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Spherical objects in dimensions two and three

Abstract. This paper classifies spherical objects in various geometric settings in dimensions two and three, including both minimal and partial crepant resolutions of Kleinian singularities, as well as arbitrary flopping 3-fold contractions with only Gorenstein terminal singularities. The main result is much more general: in each such setting, we prove that all objects in the associated null category $\mathcal{C}$ with no negative Ext groups are the image, under the action of an appropriate braid or pure braid group, of some object in the heart of a bounded t-structure. The corollary is that all objects x which admit no negative Exts, and for which $\mathrm{Hom}_{\mathcal{C}}(x, x) = \mathbb{C}$, are the images of the simples. A variation on this argument goes further and classifies all bounded t-structures on $\mathcal{C}$. There are multiple geometric, topological and algebraic consequences, primarily to autoequivalences and stability conditions. Our main new technique also extends into representation theory, and we establish that in the derived category of a finite-dimensional algebra which is silting discrete, every object with no negative Ext groups lies in the heart of a bounded t-structure. As a consequence, every semibrick complex can be completed to a simple-minded collection.

Keywords: spherical objects, Kleinian singularities, 3-fold flops, semibricks, contraction algebras, simple-minded collections.

1. Introduction

In reasonable categories, an object is *spherical* if its self-extension groups behave like the cohomology of a sphere. These objects have attracted interest across symplectic geometry, algebraic geometry and representation theory, since each spherical object generates a symmetry in the form of a twist autoequivalence [42]. It turns out that these objects control much of the structure of autoequivalence groups, at least in small dimension.

The question of classifying spherical objects has been approached by various authors [10, 11, 29, 30, 42, 43], not least because of the resulting topological and geometric consequences. The viewpoint of this paper is that any such classification must necessarily be

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the consequence of something more general. Spherical objects are not strictly speaking the correct objects in either dimension two [34, §10] or dimension three [23, 45], and furthermore birational geometry requires us to work with singular varieties, for which the self-extension groups are usually unbounded.

Our main new insight is that a more general classification is indeed possible, and we identify the homological condition for which such a result exists. In the happy situation when spherical objects exist and are relevant, their classification is a consequence.

1.1. Motivation and setting

This paper is concerned mainly with the following three geometric settings, but later in Section 1.5 it also works much more generally:

(1) Surfaces: The minimal resolution $\mathcal{Z} \to \mathbb{C}^2/\Gamma$ of a Kleinian singularity.

(2) Surfaces: A crepant partial resolution $\mathcal{Y} \to \mathbb{C}^2/\Gamma$ of a Kleinian singularity.

(3) 3-folds: A 3-fold flopping contraction $\mathcal{X} \to \operatorname{Spec} \mathcal{R}$, where $\mathcal{X}$ has at worst Gorenstein terminal singularities, e.g., $\mathcal{X}$ is smooth, and $\mathcal{R}$ is complete local.

The case when $\mathcal{Z} \to \mathbb{C}^2/\Gamma$ is the minimal resolution has attracted the most attention [11, 30]. This situation is unnaturally easy for two separate reasons. First, there are very few examples: just two infinite families and three sporadic cases. Second, the category $\mathcal{C}$ below is then controlled by an intrinsically formal DG-algebra [42], which makes many computations easier, and in particular shows that $\mathcal{C}$ is equivalent to many other categories in the literature. In the other geometric settings, we should expect neither, as the controlling DG-algebra is usually unbounded, and is far from formal.

Nonetheless, for any $X = \mathcal{X}, \mathcal{Y}$ or $\mathcal{Z}$ above, consider

$$\mathcal{C} := \{ x \in \mathrm{D}^b(\operatorname{coh} X) \mid \mathbf{R} f_* x = 0 \} \subset \mathrm{D}^b(\operatorname{coh} X). \tag{1.1}$$

The category $\mathcal{C}$ has finite-dimensional Hom spaces, and further is d-CY, where $d = \dim X$, if X is smooth. Our interest will be in spherical objects, namely those $x \in \mathcal{C}$ for which

$$\operatorname{Hom}_{\mathcal{C}}(x, x[i]) = \begin{cases} \mathbb{C} & \text{if } i = 0, d, \\ 0 & \text{else.} \end{cases}$$

In the case of the minimal resolution $\mathcal{Z} \to \mathbb{C}^2/\Gamma$, and also for some very special smooth flopping contractions, these objects are intimately related to generation questions regarding the autoequivalence group of $\mathcal{C}$, and to the existence of Lagrangians in plumbings of spheres.

However, in general, autoequivalences are controlled by more complicated objects [13, 23, 45], and what makes these objects interesting is that they are *not* determined by their self-extension groups, even in the lucky case when those groups are bounded. This fact has, until now, hampered any attempt to classify them, since it is not even clear what we are supposed to classify.

The main results below show that, rather remarkably, the only conditions that matter are that $\mathrm{Hom}_{\mathcal{C}}(x, x[i]) = 0$ for all $i < 0$, and $\mathrm{Hom}_{\mathcal{C}}(x, x) = \mathbb{C}$. These would seem to be rather weak conditions, however we will prove that these objects are precisely the orbits of the simples under the natural action of the mutation functors, even in very singular settings. In particular, there are actually very few of them.

In fact, and much more generally, it is possible to completely characterise those objects which satisfy only the condition $\mathrm{Hom}_{\mathcal{C}}(x, x[i]) = 0$ for all $i < 0$. These, it turns out, are precisely the objects in the orbits of the standard t-structure under the action of the mutation functors. Furthermore, all bounded t-structures on $\mathcal{C}$ can be similarly classified. From these results, and others, we extract geometric, topological and algebraic corollaries.

1.2. Main results

For partial resolutions $\mathcal{Y}$, or for 3-fold flopping contractions $\mathcal{X}$, it is the lack of uniqueness that makes these settings complicated, not the fact that they are singular. Whilst there is only one minimal resolution $\mathcal{Z}$, there are plenty of partial resolutions $\mathcal{Y}$, and furthermore the lack of uniqueness is the defining feature of the 3-fold flops setting. To compensate for this requires us to work, at least initially, in groupoids.

As is well known, and is recalled in Section 2.2, to each $\mathcal{X}$, $\mathcal{Y}$ or $\mathcal{Z}$, there is an associated finite simplicial hyperplane arrangement $\mathcal{H}$. In the case of the minimal resolution, this is simply the associated ADE root system, and in general the construction is roughly similar. To each chamber C in $\mathcal{H}$, we assign a category $\mathcal{C}_\mathsf{C}$, defined in Section 2.4 in terms of an associated noncommutative resolution (or variant). To each path in the Deligne groupoid $\beta\colon \mathsf{C} \to \mathsf{D}$, we assign a composition of mutation functors $\Phi_\beta\colon \mathcal{C}_\mathsf{C} \to \mathcal{C}_\mathsf{D}$.

Theorem 1.1 (Theorem 3.5). *If $x \in \mathcal{C} = \mathcal{C}_\mathsf{C}$ satisfies $\mathrm{Hom}_{\mathcal{C}}(x, x[i]) = 0$ for all $i < 0$, then there exists $\beta\colon \mathsf{C} \to \mathsf{D}$ such that $\Phi_\beta(x) \cong y$, where y is an object in homological degree zero.*

The proof is both short and elementary, and there are only two key points. First, cohomology spread should be measured with respect to the noncommutative (equivalently, perverse) t-structure, not with respect to coherent sheaves. Second, if say x has cohomology lying within some bounded region $[a, b]$, then the proof finds some $\Phi_\alpha(x)$ whose cohomology lies in a strictly smaller region $[a + 1, b]$. This is achieved by using the fact that any finite poset has a maximal element.

The first corollary is the following. To set notation, write $\mathcal{S}_1, \ldots, \mathcal{S}_n$ for the simple modules in the heart of the standard algebraic t-structure (see Section 2.6).

Corollary 1.2 (Corollary 3.6). *Suppose that $x \in \mathcal{C}_\mathsf{C}$ satisfies $\mathrm{Hom}(x, x[i]) = 0$ for all $i < 0$.*

(1) *If $\dim_{\mathbb{C}} \mathrm{Hom}(x_i, x_j) \cong \mathbb{C}\delta_{ij}$ for all indecomposable summands x_i, x_j of x, then there exist a subset $\mathcal{I} \subseteq \{1, \ldots, n\}$ and a path $\gamma\colon \mathsf{C} \to \mathsf{E}$ such that $\Phi_\gamma(x) \cong \bigoplus_{i \in \mathcal{I}} \mathcal{S}_i$.*

(2) *If $\dim_{\mathbb{C}} \mathrm{Hom}(x, x) \cong \mathbb{C}$, then there exist $\gamma\colon \mathsf{C} \to \mathsf{E}$ and i such that $\Phi_\gamma(x) \cong \mathcal{S}_i$.*

Given the control of the mutation functor orbit obtained in Corollary 1.2, a standard variation on results of [11, 29] can then be used to show that the space of stability conditions on $\mathcal{C}$ is connected, and thus $\mathrm{Stab}^\circ \mathcal{C} = \mathrm{Stab}\, \mathcal{C}$. However, a much stronger statement holds.

The following result classifies all bounded t-structures on $\mathcal{C}$ in terms of the standard hearts. These are, by definition, the extension closures of the simples $\mathcal{S}_1, \ldots, \mathcal{S}_n$ (see Section 2.5). This is a purely homological result, which is new in all cases, and its proof does not require stability techniques.

Corollary 1.3 (Theorem 4.6). *If $\mathcal{H}$ is the heart of a bounded t-structure on $\mathcal{C}_C$, then there exists Φ_β for some $\beta \colon C \to D$ such that $\Phi_\beta(\mathcal{H})$ is the standard heart on $\mathcal{C}_D$. In particular, $\mathcal{H}$ is a finite length category, with finitely many simples.*

From this, in Corollary 5.8 the statement $\mathrm{Stab}^\circ \mathcal{C} = \mathrm{Stab}\, \mathcal{C}$ immediately follows and avoids the use of any fancy deformation argument. In Corollary 5.9, we also give an intrinsic characterisation of the group $\mathrm{Auteq}\, \mathcal{C}$ in terms of certain FM transforms.

1.3. Geometric corollaries

By removing the noncommutative resolutions technology implicit in the previous subsection, in particular the use of the mutation functors, it is possible to translate the above results into more standard geometric language. We translate some results here, with more details and results given in Section 5.

In the case of the minimal resolution $\mathcal{Z}$, the mutation functors are functorially isomorphic to the Seidel–Thomas twist functors. Recall that $\mathcal{C}$ is the category geometrically defined in (1.1), and to each exceptional curve C_i in $\mathcal{Z}$, the associated $\mathcal{O}_{C_i}(-1)$ is spherical, so it gives rise to a spherical twist functor t_i. These functors generate a subgroup Br of $\mathrm{Auteq}\, \mathcal{C}$.

Part (2) of the following statement recovers the main result of [11].

Corollary 1.4 (Corollary 5.1). *Consider $\mathcal{Z} \to \mathbb{C}^2/\Gamma$, and let $x \in \mathcal{C}$. Then the following hold:*

(1) *If $\mathrm{Hom}_\mathcal{C}(x, x[j]) = 0$ for all $j < 0$, then there exists $T \in \mathsf{Br}$ such that $T(x)$ is a concentrated in homological degree zero.*

(2) *Every spherical object in $\mathcal{C}$ belongs to the orbit, under the action of the braid group, of the objects $\mathcal{O}_{C_1}(-1), \ldots, \mathcal{O}_{C_n}(-1)$.*

Furthermore, the heart of every bounded t-structure on $\mathcal{C}$ is the image, under the action of the group Br, of the module category of the preprojective algebra of (finite) ADE type.

The case of partial resolutions $\mathcal{Y} \to \mathbb{C}^2/\Gamma$ is similar, but mildly more difficult to state, so details are left to Corollary 5.2.

For 3-fold flopping contractions $\mathcal{X} \to \mathrm{Spec}\, \mathcal{R}$, the mutation functor Φ_i is functorially isomorphic to the inverse of the Bridgeland–Chen flop functor [48, Theorem 4.2], which will be denoted by Flop_i. These are the square roots of the spherical twist functors, suit-

ably interpreted (see Remark 2.13). The following generalises the particular example in [43, Corollary 6.13 (1)] to all 3-fold flops.

Corollary 1.5 (Corollary 5.5). *Let* $\mathfrak{X} \to \operatorname{Spec} \mathcal{R}$ *be a 3-fold flopping contraction, where* $\mathfrak{X}$ *has at worst Gorenstein terminal singularities, and consider* $x \in \mathcal{C}$.

(1) *The following statements are equivalent:*

    (a) $\operatorname{Hom}_{\mathcal{C}}(x, x[j]) = 0$ *for all* $j < 0$.

    (b) *There exists a sequence of flop functors such that* $\mathsf{Flop}_{\beta}(x)$ *belongs to perverse sheaves on a possibly different* $\mathfrak{X}^+ \to \operatorname{Spec} \mathcal{R}$.

(2) *Furthermore, the following are equivalent:*

    (a) $\operatorname{Hom}_{\mathcal{C}}(x, x[j]) = 0$ *for all* $j < 0$ *and* $\operatorname{Hom}_{\mathcal{C}}(x, x) \cong \mathbb{C}$.

    (b) *There exists* β *such that* $\mathsf{Flop}_{\beta}(x) \cong \mathcal{O}_{C_i}(-1)$ *for some curve* C_i *on a possibly different* $\mathfrak{X}^+ \to \operatorname{Spec} \mathcal{R}$.

Furthermore, the heart of every bounded t-structure on $\mathcal{C}$ *is the image, under the action of the group generated by the flop functors, of the module category of the contraction algebra of some* $\mathfrak{X}^+ \to \operatorname{Spec} \mathcal{R}$ *obtained from* $\mathfrak{X} \to \operatorname{Spec} \mathcal{R}$ *by iterated flop.*

In particular, when $\mathfrak{X}$ is smooth, $x \in \mathcal{C}$ is spherical if and only if there exist a $(-1, -1)$-curve C_i on a possibly different $\mathfrak{X}^+$, and a composition $\beta \colon \mathsf{D} \to \mathsf{C}$ in the Deligne groupoid such that

$$\mathsf{Flop}_{\beta}(\mathcal{O}_{C_i}(-1)) \cong x.$$

Perhaps of more interest is when the groupoid result (Corollary 1.5) is used to prove statements regarding the orbit of the pure braid group PBr on a fixed $\mathcal{C}$. In Corollary 5.6, we describe the spherical objects in terms of the orbit under this action, generalising [43, Corollary 6.13 (2)] to all 3-fold flops.

1.4. Algebraic corollaries

A more algebraic corollary of Corollary 1.2 is a generalisation of a result of Crawley–Boevey [19, Lemma 1] which states that every brick for the preprojective algebra Π of type ADE has dimension vector equal to a root. The categories $\mathcal{C}$ are either controlled by algebras of the form $e_{\mathfrak{J}} \Pi e_{\mathfrak{J}}$, or by contraction algebras, depending on the setting. The underlying combinatorial structure of these are *restricted roots*, recalled in Section 2.2 and illustrated in Example 2.2.

Corollary 1.6 (Corollary 3.7). *Consider the projective algebra* Π *of type* ADE*, and* $\mathfrak{J} \subseteq \Delta$. *Then every brick in* $\operatorname{mod} e_{\mathfrak{J}} \Pi e_{\mathfrak{J}}$ *has dimension vector equal to a primitive restricted root.*

A version of the above statement is proved in Corollary 3.9 for contraction algebras Λ_{con}, which are recalled in Section 2.5. These algebras control much of the birational geometry of 3-folds, and their representation theory is usually wild.

1.5. Silting discrete derived categories

The above techniques extend further. Recall that a finite-dimensional algebra A is called silting discrete if the category $\mathrm{K}^b(\mathrm{proj}\, A)$ is a silting discrete triangulated category (see Section 6.1).

This class of silting discrete finite-dimensional algebras is surprisingly rich: it contains all local algebras, Erdmann's algebras of dihedral, semidihedral and quaternion type, various preprojective algebras and Brauer graph algebras, together with more obvious representation-finite examples (see, e.g., [3, p. 1]). From the viewpoint of this paper, the important point is that the full list of examples also includes all 3-fold contraction algebras Λ_{con}. The resulting categories $\mathrm{D}^b(\Lambda_{\mathrm{con}})$ are related to, but are very different from, the categories $\mathcal{C}$ in the previous sections.

The analogy between $\mathcal{C}$ and $\mathrm{D}^b(\Lambda_{\mathrm{con}})$ allows us to push the previous techniques into the general silting discrete setting. The following is new in all cases.

Theorem 1.7 (Theorem 6.11). *If A is silting discrete, $\mathcal{T} = \mathrm{D}^b(A)$ and $x \in \mathcal{T}$, then the following statements are equivalent:*

(1) $\mathrm{Hom}_{\mathcal{T}}(x, x[i]) = 0$ *for all $i < 0$.*

(2) *x belongs to the heart of a bounded t-structure.*

In this level of generality, there are fewer autoequivalences, so the strategy for the proof of Theorem 1.7 minimises the cohomology with respect to the bounded t-structures on the fixed $\mathcal{T}$. This avoids groupoids. The object x has cohomology lying within some bounded region $[a, b]$ with respect to the standard t-structure. The proof first finds a new t-structure whose cohomology functors applied to x lie in a strictly smaller region $[a + 1, b]$. A simple induction argument then shows that there is a bounded t-structure such that $x[b]$ belongs to its heart.

The following are immediate consequences, which may be of independent interest. The definition of brick and semibrick complexes is recalled in Definition 6.5.

Corollary 1.8 (Corollary 6.12). *Let A be silting discrete, $\mathcal{T} := \mathrm{D}^b(A)$ and $x \in \mathcal{T}$.*

(1) *x is a sum of simples in the heart of a bounded t-structure if and only if x is a semibrick.*

(2) *x is a simple in the heart of a bounded t-structure if and only if x is a brick.*

In particular, if $x \in \mathcal{T}$ is a semibrick complex, then there exists a simple-minded collection $\mathcal{U}$ such that x is contained in $\mathcal{U}$.

It then follows in Corollary 6.13 that for silting discrete algebras, every semibrick pair forms a subset of a simple-minded collection, and further that every semibrick pair of maximal rank is itself a simple-minded collection. The slightly subtle point is that in general the simple-minded collection containing the semibrick pair need not be 2-term with respect to the standard t-structure, which explains why Corollary 1.8 does not contradict [12, 26]. We give some explicit examples (Examples 6.16 and 6.17) which illustrate this phenomenon.

The above general results apply to 3-fold contraction algebras Λ_{con}, where more can be said. The autoequivalence group of $\mathrm{D}^b(\Lambda_{\text{con}})$ is large, and so the above simplifies. We show in Corollary 6.14 that for contraction algebras, the brick and semibrick complexes can be characterised as orbits of the brick modules under the action of the autoequivalence group. The result further improves if we instead pass to the groupoid, but details are left to Section 6.3.

Conventions

To be consistent between the algebraic and geometric settings considered, all objects will be taken over $\mathbb{C}$. Furthermore, since, e.g., $\mathrm{Hom}(\mathrm{H}^i(\Phi_j(y)), \mathrm{H}^i(x))$ is slightly painful, brackets will often be dropped when applying functors to objects. Doing this, the above simplifies to $\mathrm{Hom}(\mathrm{H}^i(\Phi_j y), \mathrm{H}^i x)$.

2. Preliminaries

This section briefly recalls the known properties of the category $\mathcal{C}$ and sets notation. This is largely a summary of [28, 34] suitable for our purposes.

2.1. Noncommutative resolutions and variants

Since by definition $\mathcal{C}$ is a full subcategory of $\mathrm{D}^b(\mathrm{coh}\, X)$, it is possible to take cohomology of objects in $\mathcal{C}$ with respect to the standard t-structure on $\mathrm{D}^b(\mathrm{coh}\, X)$. These cohomology sheaves were used heavily, e.g., in [30]. However, as in [43, §6], it turns out to be easier to control the cohomology with respect to a different t-structure, namely perverse sheaves.

Throughout this paper, we will abuse notation and write f for either the minimal resolution $\mathcal{Z} \to \mathbb{C}^2/\Gamma$, a fixed partial crepant resolution $\mathcal{Y} \to \mathbb{C}^2/\Gamma$, or a fixed 3-fold flopping contraction $\mathcal{X} \to \mathrm{Spec}\,\mathcal{R}$, where $\mathcal{X}$ has only Gorenstein terminal singularities and $\mathcal{R}$ is complete local. It is well known that the reduced fibre above the unique singular point is a tree of curves, with each irreducible component $\mathrm{C}_i \cong \mathbb{P}^1$. Furthermore, X will denote the choice of fixed $\mathcal{Z}, \mathcal{Y}$ or $\mathcal{X}$, and from that choice the category $\mathcal{C}$ is defined to be

$$\mathcal{C} = \{a \in \mathrm{D}^b(\mathrm{coh}\, X) \mid \mathbf{R}f_* a = 0\}.$$

By [47], there exists a tilting bundle $\mathcal{O} \oplus \mathcal{N}$ on X, where $\mathcal{N}^*$ is generated by global sections, which induces an equivalence of categories

$$\Psi_X \colon \mathrm{D}^b(\mathrm{coh}\, X) \xrightarrow{\ \mathbf{R}\mathrm{Hom}_X(\mathcal{O} \oplus \mathcal{N}, -)\ } \mathrm{D}^b(\mathrm{mod}\, \Lambda), \tag{2.1}$$

where $\Lambda = \mathrm{End}_X(\mathcal{O} \oplus \mathcal{N})$. The algebra Λ admits an idempotent e corresponding to the summand $\mathcal{O}$, and across equivalence (2.1), $\mathcal{C}$ corresponds to those complexes whose cohomology groups are all annihilated by e, or in other words, those complexes whose cohomology groups are $\Lambda/(e)$-modules.

2.2. Intersection arrangements

To any X equal to $\mathcal{Z}$, $\mathcal{Y}$ or $\mathcal{X}$ in Section 2.1, it is possible to associate Dynkin data in the form of a subset $\mathcal{I} \subset \Delta$ with Δ ADE.

(1) For the minimal resolution $\mathcal{Z} \to \mathbb{C}^2/\Gamma$, Δ is the ADE Dynkin diagram corresponding to Γ via McKay correspondence, and $\mathcal{I} = \emptyset$.

(2) For a partial crepant resolution $\mathcal{Y} \to \mathbb{C}^2/\Gamma$, Δ is the ADE Dynkin diagram corresponding to Γ via McKay correspondence, and $\mathcal{I}$ is the subset of curves that are contracted by the morphism $\mathcal{Z} \to \mathcal{Y}$.

(3) For a flopping contraction $\mathcal{X} \to \mathrm{Spec}\,\mathcal{R}$, a generic hyperplane section slices to a partial crepant resolution as in (2) above [40], which thus associates some $\mathcal{I} \subset \Delta$.

Remark 2.1. The exceptional curves in $\mathcal{Z}$ will be denoted by C_i with $i \in \Delta$. For partial resolutions, the subset $\mathcal{I}$ is the choice of curves that get contracted from $\mathcal{Z}$. We will abuse notation and write C_i with $i \in \mathcal{I}^c = \Delta \setminus \mathcal{I}$ for the exceptional curves that survive, namely the curves in $\mathcal{Y}$. Further, we will also write C_i with $i \in \mathcal{I}^c$ for the exceptional curves in $\mathcal{X}$.

The ADE diagram Δ has an associated root system $\mathfrak{h} = \bigoplus \mathbb{R}\alpha_i$, where α_i are the simple roots. For any $\mathcal{I} \subset \Delta$, define $\mathfrak{h}_{\mathcal{I}}$ to be the quotient of $\mathfrak{h}$ by the subspace spanned by $\{\alpha_i \mid i \in \mathcal{I}\}$. The associated quotient map will be written as

$$\pi_{\mathcal{I}} \colon \mathfrak{h} \to \mathfrak{h}_{\mathcal{I}},$$

and note that $\mathfrak{h}_{\mathcal{I}}$ has basis $\{\pi_{\mathcal{I}}(\alpha_i) \mid i \notin \mathcal{I}\} = \{\pi_{\mathcal{I}}(\alpha_j) \mid j \in \mathcal{I}^c\}$. The *restricted positive roots* in $\mathfrak{h}_{\mathcal{I}}$ are precisely the non-zero images of positive roots under $\pi_{\mathcal{I}}$. A restricted positive root is *primitive* if it is not a multiple of another positive restricted root.

Example 2.2. Consider the Dynkin data ●○●●●, where $\mathcal{I}$ is the set of black nodes. Projecting all twenty positive roots of D_5 via $\pi_{\mathcal{I}}$, and discounting the zeros, gives the set

$$\{10, 01, 11, 21, 22\},$$

where, for example, 11 is shorthand for the coordinate $(1, 1)$ in $\mathfrak{h}_{\mathcal{I}}$ under the prescribed basis above. These are the restricted roots. The primitive restricted roots are $\{10, 01, 11, 21\}$.

Now write $\Theta_{\mathcal{I}}$ for the dual of the real vector space $\mathfrak{h}_{\mathcal{I}}$. For each restricted positive root

$$0 \neq \beta = \pi_{\mathcal{I}}(\alpha) \in \mathfrak{h}_{\mathcal{I}},$$

consider the dual hyperplane

$$\mathsf{H}_\beta := \{(\vartheta_i) \mid \textstyle\sum \beta_i \vartheta_i = 0\} \subseteq \Theta_{\mathcal{I}}.$$

Since there are only finitely many restricted roots, the collection of H_β forms a finite hyperplane arrangement in $\Theta_{\mathcal{I}}$, which we refer to as the *intersection arrangement*. In general, this need not be Coxeter.

Example 2.3. Continuing Example 2.2, for instance, the restricted root 21 gives the dual hyperplane $2x + y = 0$. The full intersection arrangement is the following:

Restricted root	
01	—
11, 22	—
21	—
10	│
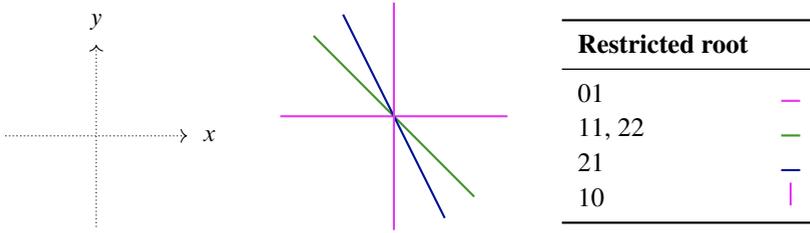
2.3. Deligne groupoid

Write $\mathcal{H}$ for the arrangement obtained from the Dynkin data $\mathcal{I} \subset \Delta$ as defined in Section 2.2, which is a finite simplicial hyperplane arrangement. This subsection summarises well-known constructions associated to $\mathcal{H}$.

Definition 2.4. Consider the graph $\Gamma_{\mathcal{H}}$ of oriented arrows, which has as vertices the chambers (i.e., the connected components) of $\mathbb{R}^n \setminus \mathcal{H}$, and there is a unique arrow $a\colon \mathsf{C} \to \mathsf{D}$ from chamber C to chamber D if the chambers are adjacent, otherwise there is no arrow. For an arrow $a\colon \mathsf{C} \to \mathsf{D}$, write $s(a) := \mathsf{C}$ and $t(a) := \mathsf{D}$.

By definition, if there is an arrow $a\colon \mathsf{C} \to \mathsf{D}$, there is a unique arrow $b\colon \mathsf{D} \to \mathsf{C}$.

Definition 2.5. Given C, then a chamber $\mathsf{op}(\mathsf{C})$ is said to be *opposite* C if there is a line in $\mathbb{R}^n$ passing through C, $\mathsf{op}(\mathsf{C})$, and the origin. The opposite chamber $\mathsf{op}(\mathsf{C})$ is unique.

As for quivers, a *positive path of length n* in $\Gamma_{\mathcal{H}}$ is defined to be a formal symbol

$$p = a_n \circ \cdots \circ a_2 \circ a_1,$$

whenever there exists a sequence of vertices $v_0, \ldots, v_n$ of $\Gamma_{\mathcal{H}}$ and arrows $a_i\colon v_{i-1} \to v_i$ in $\Gamma_{\mathcal{H}}$. Set

$$s(p) := v_0 \quad \text{and} \quad t(p) := v_n,$$

and write $p\colon s(p) \to t(p)$. The notation $\circ$ is composition, but we will often drop the $\circ$'s in the future. If $q = b_m \circ \cdots \circ b_2 \circ b_1$ is another positive path with $t(p) = s(q)$, consider the formal symbol

$$q \circ p := b_m \circ \cdots \circ b_2 \circ b_1 \circ a_n \circ \cdots \circ a_2 \circ a_1,$$

and call it the composition of p and q.

Definition 2.6. A positive path is called *reduced* if it does not cross any hyperplane twice.

Since $\mathcal{H}$ is finite, reduced positive paths coincide with shortest positive paths [38, Lemma 4.2], in the sense that there is no positive path in $\Gamma_{\mathcal{H}}$ of smaller length, with the same endpoints.

Example 2.7. Consider the following hyperplane arrangement $\mathcal{H}$ with 6 chambers, and with chamber C marked. Then $\mathsf{op}(\mathsf{C})$, and all reduced paths leaving C, are illustrated below:
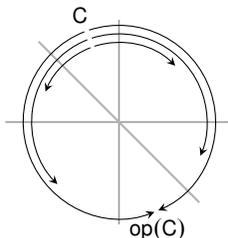
As in [21, p. 7], let $\sim$ denote the smallest equivalence relation, compatible with composition, that identifies all morphisms that arise as positive reduced paths with the same source and target. Write $\mathrm{Free}(\Gamma_{\mathcal{H}})$ for the free category on the graph $\Gamma_{\mathcal{H}}$, where morphisms are directed paths. The quotient category

$$\mathcal{G}_{\mathcal{H}}^{+} := \mathrm{Free}(\Gamma_{\mathcal{H}})/\!\sim$$

is called the category of positive paths.

Definition 2.8. The elements of $\mathcal{G}_{\mathcal{H}}^{+}$ which are reduced are called *atoms*. That is, atoms are the (now unique) shortest positive paths between chambers.

The set of atoms beginning from a fixed chamber admits a partial order.

Definition 2.9. If α, β are atoms that both begin at C, write $\alpha \geq \beta$ if there exists an atom γ such that $\alpha = \gamma \circ \beta$.

The identity is minimal, and any atom $\ell\colon \mathsf{C} \to \mathsf{op}(\mathsf{C})$ is the maximal element. This convention is chosen to match the Bruhat order, but has the rather unfortunate consequence of being opposite to the order on tilting and silting complexes. This explains why maximal elements in Section 3 switch to minimal elements in Section 6.

Definition 2.10. The *arrangement (= Deligne) groupoid* $\mathcal{G}_{\mathcal{H}}$ is the groupoid defined as the groupoid completion of $\mathcal{G}_{\mathcal{H}}^{+}$, that is, a formal inverse is added for every morphism in $\mathcal{G}_{\mathcal{H}}^{+}$.

Paths in the groupoid $\mathcal{G}_{\mathcal{H}}$ can be positive, or negative, or mixed.

2.4. Categorical representations and HomMMP

In each of the settings $X = \mathcal{X}$, $\mathcal{Y}$ or $\mathcal{Z}$ of Section 2.1, it is possible to produce a categorical representation of the Deligne groupoid by associating categories to each chamber, and certain equivalences to the wall crossings. This subsection summarises this construction, mainly to set notation.

Below, consider the preprojective algebra Π of an extended ADE Dynkin diagram Δ^{aff}. By convention, the vertices will be labelled $i = 0, 1, \ldots, n$, with 0 being the extended

vertex, and thus the vertex idempotents are labelled $e_0, \ldots, e_n$. For a subset $\mathfrak{I} \subset \Delta \subset \Delta^{\mathsf{aff}}$, to match the notation in [34], set

$$e_{\mathfrak{I}} = 1 - \sum_{i \in \mathfrak{I}} e_i \tag{2.2}$$

and consider the associated contracted preprojective algebra $e_{\mathfrak{I}} \Pi e_{\mathfrak{I}}$. The notation creates one ambiguity, namely that $e_{\{i\}}$ and e_i are different, since $e_{\{i\}} = 1 - e_i$.

To each geometric setting (1)–(3), Section 2.2 assigns a subset $\mathfrak{I} \subset \Delta$ and thus a corresponding intersection arrangement $\mathcal{H}$.

(1) For the minimal resolution $\mathcal{Z} \to \mathbb{C}^2 / \Gamma$, consider the preprojective algebra Π associated to Δ^{aff}. To every chamber C, assign the following subcategory of $\mathrm{D}^b(\operatorname{mod} \Pi)$:

$$\mathcal{C}_{\mathsf{C}} := \{ a \in \mathrm{D}^b(\operatorname{mod} \Pi) \mid e_0 \, \mathrm{H}^*(a) = 0 \},$$

where as above e_0 is the idempotent corresponding to the extended vertex. Note that $\mathcal{C}_{\mathsf{C}}$ does not depend on C. To each simple wall crossing $s_i \colon \mathsf{C} \to \mathsf{D}$ with $i = 1, \ldots, n$, the mutation functor $\Phi_i \cong \mathbf{R}\mathrm{Hom}_{\Pi}(\mathrm{I}_i, -)$ is assigned, where I_i is the bimodule kernel of the map $\Pi \to \Pi/(e_i)$.

(2) For a partial crepant resolution $\mathcal{Y} \to \mathbb{C}^2 / \Gamma$, again consider the preprojective algebra Π associated to Δ^{aff}. Now chambers in $\mathcal{H}$ are indexed by certain pairs $(x, \mathfrak{J})$, where x belongs to the Weyl group of Δ, and $\mathfrak{J} \subset \Delta$ [34, Theorem 1.12]. To a chamber $\mathsf{C} = (x, \mathfrak{J})$, assign the subcategory of $\mathrm{D}^b(\operatorname{mod} e_{\mathfrak{J}} \Pi e_{\mathfrak{J}})$ defined to be

$$\mathcal{C}_{\mathsf{C}} := \{ a \in \mathrm{D}^b(\operatorname{mod} e_{\mathfrak{J}} \Pi e_{\mathfrak{J}}) \mid e_0 \, \mathrm{H}^*(a) = 0 \},$$

where again e_0 is the idempotent corresponding to the extended vertex. To each simple wall crossing $\omega_i \colon \mathsf{C} \to \mathsf{D}$ there is an associated mutation functor Φ_i, defined generally in [33, §6], and in the setting here in [34, §5.6].

(3) For a flopping contraction $\mathcal{X} \to \operatorname{Spec} \mathcal{R}$, by [34, 48] there is a bijection $\mathsf{C} \mapsto M_{\mathsf{C}}$ between chambers in $\mathcal{H}$ and certain rigid objects in the category of Cohen–Macaulay modules $\operatorname{CM} \mathcal{R}$. Necessarily, M_{C} has $\mathcal{R}$ as a summand. Thus to a chamber C, set $\Lambda_{\mathsf{C}} := \operatorname{End}_{\mathcal{R}}(M_{\mathsf{C}})$, and assign the subcategory

$$\mathcal{C}_{\mathsf{C}} := \{ a \in \mathrm{D}^b(\operatorname{mod} \Lambda_{\mathsf{C}}) \mid e \, \mathrm{H}^*(a) = 0 \},$$

where e is the idempotent corresponding to the summand $\mathcal{R}$. To each simple wall crossing $s_i \colon \mathsf{C} \to \mathsf{D}$, assign the mutation functor $\Phi_i \cong \mathbf{R}\mathrm{Hom}_{\Lambda_{\mathsf{C}}}(\operatorname{Hom}_{\mathcal{R}}(M_{\mathsf{C}}, M_{\mathsf{D}}), -)$.

In each case (1)–(3) above, the category $\mathcal{C}_{\mathsf{C}}$ corresponds to the more geometric $\mathcal{C}$ defined in (1.1) across the derived equivalence Ψ in (2.1). The above assignments generate a groupoid $\mathbb{G}$, since all functors are equivalences, and in all cases (1)–(3) there is a homomorphism

$$\mathbb{G}_{\mathcal{H}} \to \mathbb{G}. \tag{2.3}$$

In the case of the minimal resolution, and in the notation here, this homomorphism was established in [32, Theorem 6.6] (but really first in [42]), for partial resolutions this is

[34, Theorem 5.30], and for 3-fold flopping contractions the homomorphism is [24, Definition 3.22].

Notation 2.11. For $\alpha \in \mathcal{G}_{\mathcal{H}}$, write Φ_{α} for the image of α under (2.3).

From (2.3), it is immediate that there is an induced homomorphism from the vertex group of the Deligne groupoid to Auteq $\mathcal{C}_{\mathsf{C}}$ for any fixed $\mathcal{C}_{\mathsf{C}}$. Since the vertex group is always $\pi_1(\mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}})$ [20, 38, 41], it follows that there is an induced homomorphism

$$\pi_1(\mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}}) \to \text{Auteq}\, \mathcal{C}_{\mathsf{C}}. \tag{2.4}$$

This is known as the pure braid group action. Write PBr for the image of this homomophism.

Example 2.12. Continuing Example 2.7, label the six chambers as in the left-hand side of the following diagram. The right-hand side contains the categorical representation, where each chamber has been replaced by a category, and each wall crossing by a mutation functor:
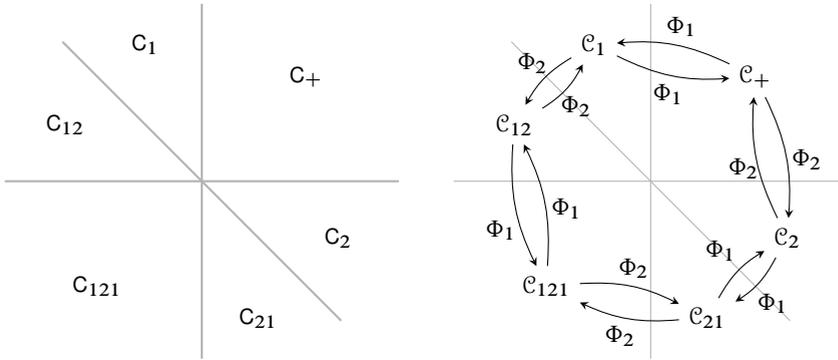
The precise definition of the categories $\mathcal{C}_{\mathsf{C}}$ and the mutation functors depends on settings (1)–(3). In this example, the existence of homomorphism (2.3) is simply the statement that there is a functorial isomorphism

$$\Phi_1 \Phi_2 \Phi_1 \cong \Phi_2 \Phi_1 \Phi_2$$

whenever that makes sense.

In general, higher length braid relations hold between the mutation functors [24].

Remark 2.13. The following calibration is important. In the setting of the minimal resolution (1), Φ_i is functorially isomorphic to the spherical twist of [42] around the sheaf $\mathcal{O}_{C_i}(-1)$. In setting (3), Φ_i is functorially isomorphic to the inverse of the Bridgeland–Chen flop functor [14, 18], and further $\Phi_i \Phi_i$ is the twist functor around the NC deformations of $\mathcal{O}_{C_i}(-1)$ [23, 24]. When $\mathcal{O}_{C_i}(-1)$ is spherical, namely C_i is a $(-1,-1)$-curve, then $\Phi_i \Phi_i$ is the standard spherical twist. In particular, in the 3-fold setting the functor Φ_i should be thought of as the square root of twist functors.

Remark 2.14. In the case of the minimal resolution $\mathcal{Z} \to \mathbb{C}^2/\Gamma$, the categories in each chamber are equal, not just equivalent. As such, they can be identified, using the Weyl group W, and so (2.4) can be improved to a homomorphism

$$\pi_1((\mathbb{C}^n \setminus \mathcal{H}_C)/W) \to \text{Auteq } \mathcal{C}_C.$$

This is precisely the statement that the braid group acts on any fixed $\mathcal{C}_C$, generated by the mutation functors Φ_i. In the other settings (2) and (3), usually the categories in different chambers are not equal, and so we cannot make this identification. As such, usually in setting (2) and always in setting (3), homomorphism (2.4) is the best possible.

The main technical point below is that in all settings (1)–(3), the mutation functor $\Phi_i \cong \mathbf{R}\text{Hom}(\mathrm{T}_i, -)$ where T_i is a tilting module of projective dimension one. This fact is what allows all cases to be treated uniformly.

2.5. Standard hearts and deformation algebras

The categories $\mathcal{C}_C$ inherit a standard t-structure. Maintaining the notation in Section 2.4, consider the following:

(1) For the minimal resolution $\mathcal{Z} \to \mathbb{C}^2/\Gamma$, the standard t-structure on $\mathrm{D}^b(\text{mod}\,\Pi)$ restricts to give a bounded t-structure on $\mathcal{C}_C$ with heart $\text{mod}(\Pi/(e_0))$. Note that $\Pi/(e_0)$ is the preprojective algebra of (finite) type ADE.

(2) For a partial resolution $\mathcal{Y} \to \mathbb{C}^2/\Gamma$, the standard t-structure on $\mathrm{D}^b(\text{mod}\,e_{\mathcal{J}}\Pi e_{\mathcal{J}})$ restricts to give a bounded t-structure on $\mathcal{C}_C$ with heart $\text{mod}(e_{\mathcal{J}}\Pi e_{\mathcal{J}}/(e_0))$.

(3) For a flopping contraction $\mathcal{X} \to \text{Spec }\mathcal{R}$, the standard t-structure on $\mathrm{D}^b(\text{mod}\,\Lambda_C)$ restricts to give a bounded t-structure on $\mathcal{C}_C$ with heart $\text{mod}(\Lambda_C/(e))$. Note that $\Lambda_{C,\text{con}} := \Lambda_C/(e) \cong \underline{\text{End}}_{\mathcal{R}}(M_C)$ is known as the contraction algebra.

Notation 2.15. Write $\text{mod}\,\mathrm{A}_{C,\text{con}}$ for the standard heart in $\mathcal{C}_C$, where $\mathrm{A}_{C,\text{con}}$ varies depending on settings (1)–(3), and so equals $\Pi/(e_0)$, $e_{\mathcal{J}}\Pi e_{\mathcal{J}}/(e_0)$ or $\Lambda_C/(e)$.

Although the precise algebra $\mathrm{A}_{C,\text{con}}$ varies, it is always finite-dimensional. Furthermore, in all cases $\mathrm{A}_{C,\text{con}}$ represents noncommutative deformations of the reduced fibre.

2.6. Simples

All the standard hearts $\text{mod}\,\mathrm{A}_{C,\text{con}}$ above have finite length, with finitely many simples. As such, recording the action of functors on the simples becomes important.

Notation 2.16. For any chamber C, write $\mathcal{S}_{1,C}, \ldots, \mathcal{S}_{n,C}$ for the simple $\mathrm{A}_{C,\text{con}}$-modules, and consider

$$\mathcal{S}_C := \bigoplus_{i=1}^{n} \mathcal{S}_{i,C}.$$

If it is implicitly clear to which category the simples belong, we will drop the subscript C and write $\mathcal{S}_1, \ldots, \mathcal{S}_n$ for the simples and $\mathcal{S} = \bigoplus \mathcal{S}_i$ for their sum.

By [47, Proposition 3.5.8], $\mathcal{S}_{i,\mathsf{C}}$ corresponds to the sheaf $\mathcal{O}_{C_i}(-1)$ under the derived equivalence (2.1). The labelling is consistent so that mutation Φ_i always shifts $\mathcal{S}_i$ to the right by one homological degree, in all three settings.

Lemma 2.17. *For a chamber* C, *consider a length one wall crossing* $s_i\colon \mathsf{C} \to \mathsf{D}$, *and also the 'longest element' atom* $\ell\colon \mathsf{C} \to \mathrm{op}(\mathsf{C})$. *Then the following statements hold:*

(1) $\Phi_i(\mathcal{S}_i) = \mathcal{S}_i[-1]$.

(2) *There exists a permutation* σ *such that* $\Phi_\ell(\mathcal{S}_i) \cong \mathcal{S}_{\sigma(i)}[-1]$ *for all* $i = 1, \ldots, n$.

(3) *If* $x \in \mathrm{mod}\, A_{\mathsf{C},\mathrm{con}}$, *then* $\Phi_\ell(x) \cong y[-1]$ *for some* $y \in \mathrm{mod}\, A_{\mathrm{op}(\mathsf{C}),\mathrm{con}}$.

Proof. (1) and (2) are contained in [48, Lemma 6.3]. Part (3) follows easily from (2) since all such x are filtered by simples. ∎

3. Classification of spherical objects

Throughout this section, let $\mathcal{C}_{\mathsf{C}}$ be the category defined in Section 2.4, and we will freely use the equivalent categories $\mathcal{C}_{\mathsf{D}}$ and the functors Φ_α explained in Sections 2.4 and 2.11.

3.1. Bounded regions

Given $x \in \mathcal{C}_{\mathsf{C}}$, write $\mathrm{H}^*(x)$ or sometimes more simply $\mathrm{H}^* x$ for the cohomology of x with respect to the standard heart $\mathrm{mod}\, A_{\mathsf{C},\mathrm{con}}$ on $\mathcal{C}_{\mathsf{C}}$. So, $\mathrm{H}^*(x)$ always denotes cohomology with respect to the standard heart of the category to which x belongs. To illustrate this, if $x \in \mathcal{C}_{\mathsf{C}}$ and $\beta\colon \mathsf{C} \to \mathsf{D}$, then $\Phi_\beta(x) \in \mathcal{C}_{\mathsf{D}}$ and so $\mathrm{H}^*(\Phi_\beta x)$ denotes the cohomology of $\Phi_\beta(x)$ with respect to the standard t-structure on $\mathcal{C}_{\mathsf{D}}$.

Notation 3.1. Write $x \in [a, b]$, where $a \leq b$, to mean $\mathrm{H}^i(x) = 0$ if $i < a$ and $i > b$, and write $x \in [\![a, b]\!]$ to mean $x \in [a, b]$, where furthermore $\mathrm{H}^a(x) \neq 0$ and $\mathrm{H}^b(x) \neq 0$.

There are obvious self-documenting variations, such as $[\![a, b]$. The above notation can be interpreted using vanishings of Hom against the direct sum of the simples $\mathcal{S} = \bigoplus \mathcal{S}_i$. The following is an easy consequence of the fact that every bounded complex admits a morphism from its (shifted) bottom cohomology, and a non-zero map to its (shifted) top cohomology, together with the fact that two objects in a heart admit no negative Exts. A more general version of the following, and proof, can be found in Lemma 4.3.

Lemma 3.2. *If* $x \in \mathcal{C}$, *then the following are equivalent:*

(1) $x \in [a, b]$.

(2) $\mathrm{Hom}(\mathcal{S}, x[i]) = 0$ *for all* $i < a$ *and* $\mathrm{Hom}(x, \mathcal{S}[i]) = 0$ *for all* $i < -b$.

Under these conditions, $x \in [\![a, b]$ *if and only if* $\mathrm{Hom}(\mathcal{S}, x[a]) \neq 0$, *and* $x \in [a, b]\!]$ *if and only if* $\mathrm{Hom}(x, \mathcal{S}[-b]) \neq 0$.

The following lemma is a summary of known results, and is ultimately an easy consequence of the mutation functors being induced by tilting modules of projective dimen-

sion one. The main point is that for atoms α, the functor Φ_α can at worst move cohomology to the right.

Lemma 3.3. *Let $x \in \mathcal{C}_C$ with $x \in [\![a, b]\!]$, and let $\alpha \colon C \to D$ be an atom. Then the following statements hold:*

(1) $\Phi_\alpha \cong \mathbf{R}\mathrm{Hom}(T_\alpha, -)$ *for some* T_α *satisfying* $\mathrm{pd}\, T_\alpha \le 1$.

(2) $\Phi_\alpha(x) \in [a, b+1]$.

(3) *If* $\Phi_\alpha(x) \in [a, b]$, *then* $\Phi_\alpha(x) \in [\![a, b]\!]$.

(4) *If there exists* $\mathcal{S}_i$ *with* $\mathrm{Hom}_{\mathcal{C}}(\mathrm{H}^b x, \mathcal{S}_i) = 0$, *then* $\Phi_i(x) \in [a, b]$.

(5) $s_i \circ \alpha$ *is an atom if and only if* $\Phi_i \circ \Phi_\alpha(\mathcal{S}) := \Phi_{s_i \circ \alpha}(\mathcal{S}) \in [0, 1]$.

(6) *Consider the longest element atom* $\ell \colon C \to \mathsf{op}(C)$. *Then* $\Phi_\ell(x) \in [\![a+1, b+1]\!]$.

Proof. (1) In the flops setting, this is [27, Theorem 4.6] when $\mathcal{X}$ is smooth, and [34, Proposition 9.34] in general. The case of partial resolutions, or the minimal resolution, is covered by [34, Theorem 5.30].

(2) Consider the spectral sequence

$$E_2^{p,q} = \mathrm{H}^p(\Phi_\alpha(\mathrm{H}^q x)) \Rightarrow \mathrm{H}^{p+q}(\Phi_\alpha x).$$

By (1), the only non-zero E_2 terms are $E_2^{0,q}$ and $E_2^{1,p}$. Thus the spectral sequence degenerates immediately, and the statement follows.

(3) If $\mathrm{H}^b(\Phi_\alpha x) = \mathrm{H}^{b+1}(\Phi_\alpha x) = 0$, then in particular the whole top row of the spectral sequence in (2) must be zero, namely $\mathrm{H}^0(\Phi_\alpha(\mathrm{H}^b x)) = \mathrm{H}^1(\Phi_\alpha(\mathrm{H}^b x)) = 0$. It follows that $\Phi_\alpha(\mathrm{H}^b x) = 0$, and so $\mathrm{H}^b x = 0$, contradicting $x \in [\![a, b]\!]$.

(4) As $y = \mathrm{H}^b x$ has finite length, the assumption $\mathrm{Hom}(y, \mathcal{S}_i) = 0$ immediately implies that $\mathrm{H}^j(\Phi_i y) = 0$ for all $j \ne 0$, by, e.g., [48, Corollary 5.10]. Thus $\mathrm{H}^1(\Phi_i(\mathrm{H}^b x)) = 0$, from which the spectral sequence in (2) implies that $\mathrm{H}^{b+1}(\Phi_i x) = 0$ and so $\Phi_i(x) \in [a, b]$.

(5) By [27, Proposition 6.3] in the smooth case, and [34, Proposition 10.17] in general, $s_i \circ \alpha$ is an atom if and only if the maximum p with $\mathrm{Ext}^p(\mathcal{S}, \Phi_{s_i \alpha}(\Lambda)) \ne 0$ is $p = 1 + d$. Since Λ is d-singular CY (see, e.g., [33, §2.4]), this holds if and only if the maximum p for which $\mathrm{Ext}^{d-p}(\Phi_{s_i \alpha}(\Lambda), \mathcal{S}) \cong \mathrm{H}^{d-p}(\Phi_{s_i \alpha}^{-1}\mathcal{S}) \ne 0$ is $p = 1 + d$. In other words, if and only if the minimum q for which $\mathrm{H}^q(\Phi_{s_i \alpha}^{-1}\mathcal{S}) \ne 0$ is $q = -1$, which holds if and only if $\Phi_{s_i \alpha}^{-1}(\mathcal{S}) \in [-1, 0]$. It is clear, using (2), that this is equivalent to $\Phi_{s_i \alpha}(\mathcal{S}) \in [0, 1]$.

(6) This is an easy consequence of Lemma 2.17 (3). ∎

The following technical lemma is a very mild generalisation of [43, Remark 6.8]. A more general statement and proof, which also avoids spectral sequences, is given in Lemma 4.2 later.

Lemma 3.4. *Let $x, y \in \mathcal{C}$ with $x \in [\![a, b]\!]$ and $y \in [c, d]$.*

(1) *If* $\mathrm{Hom}_{\mathcal{C}}(y, x[a - d]) = 0$, *then* $\mathrm{Hom}_{\mathcal{C}}(\mathrm{H}^d y, \mathrm{H}^a x) = 0$.

(2) *If* $x \in [\![a, b]\!]$, $a < b$, $\mathrm{Hom}_{\mathcal{C}}(x, x[i]) = 0$ *for all* $i < 0$, *then* $\mathrm{Hom}_{\mathcal{C}}(\mathrm{H}^b x, \mathrm{H}^a x) = 0$.

Proof. (1) Since $\mathcal{C} \subset \mathsf{D}^b(\mathrm{mod}\,\Lambda)$, by [30, §4.1] there is a spectral sequence

$$\mathrm{E}_2^{p,q} = \bigoplus_i \mathrm{Hom}_{\mathcal{C}}^p(\mathrm{H}^i y, \mathrm{H}^{i+q} x) \Rightarrow \mathrm{Hom}_{\mathcal{C}}^{p+q}(y, x).$$

Since $x \in [\![a, b]\!]$ and $y \in [c, d]\!]$, $\mathrm{E}_2^{p,q} = 0$ for all $q < a - d$. Further, since modules have no negative Exts, $\mathrm{E}_2^{p,q} = 0$ for all $p < 0$. It follows that the bottom left term is $\mathrm{E}_2^{0,a-d} = \mathrm{Hom}_{\mathcal{C}}(\mathrm{H}^d y, \mathrm{H}^a x)$, which survives to give $\mathrm{Hom}_{\mathcal{C}}(y, x[a - d])$. Since this is zero by assumption, necessarily $\mathrm{Hom}_{\mathcal{C}}(\mathrm{H}^d y, \mathrm{H}^a x) = 0$.

(2) This is an immediate consequence of (1). ∎

3.2. Main results

The following is the first main result of this paper. It asserts that any object $x \in \mathcal{C}$ which behaves like it lies in the heart of a t-structure, namely it admits no negative self-extension groups, does lie in the heart of a t-structure.

Theorem 3.5. *Suppose that $x \in \mathcal{C}_C$ satisfies $\mathrm{Hom}_{\mathcal{C}}(x, x[i]) = 0$ for all $i < 0$. Then there exists $\beta \colon C \to D$ such that the corresponding $\Phi_\beta \colon \mathcal{C}_C \to \mathcal{C}_D$ satisfies $\Phi_\beta(x) \in \mathrm{mod}\,\mathrm{A}_{\mathsf{D,con}}$.*

Proof. Since x is a bounded complex, there exist $a \leq b$ such that $x \in [\![a, b]\!]$. If $a = b$, then x is the shift of a module. Consider the longest element atom $\ell \colon C \to \mathrm{op}(C)$, then Lemma 2.17 (3) shows that applying either Φ_ℓ or Φ_ℓ^{-1} repeatedly results in a module, over the relevant $\mathrm{A}_{\mathsf{D,con}}$, in degree zero.

Hence we can assume that $a < b$. The trick is to consider the set

$$\Delta := \{\alpha \mid \alpha \text{ is an atom, and } \Phi_\alpha(x) \in [a, b]\},$$

which in light of Lemma 3.3 (2) is the set of atoms which do not make x homologically worse. Clearly, the trivial atom $C \to C$ belongs to Δ, and the longest atom $\ell \colon C \to \mathrm{op}(C)$ does not belong to Δ since $\Phi_\ell(x) \in [\![a + 1, b + 1]\!]$ by Lemma 3.3 (6).

The set Δ admits a partial order, since the set of all atoms does (see Definition 2.9), and further the set Δ is finite, since the set of all atoms is finite. Every finite poset has at least one maximal element, so consider a maximal element $\beta \in \Delta$. By definition $\Phi_\beta(x) \in [a, b]$, which by Lemma 3.3 (3) forces $\Phi_\beta(x) \in [\![a, b]\!]$.

We claim that $\Phi_\beta(x) \in [\![a + 1, b]\!]$. This implies that β has improved the complex x, which in turn will allow us to induct. Since $\Phi_\beta(x) \in [\![a, b]\!]$, the claim is simply that $\mathrm{H}^a(\Phi_\beta x) = 0$. This will be proved by assuming that $\mathrm{H}^a(\Phi_\beta x) \neq 0$, and deriving a contradiction.

Well, if $\mathrm{H}^a(\Phi_\beta x) \neq 0$, then $\Phi_\beta(x) \in [\![a, b]\!]$ and so by Lemma 3.4 (2)

$$\mathrm{Hom}(\mathrm{H}^b(\Phi_\beta x), \mathrm{H}^a(\Phi_\beta x)) = 0, \tag{3.1}$$

since $a < b$. Since $\mathrm{H}^a(\Phi_\beta x) \neq 0$, pick a simple $\mathcal{S}_i \hookrightarrow \mathrm{H}^a(\Phi_\beta x)$. From (3.1), it follows that

$$\mathrm{Hom}(\mathrm{H}^b(\Phi_\beta x), \mathcal{S}_i) = 0.$$

Applying Lemma 3.3 (4) to $\Phi_\beta(x)$ shows that $\Phi_i \Phi_\beta(x) \in [a, b]$, i.e., $\Phi_{s_i \beta}(x) \in [a, b]$. Now if $\beta = \mathrm{Id}$, then clearly $s_i \circ \beta$ is an atom, and so $\Phi_{s_i \beta}(x) \in [a, b]$ contradicts the maximality of β. Hence we can assume that $\beta \neq \mathrm{Id}$.

Since the longest atom does not belong to Δ, and $\beta \neq \mathrm{Id}$, necessarily $\Phi_\beta(\mathcal{S}) \in [\![0, 1]\!]$. The assumption $x \in [\![a, b]\!]$ implies that

$$\mathrm{Hom}(\Phi_\beta(\mathcal{S}), \Phi_\beta(x)[a-1]) \cong \mathrm{Hom}(\mathcal{S}, x[a-1]) = 0$$

by Lemma 3.2, and so it follows from Lemma 3.4 that

$$0 = \mathrm{Hom}(\mathrm{H}^1(\Phi_\beta \mathcal{S}), \mathrm{H}^a(\Phi_\beta x)). \tag{3.2}$$

The injection $\mathcal{S}_i \hookrightarrow \mathrm{H}^a(\Phi_\beta(x))$ together with (3.2) implies that

$$\mathrm{Hom}(\mathrm{H}^1(\Phi_\beta \mathcal{S}), \mathcal{S}_i) = 0.$$

Applying Lemma 3.3 (4) to $\Phi_\beta(\mathcal{S})$ shows that $\Phi_i \Phi_\beta(\mathcal{S}) \in [0, 1]$, so by Lemma 3.3 (5) $s_i \circ \beta$ is an atom. Again, $\Phi_{s_i \beta}(x) \in [a, b]$ contradicts the maximality of β. Thus

$$\mathrm{H}^a(\Phi_\beta(x)) = 0,$$

as claimed.

We conclude that $\Phi_\beta(x) \in [\![a+1, b]\!]$, and so the complex $\Phi_\beta(x)$ is strictly shorter than that of x. Replacing x by $\Phi_\beta(x)$, we can then repeat the above argument, and by doing this find a composition γ such that $\Phi_\gamma(x)$ lies in precisely one homological degree. As in the first paragraph, repeatedly applying either Φ_ℓ or Φ_ℓ^{-1} results in a module, over the relevant $A_{\mathrm{D,con}}$, in degree zero. ∎

From the viewpoint of our applications in Section 5, the following consequence of Theorem 3.5 is important. The power in the statement, and the reason that the proof is a little involved, is precisely since part (1) considers direct sums. We explain in Remark 3.10 that for the minimal resolution, and other special cases, an easier proof of (2) exists.

Corollary 3.6. *Suppose that $x \in \mathcal{C}_\mathsf{C}$ satisfies $\mathrm{Hom}(x, x[i]) = 0$ for all $i < 0$.*

(1) *If $\dim_\mathbb{C} \mathrm{Hom}(x_i, x_j) \cong \mathbb{C}\delta_{ij}$ for all indecomposable summands x_i, x_j of x, then there exist a subset $\mathcal{I}$ and a path $\gamma \colon \mathsf{C} \to \mathsf{E}$ such that $\Phi_\gamma(x) \cong \bigoplus_{i \in \mathcal{I}} \mathcal{S}_i$.*

(2) *If $\dim_\mathbb{C} \mathrm{Hom}(x, x) \cong \mathbb{C}$, then there exist $\gamma \colon \mathsf{C} \to \mathsf{E}$ and i such that $\Phi_\gamma(x) \cong \mathcal{S}_i$.*

Proof. We prove (1), since (2) is a special case. Consider first the 3-fold setting.

By Theorem 3.5, there exists $\beta \colon \mathsf{C} \to \mathsf{D}$ such that $\Phi_\beta(x) \cong y$, where y is a $\Lambda_{\mathrm{D,con}}$-module in homological degree zero. Now by assumption, x is a direct sum of Hom-orthogonal indecomposable summands, hence so too is y. Thus in the language of Section 6.1, y is a semibrick in the category $\mathrm{mod}\,\Lambda_{\mathrm{D,con}}$. Since $\Lambda_{\mathrm{D,con}}$ is silting discrete [7, Theorem 3.12], by [6] y is a subset of a 2-simple-minded collection, which by definition and the Koenig–Yang bijections implies that y is a subset of simples in the heart $\mathcal{H}$ of an intermediate bounded t-structure in $\mathrm{D}^b(\mathrm{mod}\,\Lambda_{\mathrm{D,con}})$.

Now by [8, Theorem 1.1], for every atom $\alpha\colon \mathsf{D} \to \mathsf{E}$ there exists an autoequivalence F_α such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{D}^b(\operatorname{mod}\Lambda_{\mathsf{D,con}}) & \xrightarrow{\ \mathrm{res}\ } & \mathrm{D}^b(\operatorname{mod}\Lambda_{\mathsf{D}}) \\
F_\alpha \downarrow & & \downarrow \Phi_\alpha \\
\mathrm{D}^b(\operatorname{mod}\Lambda_{\mathsf{E,con}}) & \xrightarrow{\ \mathrm{res}\ } & \mathrm{D}^b(\operatorname{mod}\Lambda_{\mathsf{E}}),
\end{array}
\tag{3.3}
$$

and further by [8], the heart of every intermediate bounded t-structure in $\mathrm{D}^b(\operatorname{mod}\Lambda_{\mathsf{D,con}})$ equals $F_\alpha^{-1}(\operatorname{mod}\Lambda_{\mathsf{E,con}})$ for some atom $\alpha\colon \mathsf{D} \to \mathsf{E}$.

Thus there exists a subset $\mathcal{J}$ such that $F_\alpha^{-1}(\bigoplus_{i\in\mathcal{J}}\mathcal{S}_{i,\mathsf{E}}) = y$. Since diagram (3.3) commutes, necessarily $\Phi_\alpha(y) \cong \bigoplus_{i\in\mathcal{J}}\mathcal{S}_{i,\mathsf{E}}$, and so the path $\alpha \circ \beta\colon \mathsf{C} \to \mathsf{E}$ satisfies

$$
\Phi_{\alpha\circ\beta}(x) = \Phi_\alpha\Phi_\beta(x) = \Phi_\alpha(y) = \bigoplus_{i\in\mathcal{J}}\mathcal{S}_{i,\mathsf{E}},
$$

as required. This completes the proof in the 3-fold setting.

Consider now the surfaces setting. By Theorem 3.5, there exists $\beta\colon \mathsf{C} \to \mathsf{D}$ such that $\Phi_\beta(x) \cong y$, where y is an $(e_\mathcal{J}\Pi e_\mathcal{J})_{\mathsf{con}}$-module in homological degree zero. Now $\mathcal{C}_{\mathsf{D}} \subset \mathrm{D}^b(\operatorname{mod} e_\mathcal{J}\Pi e_\mathcal{J})$ say, where Π is the preprojective algebra of affine ADE, and $e_\mathcal{J}$ is some idempotent. As in [45, §3] or [36, §4.3], we can construct a 3-fold flopping contraction $\mathcal{X} \to \operatorname{Spec}\mathcal{R}$, where $\mathcal{X}$ has Gorenstein terminal singularities (where $\mathcal{X}$ tilts to Λ_{D}), such that the generic hyperplane section $g \in \mathcal{R}$ satisfies

$$
(\Lambda_{\mathsf{D}})/g \cong e_\mathcal{J}\Pi e_\mathcal{J}.
$$

Furthermore, for any atom $\alpha\colon \mathsf{D} \to \mathsf{E}$, [34] shows that the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{D}^b(\operatorname{mod} e_\mathcal{J}\Pi e_\mathcal{J}) & \xrightarrow{\ \mathrm{res}\ } & \mathrm{D}^b(\operatorname{mod}\Lambda_{\mathsf{D}}) \\
\Phi_\alpha \downarrow & & \downarrow \Phi_\alpha \\
\mathrm{D}^b(\operatorname{mod} e_\mathcal{K}\Pi e_\mathcal{K}) & \xrightarrow{\ \mathrm{res}\ } & \mathrm{D}^b(\operatorname{mod}\Lambda_{\mathsf{E}}).
\end{array}
$$

Since y is a module in homological degree zero, $\operatorname{res}(y) \in \mathcal{C}_{\mathsf{D}}$ is likewise still a module, and thus it has no negative Exts. Further, by adjunction

$$
\operatorname{Hom}_{\mathcal{C}}(\operatorname{res}(y_i), \operatorname{res}(y_j)) \cong \mathbb{C}\delta_{ij}
$$

for all indecomposable summands y_i, y_j, and thus $\operatorname{res}(y)$ is a semibrick. The 3-fold proof above shows that there exists some atom α such that

$$
\Phi_\alpha(\operatorname{res}(y)) \cong \mathcal{U},
$$

where $\mathcal{U} = \bigoplus_{i\in\mathcal{J}}\mathcal{S}_i$ for some subset $\mathcal{J}$. The commutativity of the diagram implies that $\operatorname{res}(\Phi_\alpha(y)) \cong \mathcal{U}$. But also $\operatorname{res}(\mathcal{U}) = \mathcal{U}$, so since res reflects isomorphisms when restricted to the module category, $\Phi_\alpha(y) \cong \mathcal{U}$ follows. ∎

3.3. Algebraic corollaries

It is a result of Crawley–Boevey [19, Lemma 1] that every brick for the preprojective algebra Π of type ADE has dimension vector equal to a root of the corresponding Dynkin diagram. This subsection generalises this to contracted preprojective algebras $\Gamma = e_{\mathcal{J}}\Pi e_{\mathcal{J}}$, and also to contraction algebras Λ_{con}.

Recall the notion of restricted roots from Section 2.2, and the conventions in Remark 2.1 and (2.2). The following is then a consequence of Corollary 3.6.

Corollary 3.7. *Consider the projective algebra Π of type ADE and $\mathcal{J} \subset \Delta$. Then every brick in* $\mathrm{mod}\, e_{\mathcal{J}}\Pi e_{\mathcal{J}}$ *has dimension vector equal to a primitive restricted root.*

Proof. If $x \in \mathrm{mod}\,\Lambda_{\mathrm{con}}$ is a brick, then by Corollary 3.6 there is β such that $\Phi_\beta(\mathcal{S}_i) \cong x$. Writing φ_β for the action of Φ_β on $K_0(\mathcal{C})$, it follows that $\varphi_\beta(\mathbf{e}_i) = \underline{\dim}\, x$. But, as explained in, e.g., [37, §5.1, Remark 5.12], φ_β can be written as a composition of matrices N_i, each of which takes primitive restricted roots to primitive restricted roots. Since $\mathbf{e}_i$ is clearly a primitive restricted root, so too is $\underline{\dim}\, x$. ∎

Remark 3.8. It should also be possible to prove Corollary 3.7 using the fact that c-vectors are the dimension vectors of bricks, and these are dual to g-vectors [46]. The g-vectors for some $e_{\mathcal{J}}\Pi e_{\mathcal{J}}$ are known [34, §6], but these results currently require some restrictions on $\mathcal{J}$.

Recall that to every contraction algebra Λ_{con}, there exists a central element g in the radical of Λ_{con} such that $\Lambda_{\mathrm{con}}/(g) \cong e_{\mathcal{J}}\Pi e_{\mathcal{J}}$ for some $\mathcal{J} \subset \Delta$, where Δ is ADE [23, 24].

Corollary 3.9. *If Λ_{con} is the contraction algebra associated to a 3-fold flopping contraction $\mathcal{X} \to \mathrm{Spec}\,\mathcal{R}$ where $\mathcal{X}$ is at worst Gorenstein terminal, then every brick in* $\mathrm{mod}\,\Lambda_{\mathrm{con}}$ *has dimension vector equal to a primitive restricted root.*

Proof. This is an immediate consequence of Corollary 3.7, since by the standard reduction theorems [25, §4] and [6, Proposition 2.29], bricks on Λ_{con} are the same as bricks for $\Lambda_{\mathrm{con}}/(g)$. ∎

Remark 3.10. The proof of Corollary 3.6 (2), but alas not Corollary 3.6 (1), can be simplified in the case of the minimal resolution, or in the case of 3-fold crepant resolutions which slice to the minimal resolution. Indeed, in the case of the minimal resolution Crawley–Boevey [19, Lemma 1] implies that every brick x has dimension vector equal to a real root. Hence there exists a sequence of Weyl reflections such that $s_{i_t} \cdots s_{i_1} \cdot \underline{\dim}(x) = \mathbf{e}_i$ for some i.

Now, since x is a brick, either $\mathrm{Hom}(x, \mathcal{S}_{i_1}) = 0$ or $\mathrm{Hom}(\mathcal{S}_{i_1}, x) = 0$. Set $F_1 := \Phi_{i_1}$ in the first case, or $F_1 := \Phi_{i_1}^{-1}$ in the second, then $F_1(x)$ is a module, of dimension vector $s_{i_1} \cdot \underline{\dim}(x)$. Clearly, $F_1(x)$ is still a brick, so we can repeat the argument. At each stage, we can choose F_j equal to either Φ_{i_j} or $\Phi_{i_j}^{-1}$ to guarantee that $F_j \cdots F_1(x)$ remains a module, and by doing this $F_t \cdots F_1(x)$ is a module of dimension vector equal to $\mathbf{e}_i$, a simple root. Thus $F_t \cdots F_1(x) \cong \mathcal{S}_i$. If the analogue of Crawley–Boevey's result Corollary 3.7

can be proved first, then the above argument simplifies the proof of Corollary 3.6 (2) in all cases.

4. Classification of t-structures

The previous section proved that for every object $x \in \mathcal{C}$ with no negative Exts, there exists Φ_β such that $\Phi_\beta(x)$ is concentrated in homological degree zero. In particular, $x \in \mathcal{C}$ belongs to a standard heart in the groupoid. A second and related problem is whether, given the heart $\mathcal{H}$ of an arbitrary bounded t-structure, there exists Φ_γ such that $\Phi_\gamma(\mathcal{H})$ is a standard heart. Whilst the problems are similar, one does not obviously imply the other. A priori, the heart $\mathcal{H}$ does not have finite length, and so the element-wise arguments of the previous section do not apply.

This section strengthens the results of the previous section to also solve the second problem, and thus fully classifies all bounded t-structures on $\mathcal{C}$ in terms of the orbits under the groupoid action. This is new in all cases, even for the minimal resolution $\mathcal{Z} \to \mathbb{C}^2/\Gamma$.

4.1. t-structure generalities

Let $\mathcal{T}$ be a triangulated category with full additive subcategory $\mathcal{H}$. Then, as explained, e.g., in [15, Lemma 2.3], $\mathcal{H}$ is the heart of a bounded t-structure on $\mathcal{T}$ if and only if

(1) $\mathrm{Hom}_\mathcal{T}(x, y[j]) = 0$ for all $x, y \in \mathcal{H}$ and all $j < 0$.

(2) For every non-zero object $x \in \mathcal{T}$, there exist integers $a \leq b$ and triangles

$$0 = x_{a-1} \longrightarrow x_a \longrightarrow x_{a+1} \longrightarrow \cdots \longrightarrow x_{b-1} \longrightarrow x_b = x$$

$$h_a \qquad\qquad h_{a+1} \qquad\quad h_{a+2} \qquad\qquad\qquad h_b$$

such that $h_i[i] \in \mathcal{H}$ for all i.

As is standard, the objects $h_i[i] \in \mathcal{H}$ are called the cohomology objects of x with respect to $\mathcal{H}$, and will be denoted $\mathrm{H}^i_{\mathcal{H}}(x) := h_i[i]$. The following mirrors Notation 3.1.

Notation 4.1. Let $\mathcal{H}$ be the heart of a bounded t-structure in a triangulated category $\mathcal{T}$. For integers $c \leq d$, write $x \in [c, d]_{\mathcal{H}}$ to mean $\mathrm{H}^i_{\mathcal{H}}(x) = 0$ for all $i < c$ and for all $i > d$. Write $x \in [\![c, d]\!]_{\mathcal{H}}$ if $x \in [c, d]_{\mathcal{H}}$ and furthermore $\mathrm{H}^c_{\mathcal{H}}(x) \neq 0$.

As in Section 3.1, we will use other self-documenting variations, such as $[c, d[\![_{\mathcal{H}}$ and $[\![c, d[\![_{\mathcal{H}}$. The following generalises Lemma 3.4, and satisfyingly its proof avoids the use of spectral sequences.

Lemma 4.2. *Let $\mathcal{H}$ be the heart of a bounded t-structure in a triangulated category $\mathcal{T}$. If $x, y \in \mathcal{T}$ with $x \in [\![a, b[\![_{\mathcal{H}}$ and $y \in [\![c, d]\!]_{\mathcal{H}}$, then*

$$\mathrm{Hom}_\mathcal{T}(y, x[a - d]) \cong \mathrm{Hom}_\mathcal{T}(\mathrm{H}^d_{\mathcal{H}}(y), \mathrm{H}^a_{\mathcal{H}}(x)).$$

Proof. We prove the case when $a < b$ and $c < d$, since the other cases are degenerate. To ease notation, set $H^i = H^i_{\mathcal{H}}$, and consider the sequence of triangles (2) for x, and the analogous sequence

$$0 = y_{c-1} \longrightarrow y_c \longrightarrow y_{c+1} \longrightarrow \cdots \longrightarrow y_{d-1} \longrightarrow y_d = y \tag{4.1}$$

with h'_c, h'_{c+1}, h'_{c+2}, ..., h'_d as the connecting maps.

for y. Using the vanishing of negative Ext groups (1), applying $\mathrm{Hom}_{\mathcal{T}}(-, \mathcal{H})$, it is easy to verify that

$$\mathrm{Hom}_{\mathcal{T}}(x_t, \mathcal{H}[-t-j]) = 0 = \mathrm{Hom}_{\mathcal{T}}(y_t, \mathcal{H}[-t-j]) \quad \text{for all } j \geq 1. \tag{4.2}$$

Now applying $\mathrm{Hom}_{\mathcal{T}}(y[d], -)$ to the sequence of triangles (2) for x, and temporarily dropping Hom from the notation, gives the following exact sequences:

$$
\begin{array}{l}
\mathcal{T}(y[d], h_b[a-1]) \longrightarrow \mathcal{T}(y[d], x_{b-1}[a]) \longrightarrow \mathcal{T}(y[d], x[a]) \longrightarrow \mathcal{T}(y[d], h_b[a]), \\
\mathcal{T}(y[d], h_{b-1}[a-1]) \longrightarrow \mathcal{T}(y[d], x_{b-2}[a]) \longrightarrow \mathcal{T}(y[d], x_{b-1}[a]) \longrightarrow \mathcal{T}(y[d], h_{b-1}[a]), \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
\mathcal{T}(y[d], h_{a+2}[a-1]) \longrightarrow \mathcal{T}(y[d], x_{a+1}[a]) \longrightarrow \mathcal{T}(y[d], x_{a+2}[a]) \longrightarrow \mathcal{T}(y[d], h_{a+2}[a]), \\
\mathcal{T}(y[d], h_{a+1}[a-1]) \longrightarrow \mathcal{T}(y[d], x_a[a]) \longrightarrow \mathcal{T}(y[d], x_{a+1}[a]) \longrightarrow \mathcal{T}(y[d], h_{a+1}[a]).
\end{array}
$$

Since $y = y_d$, it follows from (4.2) that all Hom sets within the dotted boxes are zero. Hence, since further $x_a \cong h_a$, there are thus isomorphisms

$$\mathrm{Hom}_{\mathcal{T}}(y[d], x[a]) \cong \cdots \cong \mathrm{Hom}_{\mathcal{T}}(y[d], x_a[a]) \cong \mathrm{Hom}_{\mathcal{T}}(y[d], H^a(x)).$$

Now applying $\mathrm{Hom}_{\mathcal{T}}(-, H^a(x))$ to the rightmost triangle in (4.1) gives an exact sequence

$$\mathcal{T}(y_{d-1}[d+1], H^a(x)) \to \mathcal{T}(H^d(y), H^a(x)) \to \mathcal{T}(y[d], H^a(x)) \to \mathcal{T}(y_{d-1}[d], H^a(x)),$$

where the outer two sets are again zero by (4.2). It follows that

$$\mathrm{Hom}_{\mathcal{T}}(y[d], x[a]) \cong \mathrm{Hom}_{\mathcal{T}}(y[d], H^a(x)) \cong \mathrm{Hom}_{\mathcal{T}}(H^d(y), H^a(x)). \qquad \blacksquare$$

It is also possible to detect membership of $[a, b]_{\mathcal{A}}$ using Hom vanishings. The following result is the more general version of Lemma 3.2.

Lemma 4.3. *Let $\mathcal{A}$ be the heart of a bounded t-structure, $x \in \mathcal{T}$ an object, and let $a \leq b$ be integers. Then the following are equivalent:*

(1) $x \in [a, b]_{\mathcal{A}}$.

(2) $\mathrm{Hom}(\mathcal{A}, x[i]) = 0$ *for all $i < a$ and $\mathrm{Hom}(x, \mathcal{A}[i]) = 0$ for all $i < -b$.*

Furthermore, $x \in [\![a, b]$ if and only if $\mathrm{Hom}(\mathcal{A}, x[a]) \neq 0$, and $x \in [a, b]\!]$ if and only if $\mathrm{Hom}(x, \mathcal{A}[-b]) \neq 0$.

Proof. If $x \in [\![c, d]\!]$, then there is an exact triangle

$$\tau_{\leq 0}(x[c]) \to x[c] \to \tau_{>0}(x[c]) \to .$$

Now $\mathrm{H}^c(x) = (\tau_{\leq 0}\tau_{\geq 0})(x[c])$, so since $x[c] = \tau_{\geq 0}(x[-c])$ by the assumption $x \in [\![c, d]\!]$, the above triangle gives a non-zero morphism $\mathrm{H}^c(x) \to x[c]$. In a similar way, there exists non-zero map $x \to \mathrm{H}_{\mathcal{A}}^d(x)[-d]$.

(2) $\Rightarrow$ (1) The above paragraph shows that if $\mathrm{Hom}(\mathcal{A}, x[i]) = 0$ for $i < a$, then the lowest non-zero cohomology of x must be $\geq a$. Similarly, if $\mathrm{Hom}(x, \mathcal{A}[i]) = 0$ for $i < -b$ hold, then the highest non-zero cohomology of x must be $\leq b$.

(1) $\Rightarrow$ (2) Conversely, let $x \in \mathcal{T}$ and assume that $x \in [\![c, d]\!]_{\mathcal{A}}$ for some $a \leq c \leq d \leq b$. To prove the vanishings, we use induction on $d - c$. If $d = c$, then x is an object in $\mathcal{A}$ up to shift, and the vanishings are trivial. When $d - c > 0$, consider the triangle

$$\mathrm{H}_{\mathcal{A}}^c(x)[-c] \to x \to x' \to \mathrm{H}_{\mathcal{A}}^c(x)[-c + 1]$$

with $x' \in [c + 1, d]\!]_{\mathcal{A}}$. Applying the inductive hypothesis to x' implies that for all $i < a(\leq c)$,

$$\mathrm{Hom}(\mathcal{A}, x[i]) \cong \mathrm{Hom}(\mathcal{A}, \mathrm{H}_{\mathcal{A}}^c(x)[-c + i]) = 0,$$

which proves one of the desired vanishings. The other vanishing is similar.

The final statement follows from the first paragraph. ∎

By a slight abuse of notation, if $\mathcal{A}$ and $\mathcal{H}$ are the hearts of two bounded t-structures, write $\mathcal{H} \in [a, b]_{\mathcal{A}}$ if $x \in [a, b]_{\mathcal{A}}$ for all $x \in \mathcal{H}$.

Corollary 4.4. *Let $\mathcal{A}$ and $\mathcal{H}$ be the hearts of two bounded t-structures, and let $a \leq b$ be integers. Then the following are equivalent:*

(1) $\mathcal{H} \in [a, b]_{\mathcal{A}}$.

(2) $\mathrm{Hom}(\mathcal{A}, \mathcal{H}[i]) = 0$ *for all* $i < a$ *and* $\mathrm{Hom}(\mathcal{H}, \mathcal{A}[i]) = 0$ *for all* $i < -b$.

In particular, $\mathcal{H} \in [a, b]_{\mathcal{A}}$ if and only if $\mathcal{A} \in [-b, -a]_{\mathcal{H}}$.

Proof. (1) $\Leftrightarrow$ (2) follows by applying Lemma 4.3 to all objects x of $\mathcal{H}$. The final statement follows from the symmetry of $\mathcal{A}$ and $\mathcal{H}$ in condition (2). ∎

Corollary 4.5. *Let $\mathcal{A}$ and $\mathcal{H}$ be the hearts of two bounded t-structures, and suppose that $\mathcal{A}$ is a length heart with finitely many simples, where $\mathcal{S}$ is their direct sum. Then there exist integers $a \leq b$ such that $\mathcal{H} \in [a, b]_{\mathcal{A}}$.*

Proof. The boundedness of the t-structure giving $\mathcal{H}$ implies that the cohomology of the single object $\mathcal{S}$ with respect to $\mathcal{H}$ is bounded, and hence $\mathcal{S} \in [c, d]_{\mathcal{H}}$ for some $c \leq d$. Since $\mathcal{A}$ is the extension closure of $\mathcal{S}$ in $\mathcal{T}$, the above bound on $\mathcal{S}$ implies that

$$\mathcal{A} \in [c, d]_{\mathcal{H}}.$$

By Corollary 4.4, this is equivalent to $\mathcal{H} \in [-d, -c]_{\mathcal{A}}$. ∎

4.2. Main results

The above results allow us to easily extend Theorem 3.5. The proof of the following reverts to the conventions of Section 3.1, namely that unadorned $[a, b]$ are by default with respect to cohomology of the standard t-structure on $\mathcal{C}_C$.

Theorem 4.6. *If $\mathcal{H}$ is the heart of a bounded t-structure on $\mathcal{C}_C$, then there exists $\beta \colon C \to D$ such that $\Phi_\beta(\mathcal{H})$ is the standard heart on $\mathcal{C}_D$. In particular, $\mathcal{H}$ is a finite length category, with finitely many simples.*

Proof. Since the standard heart on $\mathcal{C}_C$ is finite length with finitely many simples, by Corollary 4.5, there exist integers $a \leq b$ such that $\mathcal{H} \in [\![a, b]\!]$. If $a = b$, then $\mathcal{H}$ is contained in a shifted standard heart, and thus it equals that shifted standard heart since if the heart of a bounded t-structure includes inside another, they must be equal. The result then follows from Lemma 3.3 (6), since applying either Φ_ℓ or Φ_ℓ^{-1} repeatedly takes this to the standard heart in degree zero.

Now assume $a < b$ and consider the set

$$\Delta := \{\alpha \mid \alpha \text{ is an atom, and } \Phi_\alpha(\mathcal{H}) \in [a, b]\}.$$

Since Δ contains the trivial atom, this is a non-empty partially ordered finite set, and hence there exists a maximal element $\beta \in \Delta$. For this β, we claim that $\Phi_\beta(\mathcal{H}) \in [a + 1, b]$. We assume the contrary, namely there exists an object $x \in \mathcal{H}$ such that $\Phi_\beta(x) \in [\![a, b]\!]$, and will derive a contradiction. Since $H^a(\Phi_\beta x) \neq 0$, fix a simple $\mathcal{S}_i \hookrightarrow H^a(\Phi_\beta x)$.

Let $y \in \mathcal{H}$ be any object, then we claim that $\Phi_i \Phi_\beta(y) \in [a, b]$. If $\Phi_\beta(y) \in [a, b-1]$, then this follows from Lemma 3.3 (2), hence assume that $\Phi_\beta(y) \in [\![a, b]\!]$, so $H^b(\Phi_\beta y) \neq 0$. Since x and y are objects of the heart $\mathcal{H}$, the vanishing $\text{Hom}(\Phi_\beta y, \Phi_\beta x[j]) = 0$ holds for all $j < 0$. Thus it follows from Lemma 3.4 (or Lemma 4.2) that

$$0 = \text{Hom}(H^b(\Phi_\beta y), H^a(\Phi_\beta x)). \tag{4.3}$$

The injection $\mathcal{S}_i \hookrightarrow H^a(\Phi_\beta(x))$ together with (4.3) implies that

$$\text{Hom}(H^b(\Phi_\beta y), \mathcal{S}_i) = 0.$$

Applying Lemma 3.3 (5) to $\Phi_\beta(y)$ gives $\Phi_i \Phi_\beta(y) \in [a, b]$. Since this holds for all $y \in \mathcal{H}$, it follows that $\Phi_i \Phi_\beta(\mathcal{H}) \in [a, b]$.

Since $\Phi_\beta(x) \in [\![a, b]\!]$, exactly the same argument as in the proof of Theorem 3.5 shows that $s_i \circ \beta$ is an atom, and thus $s_i \circ \beta$ remains an element of Δ. This is the desired contradiction, as β is maximal in Δ, and thus $\Phi_\beta(\mathcal{H}) \in [a + 1, b]$ holds, as claimed.

Repeating the above argument, there exists $\beta \colon C \to D$ such that $\Phi_\alpha(\mathcal{H})$ is contained in a shifted standard heart. As in the first paragraph, applying either Φ_ℓ or Φ_ℓ^{-1} repeatedly takes this to degree zero. ∎

Remark 4.7. It is also possible to prove Theorem 4.6 by adapting the proof of [39, Lemma 9]. The main benefit of the cohomology argument here is that it is uniform:

the same technique that works for a single brick also works for the heart of a bounded t-structure.

The key fact required to adapt the proof of [39, Lemma 9] to the geometric setting $\mathcal{C}$ is that *all* standard hearts from the groupoid (and thus all the hearts that appear in the [39] induction) admit only finitely many torsion theories. In the 3-fold setting all contraction algebras are silting discrete, so the fact there are only finitely many torsion theories is the last statement in [22, Theorem 3.8]. In the surfaces setting(s), by Asai [6, Proposition 2.29] bricks modules on $e_\mathcal{J}\Pi e_\mathcal{J}$ are the same as on Λ_{con}, and so in particular there are only finitely many. Thus by [22, Theorem 1.4], each $e_\mathcal{J}\Pi e_\mathcal{J}$ is τ-tilting finite, and so by [22, Theorem 3.8] each mod $e_\mathcal{J}\Pi e_\mathcal{J}$ has only finitely many torsion theories. See also [2].

5. Geometric corollaries

This section translates the algebraic results of Sections 3 and 4 into geometric language, then uses these results to create new geometric corollaries.

5.1. Surfaces

For the minimal resolution $\mathcal{Z} \to \mathbb{C}^2/\Gamma$, the braid group Br acts on $\mathcal{C}$ defined in (1.1), generated by spherical twists in the objects $\mathcal{O}_{C_1}(-1), \ldots, \mathcal{O}_{C_n}(-1)$.

Corollary 5.1. *Consider* $\mathcal{Z} \to \mathbb{C}^2/\Gamma$, *and let* $x \in \mathcal{C}$. *Then the following hold:*

(1) *If* $\mathrm{Hom}_\mathcal{C}(x, x[j]) = 0$ *for all* $j < 0$, *then there exists* $T \in \mathsf{Br}$ *such that* $T(x)$ *is concentrated in homological degree zero.*

(2) *Every spherical object in* $\mathcal{C}$ *belongs to the orbit, under the action of the braid group, of the objects* $\mathcal{O}_{C_1}(-1), \ldots, \mathcal{O}_{C_n}(-1)$.

Furthermore, the heart of every bounded t-structure on $\mathcal{C}$ *is the image, under the action of the group* Br, *of the module category of the preprojective algebra of (finite) ADE type.*

Proof. Under the derived equivalence $\Psi_\mathcal{Z}$ in (2.1), the geometric category $\mathcal{C}$ defined in (1.1) corresponds to the $\mathcal{C}_\mathbb{C}$ in Section 2.4. Furthermore, the mutation functors are functorially isomorphic to spherical twist, via the following commutative diagram:

$$
\begin{array}{ccc}
\mathrm{D}^b(\mathrm{coh}\,\mathcal{Z}) & \xrightarrow{\ \Psi_\mathcal{Z}\ } & \mathrm{D}^b(\mathrm{mod}\,\Pi) \\
{\scriptstyle t_i}\downarrow & & \downarrow{\scriptstyle \Phi_i} \\
\mathrm{D}^b(\mathrm{coh}\,\mathcal{Z}) & \xrightarrow{\ \Psi_\mathcal{Z}\ } & \mathrm{D}^b(\mathrm{mod}\,\Pi).
\end{array}
$$

In this setting, all the categories $\mathcal{C}_\mathsf{C}$, as C varies, are equal. Further, all functors in the above diagram restrict to the relevant $\mathcal{C}$, respectively $\mathcal{C}_\mathsf{C}$. Part (1) then follows from Theorem 3.5, part (2) by Corollary 3.6 (2) since $\mathcal{S}_i$ corresponds to $\mathcal{O}_{C_i}(-1)$, and the final statement is Theorem 4.6. ∎

The following result is slightly more difficult to state, firstly due to the non-uniqueness of partial resolutions, and secondly since unlike the case of $\mathcal{Z}$ or $\mathcal{X}$, the functor $\Psi_{\mathcal{Y}'}^{-1}\Phi_{\beta}\Psi_{\mathcal{Y}}$ below does not yet have a good geometric interpretation. We further remark that in general, the following is the best possible, but afterwards in Examples 5.3 and 5.4 we illustrate how in some situations the result can be improved.

Corollary 5.2. *Consider a partial crepant resolution $\mathcal{Y} \to \operatorname{Spec}\mathbb{C}^2/\Gamma$, and let $x \in \mathcal{C}$.*

(1) *The following statements are equivalent:*

   (a) $\operatorname{Hom}_{\mathcal{C}}(x, x[j]) = 0$ *for all $j < 0$.*

   (b) *There exist β and a partial resolution $\mathcal{Y}' \to \mathbb{C}^2/\Gamma$ obtained from $\mathcal{Y}$ by iterated wall crossing, such that $\Psi_{\mathcal{Y}'}^{-1}\Phi_{\beta}\Psi_{\mathcal{Y}}(x)$ belongs to perverse sheaves.*

(2) *Furthermore, the following are equivalent.*

   (a) $\operatorname{Hom}_{\mathcal{C}}(x, x[j]) = 0$ *for all $j < 0$ and $\operatorname{Hom}_{\mathcal{C}}(x, x) \cong \mathbb{C}$.*

   (b) *There exist β, a partial resolution $\mathcal{Y}' \to \mathbb{C}^2/\Gamma$ obtained from $\mathcal{Y}$ by iterated wall crossing, and some $\mathsf{C}_i \subset \mathcal{Y}'$ such that*

$$\Psi_{\mathcal{Y}'}^{-1}\Phi_{\beta}\Psi_{\mathcal{Y}}(x) \cong \mathcal{O}_{\mathsf{C}_i}(-1).$$

Proof. (1) is an immediate consequence of Theorem 3.5, and (2) follows from Corollary 3.6 (2) since $\mathcal{S}_i$ corresponds to $\mathcal{O}_{\mathsf{C}_i}(-1)$. ∎

Example 5.3. Consider the partial resolution $\mathcal{Y} \to \mathbb{C}^2/D_4$ given by the Dynkin data ●●●, where $\mathcal{I}$ is the set of black nodes. That is, $\mathcal{Y}$ is obtained from the D_4 minimal resolution by contracting the two curves corresponding to the black nodes. The following illustrates the associated intersection arrangement $\mathcal{H}$:

	Restricted root
10	—
11	—
12	—
01	│
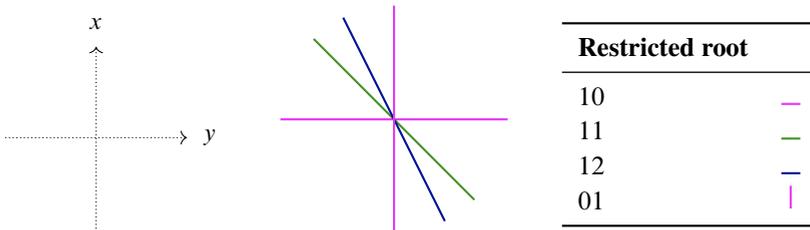
In this example, the wall crossing rule of [34, Definition 1.16] returns the same indexing set ●●● in each chamber and thus in Section 2.4 the *identical* category $\mathcal{C}_C$ gets assigned to each chamber. Further, each wall crossing is a twist autoequivalence [34, Proposition 10.5] over noncommutative deformations of the corresponding curve. Thus, as in the case of the minimal resolution, the categories can be identified and thus there is a braid group action of B_2 on $D^b(\operatorname{coh}\mathcal{Y})$. The result of Corollary 5.2 (1) then translates (since always $\mathcal{Y}' = \mathcal{Y}$) to give the analogue of Corollary 5.1 (1) for the braid group of B_2, and Corollary 5.2 (2) gives the analogue of Corollary 5.1 (2).

Example 5.4. Continuing Example 2.2, for the Dynkin data ●●●● the intersection arrangement for $\mathcal{Y} \to \mathbb{C}^2/D_5$ is given in Example 2.3.

Now the wall crossing rule of [34, Definition 1.16] returns different indexing sets to the chambers as follows:
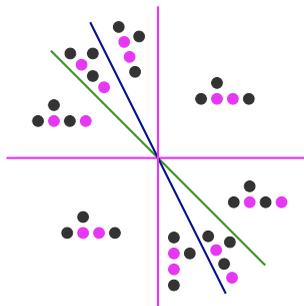
Thus, as the chambers vary, the categories $\mathcal{C}_C$ are sometimes equal, and sometimes are not. This translates into a mixed braid group action on $D^b(\text{coh }\mathcal{Y})$.

5.2. 3-folds

In the 3-fold setting of flopping contractions $\mathcal{X} \to \text{Spec }\mathcal{R}$, our main interest is in establishing Corollary 5.6 below. However, there seems to be no method of proving that without taking square roots, and first establishing the following result.

Corollary 5.5. *Let* $\mathcal{X} \to \text{Spec }\mathcal{R}$ *be a 3-fold flopping contraction, where* $\mathcal{X}$ *has at worst Gorenstein terminal singularities, and consider* $x \in \mathcal{C}$.

(1) *The following statements are equivalent:*

    (a) $\text{Hom}_\mathcal{C}(x, x[j]) = 0$ *for all* $j < 0$.

    (b) *There exists a sequence of flop functors such that* $\text{Flop}_\beta(x)$ *belongs to perverse sheaves, on a possibly different* $\mathcal{X}^+ \to \text{Spec }\mathcal{R}$.

(2) *Furthermore, the following are equivalent:*

    (a) $\text{Hom}_\mathcal{C}(x, x[j]) = 0$ *for all* $j < 0$ *and* $\text{Hom}_\mathcal{C}(x, x) \cong \mathbb{C}$.

    (b) *There exists* β *such that*

$$\text{Flop}_\beta(x) \cong \mathcal{O}_{C_i}(-1)$$

    for some curve C_i *on a possibly different* $\mathcal{X}^+ \to \text{Spec }\mathcal{R}$.

Furthermore, the heart of every bounded t-structure on $\mathcal{C}$ *is the image, under the action of the group generated by the flop functors, of the module category of the contraction algebra of some* $\mathcal{X}^+ \to \text{Spec }\mathcal{R}$ *obtained from* $\mathcal{X} \to \text{Spec }\mathcal{R}$ *by iterated flop.*

Proof. Under the derived equivalence $\Psi_\mathcal{X}$ in (2.1), the geometric category $\mathcal{C}$ defined in (1.1) corresponds to the $\mathcal{C}_C$ in Section 2.4. Furthermore, by [48, Theorem 4.2], the

mutation functors are functorially isomorphic to inverse of the flop functor, via the following commutative diagram:

$$
\begin{array}{ccc}
\mathrm{D}^b(\mathrm{coh}\,\mathcal{X}) & \xrightarrow{\ \Psi_{\mathcal{X}}\ } & \mathrm{D}^b(\mathrm{mod}\,\Lambda) \\
{\scriptstyle \mathrm{Flop}_i^{-1}}\Big\downarrow & & \Big\downarrow{\scriptstyle \Phi_i} \\
\mathrm{D}^b(\mathrm{coh}\,\mathcal{X}_i^+) & \xrightarrow{\ \Psi_{\mathcal{X}_i^+}\ } & \mathrm{D}^b(\mathrm{mod}\,\nu_i\Lambda).
\end{array}
$$

Part (1) then follows from Theorem 3.5, part (2) by Corollary 3.6 (2) since $\mathcal{S}_i$ corresponds to $\mathcal{O}_{C_i}(-1)$, and the final statement is Theorem 4.6. ∎

In terms of applications, fixing a category $\mathcal{C}$, it is desirable to know for example the spherical objects under the action of the autoequivalence group. The following generalises [43, Corollary 6.13 (2)] to all 3-fold flops.

Corollary 5.6. *Let* $\mathcal{X} \to \mathrm{Spec}\,\mathcal{R}$ *be a 3-fold flopping contraction, where* $\mathcal{X}$ *is at worst Gorenstein terminal, with contraction algebra* Λ_{con}. *For* $x \in \mathcal{C}$, *the following are equivalent:*

(1) $\mathrm{Hom}_{\mathcal{C}}(x, x[i]) = 0$ *for all* $i < 0$ *and* $\mathrm{Hom}_{\mathcal{C}}(x, x) \cong \mathbb{C}$.

(2) *Under the action of* PBr *on* $\mathcal{C}$, x *is in the orbit of the (finitely many) brick* Λ_{con}-*modules, or their shifts by* [1].

All such x *noncommutatively deform to give a spherical twist autoequivalence.*

Proof. (1) $\Rightarrow$ (2) Consider the $\beta\colon \mathsf{D} \to \mathsf{C}$ from Corollary 5.5 (2) and also the atom $\alpha\colon \mathsf{D} \to \mathsf{C}$. By torsion pairs (see, e.g., [27, §5]), each $\Phi_{\alpha}(\mathcal{S}_i)$ is either a brick module, or the shift [1] applied to a brick module. Hence applying $\Phi_{\beta} \circ \Phi_{\alpha}^{-1}$ to either this brick or its shift shows that x is in the desired orbit. The reverse direction (2) $\Rightarrow$ (1) is trivial. The final statement follows from Corollary 5.5 (2) since the sheaves $\mathcal{O}_{C_i}(-1)$ satisfy the conclusion [23, 24]. ∎

Remark 5.7. As demonstrated in the example [43, Corollary 6.13 (2)], it is possible to construct a subset of bricks, for which Corollary 5.6 (2) can be improved to the statement that x is in the orbit under PBr of this subset, and their shifts by [1]. In general, it is not clear how to construct a subset of minimal size for which Corollary 5.6 (2) remains true.

5.3. Topological corollaries

For any triangulated category $\mathcal{T}$, there is an associated complex manifold $\mathrm{Stab}\,\mathcal{T}$, whose points are Bridgeland stability conditions on $\mathcal{T}$. Every exact equivalence $\Phi\colon \mathcal{T} \to \mathcal{T}'$ induces a homeomorphism $\Phi_*\colon \mathrm{Stab}\,\mathcal{T} \xrightarrow{\sim} \mathrm{Stab}\,\mathcal{T}'$.

Again, let $\mathcal{C} = \mathcal{C}_{\mathsf{C}}$ be the category defined in Section 2.4, and $\mathsf{U}_{\mathsf{C}} \subset \mathrm{Stab}\,\mathcal{C}_{\mathsf{C}}$ be the subset of stability conditions with the standard heart. Since $\mathsf{U}_{\mathsf{C}} \cong \mathbb{H}^{\oplus n}$, where n is the number of simples, which is connected, there exists a connected component $\mathrm{Stab}^{\circ}\mathcal{C} \subset \mathrm{Stab}\,\mathcal{C}$

containing U_C. For a path $\beta\colon D \to C$, write $U_\beta \subset \mathsf{Stab}\,\mathcal{C}_C$ for the image of U_D under $(\Phi_\beta)_*$, and further write $\mathsf{Term}(C)$ for all morphisms in the Deligne groupoid ending at C.

Corollary 5.8. *The set $\{U_\beta \mid \beta \in \mathsf{Term}(C)\}$ covers* $\mathsf{Stab}\,\mathcal{C}$. *In particular,* $\mathsf{Stab}^\circ\mathcal{C} = \mathsf{Stab}\,\mathcal{C}$.

Proof. For a simple mutation $s_i\colon C \to C_i$, the U_C and U_{C_i} share a codimension one boundary (see, e.g., [27, Proposition 6.3], [16, Example 5.5]). In particular, the associated homeomorphism $(\Phi_i)_*$ restricts to

$$(\Phi_i)_*\colon \mathsf{Stab}^\circ\mathcal{C}_C \to \mathsf{Stab}^\circ\mathcal{C}_{C_i}.$$

Let $\sigma = (Z, \mathcal{A})$ be an arbitrary point in $\mathsf{Stab}\,\mathcal{C}_C$. Then by Theorem 4.6, there exists a path $\beta\colon C \to D$ such that $\Phi_\beta(\mathcal{A})$ is the standard heart of $\mathcal{C}_D$. Now Φ_β is a composition of simple mutations Φ_i or their inverses, and thus this also preserves the components $\mathsf{Stab}^\circ\mathcal{C}_C$ and $\mathsf{Stab}^\circ\mathcal{C}_D$. Since $(\Phi_\beta)_*(\sigma) \in U_D \subset \mathsf{Stab}^\circ\mathcal{C}_D$, applying $\Phi_{\beta^{-1}} = \Phi_\beta^{-1}$ gives $\sigma \in U_{\beta^{-1}} \cap \mathsf{Stab}^\circ\mathcal{C}_C$, where $\beta^{-1}\colon D \to C$. Since $U_{\beta^{-1}}$ is connected and $\mathsf{Stab}^\circ\mathcal{C}_C$ is a connected component, necessarily $\sigma \in U_{\beta^{-1}} \subset \mathsf{Stab}^\circ\mathcal{C}_C$, which proves the result. ∎

Consider the group $\mathsf{Auteq}^{\mathrm{FM}}\,\mathcal{C}$ consisting of those $\Phi|_\mathcal{C}$, where Φ is a Fourier–Mukai equivalence $\mathrm{D}^b(\mathrm{coh}\,X) \to \mathrm{D}^b(\mathrm{coh}\,X)$ that commutes with $\mathbf{R}f_*$. Automatically,

$$\Phi|_\mathcal{C}\colon \mathcal{C} \to \mathcal{C}.$$

Corollary 5.9. *Suppose that* $X \to \mathrm{Spec}\,\mathcal{R}$ *is a 3-fold flop, where X has at worst terminal singularities. Then* $\mathsf{Auteq}^{\mathrm{FM}}\,\mathcal{C} \cong \mathsf{PBr}\,\mathcal{C}$.

Proof. Consider the subgroup $\mathsf{Aut}^\circ\mathcal{C}$ of $\mathsf{Auteq}^{\mathrm{FM}}\,\mathcal{C}$ that preserves $\mathsf{Stab}^\circ\mathcal{C}$. By [28, Theorem 7.1], it is known that $\mathsf{Aut}^\circ\mathcal{C} \cong \mathsf{PBr}\,\mathcal{C}$. But now using Corollary 5.8, $\mathsf{Stab}^\circ\mathcal{C} = \mathsf{Stab}\,\mathcal{C}$, hence every autoequivalence preserves $\mathsf{Stab}^\circ\mathcal{C}$ and thus $\mathsf{Aut}^\circ\mathcal{C} = \mathsf{Auteq}^{\mathrm{FM}}\,\mathcal{C}$. ∎

6. Silting discrete derived categories

This section generalises the argument in Section 3 to the setting of derived categories $\mathrm{D}^b(\mathrm{mod}\,A)$, where A is a silting discrete finite-dimensional algebra. This gives applications to representation theory.

6.1. Silting discrete preliminaries

Throughout this section, A will be a finite-dimensional k-algebra, which to be consistent with the earlier parts of this paper requires $k = \mathbb{C}$. As is standard, this can be easily generalised to other fields by weakening the condition on the endomorphism ring of a brick to be a division ring, rather than the base field.

We briefly recall silting objects and simple-minded collections, following [4, 35]. Set $\mathcal{K} = \mathrm{K}^b(\mathrm{proj}\,A)$, $\mathcal{T} = \mathrm{D}^b(\mathrm{mod}\,A)$. An object $x \in \mathcal{K}$ is called silting if $\mathrm{Hom}_\mathcal{K}(x, x[i]) = 0$

for all $i > 0$, and further $\mathrm{thick}(x) = \mathcal{K}$. By contrast, a collection $\{y_1, \ldots, y_n\}$ of objects of $\mathcal{T}$ are called a *simple-minded collection* (smc) if $\mathrm{Hom}_{\mathcal{T}}(y_i, y_j[t]) = 0$ for all i, j and all $t < 0$, $\mathrm{Hom}_{\mathcal{T}}(y_i, y_j) \cong \mathbb{C}\delta_{ij}$, and $\mathrm{thick}(y_1, \ldots, y_n) = \mathcal{T}$. We will often blur the distinction between the set $\{y_1, \ldots, y_n\}$ and the single object $y = \bigoplus y_i$.

The standard example of a silting object is A, considered as a complex in degree zero, and the standard example of an smc is $\mathcal{S} = \bigoplus \mathcal{S}_i$, the sum of the simple A-modules.

Below, we will often use the Koenig–Yang bijections. For the categories $\mathcal{K}$ and $\mathcal{T}$ above, there are bijections [35, §5]

$$\{\text{basic silting objects in } \mathcal{K}\} \leftrightarrow \{\text{bounded t-structures in } \mathcal{T} \text{ with length heart}\}$$
$$\leftrightarrow \{\text{smcs in } \mathcal{T}\}, \tag{6.1}$$

where an object is *basic* if there are no repetitions in its Krull–Schmidt decomposition. The following is the standard definition, translated into Notation 4.1.

Definition 6.1. Let $\mathcal{H}$ be the heart of a bounded t-structure on a triangulated category $\mathcal{T}$. An smc $\mathcal{U}$ is called 2-*term* with respect to $\mathcal{H}$ if $\mathcal{U} \in [-1, 0]_{\mathcal{H}}$. In $\mathrm{D}^b(A)$, the set of 2-term smcs with respect to the standard heart of $\mathrm{D}^b(A)$ will be denoted by $2\text{-smc } A$. Under (6.1), the corresponding set of silting objects will be denoted by $2\text{-silt } A$.

Notation 6.2. If $\mathcal{V}$ is an smc, we will abuse Notation 4.1 and write $[a, b]_{\mathcal{V}} = [a, b]_{\mathcal{H}_{\mathcal{V}}}$, where $\mathcal{H}_{\mathcal{V}}$ is the heart of the bounded t-structure corresponding to $\mathcal{V}$ under the Koenig–Yang bijection (6.1). Note that with this abuse, $\mathcal{U} \in 2\text{-smc } A$ if and only if $\mathcal{U} \in [-1, 0]_{\mathcal{S}}$, which by Corollary 4.4 is equivalent to $\mathcal{S} \in [0, 1]_{\mathcal{U}}$. This is the formulation used in Theorem 6.11 below.

As in Lemmas 3.2 and 4.3, Notation 6.2 can be interpreted in terms of the Hom vanishings.

Lemma 6.3. *Let $\mathcal{U}$ be an smc, and let $x \in \mathcal{T} = \mathrm{D}^b(A)$. Then the following are equivalent:*

(1) $x \in [a, b]_{\mathcal{U}}$.

(2) $\mathrm{Hom}(\mathcal{U}, x[i]) = 0$ *for all $i < a$ and* $\mathrm{Hom}(x, \mathcal{U}[i]) = 0$ *for all $i < -b$.*

Definition 6.4 ([5, Theorem 2.4]). A finite-dimensional algebra A is called τ-*tilting finite* if the set $2\text{-smc } A$, equivalently $2\text{-silt } A$, is finite. If for all smcs $\mathcal{U}$ there are only finitely many smcs $\mathcal{V}$ for which $\mathcal{V} \in [0, 1]_{\mathcal{U}}$, then A is called *silting discrete*.

It is clear that any silting discrete algebra is τ-tilting finite.

Definition 6.5. A basic $x \in \mathrm{D}^b(A)$ is a *semibrick complex* if $\mathrm{Hom}_{\mathrm{D}^b(A)}(x, x[j]) = 0$ for all $j < 0$ and further $\mathrm{Hom}_{\mathrm{D}^b(A)}(x_i, x_j) \cong \mathbb{C}\delta_{ij}$ for all indecomposable summands x_i, x_j of x. A *brick complex* is a semibrick complex which is indecomposable. If x happens to be an A-module in homological degree zero, we will emphasise this by calling x a *semibrick module*, or *brick module*, respectively.

The following is known [31, 35, 39].

Lemma 6.6. *Let A be a silting discrete finite-dimensional algebra, and $\mathcal{H}$ be the heart of a bounded t-structure of $D^b(A)$. Then the following statements hold:*

(1) *There exists a finite-dimensional algebra Γ such that $\mathcal{H} \simeq \operatorname{mod} \Gamma$.*

(2) *The realisation functor* real$\colon D^b(\Gamma) \to D^b(A)$ *induces an injective map from the set* 2 -smc Γ *to the set of smcs in $D^b(A)$ which are 2-term with respect to $\mathcal{H}$.*

(3) *Γ is a τ-tilting finite algebra.*

Proof. (1) Since A is silting discrete, any heart of a bounded t-structure has finite length [39, Theorem A]. Under the Koenig–Yang bijection [35, Corollary 6.2], it follows that $\mathcal{H} = \operatorname{mod} \Gamma$ for some $\Gamma = \operatorname{End}_{K^b(\operatorname{proj} A)}(M)$, where $M \in K^b(\operatorname{proj} A)$ is silting.

(2) As explained in [35, Lemma 7.8], the induced map

$$\operatorname{Hom}_\Gamma(x, y[i]) \to \operatorname{Hom}_{D^b(A)}(\operatorname{real}(x), \operatorname{real}(y)[i])$$

is bijective for all $x, y \in \operatorname{mod} \Gamma$ and $i \le 1$. Let $\mathcal{U} \in$ 2 -smc Γ, say $\mathcal{U} = \{y_1, \ldots, y_r\}$. Then by [17, Remark 4.11], each y_i is either in $\operatorname{mod} \Gamma$ or $(\operatorname{mod} \Gamma)[1]$. Therefore, setting $z_i := \operatorname{real}(y_i)$, the direct sum $z = \bigoplus_{i=1}^r z_i$ remains a semibrick complex. Since each z_i is either in $\mathcal{H}$ or $\mathcal{H}[1]$, clearly z is 2-term with respect to $\mathcal{H}$.

Since $\mathcal{U}$ is an smc, by definition thick$(\mathcal{U}) = D^b(\Gamma)$ and thus $\operatorname{mod} \Gamma \subset$ thick$(\mathcal{U})$. Applying the triangulated functor real then gives

$$\mathcal{H} = \operatorname{real}(\operatorname{mod} \Gamma) \subset \operatorname{real}(\operatorname{thick}(\mathcal{U})) \subseteq \operatorname{thick}(\operatorname{real}(\mathcal{U})) := \operatorname{thick}(z).$$

Thus thick(z) contains the heart of a bounded t-structure on $D^b(A)$, and so necessarily thick$(z) = D^b(A)$. This shows that z is also an smc, and so real has induced the claimed map out of 2 -smc Γ. Since real is an equivalence on $\operatorname{mod} \Gamma \to \mathcal{H}$, and thus on $(\operatorname{mod} \Gamma)[1] \to \mathcal{H}[1]$, it follows that the induced map is injective.

(3) Consider the silting object M in (1), so that $\Gamma = \operatorname{End}_{K^b(\operatorname{proj} A)}(M)$. By [31, Theorem 0.2], there is a bijection between silting objects in $M * M[1]$ and the set of support τ-tilting modules sτ-tilt Γ (see, e.g., [1]). But as is standard [39, §1.2], a silting object $T \in M * M[1]$ if and only if $M \ge T \ge M[1]$, and so since A is silting discrete, this set is finite. Hence the set sτ-tilt Γ is finite. ∎

The following consequence of Lemma 6.6 is also somewhat implicit in the literature. As notation, write sbrick A for the set of all (basic) semibrick modules.

Corollary 6.7. *Let A be silting discrete, and suppose that x is a semibrick complex in $D^b(A)$ that is contained in the heart $\mathcal{H}$ of a bounded t-structure. Then there exists a simple-minded collection $\mathcal{U}$ in $D^b(A)$ that contains all indecomposable summands of x and is 2-term with respect to $\mathcal{H}$.*

Proof. By Lemma 6.6, $\mathcal{H} \simeq \operatorname{mod} \Gamma$, where Γ is τ-tilting finite. Let $y \in \operatorname{mod} \Gamma$ be the semibrick corresponding to $x \in \mathcal{H}$ under the equivalence. Since Γ is τ-tilting finite, every torsion class in $\operatorname{mod} \Gamma$ is functorially finite [22, Theorem 1.2]. Thus the 'functorially

finite' hypothesis can be dropped from the statement in Asai [6, Theorem 2.3], so that [6, Theorem 2.3] then asserts that there is a bijection

$$2\text{-smc}\,\Gamma \to \text{sbrick}\,\Gamma$$

given by $\mathcal{X} \to \mathcal{X} \cap \text{mod}\,\Gamma$. Thus there exists $\mathcal{U} \in 2\text{-smc}\,\Gamma$ such that $\mathcal{U}$ contains all the indecomposable summands of y. It follows from Lemma 6.6 (2) that $\text{real}(\mathcal{U})$ gives the desired smc. ∎

The challenge, and the content of the next subsection, is to prove that any semibrick complex in $D^b(A)$ is automatically contained in the heart $\mathcal{H}$ of a bounded t-structure. In fact, this works much more generally.

6.2. Main result

Throughout this section, as above, A is a finite-dimensional $\mathbb{C}$-algebra.

Definition 6.8. If $\mathcal{U}$, $\mathcal{V}$ are smcs in $D^b(A)$, write $\mathcal{U} \geq \mathcal{V}$ if $\text{Hom}(\mathcal{V}, \mathcal{U}[j]) = 0$ for all $j < 0$.

Note that, by Lemma 6.3, $\mathcal{U} \geq \mathcal{V}$ if and only if $\mathcal{V} \in [a, 0]_{\mathcal{U}}$ for some $a \leq 0$. The relation $\geq$ endows the set $\text{smc}\,D^b(A)$ with the structure of a partially ordered set [35, Proposition 7.9].

Proving the main result requires the ability to mutate simple-minded collections.

Definition 6.9 ([35]). Let $\mathcal{U} = \{y_1, \ldots, y_r\}$ be an smc in the derived category $D^b(A)$ of a finite-dimensional algebra A, and choose y_i in $\mathcal{U}$. The *left mutation* $\nu_i\mathcal{U} := \{y_1', \ldots, y_r'\}$ of $\mathcal{S}$ at y_i is defined as follows:

(1) If $j \neq i$, then y_j' is defined to be the cone of the minimal left approximation $y_j[-1] \to y_{ij}$ in the extension closure of y_i.

(2) If $j = i$, then $y_i' := y_i[1]$.

By [35, Lemma 7.8], the new collection $\nu_i\mathcal{U}$ is again an smc. Note that $\mathcal{U} > \nu_i\mathcal{U}$ holds since $\nu_i\mathcal{U} \in [-1, 0]_{\mathcal{U}}$ by construction. Right mutations are defined similarly, but below we will only require left mutations.

Lemma 6.10. *Set $\mathcal{T} = D^b(A)$, let $\mathcal{U} = \{y_1, \ldots, y_r\} \subset \mathcal{T}$ be an smc, and suppose that $x \in \mathcal{T}$ satisfies $x \in [a, b]_{\mathcal{U}}$. Then the following statements hold:*

(1) $x \in [a, b+1]_{\nu_i\mathcal{U}}$ *for all i.*

(2) *If $\text{Hom}_{\mathcal{T}}(H^b_{\mathcal{U}}(x), y_i) = 0$, then $x \in [a, b]_{\nu_i\mathcal{U}}$.*

Proof. (1) Fix any i and put $\nu_i\mathcal{U} = \{y_1', \ldots, y_r'\}$. According to Lemma 6.3, proving $x \in [a, b+1]_{\nu_i\mathcal{U}}$ is equivalent to showing that

(a) $\text{Hom}_{\mathcal{T}}(\bigoplus_{j=1}^r y_j', x[a+k]) = 0$ for all $k < 0$, and

(b) $\text{Hom}_{\mathcal{T}}(x, \bigoplus_{j=1}^r y_j'[-b-1+k]) = 0$ for all $k < 0$.

Since y_j' is contained in the extension closure of $y = \bigoplus_{l=1}^r y_l$ for all $j \neq i$, and since $y_i' = y_i[1]$, the first vanishing (a) follows from the assumption $x \in [a, b]_{\mathcal{U}}$. Again, since y_j' is contained in the extension closure of y for all $j \neq i$, the vanishing

$$\mathrm{Hom}_{\mathcal{T}}\left(x, \bigoplus_{j \neq i} y_j[-b + k]\right) = 0$$

holds for any $k < 0$. For the remaining case, there is an equality

$$\mathrm{Hom}_{\mathcal{T}}(x, y_i'[-b - 1 + k]) = \mathrm{Hom}_{\mathcal{T}}(x, y_i[-b + k])$$

by definition, and the right-hand side is zero for all $k < 0$ by the assumption $x \in [a, b]_{\mathcal{U}}$.

(2) The proof of (1) also shows that $x \in [a, b]_{\mathsf{v}_i \mathcal{U}}$ if and only if $\mathrm{Hom}(x, y_i[-b]) = 0$. Since $x \in [a, b]_{\mathcal{U}}$ and y is an object in the heart corresponding to $\mathcal{U}$, applying Lemma 4.2 yields an isomorphism $\mathrm{Hom}(x, y_i[-b]) \cong \mathrm{Hom}(\mathrm{H}_{\mathcal{U}}^b(x), y_i)$, which proves the result. ∎

The following is the main result of this section.

Theorem 6.11. *If A is silting discrete, $\mathcal{T} = \mathrm{D}^b(A)$ and $x \in \mathcal{T}$, then the following statements are equivalent:*

(1) $\mathrm{Hom}_{\mathcal{T}}(x, x[i]) = 0$ *for all $i < 0$.*

(2) *x belongs to the heart of a bounded t-structure.*

Proof. The only non-trivial direction is that (1) implies (2). Choose an smc $\mathcal{U}$, then there exist integers $a \leq b$ such that $x \in [\![a, b]\!]_{\mathcal{U}}$. If $a = b$, then there is nothing to prove. Thus we assume $a < b$ and claim that there is another smc $\mathcal{V}$ such that $x \in [a + 1, b]_{\mathcal{V}}$. If this is true, then by induction we can find an smc whose corresponding heart contains the object x.

To prove the claim, consider

$$\Delta = \{\mathcal{W} \in \mathrm{smc}\, \mathrm{D}^b(A) \mid \mathcal{U} \in [0, 1]_{\mathcal{W}} \text{ and } x \in [a, b]_{\mathcal{W}}\}.$$

This is a poset with respect to $\geq$. Further, the poset Δ is finite since A is silting discrete (see Definition 6.4), and thus there exists a minimal element $\mathcal{V} = \{y_1, \ldots, y_r\}$. We will show that this $\mathcal{V}$ satisfies the desired property. Assume, for the aid of a contradiction, that $\mathrm{H}_{\mathcal{V}}^a(x) \neq 0$, and choose a simple $y_i \hookrightarrow \mathrm{H}_{\mathcal{V}}^a(x)$. By Lemma 4.2, since $x \in [\![a, b]\!]_{\mathcal{V}}$ and $\mathcal{U} \in [0, 1]_{\mathcal{V}}$, necessarily

$$\mathrm{Hom}(\mathcal{U}, x[a - 1]) \cong \mathrm{Hom}(\mathrm{H}_{\mathcal{V}}^1(\mathcal{U}), \mathrm{H}_{\mathcal{V}}^a(x)).$$

The left-hand side is zero by Lemma 6.3 since $x \in [\![a, b]\!]_{\mathcal{U}}$. Similarly, since $a < b$ and x has no negative Exts, it also follows from Lemma 4.2 that $\mathrm{Hom}(\mathrm{H}_{\mathcal{V}}^b(x), \mathrm{H}_{\mathcal{V}}^a(x)) = 0$. Then our choice of y_i together with these vanishings yields

$$\mathrm{Hom}(\mathrm{H}_{\mathcal{V}}^1(\mathcal{U}), y_i) = 0 \quad \text{and} \quad \mathrm{Hom}(\mathrm{H}_{\mathcal{V}}^b(x), y_i) = 0.$$

By Lemma 6.10, these imply that $\mathcal{U} \in [0, 1]_{\mathsf{v}_i \mathcal{V}}$ and $x \in [a, b]_{\mathsf{v}_i \mathcal{V}}$, respectively. But then $\mathsf{v}_i \mathcal{V}$ is an smc in Δ with $\mathcal{V} > \mathsf{v}_i \mathcal{V}$, which contradicts the minimality of $\mathcal{V}$. ∎

Recall that a semibrick complex x is said to have maximal rank if the number of indecomposable summands of x equals the number of simple A-modules. The following is immediate from Theorem 6.11, where part (3) generalises [22].

Corollary 6.12. *If A is silting discrete, then the following statements hold:*

(1) *Any semibrick complex is a subcollection of a simple-minded collection.*

(2) *Any semibrick complex of maximal rank is a simple-minded collection.*

Proof. (1) Let x be a semibrick complex. Then Theorem 6.11 implies that there exists the bounded heart $\mathcal{H}$ that contains x, so the result is a special case of Corollary 6.7.

(2) Let x be a semibrick complex of maximal rank. By (1), x is a subcollection of a simple-minded collection. But by [35, Corollary 5.5], every simple-minded collection has maximal rank, thus x itself is a simple-minded collection. ∎

For the next corollary, recall that sbrick A is the set of all semibrick modules. If $x, y \in$ sbrick A, then $x \oplus y[1]$ is called a semibrick pair if

$$\mathrm{Hom}_A(x, y) = 0 = \mathrm{Ext}_A^1(x, y).$$

Note that automatically a semibrick pair $x \oplus y[1]$ is a semibrick complex.

Part (3) of the following answers a general question of [12, 26].

Corollary 6.13. *If A is silting discrete, then the following statements hold:*

(1) *Every semibrick pair is a semibrick in the heart $\mathcal{H}$ of a bounded t-structure satisfying $\mathcal{H} \in [-1, 0]$ with respect to the standard t-structure.*

(2) *Every semibrick pair is a subset of the simples in the heart $\mathcal{A}$ of a bounded t-structure satisfying $\mathcal{A} \in [-2, 0]$ with respect to the standard t-structure.*

(3) *Every semibrick pair of maximal rank is a 2-term simple-minded collection.*

Proof. Let $x \oplus y[1]$ be a semibrick pair.

(1) Let $\mathcal{S}$ be the smc consisting of simple A-modules, so that the heart corresponding to $\mathcal{S}$ is the standard heart. Consider the finite partially ordered set

$$\Delta = \{\mathcal{W} \in \mathrm{smc}\, D^b(A) \mid \mathcal{S} \in [0, 1]_{\mathcal{W}} \text{ and } x \oplus y[1] \in [-1, 0]_{\mathcal{W}}\}.$$

Then the proof of Theorem 6.11 shows that a minimal element $\mathcal{U}$ of Δ satisfies $x \in [0, 0]_{\mathcal{U}} = \mathcal{H}_{\mathcal{U}}$, where $\mathcal{H}_{\mathcal{U}}$ is the heart of the corresponding bounded t-structure. Since $\mathcal{S} \in [0, 1]_{\mathcal{H}_{\mathcal{U}}}$ by definition, Notation 6.2 shows that $\mathcal{H}_{\mathcal{U}} \in [-1, 0]$ with respect to the standard t-structure.

(2) Let $\mathcal{H}$ be the heart of the t-structure in (1). Then Corollary 6.7 allows us to find the heart $\mathcal{A}$ of another t-structure such that $\mathcal{A} \in [-1, 0]_{\mathcal{H}}$, and the smc corresponding to $\mathcal{A}$ contains all indecomposable summands of $x \oplus y[1]$. Each $a \in [-1, 0]_{\mathcal{H}}$ and so has a 2-term filtration as in (4.1), thus applying H^* with respect to the standard t-structure and using $\mathcal{H} \in [-1, 0]$, it is easy to see that $\mathcal{A} \in [-2, 0]$.

(3) This follows from (1) and Corollary 6.12. ∎

6.3. Special case: Contraction algebras

A large class of silting discrete algebras are the contraction algebras Λ_{con} from Section 2.5. The associated category $D^b(\Lambda_{con})$ is of particular interest since it conjecturally is the classifying object of smooth flops. As such, $D^b(\Lambda_{con})$ should not exhibit any exotic behaviour, nor contain any 'unexpected' objects. The following result confirms this, adding to the existing evidence in [8].

In what follows, F_β are compositions of the standard equivalences in (3.3), which fixing the same start and end point, generate a subgroup PBr of Auteq $D^b(\Lambda_{con})$.

Corollary 6.14. *Let Λ_{con} be the contraction algebra associated to a 3-fold flopping contraction $\mathcal{X} \to \mathrm{Spec}\,\mathcal{R}$, where $\mathcal{X}$ has at worst Gorenstein terminal singularities, and let $x \in D^b(\Lambda_{con})$. Then the following statements hold:*

(1) *If x is a semibrick complex, then there exist β and a sum of simple modules $\bigoplus_{i \in \mathfrak{I}} S_i$ on a possibly different contraction algebra Γ_{con}, such that $F_\beta(\bigoplus_{i \in \mathfrak{I}} S_i) \cong x$.*

(2) *Every semibrick complex in $D^b(\Lambda_{con})$ is in the orbit, under the action of PBr, of the (finitely many) semibrick modules or their shift by [1].*

Proof. (1) By Corollary 6.12 (1), the object x is a subcollection of a simple-minded collection, and thus by the Koenig–Yang bijections, x is a sum of simples in the heart of a bounded t-structure. But by [7, Corollary 3.16], all such t-structures are of the form $F_\beta(\mathrm{mod}\,\Gamma_{con})$ for some β.

(2) This follows from (1), in the same way as in Corollary 5.6. Say $\beta \colon \mathsf{D} \to \mathsf{C}$, then consider the atom $\alpha \colon \mathsf{D} \to \mathsf{C}$. By torsion pairs, each $F_\alpha(S_i)$ is either a brick module, or the shift [1] applied to a brick module. Hence $F_\beta \circ F_\alpha^{-1}$ shows that x is in the desired orbit. ∎

Remark 6.15. In the flops setting above, the previous results (Corollaries 5.5 and 5.6) classify the semibrick complexes in the category $\mathcal{C}$, whilst Corollary 6.14 classifies the semibrick complexes in the category $D^b(\Lambda_{con})$. The fact that these results are so similar is all the more remarkable, given the categories $\mathcal{C}$ and $D^b(\Lambda_{con})$ are never equivalent.

Indeed, it follows from the description of all bounded t-structures in [8] that the category $D^b(\Lambda_{con})$ enjoys the following property: the heart $\mathcal{H}$ of every bounded t-structure on $D^b(\Lambda_{con})$ satisfies $\mathrm{Ext}^t_{\mathcal{H}}(a, b) \cong \mathrm{Hom}_{D^b(\Lambda_{con})}(a, b[t])$ for all $a, b \in \mathcal{H}$ and all $t \in \mathbb{Z}$. We claim that $\mathcal{C}$ does not satisfy this property, and so $\mathcal{C}$ and $D^b(\Lambda_{con})$ are not equivalent.

As before, consider the bounded heart $\mathcal{H} := \mathrm{mod}\,\Lambda_{con}$ on $\mathcal{C}$. Writing S for the sum of the simple Λ_{con}-modules, then since Λ_{con} is projective,

$$\mathrm{Ext}^3_{\mathcal{H}}(\Lambda_{con}, S) = \mathrm{Ext}^3_{\mathrm{mod}\,\Lambda_{con}}(\Lambda_{con}, S) = 0.$$

But as in the setup (Section 2.4), $\mathcal{C} \subseteq D^b(\Lambda)$, thus

$$\mathrm{Hom}_{\mathcal{C}}(\Lambda_{con}, S[3]) = \mathrm{Hom}_{D^b(\Lambda)}(\Lambda_{con}, S[3]) = \mathrm{Ext}^3_{\Lambda}(\Lambda_{con}, S),$$

which is non-zero by [24, Proposition 4.7], as a consequence of the 3-sCY property.

6.4. Further remarks

The question of whether a semibrick complex can be completed to a 2-simple-minded collection has been addressed elsewhere (e.g., [12, 26]), but the appearance of the interval $[-2, 0]$ in Corollary 6.13 gives some evidence that this is the wrong question.

Example 6.16. Consider the projective algebra Π of type D_4, which is silting discrete by [9], and the explicit semibrick pair $(M \oplus N) \oplus E[1]$ constructed in [12, Remark 4.0.8]. As shown in *loc. cit.*, this does not complete to a 2-smc. However, by Corollary 6.13 (2) it does complete to an smc in the region $[-2, 0]$. This shows that the region $[-2, 0]$ cannot be improved to $[-1, 0]$.

In fact, the proof of Corollary 6.13 (2) shows that the full strength of silting discrete is not required to answer the question of whether a semibrick pair completes to an smc. The following example illustrates this in the case of the preprojective algebra Π of type A, where it is still not known whether Π is silting discrete.

Example 6.17. Consider the preprojective algebra Π of type A_3, namely the path algebra of the quiver
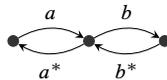
subject to the relations $aa^* = 0$, $a^*a = bb^*$ and $b^*b = 0$. Set $\mathcal{T} = \mathrm{D}^b(\mathrm{mod}\,\Pi)$ and

$$x := \left(\mathbb{C} \overset{1}{\underset{0}{\rightleftarrows}} \mathbb{C} \rightleftarrows 0\right) \oplus \left(\mathbb{C} \overset{1}{\underset{0}{\rightleftarrows}} \mathbb{C} \overset{1}{\underset{0}{\rightleftarrows}} \mathbb{C}\right)[1].$$

As can be checked by hand, using the fact that Π has finite representation type, or by using GAP-package [44], x satisfies $\mathrm{Hom}_{\mathcal{T}}(x, x[i]) = 0$ for all $i < 0$, and is a semibrick pair. It does not complete to a 2-smc. But consider the 2-smc

$$\mathbb{S}' := \left(\mathbb{C} \overset{1}{\underset{0}{\rightleftarrows}} \mathbb{C} \rightleftarrows 0\right) \oplus \left(\mathbb{C} \rightleftarrows 0 \rightleftarrows 0\right)[1] \oplus \left(0 \rightleftarrows \mathbb{C} \overset{1}{\underset{0}{\rightleftarrows}} \mathbb{C}\right)[1].$$

In particular, $\mathbb{S}'$ is the sum of the simples in a certain heart $\mathcal{H}$ of a bounded t-structure, which is intermediate between $\mathrm{mod}\,\Pi$ and $\mathrm{mod}\,\Pi[1]$. Again, either by hand or by GAP [44], it can be seen that

$$\mathrm{Hom}_{\mathcal{T}}(\mathbb{S}', x[i]) = 0 = \mathrm{Hom}_{\mathcal{T}}(x, \mathbb{S}'[i])$$

for all $i < 0$, and so since every object of $\mathcal{H}$ is filtered by $\mathbb{S}'$, it follows that

$$\mathrm{Hom}_{\mathcal{T}}(\mathcal{H}, x[i]) = 0 = \mathrm{Hom}_{\mathcal{T}}(x, \mathcal{H}[i])$$

for all $i < 0$. Since $\mathcal{H}$ is the heart of a bounded t-structure, $x \in \mathcal{H}$. Thus, x is a semibrick in the heart of a bounded t-structure which is intermediate between $\mathrm{mod}\,\Pi$ and $\mathrm{mod}\,\Pi[1]$.

Whilst x is not a sum of simples in this heart, if this heart is τ-tilting finite (this seems to be conjecturally true in the literature), then the proof of Corollary 6.7 can be used to show that x is the simples in a different heart $\mathcal{H}'$, for which $\mathcal{H}' \in [-2, 0]$.

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