Diagonal Chain Approximation for the torus.

Mark Powell

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1 Abstract

We write down the chain complex C_* of the universal cover of the torus, and indicate how to do likewise for any 2-dimensional CW complex arising as the 2-skeleton of a $K(G, 1)$. We exhibit a contracting chain homotopy for this chain complex, and again explain how to do so for the chain complex of the universal cover of our more general CW complex. This yields a contracting homotopy for the tensor product $C_* \otimes C_*$, and formulae of Davis ([2]) allow us to construct a diagonal chain approximation $\Delta_0 \colon C_* \to C_* \otimes C_*$, close to the diagonal map $x \mapsto (x, x)$. Formulae of Trotter ([3]) exist for this, but include the possibility of a 3-cell. In the case of a 2-dimensional group such as $\mathbb{Z} \oplus \mathbb{Z}$, the fundamental group of the torus, Trotter's formulae are unnecessarily long. He remarks that the solution of the word problem is pre-requisite to using the method of contracting chain homotopies; in this case, however, a canonical form for a word is readily available, and so we follow the method through.

2 Cellular Chain Complex of $S^1 \times S^1$

The torus has a cell decomposition with one 0-cell e^0, two 1-cells e^1_a and e^1_b corresponding to a meridian and longitude respectively, and one 2-cell e^2. The 0-skeleton is a single point, the 1-skeleton is $S^1 \vee S^1$. $\pi_1(S^1 \vee S^1, e^0) \cong \mathbb{Z} * \mathbb{Z} \cong F_2 \cong F(a, b)$, the non-abelian free group on two letters. A word in F_2 specifies an attaching map for a 2-cell, and the two cell is attached via the commutator word $[a, b] = aba^{-1}b^{-1}$. The fundamental group is therefore:

$$\pi_1(S^1 \times S^1) \cong \frac{F(a,b)}{\langle aba^{-1}b^{-1} \rangle} \cong \mathbb{Z} \oplus \mathbb{Z}.$$

The chain complex $C_*(S^1 \times S^1)$ is given by:

$$\mathbb{Z} \xrightarrow{(0,0)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(0,0)^T} \mathbb{Z}.$$

Of course this does not tell the whole story. We have to look at the chain complex of the universal cover if we want to extract enough algebraic information to be able to distinguish the torus from

$S^1 \vee S^1 \vee S^2$, for example. The universal cover of S^1 is $\mathbb{R}$, and so the universal cover of the torus is $\mathbb{R} \times \mathbb{R}$.

The unit square $[0,1] \times [0,1]$ is a fundamental domain for the torus, and the deck transformation group is $\mathbb{Z} \oplus \mathbb{Z}$. Integer lattice points are lifts of the 0-cell, while the grid with one coordinate integral is the lift of the 1-skeleton, so the interiors of each square in this grid completes the picture. We choose certain lifts:

$$\widetilde{e^0} := \{(0,0)\} \subset \mathbb{R} \times \mathbb{R}$$
$$\widetilde{e^1_a} := [0,1] \times \{0\} \subset \mathbb{R} \times \mathbb{R}$$
$$\widetilde{e^1_b} := \{0\} \times [0,1] \subset \mathbb{R} \times \mathbb{R}$$
$$\widetilde{e^2} := [0,1] \times [0,1] \subset \mathbb{R} \times \mathbb{R}$$

From now on, however, we drop the $\widetilde{\ }$ notation, taking it as understood. Any other lift of the cells in the universal cover of the torus will be written as translates of these cells by elements of $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$, which we represent by words in the generators a, b.

Geometrically, we can see, by considering the boundary of the cells of the fundamental domain, that the chain complex of the universal cover of the torus is:

$$\mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}] \xrightarrow{(1-b,a-1)} \mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}] \oplus \mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}] \xrightarrow{(a-1,b-1)^T} \mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}]$$

We can generalise this. Suppose we have a wedge of circles $\bigvee_c S^1$, the 1-skeleton of a CW complex. This has fundamental group $\pi_1(\bigvee_c S^1) \cong F(g_1, .., g_c) \cong F_c$, and chain complex of universal cover:

$$\bigoplus_c \mathbb{Z}[F_c] \xrightarrow{(g_1-1,..,g_c-1)^T} \mathbb{Z}[F_c]$$

. This universal cover is a *tree* - in the case $c = 2$ it is the Cayley graph. At each vertex there are $2c$ edges, corresponding to $g_1, ..g_c, g_1^{-1}, .., g_c^{-1}$, and it extends in this way infinitely in every direction without any back edges.

If we attach a 2-cell via a word w in this free group, then the fundamental group of the space $Y = S^1 \vee S^1 \cup_w D^2$ is $\pi := \pi_1(Y) = \frac{F_c}{\langle w \rangle}$. Instead of considering the universal cover of $\bigvee_c S^1$, we now must consider the π-cover, which is the pull back of the universal cover $\widetilde{Y}$ with respect to the inclusion map $\bigvee_c S^1 \hookrightarrow Y$. We denote the π-cover of $S^1 \vee S^1$ by $\widetilde{S^1 \vee S^1}$.

The 2-cell has boundary map given by the free derivative:

$$\partial_2(e^2) = \left(\frac{\partial w}{\partial g_1}, .., \frac{\partial w}{\partial g_c} \right)$$

. The free derivative of Fox tells us how to convert words into chains. It does this by examining the path in $\widetilde{S^1 \vee S^1}$ which the word represents *before* the occurrence of a letter g_i. This tell us which lift of e^1_i (the 1-cell associated to letter g_i.) occurs as the boundary of e^2. In this way we know in general the chain complex of the universal cover of a 2-dimensional cell complex.

It is necessary to check that the composite $\partial_1 \circ \partial_2 = 0$. In this instance, we require the formula called by Fox "fundamental" to his calculus. For a word $w \in F_c$:

$$w - 1 = \sum_{i=1}^{c} \frac{\partial w}{\partial g_i}(g_i - 1). \tag{1}$$

This can be proved inductively: take any word w, and try to factorise $w-1$ as a sum of elements of $\mathbb{Z}[F_c]$ multiplied by $(g_i - 1)$ on the right, as in the fundamental formula. Quickly one arrives at the free calculus definition and the derivation rule

$$d(uv) = du + udv.$$

When passing to the fundamental group of Y, we set $w = 1$, and so the LHS $= 0$. The RHS, on the other hand, is precisely the composition $\partial_1 \circ \partial_2$ which we wanted to find.

In the case of the torus, $w = aba^{-1}b^{-1}$, and so the derivatives are:

$$\left(\frac{\partial w}{\partial a}, \frac{\partial w}{\partial b}\right) = (1 - ab^{-1}, a - aba^{-1}b^{-1}) \xrightarrow{\alpha} (1 - b, a - 1)$$

where α is the abelianisation i.e. taking the quotient $\frac{F_2}{\langle aba^{-1}b^{-1}\rangle}$. A quick check verifies that $(1 - b)(a-1) + (a-1)(b-1) = 0 \in \mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}]$.

3 Contracting Chain Homotopies

Again we will start with the case of $\bigvee_c S^1$. We shall then include the 2-cell. The beauty of this program is that the algebra can be built up dimension by dimension just as a CW complex is defined inductively.

Definition 3.1. Let $(C_*, \partial)_{(n\geq 0)}$ be a chain complex with augmentation $\varepsilon\colon C_0 \to \mathbb{Z}$. Then a contracting chain homotopy is a set of maps $\delta(= s_0), s_1, s_2, ...$ such that $\partial_1 s_1 + \delta\varepsilon = \mathrm{Id}_{C_0}$, and $\partial_{i+1}s_{i+1} + s_i\partial_i = \mathrm{Id}_{C_i}$.

$$\cdots\ C_2 \underset{s_2}{\overset{\partial_2}{\rightleftarrows}} C_1 \underset{s_1}{\overset{\partial_1}{\rightleftarrows}} C_0 \underset{\delta}{\overset{\varepsilon}{\rightleftarrows}} \mathbb{Z}$$

This is the same as a chain homotopy of chain maps $1 \simeq 0$, and so algebraically it says that the chain complex is as that of a point. The chain contracting maps tell one, thinking geometrically for a second, how to construct a null homotopy - the image of an i-chain $\mathcal{U}$ ought to be the cells of dimension $i+1$ across which $\mathcal{U}$ must be contracted in order to make a null-homotopy. We return to this idea in the sequel.

Suppose that C_* is the chain complex of the universal cover of an Eilenberg-MacLane space $\widetilde{K(G,1)}$. In $K(G,1)$ there was only one non-trivial homotopy group; in the universal cover they are all trivial. By the Hurewicz theorem, all the homology groups of $\widetilde{K(G,1)}$ are also trivial. The chain complex C_* is thus contractible. The CW complex associated to this, by Whitehead's theorem, is therefore homotopy equivalent to a point i.e. contractible. Furthermore, since the chain groups are free they are certainly projective, and so the chain complex ought to be chain contractible.

With augmentation, we obtain a free $\mathbb{Z}[G]$-module resolution of $\mathbb{Z}$, often referred to as the "standard" $\mathbb{Z}[G]$-module free resolution of $\mathbb{Z}$. In the case of the torus, we take $\pi \cong \pi_1(S^1 \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$, and the chain complex given in the previous section, augmented, is the standard free $\mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}]$-module resolution of $\mathbb{Z}$.

There is an important distinction. For any space with contractible universal cover, the cellular chain complex of the universal cover is chain contractible. Whence there exist chain contracting $\mathbb{Z}$-*module* homomorphisms. However, the universal cover is not π-equivariantly[1] contractible. This is because π acts trivially on point - it has no other way to act - and π acts freely on the universal cover. This is rather elegantly algebraically reconcilable in the fact that there does not exist a $\mathbb{Z}[\pi]$-module chain contraction.

Consider the chain complex of the universal cover of S^1, that is $\mathbb{R}$. S^1 has an obvious cell decomposition with one 0-cell e^0, and one 1-cell e^1. $\pi_1(S^1) \cong \mathbb{Z} \cong \langle t \rangle$. The chain complex is therefore given by:

$$\mathbb{Z}[\mathbb{Z}] \xrightarrow{t-1} \mathbb{Z}[\mathbb{Z}].$$

We seek a chain contraction. First, try to do so via a $\mathbb{Z}[\mathbb{Z}]$-module homomorphism. Now, for $\mathbb{Z}[\mathbb{Z}]$-module homomorphisms, we only need define them on the basis element $1 \in \mathbb{Z}[\mathbb{Z}]$. So, we seek a homomorphism such that:

$$\partial_1 s_1 = 1$$

In other words we need a polynomial in $\mathbb{Z}[t, t^{-1}]$ which is equal to $\frac{1}{t-1}$. Such a polynomial *would* be $\sum_{k=0}^{\infty} -t^k$, except that infinite polynomials are *not* elements of the group ring $\mathbb{Z}[\mathbb{Z}]$. If we include the augmentation, then matters are no better.

$$\mathbb{Z}[\mathbb{Z}] \xrightarrow{t-1} \mathbb{Z}[\mathbb{Z}] \xrightarrow{\varepsilon} \mathbb{Z}.$$

Since we only define our homomorphisms on the basis element 1, we have $\delta\varepsilon(1) = 1$, and so we seek s_1 such that $\partial_1 s_1(1) = 0$, and are forced to have $s_1 = 0$. This does not work, however, since $(\partial_1 s_1 + \delta\varepsilon)(t^i) = 1$ for all i, and so is only the identity homomorphism for $i = 0$. We can remedy the situation by looking at each $\mathbb{Z}$-basis element of $\mathbb{Z}[\mathbb{Z}]$ individually, and making $\mathbb{Z}$-module homomorphisms. Define:

$$\delta(1) = 1; \quad s_1(t^i) = 1 + t + t^2 + .. + t^{i-1}.$$

Then

$$(\partial_1 s_1 + \delta\varepsilon)(t^i) = \partial_1(1 + t + .. + t^{i-1}) + \delta(1) = (t-1)(1 + t + .. + t^{i-1}) + 1 = t^i - 1 + 1 = t^i.$$

[1] As ever π denotes the deck transformation group, $\pi = G$ in the case of universal cover of a $K(G,1)$; it is the fundamental group.

So we have a contracting chain homotopy. (We have assumed that $i > 0$ here for simplicity. The reader can check that similar formulae can be used in the other cases.)

We now generalise the case of a circle to the case of a wedge sum of circles. Recall that the chain complex of the universal cover is:

$$\bigoplus_c \mathbb{Z}[F_c] \xrightarrow{(g_1-1,..,g_c-1)^T} \mathbb{Z}[F_c]$$

and define as will always be the case $\delta\colon 1 \mapsto 1$.

Let w be a word in F_c, a basis element for the $\mathbb{Z}$-*module* $\mathbb{Z}[F_c]$. Then in order to define s_1, we need elements $u_1, ..u_c \in \mathbb{Z}[F_c]$ such that:

$$\left(\sum_{i=1}^{c} u_i(g_i - 1)\right) + 1 = w.$$

By the fundamental formula of Fox, (1), we can take $u_i = \frac{\partial w}{\partial g_i}$. Note in particular that $\frac{\partial(t^i)}{\partial t} = 1 + t + t^2 + ... + t^{i-1}$, so the special case $c = 1$ as described above is included here.

Geometrically, this means that in order to "contract" a 0-cell we^0 of our Cayley graph we must move it back to the origin along the path between the two points. There is a unique such path, specified as a word w in F_c. But our chain contraction s_1 has image in $C_1\left(\widetilde{\bigvee_c S^1}\right)$, and so we need to convert the path w into a chain in this group, which is precisely what the free calculus achieves. Whichever way we look at it, everything fits!

Now comes the interesting part. Specialise to the case $c = 2$. When we include the 2-cell, the group changes by factoring out the normal subgroup generated by the attaching word w, and we consider again the π-cover of $S^1 \vee S^1$. Now for a given element $ve^0 \in C_0$, there is a choice of how we represent it as a word in F_c. Introducing relations creates back edges on the graph that is the π-cover of $S^1 \vee S^1$. An element of π corresponds to a vertex of this graph, i.e. a lift of e^0, whereas a word v in F_c specifies more; it tells us a *path* from 1 to v. A different choice of word yields a different free derivative corresponding to a different choice of contracting homotopy for that vertex. Therefore, in order to define s_1, due to this non-uniqueness, before we take the free derivative we must first make a choice of word to represent a given element. We must decide on a *canonical form* for a word representing a coset i.e. an element of π. This is equivalent to solving the word problem for the presentation of π given by $\langle\, g_1, g_2 \,|\, w \,\rangle$

In the case of the torus, $\pi \cong \mathbb{Z} \oplus \mathbb{Z}$, and we choose the canonical form:

$$v = a^n b^m.$$

This means that, for $n, m \neq 0$:

$$s_1(v) = \left((a^{n\xi_n})(1 + ... + a^{|n|-1}), a^n(b^{m\xi_m})(1 + ... + b^{|m|-1})\right).$$

$$\xi_k = \frac{1}{2}\left(\frac{k}{|k|} - 1\right)$$

When either n, m is equal to zero then the corresponding free derivative will vanish.

We now turn to defining s_2. This has to satisfy, for $v \in \pi$, and $x = a, b$:

$$(\partial_2 s_2 + s_1 \partial_1)(ve_x^1) = ve_x^1$$

$$\Leftrightarrow \partial_2 s_2(ve_x^1) = ve_x^1 - s_1(vxe^0) + s_1(ve^0)$$

This is best interpreted pictorially (see Figure 1). The RHS corresponds to a 1-chain in C_1. It can be thought of as a path in the π-cover of $S^1 \vee S^1$, which is the square grid in $\mathbb{R}^2$ of side length 1. The path starts at the origin (it doesn't have to - in the abelian setting it is really just the formal sum of 1-cells, but heuristically we can think of it as a path) and travels to the vertex $ve^0 = a^n b^m e^0$, represented by the grid point (n, m). It does so by following $s_1(ve_x^1)$ - due to the canonical form chosen it goes horizontally first to the point $(n, 0)$, and then travels vertically. Once it has arrived at the start-point ve^0 of ve_x^1, at then goes along ve_x^1 to its end-point vxe_x^1 which is $(n+1, m)$ in the case $x = a$ and $(n, m+1)$ if $x = b$. It then returns to 1, along the path $s_1(vxe_x^1)$, which goes in reverse, i.e. vertically first and then horizontally.

Now, large parts of this path may well cancel; in the chain group setting if a path and its reverse appear then they can be cancelled. In particular if $x = b$ then the whole path is null. We can therefore define $s_2(ve_b^1) = 0 \;\; \forall\, v \in \pi_1(S^1 \times S^1)$.

If $x = a$, however, then there are 2-cells crossed over when moving the 1-cell $a^n b^m e_a^1$ either up or down, whichever is necessary, until it coincides with the 1-cell $a^n e_a^1$. These 2-cells tell us how to contract the 1-cell, and hence should be (with an appropriate sign) the image $s_2(a^n b^m e_a^1)$. Therefore define:

$$s_2(a^n b^m e_a^1) = \begin{cases} a^n(1 + b + \ldots + b^{m-1}) & (m > 0) \\ 0 & (m = 0) \\ -a^n(b^{-1} + b^{-2} + \ldots + b^m) & (m < 0) \end{cases}$$

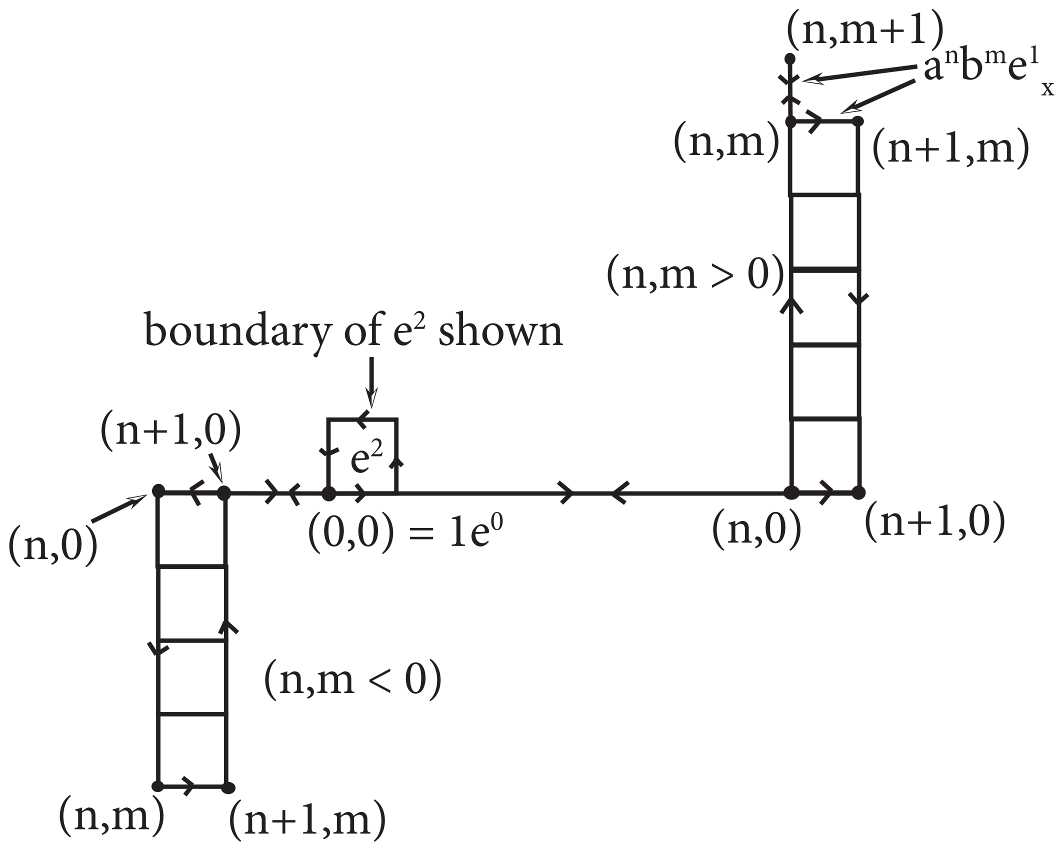

Figure 1: Defining the contracting chain homotopy maps s_1 and s_2.

4 The diagonal chain approximation

We can now use this information to work out the diagonal chain approximations of universal cover of the torus. We begin with S^1, and from there $S^1 \vee S^1$ follows immediately. The real task is then to find $\Delta_0(e^2)$, with e^2 the 2-cell of the torus, as then tensoring with $\mathbb{Z}$ over $\mathbb{Z}[\pi]$ and applying the slant map yields the duality chain equivalence, since e^2 represents the fundamental class (see the author's 1st year report for details).

Let C_* as before be $C_*(\widetilde{S^1 \times S^1})$, the chain complex of the universal cover of the torus, and let $\pi = \pi_1(S^1 \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$. $\Delta_0 \colon C_* \to C_* \otimes_{\mathbb{Z}} C_*$ is a chain map, and so fits into the commutative diagram below. Both the domain and codomain of Δ_0 are $\mathbb{Z}[\pi]$ modules; the action on $C_* \otimes C_*$ is by the diagonal action $g \cdot (x \otimes y) = gx \otimes gy$). We define Δ_0 inductively, beginning with the obvious choice $e^0 \mapsto e^0 \otimes e^0$ on C_0.

$$
\begin{array}{ccc}
\mathbb{Z} & & \mathbb{Z} \\
\uparrow \varepsilon & & \uparrow \varepsilon\otimes\varepsilon \\
C_0 & \xrightarrow{\Delta_0\ :\ e^0 \mapsto e^0\otimes e^0} & C_0 \otimes C_0 \\
\uparrow \partial_1 & & \uparrow (1\otimes\partial_1)\oplus(\partial_1\otimes 1) \\
C_1 & \xrightarrow{\Delta_0} & (C_0 \otimes C_1) \oplus (C_1 \otimes C_0) \\
\uparrow \partial_2 & & \uparrow (1\otimes\partial_2)\oplus((\partial_1\otimes 1)\oplus(1\otimes\partial_1))\oplus(\partial_2\otimes 1) \\
C_2 & \xrightarrow{\Delta_0} & (C_0 \otimes C_2) \oplus (C_1 \otimes C_1) \oplus (C_2 \otimes C_0)
\end{array}
$$

Note that in applying the maps $1 \otimes \partial_i$, if the homomorphism ∂_i, which is of degree -1, since it lowers the grading by 1, commutes past a chain of degree j, then a sign $(-1)^{-j} = (-1)^j$ must be introduced. In our case, this is relevant when C_1 is in the first position of the tensor product.

Proposition 4.1. *(Bredon chapter: Duality [1]) If (s, δ) is a chain contraction for C_*, then a contraction for $C_* \otimes C_*$ is given by $(s \otimes 1 + \delta\varepsilon \otimes s)$. Here $\varepsilon\colon C_0 \to \mathbb{Z}$, and so ε is defined to be zero on C_i for $i > 0$.*

Proposition 4.2. *Davis ([2]) gives the following formula for defining Δ_0 inductively. This formula guarantees that Δ_0 satisfies his condition* **(ii)***, which is that $(\varepsilon \otimes 1) \circ \Delta_0 = (1 \otimes \varepsilon) \circ \Delta_0 = 1$. This ensures that with this diagonal chain map the basic property $x \cup 1 = 1 \cup x = x$ is satisfied by the cup product, for all chains $x \in C_*$, with $1 \in C_0$. Assume that Δ_0 is defined on $C_0, .., C_{i-1}$. Then for an i-cell e^i:*

$$\Delta_0(e^i) := e^i \otimes e^0 + e^0 \otimes e^i + (s \otimes 1 + \delta\varepsilon \otimes s)(\Delta_0(\partial_i e^i) - \partial_i e^i \otimes e^0 - e^0 \otimes \partial_i e^i)$$

For the circle:

$$
\begin{aligned}
\Delta_0(e^1) &= e^1 \otimes e^0 + e^0 \otimes e^1 + (s_1 \otimes 1 + \delta\varepsilon \otimes s_1)(\Delta_0(\partial e^1) - \partial e^1 \otimes e^0 - e^0 \otimes \partial e^1) \\
&= e^1 \otimes e^0 + e^0 \otimes e^1 + (s_1 \otimes 1 + \delta\varepsilon \otimes s_1)(\Delta_0(te^0 - e^0) - (te^0 - e^0) \otimes e^0 - e^0 \otimes (te^0 - e^0)) \\
&= e^1 \otimes e^0 + e^0 \otimes e^1 \\
&\quad + (s_1 \otimes 1 + \delta\varepsilon \otimes s_1)(te^0 \otimes te^0 - e^0 \otimes e^0 - te^0 \otimes e^0 + e^0 \otimes e^0 - e^0 \otimes te^0 + e^0 \otimes e^0) \\
&= e^1 \otimes e^0 + e^0 \otimes e^1 + (s_1 \otimes 1 + \delta\varepsilon \otimes s_1)(te^0 \otimes te^0 - te^0 \otimes e^0 + e^0 \otimes e^0 - e^0 \otimes te^0)
\end{aligned}
$$

Now, $\delta\varepsilon(e^0) = \delta\varepsilon(te^0) = 1e^0$, while $s_1(e^0) = 0e^1$ and $s_1(te^0) = 1e^1$. We therefore have:

$$\begin{aligned}
\Delta_0(e^1) &= e^1 \otimes e^0 + e^0 \otimes e^1 \\
&\quad +(e^1 \otimes te^0 - e^1 \otimes e^0 + 0e^1 \otimes e^0 - 0e^1 \otimes te^0) \\
&\quad +(e^0 \otimes e^1 - e^0 \otimes 0e^1 + e^0 \otimes 0e^1 - e^0 \otimes e^1) \\
&= e^1 \otimes e^0 + e^0 \otimes e^1 + e^1 \otimes te^0 - e^1 \otimes e^0 + e^0 \otimes e^1 - e^0 \otimes e^1 \\
&= e^1 \otimes te^0 + e^0 \otimes e^1
\end{aligned}$$

This corresponds to a staircase approximation to the diagonal line in $\mathbb{R} \times \mathbb{R}$.

For $S^1 \vee S^1$ we therefore have the formulae, for $x = a, b$:

$$\Delta_0(e^1_x) = e^1_x \otimes xe^0 + e^0 \otimes e^1_x.$$

We now turn to the torus.

$$\Delta_0(e^2) = e^2 \otimes e^0 + e^0 \otimes e^2 + (s \otimes 1 + \delta\varepsilon \otimes s)(\Delta_0(\partial e^2) - \partial e^2 \otimes e^0 - e^0 \otimes \partial e^2)$$

Now, $\partial(e^2) = e^1_a - be^1_a + ae^1_b - e^1_b$, so we have:

$$\begin{aligned}
\Delta_0(e^2) &= e^2 \otimes e^0 + e^0 \otimes e^2 \\
&\quad +(s \otimes 1 + \delta\varepsilon \otimes s)(\Delta_0(e^1_a - be^1_a + ae^1_b - e^1_b) - (e^1_a - be^1_a + ae^1_b - e^1_b) \otimes e^0 \\
&\quad -e^0 \otimes (e^1_a - be^1_a + ae^1_b - e^1_b)) \\
&= e^2 \otimes e^0 + e^0 \otimes e^2 \\
&\quad +(s \otimes 1 + \delta\varepsilon \otimes s)(e^1_a \otimes ae^0 + e^0 \otimes e^1_a - be^1_a \otimes bae^0 - be^0 \otimes be^1_a \\
&\quad +ae^1_b \otimes abe^0 + ae^0 \otimes ae^1_b - e^1_b \otimes be^0 - e^0 \otimes e^1_b \\
&\quad -(e^1_a - be^1_a + ae^1_b - e^1_b) \otimes e^0 \\
&\quad -e^0 \otimes (e^1_a - be^1_a + ae^1_b - e^1_b))
\end{aligned}$$

The following apply here:

$$s_2(e^1_a) = s_1(e^0) = s_2(a^n b^m e^1_b) = \delta\varepsilon(x) = 0$$

for $x \in C_k$, $k > 0$, and for any $(n, m) \in \mathbb{Z}^2$.

$$s_2(be^1_a) = e^2; \;\; s_1(be^0) = e^1_b; \;\; s_1(ae^0) = e^1_a; \;\; \delta\varepsilon(a^n b^m e^0) = e^0$$

again for any $(n, m) \in \mathbb{Z}^2$.

Applying all these so the final formula above yields the final result:

$$\Delta_0(e^2) = (e^0 \otimes e^2) + (e^2 \otimes bae^0) - (e^1_b \otimes be^1_a) + (e^1_a \otimes ae^1_b).$$

5 Duality chain equivalence.

We now follow the rest of the program as indicated in my 1st year report. First, tensor with $\mathbb{Z}$:

$$\Delta_0 \colon \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}\oplus\mathbb{Z}]} C(\widetilde{S^1 \times S^1}) \to \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}\oplus\mathbb{Z}]} (C(\widetilde{S^1 \times S^1}) \otimes_{\mathbb{Z}} C(\widetilde{S^1 \times S^1})).$$

Since all the interesting $\mathbb{Z}[\mathbb{Z}\oplus\mathbb{Z}]$ coefficients are already acting on the second position in each of the tensor products we can safely leave the formula at the end of the last section alone. The difference is that this e^2 on which Δ_0 acts is really e^2 now and not $\widetilde{e^2}$. That is, it belongs to $C_2(S^1 \times S^1)$ and not $C_2(\widetilde{S^1 \times S^1})$.

We then apply the slant map: an element $x \otimes y \in C_i \otimes C_{2-i}$ corresponds to a map in $\mathrm{Hom}(C_i^*, C_{2-i})$. We have to introduce some signs according to a convention in order to make the resulting maps of chains $C^* \to C_*$ a chain map. We adopt the convention that the boundary maps in the dual complex $\partial^* \colon C^j \to C^{j+1}$ have a sign $(-1)^{j+1}$ (See Bredon [1]). The resulting commutative diagrams which show the duality chain equivalences for the circle and the torus are as follows. For the circle:

$$\begin{array}{ccc}
C_1(\widetilde{S^1}) & \xrightarrow{(t-1)} & C_0(\widetilde{S^1}) \\
\uparrow{\scriptstyle (1)} & & \uparrow{\scriptstyle (t)} \\
C_0^*(\widetilde{S^1}) & \xrightarrow[-(t^{-1}-1)]{} & C_1^*(\widetilde{S^1})
\end{array}$$

and for the torus:

$$\begin{array}{ccccc}
C_2(\widetilde{S^1 \times S^1}) & \xrightarrow{(1-b,a-1)} & C_1(\widetilde{S^1 \times S^1}) & \xrightarrow{(a-1,b-1)^T} & C_0(\widetilde{S^1 \times S^1}) \\
\uparrow{\scriptstyle (1)} & & \uparrow{\scriptstyle \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix}} & & \uparrow{\scriptstyle (ba)} \\
C_0^*(\widetilde{S^1 \times S^1}) & \xrightarrow[-(a^{-1}-1,b^{-1}-1)]{} & C_1^*(\widetilde{S^1 \times S^1}) & \xrightarrow[(1-b^{-1},a^{-1}-1)^T]{} & C_2^*(\widetilde{S^1 \times S^1})
\end{array}$$

Keeping in mind that these are row vectors with arrows indicating matrices to be multiplied on the right (although everything is abelian so the order of elements doesn't matter here) the reader can check that these diagrams are commutative.

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