

Splitting symplectic monodromy

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ABSTRACT: We discuss some examples in which symplectic monodromy (provably or conjecturally) splits off the symplectic mapping class group, hoping to illustrate different techniques and inputs to the arguments. Along the way we formulate several open questions and conjectures.

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1 Introduction

Let (X, ω) be a symplectic manifold; we will typically be interested in non-compact manifolds well-behaved (e.g. convex) at infinity. Suppose X moves in a moduli space $\mathcal{M}_X$ which carries a universal family $\mathcal{C} \rightarrow \mathcal{M}_X$ which is a locally Hamiltonian fibre bundle. Symplectic parallel transport gives rise to a monodromy representation

$$(1.A) \quad \rho : \pi_1(\mathcal{M}_X) \rightarrow \pi_0 \operatorname{Symp}(X, \omega).$$

This may be valued in $\pi_0 \operatorname{Symp}_{ct}(X, \omega)$ depending on the geometry of the family at infinity. Such representations provide arguably the primary source of constructions of interesting symplectomorphisms. This suggests that a fundamental question is to understand the map ρ .

To understand a group one should make it act on something. The group $\pi_0 \operatorname{Symp}(X, \omega)$ acts by autoequivalences of the (derived) Fukaya category $\mathcal{F}(X)$, defining a representation

$$(1.B) \quad \pi_0 \operatorname{Symp}(X, \omega) \rightarrow \operatorname{Auteq}(\mathcal{F}(X)).$$

This may be up to shift; we will consider gradings, and the question of compact or non-compact support of maps, more carefully in the main text. *A priori* the symplectic topology of X has no reason to know anything about its incarnation as a special fibre of $\mathcal{C} \rightarrow \mathcal{M}_X$ (for instance, its incarnation as an algebro-geometric complete intersection), and yet remarkably, in a few non-trivial cases, one can relate the autoequivalence group to $\pi_1(\mathcal{M}_X)$ itself, and infer that ρ *splits*.

This note illustrates that phenomenon in some basic cases: certain Milnor fibres (Theorem 3.4), quiver 3-folds (Theorem 4.4) and mirrors to 3-fold flopping contractions (Theorem 5.1). The results in Section 7 have a different character, morally arising from monodromy in a space of integral affine structures on an SYZ base. The point of gathering examples together is to emphasise that it is an interesting question

to understand how widespread this phenomenon is, and to show that establishing it in any given setting typically requires non-trivial machinery. The extent to which the collection of cases gathered here is prototypical is itself an interesting question.

Remark 1.1 The paper does not discuss an exhaustive list of splittings of symplectic monodromy, but we gather a few other examples in Section 8, for completeness and to help establish the broader scene for the reader.

Acknowledgements. A.K. is partially supported by EPSRC Open Fellowship EP/W001780/1. I.S. is partially supported by UKRI Frontier Research grant EP/X030660/1 (in lieu of an ERC Advanced Grant). M.W. is partially supported by ERC Consolidator Grant 101001227.

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2 Flavours of Fukaya category

Let (X, ω) be a connected symplectic manifold with $2c_1(X) = 0$. We will fix a Maslov cover of the Lagrangian Grassmannian, meaning that there is a well-defined group $\text{Symp}^{gr}(X)$ of graded symplectomorphisms, and a subgroup $\mathbb{Z} \leq \text{Symp}^{gr}(X)$ generated by the shift (i.e. the fibrewise deck transformation). Recall from [Sei00, Lemma 2.4]:

Lemma 2.1 *There is an exact sequence*

$$1 \rightarrow \mathbb{Z} \rightarrow \text{Symp}^{gr}(X) \rightarrow \text{Symp}(X) \rightarrow H^1(X; \mathbb{R}).$$

The final map is not a homomorphism but its kernel is a subgroup.

Lemma 2.2 *The forgetful map $\text{Symp}^{gr}(X) \rightarrow \text{Symp}(X)$ is split over the subgroup $\text{Symp}_{ct}(X)$.*

Proof Take the unique gradings for symplectomorphisms which act trivially on the fibres of the Maslov cover near infinity. $\square$

Let (X, ω) be a Liouville manifold, with contact type boundary, and assume $2c_1(X) = 0$. We will be interested in all of

- (1) the compact Fukaya category $\mathcal{F}(X)$, with objects compact graded exact spin Lagrangians;
- (2) the wrapped Fukaya category $\mathcal{W}(X)$, with objects graded exact spin Lagrangians which are cones over Legendrian submanifolds near infinity;
- (3) the Rabinowitz Fukaya category $\mathcal{R}(X) := \mathcal{W}(X)/\mathcal{F}(X)$, which is the indicated Verdier quotient.

All three categories are defined over $\mathbb{Z}$, and $\mathcal{F}(X) \subset \mathcal{W}(X)$ is a full and faithful subcategory. (In Section 7 it will be important to work over $\mathbb{C}$, to incorporate $\mathbb{C}^*$ -valued local systems on Lagrangians.) Any autoequivalence of $\mathcal{W}(X)$ arising from a symplectomorphism of X will preserve the compact category $\mathcal{F}(X)$, and hence induce an equivalence of $\mathcal{R}(X)$. For background on $\mathcal{F}(X)$ see [Sei08a] and for background on $\mathcal{W}(X)$ see [AS10]; for background on $\mathcal{R}(X)$ see [Dri04, GGV22].

We will be interested in actions of symplectomorphism groups on Fukaya categories. Especially for actions on the wrapped category, one needs some control of maps at infinity to ensure that (perhaps after isotopy) they preserve the class of Lagrangians which are conical¹ at infinity. We say a symplectomorphism ϕ of a Liouville manifold $(X, d\theta)$ is *exact* if $\phi^*\theta = \theta + df$ for some $f : X \rightarrow \mathbb{R}$, and *strongly exact* if one can take f to have compact support. Write $\text{Symp}_{ex}(X)$ and $\text{Symp}_{str}(X)$ respectively for the corresponding groups. Both notions have obvious graded analogues. It is classical that strongly exact maps act on the wrapped category.

Lemma 2.3 *Assume that the Liouville manifold X has finite type, so can be written as the completion of a Liouville domain $\hat{X}$. Then*

- (1) *There is a homomorphism $\pi_0 \text{Symp}_{ct}(X) \rightarrow \pi_0 \text{Symp}_{ex}(\hat{X})$;*
- (2) *The group $\pi_0 \text{Symp}(\hat{X})$ acts on both $\mathcal{F}(X)$ and $\mathcal{W}(X)$.*

Proof For the first statement, conjugation by a suitably large Liouville flow will bring the support into $\hat{X}$. We claim we can then deform to obtain an exact representative of the map. Let $\omega = d\lambda$ be the Liouville form on X , and suppose we are given $\psi \in \text{Symp}_{ct}(X)$. We have $\psi^*\lambda - \lambda = \theta$, where θ is a closed one-form with compact support. Let V be the vector field which is symplectically dual to $-\theta$. This has compact support, and so is integrable for all time; let σ_t denote its flow. Then applying Cartan's formula, we get that $\psi \circ \sigma_1$ is an exact symplectomorphism. (Note that the proof of [BEE12, Lemma 1.1] specialises to this argument in this case.) To see this induces a group homomorphism after passing to Hamiltonian isotopy equivalence classes, it's enough to note that $(\varphi \circ \sigma_2) \circ (\psi \circ \sigma_1) = (\varphi \circ \psi) \circ ((\psi^{-1} \circ \sigma_2 \circ \psi) \circ \sigma_1)$.

The second statement is exactly [KS24, Corollary 2.4] (cf. [KS25], which in particular involves deforming a map of the Liouville domain to be exact). $\square$

Lemma 2.4 *In the setting of Lemma 2.3, assume moreover that $2c_1(\hat{X}) = 0$. Then there are compatible representations*

$$\begin{array}{ccc} \pi_0 \text{Symp}_{ct}^{gr}(X) & \longrightarrow & \text{Auteq}(\mathcal{F}(X)) \\ \downarrow & & \uparrow \\ \pi_0 \text{Symp}^{gr}(\hat{X}) & \longrightarrow & \text{Auteq}(\mathcal{W}(X)) \end{array}$$

The top line is classical; the bottom line is [KS24, Corollary 2.9] (see also [KS25]).

Although the Rabinowitz category is not a priori determined by the contact boundary of X , one still has:

Lemma 2.5 *The group $\pi_0 \text{Symp}_{ct}^{gr}(X)$ acts trivially on the Rabinowitz category.*

Proof Let L, K be Lagrangians which are cones over Legendrian submanifolds L_∞ and K_∞ . The morphism group in $\mathcal{R}(X)$ between L and K is the chain-level mapping cone of a natural continuation map $CW_*(L, K) \rightarrow CW^*(L, K)$ coming from a monotone homotopy of Hamiltonians from ‘infinite negative wrapping to infinite positive wrapping’. This factors through Hamiltonians $H_{-\varepsilon}$ and $H_{+\varepsilon} = -H_{-\varepsilon}$.

¹Or more generally geometrically bounded for a dissipative almost complex structure in the sense of Groman [Gro23].

Changing the sign of the Hamiltonian function exchanges homology and cohomology, and the continuation from small negative to small positive wrapping is an isomorphism realising the Poincaré duality isomorphism between homology relative boundary and cohomology; compare to [CFO10, Proposition 1.3] and [Rit13, Theorem 13.1]. Therefore, dividing out by an acyclic complex, one sees that the morphism group can be defined by a chain complex all of whose generators are Reeb chords between L_∞ and K_∞ . Any compactly supported map then acts trivially on the underlying chain group, hence on the whole A_∞ -category. $\square$

3 Type A Milnor fibres

We consider the $A_k^{(n)}$ -Milnor fibre of complex dimension n , that is the affine hypersurface

$$(3.A) \quad \{x_1^2 + \dots + x_n^2 + p_{k+1}(z) = 0\} \subset \mathbb{C}^{n+1}, \omega_{st}$$

with p_{k+1} a degree $k + 1$ polynomial with distinct zeroes. This is symplectically a plumbing of k copies of T^*S^n , associated to the A_k -quiver:

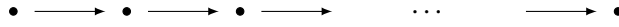


Figure 1: The A_k -quiver

It comes in a family over the configuration space $\text{Conf}_{k+1}(\mathbb{C})$ parametrizing polynomials of degree $k + 1$ with distinct zeroes, which yields a representation

$$\rho : Br_{k+1} \rightarrow \pi_0 \text{Symp}_{ct}(A_k^{(n)})$$

from the classical braid group on $k + 1$ strings. Combining the work of many authors one has:

Proposition 3.1 [KS02, Corollary 1.4] *ρ is injective for every k, n .*

Proposition 3.2 [Eva11, Theorem 1.4] and [Wu14, Theorem 1.2]. *When $n = 2$, ρ is an isomorphism.*

Proposition 3.3 [Sei08a, Corollary 17.17] and [DRE15, Theorem A]. *For some pairs k, n the monodromy ρ is not surjective.*

The first result is based on direct computations of Floer cohomology; the second relies on special features of holomorphic curves in four dimensions; the third result uses ‘reframings’ of twists by exotic diffeomorphisms of spheres (for instance, it applies when $n = 8$ and $k \in \{1, 2\}$). This last result makes the following interesting:

Theorem 3.4 *For $n \geq 3$ and arbitrary k , the map ρ splits.*

We first establish the slightly simpler:

Proposition 3.5 *There is a map $p : \pi_0 \text{Symp}_{ct}(A_k^{(n)}) \rightarrow Br_{k+1}/Z(Br_{k+1})$ for which the composite $p \circ \rho$ is the natural projection.*

Proof Since $\pi_0 \operatorname{Symp}_{ct}(A_k^{(n)})$ maps canonically to $\pi_0 \operatorname{Symp}^{gr}(A_k^{(n)})$, it acts on the space of stability conditions on the compact Fukaya category $\mathcal{F}$. This has a distinguished component in the stability space coming from tilts of chambers associated to the standard heart. [QW18, Theorem B] (building on references in *loc. cit.*) proves that in fact the space of stability conditions is connected, so the whole of $\operatorname{Auteq}(\mathcal{F})$ acts on this unique component. The autoequivalences preserving the given component have been computed in [BS15, Theorem 9.9] when $n = 3$ and in general in [Ike17, Theorem 7.16] and [QW18, Theorem B]. The complete answer is as follows. We can identify $\mathcal{F}$ with the category of modules over the endomorphism algebra of the A_k -chain of Lagrangian ‘core’ spheres. Ikeda proves that

$$\operatorname{Auteq}(\mathcal{F}(A_k^{(n)})) = (Br_{k+1} \times \mathbb{Z})/\mathbb{Z}$$

where the $\mathbb{Z}$ -factor in the numerator is generated by the shift, and the embedding of the denominator sends the generator to $(\Delta, d_{k,n})$ where Δ generates the centre and

$$d_{k,n} = (k+1)(n-2) + 2.$$

There are thus exact sequences

$$(3.B) \quad 1 \rightarrow Br_{k+1} \rightarrow \operatorname{Auteq}(\mathcal{F}(A_k^{(n)})) \rightarrow \mathbb{Z}/d_{k,n}\mathbb{Z} \rightarrow 1;$$

and

$$(3.C) \quad 1 \rightarrow \mathbb{Z} \rightarrow \operatorname{Auteq}(\mathcal{F}(A_k^{(n)})) \rightarrow Br_{k+1}/Z(Br_{k+1}) \rightarrow 1$$

and the latter admits a map from the sequence

$$1 \rightarrow \mathbb{Z} \rightarrow \pi_0 \operatorname{Symp}_{ct}^{gr}(A_k^{(n)}) \rightarrow \pi_0 \operatorname{Symp}_{ct}(A_k^{(n)}) \rightarrow 1$$

where we are using the fact that the forgetful map $\pi_0 \operatorname{Symp}_{ct}^{gr}(A_k^{(n)}) \rightarrow \pi_0 \operatorname{Symp}_{ct}(A_k^{(n)})$ splits by Lemma 2.2. The existence of a map $\pi_0 \operatorname{Symp}_{ct}(A_k^{(n)}) \rightarrow Br_{k+1}/Z(Br_{k+1})$ follows. $\square$

Remark 3.6 Since the centre of the braid group acts by a shift on $\mathcal{F}(X)$, it acts trivially on any simplicial complex constructed from the set of (ungraded) Lagrangian spheres in X . Note that the space of stability conditions on a partially wrapped Fukaya category, such as that considered in [Sei08a, Section 20], has not been computed.

Via the splitting of the map from graded to ungraded symplectomorphisms over the compactly supported subgroup of Lemma 2.2, we have a representation

$$\rho : \pi_0 \operatorname{Symp}_{ct}(A_k^{(n)}) \rightarrow \operatorname{Auteq}(\mathcal{F}(A_k^{(n)}))$$

and to prove Theorem 3.4 it remains to show that the composite $\chi \circ \rho : \pi_0 \operatorname{Symp}_{ct}(A_k^{(n)}) \rightarrow \mathbb{Z}/d_{k,n}\mathbb{Z}$ vanishes. This will follow directly from a geometric picture of the target.

Recall the cluster category $\mathcal{C}_{k,n}$ of type (k, n) is by definition a certain orbit category of the path category of the underlying quiver of $A_k^{(n)}$, given by dividing out by a composition of the shift and the Auslander-Reiten translation. It is naturally a CY_{n-1} -category. Let

$$d_{k,n} = (n-2)(k+1) + 2.$$

Let (S, M) be the marked bordered surface which is the disc S with a collection of $d_{k,n}$ boundary marked points, equivalently of a regular $d_{k,n}$ -gon. We think of the group $\mathbb{Z}/d_{k,n}\mathbb{Z}$ as the oriented mapping class group $\Gamma(S, M)$ of (S, M) .

Lemma 3.7 *There is a natural homomorphism*

$$\mathrm{Auteq}(\mathcal{C}_{k,n}) \longrightarrow \Gamma(S, M) = \mathbb{Z}/d_{k,n}\mathbb{Z}.$$

Proof This is a variation on the proof of [BS15, Theorem 9.9], see also [Ike17, Theorem 7.15], and is exactly the result [BQ15, Theorem 4.7] in the special case when the cluster category is CY_2 . The paper [BM08] gives a combinatorial description of the cluster category. Its indecomposable objects are given by the n -diagonals of a $d_{k,n}$ -gon (i.e. diagonals which decompose the polygon into a n -gon and a remainder) and the cluster tilting objects biject with n -angulations of the $d_{k,n}$ -gon (maximal sets of disjoint n -diagonals). See [BM08, Proposition 5.4] and [JM22, Theorem 4.2]. (In the special case where the cluster category is CY_2 one studies actual triangulations of the polygon.)

The cluster exchange graph of $\mathcal{C}_{k,n}$ (with vertices being simple hearts and edges arising from tilts) then has a purely combinatorial description in terms of flips of n -angulations, cf. [BM08, Theorem 5.6] or [Ike17, Theorem 6.3]. The orientation-preserving mapping class group of the marked disc acts freely on n -angulations and hence on the cluster exchange graph.

In [KQ15, Theorem 8.6] the authors show that the braid group acts transitively on the fibres of the natural map from the exchange graph of the Ginzburg CY_n -category associated to the quiver underlying $A_k^{(n)}$ to that of the exchange graph of $\mathcal{C}_{k,n}$. This then establishes all the conditions needed for the argument of [BS15, Theorem 9.9]. $\square$

Lemma 3.8 *There is an equivalence $\mathcal{R}(A_k^{(n)}) \simeq \mathcal{C}_{k,n}$.*

Proof The cluster category and Rabinowitz Fukaya category for a Dynkin quiver are identified in [BJK22, Corollary 1.4]. $\square$

Lemma 3.9 *There is a commuting diagram*

$$\begin{array}{ccc} \mathrm{Auteq}(\mathcal{F}(A_k^{(n)})) & \xrightarrow{\text{Ikeda}} & \mathbb{Z}/d_{k,n}\mathbb{Z} \\ \uparrow & & \parallel \\ \mathrm{Auteq}(\mathcal{W}(A_k^{(n)})) & \longrightarrow \mathrm{Auteq}(\mathcal{R}(A_k^{(n)})) \longrightarrow & \mathbb{Z}/d_{k,n}\mathbb{Z} \end{array}$$

Proof Any autoequivalence of the wrapped category preserves the compact category. Ikeda [Ike17, Section 7] constructs the top arrow using the action on the heart exchange graph. The quotient of the heart exchange graph by the braid group action is identified with the cluster exchange graph of $\mathcal{R}(A_k^{(n)})$ by [KQ15]. Furthermore the cluster exchange graph is purely combinatorial, as discussed above, and the bottom right map is constructed using the action on the cluster exchange graph. $\square$

Remark 3.10 It seems likely that the simplicial automorphism group of the cluster exchange graph is given by the mapping class group of the marked surface, which in this case is just the given cyclic group. This is known in the case of triangulations (here corresponding to the 3-fold case), but does not seem to have been written down for m -angulations in general. That would give a more direct argument for the previous result.

Since compactly supported symplectomorphisms of X act on both $\mathcal{F}$ and $\mathcal{W}$ and act trivially on the quotient category $\mathcal{R}(X)$, we immediately infer that the composite map

$$\pi_0 \operatorname{Symp}_{ct}(X) \rightarrow \operatorname{Auteq}(\mathcal{F}(A_k^{(n)})) \rightarrow \mathbb{Z}/d_{k,n}\mathbb{Z}$$

vanishes, hence the first arrow lifts to Br_{k+1} , as required.

The case of type A Milnor fibres raises an obvious question:

Question 3.11 *Let X be the Milnor fibre of an isolated hypersurface singularity of dimension ≥ 2 . Let $\mathcal{M}_X$ denote the universal unfolding of the singularity. Does symplectic monodromy split over $\mathcal{M}_X$?*

Remark 3.12 The dimension hypothesis is necessary. Indeed, already in type A in the lowest-dimensional case $n = 1$ in (3.A), the monodromy representation defines an embedding of the braid group into the mapping class group of a punctured surface. Such embeddings do not split (for instance, the mapping class group typically has trivial abelianisation, and the braid group has abelianisation $\mathbb{Z}$; there are many more subtle obstructions, e.g. the induced map on stable homology is trivial in positive degrees, cf. [ST07]).

4 Quivers and threefolds from marked bordered surfaces

The type A Milnor fibres $A_k^{(n)}$ have a quite different generalisation from that of Question 3.11, namely affine Calabi-Yau varieties which are Lefschetz fibrations with general fibre T^*S^{n-1} over a non-compact surface. We will focus on the case of 3-folds fibred in T^*S^2 . In this case, the space of stability conditions is not known to be connected, but we can access an analogue of Proposition 3.5 and Theorem 3.4 using knowledge of spherical objects. (Note that in more complicated cases the relevant braid-type group may have no center, in which case these two results are essentially equivalent.)

Let S be a surface of genus g with a non-empty set of d marked points $\{p_1, \dots, p_d\}$. Fix an integer $d_i > 2$ for each p_i ; we encode this information by a vector $\mathbf{d}$. We take the real blow-up of S at the $\{p_i\}$ and mark $d_i - 2$ points on the i -th boundary component; let M denote the resulting collection of marked points. We consider ideal triangulations of S in the sense of [FST08], i.e. maximal collections of disjoint arcs, where an arc is a smooth embedded path connecting points of $M \subset \partial S$, with interior in $S \setminus \partial S$. [To get something visibly resembling a triangulation one should include arcs in ∂S joining points of M , but these are *not* included in the set of arcs.] A triangle is ‘interior’ if its closure meets ∂S only in M . Any ideal triangulation has

$$(4.A) \quad 6g - 6 + \sum_i (d_i + 1) \text{ edges} \quad \text{and} \quad 4g - 4 + \sum_i d_i \text{ faces}$$

(we are in the case with ‘no punctures’ in the terminology of [BS15], and we are not including the boundary arcs between points of M as edges).

We will be interested in the cellulation which is dual to the given ideal triangulation, whose vertices will correspond to simple zeroes of a quadratic differential below.

There is a quiver associated to the ideal triangulation, with vertices the mid-points of the edges and an arrow when two open arcs share a vertex (in the direction in which one is anticlockwise from the other). The resulting collection of arrows in particular inscribes a clockwise 3-cycle in each interior triangle,

so comes with an obvious collection of 3-cycles: the triangles just inscribed. This defines a canonical potential

$$W = \sum (\text{Q triangles})$$

and hence via the Ginzburg construction a CY_3 -category $\mathcal{C}(Q, W)$, which one can show depends only on the pair (S, M) or equivalently on $(g, \mathbf{d})$. More precisely, flips of the triangulation induce mutations of quivers-with-potential, which yield quasi-equivalent categories by [KY11]. We may write $\mathcal{C}(S, M)$ or $\mathcal{C}(g, \mathbf{d})$ for a category in this equivalence class.

Let ϕ be a meromorphic quadratic differential on a surface S with poles of order d_i at p_i and with distinct zeroes. Let $D \subset S$ be the set of poles. Let $Y = Y_\phi \rightarrow S \setminus D$ be the Lefschetz fibration with nodal singular fibres over the zeroes of ϕ . This is a non-compact affine Calabi-Yau 3-fold, which moves in a family over the complex orbifold $\text{Quad}(g, \mathbf{d})$ of quadratic differentials with the prescribed pole orders and with simple zeroes. An embedded ‘matching path’ in S between zeroes of ϕ defines a Lagrangian 3-sphere in Y_ϕ .

Lemma 4.1 *An ideal triangulation defines a collection of matching paths, which form the edges of the dual cellulation.*

See [Smi15] for a proof, and further background.

Let $\Gamma(S, M)$ denote the mapping class group of the associated marked bordered surface². Let $\text{Sph}(S, M)$ denote the group generated by spherical twists in matching paths dual to the edges of the ideal triangulation.

Theorem 4.2 [BS15, Smi15] *There is a distinguished connected component $\text{Stab}^\dagger(\mathcal{C}(S, M))$ of the space of stability conditions on $\mathcal{C}(S, M)$, and a subgroup $\text{Auteq}^\dagger(\mathcal{C}) \subset \text{Auteq}(\mathcal{C})$ of autoequivalences preserving that component, with the following properties:*

- $\text{Stab}^\dagger(\mathcal{C}(S, M)) / \text{Auteq}^\dagger(\mathcal{C}) \cong \text{Quad}(g, \mathbf{d})$ is the complex orbifold whose points parametrise biholomorphism classes of pairs of a Riemann surface S of genus g and a meromorphic quadratic differential on S with poles of orders $d_i > 2$ and with simple zeroes.
- There is an exact sequence

$$(4.B) \quad 1 \rightarrow \text{Sph}(S, M) \rightarrow \text{Auteq}^\dagger(\mathcal{C}(S, M)) \rightarrow \Gamma(S, M) \rightarrow 1.$$
- There is an embedding $\mathcal{C}(Q, W) \hookrightarrow \mathcal{F}(Y_\phi)$ for (Q, W) the quiver with potential associated to any ideal triangulation of (S, M) .

Remark 4.3 Let $Z \subset \text{interior}(S) \subset S \setminus M$ denote a set of points in bijection with the (assumed simple) zeroes of a quadratic differential ϕ of the prescribed polar type. We can consider the mapping class group $\Gamma(S, M; Z)$ of mapping classes which fix M setwise and which preserve the finite set Z setwise. This fits into a short exact sequence

$$1 \rightarrow \text{SBr}(S, Z) \rightarrow \Gamma(S, M; Z) \rightarrow \Gamma(S, M) \rightarrow 1$$

where the first term is the ‘spherical braid group’ of (S, Z) , i.e. the fundamental group of the space of configurations of unordered configurations of $|Z|$ -many points on S . The spherical braid group has

²For comparison to [BS15, Smi15], note that we are in the case of an unpunctured surface with non-empty boundary.

two kinds of generators: braid-type maps associated to a matching path between two distinct points of Z , and point-pushing maps associated to a non-trivial loop on S and a single point of Z . [Qiu16] has shown that $\text{Sph}(S, M)$ is the subgroup of $\text{SBr}(S, Z)$ generated by the first type of map.

Let $\Sigma \rightarrow S$ be the spectral curve, which is the double cover branched at $Z \subset S$, and let $\iota : \Sigma \rightarrow \Sigma$ denote the covering involution. Let $\Gamma(\Sigma, \iota)$ denote the hyperelliptic-type mapping class group of isotopy classes of diffeomorphisms of Σ which commute with ι . Associating to a braid-twist in a matching path γ on S with end-points in Z the Dehn twist in the ι -invariant simple closed curve $\hat{\gamma} \subset \Sigma$ which is the lift of γ shows that there is an embedding

$$\text{Sph}(S, M) \hookrightarrow \Gamma(\Sigma, \iota)$$

According to the work of [BH73] on symmetric mapping class groups, cf. also [MW21], and noting that we might as well restrict to the part of the autoequivalence group which maps to $\Gamma(S, M_{=})$ which fixes M and hence ∂S pointwise, we find that when $\chi(S) < 0$, then $\text{Sph}(S, M)$ is characterised as the centraliser of the involution ι in $\Gamma(\Sigma, \partial\Sigma)$. [We don't need to divide by ι since the mapping class group of a surface with non-empty boundary, fixing the boundary pointwise, is torsion-free, so ι does not define an element of $\Gamma(\Sigma, \partial\Sigma)$.]

The group $\text{Sph}(S, M)$ acts via spherical twists which are geometrically realised by Dehn twists in Lagrangian matching spheres, so there is a natural representation

$$\text{Sph}(S, M) \rightarrow \pi_0 \text{Symp}_{ct}(Y)$$

The following generalises (the 3-dimensional instance of) Proposition 3.5.

Theorem 4.4 *The monodromy map $\rho : \text{Sph}(S, M) \rightarrow \pi_0 \text{Symp}_{ct}(Y)$ splits.*

Recall that a spherical object is reachable if it is one of the simples in a reachable heart, i.e. one obtained from the standard heart by a finite sequence of tilts.

Lemma 4.5 *The set of reachable spherical objects in $\mathcal{C}(Q, W)$ is in bijection with the set of Lagrangian matching spheres.*

Proof This is proved in [KQ20] (note we are in the ‘unpunctured case’ which they treat). $\square$

Lemma 4.6 *Every Lagrangian 3-sphere in Y_ϕ is quasi-isomorphic to the Lagrangian 3-sphere associated to some (unique up to isotopy) embedded matching path.*

Proof Let $L \subset Y_\phi$ be a Lagrangian 3-sphere. Then L lifts to a Lagrangian $\tilde{L}$ in the universal cover $\pi : \tilde{Y} \rightarrow Y$, which is a Lefschetz fibration $\tilde{Y} \rightarrow \mathbb{C}$ with fibre T^*S^2 and infinitely many Lefschetz singular fibres. This is an ‘infinite’ Milnor fibre of type A in dimension 3. Being compact, any lift of L lives in a finite subspace which is the total space of a fibration $A_k^{(3)}$ for some $k \gg 0$. Spherical objects in $A_k^{(3)}$ are classified; for instance, this follows from the result of [KQ20] together with the fact that the heart exchange graph is connected in type A [Ike17], which means that all hearts are reachable. It follows that there is a unique embedded matching path γ , with associated Lagrangian sphere $\tilde{L}_\gamma$, such that $\tilde{L}$ is quasi-isomorphic to $\tilde{L}_\gamma$ in the Fukaya category of $A_k^{(3)}$.

Let σ denote the immersed matching path downstairs which is the image of γ , and let L_σ be the corresponding immersed matching sphere, which is the image of $\tilde{L}_\gamma$ under π . We want to show that

L and L_σ are quasi-isomorphic and that σ is embedded. In order to do this, we will construct a quasi-isomorphism upstairs and descend it.

Let $\tilde{L}'_\gamma$ be a small Hamiltonian perturbation of $\tilde{L}_\gamma$. By a small perturbation of L pulled back to $\tilde{L}$, we can assume that the pairs $(\tilde{L}, \tilde{L}_\gamma)$ and $(\tilde{L}, \tilde{L}'_\gamma)$ both intersect transversally. Let σ denote the immersed matching path downstairs which is the image of γ , and let L_σ respectively L'_σ be the images under π of $\tilde{L}_\gamma$, respectively $\tilde{L}'_\gamma$. Note that γ being embedded means that σ bounds no teardrops (disjoint from the marked points), which implies the associated immersed (cleanly self-intersecting) Lagrangian 3-sphere $L_\sigma \subset Y_\phi$ is unobstructed.

Pick a regular J on Y_ϕ and pull it back to $\tilde{J}$ on $\tilde{Y}$. By unique lifting of holomorphic strips based at a point, the projection map π defines a chain map

$$(4.C) \quad CF_{\tilde{Y}}(\tilde{L}_\gamma, \tilde{L}; \tilde{J}) \rightarrow CF_Y(L_\sigma, L; J)$$

and similarly for $(\tilde{L}, \tilde{L}_\gamma)$ and $(\tilde{L}, \tilde{L}'_\gamma)$.

Pick cycles $\alpha \in CF_{\tilde{Y}}(\tilde{L}, \tilde{L}'_\gamma)$, $\beta \in CF_{\tilde{Y}}(\tilde{L}_\gamma, \tilde{L})$ such that

$$\mu_2(\alpha, \beta) = e \in CF_{\tilde{Y}}(\tilde{L}_\gamma, \tilde{L}'_\gamma)$$

where e represents the cohomological unit. The images of α, β and e under π all represent cycles by 4.C, say $\pi(\alpha), \pi(\beta)$ and $\pi(e)$. By unique lifting for based holomorphic triangles, there is a non-trivial Floer product

$$\mu_2(\pi(\alpha), \pi(\beta)) = \pi(e) \in CF_Y(L_\sigma, L'_\sigma)$$

and $\pi(e)$ again represents the cohomological unit.

Taking the product in the other order (and using L, L_σ and a small Hamiltonian perturbation of L), we deduce that L and L_σ are quasi-isomorphic.

Finally, if σ is not embedded, then by the arguments of [KS02], $HF(L_\sigma, L_\sigma)$ has rank > 2 , contradiction. $\square$

Corollary 4.7 *$\mathcal{C}(Q, W)$ is a symplectomorphism-invariant subcategory of $\mathcal{F}(Y_\phi)$, and all Lagrangian spheres are reachable spherical objects.*

We now know that $\pi_0 \text{Symp}_{ct}$ preserves reachable spherical objects, but not at this point reachable hearts.

Lemma 4.8 *Let L and L' be two Lagrangian spheres in Y_ϕ . Then $HF^*(L, L')$ has rank one exactly when the two corresponding matching paths share one end-point.*

Proof Let γ and γ' denote the relevant matching paths. By isotopy of these (which induces a Hamiltonian isotopy of the corresponding matching Lagrangian 3-spheres), we can assume that the two paths meet minimally. If two sub-arcs of $\gamma \cup \gamma'$ bound a loop which is non-trivial in $\pi_1(S)$ then there can clearly be no Floer strips with those sub-arcs as boundary. Suppose two sub-arcs of the paths bound a region which contains a Lefschetz critical point of the fibration. Then there are no holomorphic strips which project to that region; this follows from the same vanishing result that underlies the proof of the long exact triangle in Floer theory (cf. [Sei08a] and compare to the rank computations in [KS02]). Given this, the result is straightforward. $\square$

Lemma 4.9 *Let L and L' be two Lagrangian spheres in Y_ϕ which are not quasi-isomorphic. Then $HF^*(L, L')$ has rank two exactly when*

- (1) either the two associated matching paths share both endpoints and are otherwise disjoint;
- (2) or the two associated matching paths meet transversely at exactly one interior point.

Proof Since $L \not\sim L'$ the associated matching paths γ and γ' represent different relative homotopy classes in $\pi_1(S, \text{Zeroes})$. If they meet only at both end-points, then since the paths are not homotopic their union traces a non-trivial loop, which means that the union $L_\gamma \cup L_{\gamma'}$ bounds no rigid holomorphic strips, so $HF(L, L')$ is indeed rank two. If they meet transversely at an interior point, then L_γ and $L_{\gamma'}$ meet cleanly along an S^2 lying above that point. Putting the matching paths in minimal position and arguing as in the proof of Lemma 4.8 shows that any such S^2 -component of the intersection locus contributes a two-dimensional vector space to Floer cohomology. $\square$

Lemma 4.10 *Suppose L, L' are not quasi-isomorphic and $HF^*(L, L')$ has rank two. The two cases of Lemma 4.9 cannot be related by a global symplectomorphism.*

Proof Assume first that our differentials ϕ have at least five zeroes. We claim that the two cases are distinguished by whether or not there is a third Lagrangian sphere L'' with $HF^*(L, L'')$ of rank one and $HF^*(L', L'')$ of rank zero. To see this, suppose $p \in \gamma$ is an end-point of γ and $p \notin \gamma'$. Then we are necessarily in the second case, and the union $\gamma \cup \gamma'$ lies in a disc in S and in particular does not separate. Then one can take a third matching path σ which meets γ at p but is disjoint from γ' . Suppose then ϕ has exactly four zeroes (so the original ideal triangulation had four faces). By (4.A) S is a disc, annulus or a torus with one boundary component. The first two cases correspond to the A_4 or affine A_4 spaces, where the result is well-known (cf. the proof of Lemma 4.8). In the final case either the two matching paths sweep a non-trivial loop in $\pi(T^2)$, and the result is straightforward, or the whole configuration lies in a disc or annulus, reducing one to a previously considered case. $\square$

The set $\text{Tri}(S, M)$ of ideal triangulations of (S, M) forms a graph, where two triangulations are connected by an edge if they are related by a flip. (Since we have no punctures, there are no self-folded triangles.)

Lemma 4.11 $\pi_0 \text{Symp}_{ct}(Y_\phi)$ acts on the graph of ideal triangulations of (S, M) .

Proof Let $f \in \pi_0 \text{Symp}_{ct}(Y_\phi)$. Fix an ideal triangulation Δ , which defines a collection of matching spheres $\{L_e\}$ indexed by the edges of Δ . We know that for each e the Lagrangian sphere $f(L_e)$ is associated to some new matching path $\gamma_f(e)$. Moreover, Lemma 4.8, 4.9 and Lemma 4.10 imply that $\gamma_f(e)$ and $\gamma_f(e')$ share exactly one respectively two end-points exactly when e and e' share exactly one respectively two end-points. The collection $\{\gamma_f(e)\}_{e \in \Delta}$ form a configuration of $4g - 4 + \sum k_i$ matching paths with no interior intersections, so these must form a maximal set of such arcs, hence are dual to the edges of some ideal triangulation. $\square$

Corollary 4.12 $\pi_0 \text{Symp}_{ct}(Y_\phi)$ preserves $\text{Stab}^\dagger(\mathcal{C}(S, M))$.

Proof It acts on the graph $\text{Tri}(S, M)$, which is connected. This graph is identified [KQ20] with the cluster exchange graph of reachable hearts. Therefore the group sends reachable hearts to reachable hearts, and hence preserves the distinguished component of the space of stability conditions. $\square$

If we fix a generic ideal triangulation with no self-folded triangles, we get a finite collection of Lagrangian 3-spheres L_e indexed by the dual 1-cycles to the edges e of the triangulation, and these meet pairwise transversely. Let Y_{pl} denote the plumbing of copies of T^*S^3 according to the graph defined by the dual cellulation to the ideal triangulation. A Stein subdomain of the compact core of Y_{pl} embeds into Y_ϕ , inducing an equivalence on compact and wrapped Fukaya categories. Let T_e^* denote the dual cocore to L_e (so T_e^* is disjoint from $L_{e'}$ for $e' \neq e$, i.e. we are not using Lefschetz thimbles but ascending manifolds of a suitable plurisubharmonic function, or cotangent fibres in a ‘plumbing model’).

Lemma 4.13 *In the previous situation:*

- (1) Y_ϕ admits a grading structure for which the algebra $\bigoplus_{e,e'} HF^*(L_e, L_{e'})$ is concentrated in non-negative degree and $\bigoplus_{e,e'} HW^*(T_e^*, T_{e'}^*)$ is concentrated in non-positive degree.
- (2) The inclusion $\mathcal{C}(Q, W) \hookrightarrow \mathcal{F}(Y_\phi)$ is an equivalence, i.e. the compact cores generate the compact category. Thus $\mathcal{F}(Y_\phi)$ is equivalent to the dg-category of finite-dimensional modules over the Ginzburg algebra $J(Q, W)$.
- (3) The compact category $\mathcal{F}(Y_\phi)$ and wrapped category $\mathcal{W}(Y_\phi)$ are Koszul dual, and $\mathcal{W}(Y_\phi)$ is equivalent to the dg-category of all modules over $J(Q, W)$.

Proof The first two results follow from [AS12]. Finite-dimensionality of HW^0 follows from the explicit computation in [KL24] using that around any puncture, the sum of the corresponding d_e is strictly negative, because the underlying quadratic differential has a pole of order $d_i > 2$. Given that, the argument in [AS12] shows that the compact cores generate the compact category. (Note that the grading argument in that paper is local near each intersection point, providing one chooses gradings for all core components which are symmetric in the sense of *op. cit.*) Koszul duality then follows from [KY16]. The equivalence to the given Ginzburg algebras is established in [Smi15] for the compact category and [Chr22, KL24] for the wrapped category. $\square$

Lemma 4.14 *The Rabinowitz Fukaya category of Y_ϕ is the cluster category $\text{Clust}(Q, W)$ of the quiver with potential (Q, W) . Moreover, the cluster tilting objects are in bijection with ideal triangulations.*

Proof The first statement follows from the previous result, and the definition of the cluster category for a quiver with potential with finite-dimensional Jacobi algebra, cf. [Ami09]. In the case of a marked bordered surface, the cluster-tilting objects are classified in [BZ11]. $\square$

Lemma 4.15 *There is a natural surjection $\text{Auteq}(\text{Clust}(Q, W)) \longrightarrow \Gamma(S, M)$*

Proof We have an identification between the triangulation graph $\text{Tri}(S, M)$ and the cluster exchange graph, so the given autoequivalence group acts by simplicial automorphisms of the triangulation graph. The group of such simplicial automorphisms was computed in [Dis15] and is exactly $\Gamma(S, M)$. $\square$

We now have a map

$$\text{Sph}(S, M) \rightarrow \pi_0 \text{Symp}_{ct}(Y_\phi) \rightarrow \text{Auteq}^\dagger(\mathcal{C}(Q, W)).$$

Our main result is completed on showing:

Lemma 4.16 *The composite map*

$$\pi_0 \operatorname{Symp}_{ct}(Y_\phi) \rightarrow \operatorname{Auteq}^\dagger(\mathcal{C}(Q, W)) \rightarrow \Gamma(S, M)$$

vanishes.

Proof The map $\operatorname{Auteq}^\dagger(\mathcal{C}(Q, W)) \rightarrow \Gamma(S, M)$ factors through autoequivalences of the cluster category, hence the composite map factors through the autoequivalence group of the Rabinowitz category, and so the result follows from Lemma 2.5. $\square$

5 The double bubble and multi-bubbles

Arguably the simplest quiver admitting a non-trivial potential is the oriented 2-cycle:

$$\bullet \begin{array}{c} \xrightarrow{e} \\ \xleftarrow{f} \end{array} \bullet, \text{ potential } W = (ef)^2$$

(Any potential has the form $W = \sum_j a_j (ef)^j = (ef)^k (a_k + \dots) \simeq (ef)^k$, up to formal change of variable and working over $\mathbb{C}$.) We write $\mathcal{C}$ for the corresponding CY_3 -category.

A symplectic model X is given by the affine $(1, 1)$ -hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$; this is also a plumbing of two Lagrangian S^3 s L_1, L_2 along an embedded unknotted circle S^1 such that the Morse-Bott Lagrangian surgery K of L_1, L_2 is again a 3-sphere. Considering suitable pencils and nets of hypersurfaces shows that there is a monodromy representation

$$PBr_3 \rightarrow \pi_0 \operatorname{Symp}_{ct}(X, \omega)$$

with the pure braid group generated by the spherical twists $\tau_{L_1}, \tau_{L_2}, \tau_K$; this can be seen more concretely from the realisation of X as a Morse-Bott-Lefschetz fibration over $\mathbb{C}$, which comes in a family of such where one moves the critical values of the fibration. (Since the vanishing cycles differ at each singularity, only pure braids show up.)

Theorem 5.1 *The natural monodromy map $PBr_3 \rightarrow \pi_0 \operatorname{Symp}_{ct}(X, \omega)$ splits.*

En route to showing this, it will be helpful to know:

Lemma 5.2 *Any compactly supported symplectomorphism of X acts trivially on $K_{num}(\mathcal{F}(X)) \simeq \mathbb{Z}^2$.*

Proof One checks that $K_{num}(\mathcal{F}(X)) \cong H_3(X; \mathbb{Z})$, and that $H^3(X) \rightarrow H^3(\partial X)$ is an injection. The latter follows from considering the exact sequence for the pair $(X, \partial X)$ and computing the cohomology of ∂X from the Gysin sequence, remembering that X is the complement of a generic $(1, 1)$ -divisor in the flag 3-fold (which is itself a $(1, 1)$ -divisor in $\mathbb{P}^2 \times \mathbb{P}^2$), in particular $\partial X \rightarrow dP_3$ is a circle bundle over the del Pezzo surface $\mathbb{P}^2 \# 3\overline{\mathbb{P}}^2$. Compare to the directly analogous argument in [KS24]. $\square$

There is also a geometric model in algebraic geometry, taking Y to be the crepant resolution $f : Y \rightarrow \operatorname{Spec}(R)$ of the isolated singularity given by $R = \mathbb{C}[[u, v, x, y]] / \langle uv - xy(x + y) \rangle$; then Y is a neighbourhood of two floppable $(-1, -1)$ -curves meeting once. We consider the subcategory $\mathcal{C}_Y$ of the derived category of Y of objects $\mathcal{E}$ supported on the exceptional locus and with $Rf_* \mathcal{E} = 0$ (this is split-generated by the $\mathcal{O}_{\mathbb{P}^1}(-1)$ for the two exceptional $\mathbb{P}^1$'s).

There are actions of the pure braid group on X and on $D(Y)$ via symplectic parallel transport respectively via flop-flop functors. Mirror symmetry in this example states:

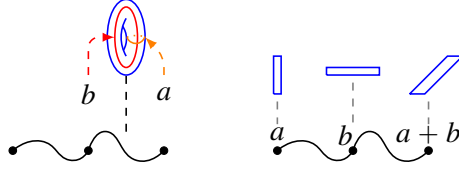


Figure 2: Vanishing cycles for Morse-Bott-Lefschetz fibration $X \rightarrow \mathbb{C}$, fibre $(\mathbb{C}^*)^2$

Theorem 5.3 [SW23, Proposition 4.27(1), Corollary 5.7] *There are equivalences*

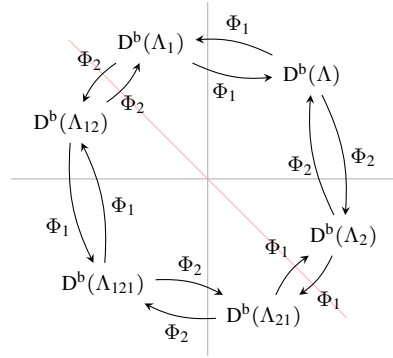
$$\mathcal{F}(X) \simeq \mathcal{C}_Y \quad \text{and} \quad \mathcal{W}(X) \simeq D(Y)/\langle \mathcal{O}_Y \rangle$$

which entwine the action of the pure braid group.

Remark 5.4 It seems likely that $D(Y)$ itself is a ‘partially wrapped’ category of X ; however, it seems hard to characterise $D(Y)$ in a way which lets one pick out (and prove the equivalence to) a distinguished such. Note that $\mathcal{W}(X)$ is not a categorical crepant resolution of $\text{Spec}(R)$ (contrast with the setting of [Pom21]); since X admits a projective compactification with smooth boundary divisor and $SH^*(X)$ has linear growth, $\text{Spec } SH^0(X)$ is at most 1-dimensional.

The space of stability conditions on $\mathcal{C}_Y = \mathcal{C}$ was shown to be connected in [HW22], so the ‘distinguished’ component is preserved by all autoequivalences. They also proved that the ‘Fourier-Mukai’ equivalences (namely those induced from equivalences of $D(Y)$ commuting with Rf_*) define a group $\text{Auteq}^{FM}(\mathcal{C}) \simeq PBr_3$. Unfortunately, the symplectic counterpart of commuting with Rf_* is not immediate. We circumvent this by using the classification of all hearts of bounded t -structures on $\mathcal{C}$.

We briefly recall some results on the Deligne groupoid that appears in this setting. One can place the six different crepant resolutions of the singularity of $\text{Spec}(R)$ in the chambers of a hyperplane arrangement, where they are related by *mutation functors* Φ_1 and Φ_2 corresponding to mutation between the NCCRs of R at vertex 1, and vertex 2 respectively. These form a representation of the Deligne groupoid, as follows:



where the algebras $\Lambda, \Lambda_1, \Lambda_2, \Lambda_{12}, \Lambda_{21}, \Lambda_{121}$ are the six NCCRs corresponding to the six crepant resolutions of $\text{Spec} R$. As explained in [HW23, Remark 4.7], there is a functorial isomorphism $\Phi_1 \Phi_2 \Phi_1 \cong \Phi_2 \Phi_1 \Phi_2$. By slight abuse of notation, we write $\mathcal{S}_1, \mathcal{S}_2$ for the simple modules for the NCCRs, regardless of which NCCR is under consideration. The labelling is consistent, so that mutation Φ_i always shifts $\mathcal{S}_i$.

Write C_+ for the top right chamber in the hyperplane arrangement.

Lemma 5.5 *There is a bijection between morphisms in the Deligne groupoid ending at C_+ and bounded t -structures on $\mathcal{C}$.*

Proof This is the analogue of [AW22, Corollary 5.3] but dealing with bounded t -structures in place of tilting complexes. One considers the map $\alpha \mapsto \Phi_\alpha(\mathbb{T})$, with $\mathbb{T}$ the standard t -structure, as a map from morphisms in the groupoid to t -structures. Comparing to the proof in *op. cit.* the injectivity argument goes through exactly as written (using that the pure braid group action on $\mathcal{C}$ is faithful, by e.g. [AW22, Corollary 5.5]), whilst the surjectivity argument follows from [HW22, Theorem 4.6]. $\square$

Lemma 5.6 *If f is a compactly supported symplectomorphism of X inducing an autoequivalence F , there is a path α so that $F(S_i) = \Phi_\alpha(S_i)$ for all the simple objects in the standard heart.*

Proof We know that F applied to the standard t -structure $\heartsuit$ yields some bounded heart, so by the previously established bijection there is some α so that $F(\heartsuit) = \Phi_\alpha(\heartsuit)$. Now $F \circ \Phi_\alpha^{-1}$ fixes $\heartsuit$ and hence permutes the simples. By Lemma 5.2, F acts trivially on K -theory so induces the identity permutation. On the other hand, the combinatorics of [HW23, Proposition 4.8] shows that Φ_α can only act on K -theory via one of the six possible matrices realised by the compositions of the standard reflection functors, and one checks directly that only the identity map acts by a permutation matrix. The result follows. $\square$

Lemma 5.7 *If $\Phi_i^\pm$ denote the left and right tilts at the simple S_i , then $F \circ \Phi_i^\pm$ and $\Phi_\alpha \circ \Phi_i^\pm$ send the standard heart to the same heart.*

Proof The definition of the right and left tilts depends only on the heart and choice of simple. The result then follows from Lemma 5.6, which implies that $F \circ \Phi_i^\pm$ and $\Phi_\alpha \circ \Phi_i^\pm$ act in the same way on the torsion theory defining the tilt. $\square$

Lemma 5.8 *All bounded hearts in $\mathcal{C}$ are obtained from a sequence of tilts starting from the standard heart (i.e. the heart exchange graph is connected).*

Proof By Lemma 5.5 every bounded t -structure is of the form $\Phi_\alpha(\mathbb{T})$, where Φ_α is a composition of mutation functors and their inverses. The statement follows using the fact that the mutation functors (and their inverses) give the left and right tilts at simples (see e.g. [HW23, Lemma 5.5]). $\square$

We now induct over the number of tilts needed to reach some given heart from the standard heart (this is a careless induction: we are not appealing to any normal form). By this induction, given the α from Lemma 5.6 we conclude that F and Φ_α agree on the t -structure obtained from $\heartsuit$ by applying any element of the braid group, i.e.

$$F\Phi_\gamma(\heartsuit) = \Phi_\alpha(\Phi_\gamma(\heartsuit))$$

for every path γ in the Deligne groupoid ending in the chamber C_+ . Using the bijection from Lemma 5.5, we see that F and Φ_α agree on all bounded hearts. The following now completes the proof of Theorem 5.1.

Corollary 5.9 *The association $f \mapsto F$ of Lemma 5.6 defines a homomorphism $\pi_0 \text{Symp}_{ct}(X) \rightarrow PBr_3$.*

Proof Given symplectomorphisms f, g yielding autoequivalences F, G which act trivially on K -theory, we define pure braids α and β by the property that

$$F(\heartsuit) = \Phi_\alpha(\heartsuit), \quad G(\heartsuit) = \Phi_\beta(\heartsuit).$$

So then $FG(\heartsuit) = F(\Phi_\beta(\heartsuit))$ and we wish to show this equals $\Phi_\alpha\Phi_\beta(\heartsuit)$. But this follows immediately from the discussion above, which established that F and Φ_α act the same way on all bounded hearts. $\square$

The categorical / stability condition aspects of the above discussion were not particularly special to the double bubble, but apply more generally to a wide class of 3-fold flopping contractions (the fact that the map Φ_α could not act by an interesting permutation matrix on K -theory also applies when there are more simples, by e.g. the analogue of [AW22, 2.12]).

As a concrete example, one can consider a ‘multi-bubble’ plumbing W_f , which is the Weinstein total space of a Morse-Bott-Lefschetz fibration with fibre $(\mathbb{C}^*)^2 = T^*T^2$, and with $n + 1$ singular fibres having vanishing cycles

$$k_i m + l_i \ell, \quad k_i = 1 \text{ or } l_i = 1; \quad (k_i, \pm l_i) \neq (k_j, \pm l_j) \text{ for } 0 \leq i \neq j \leq n$$

(where m, ℓ are a meridian and a longitude in the torus fibre). These spaces were studied by Xie and Li in [XL25]; the hypotheses on the vanishing cycles imply that (i) the core Lagrangians are Lens spaces, and (ii) all the possible surgeries amongst those cores are homology Lens spaces, and not copies of $S^1 \times S^2$. If

$$f(x, y) = \prod f_j(x, y), \quad f_j(x, y) = x^{l_j} \pm y^{k_j}$$

and $R_f = \mathbb{K}[u, v, x, y] / \langle uv - f(x, y) \rangle$ then $\text{Spec}(R_f)$ has an isolated Gorenstein cA_n -singularity, which admits a crepant resolution by a chain of n small resolution curves, each with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1, -1)$ or $\mathcal{O}_{\mathbb{P}^1}(0, -2)$.

Let $D_n \subset \mathbb{C}$ denote the set of $n + 1$ critical values of the MBL fibration. There is a relative wrapped category for the pair $(\mathbb{C}, D_n)$, whose objects are Lagrangians in $\mathbb{C} \setminus D_n$, where morphism groups are defined using wrapping at the punctures, but where holomorphic curves may cross the divisor D_n and are then counted with weights, yielding a category defined over $R = \mathbb{K}[t_0, \dots, t_n]$.

In part building on ideas from [LS23], Xie and Li establish equivalences over any field $\mathbb{K}$

$$\text{Perf } \mathcal{W}(W_f; \mathbb{K}) \simeq \text{Perf } \mathcal{W}(\mathbb{C}, D_n) \otimes_R \mathbb{K}[x, y] \simeq \text{DCoh}(Y_f) / \langle \mathcal{O}_{Y_f} \rangle$$

where $R \rightarrow \mathbb{K}[x, y]$ sends $t_i \mapsto f_i(x, y)$ and where the $\mathbb{K}[x, y]$ -linear structure on $\mathcal{W}(W_f)$ arises from the map $\mathbb{K}[x, y] \rightarrow SH^0(W_f)^\times$ sending x, y to the Seidel elements of the fibrewise Hamiltonian circle actions on $W_f \rightarrow \mathbb{C}$. (The fact that this equivalence holds before any completion is a little delicate, and carefully addressed in *op. cit.*)

Exactly as in [SW23], the relative singularity category is Koszul dual to the ‘small subcategory’ $\mathcal{C}_f \subset D(\text{Coh } Y_f)$ of objects supported on the exceptional locus and with trivial derived push-forward

$$\text{DCoh}(Y_f) / \langle \mathcal{O}_{Y_f} \rangle \simeq \mathcal{C}_f^!.$$

One thus obtains maps

$$\text{Auteq}_{\mathbb{K}}(\mathcal{W}(W_f; \mathbb{K})) \rightarrow \text{Auteq}_{\mathbb{K}}(\mathcal{C}_f).$$

One can then use the classification of hearts, and of objects with no negative Ext ’s, from [HW22] to see that the symplectic monodromy map

$$PBr_{n+1} \rightarrow \pi_0 \text{Sym}^{gr}(W_f)$$

splits, just as for double bubbles.

Remark 5.10 One feature that is special to the double bubble plumbing X (in which the cores and surgeries are actually S^3 's with no lens spaces around) is that the action of the pure braid group by symplectomorphisms is known to lift to the compactly supported symplectic mapping class group $\pi_0 \text{Symp}_{ct}(X)$; in more complicated geometries (e.g. longer linear plumbing chains) even when the spherical twist generators are compactly supported, it is less clear that the whole representation lifts.

Remark 5.11 It would be interesting to directly infer splitting from the identification of $\mathcal{W}(W_f)$ with the category $\mathcal{W}(\mathbb{C}, D_n) \otimes_R \mathbb{K}[x, y]$, which gives a trivial proof of *faithfulness* of the pure braid action.

In all these cases, the underlying monodromy groups are classical (pure) braid groups. We know that more interesting, e.g. non-Coxeter, groups arise from more interesting flopping contractions, though we do not always have candidate geometric mirrors.

Question 5.12 Let X be mirror to the small resolution of a 3-fold flopping contraction $Y \rightarrow \text{Spec}(R)$, and let $\mathcal{M}_X$ denote the complexified hyperplane complement with fundamental groupoid the pure braid group $PBr(\mathcal{C})$ associated to the small category $\mathcal{C}$. Does symplectic monodromy split over $\pi_1 \mathcal{M}_X$?

6 The conifold and its inexact deformations

The double bubble has a cousin, given by the ‘conifold quiver’

$$\bullet \begin{array}{c} \xrightarrow{e, e'} \\ \xleftarrow{f, f'} \end{array} \bullet, \quad W = ef'e'f' - ef'e'f$$

with two vertices and two arrows in each direction. On the B -side this again is well-known as describing the derived category of sheaves on the small resolution of the ordinary double point, supported on the exceptional curve $\mathbb{P}^1$. A better mirror geometry is provided by the space $Y = (\mathcal{O}(-1) \oplus \mathcal{O}(-1)) \setminus C$, where $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ is the small resolution of the threefold ordinary double point and C is a smooth conic in the complement of the exceptional curve. The space Y still admits a morphism $f : Y \rightarrow \mathbb{P}^1$. To that one associates two categories

$$\mathcal{C} = \{E^\bullet \in D(Y) \mid Rf_*(E^\bullet) = 0\} \quad \text{and} \quad \mathcal{D} = \{E \in D(Y) \mid \text{supp } H^*(E^\bullet) \subset \mathbb{P}^1\}$$

For a symplectic model we now plumb two T^*S^3 's along a Hopf link; this is again an affine variety X , namely the complete intersection

$$(6.A) \quad X = \{(u_1, v_1, u_2, v_2, z) \in \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^* \mid u_1 v_1 = z - 1, u_2 v_2 = z + 1\}$$

Write $\mathcal{F}_{\text{core}}(X)$ for the subcategory of the compact Fukaya category split-generated by the core Lagrangian spheres of the plumbing. Mirror symmetry now states

Theorem 6.1 [CPU16, Theorems 1.2 and 1.4] *There are equivalences $\mathcal{F}_{\text{core}}(X) \simeq \mathcal{C}$ and $\mathcal{W}(X) \simeq \mathcal{D}$.*

X admits a MBL fibration over $\mathbb{C}^*$, by mapping to the z coordinate, with fibre $\mathbb{C}^* \times \mathbb{C}^*$; singular fibres at ± 1 ; and core matching spheres associated (up to isotopy) the matching paths given by the upper and

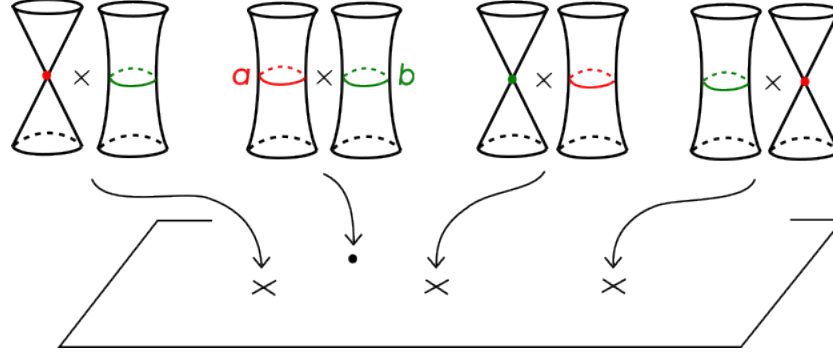


Figure 3: Morse–Bott–Lefschetz fibration on (W_0, ω_0) . The crosses in the base denote singular values, and the dot a regular one.

lower half-circles of the unit circle in $\mathbb{C}^*$. Varying the fibration gives a family $\mathcal{C} \rightarrow \text{Conf}_2(\mathbb{C}^*)$ and hence a representation

$$PBr_3 \rightarrow \pi_0 \text{Symp}(X).$$

This no longer has image in $\text{Symp}_{ct}(X)$. Forgetting strands defines a natural map

$$PBr_3 \rightarrow PBr_2 \times PBr_2$$

with kernel $PBr_3^c = \mathbb{Z}^{*\infty}$, a countably infinite rank free group. For details, see [KS24, Section 3].

Theorem 6.2 (K., S.) *The monodromy homomorphism*

$$\rho : PBr_3^c \rightarrow \pi_0 \text{Symp}_{ct}(X)$$

is split.

Proof This is spelled out explicitly in [KS24, Theorem 1.1]. □

This example illustrates that splitting can have non-trivial implications, e.g. here it yields a Stein manifold with infinitely generated symplectic mapping class group.

There is a geometry which mediates between the double bubble and the conifold. Let (W_0, ω_0) be the Stein manifold which is a plumbing of two copies of T^*S^3 plumbed along a circle so that the Lagrange surgery of the core spheres is an $S^1 \times S^2$. This is the Morse–Bott–Lefschetz fibration with fibre $(\mathbb{C}^*)^2$ and, in the notation from Figure 2, vanishing cycles a, b, a respectively. See Figure 3.

Remark 6.3 The analysis from [SW23] did not yield a classification of Lagrangian 3-spheres in W_0 , in contrast to the case for the closely related double bubble plumbing from section 5 (called W_1 in *op.cit.*). The space W_0 is an affine quartic, and the total space of a conic fibration over T^*S^2 , which in turn is a conic fibration over a plane. Note that W_0 admits a $\mathbb{C}^*$ -action, and any Lagrangian sphere defines an equivariant module, cf. [AS21, Proposition 6.5] which follows [Sei15, Corollary A.12]. Recent speculations of Lekili and Segal [LS23] give a description of the $(\mathbb{C}^*)^2$ -equivariant Fukaya category of W_0 , for the fibrewise action, in terms of a partially wrapped Fukaya category of a thrice-punctured plane. Starting from this, and using classification results for objects in Fukaya categories of surfaces, one could hope to classify Lagrangian 3-spheres in W_0 . This would be a rather different route to classification than that taken for W_1 in [SW23, HW22], closer to the recent arguments of [XL25] (cf. Remark 5.11 above).

One easily checks that $H^2(W_0; \mathbb{Z}) \cong \mathbb{Z}$, so there is an inexact deformation ω_t of the symplectic form ω_0 on W_0 . In fact, this can be realised geometrically. Note that the original Morse-Bott-Lefschetz fibration on W_0 has monodromies on $\mathbb{C}^* \times \mathbb{C}^*$ which are given by

$$\tau_a \times \{id\}, \{id\} \times \tau_b, \tau_a \times \{id\}$$

respectively, where $\tau_\bullet$ denotes the Dehn twist in the curve $\bullet$. The corresponding vanishing cycles, for appropriate vanishing paths, are $\{a, b, a\}$ in a hopefully obvious shorthand. Let ϕ_x denote the symplectomorphism of $\mathbb{C}^* = \mathbb{R} \times S^1$ which is given by a translation by x in the $\mathbb{R}$ -factor, i.e. the symplectomorphism (unique up to Hamiltonian isotopy) with flux $x \in H^1(\mathbb{C}^*; \mathbb{R}) \cong \mathbb{R}$.

Lemma 6.4 *The space (W_0, ω_t) admits a Morse-Bott-Lefschetz fibration with the same monodromies as before, but the vanishing cycles associated to the previously chosen vanishing paths are now $\{a, b, (\phi_t \times id)(a)\}$.*

Proof Cut W_0 along the codimension one hypersurface which is the preimage of a vertical line in $\mathbb{C}$ disjoint from the critical values of the MBL fibration, and such that the two critical values with vanishing cycle a lie on either side of it; and reglue by the map given by translation on the first factor of the fibre and the identity on the second. This is a symplectomorphism, so the result is again a symplectic manifold, manifestly diffeomorphic to W_0 (we have only changed the locally Hamiltonian structure of the fibration; the re-gluing is by a map smoothly isotopic to the identity). Since the change in symplectic structure is localised away from the critical submanifolds, the local monodromies are unchanged, but at least one of the vanishing paths will cross the cut locus and the corresponding vanishing cycle will therefore change as indicated. $\square$

Corollary 6.5 *Let $0 \neq t \in \mathbb{R}$. In the inexact deformation (W_0, ω_t) , the two core Lagrangian 3-spheres are disjoint.*

Proof The two matching spheres now meet the central singular fibre in copies of a circle $S^1 \times \{pt\} \subset \mathbb{C}^* \times \mathbb{C}^*$, where the circles for the two different matching spheres differ by a translation of flux t . $\square$

Note that the Lagrangian $S^1 \times S^2 \subset (W_0, \omega_0)$ which was fibred over a matching path between the outermost critical values is no longer Lagrangian, since the two-sphere factor has acquired positive area. The space W_0 carries an action by symplectic parallel transport of the mixed braid group MBr_3 . Consider the generator given by the lift of a half-twist, in $\mathbb{C}$, along a matching path between the two critical values of type a . Observe that this is also naturally a lift under a map $W_0 \rightarrow T^*S^2$ (forgetting the second coordinate on each fibre) of the Dehn twist in the S^2 , which no longer has infinite order after the inexact deformation of ω . Similarly, the twists in the core spheres now commute, since these are now disjoint.

The conifold mirror X introduced in (6.A) has $H^2(X; \mathbb{R}) \cong \mathbb{R}_{s,t}^2$. Let $(X, \omega_{s,t})$ be a corresponding inexact deformation of the Weinstein manifold X . We obtain a geometric mirror for $(X, \omega_{s,t})$ by considering the Morse-Bott-Lefschetz fibration over $\mathbb{C}^*$ with two singular fibres with monodromies

$$\tau_a \times id, \{\phi_s\} \times \{\phi_t\} \text{ and } id \times \tau_b$$

around $-1, 0$ and 1 respectively, for the obvious three loops based at $i \in \mathbb{C}^*$. Note that, as one iterates monodromies around the critical fibres, the fluxes add, so one finds:

Corollary 6.6 *Let s, t both be non-zero. The deformation $(X, \omega_{s,t})$ of the conifold mirror contains infinitely many pairwise disjoint Lagrangian 3-spheres.*

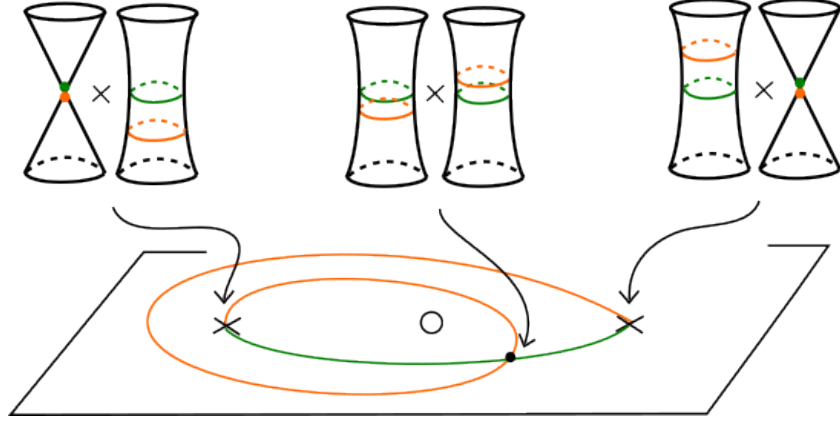


Figure 4: Example of disjoint Lagrangian matching spheres in $(X, \omega_{s,t})$.

See Figure 4.

Remark 6.7 The Lagrangian spheres associated to matching paths are still the only Lagrangian spheres we can ‘see’ after deformation, but we no longer have even a tentative route to a classification guaranteeing that these are the only spherical objects.

Each Lagrangian matching sphere has an associated Dehn twist. Disjointness implies these twists now commute. By considering the action on the set of graded Lagrangian spheres, and using that $\tau_L(L) = L[-1]$ but τ_L does not shift K if $K \cap L = \emptyset$, one sees that the homomorphism from $\mathbb{Z}^{\oplus \infty} \rightarrow \pi_0 \text{Symp}_c^{gr}(X, \omega_{s,t})$ will be injective when $s, t \neq 0$.

Question 6.8 Does the resulting homomorphism $\mathbb{Z}^{\oplus \infty} \rightarrow \pi_0 \text{Symp}_c(X, \omega_{s,t})$ split?

Remark 6.9 The approach suggested in Remark 6.3 does not obviously survive the deformation of the symplectic form, since this has no obvious interpretation in the model for the equivariant category formulated on the reduced space of the $\mathbb{C}^*$ -actions.

As a digression, we note that one can ‘compactify’ the preceding example:

Proposition 6.10 *There is a compact symplectic six-manifold which contains infinitely many pairwise disjoint Lagrangian 3-spheres.*

Proof We consider a Morse-Bott-Lefschetz (MBL) fibration with fibre $T^2 \times T^2$ constructed as follows. Begin with the four-manifold which is a symplectic T^2 -fibration over a torus with monodromies id and ϕ_s for some flux s . This is locally Hamiltonian, so admits a symplectic structure. We fibre sum this with a rational elliptic surface, to obtain a symplectic fibration $Z \rightarrow T^2$ over the two-torus with a dozen nodal singular fibres lying over a disc $D \subset T^2$, still with monodromies $\{\text{id}, \phi_s\}$.

We now take two copies Z and Z' of this four-manifold, where Z has monodromies $\{\text{id}, \phi_s\}$ and Z' has monodromies $\{\phi_t, \text{id}\}$ (so the flux has switched to the other factor of the base). We also insist that the singular fibres for Z respectively Z' lie over disjoint discs $D, D' \subset T^2$. Now take the fibre product $Z \times_{T^2} Z'$. The disjointness of D and D' means that at most one of the two input fibrations has a singular

point over a given $p \in T^2$, which ensures that the fibre product is the total space of a MBL fibration with general fibre T^4 .

We now impose the condition that the non-trivial fluxes $s, t \in H^1(T^2; \mathbb{R}) \cong \mathbb{R}^2$ are irrational, so have infinite order in the symplectomorphism group of the fibre. One can construct Lagrangian matching spheres in the total space exactly as in the inexact conifold. The irrationality of the fluxes ensures that the spheres associated to matching paths which meet in the base, as in Figure 4, are still disjoint in the total space. $\square$

Such examples don't seem to have been observed previously.

7 The birational projective plane

Let X denote the ‘universal mirror’ to log CY surfaces constructed in [KW24, Section 2.5] (where it is denoted M_{univ}). It is a Stein manifold of infinite type. In abuse of notation, we will write

$$\pi_0 \text{Symp}_{\text{ex}}(X) = \text{Symp}_{\text{ex}}(X) / \text{Ham}_{\text{ct}}(X)$$

for the quotient of *exact* (but not necessarily compactly supported) symplectomorphisms by compactly supported Hamiltonian isotopy. Let $\Omega = dx/x \wedge dy/y$ denote the standard meromorphic volume form on $\mathbb{P}^2$. The volume-preserving Cremona group $\text{Bir}(\mathbb{P}^2, \Omega)$ is the group of birational maps which preserve Ω . Let $\text{Bir}_e(\mathbb{P}^2, \Omega)$ be the subgroup of the volume-preserving Cremona group which is generated by $SL_2(\mathbb{Z})$ together with the basic birational cluster transformation E , with

$$E : (x, y) \mapsto (x, y(1 + x)^{-1}).$$

(The whole $\text{Bir}(\mathbb{P}^2, \Omega)$ is generated by this subgroup and $(\mathbb{C}^*)^2$, acting by multiplication coordinate-wise.) [KW24, Section 3.2] constructs a natural map $\text{Bir}_e(\mathbb{P}^2, \Omega) \rightarrow \pi_0 \text{Symp}_{\text{ex}}(X)$.

Remark 7.1 As Blanc points out in [Bla13], the subgroup $\text{Bir}_e(\mathbb{P}^2, \Omega)$ is defined over any field.

The space X is the total space of a Lagrangian torus fibration over $\mathbb{R}^2$ with infinitely many singular fibres. Let $\text{Symp}_{\text{fib}}(X) \subset \text{Symp}_{\text{ex}}(X)$ denote the subgroup of maps which preserve the set of Hamiltonian isotopy classes of smooth Lagrangian torus fibres near infinity, i.e. which take such fibres to tori which are Hamiltonian isotopic to fibres. Viewing the fibres near infinity as the pre-Lagrangian tori which live inside the mapping torus boundary of a large circle of regular values, this can be seen as a version of being ‘contact type’ at infinity in this special case; compactly-supported maps, linear maps and the cluster transformation E naturally lift to $\text{Symp}_{\text{fib}}(X)$.

Conjecture 7.2 *The map $\text{Bir}_e(\mathbb{P}^2, \Omega) \rightarrow \pi_0 \text{Symp}_{\text{fib}}(X)$ splits.*

We will outline an argument for this, but a full proof would need more care because we use inexact Lagrangians (general Lagrangian torus fibres of an almost toric fibration), including some properties expected by experts – whereas the set-up and HMS proofs of [KW24] restrict to the exact Fukaya category. We have packaged these properties into Assumption 7.5, and sketch a proof of Conjecture 7.2 subject to the assumption.

The space X admits a handle decomposition in which countably many two-handles are attached to T^*T^2 . The zero-section of that T^*T^2 defines an exact Lagrangian torus $T \subset X$. The space X is

constructed as the direct limit of finite-type subdomains X_ξ , where the ξ run over a directed system Ξ . This indexes log Calabi-Yau surfaces (Y_ξ, D_ξ) with maximal boundary and split mixed Hodge structure, equipped with explicit toric models (see [KW24, Definition 2.4]). We have

$$X = \varinjlim_{\xi \in \Xi} X_\xi$$

and the mirror is a corresponding limit of open log CY surfaces:

$$U = \varinjlim_{\xi \in \Xi} U_\xi$$

where $U_\xi = Y_\xi \setminus D_\xi$.

Lemma 7.3 *For any $\xi \in \Xi$, there is a quasi-isomorphism of A_∞ -categories*

$$\Upsilon : \text{coh}(U_\xi; \mathbb{C}) \simeq \text{tw } \mathcal{W}(X_\xi; \mathbb{C}).$$

Here $\text{coh}(U_\xi; \mathbb{C})$ denotes a dg enhancement of the derived category of coherent sheaves $D^b \text{Coh}(U_\xi; \mathbb{C})$ (recall these are unique up to quasi-isomorphism [LO10]), and $\text{tw } \mathcal{W}(X_\xi; \mathbb{C})$ is the category of twisted complexes enlarging $\mathcal{W}(X_\xi; \mathbb{C})$. Moreover, for any choice of brane structure ρ on T , Υ maps (T, ρ) to the skyscraper sheaf of a closed point in U_ξ (up to quasi-isomorphism).

Proof See the equivalence of categories [KW24, Section 2.3], which in turn imports this from [HK22]. The claim about the mirror to (T, ρ) is in [HK25, Lemma 2.24]. $\square$

For what follows, it will be important to pass from the exact Fukaya category over $\mathbb{C}$ to the more general tautologically unobstructed Fukaya category, which is defined over the one-variable Novikov field $\Lambda_{\mathbb{C}}$. This category admits all Maslov zero Lagrangians as objects, since these are tautologically unobstructed for generic J for virtual dimension reasons. We remind the reader that $\Lambda_{\mathbb{C}}$ is an algebraically closed field of characteristic zero.

The log Calabi-Yau surfaces U_ξ are characterised by a split mixed Hodge structure condition which is naturally formulated over $\mathbb{C}$.

Lemma 7.4 *The log CY pairs (Y_ξ, D_ξ) , U_ξ , and the direct system of inclusions of the U_ξ indexed by $\xi \in \Xi$, are all naturally defined over $\Lambda_{\mathbb{C}}$.*

Proof Base-change for the inclusion $\mathbb{C} \rightarrow \Lambda_{\mathbb{C}}$. $\square$

Assumption 7.5 *There is an equivalence*

$$\psi : \text{tw } \mathcal{W}(X_\xi; \Lambda_{\mathbb{C}}) \xrightarrow{\sim} \text{coh}(U_\xi; \Lambda_{\mathbb{C}})$$

with the following properties:

- (i) Under ψ , the local systems on T define a chart $(\Lambda_{\mathbb{C}}^*)^2 \hookrightarrow U_\xi$.
- (ii) For any sufficiently large compact subset $Z \subset \mathbb{R}^2$, under ψ each smooth Lagrangian torus fibre of $X_\xi \rightarrow \mathbb{R}^2$ lying over $\mathbb{R}^2 \setminus Z$ is taken to the skyscraper sheaf of a point in U_ξ . Furthermore, such points cover a Zariski open subset of a punctured neighbourhood of the boundary $D_\xi = Y_\xi \setminus U_\xi$.

Discussion For (i), note that by [GPS20], we can pick a finite set of split-generators for $\mathcal{W}(X_\xi)$. By expressing the exact torus T in terms of those split-generators, T gives rise to an algebraic family of modules over the Fukaya category parametrised by $H^1(T; \mathbb{G}_m)$; see [Sei14a] and [Kar21, Section 2.3]. The conclusion of the part (i) is then essentially [Kar21, Note 1.12] (though this is only sketched). Here is a brief outline: the mirror equivalence takes the algebraic family of modules over $\text{tw } \mathcal{W}(U_\xi)$ to an algebraic map from $H^1(T; \mathbb{G}_m)$ to the Artin stack parametrising the moduli of objects of $\text{coh}(U_\xi)$. In general the moduli of objects in such a dg -category is only locally of finite type [TV07], and can be quite pathological. The content of [Kar21] is to nonetheless identify non-unitary deformations with flux images, by crucially using the group-like property of the family. $\square$

The rest of this section is contingent on Assumption 7.5, but we will not spell this out in every statement.

We now wish to consider the corresponding charts for tori T and $\phi(T)$, where $\phi \in \pi_0 \text{Symp}_{\text{ex}}(X)$.

Lemma 7.6 *Let $T' \subset X$ be any exact Lagrangian torus. Then there is some $\xi = \xi(T')$ such that $T' \subset X_\xi \subset X$.*

Proof [KW24] constructs an almost-toric torus fibration $X \rightarrow \mathbb{R}^2$ (called π_{univ}), which is a limit of almost toric fibrations from the X_ξ to increasingly large subsets of $\mathbb{R}^2$ (see [KW24, Sections 2.2 and 2.3] for details). As T' projects to a compact subset, it must lie inside some X_ξ . $\square$

Exactness implies that each of T and T' support a family of objects of $\mathcal{W}(X_\xi; \Lambda_{\mathbb{C}})$ parametrized by $(\Lambda_{\mathbb{C}}^*)^2$, since their self Floer cohomology is non-trivial for any rank one Novikov-valued local system. Under the given equivalence $\text{tw } \mathcal{W}(X_\xi; \Lambda_{\mathbb{C}}) \rightarrow \text{coh}(U_\xi; \Lambda_{\mathbb{C}})$, the family supported on T defines a chart $(\Lambda_{\mathbb{C}}^*)^2 \hookrightarrow U_\xi$, by Assumption 7.5. The corresponding family of objects parametrised by T' are in principle only objects of the derived category: they need not *a priori* be realised as another chart.

Recall that $U_\xi = Y_\xi \setminus D_\xi$, where (Y_ξ, D_ξ) is a log CY surface with maximal boundary. Maximality of the boundary implies that the components are homologically non-torsion, so K_{Y_ξ} is non-torsion in Chow. We have:

Lemma 7.7 *Fix an algebraically closed characteristic zero field k . If Y is a projective surface over k with $K_Y \neq 0 \in CH^*(Y)_{\mathbb{Q}}$ then a point-like object in $D(Y)$ is either the skyscraper sheaf of a point up to shift, or is supported on a (possibly reducible) connected closed curve C with $K_Y \cdot C = 0 \in \mathbb{Z}$.*

Proof This is part of [HK24, Lemma 2.2] (note that the proof is written over $\mathbb{C}$, but works identically over k). $\square$

We're now ready to sketch the proof of Conjecture 7.2. Suppose we start with a class $[\phi] \in \pi_0 \text{Symp}_{\text{fib}}(X)$, with representative ϕ , a symplectomorphism assumed to preserve fibres at infinity. Set $T' = \phi(T)$. The key difficulty in establishing any version of Conjecture 7.2 is to rule out the case in which the complexes associated to the family of objects on T' have support containing a (-2) -curve in U_ξ ; this will be where fibred-ness enters.

Continuing to work over $\Lambda_{\mathbb{C}}$, we have:

Lemma 7.8 *The objects of $D(U_\xi)$ associated to local systems on T' have compact support contained in U_ξ .*

Proof We know T' has trivial Floer cohomology with all the Lagrangian torus fibres of $X_\xi \rightarrow \mathbb{R}^2$ outside some compact subset $Z \subset \mathbb{R}^2$. The result then follows from the Assumption 7.5 part (ii). $\square$

Corollary 7.9 *The tori T and $T' = \phi(T)$ define two families of point-like objects in $\text{Coh}(U_\xi)$ associated to two charts $(\Lambda_\mathbb{C}^*)^2 \hookrightarrow U_\xi$, which differ by a well-defined birational automorphism $\Psi_\xi(\phi)$.*

Sketch By hypothesis ϕ is “fibred at infinity”, meaning that its mirror automorphism sends skyscraper sheaves of points near infinity to skyscraper sheaves of points. Taking a point p not lying on one of the countably many (-2) -curves, one sees that the family of point-like objects associated to $\phi(T)$ has a member whose support does not contain any (-2) -curve, so is in fact a point, and is thus quasi-isomorphic to the skyscraper sheaf $\mathcal{O}_p$ (cf. the discussion of Assumption 7.5 part (i)). For any (-2) -curve $C \subset U_\xi$, we have $\text{Ext}^*(\mathcal{O}_p, \mathcal{O}_C) = 0$. Semicontinuity of the rank of Ext-groups, and the algebraicity of the family of modules associated to T' as constructed in [Kar21], means that the rank of $\text{Ext}^*(\mathcal{J}, \mathcal{O}_C)$ jumps up on at most a Zariski-closed subset of $(\Lambda_\mathbb{C}^*)^2$, where this torus parametrises the family of modules $\mathcal{J}$ defined by T' .

Suppose now some member $\mathcal{J}$ of our family of modules does have support a (possibly reducible) (-2) -curve C . Pick a point $q \in C$; then $\text{Ext}^*(\mathcal{J}, \mathcal{O}_q) \neq 0$, so by semicontinuity, it remains non-zero on the whole family. But this contradicts the previous discussion. It follows that the two Lagrangian tori define two charts of the given form. Since U_ξ is irreducible, these necessarily differ by a birational map.

This birational map is only well-defined up to the ambiguity coming from a choice of identification of the local systems on each of the tori with $(\Lambda^*)^2$, which amounts to the ambiguity of composition with a linear map. The hypothesis that $T' = \phi(T)$ for $\phi \in \text{Symp}(X)$ gives a distinguished identification between local systems on T and on T' . This implies that the birational map $\Psi_\xi(\phi)$ between the two charts is uniquely defined. $\square$

Lemma 7.10 $\Psi_\xi(\phi)$ preserves Ω .

Proof This is a consequence of Poincaré duality in Floer cohomology, compare to [Sei14b, Section 11]. To spell it out somewhat, suppose we have two exact Lagrangian tori $T, T' \subset X$, and an isomorphism in the Fukaya category between Zariski open subsets of the families of local systems on T and T' . The tangent space to the family of local systems can be identified at a given point with HF^1 , and the isomorphism $HF^*(T, T) \cong HF^*(T', T')$ is one of rings. Poincaré duality implies that the isomorphism is compatible with the trace map $\int_{T'} : H^2(T', T') = HF^2(T', T') \rightarrow \Lambda_\mathbb{C}$, since if $x \in HF(T, T')$ and $y \in HF(T', T)$ are mutually inverse isomorphisms, then duality says

$$\int_T xay = \int_{T'} yxa = \int_{T'} a$$

for $a \in HF^2(T', T')$ (and similarly with the roles of T, T' reversed, or for the isomorphisms at a pair of non-trivial local systems). This in turns means that the isomorphism on $HF^1 = H^1$ has determinant one, which on the B -side is the volume-preserving condition. $\square$

Up to this point one can take any ‘sufficiently large’ subdomain X_ξ of X or sufficiently large log CY U_ξ .

Lemma 7.11 *The birational map $\Psi_\xi(\phi)$ depends only on ϕ , i.e. is unchanged under embedding $U_\xi \hookrightarrow U_{\xi'}$.*

Proof For local systems $\zeta \rightarrow T$ and $\zeta' \rightarrow T'$, the (non-)existence of a quasi-isomorphism between (T, ζ) and (T', ζ') is independent of further embedding $X_\xi \hookrightarrow X_{\xi'}$. $\square$

We therefore just write $\Psi(\phi)$ for the birational map $\Psi_\xi(\phi)$ for any choice of sufficiently large ξ .

Lemma 7.12 *The association $\phi \mapsto \Psi(\phi)$ is a homomorphism $\pi_0 \text{Symp}_{fib}(X) \rightarrow \text{Bir}_e(\mathbb{P}^2, \Omega)$.*

Proof The map descends to $\pi_0 \text{Symp}_{fib}(X)$ using the invariance of Floer cohomology under Hamiltonian isotopy. Given ϕ_1 and ϕ_2 we can choose ξ large enough that $T, \phi_1(T), \phi_2(T)$ all lie inside X_ξ ; by arguments analogous to those above, we get three corresponding $(\Lambda_{\mathbb{C}}^*)^2$ -charts of point-like objects on U_ξ . The homomorphism property is then immediate. $\square$

We remind the reader that all these results are contingent on Assumption 7.5. But granted that, by construction, the map of Lemma 7.12 would split the homomorphism $\text{Bir}_e(\mathbb{P}^2, \Omega) \rightarrow \pi_0 \text{Symp}_{fib}(X)$ given in [KW24].

8 Further examples

8.1 Tori

For the standard symplectic structure on the torus $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$, there is a linear action by symplectomorphisms, which is split by considering the action on first homology of the torus.

8.2 Two-dimensional surfaces

For a compact symplectic surface Σ_g , [AS21] shows that $\Gamma_g \rightarrow \text{Symp}(\Sigma_g)/\text{Ham}(\Sigma_g)$ splits by considering the composite of the monodromy homomorphism with the representation of Symp/Ham on $\text{Auteq}(\mathcal{F}(\Sigma_g))$, where we work over the Novikov field. That composite homomorphism splits; this is established by classifying spherical objects in the Fukaya category with non-trivial Chern character, and thereby reconstructing the ‘Schmutz graph’ of the surface [SS00].

8.3 (Log) Calabi-Yau surfaces

Gromov [Gro85] showed that $\text{Symp}_c(\mathbb{C}^2, \omega_{\text{std}})$ is contractible. Later, Wendl [Wen10] and Evans [Eva11] proved contractibility of $\text{Symp}_c(\mathbb{C}^* \times \mathbb{C}^*)$ and $\text{Symp}_c(\mathbb{C} \times \mathbb{C}^*)$ respectively.

Seidel [Sei98, Sei08b] showed $\text{Symp}_c T^*S^2$ is weakly homotopy equivalent to $\mathbb{Z}$, and Evans *op. cit.* proved that $\text{Symp}_c(T^*\mathbb{RP}^2) \simeq \mathbb{Z}$, in both cases generated by the corresponding (spherical respectively projective) Dehn twist.

The symplectomorphism groups of log Calabi-Yau surfaces were studied systematically via homological mirror symmetry in [HK22, HK21]. We summarise the relevant results.

Let (Y, D) be a log Calabi-Yau surface with maximal boundary and split mixed Hodge structure, and let M be its mirror. Let $U = Y \setminus D$. We have a mirror equivalence of categories $D(U) \cong D^b(\text{tw } \mathcal{W}(M))$.

In this context, there is a natural subgroup of ‘compactly supported’ auto-equivalences of $D(U)$, $\text{Auteq}_c D(U)$ (see [HK21, Definition 2.16]). Via the mirror equivalence, we have

$$\pi_0 \text{Symp}_c(M) \longrightarrow \text{Auteq}_c D(U).$$

There exist conjectural descriptions of $\text{Auteq} D(Y)$ and of $\text{Auteq}_c D(U)$, both in terms of generators and as fundamental groups of suitable moduli spaces of complex structures, see [Ueh19, Conjecture 1.2], [HK21, Conjectures 1.5 and 1.7]. One can think of these as relative versions of the better-known conjecture of Bridgeland on autoequivalences of derived categories of $K3$ surfaces. In particular, we expect $\text{Auteq}_c D(U) = \langle Q, \mathcal{T} \rangle \rtimes \text{Aut}(Y, D)$. Here $Q = \langle D_1, \dots, D_k \rangle^\perp \subset \text{Pic}(Y)$, where $D_1, \dots, D_k$ are the irreducible components of D , is the subgroup of line bundles $\mathcal{L}$ on Y such that $\mathcal{L}|_D = \mathcal{O}_D$ (acting by tensor product); $\mathcal{T}$ is the collection of spherical twists in objects of the form $i_* \mathcal{O}_C(k)$, for $C \subset U$ a (-2) curve and $k \in \mathbb{Z}$; and $\text{Aut}(Y, D)$ is the group of automorphisms of Y fixing D pointwise (acting by pushforward). By the Torelli theorem in [GHK15], $\text{Aut}(Y, D)$ is identified with the quotient of the group Adm of admissible lattice automorphisms of $\text{Pic}(Y)$ by the Weyl group W .

The spherical objects $i_* \mathcal{O}_C(k)$ are mirror to embedded Lagrangian spheres [HK21, Proposition 5.2], with spherical twists mirror to Dehn twists in the usual manner. Moreover, there is a faithful group homomorphism [HK21, Theorem 7.1]

$$(8.A) \quad \iota : Q \rtimes \text{Adm}/W \hookrightarrow \pi_0 \text{Symp}_c(M).$$

The image symplectomorphisms are described in terms of the SYZ (almost-toric) Lagrangian fibration on M . Under ι , a line bundle $\mathcal{L}$ acts as a ‘Lagrangian translation’, with mirror autoequivalence $\otimes \mathcal{L} \in \text{Auteq} D(U)$; and an element $\phi \in \text{Adm}/W$ acts as a ‘nodal slide recombination’, with mirror autoequivalence $\phi_* \in \text{Auteq} D(U)$. In particular, combining back with Dehn twists, one has precise descriptions of the monodromy representation of the (conjectural) relevant moduli space; and, subject to the conjecture on $\text{Auteq}_c D(U)$, we expect the representation (8.A) to split.

For projective Calabi-Yau surfaces, a general splitting result is expected to follow from Bridgeland’s well-known conjecture on spaces of stability conditions on $K3$ surfaces. Some Picard rank one $K3$ surfaces were studied via HMS in [SS20]. For a concrete example, one can take the ‘mirror quartic’ (X, ω) , which is a crepant resolution of the quotient of the Fermat quartic $\{\sum z_j^4 = 0\} \subset \mathbb{CP}^3$ by an action of $(\mathbb{Z}/4)^2$. The crepant resolution resolves six A_3 -singularities, yielding 18 exceptional curves. Take a Kähler form which is ‘sufficiently irrational’ in the sense that the areas of these 18 curves, and of a line, are all linearly independent over $\mathbb{Q}$. The moduli space of complex structures is an orbifold curve with fundamental group

$$\pi_1(\mathcal{M}_X) = \mathbb{Z} * \mathbb{Z}/4.$$

The natural map $\pi_1(\mathcal{M}_X) \rightarrow \pi_0 \text{Symp}(X, \omega)$ indeed splits by [SS20, Section 7.2]. Similar results are obtained for the crepant resolution of the Fermat sextic in $\mathbb{P}(1, 1, 1, 3)$, again with a sufficiently irrational Kähler form (meaning that the relevant complex structure moduli spaces is again only one-dimensional).

8.4 del Pezzo surfaces

We consider rational surfaces which admit a monotone symplectic form, so the surfaces X_k which are monotone blow-ups of the projective plane at $0 \leq k \leq 8$ points (the del Pezzo of degree $9 - k$) and the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$. The symplectomorphism groups in these cases have been extensively studied, and we summarise some of the results below. We write Symp_0 for the Torelli group of

symplectomorphisms acting trivially on homology. The algebro-geometric statements are taken from [Loo08, Dol12, OSS16].

- (1) $\mathrm{Symp}_c(\mathbb{P}^2, \omega_{\mathrm{FS}}) \simeq \mathbb{P}U(3)$;
- (2) $\mathrm{Symp}_c(\mathbb{P}^1 \times \mathbb{P}^1) \simeq (SO(3) \times SO(3)) \rtimes \mathbb{Z}/2$, where $\mathbb{Z}/2$ acts by swapping the factors.
- (3) (degree 8) $\mathrm{Symp}(\mathrm{Bl}_1(\mathbb{P}^2), \omega) \simeq U(2)$.
- (4) (degree 7) $\mathrm{Symp}(\mathrm{Bl}_2(\mathbb{P}^2), \omega) \simeq T^2 \rtimes \mathbb{Z}/2$, see [LP04, Pin08].
- (5) (degree 6) $\mathrm{Symp}_0(\mathrm{Bl}_3(\mathbb{P}^2), \omega) \simeq T^2$ [Eva11]; the moduli space of complex structures is given by triples of distinct points in $\mathbb{P}^2 \bmod \mathbb{P}GL_3(\mathbb{C})$. The holomorphic automorphism group of each surface is positive dimensional $(\mathbb{C}^*)^2 \rtimes (S_3 \times S_2)$, with the finite part acting by permutations and quadratic Cremona transformations.
- (6) (degree 5) $\mathrm{Symp}_0(\mathrm{Bl}_4(\mathbb{P}^2)) \simeq \{pt\}$ [Eva11]. The moduli space is $\{pt\}/S_5$ (here $S_5 = \hat{A}_4$ also arises as a particular Artin group quotient in [Loo08]). The monodromy action is split by taking the action on homology. See also [Eva11, LW12, LLW15, LLW22a].
- (7) (degree 4, intersections of two quadrics in $\mathbb{P}^4$) We have that $\mathrm{Symp}_0(\mathrm{Bl}_5(\mathbb{P}^2)) \simeq \mathrm{Diff}^+(S^2, 5) \simeq PBr(S^2, 5)/\langle \tau \rangle$ where τ is the full twist, which has order two. Seidel [Sei08b] showed that the monodromy map for the universal homologically marked family defines a splitting
$$PBr(S^2, 5)/\langle \tau \rangle \longrightarrow \pi_0 \mathrm{Symp}_0(\mathrm{Bl}_5(\mathbb{P}^2)) \longrightarrow PBr(S^2, 5)/\langle \tau \rangle.$$
- (8) (degree 3, cubic surfaces) The fundamental group of the moduli space is a certain Artin group $\hat{E}_6$. As far as we know, this case has not been fully analysed, and splitting is unknown.
- (9) (degree 2, quartics in $\mathbb{P}^2$) Again, this case seems not to have been fully analysed.

Separately, the reader may be interested in the related work [LLW22b], which inter alia applies to some non-monotone del Pezzo surfaces.

8.5 A higher-dimensional non-splitting example?

Let $X := Q_0 \cap Q_1 \subset \mathbb{P}^{2g+1}$ be the intersection of two smooth quadric hypersurfaces, with its canonical monotone symplectic form. Let $\mathcal{M}_X$ denote the moduli space of complex structures on X ; these are all realised as $(2, 2)$ -complete intersections, by [Ion80, Theorem 1].

Conjecture 8.1 *For $g \geq 3$ the symplectic monodromy*

$$\pi_1(\mathcal{M}_X) \rightarrow \pi_0 \mathrm{Symp}(X, \omega)$$

is not virtually split.

The two quadrics generate a pencil, whose singular members are generically parametrized by $2g + 2$ points in $\mathbb{P}^1$. This sets up an identification between the moduli space of complex structures on X and a space of hyperelliptic curves, and $\mathcal{M}_X$ has fundamental group an extension $H^1(\Sigma_g; \mathbb{Z}/2) \rtimes \Gamma_g^{\mathrm{hyp}}$ of the hyperelliptic mapping class group of a genus g surface, see [Rei, Smi12]. [Smi12] constructs a natural map [Smi12] $\pi_0 \mathrm{Symp}(X) \rightarrow \Gamma_g$ via autoequivalences of the Fukaya category. When $g = 2$ the monodromy thus virtually splits, but for $g \geq 3$ the non-existence of a map $\Gamma_g \rightarrow \Gamma_g^{\mathrm{hyp}}$ virtually splitting the inclusion suggests that the symplectic monodromy does *not* split (more precisely, no splitting could factor through the autoequivalences of the Fukaya category). It would be interesting to investigate other Fano complete intersections.

Remark 8.2 For degrees $d \geq 5$, the monodromy of the moduli space of plane curves is (at most) a proper subgroup of the mapping class group, corresponding to automorphisms which preserve an r -spin structure [Sal25]; conjecturally, the monodromy group is as large as this constraint permits, so whilst there is no splitting of the monodromy viewed as mapping to the classical mapping class group, there is a virtual splitting (one to a finite index subgroup). This example, as well as the case of the A_k -Milnor fibre, motivate considering virtual as well as actual splittings.

8.6 The C^0 -category

We end with a question. Following the work of Jannaud [Jan21], in many cases one can show that a monodromy map $G \rightarrow \pi_0 \operatorname{Symp}(X)$, whose injectivity is detected by computations of ranks of Floer cohomology groups, remains injective on further composition with the natural map

$$\pi_0 \operatorname{Symp}(X) \rightarrow \pi_0 \overline{\operatorname{Symp}}(X).$$

Here $\overline{\operatorname{Symp}}(X)$ refers to the group of symplectic homeomorphisms, i.e. the C^0 -closure of $\operatorname{Symp}(X)$ inside $\operatorname{Homeo}(X)$.

Such results are proved by exploiting the existence of a map $\overline{\operatorname{Symp}}(X) \rightarrow \mathcal{B}$ where $\mathcal{B}$ is a suitable space of (tuples of) barcodes with the bottleneck distance, and the homomorphism associates to a smooth symplectomorphism ϕ the barcode(s) associated to some particular sum of Floer complexes $\bigoplus_i CF(L_i, \phi(L_i))$ (where, for instance, the $\{L_i\}$ form a set of split-generators). The set of components $\pi_0 \mathcal{B}$ is not naturally a group.

Question 8.3 *In the examples studied previously, does the monodromy homomorphism to the C^0 -symplectic mapping class group ever split?*

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