy spheres. We now show that there are no additional sources of trouble.

Theorem 18.4.
$$\pi_k\left(^{\mathrm{TOP}}/_{O}\right) \cong \pi_k\left(^{\mathrm{TOP}}/_{\mathrm{PL}}\right)$$
for $0 \leq i \leq 4$*, while*
$$\pi_k\left(^{\mathrm{TOP}}/_{O}\right) \cong \pi_k\left(^{\mathrm{PL}}/_{O}\right)$$
for $i \geq 5$*.*

Proof. Apply Theorem 17.19, Theorem 18.3, and the long exact seqeunce in homtopy groups associated to the fibre sequence
$$^{\mathrm{PL}}/_{O} \longrightarrow {}^{\mathrm{TOP}}/_{O} \longrightarrow {}^{\mathrm{TOP}}/_{\mathrm{PL}},$$
to see that we have
$$\pi_k\left(^{\mathrm{TOP}}/_{O}\right) \cong \begin{cases} \pi_k\left(^{\mathrm{TOP}}/_{\mathrm{PL}}\right) \cong \pi_k(K(\mathbb{Z}/2,3)) & 0 \leq k \leq 4 \\ \pi_k\left(^{\mathrm{PL}}/_{O}\right) \cong \Theta_k & k \geq 5. \end{cases}$$
□

Corollary 18.5. *Every compact topological manifold of dimension at least* 6 *admits finitely many smooth/PL structures (including possibly zero).*

Proof. This follows from obstruction theory and the theorem of Kervaire-Milnor that $|\Theta_k| < \infty$, together with Theorem 17.19 that $\pi_k(\mathrm{TOP}\,/\,\mathrm{PL})$ is finite for $k \leq 4$. □

The corollary holds for compact 5-manifolds as well, provided we fix a CAT structure on the 4-dimensional boundary.

18.3 The homotopy groups of TOP / PL

In this section we give the proof of Theorem 17.19. Here is the statement again.

Theorem 18.6. $^{\mathrm{TOP}}/_{\mathrm{PL}} \simeq K(\mathbb{Z}/2, 3)$.

This means, remarkably, that the difference between the topological and piecewise-linear categories, is rather small. From the point of view of obstruction theory, there is just a single $\mathbb{Z}/2$ obstruction. The proof we are going to present is from [KS77, Essay IV chapter 10 and Essay V Theorem 5.3].

By Lemma 18.1 it remains to compute $\pi_k\left(^{\mathrm{TOP}}/_{\mathrm{PL}}\right)$ for $0 \leq k \leq 4$. First of all we will show that $\pi_k\left(^{\mathrm{TOP}}/_{\mathrm{PL}}\right) = 0$ for $i = 0, 1, 2,$ and 4, and that $\pi_3\left(^{\mathrm{TOP}}/_{\mathrm{PL}}\right) \leq \mathbb{Z}/2$. To do this we shall define, for each $0 \leq k \leq 4$, a map
$$\psi_k \colon \pi_k\left(^{\mathrm{TOP}}/_{\mathrm{PL}}\right) \to \mathcal{S}^*_{\mathrm{PL}}(D^k \times T^{6-k}, \partial)$$
We will define $\mathcal{S}^*_{\mathrm{PL}}(D^k \times T^{6-k}, \partial)$ in detail below.

Our overall aims for this computation are as follows. We will show that ψ_k is injective and that the right hand side is zero for $k = 0, 1, 2, 4$ and is $\mathbb{Z}/2$ for $k = 3$. Once we have shown all of this we will show separately that $\pi_3(\mathrm{TOP}\,/\,\mathrm{PL})$ is nontrivial.

The set $\mathcal{S}^*_{\mathrm{PL}}(D^k \times T^{6-k}, \partial)$ is by definition the subset of the structure set

$$\mathcal{S}_{\mathrm{PL}}(D^k \times T^{6-k}, \partial) := \left\{ \begin{array}{ccc} M & \xrightarrow{\simeq} & D^k \times T^{6-k} \\ \uparrow & & \uparrow \\ \partial M & \xrightarrow{\cong_{\mathrm{PL}}} & S^{k-1} \times T^{6-k} \end{array} \right\} / \text{ PL homeo over } D^k \times T^{6-k}$$

consisting of those elements which are invariant under passing to λ^{6-k} covers for all $\lambda \in \mathbb{N}$. That is, passing to a λ^{6-k} cover

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\simeq} & D^k \times \widetilde{T^{6-k}} \\ \uparrow & & \uparrow \\ \widetilde{\partial M} & \xrightarrow{\cong_{\mathrm{PL}}} & S^{k-1} \times \widetilde{T^{6-k}} \end{array}$$

yields an equivalent element in $\mathcal{S}_{\mathrm{PL}}(D^k \times T^{6-k}, \partial)$. We are considering the rel. boundary structure set. The equivalence relation stipulates that

$$\begin{array}{ccc} M & \xrightarrow[F]{\simeq} & D^k \times T^{6-k} \\ \uparrow & & \uparrow \\ \partial M & \xrightarrow[\partial F]{\cong_{\mathrm{PL}}} & S^{k-1} \times T^{6-k} \end{array} \quad \text{and} \quad \begin{array}{ccc} M' & \xrightarrow[F']{\simeq} & D^k \times T^{6-k} \\ \uparrow & & \uparrow \\ \partial M' & \xrightarrow[\partial F']{\cong_{\mathrm{PL}}} & S^{k-1} \times T^{6-k} \end{array}$$

are equivalent if there are PL homeomorphisms

$$\begin{array}{ccc} M & \xrightarrow[G]{\cong_{PL}} & M' \\ \uparrow & & \uparrow \\ \partial M & \xrightarrow[\partial G]{\cong_{\mathrm{PL}}} & \partial M' \end{array}$$

such that $\partial F' \circ \partial G = \partial F$ and $F' \circ G \sim F$. So the commutativity of the triangles

$$\begin{tikzcd} M \arrow[rr, "G"] \arrow[dr, "F"'] & & M' \arrow[dl, "F'"] \\ & D^k \times T^{6-k} & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \partial M \arrow[rr, "\partial G"] \arrow[dr, "\partial F"'] & & M' \arrow[dl, "\partial F'"] \\ & S^{k-1} \times T^{6-k} & \end{tikzcd}$$

is up to homotopy for the first triangle and precise commutativity for the second triangle.

18.3.1 Definition of ψ_k.. To define

$$\psi_k \colon \pi_k \left(\mathrm{TOP}/\mathrm{PL}\right) \to \mathcal{S}^*_{\mathrm{PL}}(D^k \times T^{6-k}, \partial),$$

fix a basepoint $* \in \mathrm{TOP}/\mathrm{PL}$ and represent $x \in \pi_k(\mathrm{TOP}/\mathrm{PL}, *)$ by a diagram:

$$\begin{array}{ccc} D^k & \longrightarrow & \mathrm{TOP}/\mathrm{PL} \\ \uparrow & & \uparrow \\ S^{k-1} & \longrightarrow & * \end{array}$$

Combining these maps with the projection maps pr_1 onto the first factor, we obtain a diagram

$$\begin{array}{ccccc} D^k \times T^{6-k} & \xrightarrow{\mathrm{pr}_1} & D^k & \longrightarrow & \mathrm{TOP}/\mathrm{PL} \\ \uparrow & & \uparrow & & \uparrow \\ S^{k-1} \times T^{6-k} & \xrightarrow{\mathrm{pr}_1} & S^{k-1} & \longrightarrow & * \end{array}$$

Write $\overline{x}$ for the resulting map of pairs

$$\overline{x}\colon (D^k \times T^{6-k}, S^{k-1} \times T^{6-k}) \to ({}^{\mathrm{TOP}}\!/\!_{\mathrm{PL}}, *).$$

We know from Theorem 17.17 that the set of homotopy classes of such maps acts freely and transitively on the concordance classes of PL structures on $(D^k \times T^{6-k}, S^{k-1} \times T^{6-k})$.

Let $(M, \partial M)$ denote $(D^k \times T^{6-k}, S^{k-1} \times T^{6-k})$ with the PL structure obtained by acting on the standard structure by $\overline{x}$. It does not change the PL structure on the boundary. This maps by the identity to $D^k \times T^{6-k}$. Thus we obtain

$$(F, \partial F)\colon (M, \partial M) \xrightarrow{\simeq, \cong_{PL}} (D^k \times T^{6-k}, S^{k-1} \times T^{6-k})$$

since the identity map on the underlying topological manifolds is in particular a homotopy equivalence.

The induced structure on a λ^{6-k} cover is that induced by

$$D^k \times T^{6-k} \xrightarrow{\mathrm{Id} \times \lambda^{6-k}} D^k \times T^{6-k} \xrightarrow{\mathrm{pr}_1} D^k \xrightarrow{x} \mathrm{TOP} \,/\, \mathrm{PL}$$

Since this map equals the original map $\overline{x}\colon D^k \times T^{6-k} \to \mathrm{TOP} \,/\, \mathrm{PL}$, we see that element of the structure set $(F, \partial F)\colon (M, \partial M) \to (D^k \times T^{6-k}, S^{k-1} \times T^{6-k})$ is invariant under passing to finite covers and therefore determines an element of $\mathcal{S}^*_{\mathrm{PL}}(D^k \times T^{6-k}, \partial)$.

18.3.2 Injectivity of ψ_k. Having defined the map ψ_k, we now show that it is injective.

It will be useful to recall the definition of an isotopy of PL-structures from Definition 17.3.

Definition 18.7. An *isotopy* between PL structures Σ and Σ' on a manifold M is a path of homeomorphisms $h_t\colon M \to M$ from $h_0 = \mathrm{Id}_M$ to a PL homeomorphism $h_1\colon M_\Sigma \to M_{\Sigma'}$.

Each h_t can be used to pullback Σ', so an isotopy gives a continuous family of PL structures on M, starting with Σ' and ending with Σ.

Lemma 18.8. *For $k = 0, 1, 2, 3, 4$, the map $\psi_k\colon \pi_k \left({}^{\mathrm{TOP}}\!/\!_{\mathrm{PL}}\right) \to \mathcal{S}^*_{\mathrm{PL}}(D^k \times T^{6-k}, \partial)$ is injective.*

Remark 18.9. In Kirby-Siebenmann, it is only shown that the inverse image of the trivial element is the trivial element. It is implicitly assumed that the structure set is a group, and that ψ_k is a homomorphism, but this is not discussed, although it seems to be true. We will avoid this question by showing that the map is injective as a map of sets.

Proof. Suppose that $\psi_k(x) = \psi_k(y)$ in the structure set. That is, $\psi_k(x)$ and $\psi_k(y)$ give rise to PL structure Σ and Σ' on $D^k \times T^{6-k}$, and there exists a PL homeomorphism

$$h\colon [D^k \times T^{6-k}]_{\Sigma'} \xrightarrow{\cong_{PL}} [D^k \times T^{6-k}]_{\Sigma}$$

which the identity near the boundary, and moreover $h \sim \mathrm{Id}$ rel. boundary. We want to show that $x = y \in \pi_k(\mathrm{TOP} \,/\, \mathrm{PL})$.

Lifting to a λ^{6-k} cover, for some $\lambda \in \mathbb{N}$, gives structures $[D^k \times T^{6-k}]_{\Sigma_\lambda}$ and $[D^k \times T^{6-k}]_{\Sigma'_\lambda}$ defined to make $\mathrm{Id} \times \lambda^{6-k}$ a PL map. We obtain a diagram of PL maps:

$$\begin{tikzcd}
(D^k \times T^{6-k})_{\Sigma'_\lambda} \arrow[r, "h_\lambda", "\cong"'] \arrow[d, "\mathrm{Id} \times \lambda^{6-k}"'] & (D^k \times T^{6-k})_{\Sigma_\lambda} \arrow[d, "\mathrm{Id} \times \lambda^{6-k}"] \\
(D^k \times T^{6-k})_{\Sigma'} \arrow[r, "h", "\cong"'] & (D^k \times T^{6-k})_{\Sigma}
\end{tikzcd}$$

Observe that for λ sufficiently large, we can make h_λ arbitrarily close to the identity on T^{6-k}. In addition, extend h_λ by the identity to

$$\overline{h}_\lambda\colon \mathbb{R}^k \times T^{6-k} \to \mathbb{R}^k \times T^{6-k}.$$

Let $H_t\colon \mathbb{R}^k \to \mathbb{R}^k$ be an isotopy shrinking D^k to be very small, with H_0 the identity, and H_1 the result of this shrink. Define

$$G_t := (H_t \times \mathrm{Id}) \circ \overline{h}_\lambda \circ (H_t \times \mathrm{Id})^{-1}\colon [D^k \times T^{6-k}]_{\Sigma'_\lambda} \to [D^k \times T^{6-k}]_{\Sigma_\lambda}.$$

This is an isotopy on $D^k \times T^{6-k}$ with G_0 the identity and with G_1 arbitrarily close to the identity.

Now by the local path connectedness of $\mathrm{Homeo}_\partial(D^k \times T^{6-k})$ (Theorem 14.1), G_1 is isotopic to the identity, which implies that h_λ is isotopic to the identity. Therefore we have an isotopy of homeomorphisms from a PL homeomorphism

$$h_\lambda\colon [D^k \times T^{6-k}]_{\Sigma'_\lambda} \to [D^k \times T^{6-k}]_{\Sigma_\lambda}$$

to the identity map. Therefore Σ_λ and Σ'_λ are isotopic PL structures, which means that the maps $D^k \times T^{6-k} \to \mathrm{TOP}\,/\,\mathrm{PL}$ which produced them are homotopic. Now consider the diagram:

$$\begin{tikzcd}
& D^k \times T^{6-k} \arrow[d, "\lambda^{6-k}"] \arrow[dr, "\Sigma_\lambda{,}\Sigma'_\lambda"] & \\
D^k \times \{\mathrm{pt}\} \arrow[ur] \arrow[r] \arrow[dr, "\mathrm{Id}"'] & D^k \times T^{6-k} \arrow[r, "\overline{x}{,}\overline{y}"] \arrow[d, "\mathrm{pr}_1"] & \mathrm{TOP}/\mathrm{PL} \\
& D^k \arrow[ur, "x{,}y"'] &
\end{tikzcd}$$

Here the maps with codomain $\mathrm{TOP}\,/\,\mathrm{PL}$ indicate two maps. We label the maps that determine the structures Σ_λ and Σ'_λ by the structure. The diagram commutes by definition of the maps involved. We have seen that there is a homotopy between the maps Σ_λ and Σ'_λ. This induces a homotopy between the two maps $D^k \times \{\mathrm{pt}\} \to \mathrm{TOP}/\mathrm{PL}$ via the top route. By commutativity of the diagram this induces a homotopy between the two maps via the bottom route. Hence x is homotopic to y as desired. □

18.3.3 Computation of $\mathcal{S}^*_{PL}(D^3 \times T^n, \partial)$. So far we did not apply any surgery theory computations. Now we need to appeal to them. Recall we discussed the surgery classification of PL homotopy tori in Section 13. The results discussed there generalise to the following. Before, we focused on the case $k = 0$ that we needed for the proof of the stable homeomorphism theorem.

Theorem 18.10 ([HS69, Wal69]). *There is an isomorphism*

$$\mathcal{S}_{PL}(D^k \times T^n, S^{k-1} \times T^n) \cong H^{3-k}(T^n, \mathbb{Z}/2)$$

with $n + k \geq 5$ and this bijection is natural under finite covers. In particular, if $k = 0$, we have $H^3(T^n; \mathbb{Z}/2) = (\bigwedge^{n-3} \mathbb{Z}^n) \otimes \mathbb{Z}/2$ from before.

Corollary 18.11. *The subset of the structure set $\mathcal{S}_{PL}(D^k \times T^n, S^{k-1} \times T^n)$ that is invariant under finite covers is trivial unless $k = 3$. For $k = 3$ we have*

$$\mathcal{S}^*_{PL}(D^3 \times T^n, \partial) \cong H^0(T^n; \mathbb{Z}/2) \cong \mathbb{Z}/2.$$

So we have that

$$\psi_k\colon \pi_k\left(\mathrm{TOP}/\mathrm{PL}\right) \hookrightarrow \mathcal{S}^*_{\mathrm{PL}}(D^k \times T^{6-k}, \partial)$$

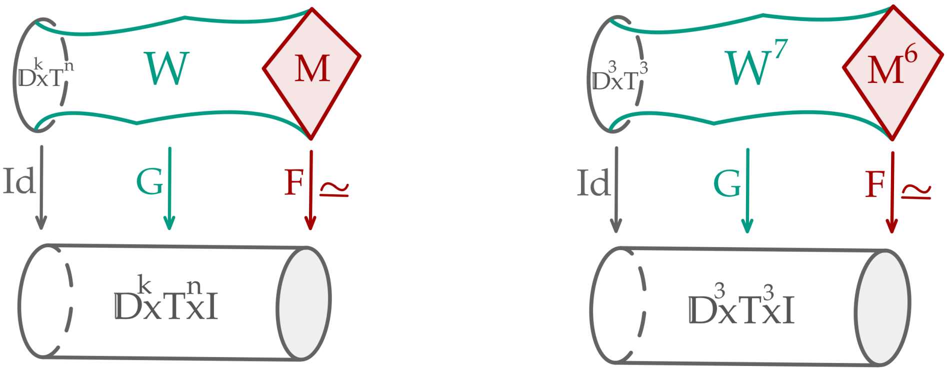
has trivial right hand side for $k = 0, 1, 2, 4$. We see that $\pi_k\left(\mathrm{TOP}/\mathrm{PL}\right) \cong \pi_k\left(\mathrm{TOP}/O\right) = 1$ for $k = 0, 1, 2, 4$ and we have an injective map $\psi_3\colon \pi_3\left(\mathrm{TOP}/\mathrm{PL}\right) \to \mathbb{Z}/2$. It therefore just remains to show that $\pi_3\left(\mathrm{TOP}/\mathrm{PL}\right)$ is nontrivial.

To do this, first we construct an element of $\mathcal{S}^*_{PL}(D^3 \times T^n, \partial)$ and show it is nontrivial.

For $k+n \geq 6$, a fake $D^k \times T^n$, i.e. a manifold homotopy equivalent to $D^k \times T^n$, with PL-homeomorphic boundary but not PL homeomorphic to $D^k \times T^n$, arises from elements of

$$L_{n+k+1}(\mathbb{Z}[\mathbb{Z}^n]) \big/ \mathcal{N}(D^k \times T^n \times I, D^k \times T^n \times \{0,1\}).$$

To create a fake $D^k \times T^n$, for $k+n \geq 6$, choose $y \in L_{n+k+1}(\mathbb{Z}[\mathbb{Z}^n])$ such that y does not lie in the image of $\mathcal{N}(D^k \times T^n \times I, D^k \times T^n \times \{0,1\})$, and realise y, using Wall realisation, by a normal bordism starting with the identity of $D^k \times T^n$, and ending with a new homotopy equivalence $F \colon M \to D^k \times T^n$. See Fig. 47a. The new pair (M,F) is a degree one normal map with F a homotopy equivalence, but F is not homotopic rel. boundary to a PL homeomorphism.



(A) A normal bordism produces a pair (M,F). (B) A normal bordism for $M \simeq D^3 \times T^3$.

We now consider an invariant that can be used to detect if M is PL homeomorphic to $D^k \times T^n$. Let us focus on the case of interest: $k = n = 3$. Given a PL manifold M and $F \colon M^6 \xrightarrow{\simeq} D^3 \times T^3$, we choose a PL normal bordism

$$(G, F, \mathrm{Id}) \colon (W^7; M^6, D^3 \times T^3) \to D^3 \times T^3 \times (I; \{0\}, \{1\}).$$

We then cross this bordism with $\mathbb{CP}^2$, so that we can apply results from the high dimensional theory, avoiding 4-manifolds. We obtain a map

$$G \times \mathrm{Id} \colon\ W \times \mathbb{CP}^2 \to D^3 \times T^3 \times I \times \mathbb{CP}^2$$

between 11-manifolds. By PL transversality, the inverse image

$$W' := (G \times \mathrm{Id})^{-1}(D^3 \times \{\mathrm{pt}\} \times I \times \mathbb{CP}^2)$$

is a PL 8-manifold with boundary, over $D^4 \times \mathbb{CP}^2$, such that the map of its boundary to $S^3 \times \mathbb{CP}^2$ is a PL homeomorphism. Take its (simply-connected) surgery obstruction in $L_8(\mathbb{Z})$. We know that $L_8(\mathbb{Z}) \cong \mathbb{Z}$ where the map takes the signature divided by 8, and we take the modulo 2:

$$L_8(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z} \longrightarrow \mathbb{Z}/2$$

We claim that this is a well-defined obstruction to the original map G being normally bordant to a homotopy equivalence. This follows from Farrell's fibering theorem, which implies that if G were bordant to a homotopy equivalence, then the surgery obstruction of

$$W' \to D^4 \times \mathbb{CP}^2$$

would be trivial. Finally, note that $\sigma(X) = \sigma(X \times \mathbb{CP}^2)$ for a 4-manifold X, since signature multiplies under products and $\sigma(\mathbb{CP}^2) = 1$.

Theorem 18.12 (Rochlin)**.** *If X a PL spin closed 4-manifold, then $\sigma(X)$ is divisible by 16.*

The obstruction $\sigma(W')\big/8$ mod 2 cannot be killed by a change in normal bordism, because that would change the inverse image by a closed, spin 4-manifold crossed with $\mathbb{CP}^2$, and by the Rochlin theorem this changes the signature obstruction in $L_8(\mathbb{Z})$ by a multiple of 16.

18.3.4 Nontriviality of $\pi_3(\mathrm{TOP}\,/\,\mathrm{PL})$. We showed $\pi_k(\mathrm{TOP}/\mathrm{PL}) = 0$ for $k \neq 3$, and that

$$\pi_3(\mathrm{TOP}/\mathrm{PL}) \subseteq \mathcal{S}_3^*(D^3 \times T^3, \partial) \cong \mathbb{Z}/2.$$

Now we show that this inclusion is equality, i.e. that $\pi_3(\mathrm{TOP}/\mathrm{PL})$ is nontrivial.

Remark 18.13. If we knew that all the fake tori in dimensions at least 5 are homeomorphic to one another, instead of just homotopy equivalent, then we would be done. However while this is true, it is harder to establish, and the proof might even use this result by comparing with the PL case. The method we are about to explain has the advantage that it uses machinery that we have already proven, or are assuming from the DIFF/PL development.

Definition 18.14. Let $\mathrm{PL}(D^3 \times T^3, \partial)$ be the set of PL structures on $D^3 \times T^3$, restricting to standard structure on $S^2 \times T^3$, considered up to isotopy.

Lemma 18.15. *There is an isomorphism* $\phi\colon \pi_3\left(\mathrm{TOP}/\mathrm{PL}\right) \xrightarrow{\cong} \mathrm{PL}(D^3 \times T^3, \partial)$*.*

Proof. By the Product Structure Theorem (Theorem 17.8) we have

$$\begin{aligned}\mathrm{PL}(D^3 \times T^3, \partial) &\cong [(D^3 \times T^3, \partial), \left(\mathrm{TOP}/\mathrm{PL}, *\right)] \\ &\cong H^3\left(D^3 \times T^3; \pi_3\left(\mathrm{TOP}/\mathrm{PL}\right)\right).\end{aligned}$$

We do not yet know whether $\pi_3\left(\mathrm{TOP}/\mathrm{PL}\right)$ is trivial or $\mathbb{Z}/2$, but we do not mind. Now by Poincare-Lefschetz duality this is

$$H^3\left(D^3 \times T^3; \pi_3\left(\mathrm{TOP}/\mathrm{PL}\right)\right) \cong H_3\left(T^3; \pi_3\left(\mathrm{TOP}/\mathrm{PL}\right)\right) \cong 2\pi_3\left(\mathrm{TOP}/\mathrm{PL}\right).$$

We denote the inverse of this chain of isomorphisms by ϕ. □

We have maps

$$\pi_3(\mathrm{TOP}/\mathrm{PL}) \xrightarrow[\cong]{\phi} \mathrm{PL}(D^3 \times T^3, \partial) \xrightarrow{\theta} \mathcal{S}_3^*(D^3 \times T^3, \partial)$$

where by definition $\theta((D^3 \times T^3)_\Sigma) = \mathrm{Id}\colon (D^3 \times T^3)_\Sigma \xrightarrow{\cong} D^3 \times T^3$. We have to show that θ is onto. This means that we take the nontrivial element of the codomain, which might be represented by some other manifold M that is homotopy equivalent to $D^3 \times T^3$, rel. boundary and we try to show that the M is in fact in the image of $\mathrm{PL}(D^3 \times T^3, \partial)$, so it is homeomorphic to $D^3 \times T^3$, but is perhaps not PL-homeomorphic.

Proposition 18.16. *The map* $\theta\colon \mathrm{PL}(D^3 \times T^3, \partial) \to \mathcal{S}_3^*(D^3 \times T^3, \partial)$ *is onto.*

Proof. Consider the nontrivial element of the structure set $\mathcal{S}_3^*(D^3 \times T^3, \partial)$

$$f\colon (M, \partial M) \xrightarrow{\simeq, \cong_{\mathrm{PL}}} (D^3 \times T^3, \partial)$$

We will construct a topological homeomorphism $h\colon M \to D^3 \times T^3$ and show that f is homotopic to h rel. boundary. This will show that f is in the image of θ.

For the rest of proof let us identify $D^3 \cong I^3 = [0,1]^3$. Consider the triple

$$(M; f^{-1}(\{0\} \times I^2 \times T^3), f^{-1}(\{1\} \times I^2 \times T^3).$$

By the rel. boundary s-cobordism theorem, there is a PL homeomorphism $f'\colon M \xrightarrow{\cong_{\mathrm{PL}}} I^3 \times T^3$ with $f' = f$ on $f^{-1}(\{1\} \times I^2 \cup I \times \partial I^2) \times T^3$. Next we investigate the failure of f' to equal f on the remaining part of the boundary, $\{0\} \times I^2 \times T^3$. Namely, consider the PL homeomorphism

$$g := f' \circ f|^{-1}\colon \{0\} \times I^2 \times T^3 \to \{0\} \times I^2 \times T^3$$

Lemma 18.17. g *is* TOP *isotopic to the identity* $\mathrm{Id}_{I^2 \times T^3}$*.*

Assuming for a moment such an isotopy exists, we can glue it in a collar neighbourhood of $\{0\} \times I^2 \times T^3$ to alter f', see Fig. 48. This produces the desired homeomorphism $h\colon M \to I^3 \times T^3$ which is equal to f near ∂M, and it remains to check that h is homotopic to f.

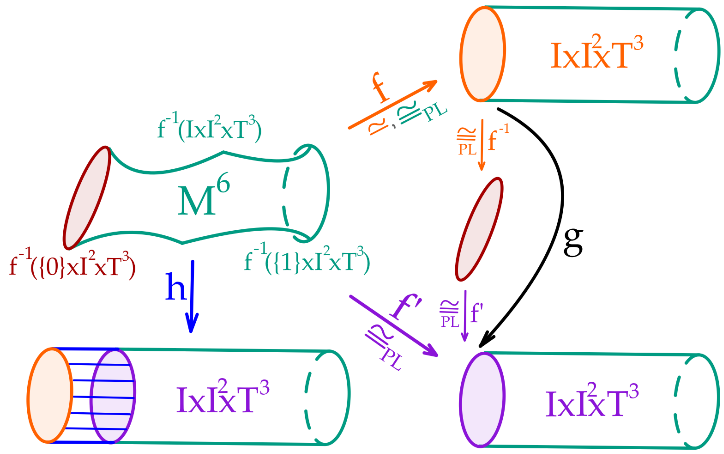


FIGURE 48. Modifying the PL homeomorphism f' by attaching into the collar of $\{0\} \times I^2 \times T^3$ an isotopy of g to the identity.

Lemma 18.18. *If two homeomorphisms h and f from M to $I^3 \times T^3$ agree near ∂M, then they are homotopic rel. boundary.*

Proof. The obstructions to extending

$$h \cup h \times \mathrm{Id}_I \cup f\colon M \times \{0\} \cup \partial M \times I \cup M \times \{1\} \to I \times I^3 \times T^3$$

to the homotopy $M \times I \to I^3 \times T^3$ lie in $H^{j+1}(I^4 \times T^3, \partial; \pi_j(I^3, \times T^3)) \cong H_{7-j-1}(T^3; \pi_j(T^3))$. This is always zero, since $\pi_j(T^3) \neq 0$ implies that $j = 0, 1$, so that $7 - j - 1$ is 5 or 6. But the cohomology of T^3 is trivial above degree 3. □

This finishes the proof of the proposition, modulo the proof of Lemma 18.17. □

Proof of Lemma 18.17. We use a similar method to that used in the injectivity of ψ_k proof: we note that M can be replaced by a large finite λ^3-fold cover M_λ, and similarly f by f_λ. We have that

$$[(M, f)] = [(M_\lambda, f_\lambda)] \in \mathcal{S}^*_{\mathrm{PL}}(I^3 \times D^3, \partial)$$

by invariance under finite covers. By the procedure above, we obtain analogous maps

$$f'_\lambda\colon M_\lambda \to I \times I^2 \times T^3_\lambda \quad \text{and} \quad g_\lambda\colon I^2 \times T^3_\lambda \to I^2 \times T^3_\lambda.$$

Lemma 18.19. *There is a finite λ^3 cover such that the map g_λ is* TOP *isotopic to* $\mathrm{Id}_{I^2 \times T^3_\lambda}$.

Proof. Passing to a large λ^3 finite cover, and squeezing in the I^2 coordinate, we may obtain a map that is as close to the identity as we please, which is therefore isotopic to the identity by local contractibility. More details follow.

Observe that for λ sufficiently large, g_λ is arbitrarily close to the identity on T^3_λ. In addition, extend g_λ by the identity to

$$\overline{g}_\lambda\colon \mathbb{R}^2 \times T^3_\lambda \to \mathbb{R}^2 \times T^3_\lambda.$$

Let $H_t\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an isotopy shrinking D^2 to be very small, with H_0 the identity, and H_1 the result of this shrink. Define

$$G_t := (H_t \times \mathrm{Id}) \circ \overline{g}_\lambda \circ (H_t \times \mathrm{Id})^{-1}\colon D^2 \times T^3_\lambda \to D^2 \times T^3_\lambda.$$

This is an isotopy on $D^2 \times T^3_\lambda$ with G_0 the identity and with G_1 arbitrarily close to the identity.

Now by the local path connectedness of $\mathrm{Homeo}_\partial(D^2 \times T^3_\lambda)$ (Theorem 14.1), G_1 is isotopic to the identity, which implies that g_λ is isotopic to the identity. □

The argument from above then allows us, for λ coming from Lemma 18.19, to improve f'_λ to $h_\lambda\colon M_\lambda \xrightarrow{\cong} I^3 \times T^3$, a homeomorphism (not a PL homeomorphism), with

$$f_\lambda|_\partial = h_\lambda|_\partial \colon \partial M_\lambda \to \partial I^3 \times T^3_\lambda.$$

Also f_λ is homotopic to h_λ by Lemma 18.18. Therefore indeed

$$[(M,f)] = [(M_\lambda, f_\lambda)] \in \mathcal{S}^*_{\mathrm{PL}}(I^3 \times D^3, \partial)$$

is in the image of $\theta\colon\ \mathrm{PL}(D^3 \times T^3, \partial) \to \mathcal{S}^*_3(D^3 \times T^3, \partial)$, as desired. □

This concludes our computation of the homotopy type of TOP / PL, and of the homotopy groups of TOP $/O$.

19 Concordance implies isotopy

First we will prove CAT handle straightening, then we will apply it to prove the concordance implies isotopy theorem. Most of the work is in proving the handle straightening theorem.

19.1 CAT handle straightening

Consider a topological embedding $h\colon B^k \times \mathbb{R}^n \hookrightarrow V^{k+n}_{\mathrm{CAT}}$ which is a CAT embedding near $(\partial B^k) \times \mathbb{R}^n$. Then we say handle h can be (PL-)straightened/smoothed if there exists an isotopy $h_t\colon B^k \times \mathbb{R}^n \hookrightarrow V^{k+n}$ such that

(1) $h_0 = h$;
(2) h_1 is a CAT embedding near $B^k \times B^n$;
(3) $h_t = h$ for $t \in [0,1]$ outside a compact set and near $(\partial B^k) \times \mathbb{R}^n$.

We will show that handles can be straightened assuming that a handle problem is concordant to a solution, and in fact this will imply that an entire concordance can be straightened.

Recall that for M a manifold with boundary, the symbol $\sqsupset(I \times M)$, sometimes just called $\sqsupset$, denotes the edges $I \times \partial M \cup \{1\} \times M$. The next theorem and its proof are from [KS77, Essay I.3].

Theorem 19.1. *Let X be a CAT manifold and $h\colon I \times B^k \times \mathbb{R}^n \to X$ a (TOP) homeomorphism, and a CAT embedding near $\sqsupset$. Suppose $m := k + n \geq 5$. Then there is an isotopy*

$$h_t\colon I \times B^k \times \mathbb{R}^n \to X, \quad t \in [0,1]$$

such that $h_0 = h$, and h_1 a CAT embedding near $I \times B^k \times B^n$, and there is $r > 0$ such that for all $t \in [0,1]$ we have $h_t = h$ near $\sqsupset$ and outside $I \times B^k \times rB^n$.

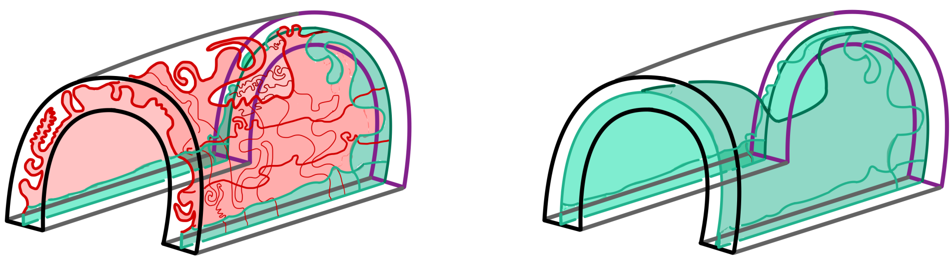


FIGURE 49. The given h is already a CAT embedding on the green region. After an isotopy we obtain h_1 which is also CAT near the blue region $I \times B^k \times B^n$.

Recall that we did handle straightening for TOP, where the condition was that a handle was "close" to a straightened one. Now we have PL/DIFF structures and the condition is given by a concordance instead.

Proof. Let us fix some notation, similarly as for the previous torus trick. Let $\rho\colon \mathbb{R}^n \to T^n$ be the standard covering and define

$$\begin{aligned} \overline{e}\colon \mathbb{R}^n &\to T^n \\ y &\mapsto \rho(y/8). \end{aligned}$$

Let $p := \overline{e}(1/2, \cdots, 1/2)$ and pick a CAT immersion $\alpha'\colon T^n \setminus \{p\} \looparrowright \mathbb{R}^n$. As before, we can arrange that $\alpha' \circ \overline{e}|_{2B^n} = \mathrm{Id}_{2B^n}$. Let i, i_1 be such that the diagram in Fig. 50 commutes. We can choose α' carefully so that the immersion

$$\alpha := \mathrm{Id}_{I \times B^k} \times \alpha'\colon I \times B^k \times T^n \setminus \{p\} \looparrowright I \times B^k \times \mathbb{R}^n$$

is one-to-one on the preimage of $i(I \times B^k \times 2\mathring{B})$. This will imply that i_3 in the diagram below is a CAT embedding. Finally, define $e := \mathrm{Id}_{I \times B^k} \times \overline{e}\colon I \times B^k \times \mathbb{R}^n \to I \times B^k \times \mathbb{R}^n$.

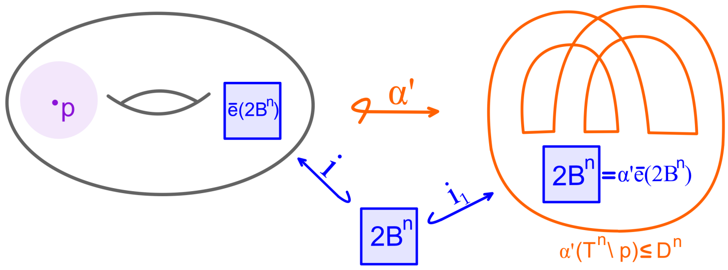


FIGURE 50. An immersion of the n-torus.

The aim is to construct the following diagram.

$$\begin{tikzcd}
 & {[I \times B^k \times \mathbb{R}^n]_{\Sigma_4}} \arrow[r, "H", "\substack{\cong_{\mathrm{CAT}, \mathrm{Id}_{\text{near } \sqsupset, \infty}} \\ I \times B^k \times B^n \subseteq H(I \times B^k \times 2\mathring{B}^n)}"'] & {[I \times B^k \times \mathbb{R}^n]_{\mathrm{std}}} \\
 & {[I \times B^k \times \mathbb{R}^n]_{\Sigma_3}} \arrow[u, "j"', "\substack{\text{CAT emb.} \\ \text{on } \mathrm{Im}(i_3)}"] \arrow[d, "e", "\text{CAT cover}"'] \arrow[r, "G", "\substack{\cong_{\mathrm{CAT}, \mathrm{Id}_{\text{near } \sqsupset}} \\ \text{bded dist. from Id}}"'] & {[I \times B^k \times \mathbb{R}^n]_{\mathrm{std}}} \arrow[u, "j", "\substack{\text{radial} \\ \text{compression}}"'] \arrow[d, "e"] \\
 & {[I \times B^k \times T^n]_{\Sigma_2}} \arrow[r, "g", "\cong_{\mathrm{CAT}, \mathrm{Id}_{\text{near } \sqsupset}}"'] & {[I \times B^k \times T^n]_{\mathrm{std}}} \\
{[I \times B^k \times 2\mathring{B}^n]_{\Sigma}} \arrow[r, hook, "i_1"] \arrow[uuur, hook, bend left, "i_4"] \arrow[uur, hook, bend left, "i_3"'] \arrow[dr, hook, bend right, "i"'] & {[I \times B^k \times T^n \setminus \{p\}]_{\Sigma_1}} \arrow[u, "\substack{\text{CAT emb.} \\ \text{on } \mathrm{Im}(i_1)}"] \arrow[d, "\alpha", "\text{CAT immersion}"'] & \\
 & {[I \times B^k \times \mathbb{R}^n]_{\Sigma}} \arrow[r, "h", "\cong_{\mathrm{CAT}}"'] & X
\end{tikzcd}$$

(1) Let Σ be the CAT structure on $I \times B^k \times \mathbb{R}^n$ obtained by pulling back the CAT structure on X via h. This induces a CAT structure on $I \times B^k \times 2\mathring{B}^n$, which we also label Σ. The map i is the inclusion map.

(2) Define a CAT structure Σ_1 on $I \times B^k \times T^n \setminus \{p\}$ so that α is a CAT immersion with respect to Σ. Since we use Σ to obtain Σ_1, in a sense it was not important that α was originally a CAT map with respect to the standard structures. We now choose the CAT structure on the domain to make α a CAT immersion.

(3) The CAT structure Σ_2 comes from extending Σ_1 on a subset away from the missing point in T^n. As before we have to use the Schoenflies theorem, and the non-compact h-cobordism theorem. This uses the dimension restriction that $k + n \geq 5$. We postpone the details of this until later in the proof.

(4) The cobordism
$$[(I; \{0\}, \{1\}) \times B^k \times T^n]_{\Sigma_2}$$
is topologically a product, but a priori we do not know that it is a CAT product. But the CAT s-cobordism theorem (recall $\mathrm{Wh}(\mathbb{Z}[\mathbb{Z}^n]) = 0$) shows that there is a CAT isomorphism g, which is the identity near $\sqsupset$. This again used the dimension restriction $k + n \geq 5$.

This is where the I coordinate helps. Recall that at the analogous stage in the proof of the stable homeomorphism theorem, we had to lift to a finite cover and apply the classification of homotopy tori. That will not work here, since we want to also be able to straighten 3-handles. But we have the extra hypothesis of a concordance to a straightened handle, and we use it crucially here.

(5) Define G to be the lift of g along e. Define Σ_3 so that G is a CAT isomorphism. Since $Ge = eg$ and g, G, and the right hand e are all CAT maps, so is the left hand e. Note that since g is homotopic to the identity, G is bounded distance from the identity.

(6) The maps i_3 and i_4 are the natural inclusion maps. The fact that $\alpha' \circ \overline{e}|_{2B^n} = \mathrm{Id}_{2B^n}$ implies that i_3 is a CAT embedding.

(7) Define the map j to be a radial compression fixing $I \times B^k \times 2B^n$ pointwise, a homeomorphism onto its image $I \times B^k \times rB^n$ for some $r > 0$. Since j does not change $I \times B^k \times 2B^n$, it follows that i_4 is a CAT embedding.

(8) Choose a map

$$\beta\colon [I \times B^k \times \mathbb{R}^n]_{\Sigma_3} \to [I \times B^k \times \mathbb{R}^n]_{\Sigma_3}$$

such that $G \circ \beta(I \times B^k \times 2B^n) \supseteq I \times B^k \times B^n$ fixing $\sqsupset$ and near ∞. So $G' := G \circ \beta$ equals G near ∞ and equals Id near $\sqsupset$. Also $G'(I \times B^k \times 2\mathring{B}^n) \supseteq I \times B^k \times B^n$.

(9) Define

$$H := \begin{cases} jG'j^{-1} & \text{on } j(I \times B^k \times \mathbb{R}^n) = I \times B^k \times r\mathring{B}^n, \\ \mathrm{Id} & \text{else.} \end{cases}$$

We use that G is bounded distance from the identity to see that $jG'j^{-1}$ limits to Id on $I \times B^k \times rS^n$, and we may therefore extend it by the identity.

(10) Choose Σ_4 to make H a CAT isomorphism.

This finishes the construction of the diagram, apart from the construction of Σ_2. We give some details on this now. Recall that we have a structure Σ_1 on $I \times B^k \times (T^n \setminus \{p\})$. Our aim is to construct a CAT structure Σ_2 on $I \times B^k \times T^n$ that is standard near $\sqsupset$, and such that

$$[I \times B^k \times (T^n \setminus \{p\})]_{\Sigma_1} \to [I \times B^k \times T^n]_{\Sigma_2}$$

is a CAT embedding near $i_1(I \times B^k \times 2B^n)$.

First, for some $\lambda < 1$, extend Σ_1 from $I \times B^k \times (T^n \setminus \{p\})$ to $(I \times B^k \times T^n) \setminus (I \times \lambda B^k \times \{p\})$. The structure is already standard near $I \times \partial B^k \times \{p\}$, so we can extend in this way. We also call the extension also Σ_1. Choose an embedding

$$\psi\colon (I \times B^k \times T^n) \setminus (I \times \{0\} \times \{p\}) \hookrightarrow (I \times B^k \times T^n) \setminus (I \times \lambda B^k \times \{p\})$$

that is the identity outside a neighbourhood of $I \times \lambda B^k \times \{p\}$. Note that $B^k \setminus \lambda B^k \cong B^k \setminus \{0\}$, so such an embedding exists. Define

$$\Sigma_1' := \psi^{-1}(\Sigma_1).$$

This CAT structure has changed nothing on $i_1(I \times B^k \times 2B^n)$, nor near $\sqsupset$. So we just need to extend it to some compatible CAT structure on all of $I \times B^k \times T^n$.

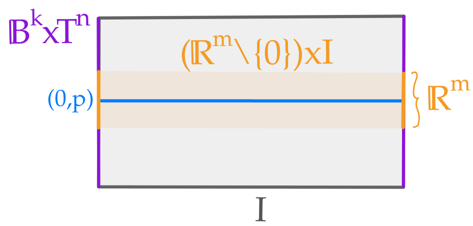


FIGURE 51. Construction of Σ_2.

Recall that $m := k + n$, and consider $I \times B^k \times T^n$ as $(B^k \times T^n) \times I$. Consider an $\mathbb{R}^m$ neighbourhood of $(0, p)$ in $B^k \times T^n$. Then $(\mathbb{R}^m \setminus \{0\}) \times I$ inherits a CAT structure from Σ_1', which we also call Σ_1', and it suffices to extend this over all of $\mathbb{R}^m \times I$. Now

$$[\mathbb{R}^m \times I \setminus (\{0\} \times I)]_{\Sigma_1}$$

is a CAT non-compact proper h-cobordism. (Here proper means that the inverse image of each compact set is compact). The proper h-cobordism theorem gives a proper CAT isomorphism

$$\phi\colon [\mathbb{R}^m \times I \setminus (\{0\} \times I)]_{\Sigma_1} \xrightarrow{\cong_{\text{CAT}}} [\mathbb{R}^m \times I \setminus (\{0\} \times I)]_{\Sigma_{\text{std}}}$$

that restricts to the identity on $(\mathbb{R}^m \setminus \{0\}) \times \{1\}$, where Σ_1' was already standard. Let

$$C := I \times \frac{1}{2} B^m.$$

Extend ϕ by $(0, 0) \mapsto (0, 0)$ and $(0, 1) \mapsto (1, 1)$. The resulting ϕ is then a homeomorphism and $\phi(\partial C)$ is a sphere S^m. By the Schoenflies Theorem 7.19, $\phi(\partial C)$ bounds a topological ball B^{m+1} in $\mathbb{R}^m \times I$. Extend ϕ over that ball by coning. Then we obtain a homeomorphism

$$\Phi\colon \mathbb{R}^m \times I \to \mathbb{R}^m \times I$$

that is a CAT embedding outside $sB^m \times I$, where $s > 0$ is large enough to encompass both C and $\phi(C)$. Then

$$\sigma := \Phi^{-1}(\Sigma_{\text{std}})$$

gives a CAT structure on $\mathbb{R}^m \times I$ that agrees with Σ_1' outside $sB^m \times I$. Patching together Σ_1' and σ, which we can do since they agree on the open set $(\mathbb{R}^m \times I) \setminus (sB^m \times I)$ of $\mathbb{R}^m \times I$, we obtain the desired structure Σ_2 on all of $I \times B^k \times T^n$. As promised we have that Σ_2 is standard near $\sqsupset$, and the inclusion

$$[I \times B^k \times (T^n \setminus \{p\})]_{\Sigma_1} \to [I \times B^k \times T^n]_{\Sigma_2}$$

is a CAT embedding near $i_1(I \times B^k \times 2B^n)$. This completes the construction of Σ_2, which was the only part missing in the construction of the main diagram in the enumerated list above.

Now we use the diagram to complete the proof. We use the existence of the CAT-isomorphism H with the properties shown in the diagram, namely that it is Id near $\sqsupset$ and near ∞, and that $I \times B^k \times B^n \subseteq H(I \times B^k \times 2\mathring{B}^n)$. We also use that $\Sigma = \Sigma_4$ on $I \times B^k \times 2\mathring{B}^n$.

Extend H by Id to homeomorphism of $[0, \infty) \times \mathbb{R}^{n+k}$. Then let $H_t\colon I \times B^k \times \mathbb{R}^n \to I \times B^k \times \mathbb{R}^n$ be an Alexander isotopy of homeomorphisms defined by

$$H_t(x) := \begin{cases} tH(\frac{x}{t}) & 0 < t \leq 1, \\ H_0(x) = x & \end{cases}$$

Finally, define

$$h_t := h \cdot H_t^{-1}$$

We have $h_0 = h$ and $h_1 = hH^{-1}$, and

$$h_t\colon I \times B^k \times \mathbb{R}^n \xrightarrow{H^{-1}} [I \times B^k \times \mathbb{R}^n]_{\Sigma_4} \xrightarrow{\text{Id}} [I \times B^k \times \mathbb{R}^n]_{\Sigma} \xrightarrow{h} X$$

Since $H^{-1}(I \times B^k \times B^n) \subseteq [I \times B^k \times 2\mathring{B}^n]_{\Sigma_4}$ and $\Sigma = \Sigma_4$ on $I \times B^k \times 2\mathring{B}^n$, we have that h_1 is a CAT embedding on $I \times B^k \times B^n$. Note that the map h from the hypotheses was used to define the various CAT structures, starting with Σ, as well as in the final step of the proof. $\square$

19.2 Proof of concordance implies isotopy

Next, as promised we shall prove that concordance implies isotopy for CAT structures Theorem 17.5. Here is the technical relative version we will prove, see Fig. 52.

Theorem 19.2 (Concordance implies isotopy, relative version)**.** *Let M^m be a topological manifold with a CAT structure Σ, and pick closed subsets $C \subseteq M$ and $D \subseteq M$, and open neighbourhoods $U \supseteq C$ and $V \supseteq D \setminus C$. We need $m \geq 6$ or $m = 5$ and $\partial M \subseteq U$.*

Let Γ be a CAT structure on $M \times I$ such that $\Gamma = \Sigma \times [0, \delta)$ near $M \times \{0\}$ and $\Gamma = \Sigma \times I|$ on $U \times I$. Fix a continuous function $\varepsilon \colon M \times I \to (0, \infty]$.

Then there exists an isotopy

$$h_t \colon M \times I \to M \times I, \quad t \in [0,1]$$

such that

(1) $h_0 = \mathrm{Id}_{M \times I}$,
(2) $h_1 \colon M_\Sigma \times I \to (M \times I)_\Gamma$ *is a CAT embedding near* $(C \cup D) \times I$,
(3) h_t *fixes a neighbourhood of* $(M \setminus V) \times I \cup M \times \{0\} \cup C \times I$,
(4) $d(h_t(x), x) < \varepsilon(x)$ *for all* $x \in M \times I$ *and* $t \in [0,1]$.

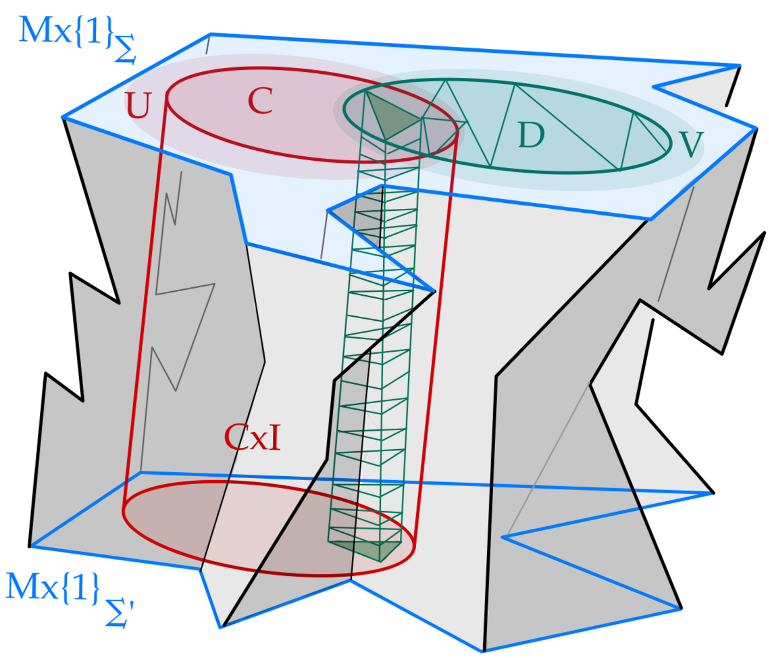


FIGURE 52. The setup of the relative version of concordance implies isotopy.

This is known as a "CUDV" theorem, which is a colloquialism for a relative statement. The roles of C, U, D, and V are as follows. There is a solution on C that we want to maintain. If the solution can be extended to U, we can solve the problem on D whilst keeping the given solution on C, and not changing anything outside V. This version with precise control, in terms of $CUDV$ and ε is what we will use in the proof of the product structure theorem. The fact that handle straightening allows us to work handle by handle means that achieving the control we desire is fairly straightforward.

Taking $C = U = \emptyset$, $D = V = M$, and $\varepsilon \equiv \infty$ yields the special case with $\partial M = \emptyset$ that was stated before as Theorem 17.5.

Proof. We have already done the hard work in proving Theorem 19.1. Relabel $I \to I$ by sending $t \mapsto 1 - t$, so it will be easier to apply handle straightening, as shown in Fig. 52.

First we triangulate V using the CAT structure. Convert to a handle structure – remember that triangulations give PL handle decompositions. If CAT=DIFF we can use Morse theory directly, and need not first obtain a triangulation. Make the handle structure fine enough so that every handle that touches C is contained in U. This might require subdividing.

Let

$$K := \{\text{handles of } K \text{ contained in } U\}$$

and let

$$L := \{\text{handles of } V \text{ that meet } D \setminus C\}.$$

Note that

$$K \cup L \supseteq (C \cup D) \cap V.$$

Induct on handles in L. Start with handles whose attaching region is contained in U, and straighten handles in the order in which they are attached. As we isotope a handle, we shall also move subsequent handles that we have not yet straightened, which are attached to the handle we are straightening.

Extend each handle $B^k \times B^n$ with $n = m - k$ to $B^k \times \mathbb{R}^n \subseteq V$, and apply the handle straightening Theorem 19.1 to the identity map

$$\mathrm{Id}\colon I \times B^k \times \mathbb{R}^n_{\mathrm{std}} \to [I \times B^k \times \mathbb{R}^n]_\Gamma.$$

The structure Γ agrees with Σ on $M \times \{1\}$, and we fix it by isotopy to agree with $\Sigma \times I$ on $I \times B^k \times B^n$. If necessary first subdivide the decomposition further to make the handle decomposition fine enough, with respect to ε, to arrange that $d(h_t(x), x) < \varepsilon(x)$ for all x, t. $\square$

20 The Product Structure theorem

We now prove the product structure theorem. This will use the technical relative version of concordance implies isotopy. Before we begin, we also need one more ingredient, about CAT structures on Euclidean space. This uses a result of Browder-Levine-Livesay on CAT-manifolds, and the stable homeomorphism theorem.

20.1 CAT structures on Euclidean spaces

Theorem 20.1 (Stallings [Sta62])**.** *Any two* CAT *structures on* $\mathbb{R}^n$ *are isotopic for* $n \geq 6$*.*

We have already stated this result as a corollary of the Product Structure Theorem 17.8, but we will actually use it in its proof, so certainly we need an independent argument. The original proof due to Stallings uses engulfing for a PL proof, which works for $n \geq 5$. Then the deduction from PL to smooth goes via the PL-to-smooth smoothing theory. But we present a different proof, which works only for $n \geq 6$ (but this will be enough). The proof we will give has the advantage that if one wants the smooth version, there is a more directly smooth proof. It uses the following ingredient.

Theorem 20.2 (Browder-Levine-Livesay [BLL65])**.** *Let X be an open* CAT *(*PL *or* DIFF*) n-manifold with $n \geq 6$, which is simply connected at infinity and H_*X are finitely generated.*

Then X is CAT *isomorphic to the interior of a compact manifold Y with simply connected boundary. Moreover, such Y is unique.*

Proof of Theorem 20.1. Let Σ be a CAT structure on $\mathbb{R}^n$. By Theorem 20.2, we have that $\mathbb{R}^n_\Sigma \cong \operatorname{Int} W$ for some compact manifold W with $\pi_1(\partial W)$ trivial.

Recall that a manifold is homotopy equivalent to its interior. Then a homology computation implies that $\partial W \simeq S^{n-1}$. To see this, we have an exact sequence of homology with $\mathbb{Z}$ coefficients

$$H_{k+1}(W, \partial W) \to H_k(\partial W) \to H_k(W).$$

For $k \geq 1$, $H_k(W) = 0$. Also $H_{k+1}(W, \partial W) \cong H^{n-k-1}(W) = 0$ for $n-k-1 > 0$, that is $k < n-1$. Therefore $H_k(\partial W) = 0$ for $1 \leq k \leq n$. For $k = n-1$ we have $H_{n-1}(\partial W) \cong H_n(W, \partial W) \cong H^0(W) \cong \mathbb{Z}$. The Hurewicz theorem and Whitehead's theorem then imply that $\partial W \simeq S^{n-1}$ as claimed.

Also, $W \setminus \mathring{D}^n$, for a small ball D^n in the interior, is an h-cobordism from ∂W to S^{n-1}. Thus, we have that they are CAT isomorphic by the h-cobordism theorem (this uses $n \geq 6$).

We can glue a disc D^m to W in such a way that $W \cup_{\partial W} D^m$ is CAT isomorphic S^n. In the smooth category this needs some care, since gluing two discs together can also produce exotic spheres. But we make the choice that yields the standard sphere. We view $S^n = \partial D^{n+1}$, and this gives an h-cobordism from W to D^m. Therefore, by the CAT h-cobordism theorem there is a CAT isomorphism $D^m \to W$, which on the interior restricts to a CAT isomorphism

$$\Theta \colon \mathbb{R}^n_{\mathrm{std}} \to \mathbb{R}^n_\Sigma.$$

By the Stable Homeomorphism Theorem, or more precisely Theorem 12.30, Θ is TOP isotopic to Id, therefore Σ is isotopic to the standard structure. □

20.2 The proof of the product structure theorem

First we recall the statement of concordance implies isotopy, since we will need it here a couple of times.

Theorem 20.3 (Concordance implies isotopy, relative version)**.** *Let M^m be a topological manifold with a CAT structure Σ, and pick closed subsets $C \subseteq M$ and $D \subseteq M$, and open neighbourhoods $U \supseteq C$ and $V \supseteq D \setminus C$. We need $m \geq 6$ or $m = 5$ and $\partial M \subseteq U$.*

Let Γ be a CAT structure on $M \times I$ such that $\Gamma = \Sigma \times [0, \delta)$ near $M \times \{0\}$ and $\Gamma = \Sigma \times I$ on $U \times I$. Moreover, fix a continuous function $\varepsilon \colon M \times I \to (0, \infty]$.

Then there exists an isotopy

$$h_t \colon M \times I \to M \times I, \quad t \in [0,1]$$

such that

(1) $h_0 = \mathrm{Id}_{M \times I}$,
(2) $h_1 \colon M_\Sigma \times I \to (M \times I)_\Gamma$ *is a CAT embedding near* $(C \cup D) \times I$,
(3) h_t *fixes a neighbourhood of* $(M \setminus V) \times I \cup M \times \{0\} \cup C \times I$,
(4) $d(h_t(x), x) < \varepsilon(x)$ *for all* $x \in M \times I$ *and* $t \in [0,1]$.

Here is the version of the product structure theorem we are going to prove. Where notation overlaps between the statement of this theorem and the statement of the previous theorem, ignore it. In the course of the proof of the product structure theorem we will apply Concordance implies Isotopy twice, with different subsets playing the role of C, U, D, and V.

Theorem 20.4 (Relative Product Structure Theorem)**.** *Let M be a manifold and fix an open subset $U \subseteq M$. Assume* $\dim M \geq 6$, *or* $\dim M = 5$ *with $\partial M \subseteq U$. Let Σ be a* CAT *structure on $M \times \mathbb{R}^q$ for some $q \geq 1$, and suppose there exists a* CAT *structure ρ on U with $\Sigma|_{U \times \mathbb{R}^q} = \rho \times \mathbb{R}^q$.*

Then ρ extends to a CAT *structure σ, and there is a concordance $(M \times \mathbb{R}^q \times I)_\Gamma$ from Σ to $(M \times \mathbb{R}^q)_{\sigma \times \mathbb{R}^q}$ relative to $U \times \mathbb{R}^q$. Moreover, any two such structures σ on M are unique up to concordance.*

We will use the stable homeomorphism theorem, the uniqueness of CAT structures on $\mathbb{R}^m$ for $m \geq 6$ up to isotopy, Theorem 20.3 that concordance implies isotopy, and a new lemma called the Windowblind Lemma, which we will explain when the time comes.

Proof. First we observe that it suffices to prove the case of $q = 1$, by induction. Also we can work chart by chart, since we have a relative theorem. We will also ignore the boundary for brevity. So we can assume that $M = \mathbb{R}^m$. Then for the general case this will play the role of a single chart in the induction.

Since $m \geq 5$, we have that $\dim(M \times \mathbb{R}) \geq 6$. Then we know that Σ is isotopic (and therefore concordant) to the standard structure on $\mathbb{R}^{m+1}$ (recall we are assuming that $M = \mathbb{R}^m$). However, this is not relative to $U \times \mathbb{R}$, so we still have work to do. The first step is to apply Theorem 20.3 with $U = C = \emptyset$, $D = M \times [1, \infty)$, $V = M \times (\frac{1}{2}, \infty)$. Then we obtain an isotopy from Σ to a structure Σ_1, where Σ_1 equals the standard structure on $M \times [1, \infty)$, so is a product structure there, and equals Σ on $M \times (-\infty, 0]$.

For the next step, on $U \times [0,1]$, we have a CAT structure $\Sigma_1|_{U \times [0,1]}$. Let $\varepsilon \colon M \times [0,1] \to [0, \infty)$ be a continuous function with $\varepsilon^{-1}((0,\infty)) = U \times [0,1]$ (this is potentially confusing, in Theorem 20.3 the codomain of ε was $(0, \infty)$, but this is not a problem, since we will only apply it to $U \times [0,1]$). Next apply Theorem 20.3 to $U \times [0,1]|_{\Sigma_1|}$, setting $C = U = \emptyset$ (where the U comes from Theorem 20.3), and $V = D = U$ (where U comes from the current statement). We obtain an isotopy $h_t \colon U \times [0,1]$ from Id to h_1 where $h_1 \colon U \times [0,1]_{\rho \times [0,1]} \to U \times [0,1]_{\Sigma_1}$.

Extend h_1 to a homeomorphism $h \colon M \times \mathbb{R} \to M \times \mathbb{R}$ by setting

$$(x, r) \mapsto \begin{cases} (x, r) & r \leq 0 \\ h_1(x, r) & (x, r) \in U \times [0,1] \\ (\mathrm{pr}_1(h_1(x, 1)), r) & x \in U, r \geq 1 \\ (x, r) & x \notin U \end{cases}$$

where pr_1 is the projection $U \times \{1\} \to U$. Then define Σ_2 such that $h \colon [M \times \mathbb{R}]_{\Sigma_2} \to [M \times \mathbb{R}]_{\Sigma_1}$ is a CAT isomorphism. The structure Σ_2 now has the property that it is still a product on $M \times [1, \infty)$, and equals Σ on $M \times (-\infty, 0]$, but now it also equals Σ on $U \times \mathbb{R}$, and is therefore also a product structure $\rho \times \mathbb{R}$ on $U \times \mathbb{R}$.

To finish off the proof, we need the next lemma.

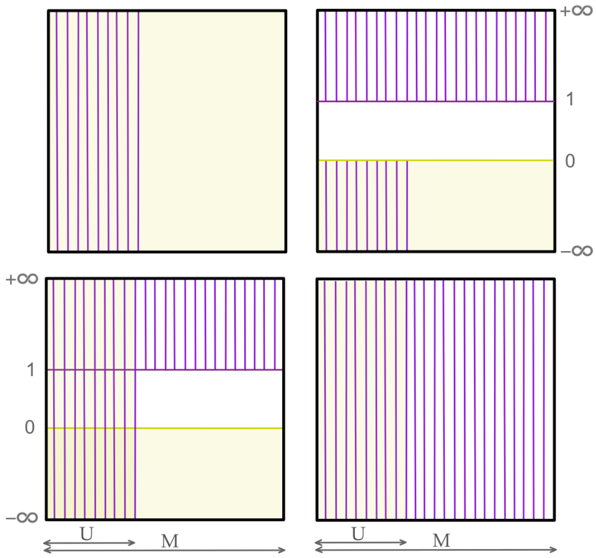


FIGURE 53. Proof of the product structure theorem. Each square depicts $M = \mathbb{R}^m$ as the horizontal axis, with $U \subseteq M$ a subset of it, while the $\mathbb{R}$ coordinate corresponds to the vertical axis. In each square, vertical lines indicate that the structure there is a product structure. Shaded yellow indicates that the structure coincides with Σ on that region. The top left square shows Σ, where we start. The top right shows Σ_1. The bottom left square depicts Σ_2. The bottom right shows the goal, $\sigma \times \mathbb{R}$, which agrees with Σ on $U \times \mathbb{R}$.

Lemma 20.5 (Windowblind lemma)**.** *Let Σ' and Σ'' be CAT structures on $M \times \mathbb{R}$. Suppose that $\Sigma' = \Sigma''$ on $M \times (a, b)$ for some $-\infty \leq a < b \leq \infty$ and both Σ' and Σ'' are products on $U \times \mathbb{R}$. Then there exists a concordance from Σ' to Σ'' relative to $U \times \mathbb{R}$.*

Proof. Choose an isotopy of embeddings $h_t \colon \mathbb{R} \to \mathbb{R}$ with $h_0 = \mathrm{Id}_{\mathbb{R}}$ and $h_1 \colon \mathbb{R} \to (a, b)$ an onto embedding. Define

$$H \colon I \times M \times R \to T \times M \times \mathbb{R}$$
$$(t, x, r) \mapsto (t, x, h_t(r)) =: H_t(x, r).$$

Then $H^{-1}(I \times \Sigma')$ is a structure on $I \times M \times \mathbb{R}$ so that $H \colon I \times M \times \mathbb{R}_{H^{-1}(I\times\Sigma')} \to I \times M \times \mathbb{R}_{\Sigma'}$ is a CAT embedding.

Then $[I \times M \times \mathbb{R}]_{H^{-1}(I\times\Sigma')}$ is a concordance from Σ' to $H_1^{-1}(\Sigma')$ relative to $U \times \mathbb{R}$, since Σ' is already a product on $U \times \mathbb{R}$. Similarly, there exists a concordance from Σ'' to $H_1^{-1}(\Sigma'')$. But $\Sigma' = \Sigma''$ on $M \times (a, b)$, so we know that $H_1^{-1}(\Sigma') = H_1^{-1}(\Sigma'')$, and therefore Σ' is concordant to Σ'' relative to $U \times \mathbb{R}$. □

Returning to the proof of the product structure theorem, choose some $r \in (1, \infty)$ and let σ be the CAT structure on $M \times \{r\}$, with $(a, b) \subseteq (1, \infty)$. Apply the Windowblind lemma to Σ_2 and $\sigma \times \mathbb{R}$ to get a concordance form Σ_2 to $\sigma \times \mathbb{R}$.

Next apply the lemma to Σ and Σ_2 with $(a, b) \subseteq (-\infty, 0)$ to get a concordance from Σ to Σ_2 relative to $U \times \mathbb{R}$.

Putting these together we get the desired concordance from Σ to $\sigma \times \mathbb{R}$. This completes the proof in the case that $M = \mathbb{R}^m$ and $q = 1$. As stated at the start of the proof, this is sufficient by an inductions over charts and over q.

The product structure theorem also included the statement that any two such CAT structures σ and σ' on M arising in this way are unique up to concordance. We did not prove this yet, so let us do so now. We have concordances

$$\sigma \times \mathbb{R}^q \sim \Sigma \sim \sigma' \times \mathbb{R}^q$$

relative to $U \times \mathbb{R}^q$. Gluing them together gives a CAT structure Γ on $I \times M \times \mathbb{R}^q$, between $\sigma \times \mathbb{R}^q$ and $\sigma' \times \mathbb{R}^q$, and we extend it to a CAT structure, also called Γ, on $\mathbb{R} \times M \times \mathbb{R}^q$. We may assume that the concordance is *conditioned*, i.e. it is a product near $\{i\} \times M \times \mathbb{R}^q$ for $i = 0, 1$, so that it can be extended over $\mathbb{R} \times M \times \mathbb{R}^q$. Let

$$U' \times \mathbb{R}^q := ((\mathbb{R} \setminus [1/4, 3/4]) \times M \times \mathbb{R}^q) \cup (\mathbb{R} \times U \times \mathbb{R}^q).$$

Since Γ is conditioned, we may isotope Γ to a CAT structure that is a product $\mathbb{R} \times \theta \times \mathbb{R}^q$ on $U' \times \mathbb{R}^q$.

Apply the product structure theorem with $U' \subseteq \mathbb{R} \times M$, the CAT structure $\Gamma|_{U' \times \mathbb{R}^q} = \mathbb{R} \times \theta \times \mathbb{R}^q$ on U', and the CAT structure Γ on $\mathbb{R} \times M \times \mathbb{R}^q$. It yields a product CAT structure $\gamma \times \mathbb{R}^q$ on $\mathbb{R} \times M \times \mathbb{R}^q$ which agrees with the CAT structure $\mathbb{R} \times \theta \times \mathbb{R}^q$ on $U' \times \mathbb{R}^q$. In particular $\gamma \times \{0\}$ is a CAT structure on $\mathbb{R} \times M$ that extends $\mathbb{R} \times \theta$ on U'. Restricting to $I \times M$, γ gives a concordance between σ and σ', as desired. $\square$

20.3 Recap of PL-ing and Smoothing theory

Now that we have proven the product structure theorem, it might help to recap its place in PL-ing and smoothing theory. Recall that one of the main questions we studied was whether a topological manifold M admits a CAT structure, where CAT stands for either PL or DIFF. We will discuss the case of $\partial M = \emptyset$ in this recap for simplicity.

The first observation was that smooth manifolds admit a tangent vector bundle. This motivated us to study the question of whether something analogous exists for purely topological manifolds. Back in Section 9 we learnt about the topological tangent microbundle $\mathfrak{t}_M = (M \to M \times M \to M)$. By Kister's Theorem (Theorem 9.10) we know that $\mathfrak{t}_M$ is equivalent to a $\mathrm{TOP}(n)$-bundle, where $\mathrm{TOP}(n) := \mathrm{Homeo}_0(\mathbb{R}^n)$ is the group of homeomorphisms of $\mathbb{R}^n$ fixing the origin. There is an analogous version for the PL category, which we did not cover.

We saw in Section 16 that $\mathrm{TOP}(n)$-bundles are stably classified by homotopy classes of maps $M \to B\,\mathrm{TOP}$. We then studied the obstruction theory to lifting this map to $BCAT$, that is, finding a map $M \to B\mathrm{CAT}$, in the diagram

$$\begin{tikzcd} & B\,\mathrm{PL} \arrow[d] \\ M \arrow[r] \arrow[ur, dotted] & B\,\mathrm{TOP} \arrow[d] \\ & B(\mathrm{TOP}\,/CAT). \end{tikzcd}$$

Denote the lower map by $\delta\colon B\,\mathrm{TOP} \to B(\mathrm{TOP}\,/\,\mathrm{PL})$, where the latter space is defined by Theorem 16.13. Kirby-Siebenman proved (Theorem 17.19) that $\mathrm{TOP}\,/\,\mathrm{PL} \simeq K(\mathbb{Z}/2, 3)$. Since $B(\mathrm{TOP}\,/\,\mathrm{PL})$ is a delooping of $\mathrm{TOP}\,/\,\mathrm{PL}$, we know that $B(\mathrm{TOP}\,/\,\mathrm{PL}) \simeq K(\mathbb{Z}/2, 4)$ so that $[M, B(\mathrm{TOP}\,/\,\mathrm{PL})] \cong H^4(M; \mathbb{Z}/2)$ via a canonical map. The image of $\delta \circ \mathfrak{t}_M$ in $H^4(M; \mathbb{Z}/2)$ is by definition the *Kirby-Siebenmann invariant*. By obstruction theory, we know that it is the only obstruction to lifting $\mathfrak{t}_M$ to $B\,\mathrm{PL}$.

There are further obstructions to lifting $\mathfrak{t}_M$ to $B\,\mathrm{DIFF}$. The next potentially nontrivial obstruction lies in $H^8(M; \mathbb{Z}/28)$ corresponding to $\Theta_7 \cong \mathbb{Z}/28$. In order to see this, one should know the homotopy type of $B(\mathrm{TOP}\,/O)$, which we described in Section 18.2.

Now, from the Precursor to smoothing theory (Section 15.3) we saw that having a lift $M \to B\mathrm{CAT}$ implies that there is $q \geq 0$ such that $M \times \mathbb{R}^q$ admits a CAT structure. As before, we only showed this in the case of $CAT = \mathrm{DIFF}$ but there is an analogue in the case of $CAT = \mathrm{PL}$.

Finally, the Product Structure theorem (Theorem 17.8) tells us that if $n \geq 5$ and $\partial M = \emptyset$ then a CAT structure on $M \times \mathbb{R}^q$ can be used to equip M with a CAT structure. Observe that this is the first time we have had to restrict the dimension of M.

To summarise, via the product structure theorem, we know that the Kirby-Siebenmann invariant is the only obstruction to the existence of a PL structure on a closed topological manifold M with dimension ≥ 5. There are further obstructions to the existence of a smooth structure, with the next potentially nontrivial obstruction lying in $H^8(M, \mathbb{Z}/28)$, and more generally in $H^{k+1}(M; \Theta_k)$ for $k \geq 7$.

Remark 20.6. Since we only needed to restrict dimensions in the final step where we applied the product structure theorem, there is still something we can say in the case of $n = 4$. Specifically, given a closed topological 4-manifold M, $\mathrm{ks}(M) = 0$ then $M \times \mathbb{R}$ has a smooth (and therefore PL structure. However, there do exist nonsmoothable 4-manifolds with trivial Kirby-Siebenmann invariant, as follows. Let E_8 denote the E_8 manifold, constructed by Freedman [Fre82]. Then $E_8 \# E_8$ does not admit a smooth structure (by Donaldson's theorem) but has trivial Kirby-Siebenmann invariant.

Example 20.7. There exist non PL-able manifolds in each dimension at least 4. For example, let E_8 denote the E_8 manifold. Then $E_8 \times S^k$ for $k \geq 1$ does not admit a PL structure.

Example 20.8. Siebenmann showed that every orientable closed topological 5-manifold is triangulable. Consequently, $E_8 \times S^1$ is triangulable but not PL-able.

Example 20.9. There exist nontriangulable manifolds in each dimension at least 4. This was done by Freedman for $n = 4$ via the E_8 manifold and for $n \geq 5$ by Manolescu.

Example 20.10. There exist non-PL triangulations of PL manifolds. This follows since the double suspension of the Mazur homology sphere is S^5 as shown by Edwards and more generally by the double suspension theorem of Cannon [Can79].

Example 20.11. There exist PL manifolds with no smooth structure. This was first shown by Kervaire in 1960 in dimension 10. The lowest possible dimension is 8, shown by Ells-Kuiper (1961).

21 Fundamental tools in topological manifolds

A topological manifold is covered by charts, each of which is homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^n_+$. These admit CAT structures, so locally we can apply results about CAT structures, such as the existence of handle structures or transversality. The product structure theorem will enable us to piece together the local solutions into global solutions. This is based on [KS77, Essay III].

21.1 Handle decompositions

Definition 21.1. Let W^m be a CAT manifold, with CAT one of $\mathrm{TOP}, \mathrm{DIFF}, \mathrm{PL}$, and $M \subseteq W$ a codimension zero closed submanifold. A CAT handle decomposition of W relative to M is a filtration

$$M = M_0 \subseteq M_1 \subseteq \dots$$

such that

- $\bigcup_{i \geq 0} M_i = W$,
- for each $i \geq 0$, M_i is a closed codimension zero submanifold of W,
- for each $i \geq 0$, the set $H_i := \overline{M_i \setminus M_{i-1}}$ is a compact submanifold such that

  $$(H_i, H_i \cap M_{i-1}) \cong_{\mathrm{CAT}} (D^k, \partial D^k) \times D^{m-k}$$

  for some $0 \leq k \leq m$,
- the collection $\{H_i\}$ is locally finite.

For CAT $=$ DIFF we also smooth corners. There is essentially unique smooth structure on the result of attaching a handle. The following is obtained from Morse theory and the flow of a gradient-like vector field (for a proof see Milnor or Thom?).

Theorem 21.2. *(Relative) handle decompositions exist for smooth manifolds for all m.*

The following result uses barycentric subdivisions instead, see [Hud69, p. 223].

Theorem 21.3. *(Relative) handle decompositions exist for* PL *for all m.*

We will prove the PL analogue as a consequence of the product structure theorem and Theorem 21.2.

Theorem 21.4. *(Relative) handle decompositions exist for* TOP *for $m \geq 6$.*

For $m \leq 3$ by Rado and Moise all structures are equivalent. Handle decompositions exist for $m = 5$ by the work of Quinn [Qui82], [FQ90, Chapter 9]. However, for $m = 4$ a handle decomposition exists on W^4 if and only if W^4 is smoothable (equivalently, PL-able). Since there are non-smoothable 4-manifolds, this implies that handle decomposition do not exist for all 4-manifolds.

Sketch of proof of Theorem 21.4. The idea is to apply Product Structure theorem locally, working in charts. Assume for simplicity that $\partial W = \emptyset$ and that W is compact.

Let us cover W by compact sets $A_1, \dots, A_k$ with $A_i \subseteq U_i \cong \mathbb{R}^m$. By pulling back a smooth structure on $\mathbb{R}^m$, each U_i has a smooth structure, and therefore has a handle decomposition relative to any smooth codimension zero submanifold, by Theorem 21.2.

We will construct a filtration $M = N_0 \subseteq N_1 \subseteq \dots N_k = W$ such that each N_i is a TOP handlebody relative to N_{i-1}. This gives a handlebody decomposition of W, by taking the union of respective handlebody filtrations for all N_i.

Assume we have inductively constructed a codimension zero submanifold N_{i-1} for some $i \geq 1$, with $M \cup A_1 \cup A_2 \dots \cup A_{i-1} \subseteq N_{i-1}$. Let us define

$$P_i := U_i \cap \partial N_{i-1} \subseteq U_i,$$

see Fig. 54. This is a codimension one submanifold of W with $\dim \geq 5$, so it has a bicollar $P_i \times \mathbb{R}$ with induced smooth structure from U_i. We have seen Product Structure Theorem 20.4,

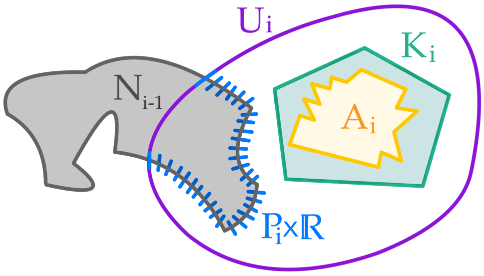


FIGURE 54. The induction step in the proof of Theorem 21.4.

but there is also the following local version: one can isotope the smooth structure on U_i relative to $(P_i \times \mathbb{R}) \setminus (P_i \times (-1,1))$ to a new structure which is a product near $P_i \times \{0\}$. This makes $P_i = P_i \times \{0\}$ into a smooth submanifold of U_i. Thus with this new smooth structure, $U_i \cap N_{i-1}$ is a codimension zero smooth manifold in this new smooth structure from U_i.

Now choose a compact submanifold K_i with $A_i \subseteq K_i \subseteq U_i$. We apply Theorem 21.2 to obtain a handle decomposition for $(U_i \cap N_{i-1}) \cup K_i$ relative to $U_i \cap N_{i-1}$. Then define $N_i \coloneqq N_{i-1} \cup K_i$. This completes the inductive step. □

A useful exercise is to consider why the previous proof does not produce a smooth structure on M. The idea is that while the smooth structure is improved on U_i in a neighbourhood of P_i, this does not respect a given smooth structure on N_{i-1}.

21.2 Transversality

There are two main versions of transversality. Map transversality perturbs a map between manifolds by a homotopy so that the inverse image of a point, or indeed a submanifold N, is again a submanifold, and teh codimension of the inverse image in the domain equals the codimension of N in the codomain.

There is also submanifold transversality, which is stronger. Given two submanifolds, it enables us to perturb one of them by a locally flat *isotopy*, fixing the other submanifold, until the intersections are transverse.

There is a subtlety that one needs normal microbundles to do both of these carefully. We will not go into this here, and instead present the following warm up version of transversality, which is all we have time for right now.

Theorem 21.5. *Let $f_0\colon M^m \to \mathbb{R}^n$ be a continuous map with M closed topological manifold and $m - n > 4$. Then f_0 is homotopic to f_1 which is transverse to $0 \in \mathbb{R}^n$, that is, $f_1^{-1}(0)$ is a topological manifold L of dimension $m - n$ and has a trivial normal bundle.*

Proof sketch. Cover M by compact sets A_i with $A_i \subseteq U_i \cong \mathbb{R}^m$. Assume for the inductive hypothesis that $f\colon M \to \mathbb{R}^n$ is transverse to $0 \in \mathbb{R}^n$ on a neighbourhood Y of $A_1 \cup \cdots \cup A_{i-1}$. That is, $(f|_Y)^{-1}(0)$ is an $(m-n)$-dimensional submanifold $L_{i-1} \subseteq Y$ with trivial normal bundle.

Then $L_{i-1} \cap U_i \eqqcolon L'$ has trivial normal bundle $L' \times \mathbb{R}^n$. By the Local Product Structure Theorem we can isotope the smooth structure on U_i such that L' is a smooth submanifold. Assume that $L' \times \mathbb{R}^n$ is a smooth normal bundle of L'.

Now apply smooth transversality to $f|_{U_i}$: we can homotope f to $f'\colon M \to \mathbb{R}^n$ which is transverse to $0 \in \mathbb{R}^n$ on a neighbourhood of A_i, and such that $f' = f$ near $(L_{i-1} \cap \cup_{j=1}^{i-1} A_j) \times \mathbb{R}^n$ and near $M \setminus U_i$. □

Here are some further consequences of the Product Structure Theorem.

- There exist TOP Morse functions.
- Simple homotopy type is well-defined. To do this we find a PL disc bundle over M embedded as a PL submanifold of a high dimensional Euclidean space. The simple

homotopy type of this disc bundle turns out to be well-defined, and it gives the simple homotopy type of the manifold M.

– High-dimensional manifolds are homeomorphic to CW complexes (open for 4-manifolds). This follows from the existence of topological handle decompositions.

Theorem 21.6 (Topological high-dimensional Poincaré conjecture)**.** *If M^m is a compact topological manifold of dimension $m \geq 5$ and $M^m \simeq S^m$, then M^m is homeomorphic to S^m.*

Sketch of proof using the work of Kirby and Siebenmann. For $m = 5$ smoothing theory applies to smooth M^m, and then we can deduce the result using the smooth resolution of the Poincaré conjecture in this dimension.

Assume now $m \geq 6$. Take out two m-balls from M and prove by ome elementary algebraic topology computations that what remains is a simply-connected h-cobordism. Then the result follows from the topological h-cobordism theorem and the Alexander trick.

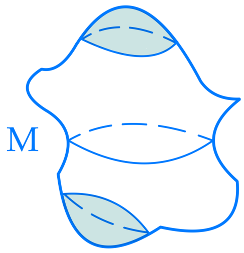


FIGURE 55. Reduction of the Poincaré conjecture to the h-cobordism theorem

To show the topological h-cobordism theorem one uses topological handle decomposition and arrange handles are in increasing order. Then cancel or trade any additional handles of index 0, 1, m and $m - 1$. This for example uses perturbing (i.e. transversality) a null-homotopy of the circle that a 1-handle generates, to produce an embedded disc, then thickening this to a cancelling pair of a 2 and a 3-handle.

Then cancel r- and $(r + 1)$-handle pairs, using Whitney trick. This again requires perturbing the pair into a general position. Once all handles have been cancelled, we must have a product, which completes the outline of the proof of the topological h-cobordism theorem. $\square$

22 Topological manifolds are like high dimensional smooth and PL manifolds, only more so.

The title of this section is a quote from slides of Andrew Ranicki. We think that what he meant was that smooth manifolds, of dimension at least five, admit a remarkably close relation to homotopy theory and algebra, via surgery theory. However there are complications in the smooth category, principally arising from exotic spheres and from Rochlin's theorem, that muddy the waters. In the topological category, the correspondence between geometry and algebra is crisper and more elegant, whence "only more so". In this section we will try to explain the slogan in a precise way.

In the topological category, the following are true. We will not explain what they mean here, but they can be thought of as suggestions for further reading.

(1) The Poincaré conjecture holds in all dimensions.
(2) The Schoenflies conjecture holds in all dimensions.
(3) Orientation preserving homeomorphisms of S^n are isotopic to the identity.
(4) The Alexander trick works.
(5) The surgery obstruction map for the sphere is a bijection.
(6) The surgery exact sequence is a sequence of abelian groups.
(7) The simply-connected surgery exact sequence is a collection of short exact sequences.
(8) Knots $S^{n-k} \subset S^n$ for $k \geq 3$ are unknotted.
(9) Sullivan periodicity: $\Omega^4(\mathbb{Z} \times G/\operatorname{TOP}) \simeq \mathbb{Z} \times G/\operatorname{TOP}$.
(10) Siebenmann periodicity.
(11) Topological Rigidity: the Borel conjecture that every homotopy equivalence between closed aspherical n-manifolds is homotopic to a homeomorphism holds in many cases.
(12) Topological surgery in dimension 4.
(13) The total surgery obstruction gives a criterion for a Poincar'e complex to be homotopy equivalent to a topological manifold.

Solutions to the Exercises

Ex. on p.9. **Solution 2.1.**(PS1.1) Solution by Ekin Ergen.

The line with two origins: Let $X = \mathbb{R} \sqcup \mathbb{R}/\sim$, where $x_i \sim y_j$ iff $x_i = y_j \neq 0$, where $i, j \in \{1, 2\}$ denote the components of the disjoint sum the element is coming from. Let this space with the quotient topology wrt the standard topologies of $\mathbb{R}$. In other words, we are gluing the two lines at corresponding points except 0.

(1) not Hausdorff at 0: There are two points that correspond to 0. These points are not separable by open subsets of X, as any open neighborhoods of 0_1 and 0_2 of X include some balls $(-\varepsilon_1, \varepsilon_1)$ resp. $(-\varepsilon_2, \varepsilon_2)$. However, these cannot be disjoint by construction.
(2) Paracompact: By quotient topology, every open cover of X can be pulled back to an open cover of $\mathbb{R} \sqcup \mathbb{R}$ by taking preimages of $p : \mathbb{R} \sqcup \mathbb{R} \to \mathbb{R} \sqcup \mathbb{R}/\sim= X$. This has a locally finite open refinement since $\mathbb{R}$ and therefore $\mathbb{R} \sqcup \mathbb{R}$ are paracompact. Again by quotient topology, the image of this refinement is locally finite.
(3) Pick $p(-\varepsilon_1, \varepsilon_1)$ for some $\varepsilon_1 \in \mathbb{R}$. This is an Euclidean open neighborhood in X due to quotient topology.

Ex. on p.9. **Solution 2.2.**(PS1.2) Solution by Christian Kremer. Let Ω be the first uncountable ordinal. This is a well-ordered set which is not countable with the property that for all $i \in \Omega$, the set $\{j \in \Omega | j \leq i\}$ is countable. Take a copy of $[0, 1)$ for each $i \in \Omega$ to define a set R. Elements are of the form (x, i) where $i \in \Omega$ and x lies in the copy of $[0, 1)$ corresponding to i. This set has a total order by $[x, i) \leq [y, j)$ if either $i < j$ or $i = j$ and $x \leq y$. Taking intervals to be open defines a topology on R. Also R has a smallest element 0. Since the set $\{j \leq i\}$ is countable, we see that $[0, (i, x)]$ is actually homeomorphic to a compact interval in $\mathbb{R}$. Define $L = R \coprod_0 R$. This is clearly a locally 1-Euclidean Hausdorff space. It is also (path-)connected, so if it were paracompact, it were second countable.

But L is not second-countable, since L has a collection of uncountably many disjoint sets open sets, namely the sets $U_i = \{x \in R \subset L | (0, i) < x < (1, i)\}$. If it were second-countable there had to exist countably many non-empty sets, each of which lying in some U_i that cover all the U_i's. This would imply that Ω would be countable.

Ex. on p.9. **Solution 2.3.**(PS1.3) Solution by Ekin Ergen.

Let M be a compact topological manifold with charts $\{U_i\}_{i \in I}$ for some finite set I. Without loss of generality, M is connected, otherwise we can embed each component of M in some $\mathbb{R}^{N_i}$. Since M is compact, there are finitely many components, which each have a compact image on $\mathbb{R}^{N_i}$, so we can take the largest of the N_i and embed the disjoint union via appropriate translations (and extensions wrt dimension of $\mathbb{R}$) of each of the embeddings. In particular, the dimension of M is well defined, say n. Choose embeddings $\iota_i : U_i \to \mathbb{R}^n$. Pick a partition of unity $\{f_i\}_{i \in I}$ subordinate to $\{U_i\}$, let A_i be the support of f_i. Define $h_i : U_i \to \mathbb{R}^n$ with

$$h_i(x) := \begin{cases} f_i(x)\iota_i(x), & x \in U_i \\ 0, & x \in X \setminus A_i \end{cases} \tag{22.1}$$

This is a well-defined continuous map because $\{f_i\}$ is a partition of unity. Finally, for $N = |I|(n + 1)$ define $F : X \to \mathbb{R}^N$ by $x \mapsto (f_1(x), f_2(x), ..., f_n(x), h_1(x), ..., h_n(x))$. This map is continuous as it is in continuous in each component. M is compact and $\mathbb{R}^N$ is Hausdorff, so by compact-Hausdorff argument, it is also open. Finally, it is injective: Let $F(x) = F(y)$. Then $f_i(x) = f_i(y)$ and $h_i(x) = h_i(y)$ for all i. Some $f_i(x)$ must be nonzero since for each x, these add up to 1, which implies that $\iota_i(x) = \iota(y)$ for some i. But ι_i is an embedding, so $x = y$.

Ex. on p.14. **Solution 4.1.**(PS6.1) Solution by Isacco Nonino. We want to show that every connected topological manifold M with empty boundary is homogeneous.

Step 1: We show that for any two points $a, b \in Int(D^n)$ there exist an homeomorphism of the disc, fixed on the boundary, sending a to b.

- First we produce a radial shrink t in order to make the radius of a the same as the radius of b (embed D^n in $\mathbb{R}^n$). WLOG suppose that the radius of a is at least the radius of b.
- Next, take the ball of radius b and rotate the boundary of the ball by a rotation r in order to send the shrinked a to b. We can extend this to the b-ball with the Alexander Trick. Then we extend on the other side of the boundary by making the rotation "die" in a continuous way:

$$r_{t+(1-t)b} \cdot x = e^{\phi(t)\pi i} \cdot x \tag{22.2}$$

  where $\phi(t)$ shrinks the angle continuously, $\phi(0) = \theta_0$ is the original rotation angle, and $\phi(1) = 0$.
- Compose $h = r \cdot t$ to get the desired homeomorphism, $h(a) = b$ and $h|_{\partial D^n} = id$.

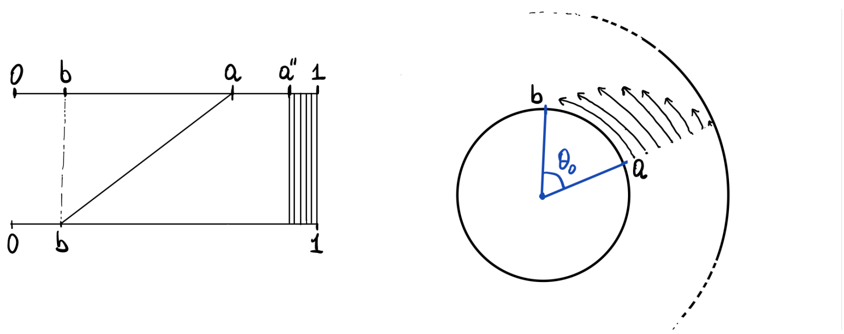


FIGURE 56. The two steps

Step 2 Now we show that the orbit of each point under the action of Homeo(M) is both open and closed. Since M is connected, this implies that the orbit is indeed M.

- First we show that the orbit is open. Take b in the orbit of x, so there is an $h \in$ Homeo(M) such that $h(x) = b$. Take a euclidean open ball B around b. We will show that this ball is contained in the orbit of x. Indeed, by composing with the chart homeomorphism ϕ, this ball becomes the interior of a disc in $\mathbb{R}^n$; we saw that given $\phi(y), y \in B$ there is an homeomorphism H of this disc, fixed on the boundary, sending $\phi(b)$ to $\phi(y)$. The composition $\phi^{-1}H\phi$ can be extended to the whole manifold M by using the pasting lemma and taking the identity outside B (this is possible because the homeomorphism on the disc fixes the boundary). The extended homeomorphism is an element of Homeo(M) sending b to y. We compose this with h to send x to y. Hence y lies in the orbit of x, and this works for each y in B. Thus, the orbit set is open.
- Now we show that the orbit is closed. Take a point c lying outside the orbit of x. Again, by considering a euclidean ball C around c and the chart homeomorphism ϕ, we can construct homeomorphisms sending c to each point of C. So, if a point in C lies in the orbit of x, then by composing with the inverse of the previous homeomorphism we would get that c itself lies in the orbit, which is a contradiction. Hence, the complement of the orbit set is open, so the orbit set is closed.

Ex. on p.19. **Solution 5.1.**(PS2.1) Solution by Ekin Ergen.

It suffices to show the latter of the two statements, as $[0,1] \subset \mathbb{R}^3$ is locally flat by definition. Using the hint, we choose all of the balls B_i to be centered at the compactification point p (granting $\cap B_i \supset \{x\}$), and their radii should be so that the ball B_i contains all but the i leftmost knots in its interior, and its boundary crosses γ in exactly one point.
To find the isotopies, we have to unknot the partial arcs. When we were working with the one-sided Fox-Artin arc, we moved thefree end to unknot all the knots one by one. This time, however, we have to keep both ends of partial arcs $\gamma \cap (B_i \setminus \operatorname{Int} B_{i+1})$ (that consist of one knot each by construction) constant throughout the isotopy in order to maintain identity on the complement. Therefore, the idea is to 'move the other end'. This is not allowed either, but we can realize this as sliding the knot through B_{i+1}. Then we will only have moved $\gamma \cap (B_i \setminus \operatorname{Int} B_{i+1})$. Each of these unknottings yield ambient isotopies $H_t^i : B_i \to B_i$ $t \in [0,1]$ that fix ∂B_i as well as B_{i+1} (e.g. by isotopy extension theorem, after slight thickening of $\gamma \cap (B_i \setminus B_{i+1})$). (I had the idea of writing this down as a conjugation where we move B_{i+1} by an isotopy of radial maps in the "middle map", but even this requires a lot of ugly math, especially when the embedding of the arc isn't specified explicitly. I hope this is enough explanation.)
Define

$$h(x) = \begin{cases} H_1^i(x), & x \in B_i \setminus \operatorname{Int} B_{i+1} \text{for some } i \\ x, & \text{elsewhere} \end{cases} \tag{22.3}$$

Clearly, h is continuous in $\mathbb{R}^3 \setminus \{p\}$. In fact, it is also continuous in p: For any $\varepsilon > 0$, we can pick $\delta = \varepsilon$ to fulfill the $\varepsilon-\delta$-criterion as points do not move away from slices under h. Therefore h is continuous. Passing from $\mathbb{R}^3$ to S^3 by compactification and using compact-Hausdorff argument, we can also see that it is open. Bijectivity can be seen restricting to $B_i \setminus \operatorname{Int} B_{i+1}$, as points do not move from one slice to another under any of the given homeomorphisms.

Ex. on p.19. **Solution 5.2.**(PS2.2)

Ex. on p.25. **Solution 6.1.**(PS2.3)

Ex. on p.27. **Solution 7.1.**(Not assigned as homework)

Ex. on p.27. **Solution 7.2.**(Not assigned as homework)

Ex. on p.35. **Solution 7.3.**(Not assigned as homework) Let $\Sigma \subseteq S^n$ be an embedded S^{n-1} and let $S^n - \Sigma = A \cup B$. If A is a smooth ball, then $\overline{A}$ is an embedded disc. By the smooth Palais' Theorem, we are able to isotope this disc to the lower hemisphere, in which case the B will be diffeomorphic to the (open) upper hemisphere. In particular, it will be a smooth ball.

Ex. on p.36. **Solution 7.4.**(Not assigned as homework)

Ex. on p.36. **Solution 7.5.**(PS3.1) Solution by Isacco Nonino. The double Fox-Artin arc is not cellular in the interior of D^3. Let α be the double Fox-Artin arc. Suppose that α is cellular. Then we have that $D^3/\alpha \cong D^3$. Now $D^3/\alpha \setminus \{pt\} \cong D^3 \setminus \alpha$ by Proposition 7.7 where $\{\text{pt}\}$ is the image of α

in D^3/α. By assumption, we have the following:

$$D^3 \setminus \{\text{pt}\} \cong D^3/\alpha \setminus \{\text{pt}\} \cong D^3 - \alpha$$

So we see that if the double Fox-Artin arc were cellular, then the complement of the double Fox-Artin arc in the disc would be homeomorphic to the complement of a point in D^3. Now this space is homeomorphic to $S^2 \times (0,1]$, which is homotopy equivalent to a sphere. Since homeomorphism preserves this property, $D^3 \setminus \alpha$ must be homotopy equivalent to a sphere. However, we saw that the complement of the double Fox Artin arc has nontrivial fundamental group, which leads to a contradiction. Therefore, the double Fox-Artin arc is not cellular in D^3.

Ex. on p.36. **Solution 7.6.**(PS3.2) Solution typed up by Arunima Ray. Check out Bing's book, Geometric Topology of 3-manifolds, Theorem V.2.C as well.

The compact set $M \setminus U_1$ is contained in U_2 and therefore is contained in (the image of) a round collared ball B_1 of large radius in U_2 (the round balls of increasing radius give a compact exhaustion of $\mathbb{R}^n$). Then the boundary $\Sigma = \partial B_1$ is a bicollared sphere in U_2. By the Schoenflies theorem, Σ bounds a ball B_2 in U_2 and we have $M = B_1 \cup B_2$ where the two balls are being glued together along the boundary. By the Alexander trick, the result of gluing two balls together along the boundary is homeomorphic to S^n.

For part (b), we know by hypothesis that each suspension point has a Euclidean neighbourhood. By the definition of a suspension, these neighbourhoods can be stretched out so that M is the union of the two neighbourhoods, which are homeomorphic to $\mathbb{R}^n$ by definition. Now apply part (a).

Ex. on p.36. **Solution 7.7.**(PS3.3) Since $\overline{U}$ is a manifold, the boundary is collared by Brown's theorem (Theorem 6.5). Then while Σ might not be bicollared, a push-off of Σ into the collar is. Let Σ' denote such a push-off. Then by the Schoenflies theorem, each component of $S^n \setminus \Sigma'$ is a ball. But then $\overline{U}$ is homeomorphic to a ball union a boundary collar, which is still a ball.

Ex. on p.36. **Solution 7.8.**(PS3.4) Solution by Isacco Nonino.

Let $f\colon D^n \to D^n$ be an embedding. We know that $f(D^n)$ is locally collared. By Brown's result, given $B \subseteq X$, with B and X compact, then locally collared implies globally collared (Theorem 6.5). So we have a global collar

$$h\colon f(S^{n-1}) \times [0,1] \to D^n$$

for $f(S^{n-1})$. Now we will prove that $f(D^n)$ is cellular in D^n. We define B_i to be $f(D^n) \cup h(f(S^{n-1}) \times [0, 1/i])$. These B_i are all homeomorphic to D^n since each is a ball with an added boundary collar. Also $\operatorname{Int} B_i \subseteq B_{i-1}$ and the intersection of all B_i is precisely $f(D^n)$ (the sequence of $1/i$ converges to zero, corresponding exactly to $f(S^{n-1})$). Hence $f(D^n)$ is cellular. We obtain:

$$D^n/f(D^n) \cong D^n$$
$$D^n \setminus f(D^n) \cong D^n/f(D^n) \setminus \{pt\} \cong D^n \setminus \{pt\} \cong S^{n-1} \times (0,1]$$

Ex. on p.50. **Solution 9.1.**(PS4.1) Solution by Isacco Nonino.

First key observation. Let $r\colon B \to \{b\}$ be the retraction to the point b. Then the assumption that B is contractible tells us that $r \cong id$.

Second key observation. Recall the following result. Given a paracompact space A, two maps $f, g\colon A \to B$ such that $f \cong g$, and a microbundle ξ over B, then $f^*\xi$ is isomorphic to $g^*\xi$. Now we can stare at the following diagram.

$$\begin{tikzcd} r^*E \arrow[r] \arrow[d] & E \arrow[d, "j"] \\ B \arrow[r, "r"] & B \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r] \arrow[d] & E \arrow[d, "j"] \\ B \arrow[r, "id"] & B \end{tikzcd}$$

Combining our observations, we see that $r * E \cong E$ for each microbundle E over B. So it suffices to show that r^*E is isomorphic to the trivial microbundle. The total space of r^*E is precisely $B \times j^{-1}(\{b\})$. Now by the local trivialization property, given $U \ni b$ an open neighbourhood of b, there is a $V \subseteq E$ such that $V \cong U \times \mathbb{R}^n$.

$$\begin{tikzcd} & V \arrow[dr, "j|_V"] \arrow[dd, "\cong"] & \\ U \arrow[ur, "i"] \arrow[dr, "\times 0"'] & & U \\ & U \times \mathbb{R}^n \arrow[ur, "pr1"'] & \end{tikzcd}$$

We consider the microbundle $(B \hookrightarrow B \times j|_V^{-1}(\{b\}) \to B)$, which is isomorphic to r^*E. Remember that we just care about what happens locally around the 'zero section' $x \to (x, i(b))$.

Consider now the microbundle $E|_{\{b\}}$, the restriction of E at the point b. There is an homeomorphism $h\colon j|_V^{-1}(\{b\}) \to \{b\} \times \mathbb{R}^n \cong \mathbb{R}^n$ coming from the local trivialization homeomorphism.

Now we conclude by the diagram

$$\begin{tikzcd} & B \times j|_V^{-1}\{b\} \arrow[dr, "pr"] \arrow[dd, "h"] & \\ B \arrow[ur, "i'"] \arrow[dr, "\times 0"] & & B \\ & B \times \mathbb{R}^n \arrow[ur, "pr"] & \end{tikzcd}$$

that r^*E, and hence E itself, is isomorphic to the trivial microbundle.

Ex. on p.50. **Solution 9.2.**(PS4.3) Solution by Ekin Ergen.

Recall that the compact-open topology of $C(X, Y)$ is generated by a subbasis $\{f | f(K) \subset U\}_{K,U}$, where K runs over compact subsets of X and U runs over open subsets of Y.

(1) The compact open topology is coarser than uniform topology. We want to see that all open subsets with respect to the compact open topology is open with respect to the uniform topology. To this end, it suffices to show this claim for the subbasis mentioned above, as all open subsets of compact open topology are generated by finite intersections of such sets. Let $B(K, U) := \{f \mid f(K) \subset U\}$ be a such open set for a fixed K and U as above. Let $f \in B(K, U)$. If we can show $B(f, \varepsilon) \subset B(K, U)$ for some ε, we are done because then we can take the union over all f as $B(K, U)$. Here, it suffices to pick $\varepsilon = d(f(K), U')$ where U' denotes the complement of U. Then any $h \in B(f, \varepsilon)$ satisfies $d(f(x), h(x)) < d(f(K), U') \leq d(f(x), U')$ for all $x \in K \Rightarrow h(x) \in U$. Note that $d(f(K), U')$ is well-defined because both are closed and $f(K)$ is compact.

(2) The uniform topology is coarser than the compact open topology. Conversely, we want to find $f \in T \subset B(f, \varepsilon)$ for given f, ε, such that T is open with respect to the compact open topology. For each $x \in X$, pick N_x such that $f(N_x)$ lies in the ε'-neighbourhood of $f(x)$ for some $\varepsilon' < \varepsilon/3$, which we call U_x to use later. In particular, $f(\overline{N_x})$ has

diameter less than $2\varepsilon/3$. Since X is compact, we can find a finite cover among N_x, say of the points $x_1, \dots, x_n$. Finally define $C_i := \overline{N_{x_i}}$ and $U_i := U_{x_i}$ that $f(C_i)$ lies in. Then $\bigcap_{i=1}^n B(C_i, U_i)$ includes f and lies in $B(f, \varepsilon)$. To see the latter, let $g \in \bigcap_{i=1}^n B(C_i, U_i)$. As $X = \bigcup C_i$, $x \in X$ means $x \in C_i$ for some i, and hence $g(x) \in U_i$ because $g \in B(C_i, U_i)$. Then $d(f(x), g(x)) \leq d(f(x), f(x_i)) + d(f(x_i), g(x)) \leq \varepsilon'/3 + 2\varepsilon'/3 < \varepsilon$.

Ex. on p.50. **Solution 9.3.**Solution by Isacco Nonino and Christian Kremer.

The result is generally attributed to unpublished work of Brown. It is sketched in [EK71, p. 85]. Alternative proofs are given in [Sie68, p. 535] and [Sie70, Corollary 5.4].

Let $h\colon X \times \mathbb{R} \to Y \times \mathbb{R}$ be a homeomorphism. The key point in this argument will be that $Y \times \mathbb{R}$ has two product structures, the intrinsic one and the one induced from $X \times \mathbb{R}$ via h.

Let X_t denote $X \times \{t\}$ for $t \in \mathbb{R}$ and let $X_{[t,u]}$ denote $X \times [t, u]$ for $[t, u] \subseteq \mathbb{R}$. Similarly, let Y_s denote $Y \times \{s\}$ for $s \in \mathbb{R}$ and let $Y_{[r,s]}$ denote $Y \times [r, s]$ for $[r, s] \subseteq \mathbb{R}$. By compactness of X and Y, there exist $a < c < e$ and $b < d$ such that

(1) Y_a, Y_c, Y_e, $h(X_b)$, and $h(X_d)$ are pairwise disjoint in $Y \times \mathbb{R}$,
(2) $h(X_b) \subseteq Y_{[a,c]}$,
(3) $Y_c \subseteq h(X_{[b,d]})$, and
(4) $h(X_d) \subseteq Y_{[c,e]}$,

as illustrated in the leftmost panel in Figure 57. This may be achieved by first fixing a, and then choosing as follows.

- Choose b so that (1) is satisfied for a and b.
- Choose $c > a$ so that (1) and (2) are satisfied for a, b, and c.
- Choose $d > b$ so that (1) and (3) are satisfied for a, b, c, and d.
- Choose $e > c$ so that (1) and (4) are satisfied.

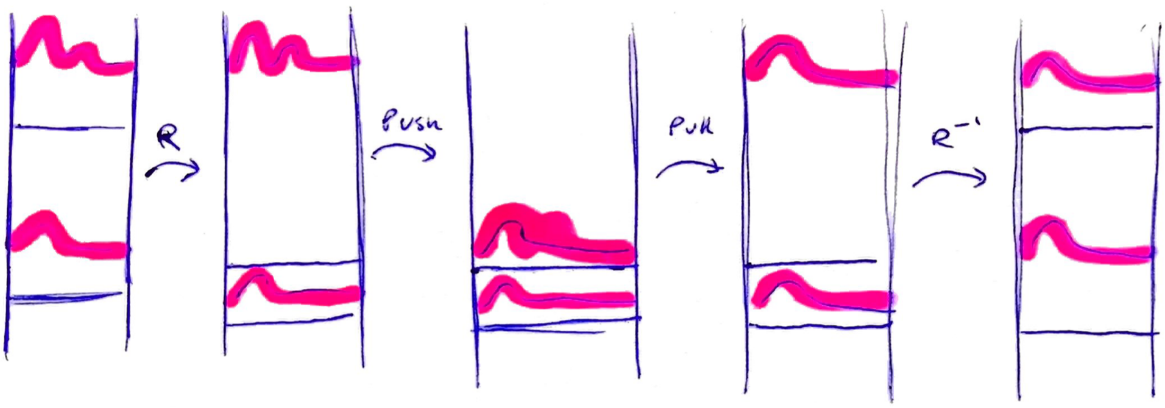


FIGURE 57. The push-pull construction. Each panel depicts the space $Y \times \mathbb{R}$. The blue and yellow regions denote $h(X_{[b,d]})$ and $Y_{[a,c]}$, respectively. Note that the regions overlap.

Now we construct a self-homeomorphism χ of $Y \times \mathbb{R}$ as the composition

$$\chi = C^{-1} \circ P_Y \circ P_X \circ C,$$

where the steps are illustrated in Figure 57. The maps P_X and P_Y will constitute the actual pushing and pulling while C, which we might call *cold storage*, makes sure that nothing is pushed or pulled unless it is supposed to be.

The maps are obtained as follows:

- The map C rescales the intrinsic $\mathbb{R}$-coordinate of $Y \times \mathbb{R}$ such that $C(Y_{[a,c]})$ lies below $h(X_b)$ and leaves $h(X_d)$ untouched. We require C to be the identity on $Y_{[c+\varepsilon,\infty)}$ and $Y_{(-\infty,a]}$, for ε small enough so that $Y_{c+\varepsilon} \subsetneq h(X_{[b,d]})$.
- The map P_X pushes $h(X_d)$ down to $h(X_b)$ along the $\mathbb{R}$-coordinate induced by h, that is, the image of the product structure of $X \times \mathbb{R}$, without moving $C(Y_{[a,c]})$.
- The map P_Y pulls $h(X_b) = (P_X \circ C \circ h)(X_d)$ up along the intrinsic $\mathbb{R}$-coordinate of $Y \times \mathbb{R}$ so that it lies above the support of C^{-1}, again without moving $C(Y_{[a,c]})$. This can be done in such a way that P_Y is supported below Y_e.

The map χ is the identity outside of $Y_{[a,e]}$. Observe that χ leaves $h(X_b)$ untouched and that $\chi(h(X_d))$ appears as a translate of $h(X_b)$ in the intrinsic $\mathbb{R}$-coordinate. In other words, for each $x \in X$ we have that $\chi h(x,d) = \tau_K(\chi h(x,b))$, where τ_K is the translation in $Y \times \mathbb{R}$ by some constant K.

Define $H := \chi \circ h \colon X \times \mathbb{R} \to Y \times \mathbb{R}$, and consider the diagram

$$\begin{tikzcd}
X \times [b,d] \arrow[r, "H|"] \arrow[d, "\pi", "b\sim d"'] & Y \times \mathbb{R} \arrow[d, "e", "t\sim t+K"'] \\
X \times S^1 \arrow[r, "g"] & Y \times S^1
\end{tikzcd}$$

where the lower horizontal map $g\colon X \times S^1 \to Y \times S^1$ is by definition the composition $e \circ H| \circ \pi^{-1}$. The map g is well defined since $H(x,d) = \tau_K H(x,b)$. Similarly g is injective since H is a homeomorphism and $e(y,t) = e(y,t')$ implies, without loss of generality, that either $(y,t) = (y,t')$ or $(y,t) = H(x,b)$ and $(y,t') = H(x.d)$ for some x. It remains only to check that g is surjective. It suffices to show that for each $(y,t) \in Y \times \mathbb{R}$ there exists $n \in \mathbb{Z}$ such that $\tau_K^n(y,t) \in H(X_{[b,d]})$. Fix some (y,t). Observe that the complement of $H(X_{[b,d]})$ in $Y \times \mathbb{R}$ has two components. Let N be the least integer such that $p := \tau_K^N(y,t)$ lies strictly above $H(X_{[b,d]})$. We now prove that $p' := \tau_K^{N-1}(y,t) \in H(X_{[b,d]})$. The line $\{y\} \times \mathbb{R} \ni p, p'$ intersects $H(X_{[b,d]})$ in a disjoint collection of intervals, that is,

$$(\{y\}\times\mathbb{R})\cap H(X_{[b,d]}) = \{y\}\times([t_1,t_2]\cup[t_3,t_4]\cdots\cup[t_L,t_1+K]\cup[t_2+K,t_3+K]\cup\cdots\cup[t_{L-1}+K,t_L+K])$$

for some odd L. In particular, $\{(y,t_i)\}$ are the intersections of $\{y\} \times \mathbb{R}$ with $H(X_b)$ and $\{(y,t_i+K)\}$ are those with $H(X_d)$. (Depending on the shape of $H(X_b)$ the intervals may not have been listed in ascending order, i.e. it might be that, e.g., $t_1+K < t_i$ for some i). Nonetheless, observe that, under the product metric, we have that $d(p,(y,t_1)) > K$ while $d(p,p') = K$. So p' lies above (y,t_1) on the line $\{y\} \times \mathbb{R}$.

If $p' \in \{y\} \times (t_{2i}, t_{2i+1})$ for some i, then

$$K = d(p,p') > d(p,(y,t_{2i+1})) > d((y,t_{2i+1}+K),(y,t_{2i+1})) = K,$$

which is a contradiction. If p' lies in the component of $(Y \times \mathbb{R}) \setminus H(X_{[b,d]})$ above $H(X_{[b,d]})$, it would contradict the minimality of N. Then either p' lies in one of the intervals of the form $\{y\} \times [t_{2i-1}, t_{2i}]$ or $\{y\} \times [t_{2i}+K, t_{2i+1}+K]$, which implies that p' lies in $H(X_{[b,d]})$ as desired.

Ex. on p.61. **Solution 10.1.**(PS7.2) Solution by Christian Kremer. The "only if" part is clear. Notice that for every $f \in \mathrm{Homeo}(M)$, the map given by postcomposition with f induces an isomorphism $f_*\colon \mathrm{Homeo}(M) \to \mathrm{Homeo}(M)$. It is continous being the restriction of the composition $\mathrm{Homeo}(M) \times \mathrm{Homeo}(M) \to \mathrm{Homeo}(M)$ to the subspace $\{f\} \times \mathrm{Homeo}(M)$ and clearly has the inverse $(f^{-1})_*$. If $U \subseteq \mathrm{Homeo}(M)$ is a contractible neighbourhood of the identity, then $f_*(U)$ is a contractible neighbourhood of f: It contains f, is open and contractible since f_* is a homeomorphism. As a side remark, it is not in general true that $\mathrm{Homeo}(M)$ is a topological group since the inversion may not be continous.

Ex. on p.61. **Solution 10.2.**(PS7.1) Solution by Christian Kremer. First, we check that the sets of the form $W(f,K,\varepsilon)$ are actually open. Let $g \in W(f,K,\varepsilon)$ be an element. Let $m = \max\{d(f(x),g(x))|x \in K\}$. Then $g \in W(g,K,\varepsilon - m) \subseteq W(f,K,\varepsilon)$, so it actually suffices to find an open neighbourhood of f in $W(f,K,\varepsilon)$, which will make notation a little easier. Cover $f(K)$ with finitely balls B_i of radius $2/3$ such that the compact sets $K_i = f^{-1}(1/2 \cdot \overline{B_i}) \cap K$ cover K. Then $f \in V(K_i,B_i)$. Suppose $g \in \bigcap_i V(K_i,B_i)$ and $x \in K$ is a point. Pick i with $x \in K_i$ and let x_i be the centre of the ball B_i. Then

$$d(g(x),f(x)) \leq d(x_0,f(x)) + d(g(x),x_0) < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon.$$

Hence $f \in \bigcap_i V(K_i,B_i) \subseteq W(f,K,\varepsilon)$.

Now we check that those sets consitute a basis of the topology. It suffices to show that for all $f \in U$ open, there is $f \in W(g,K,\varepsilon \subseteq U)$. First, we can find a finite intersection of sets of the form $V(K_i,W_i)$ which is contained in U containing f, since those sets form a subbasis. Let ε_i be the distance of $f(K_i)$ and the complement of U_i. Then $f \in W(f,K_i,\varepsilon_i) \subseteq V(K_i,W_i)$. Now notice that

$$W(f,\bigcup_i K_i, \min_i\{\varepsilon_i\}) \subseteq \bigcap_i V(K_i,W_i) \subseteq U.$$

This finishes the proof. Notice that if M is compact, $f \in W(f,M,\varepsilon) \subseteq W(f,K,\varepsilon)$ for each K. Since for any f in an open subset U we can find K and ε with $f \in W(f,K,\varepsilon) \subseteq U$, we see that actually sets of the form $W(f,M,\varepsilon)$ already form a basis of the topology. Of course, this is the topology induced by the ∞-norm.

Ex. on p.62. **Solution 10.3.**(PS6.2) Solution by Christian Kremer.

(i) The orientation-beviour of homeomorphisms defines a map $\mathrm{Homeo}(\mathbb{R}^2) \to \{+,-\}$. To see this, notice that $\mathrm{Homeo}(\mathbb{R}^2)$ is locally path-connected (for example, since it is locally contractible) and isotopic homeomorphims have the same orientation-behaviour. (A possible definition of the orientation behaviour either could be of homological flavor or by passing to the one-point compactification S^2. An isotopy of homeomorphisms of $\mathbb{R}^2$ induces an isotopy of homeomorphisms of S^2.)

(ii) We know that $\mathrm{Homeo}(\mathbb{R}^2)$ is homotopy equivalent to $\mathrm{Homeo}_0(\mathbb{R}^2)$. The map $f \mapsto (f(1,0))/|f((1,0))| \in S^1$ is continous and admits a section by $S^1 \subseteq O(1) \subseteq \mathrm{Homeo}_0(\mathbb{R}^2)$. Thus, S^1 is a retract of $\mathrm{Homeo}_0(\mathbb{R}^2)$, so $\mathrm{Homeo}_0(\mathbb{R}^2)$ can not be contractible. (For example, the inclusion $S^1 \to \mathrm{Homeo}_0(\mathbb{R}^2)$ has to induce an injection on fundamental groups and the fundamental group of S^1 is famously non-trivial.)

Ex. on p.62. **Solution 10.4.**(PS7.3) Solution by Christian Kremer. We indicate the construction of the map in the picture below. The first homology of M is freely generated by arcs γ_i around B_i. Now

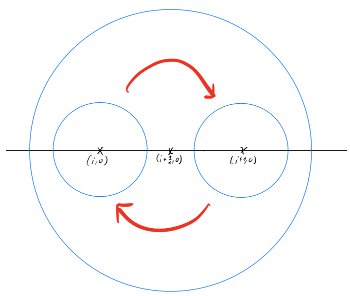


FIGURE 58. Schematic picture of the map h_i.

$H_1(h_i)(\gamma_i) = \gamma_{i+1}$ so that h_i does not induce the identity on homology. Hence it can not be homotopic to the identity. Using 10.2 we see that a neighbourhood basis of the identity is given by sets of the form $W(\mathrm{Id}, K, \varepsilon) \cap \mathrm{Homeo}(M)$. Since each K is contained in a ball around 0 of radius r and $W(\mathrm{Id}, \overline{B_r(0)}, \varepsilon) \subseteq W(\mathrm{Id}, K, \varepsilon)$, actually sets of the form $W(\mathrm{Id}, B, \varepsilon)$ where B is a closed ball around the origin. For every closed ball around the origin there is an i such that h_j is the identity on this ball for $j \geq i$, so the sequence (h_i) converges to the identity. Since every neighbourhood of the identity contains a map of the form h_i, all of which are not homotopic to the identity, no neighbourhood of the identity is path-connected, since a path in the space of homeomorphisms is a homotopy (even stronger, an isotopy).

Ex. on p.79. **Solution 12.1.**(PS8.1) Solution by Isacco Nonino.

First proof. We first prove the result using the stable homeomorphism theorem SH_n. Let $h\colon T^n \to T^n$ be an orientation preserving homeomorphism. We saw in class that such homeomorphism can be lifted to an homeomorphism $\widetilde{h}\colon \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\tilde{h}} & \mathbb{R}^n \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e} \\ T^n & \xrightarrow{h} & T^n \end{array}$$

commutes. By SH_n, since $\widetilde{h}$ is an orientation preserving homeomorphism, $\widetilde{h}$ is stable. Let $\{U_i\}$ be open subsets of $\mathbb{R}^n$ such that $\widetilde{h} = \widetilde{h}_1 \circ \cdots \circ \widetilde{h}_k$, where $\widetilde{h}_i$ agrees with identity on U_i. Define $V_i := e(U_i)$, which is open in T^n. Let $h_i := e \circ \widetilde{h}_i \circ e^{-1}$. Clearly, $h_i|_{V_i} = \mathrm{Id}$ and by definition $h = e \circ \tilde{h} \circ e^{-1} = h_1 \circ \cdots \circ h_k$. Therefore h is stable.

Second proof. We show the result without using SH_n.

- We first suppose that $h^*\colon \pi_1(T^n, x_0) \to \pi_1(T^n, x_0) = \mathrm{Id}$. (This is independent from the choice of the basepoint x_0; we can also assume that h preserves the basepoint x_0 since T^n is homogeneous). Now we lift the homeomorphism to the universal cover $\mathbb{R}^n$ as we did before.

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\tilde{h}} & \mathbb{R}^n \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e} \\ T^n & \xrightarrow{h} & T^n \end{array}$$

  Without loss of generality, suppose that $x_0 = e(0, \ldots, 0) = (1, \ldots, 1)$. Take the unit cube I^n and let $M := \max\{\|\widetilde{f}(x) - x\| \mid x \in I^n\}$. The maximum M exist since I^n is compact. The identity condition on the fundamental groups implies that each integer point on the lattice $\mathbb{Z}^n \subseteq \mathbb{R}^n$ is fixed by the lift $\widetilde{h}$. This means that each unit cube with integer vertices in $\mathbb{R}^n$ is mapped in exactly in the same way as I^n (since integer translations are deck transformations). This means $\max\{\|\widetilde{f}(x) - x\| \mid x \in \mathbb{R}^n\} = M$, i.e. $\widetilde{h}$ is at bounded distance from the identity. Thus $\widetilde{h}$ is stable and we conclude as in the previous proof.
- Suppose now that the induced map on fundamental groups is not the identity. Let A be the $n \times n$ matrix that encodes h^*. Important: A has determinant 1 since it is invertible and has $\mathbb{Z}$ entries.

  Claim: there exist a diffeomorphism $g\colon T^n \to T^n$ such that g^* has matrix expression A^{-1}.

  Proof of the claim: The matrix A^{-1} corresponds to a mapping of the integral lattice $\mathbb{Z}^n$ to itself (notice that in the previous point we were using that the mapping was the identity). A^{-1} is the product of elementary matrices with integer entries; each elementary matrix represent a diffeomorphism of $\mathbb{R}^n$. By passing to the quotient space over the integer lattice -the torus- we obtain a product of diffeomorphism of T^n, i.e. a diffeomorphism g such that when lifted acts on the integral lattice by A^{-1}. Now $g \circ h$ is the identity on the fundamental group and by our previous step this means $g \circ h$ is stable. Since $h = g^{-1} \circ (g \circ h)$, it suffices to show that the diffeomorphism g is stable itself (because product of stable is stable).

Claim: A diffeomorphism $f : T^n \to T^n$ is stable.

Proof of claim: We saw in class that every o.p. diffeomorphism of $\mathbb{R}^n$ is stable (we used the smooth isotopy extension theorem there). So we can consider a smooth structure for the torus T^n; given a diffeomorphism $\varphi : T^n \to T^n$, composing it with the atlas diffeomorphisms gives a diffeomorphism of $\mathbb{R}^n$. This is stable, and hence the original diffeomorphism is stable as well.

Now that we have this result, we deduce that h is stable as we wanted to show.

Ex. on p.79. **Solution 12.2.**(PS8.2) Solution by Isacco Nonino.

Step 1. Recall that Homeo($\mathbb{R}^n$) has two connected components. Moreover, the connected component containing the identity – call it I – consists of orientation preserving homeomorphisms. If we can prove that the space of stable homeomorphisms is both closed and open in Homeo($\mathbb{R}^n$), then it must be one of the two connected components of Homeo($\mathbb{R}^n$). But as we saw in class, stable homeomorphisms are isotopic to the identity, hence SHomeo($\mathbb{R}^n$) must be equal to I.

Step 2. We prove that SHomeo($\mathbb{R}^n$) is open.

- Claim: the identity has an open neighbourhood consisting of stable homeomorphisms. To prove the claim, let C be a compact subset of $\mathbb{R}^n$. By a previous exercise, $W(C, \varepsilon) = \{f \in \text{Homeo}(\mathbb{R}^n) \mid |h(x) - x| < \varepsilon, x \in C\}$ is an open neighbourhood of the identity for the compact-open topology. Let $h \in W(C, \varepsilon)$. Now we apply the torus trick. Namely, we construct a lift:

$$\begin{tikzcd}
T^n \arrow[r, "\bar{h}"] & T^n \\
T^n \setminus 2D^n \arrow[u, hook] \arrow[r, "\widehat{h}"] \arrow[d, "\alpha"'] & T^n \setminus D^n \arrow[u, hook] \arrow[d, "\alpha"] \\
\mathbb{R}^n \arrow[r, "h"] & \mathbb{R}^n
\end{tikzcd}$$

  where $\alpha(T^n \setminus D^n) \subseteq C$. In particular, the map $\widehat{(h)}$ is an homeomorphism $T^n \to T^n$. By Exercise 8.1, this homeomorphism is stable. But then going in the other direction we also get that $\widehat{h}$ is stable (it is just the restriction) and hence h is stable as well. Therefore $W(C, \varepsilon)$ consists of stable homeomorphisms.
- Now take another homeomorphism g in Homeo($\mathbb{R}^n$). Since the translation is a continuous map in this topological group, we can just translate the stable-open neighbourhood $W(C, \varepsilon)$ of id to a open neighbourhood of g consisting of stable homeomorphisms (just pre-compose with g).

Step 3. We prove that SHomeo($\mathbb{R}^n$) is also closed. We know that each coset of SHomeo($\mathbb{R}^n$) in Homeo($\mathbb{R}^n$) is open: this is a general fact about topological groups. Their union is again open, and also equals the complement of SHomeo($\mathbb{R}^n$). Hence SHomeo($\mathbb{R}^n$) is closed. This concludes the proof.

Ex. on p.79. **Solution 12.3.**(PS9.1) Solution by Isacco Nonino.

- Step 1: let φ, ψ the two locally collared embeddings. Let $p, q \in M$ be $\varphi(0), \psi(0)$. We have proved in a previous problem that M connected n-manifold without boundary is homogeneous. Since both embeddings land in $IntM$, we know that there is an homeomorphism h_1 of the interior that satisfies $h_1(p) = q$. Now extend this continuously to the boundary obtain an homeomorphism $h_1 \in \text{Homeo}(M)$. We can do this because we have that the images of the disks lie in interior charts and outside such charts we can just take the identity (see the proof of homogeneity!). Notice that this new embedded disc is still locally collared! Moreover, by Brown, locally collared implies globally collared.

– Step 2: Now we produce an homeomorphism h_2 as showed in Picture (59): Namely, we

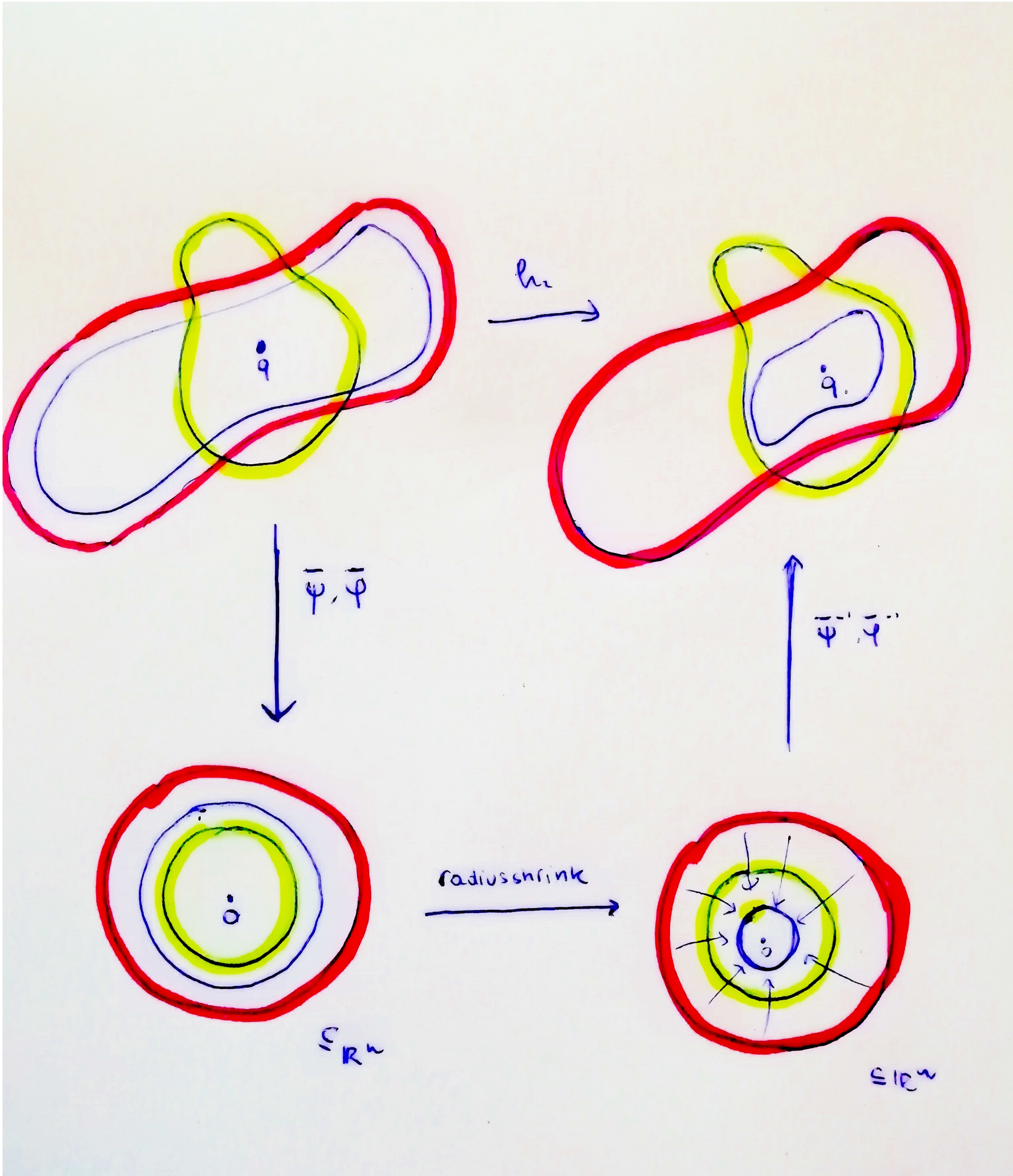


FIGURE 59. Shrinking embeddings

start by taking coordinate charts corresponding to $h_1\varphi(D^n)$ with its collar and $\psi(D^n)$ (let's call them $\bar{\varphi}$ and $\bar{\psi}$) sending the embedded disks to concentric disks in $\mathbb{R}^n$ (as in the picture). Now we produce an homeo. of $\mathbb{R}^n$ that shrinks the radius of the middle concentric disc while keeping everything fixed inside the small disc and outside the "collar" disc. Basically it's a push pull argument! Now just revert the chart maps. By concatenation, we obtained an homeomorphism of M that sends the embedded $h_1\varphi(D^n)$ inside $\psi(D^n)$, while keeping the boundary of the collared disc fixed.

– Step 3: Consider $\psi(D^n) - int(h_2 \cdot h_1(D^n))$. By the Annulus Conjecture, this is homeomorphic to $\mathbb{S}^{n-1} \times I$ via a. So now stretch $\mathbb{S}^{n-1} \times 0$ over $\mathbb{S}^{n-1} \times 1$ with s and precompose with a^-1. This composition -which we will call h_3- of homeomorphisms stretches the internal disc over the entire $\psi(D^n)$. Note that since ψ is globally collared as well, everything we do inside this disc can be extended to an homeomorphism of M. Now define $h \in \mathrm{Homeo}(M)$ by $h := h_3 \cdot h_2 \cdot h_1$. By construction, $h \cdot \varphi(D^n) = \psi(D^n)$.

– So now we have an homeomorphism h that arranges the two images to be the same. We want a final homeomorphism H of M such that ψ, $H \cdot h \cdot \phi$ are equal as maps. To do so, we work as in the picture below.

Ex. on p.90. **Solution 13.1.**(PS9.2)

Ex. on p.98. **Solution 14.1.**(PS10.1) Solution by Christian Kremer. Quick outline: Arrange $\phi(0) = \psi(0)$ (1). Using a collar, shrink ϕ unitil it has image inside the interior of $\psi(D^n)$ (2). Using the Annulus Theorem 12.1 we can blow up ϕ until $\phi(D^n) = \psi(D^n)$ (3). Using the Alexander Isotopy, and the fact that all orientation-preserving homeomorphisms of S^n are isotopic we finally arrange that ϕ and ψ are isotopic (4).

Detailed solution. First notice that Isotopy is an equivalence relation, in particular it is transitive. We will change ϕ up to isotopy unitl it coincides with ψ. Also note that the notion "being locally collared" is invariant under embeddings which are related by an isotopy from the identity to another homeomorphism $M \to M$.

(1) We want to arrange $\phi(0) = \psi(0)$. This follows from the following fact: If M is a connected manifold and p, q are points in its interior, there exists an isotopy from the identity to a self-homeomorphism of M which sends p to q. We do this by showing that the set U of points q for which exists such an isotopy is both open and closed in the interior of M which is connected.

To show that it is open, let q be a point in U. Pick a chart around q such that q is contained in the interior of the unit disc. If q' is any other point in the interior of the unit disc, we can find an obvious isotopy moving q to q' as indicated in the following picture.

To show that it is closed, let q be a point which does not lie in U. Then by the argument above, the same is true for points q' is a small disc neighbourhood of q, since if there exists an isotopy moving p to q' then there would also exists one moving p to q' since as we saw above, there exists one moving q' to q.

(2) The subspace $M \setminus \operatorname{Int} \phi(D^n)$ is a manifold with boundary since ϕ is assumed to be locally collared. Its boundary includes course ∂D^n. Attaching a collar to this, we see that we can extend ϕ to an embedding $\phi' \colon 2D^n \to M$. By a push-pull argument, as indicated in the picture below, we can isotope ϕ' relative boundary to arrange $\phi(D^n) \subseteq \psi(D^n)$. We can even arrange that $\phi(D^n)$ maps to the interior of $\psi(D^n)$.

(3) By the Annulus Theorem 12.1, $\psi(D^n) \setminus \phi(D^n)$ is an annulus, i.e. there is an embedding $\alpha \colon \partial(D^n) \times I \to M$ which maps homeomorphically into $\psi(D^n) \setminus \phi(D^n)$ such that $\alpha|_{\partial D^n \times \{0\}}$ is $\psi|_{\partial D^n}$ under the identification $\partial D^n = \partial D^n \times \{0\}$. We can extend this to

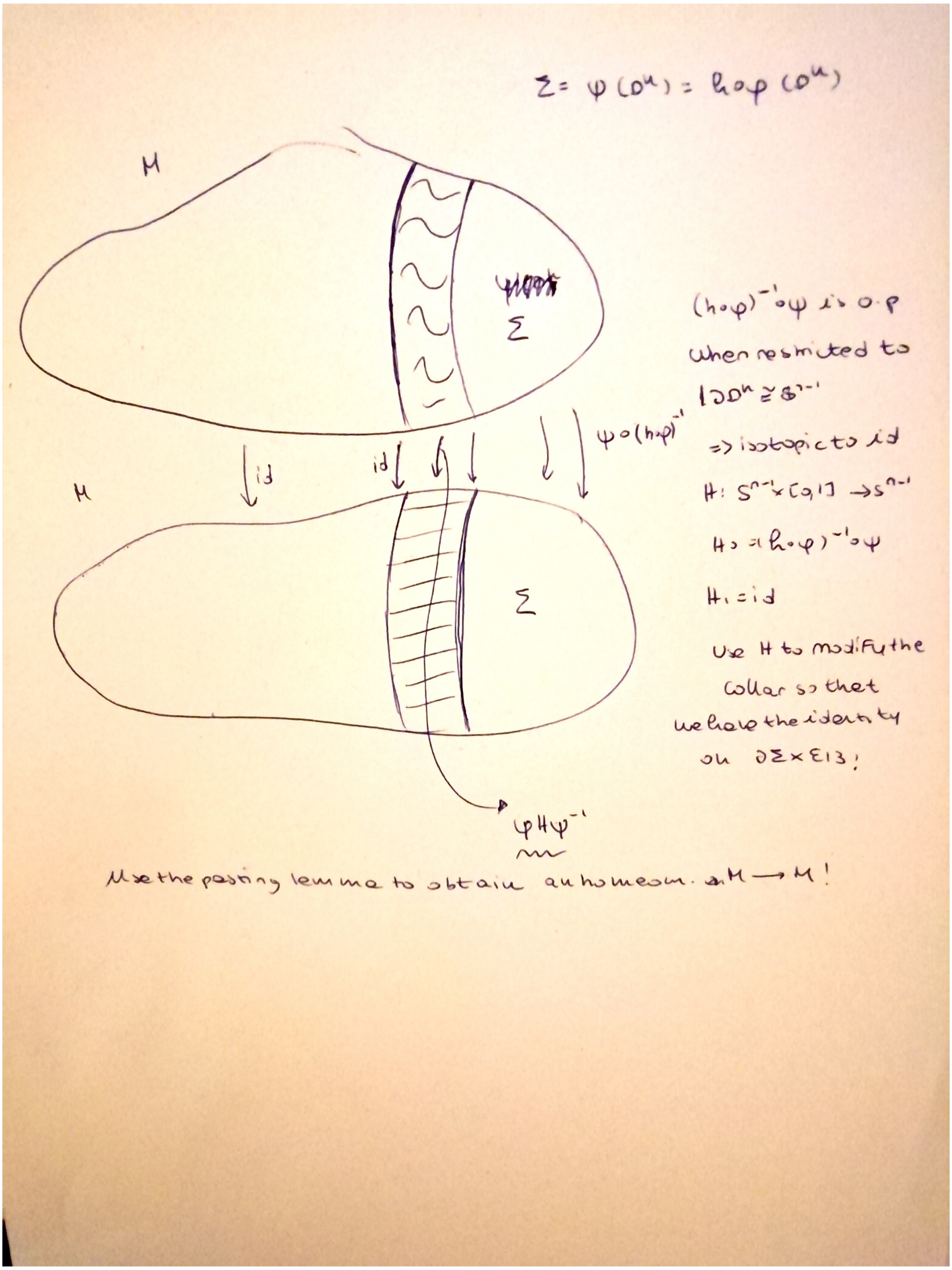


FIGURE 60. the final step

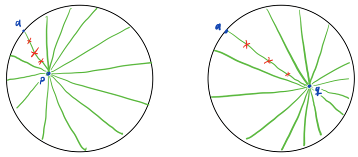


FIGURE 61. Moving points inside a disc

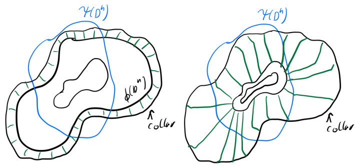


FIGURE 62. Another push-pull argument

an embedding $\beta\colon \partial D^n \times [-1,1] \to M$ using a collar. Denote by f the homeomorphism $\beta|_{\partial D^n\times\{1\}} \circ \psi^{-1}_{\partial D^n}$. Now we define an isotopy

$$H_t\colon M \to M, \quad x \mapsto \begin{cases} x: \text{ for } x \text{ not in the image of } \phi \text{ or } \beta; \\ \beta(v, s(1-\frac{t}{2})-\frac{t}{2}): \text{ for } x = \beta(v,s); \\ \beta(v, 2s(1-t)): \text{ for } x = \psi(w),\ s = |w| + \frac{t}{2} \geq 1 \text{ and } v = f(\frac{w}{|w|}); \\ \psi(v\cdot(1+\frac{t}{2})): \text{ for } x = \psi(v) \text{ and } |v| + \frac{t}{2} \leq 1. \end{cases}$$

Notice that H_0 is the identity and H_1 arranges the $\phi(D^n) = H_1 \circ \phi(D^n)$. Of course, we sketch what H_t is supposed to do in the following picture.

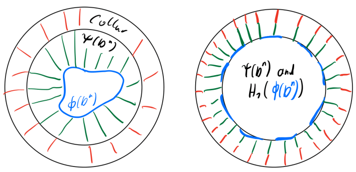


FIGURE 63. Lining up ψ and ϕ

(4) Now we are ready to do the last step. Note that $\psi^{-1}_{\partial D^n} \circ \phi_{\partial D^n}$ defines an orientation-preserving homeomorphism from the sphere to itself. We have already shown that such a homeomorphism is isotopic to the identity, say via an isotopy h_t. Using the Alexander trick, we can extend this to an isotopy of D^n to itself. Define $H_t = \psi \circ h_t$. By the isotopy extension theorem, this extends to an isotopy of the identity $M \to M$ to a self-homeomorphism carrying $\psi|_{\partial D^n}$ to $\phi|_{\partial D^n}$. At last, using the Alexander isotopy, we can isotope ψ relative ∂D^n to ϕ.

Ex. on p.98. **Solution 14.2.**(PS10.2)

Ex. on p.98. **Solution 14.3.**(PS11.1) Solution by Isacco Nonino.

If the two locally flat embeddings f, g are locally-flat isotopic (via h_t), then using the IET we can recover an ambient isotopy $H_t : \mathbb{S}^{n+2} \to \mathbb{S}^{n+2}$ such that $H_0 = id$ and $H_t \cdot h_0 = h_t$. We show that $H_1 : \mathbb{S}^{n+2} \to \mathbb{S}^{n+2}$ is the desired homeomorphism. First of all, notice that it is indeed orientation preserving: since it is the "ending point" of an isotopy connecting it to

the identity, it must lie in the orientation preserving connected component of $Homeo(\mathbb{S}^{n+2})$! Moreover, $H_1 \cdot f = H_1 \cdot h_0 = h_1 = g$, hence $H_1(f(\mathbb{S}^n)) = g(\mathbb{S}^n)$, i.e $H_1(K) = J$.Finally, we have that $(H_1 \cdot f)^{-1} \cdot g : \mathbb{S}^n \to \mathbb{S}^n$ is isotopic to the identity (via the restriction of H on K), hence it is an orientation preserving homeomorphism of $\mathbb{S}^n$. Since f,g are indeed orientation preserving, we must have $H_1|_K$ it is as well.
On the other hand, suppose we have an homeomorphism F with the said properties. We have that F is isotopic to the identity, so there exist $H_t : \mathbb{S}^{n+2} \to \mathbb{S}^{n+2}$ such that $H_0 = id$,$H_1 = F$. If we precompose the isotopy with f, we obtain $H_t \cdot f : \mathbb{S}^n \to \mathbb{S}^n$ which is an isotopy between f and $F \cdot f$.
Now, consider $(F \cdot f)^{-1} \cdot g : \mathbb{S}^n \to \mathbb{S}^n$. This is a well-defined orientation preserving homeomorphism (because we know that $F \cdot f(\mathbb{S}^n) = g(\mathbb{S}^n)$ and the restriction is orientation preserving , hence composition is again orientation preserving). Thus it is isotopic to the identity via an isotopy $h_t : \mathbb{S}^n \to \mathbb{S}^n$.
Postcompose h_t with $H_1 \cdot f$ to get an isotopy $H_1 \cdot f h_t$ between $F \cdot f$ and g. We can patch together the two isotopies to get an isotopy between f, g. For the local flatness, the only problem should arise when we attach the two isotopies, say at time $1/2$. But we can make sure that in small intervals $[1/2 - \varepsilon]$, $[1/2 + \varepsilon]$ the isotopy is constant! For times in $[1 - \varepsilon, 1 + \varepsilon]$ the isotopy is then constant, and hence locally flat.

Ex. on p.98. **Solution 14.4.**(PS11.2)

Ex. on p.104. **Solution 15.1.**(PS4.2) Solution by Isacco Nonino.

Following Milnor's idea, we start by defining the composition of two microbundles.

Step 1: composition of microbundles. Let $\xi\colon B \to E \to B$ and $\nu\colon E \to E' \to E$ be two microbundles such that the total space of ξ is equal to the base space of ν. We define a new microbundle over B with total space E' as $\xi \cdot \nu$:

$$B \xrightarrow{i' \cdot i} E' \xrightarrow{j \cdot j'} B,$$

where the inclusions and projections are the ones inherited from ξ and ν.

Step 2: the normal and tangent microbundle cases. Let $t_M\colon M \xrightarrow{\Delta} M \times M \xrightarrow{pr_1} M$ be the tangent microbundle and let $p_2^* n$ be the pullback of the normal microbundle via the projection onto second coordinate

$$\begin{tikzcd}
M \times M \arrow[dr, bend right, "i'"] & & \\
& p_2^*(U) \arrow[r] \arrow[d, "pr_1"'] & U \arrow[d, "r"] \\
& M \times M \arrow[r, "pr_2"] & M
\end{tikzcd}$$

where the total space is $p_2^*(U) = \{((m, \bar{m}), u) \mid \bar{m} = r(u)\}$. Now define the composition $t_M \cdot p_2^* n$:

$$M \xrightarrow{i' \cdot \Delta} p_2^*(U) \xrightarrow{pr_1 \cdot pr_1} M$$

(with some abuse of notation for the projections). Consider now $t_N|_M$:

$$M \xrightarrow{\Delta} M \times N \xrightarrow{pr_1} M$$

We want to show that they are isomorphic. $i' \cdot \Delta(M)\colon m \mapsto ((m, m), i(m)) \in p_2^* U$ for $i\colon M \hookrightarrow U$, while $\Delta(M)\colon m \mapsto (m, m) \in M \times N$. There is an open neighbourhood of $i' \cdot \Delta(M)$ (which we can think of as a 'cube diagonal',in some sense) which can be mapped homeomorphically to an open neighbourhood of $\Delta(M)$ in $M \times N$: take an open neighbourhood U_m of each fibre $r^{-1}(m)$

in U and then take the union on each m. This gives an open set $\bigcup_m (m,m) \times U_m$ that can be mapped to $\bigcup_m m \times U_m$, an open neighbourhood in $M \times N$. Hence the two microbundles are isomorphic.

We do a similar procedure with $pr_1^* n$. In this case the isomorphism is much clearer: the total space of the Whitney Sum is given exactly by $E(t_M \oplus n) = \{((m,m'),u) \mid m' = r(u)\}$, while the total space of $p_1^* n = \{((m,m'),u) \mid m = r(u)\}$. Hence by the following diagram:

$$\begin{tikzcd}
 & E(t_M \oplus n) \arrow[dd, "id"] \arrow[dr, "pr_1 \cdot pr_1"] & \\
M \arrow[ur, "m \to ((m,m),i(m))"] \arrow[dr, "m \to ((m,m),i(m))"'] & & M \\
 & E(pr_1^* n) \arrow[ur, "pr_1 \cdot pr_1"'] &
\end{tikzcd}$$

we see that the two microbundles $t_M \oplus n$ and $t_M \cdot pr_1^* n$ are indeed isomorphic.

Now we take D to be a neighbourhood of the diagonal in $M \times M$ such that the two projection maps are homotopic. To do so, recall that M is an ENR. Let V be the euclidean neighborhood that retracts on M. Now take D to be the set of all (m,m') such that the segment joining m, m' lies within V. Now we can construct a homotopy between the projections as $H\colon M \times M \times I \to M$ by $H((m,m'),t)) := (1-t)m + tm'$, which is continuous and $H_0 = pr_1, H_1 = pr_2$.

By the property of the induced microbundle, we see that $p_1^* n|D \cong p_2^* n|D$. Moreover:

- the microbundle $\bar{t}_M$, obtained by taking D as the total space instead of $M \times M$ and restricting the projection to D, is isomorphic to t_M, since restricting the neighbourhood of the zero section does not change the isomorphism type of the microbundle.
- the composed microbundle $\bar{t}_M \cdot p_1^* n|D\colon M \to E(p_1^* n|D) \to M$ is isomorphic to the composed microbundle $t_M \cdot p_1^* n\colon M \to E(p_1^* n) \to M$. Again, we are just taking a restricted neighbourhood of the zero section, the defining maps are just the restriction of the others. The same holds for the projection on second coordinate.

Step 3: conclusion Now we have all the ingredients in our hands to obtain the result.

$$t_M \cdot p_1^* n \cong \bar{t}_M \cdot p_1^* n|D \cong \bar{t}_M \cdot p_2^* n|D \cong t_M \cdot p_2^* n \tag{22.4}$$

By (1) plus the results obtained in the previous two steps, we eventually obtain:

$$t_M \oplus n \cong t_N|M \tag{22.5}$$

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