FAITHFUL ACTIONS FROM HYPERPLANE ARRANGEMENTS

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ABSTRACT. We show that if X is a smooth quasi-projective 3-fold admitting a flopping contraction, then the fundamental group of an associated simplicial hyperplane arrangement acts faithfully on the derived category of X. The main technical advance is to use torsion pairs as an efficient mechanism to track various objects under iterations of the flop functor (respectively, mutation functor). This allows us to relate compositions of the flop functor (respectively, mutation functor) to the theory of Deligne normal form, and to give a criterion for when a finite composition of 3-fold flops can be understood as a tilt at a single torsion pair. We also use this technique to give a simplified proof of Brav–Thomas [BT] for Kleinian singularities.

1. INTRODUCTION

Autoequivalence groups of the bounded derived categories $D^b(\mathrm{coh}\, X)$ of coherent sheaves of varieties X have been studied in many articles. On one hand, Bondal and Orlov [BO] proved that derived categories $D^b(\mathrm{coh}\, X)$ of smooth projective varieties X with K_X or $-K_X$ ample have only standard autoequivalences. On the other hand, Seidel and Thomas [ST] showed that if $\pi\colon X \to \mathbb{C}^2/G$ is a minimal resolution of a quotient singularity $\mathbb{C}^2/G$ by a finite group $G \subset SL_2(\mathbb{C})$, then the derived category $D^b(\mathrm{coh}\, X)$ has non-standard autoequivalences, called *spherical twists*. Across mirror symmetry, these correspond to autoequivalences of the derived Fukaya category of a homological mirror partner $X^\vee$ of X, which arises from generalized Dehn twists along Lagrangian spheres in $X^\vee$ [ST]. More precisely, if $C := \pi^{-1}(0) = \bigcup_{i=1}^n C_i$ with C_i irreducible, Seidel–Thomas showed that the objects $\mathcal{O}_{C_i}(-1)[1]$ induce autoequivalences $t_i \in \mathrm{Auteq}\, D^b(\mathrm{coh}\, X)$, and that these together induce a group homomorphism

$$\begin{array}{ccc}
\rho\colon & B_\Gamma & \longrightarrow & \mathrm{Auteq}\, D^b(\mathrm{coh}\, X) \\
& \cup & & \cup \\
& s_i & \longmapsto & t_i
\end{array}$$

where $B_\Gamma = \langle s_1, \ldots, s_n \rangle$ is the braid group of the dual graph of exceptional curves $\bigcup_{i=1}^n C_i$ of π, which is a Dynkin diagram of type ADE. Seidel–Thomas showed that ρ is injective when Γ is of type A, and later Brav–Thomas showed that ρ is injective in the general case [BT]. This means that there is a faithful braid group action on $D^b(\mathrm{coh}\, X)$.

Moving up one dimension, if $X \to X_{\mathrm{con}}$ is a flopping contraction between quasi-projective 3-folds, where X is smooth and each of the n irreducible exceptional curves is individually floppable, then [P4, W] associates to this data a real hyperplane arrangement $\mathcal{H} \subseteq \mathbb{R}^n$, as a certain intersection in an ADE root system. The main result of [DW3] is that this induces an action of the fundamental group on the derived category; more precisely there exists a group homomorphism

$$\varphi\colon \pi_1(\mathbb{C}^n \backslash \mathcal{H}_\mathbb{C}) \to \mathrm{Auteq}\, D^b(\mathrm{coh}\, X),$$

where $\mathcal{H}_\mathbb{C}$ denotes the complexification of the real hyperplane arrangement $\mathcal{H} \subseteq \mathbb{R}^n$. The group $\pi_1(\mathbb{C}^n \backslash \mathcal{H}_\mathbb{C})$ should be viewed as a form of *pure braid group*, since in the case $\mathcal{H}$

is a Coxeter arrangement, this is precisely what it is. However, in general, $\mathcal{H}$ need not be Coxeter. Motivated by the situation of surfaces above, and also by considerations in Bridgeland stability conditions, in this paper we prove that φ is injective, that is, the action is also faithful.

In fact, we do more, and our proof also recovers the surfaces case of [BT] in a much simpler way. Some of the techniques in [BT] are not suited to the 3-fold and more general settings, and so we are forced to develop a new approach. There are four main problems:

(1) In the 3-fold flops setting, the action φ is obtained by iterating flops. There is no 'formula' for the flop functor, unlike for spherical twists, and so tracking objects under iterated flops is much more challenging.

(2) The arrangement $\mathcal{H}$ need not be Coxeter, so there is no finite Weyl group from which we can use reduced expressions of elements, or Garside normal form.

(3) Higher length braid relations exist, making it harder to induct on path length.

(4) There is no explicit presentation of $\pi_1(\mathbb{C}^n \backslash \mathcal{H}_{\mathbb{C}})$ to work with.

It turns out that these phenomena also exist for surfaces, but we need to go to *partial* resolutions of Kleinian singularities in order to see them; most work to date only considers the minimal resolution. This is addressed further in [IW3].

To obtain our main geometric results, we restrict to the formal fibre, and manipulate tilting modules there. The following is our main result.

Theorem 1.1 (6.5, 6.6)**.** *Suppose that* $f \colon X \to \operatorname{Spec} \mathfrak{R}$ *is a complete local 3-fold flopping contraction, where X is smooth. Then the natural functor from the Deligne groupoid $\mathbb{G}_{\mathcal{H}}$ to the natural flops groupoid is faithful. In particular, the induced group homomorphism*

$$\varphi \colon \pi_1(\mathbb{C}^n \backslash \mathcal{H}_{\mathbb{C}}) \to \operatorname{Auteq} \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$$

is injective.

This immediately gives global corollaries, such as the following.

Corollary 1.2 (6.7)**.** *Suppose that* $f \colon X \to X_{\mathrm{con}}$ *is a flopping contraction between quasi-projective 3-folds, where X is smooth, and all curves in the contraction f are individually floppable. Then there is an injective group homomorphism*

$$\varphi \colon \pi_1(\mathbb{C}^n \backslash \mathcal{H}_{\mathbb{C}}) \to \operatorname{Auteq} \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X).$$

There is a similar statement for when the curves are not individually floppable, but being slightly more technical to state, we refer the reader to 6.8. We also recover in Appendix A a simplified version of Brav–Thomas in the case of minimal resolutions of Kleinian singularities.

The main technical engine in the proof is to use the order on tilting modules to control iterations. Our new main technical result is the following, which here we state slightly vaguely, leaving details to §4.

Theorem 1.3 (4.6)**.** *With the assumptions in 1.1, suppose that* $\alpha \colon C \to D$ *is a positive minimal path. Then the composition of mutation functors along this path is functorially isomorphic to a single functor induced by a tilting module.*

Since tilting modules induce torsion pairs, this allows us to use torsion pairs to control iterations. Applying this to 3-fold flops, where by [W] the flop functor is isomorphic to the inverse of the mutation functor, gives the following result. The first part is implicit in [DW3], whereas the second part is new, and may be of independent interest.

Theorem 1.4 (6.9)**.** *Consider two crepant resolutions*

$$X \qquad\qquad Y$$
$$\searrow \qquad \swarrow$$
$$\operatorname{Spec} R$$

of $\operatorname{Spec} R$*, where R is an isolated cDV singularity.*

(1) *Given two minimal chains of flops connecting X and Y, the composition of flop functors associated to each chain are functorially isomorphic.*

(2) *Perverse sheaves on Y, namely ${}^0\mathrm{Per}(Y, R)$, can be obtained from perverse sheaves on X, namely ${}^0\mathrm{Per}(X, R)$, by a single tilt at a torsion pair.*

For definitions, we refer the reader to §6.2.

1.1. **Outline of Paper.** §2 contains background on hyperplane arrangements, arrangement groupoids and Deligne Normal Form. In §3 we then relate this to tilting modules, under the general setting that we will consider. So as not to disturb the flow of the paper, proofs of some of the results in §3 appear in Appendix B. In §4 we establish in 4.6 that compositions of tilts behave well under Deligne Normal Form, and the first consequences appear in the short §5. In §6 we use this torsion pair viewpoint to prove the faithfulness in the complete local setting, and we give all the geometric corollaries. In Appendix A, which can be read independently, we give a simple direct proof of faithfulness in the case of Kleinian singularities, to demonstrate that the torsion pair viewpoint simplifies the proof.

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1.3. **Conventions.** All rings and algebras are assumed to be noetherian, and to be k-algebras, where k is some field. All modules are right modules, unless stated otherwise. When considering flopping contractions, the base field is assumed to be algebraically closed of characteristic zero. Throughout:

- For a triangulated category $\mathcal{C}$, and $a, b \in \mathcal{C}$, to match [BT] we write
$$[a, b]_t := \mathrm{Hom}_{\mathcal{C}}(a, b[t]).$$

- For an algebra Λ, we write $\mathsf{fl}\,\Lambda$ for the category of finite length right Λ-modules.
- For a noetherian ring R, $\mathrm{CM}\,R$ denotes the category of maximal Cohen-Macaulay R-modules, and $\mathrm{ref}\,R$ denotes the category of finitely generated reflexive R-modules.
- For an additive category $\mathcal{C}$, and an object $x \in \mathcal{C}$, we write $\mathrm{add}\,x \subset \mathcal{C}$ for the full subcategory consisting of direct summands of finite direct sums of x.

2. Preliminaries

2.1. **Hyperplane Arrangements.** Throughout this subsection $\mathcal{H}$ will denote a finite set of hyperplanes in $\mathbb{R}^n$, which we will refer to as a *real hyperplane arrangement*. Such an arrangement is called *Coxeter* if it arises as the set of reflection hyperplanes of a finite real reflection group.

Recall that $\mathcal{H}$ is *simplicial* if $\bigcap_{H \in \mathcal{H}} H = \{0\}$ and all chambers in $\mathbb{R}^n \backslash \mathcal{H}$ are open simplicial cones. All Coxeter arrangements are simplicial, but the converse is false. When $\mathcal{H}$ is simplicial, we will write
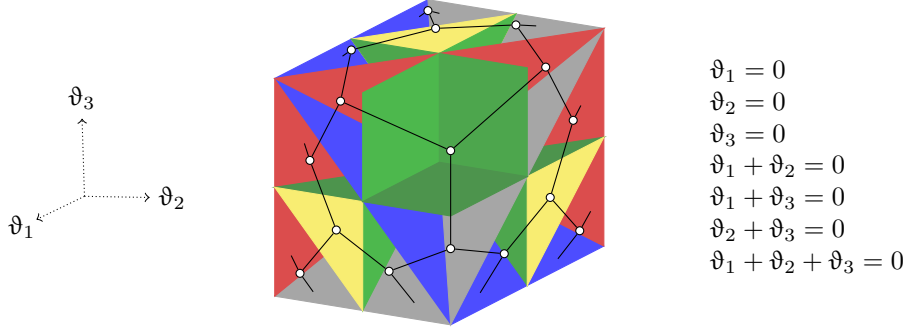$$\mathcal{H}_{\mathbb{C}} := \bigcup_{H \in \mathcal{H}} H_{\mathbb{C}},$$
where $H_{\mathbb{C}}$ denotes the complexification of H. The fundamental object of interest to us is the fundamental group $\pi_1(\mathbb{C}^n \backslash \mathcal{H}_{\mathbb{C}})$ and, as is standard, to access this combinatorially we will use the Deligne groupoid in the next subsection.

Remark 2.1. When $\mathcal{H}$ is Coxeter, it is well-known that $\pi_1(\mathbb{C}^n \backslash \mathcal{H}_{\mathbb{C}})$ is the *pure braid group* associated to the corresponding finite Coxeter group, that is, the kernel of the natural morphism from the braid group to the Weyl group. When the arrangement is simplicial but not Coxeter, there is no such description in terms of a kernel.

When $\mathcal{H}$ is a simplicial hyperplane arrangement, its 1-skeleton is defined to be the graph with vertices corresponding to the chambers, and edges joining chambers which share a codimension one wall.

Example 2.2. As an example, consider the following hyperplane arrangement $\mathcal{H}$ in $\mathbb{R}^3$, and its 1-skeleton. It has 7 hyperplanes, 32 chambers, and is not Coxeter:

$$\begin{aligned} \vartheta_1 &= 0 \\ \vartheta_2 &= 0 \\ \vartheta_3 &= 0 \\ \vartheta_1 + \vartheta_2 &= 0 \\ \vartheta_1 + \vartheta_3 &= 0 \\ \vartheta_2 + \vartheta_3 &= 0 \\ \vartheta_1 + \vartheta_2 + \vartheta_3 &= 0 \end{aligned}$$

This hyperplane arrangement appears for cD_4 singularities with three curves meeting at a point [W, 7.4]; an explicit example of such a cD_4 singularity can be found in [CS, 11.2.19].

2.2. **The Deligne Groupoid.** In this section we summarise some known combinatorial approaches to $\pi_1(\mathbb{C}^n \backslash \mathcal{H}_\mathbb{C})$. For more detailed references, see [P1, P2, D].

Recall that a groupoid is a small category $\mathcal{G}$ such that for any two objects $g, h \in \mathcal{G}$, the set of morphisms $\mathrm{Hom}(g, h)$ is non-empty and further all morphisms are invertible. We recall that a hyperplane arrangement $\mathcal{H}$ in $\mathbb{R}^n$ induces a groupoid $\mathbb{G}_\mathcal{H}$ called the *arrangement groupoid* (or *Deligne groupoid*) of $\mathcal{H}$. To define this, we first associate an oriented graph $\Gamma_\mathcal{H}$ to the hyperplane arrangement $\mathcal{H}$.

Definition 2.3. *The vertices of $\Gamma_\mathcal{H}$ are the chambers (i.e. the connected components) of $\mathbb{R}^n \backslash \bigcup_{H \in \mathcal{H}} H$. There is an arrow $a \colon v_1 \to v_2$ from chamber v_1 to chamber v_2 if the chambers are adjacent, otherwise there is no arrow. For an arrow $a \colon v_1 \to v_2$, we set $s(a) := v_1$ and $t(a) := v_2$.*

Example 2.4. Consider the following hyperplane arrangement $\mathcal{H}$ in $\mathbb{R}^2$, and its associated $\Gamma_\mathcal{H}$. We have labelled the arrows in $\Gamma_\mathcal{H}$ by abuse of notation.

$$\begin{aligned} &\vartheta_1 \\ &\vartheta_2 \\ &\vartheta_1 + \vartheta_2 \\ &\vartheta_1 + 2\vartheta_2 \end{aligned}$$
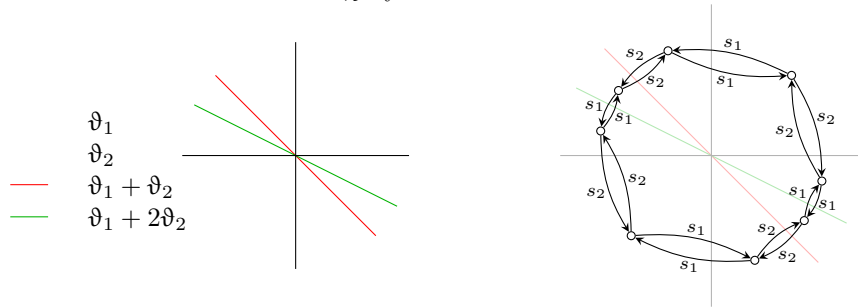
A *positive path of length n* in $\Gamma_\mathcal{H}$ is defined to be a formal symbol

$$p = a_n \circ \ldots \circ a_2 \circ a_1,$$

whenever there exists a sequence of vertices $v_0, \ldots, v_n$ of $\Gamma_\mathcal{H}$ and exist arrows $a_i \colon v_{i-1} \to v_i$ in $\Gamma_\mathcal{H}$. We define $s(p) := v_0$, $t(p) := v_n$, and $\ell(p) := n$. The notation $\circ$ should remind us of composition, but we will often drop the $\circ$'s in future sections. If $q = b_m \circ \ldots \circ b_2 \circ b_1$ is another positive path with $t(p) = s(q)$, we consider the formal symbol

$$q \circ p := b_m \circ \ldots \circ b_2 \circ b_1 \circ a_n \circ \ldots \circ a_2 \circ a_1,$$

and call it the *composition* of p and q. As usual, there are paths of length zero at each vertex v, and by abuse of notation we will also denote the length zero path at v by v, and identify the compositions $t(p) \circ p$ and $p \circ s(p)$ with p.

Definition 2.5. *A positive path is called* minimal *if there is no positive path in $\Gamma_{\mathcal{H}}$ of smaller length, and with the same endpoints. The positive minimal paths are called* atoms.

Example 2.6. In 2.4, the following are all the atoms starting in the chamber C_+.
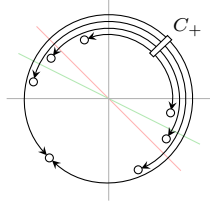
For each choice of start chamber, there is a similar picture.

Following [P1, p170], there is an equivalence relation $\sim$ on the set of paths in $\Gamma_{\mathcal{H}}$, defined as the smallest equivalence relation such that the following conditions are satisfied:
  (1) If $p \sim q$, then $s(p) = s(q)$ and $t(p) = t(q)$.
  (2) If p and q are atoms with the same source and targets, then $p \sim q$.
  (3) If $p \sim q$, then $upr \sim uqr$ for all positive paths u and r satisfying $t(r) = s(p) = s(q)$, and $s(u) = t(p) = t(q)$, .

Write $\mathsf{Path}\,\Gamma_{\mathcal{H}}$ for the set of equivalence classes of positive paths in $\Gamma_{\mathcal{H}}$ with respect to the equivalence relation $\sim$, and write $[p]$ for the equivalence class of a positive path p.

Definition 2.7. *When $\mathcal{H}$ is a simplicial hyperplane arrangement, write $\mathbb{G}_{\mathcal{H}}^+$ for the category whose objects are the vertices in $\Gamma_{\mathcal{H}}$, and whose morphisms are defined*

$$\mathrm{Hom}_{\mathbb{G}_{\mathcal{H}}^+}(v, u) := \{[p] \in \mathsf{Path}\,\Gamma_{\mathcal{H}} \mid s(p) = v \text{ and } t(p) = u\}.$$

The Deligne groupoid *(or the* arrangement groupoid*) $\mathbb{G}_{\mathcal{H}}$ is the groupoid defined as the groupoid completion of $\mathbb{G}_{\mathcal{H}}^+$, that is, adding formal inverses of all morphisms in $\mathbb{G}_{\mathcal{H}}^+$ (see e.g. [P5, §2.3.1]).*

In future sections, we will abuse notation, and refer to $[\alpha] \in \mathsf{Path}\,\Gamma_{\mathcal{H}}$ simply by α, with the equivalence relation being implicit. The following is well-known by [D, P1, P3, Sa] (see also [P2, 2.1]), and is our main reason for considering the Deligne groupoid.

Theorem 2.8. *If $\mathcal{H}$ is simplicial, any vertex group of the groupoid $\mathbb{G}_{\mathcal{H}}$ defined above is isomorphic to $\pi_1(\mathbb{C}^n \backslash \mathcal{H}_{\mathbb{C}})$.*

2.3. **Faithfulness.** The faithfulness of the action of $\pi_1(\mathbb{C}^n \backslash \mathcal{H}_{\mathbb{C}})$ on $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\,X)$ will follow from a more general faithful result on groupoids, which we briefly outline here.

Definition 2.9 ([D, Section 1]). *Assume that $\mathcal{H}$ is simplicial. Let v_i and v_j be vertices in $\Gamma_{\mathcal{H}}$, and let C_i and C_j be the corresponding chambers of $\mathbb{R}^n \setminus \bigcup_{H \in \mathcal{H}} H$. Then we say that v_j is opposite to v_i if there is a line l in $\mathbb{R}^n$ passing through C_i, C_j, and the origin. An opposite vertex of v is unique, and we denote it by $-v$.*

Lemma 2.10. *Assume that $\mathcal{H}$ is simplicial.*
  (1) *For any atom p in $\Gamma_{\mathcal{H}}$, there is an atom p' such that $s(p') = -t(p)$, $t(p') = s(p)$, and the composition pp' is also an atom.*
  (2) *Let a and b be two atoms in $\Gamma_{\mathcal{H}}$ such that $t(a) = t(b)$. Then there are atoms p and q such that $b^{-1}a = qp^{-1}$ in $\mathrm{Hom}_{\mathbb{G}_{\mathcal{H}}}(s(a), s(b))$.*

Proof. (1) This follows from [P1, Section 4, Corollary 2].
(2) By (1), there are atoms p and q such that $s(p) = s(q) = -t(a)$, and ap and bq are atoms. Since the targets and sources of ap and bq are equal, we have $ap \sim bq$. This implies $b^{-1} \circ a = q \circ p^{-1}$ in $\mathrm{Hom}_{\mathbb{G}_{\mathcal{H}}}(s(a), s(b))$. □

Since $\mathbb{G}_{\mathcal{H}}$ is obtained from $\mathbb{G}_{\mathcal{H}}^+$ by adding inverses, there is a natural functor

$$\iota \colon \mathbb{G}_{\mathcal{H}}^+ \longrightarrow \mathbb{G}_{\mathcal{H}}.$$

The following lemma is an easy analogue of [BT, Lemma 2.3], and relies on the fact that ι is faithful for simplicial $\mathcal{H}$.

Lemma 2.11. *Assume that $\mathcal{H}$ is simplicial, and let $F \colon \mathbb{G}_{\mathcal{H}} \to \mathcal{G}$ be a functor between groupoids. Then F is faithful if and only if $F \circ \iota \colon \mathbb{G}_{\mathcal{H}}^+ \to \mathcal{G}$ is faithful.*

Proof. By [D], $\iota \colon \mathbb{G}_{\mathcal{H}}^+ \longrightarrow \mathbb{G}_{\mathcal{H}}$ is faithful. Thus it immediately follows that if F is faithful, so is $F \circ \iota \colon \mathbb{G}_{\mathcal{H}}^+ \to \mathcal{G}$.

For the other direction, assume that $F \circ \iota \colon \mathbb{G}_{\mathcal{H}}^+ \to \mathcal{G}$ is faithful, and let $p, q \in \mathrm{Hom}_{\mathbb{G}_{\mathcal{H}}}(v, w)$ be morphisms. It is enough to show that if $F(p) = F(q)$ then $p = q$. At first, we consider the case when $v = w$. In this case, it is enough to show that, if $F(p) = \mathrm{id}_{F(v)}$, then $p = \mathrm{id}_v$. By repeated use of 2.10(2), there are positive paths p_1 and p_2 such that $p = p_1 \circ p_2{}^{-1}$. Since we have $F(p_1) = F(p_2)$ and $F \circ \iota$ is faithful, necessarily $p_1 = p_2$ and so $p = p_1 \circ p_2{}^{-1} = \mathrm{id}_v$. Next, we consider the general case when $F(p) = F(q)$. Then we have $F(pq^{-1}) = \mathrm{id}_{F(v_j)}$. By the above argument, we see that $pq^{-1} = \mathrm{id}_v$, and thus $p = q$. $\qquad\square$

Corollary 2.12. *Assume that $\mathcal{H}$ is simplicial, $F \colon \mathbb{G}_{\mathcal{H}} \to \mathcal{G}$ is a functor between groupoids, and for any chamber C write $\mathrm{Autgp}(FC) := \mathrm{Hom}_{\mathcal{G}}(FC, FC)$. If F is a faithful functor, then there is an injective group homomorphism*

$$\pi_1(\mathbb{C}^n \backslash \mathcal{H}_{\mathbb{C}}) \to \mathrm{Autgp}(FC).$$

Proof. If F is faithful, the induced group homomorphism $F \colon \mathrm{Hom}_{\mathbb{G}_{\mathcal{H}}}(C, C) \to \mathrm{Autgp}(FC)$ is injective for any chamber $C \in \mathbb{G}_{\mathcal{H}}$. Since $\mathrm{Hom}_{\mathbb{G}_{\mathcal{H}}}(C, C)$ is isomorphic to $\pi_1(\mathbb{C}^n \backslash \mathcal{H}_{\mathbb{C}})$ by 2.8, the result holds. $\qquad\square$

2.4. **Deligne normal form.** By 2.11 and 2.12 our problem will reduce to proving the faithfulness of a *positive part* of a groupoid action. This is a significant reduction in complexity, since every positive path has a *Deligne normal form*, which we recall here. This normal form replaces the Garside normal form in [BT], which is only defined for Coxeter arrangements. The proof of faithfulness will simply induct on the number of factors of this normal form.

For positive paths $p, q \in \Gamma_{\mathcal{H}}$ with $s(p) = s(q)$, we say that p *begins* with q if there exists a positive path r such that $s(r) = t(q)$, $t(r) = t(p)$ and $p \sim rq$. For a positive path p, write $\mathsf{Begin}(p)$ for the set of all atoms with which p begins. Similarly, we can consider the set of atoms with which p ends, which is defined in the analogous way, and we denote this set by $\mathsf{End}(p)$.

Definition 2.13. *For any path $p \in \Gamma_{\mathcal{H}}$, by [P2, 2.2] (or [D]), there exists a unique (up to equivalence) atom α_1 such that $\mathsf{Begin}(p) = \mathsf{Begin}(\alpha_1)$. Then, in particular, p begins with α_1, and so there is a positive path β with $s(\beta) = t(\alpha_1)$ and $t(\beta) = t(p)$ such that*

$$p \sim \beta \circ \alpha_1.$$

*Continuing this process with β, we decompose p into atoms*

$$p \sim \alpha_n \circ \ldots \circ \alpha_2 \circ \alpha_1,$$

which we refer to as the Deligne normal form *of p.*

The following lemma is convenient, and is well known [P1, Lemma 4.2].

Lemma 2.14. *If $p \in \Gamma_{\mathcal{H}}$, then p is an atom if and only if p does not cross any hyperplane twice.*

Example 2.15. Continuing the example and notation in 2.4, dropping the composition symbol $\circ$, the path $p = s_2 s_1 s_2 s_1 s_2 s_2 s_1 s_1 s_2 s_1$ satisfies $\mathsf{Begin}(p) = \mathsf{Begin}(s_2 s_1 s_2 s_1)$ since
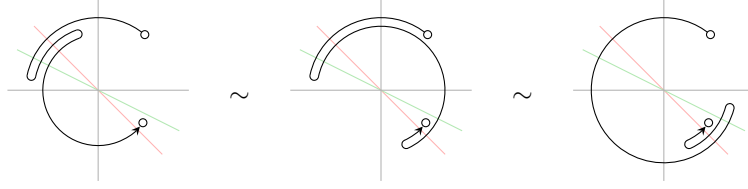
Continuing in this way, p has Deligne normal form $s_2(s_2 s_1)(s_1 s_2 s_1)(s_2 s_1 s_2 s_1)$.

3. The Tilting Order and Chambers

Our strategy to prove faithfulness of the action in the flops setting is to exploit the partial order on tilting modules, due to Riedtmann–Schofield and Happel–Unger [RS, HU]. In the case of minimal resolutions of Kleinian singularities, we can bypass this step by simply appealing to [IR, §6], and so for the proof of faithfulness in this case, the reader can skip immediately to Appendix A.

3.1. **Tilting Modules and Mutation.** Recall first that for an algebra A such that the category $\mathrm{mod}\,A$ of finitely generated A-modules is Krull–Schmidt, $M \in \mathrm{mod}\,A$ is called *basic* if there is no repetition in its Krull–Schmidt decomposition into indecomposable A-modules, and the algebra A is called *basic* if it is basic as an A-module.

Throughout this section, Λ is a basic $\mathfrak{R}$-algebra, where $\mathfrak{R}$ is a complete local domain. Note by [Sw, p566], for such rings the category $\mathrm{mod}\,\Lambda$ is Krull–Schmidt. In our geometric settings later, such Λ appear when we work on the formal fibre.

Definition 3.1. $T \in \mathrm{mod}\,\Lambda$ *is a* classical tilting module *if the following conditions hold.*

(1) $\mathrm{pd}_\Lambda T \leq 1$.
(2) $\mathrm{Ext}^1_\Lambda(T, T) = 0$.
(3) *There exists a short exact sequence* $0 \to \Lambda \to T_1 \to T_2 \to 0$, *with each* $T_i \in \mathrm{add}\,T$.

We write $\mathsf{tilt}\,\Lambda$ *for the set of basic classical tilting* Λ-*modules.*

We shall refer to classical tilting modules simply as *tilting modules*, with it being implicit that $\mathrm{pd}_\Lambda T \leq 1$. When T is a tilting module, we write $\mathsf{Fac}\,T$ for the full subcategory of $\mathrm{mod}\,\Lambda$ consisting of those modules Y such that there exists a surjection $T' \twoheadrightarrow Y$ with $T' \in \mathrm{add}\,T$. It is known, and easy to prove from 3.1(3), that

$$\mathsf{Fac}\,T = \{X \in \mathrm{mod}\,\Lambda \mid \mathrm{Ext}^1_\Lambda(T, X) = 0\}, \tag{3.A}$$

so in particular for any $X \in \mathsf{Fac}\,T$ there is an exact sequence

$$0 \to Y \to T' \to X \to 0$$

with $Y \in \mathsf{Fac}\,T$ and $T' \in \mathrm{add}\,T$. It follows immediately that

$$\mathrm{add}\,T = \{X \in \mathsf{Fac}\,T \mid \mathrm{Ext}^1_\Lambda(X, \mathsf{Fac}\,T) = 0\}. \tag{3.B}$$

The set $\mathsf{tilt}\,\Lambda$ carries the natural structure of a partially ordered set.

Notation 3.2. Let $T, U \in \mathsf{tilt}\,\Lambda$. We write $T \geq U$ if $\mathrm{Ext}^1_\Lambda(T, U) = 0$, or equivalently by (3.A), if $U \in \mathsf{Fac}\,T$. We write $T > U$ if $T \geq U$ and $\neg(U \geq T)$.

It is immediate from (3.B) and the Krull–Schmidt property that if $T, U \in \mathsf{tilt}\,\Lambda$ with $T \geq U \geq T$, then $T \cong U$. We remark that $T \geq U$ if and only if $\mathsf{Fac}\,T \supseteq \mathsf{Fac}\,U$, and that $\Lambda \in \mathsf{tilt}\,\Lambda$ is the greatest element with respect to $\geq$.

Another key property of the set $\mathsf{tilt}\,\Lambda$ is that it admits an operation called *mutation*. For $T \in \mathsf{tilt}\,\Lambda$, and an indecomposable direct summand T_i of T, there exists at most one basic tilting Λ-module $\nu_i T = (T/T_i) \oplus U_i$ such that $T_i \not\cong U_i$ (c.f. [RS]). The module $\nu_i T$ is called a *tilting mutation* of T, and in general it may or may not exist. As is standard, mutation is encoded in the *exchange graph* of $\mathsf{tilt}\,\Lambda$.

Notation 3.3. We write $\mathsf{EG}(\Lambda)$ for the *exchange graph*, where vertices are elements of $\mathsf{tilt}\,\Lambda$, and we draw an edge between T and $\boldsymbol{\nu}_i T$ for all T and i such that $\boldsymbol{\nu}_i T$ exists. Further, for a fixed projective P, let $\mathsf{EG}_P(\Lambda)$ denote the full subgraph of the exchange graph of Λ consisting of those vertices that contain P as a summand.

3.2. **Chambers Associated to Tilting Modules.** To functorially control compositions of tilting mutations requires chambers, which we now describe. We first fix notation. Let Λ be a basic $\mathfrak{R}$-algebra, where $\mathfrak{R}$ is a complete local domain, and write $K_0 := K_0(\mathrm{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda))$. It is well known that

$$K_0 \cong \mathbb{Z}^{n+1} \tag{3.C}$$

since every $P \in \mathrm{proj}\,\Lambda$ can be uniquely written as a direct sum of indecomposable projectives $P_0^{\oplus a_0} \oplus \ldots \oplus P_n^{\oplus a_n}$ for some a_i. In what follows, we will fix the $\mathbb{Z}$-basis of K_0 given by (3.C), namely $\{\mathbf{e}_0, \ldots, \mathbf{e}_n\}$ where $\mathbf{e}_i$ is the class of P_i in K_0.

We now fix a projective, which by convention will be P_0, and we will primarily be interested in $\mathsf{EG}_0(\Lambda) := \mathsf{EG}_{P_0}(\Lambda)$, and its vertex set $\mathsf{tilt}_0(\Lambda)$ consisting of all tilting Λ-modules that contain P_0 as a summand. For this purpose, consider the following factor $\mathbb{R}$-vector space of $K_0 \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{n+1}$ given by

$$\Theta_\Lambda := (K_0 \otimes_{\mathbb{Z}} \mathbb{R})/\operatorname{Span}\{\mathbf{e}_0\} \cong \mathbb{R}^n.$$

By abuse of notation, we write $\{[P_1], \ldots, [P_n]\}$ for the $\mathbb{R}$-basis of Θ_Λ induced by (3.C), with it being implicit that the $[-]$ notation works modulo $\operatorname{Span}\{\mathbf{e}_0\}$. From this, we define

$$C_+ := \left\{ \sum_{i=1}^n \vartheta_i[P_i] \ \middle| \ \vartheta_i > 0 \text{ for all } 1 \leq i \leq n \right\} \subseteq \Theta_\Lambda.$$

For $T \in \mathsf{tilt}_0\,\Lambda$, write $T = T_0 \oplus T_1 \oplus \ldots \oplus T_n$ where by convention $P_0 = T_0$, and consider

$$C_T := \left\{ \sum_{i=1}^n \vartheta_i[T_i] \ \middle| \ \vartheta_i > 0 \text{ for all } 1 \leq i \leq n \right\} \subseteq \Theta_\Lambda. \tag{3.D}$$

It is clear from the definition that $C_\Lambda = C_+$.

The following is elementary, and is very similar to the arguments of [H, DIJ]. Since the setting here does not involve Hom-finite categories, we give the proof in Appendix B.

Lemma 3.4. *Suppose that Λ is a basic $\mathfrak{R}$-algebra, where $\mathfrak{R}$ is a complete local domain. If $T, U \in \mathsf{tilt}_0\,\Lambda$ are related by a mutation at an indecomposable summand, then C_T and C_U do not overlap, and are separated by a codimension one wall.*

It is the following that will allow us to control iterations, as it relates the combinatorics of chamber structures to the homological property of the tilting order. The result seems to be folklore; for lack of a suitable reference, and since we are working slightly more generally than usual, we give the proof in B.4 in Appendix B.

Theorem 3.5. *Suppose that Λ is a basic $\mathfrak{R}$-algebra, where $\mathfrak{R}$ is a complete local domain. Suppose that $T, U \in \mathsf{tilt}_0\,\Lambda$ are related by a mutation at an indecomposable summand, so by 3.4 C_T and C_U are separated by H. Suppose that $[\Lambda] \notin H$. Then $T > U$ iff C_T lies on the same side of H as $[\Lambda]$.*

4. Compositions of Mutations and Flops

In this section we will describe compositions of mutation functors, respectively flop functors, under Deligne normal form. This, and more generally the proof of faithfulness of the group action, will be reduced to the formal fibre, and so for much of the paper we will work under the following setup.

Setup 4.1. Suppose that $f: U \to \operatorname{Spec}\mathfrak{R}$ is a complete local 3-fold flopping contraction, where U is smooth.

It is well-known [V, 3.2.8] that in this setting $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} U)$ admits a tilting bundle $\mathcal{V}$ generated by global sections, which after setting $M := f_*\mathcal{V}$, induces an equivalence $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} U) \cong \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \operatorname{End}_{\mathfrak{R}}(M))$. The algebra $\operatorname{End}_{\mathfrak{R}}(M)$ contains $\operatorname{Hom}_{\mathfrak{R}}(M, \mathfrak{R})$ as a summand, and in the following we fix $P_0 := \operatorname{Hom}_{\mathfrak{R}}(M, \mathfrak{R})$, so that $\mathsf{tilt}_0(\Lambda)$ consists of those tilting Λ-modules containing $\operatorname{Hom}_{\mathfrak{R}}(M, \mathfrak{R})$ as a summand.

4.1. **CT objects and Simple Wall Crossings.** Under the above flops setup, $M \in \operatorname{CM} \mathfrak{R}$ and $\operatorname{End}_{\mathfrak{R}}(M)$ is a NCCR [V, 3.2.9, 3.2.10]. It follows [IW, 5.4] that M is a *cluster tilting* (=CT) object of $\operatorname{CM} \mathfrak{R}$, namely there are equalities

$$\operatorname{add} M = \{X \in \operatorname{CM} \mathfrak{R} \mid \operatorname{Ext}^1_{\mathfrak{R}}(X, M) = 0\} = \{Y \in \operatorname{CM} \mathfrak{R} \mid \operatorname{Ext}^1_{\mathfrak{R}}(M, Y) = 0\}.$$

We can, and will, assume that M is basic. The class of basic cluster tilting objects carries an operation of *mutation*, which involves picking an indecomposable summand M_i of a CT module M, and uniquely replacing it with a different indecomposable summand whilst remaining CT; the resulting module will be denoted $\nu_i M$.

By the three-dimensional Auslander–McKay correspondence [W, 6.9], the number of CT $\mathfrak{R}$-modules is equal to the number of chambers of some simplicial hyperplane arrangement, described in detail in [W, 5.24, 5.25], and furthermore crossing a codimension one wall (henceforth a *simple wall crossing*) corresponds to mutating an indecomposable summand of the associated CT module. Consequently, the 1-skeleton of the arrangement equals the exchange graph of CT $\mathfrak{R}$-modules.

Under the setup of 4.1, to fix notation we will write $\mathcal{H}_\Lambda$ for the simplicial hyperplane arrangement associated to f, set $M := f_*\mathcal{V}$, which will correspond to the chamber C_+, and fix $\Lambda := \operatorname{End}_{\mathfrak{R}}(M)$.

Example 4.2. There exists [K] a cD_4 flop with the following simplicial hyperplane arrangement. Under the Auslander–McKay correspondence, the following picture illustrates the exchange graph of CT objects, where $\nu_{i_2 i_1} := \nu_{i_2} \nu_{i_1}$ etc.
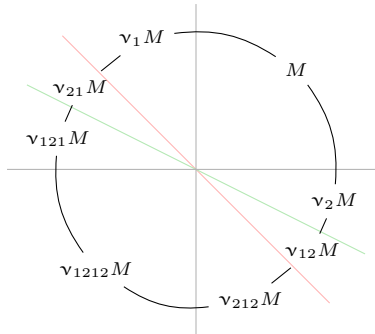
Thus, under the setup of 4.1, via [W, 5.24, 5.25] every chamber C in $\mathcal{H}_\Lambda$ has an associated CT $\mathfrak{R}$-module N_C say, and thus an associated derived category $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda_C)$, where $\Lambda_C := \operatorname{End}_{\mathfrak{R}}(N_C)$. There are natural equivalences between these categories, as follows.

Notation 4.3. Suppose that $\alpha: C \to D$ is an atom in $\Gamma_{\mathcal{H}_\Lambda}$. Then by [IW, 4.17] $T_{CD} := \operatorname{Hom}_{\mathfrak{R}}(N_C, N_D)$ is a tilting bimodule from Λ_C to Λ_D, and we consider the equivalence

$$\mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda_C) \xrightarrow{t_\alpha := \mathbf{R}\operatorname{Hom}_{\Lambda_C}(T_{CD}, -)} \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda_D).$$

When α is a simple wall crossing, mutating the ith summand of N_C say, we will write

$$t_i := \mathbf{R}\operatorname{Hom}_{\Lambda_C}(T_{CD}, -),$$

and refer to t_i as the *mutation functor*.

Remark 4.4. By [W, 4.2], the functor t_i is functorially isomorphic to the inverse of the flop functor, flopping a single curve C_i.

It is known [DW3, 3.22] that the mutation functors t_i form a representation of the Deligne groupoid, and thus they alone are enough to induce the action of the fundamental group. However, it is the existence of the additional functors t_α for every atom α that will allow us to control this action, and prove faithfulness in this paper.

Example 4.5. Continuing the example 4.2, setting $\Lambda_I := \mathrm{End}_{\mathfrak{R}}(\nu_I M)$, the mutation functors t_i are as follows:
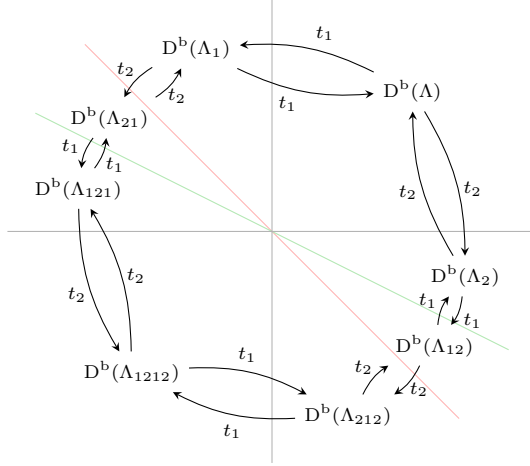
There are more direct functors, for all atoms. As in 2.6, for those out of C_+ these are
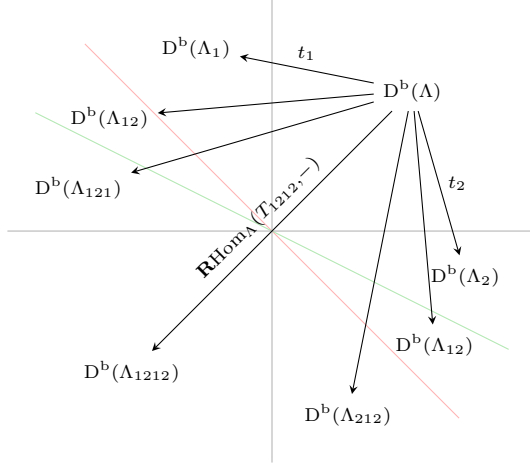
There are similar additional functors emerging from each of the other chambers.

4.2. **Atoms and the Tilting Order.** Under the flops setup 4.1, recall from the last subsection that we associate an algebra $\Lambda = \mathrm{End}_{\mathfrak{R}}(M)$, and a simplicial hyperplane arrangement $\mathcal{H}_\Lambda$. The functor

$$\mathbb{F} := \mathrm{Hom}_{\mathfrak{R}}(M, -) \colon \mathrm{mod}\, R \to \mathrm{mod}\, \Lambda$$

is fully faithful, and furthermore by [IW, 4.17, 5.11] induces an injective map

$$\mathbb{F} \colon \{\text{CT } \mathfrak{R}\text{-modules}\} \to \mathsf{tilt}_0\, \Lambda \tag{4.A}$$

where recall $\mathsf{tilt}_0\, \Lambda$ consists of all tilting Λ-modules containing $P_0 = \mathrm{Hom}_{\mathfrak{R}}(M, \mathfrak{R})$ as a summand. By [IW2, 4.5(1)] this map is compatible with mutation. But since $\mathfrak{R}$ is an isolated singularity, all possible mutations of a fixed CT $\mathfrak{R}$-module N give all possible mutations of $\mathbb{F}N$ in $\mathsf{tilt}_0\, \Lambda$, hence the finite connected mutation graph of CT $\mathfrak{R}$-modules induces, under $\mathbb{F}$, a finite connected component of $\mathsf{tilt}_0\, \Lambda$. By a result of Happel–Unger

(adapted and proved in the setting here in [IW2, 4.9]), $\mathsf{tilt}_0 \Lambda$ must equal this finite connected component, thus (4.A) is in fact a bijection compatible with mutation.

It follows that the exchange graph $\mathsf{EG}_0 \Lambda$ from §3.2 equals the exchange graph of CT $\mathfrak{R}$-modules, in a way compatible with mutation. Hence, by the last subsection, $\mathsf{EG}_0 \Lambda$ also equals the 1-skeleton of $\mathcal{H}_\Lambda$, and thus the chambers of $\mathcal{H}_\Lambda$ are indexed by tilting Λ-modules, in a manner such that two modules that share a codimension one wall are related by a mutation at an indecomposable summand, in the sense of §3.1. We refer the reader to 4.8 for an example.

The following is our main technical lemma, which uses the tilting chambers to establish in the second part that the composition of mutation functors along Deligne normal form is given by a direct tilt. To avoid confusion, write D_T for the chamber of $\mathcal{H}_\Lambda$ indexed by $T \in \mathsf{tilt}_0 \Lambda$, and write C_T for the chamber (3.D). We write $D_+ := D_\Lambda$.

Theorem 4.6. *Under the setup 4.1, for any $S \in \mathsf{tilt}_0 \Lambda$, suppose that $\alpha \colon D_+ \to D_S$ is an atom in $\Gamma_{\mathcal{H}_\Lambda}$, and choose a decomposition of α into length one positive paths*

$$\alpha = \quad D_\Lambda \xrightarrow{s_{i_1}} D_2 \to \ldots \to D_m \xrightarrow{s_{i_m}} D_{m+1}.$$

For $i = 2, \ldots, m+1$, write M_i for the CT $\mathfrak{R}$-module corresponding to the chamber D_i, so that $S = \mathbb{F} M_{m+1}$. Then the following assertions hold.

(1) *As tilting Λ-modules, $\Lambda = \mathbb{F} M > \mathbb{F} M_2 > \ldots > \mathbb{F} M_m > \mathbb{F} M_{m+1} = S$.*

(2) *There is a bimodule isomorphism*

$$\mathrm{Hom}_{\mathfrak{R}}(M_m, M_{m+1}) \overset{\mathbf{L}}{\otimes} \ldots \overset{\mathbf{L}}{\otimes} \mathrm{Hom}_{\mathfrak{R}}(M_2, M_3) \overset{\mathbf{L}}{\otimes} \mathrm{Hom}_{\mathfrak{R}}(M, M_2) \cong \mathrm{Hom}_{\mathfrak{R}}(M, M_{m+1})$$

where, reading right to left, the tensors are over $\mathrm{End}_{\mathfrak{R}}(M_i)$ for $i = 2, \ldots, m$.

(3) *$C_S := \{\sum_{i=1}^n \vartheta_i [S_i] \mid \vartheta_i > 0 \text{ for all } 1 \leq i \leq n\}$ equals D_S.*

Proof. We prove all assertions together. By induction we can assume that

$$\Lambda = \mathbb{F} M > \ldots > \mathbb{F} M_m, \tag{4.B}$$

that $C_X = D_X$ for the tilting modules in (4.B), and that there is a bimodule isomorphism

$$\mathrm{Hom}_{\mathfrak{R}}(M_{m-1}, M_m) \overset{\mathbf{L}}{\otimes} \ldots \overset{\mathbf{L}}{\otimes} \mathrm{Hom}_{\mathfrak{R}}(M, M_2) \cong \mathrm{Hom}_{\mathfrak{R}}(M, M_m) \tag{4.C}$$

since the case $m = 1$ is clear.

Certainly the hyperplanes of $\mathcal{H}_\Lambda$ cannot pass through any chamber, in particular $C_\Lambda = D_\Lambda$. Write $T := \mathbb{F} M_m$, and H' for the wall separating C_T and C_S. Since $C_T = D_T$ by induction, extending H' to a hyperplane, H' is one of the hyperplanes of $\mathcal{H}_\Lambda$. Hence since $[\Lambda] \in C_\Lambda$, and the hyperplanes of $\mathcal{H}_\Lambda$ cannot pass through C_Λ, necessarily $[\Lambda] \notin H'$.

We next crash through the wall H from D_T into D_S. If D_+ is not on the same side of H as D_T, then α would have to cross H twice, and so by 2.14 applied to $\mathcal{H}_\Lambda$, the path α would not be an atom. Hence D_+ must be on the same side of H as D_T. Since $[\Lambda] \in C_+ = D_+$ and $C_T = D_T$, we conclude that $[\Lambda]$ is on the same side of H' as C_T.

By 3.5 necessarily $T > S$, i.e. $\mathbb{F} M_m > \mathbb{F} M_{m+1}$, so combining with (4.B) proves (1). Next, the induction (4.C) gives a bimodule isomorphism

$$\mathrm{Hom}_{\mathfrak{R}}(M_m, M_{m+1}) \overset{\mathbf{L}}{\otimes} (\ldots \overset{\mathbf{L}}{\otimes} \mathrm{Hom}_{\mathfrak{R}}(M, M_2)) \cong \mathrm{Hom}_{\mathfrak{R}}(M_m, M_{m+1}) \overset{\mathbf{L}}{\otimes} \mathrm{Hom}_{\mathfrak{R}}(M, M_m),$$

so to prove (2) it suffices to show that there is a bimodule isomorphism

$$\mathrm{Hom}_{\mathfrak{R}}(M_m, M_{m+1}) \overset{\mathbf{L}}{\otimes} \mathrm{Hom}_{\mathfrak{R}}(M, M_m) \cong \mathrm{Hom}_{\mathfrak{R}}(M, M_{m+1}). \tag{4.D}$$

Applying B.1 with $T = \mathbb{F} M_m$, $\Gamma = \mathrm{End}_{\mathfrak{R}}(M_m)$ and $\nu_i \Gamma = \mathrm{Hom}_{\mathfrak{R}}(M_m, M_{m+1})$ shows that the left hand side of (4.D) is concentrated in degree zero, so to prove (2) it suffices to show that there is a bimodule isomorphism

$$\mathrm{Hom}_{\mathfrak{R}}(M_m, M_{m+1}) \otimes \mathrm{Hom}_{\mathfrak{R}}(M, M_m) \cong \mathrm{Hom}_{\mathfrak{R}}(M, M_{m+1}). \tag{4.E}$$

But there is a chain of isomorphisms

$$\mathrm{Hom}_{\mathfrak{R}}(M_m, M_{m+1}) \otimes \mathrm{Hom}_{\mathfrak{R}}(M, M_m) \xrightarrow{\sim} \mathrm{Hom}_{\Lambda}(T, \mathbb{F}M_{m+1}) \otimes T \xrightarrow{\sim} \mathbb{F}M_{m+1}$$

where the first is reflexive equivalence $g \otimes f \mapsto (g \circ -) \otimes f$, and the second is the adjunction from the derived equivalence (using the last statement in B.1), which takes $\varphi \otimes t \mapsto \varphi(t)$. Composing the above shows that there is an isomorphism (4.E), given by $g \otimes f \mapsto g \circ f$. By inspection this an isomorphism in the category of bimodules, proving (2).

Finally, to prove (3), note that the bimodule isomorphism in (2) induces a functorial isomorphism between $\mathbf{R}\mathrm{Hom}_{\Lambda}(\mathbb{F}M_{m+1}, -) = \mathbf{R}\mathrm{Hom}_{\Lambda}(S, -)$ and the composition

$$\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda) \xrightarrow{t_{i_1}} \ldots \xrightarrow{t_{i_m}} \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda_{m+1}). \tag{4.F}$$

Writing $\Lambda_{m+1} = P_0 \oplus Q_1 \oplus \ldots \oplus Q_n$, it is easy to see that tracking

$$\{\sum_i \vartheta_i [Q_i] \mid \vartheta_i > 0, i = 1, \ldots, n\} \tag{4.G}$$

through the inverse of $\mathbf{R}\mathrm{Hom}_{\Lambda}(S, -)$ gives

$$\{\sum_i \vartheta_i [S_i] \mid \vartheta_i > 0, i = 1, \ldots, n\} = C_S.$$

By the functorial isomorphism, this must give the same answer as tracking (4.G) through the inverse of (4.F). We thus claim that tracking (4.G) through the inverse of (4.F) gives D_S, as then $D_S = C_S$ and the result follows.

On one hand, by the definition of the mutation functors, tracking (4.G) through the inverse of (4.F) precisely follows the moduli-tracking rules laid out in [W, 5.14, 5.15]. On the other hand, it is known [W, 5.25] that after possibly replacing some of the

$$t_j = \mathbf{R}\mathrm{Hom}_{\mathrm{End}_{\mathfrak{R}}(N)}(\mathrm{Hom}_{\mathfrak{R}}(N, \nu_j N), -)$$

in (4.F) by

$$t_j' := - \otimes^{\mathbf{L}}_{\mathrm{End}_{\mathfrak{R}}(N)} \mathrm{Hom}_{\mathfrak{R}}(\nu_j N, N),$$

tracking (4.G) back through the inverse of the replacement chain does indeed give the simplicial cone D_S. Crucially, since the combinatorial rules for tracking through t_j and through t_j' are the same in this flops setting (see [W, 5.15]), the replacements do not matter, and so tracking (4.G) through the inverse of (4.F) also gives D_S, as required. □

Remark 4.7. We remark that the initial choice of decomposition of α in 4.6 does not matter, as the theorem shows that all choices are functorially isomorphic to t_α.

Example 4.8. Continuing the flopping contraction example in 4.5, the chambers of $\mathcal{H}_\Lambda$ can be indexed by elements of $\mathsf{tilt}_0 \Lambda$, as illustrated in the left hand side of the following picture, where $\nu_{i_2 i_1} \Lambda = \mathrm{Hom}_{\mathfrak{R}}(M, \nu_{i_2 i_1} M)$ etc. The ordering, which is illustrated in the right hand side, is forced by 4.6(1).

$\mathsf{EG}_0(\Lambda)$ Order
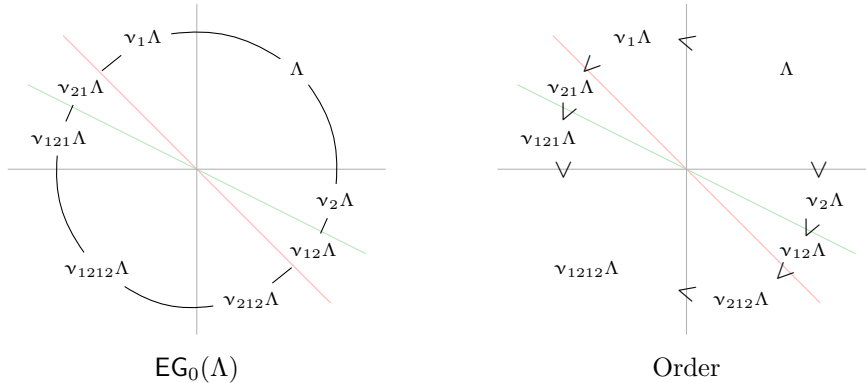
Since the positive path below corresponding to the composition $t_2 t_1 t_2$ is an atom, it also follows from 4.6(2) (applied to Λ_1) that the composition $t_2 t_1 t_2$ is functorially isomorphic to the direct functor shown
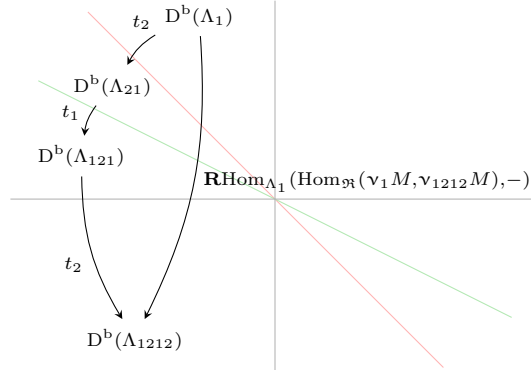
For future use, a useful corollary of 4.6 is the following.

Corollary 4.9. *Under the assumptions 4.1, let $\alpha\colon C \to D$ be an atom, and let $N \in \mod \Lambda_C$. Then $H^i(t_\alpha N) = 0$ for all $i \neq 0, 1$.*

5. TRACKING VIA TORSION PAIRS

Under the flops setup of 4.1, suppose that C is a chamber of $\mathcal{H}_\Lambda$. It follows from 4.6 applied to $\Lambda_C = \mathrm{End}_{\mathfrak{R}}(N_C)$ that if $\alpha\colon C \to D$ is an atom, then the composition of mutation functors along the path α is functorially isomorphic to $t_\alpha = \mathbf{R}\mathrm{Hom}_{\Lambda_C}(T_\alpha, -)$ where $T_\alpha := \nu_\alpha \Lambda_C \in \mathsf{tilt}_0 \Lambda_C$. We will use this implicitly from now on.

As is standard (see e.g. [SY, 2.7]), T_α induces two torsion pairs, which restrict to torsion pairs on finite length modules $\mathsf{fl}\,\Lambda_C$ and $\mathsf{fl}\,\Lambda_D$. These are $(\mathcal{T}_\alpha, \mathcal{F}_\alpha)$ and $(\mathcal{X}_\alpha, \mathcal{Y}_\alpha)$, where

$$\mathcal{T}_\alpha := \{N \in \mathsf{fl}\,\Lambda_C \mid \mathrm{Ext}^1_{\Lambda_C}(T_\alpha, N) = 0\}$$
$$\mathcal{F}_\alpha := \{N \in \mathsf{fl}\,\Lambda_C \mid \mathrm{Hom}_{\Lambda_C}(T_\alpha, N) = 0\},$$

and

$$\mathcal{X}_\alpha := \{N \in \mathsf{fl}\,\Lambda_D \mid N \otimes_{\Lambda_D} T_\alpha = 0\}$$
$$\mathcal{Y}_\alpha := \{N \in \mathsf{fl}\,\Lambda_D \mid \mathrm{Tor}_1^{\Lambda_D}(N, T_\alpha) = 0\}.$$

The Brenner–Butler Theorem for finite dimensional algebras (proved in the module-finite setting here in [SY, 2.9]) asserts that these tilting modules not only induce the above two torsion pairs, but also induce the following categorical equivalences:

$$\mathcal{T}_\alpha \; \underset{\otimes_{\Lambda_D} T_\alpha}{\overset{\mathrm{Hom}_{\Lambda_C}(T_\alpha, -)}{\rightleftarrows}} \; \mathcal{Y}_\alpha \qquad \text{and} \qquad \mathcal{F}_\alpha \; \underset{\mathrm{Tor}_1^{\Lambda_D}(-, T_\alpha)}{\overset{\mathrm{Ext}^1_{\Lambda_C}(T_\alpha, -)}{\rightleftarrows}} \; \mathcal{X}_\alpha. \qquad (5.\text{A})$$

To control the functors t_α requires us to track various objects, which we do here. The following lemma is a standard fact about Deligne normal form, which precisely mirrors the Coxeter version.

Lemma 5.1. *Suppose that α is an atom. Then*

(1) *αs_i is an atom $\iff s_i \notin \mathsf{Begin}(\alpha)$.*
(2) *$s_j \alpha$ is an atom $\iff s_j \notin \mathsf{End}(\alpha)$.*

Proof. For lack of a suitable reference, we give the proof of (1), with (2) being similar.
($\Rightarrow$) is clear, using 2.14.
($\Leftarrow$) We prove the contrapositive. Suppose that the composition

$$A \xrightarrow{s_i} B \xrightarrow{\alpha} C$$

is not an atom, and write H for the hyperplane separating A and B. By 2.14, αs_i must cross some hyperplane at least twice. But since α is an atom, again by 2.14, the hyperplanes that α crosses must be distinct. Hence the only possibility is that αs_i crosses H precisely twice.

In particular, α must cross H, and so since it cannot cross H twice by 2.14, $t(\alpha) = C$ must be on the same side of H as A. If we write β for the smallest positive path (atom) from A to C, then β cannot cross H by 2.14, since A and C lie on the same side of H. Since $s_i \colon B \to A$ obviously only crosses H, it follows again by 2.14 that the composition

$$B \xrightarrow{s_i} A \xrightarrow{\beta} C$$

is an atom. Hence $\alpha \sim \beta s_i$, since both are atoms from B to C, and so $s_i \in \mathsf{Begin}(\alpha)$. $\square$

Notation 5.2. In each chamber D of $\mathsf{EG}_0(\Lambda)$ there is an algebra Λ_D with precisely $n+1$-simples. By abuse of notation we will denote these simples $S_0, S_1, \ldots, S_n$, where S_0 always corresponds to P_0, and performing the simple wall crossing s_i corresponds to the tilting mutation at the projective cover of S_i. We will use the same notation S_i for every Λ_D, and will often consider $\mathcal{S} := \bigoplus_{i=0}^n S_i$, with it being implicit from the context which Λ_D to view this as a module over.

Lemma 5.3. *Under the assumptions 4.1, $t_i(S_i) \cong S_i[-1]$ for all $1 \le i \le n$.*

Proof. Say $s_i \colon C \to D$, so that $t_i = \mathbf{R}\mathrm{Hom}_{\Lambda_C}(\nu_i \Lambda_C, -)$. Since $\Lambda_C > \nu_i \Lambda_C$, as in Appendix B there exists a short exact sequence

$$0 \to P_i \to P' \to C_i \to 0$$

with $P' \in \mathrm{add}\, \frac{\Lambda_C}{P_i}$ such that $\nu_i \Lambda_C = \frac{\Lambda_C}{P_i} \oplus C_i$. Applying $\mathrm{Hom}_{\Lambda_C}(-, S_i)$ to the above sequence yields the result, exactly as in [W, 4.15(2)]. $\square$

For our purposes later, we require more than 5.3, namely for atoms $\alpha \colon C \to D$ we need to track all summands of $\mathcal{S}$ under the inverse functor $t_\alpha^{-1} \cong -\otimes_{\Lambda_D}^{\mathbf{L}} T_\alpha$. Since $(\mathcal{X}_\alpha, \mathcal{Y}_\alpha)$ is a torsion pair on $\mathsf{fl}\,\Lambda_D$, and each S_i is simple, either $S_i \in \mathcal{X}_\alpha$ or $S_i \in \mathcal{Y}_\alpha$. Using the categorical equivalences (5.A) it thus follows that

$$t_\alpha^{-1}(S_i) = \begin{cases} \mathrm{Tor}_1^{\Lambda_D}(S_i, T_\alpha)[1] & \text{if } S_i \in \mathcal{X}_\alpha \\ S_i \otimes_{\Lambda_D} T_\alpha & \text{if } S_i \in \mathcal{Y}_\alpha. \end{cases} \tag{5.B}$$

In the top case, $t_\alpha^{-1}(S_i)$ is the shift of a module in $\mathcal{F}_\alpha$, and in the bottom case $t_\alpha^{-1}(S_i)$ is a module in $\mathcal{T}_\alpha$.

The following is our key preparatory lemma, which says that the torsion pairs $(\mathcal{T}_\alpha, \mathcal{F}_\alpha)$ and $(\mathcal{X}_\alpha, \mathcal{Y}_\alpha)$ detect both how α starts, and how α ends.

Lemma 5.4. *Under the assumptions 4.1, suppose that $\alpha \colon C \to D$ is an atom. Then for $i \neq 0$, the following statements hold.*

(1) $S_i \in \mathcal{F}_\alpha \iff s_i \in \mathsf{Begin}(\alpha)$.
(2) $S_i \in \mathcal{T}_\alpha \iff s_i \notin \mathsf{Begin}(\alpha)$.
(3) $S_i \in \mathcal{X}_\alpha \iff s_i \in \mathsf{End}(\alpha)$.
(4) $S_i \in \mathcal{Y}_\alpha \iff s_i \notin \mathsf{End}(\alpha)$.

Proof. We prove (1), with all others being similar.

($\Leftarrow$) Suppose that α starts with s_i, and write $\alpha = \beta \circ s_i$. Then $t_\alpha(S_i) \overset{5.3}{=} t_\beta(S_i)[-1]$. Applying 4.9 to both sides, it follows that $H^j(t_\alpha(S_i)) = 0$ for all $j \neq 1$, so $S_i \in \mathcal{F}_\alpha$.
($\Rightarrow$) Suppose that α does not start with s_i, then by 5.1 $\alpha \circ s_i$ is still a reduced expression

of Deligne normal form. Hence $t_{\alpha s_i} = t_\alpha \circ t_{s_i}$, and so $t_{\alpha s_i}(S_i) \overset{5.3}{=} t_\alpha(S_i)[-1]$. Thus $t_\alpha(S_i) = t_{\alpha s_i}(S_i)[1]$, so again applying 4.9 to both sides, it follows that $H^j(t_\alpha(S_i)) = 0$ for all $j \neq 0$, so $S_i \in \mathcal{T}_\alpha$. In particular, $S_i \notin \mathcal{F}_\alpha$. $\qquad\square$

Lemma 5.5. *Under the assumptions 4.1, suppose that $\alpha\colon C \to D$ is an atom. Then $S_0 \in \mathcal{T}_\alpha$ and $S_0 \in \mathcal{Y}_\alpha$.*

Proof. The first statement holds since P_0 is a summand of T_α, so $\operatorname{Hom}_{\Lambda_C}(T_\alpha, S_0) \neq 0$. Thus $S_0 \notin \mathcal{F}_\alpha$ and so since S_0 is simple, necessarily $S_0 \in \mathcal{T}_\alpha$.

For the second statement is similar, but uses the duality on tilting modules, so we sketch the proof. To ease notation set $A := \Lambda_C$, $B := \Lambda_D$, and $T := T_\alpha$. By convention the simple right A-module S_0 corresponds to the indecomposable projective P_0 of A, so consider the idempotent e_0 such that $P_0 = e_0 A$. Similarly, $B \cong \operatorname{End}_A(T)$ has an idempotent e_0' corresponding to the summand $e_0 A$ in the decomposition $T = e_0 A \oplus X$ as right A-modules. By convention S_0 is the simple right B-module corresponding to $e_0' B$, so that the k-dual DS_0 is the simple left B-module corresponding to Be_0'. It follows that $\operatorname{Hom}_{B^{\mathrm{op}}}(Be_0', DS_0) \neq 0$.

We first claim that Be_0' is a summand of ${}_B T$. By construction, it is clear that $Be_0' = \operatorname{Hom}_A(e_0 A, T)$ as left B-modules. As in [BB, p33], the functor

$$^\star(-) := \operatorname{Hom}_A(-, T)\colon \operatorname{mod} A \to \operatorname{mod} B^{\mathrm{op}}$$

clearly takes $A_A \mapsto {}_B T$, and thus since $e_0 A$ is a summand of A_A, by applying $^\star(-)$ we see that $^\star(e_0 A) \cong Be_0'$ is a summand of ${}_B T$.

Now by [CE, VI.5.1] there is an isomorphism

$$D(S_0 \otimes_B Be_0') \cong \operatorname{Hom}_{B^{\mathrm{op}}}(Be_0', DS_0),$$

which is non-zero by above. Thus $S_0 \otimes_B Be_0' \neq 0$. Since by above $S_0 \otimes_B T$ has summand $S_0 \otimes_B Be_0'$, it follows that $S_0 \otimes_B T \neq 0$. Hence $S_0 \notin \mathcal{X}_\alpha$, so again since S_0 is simple, necessarily $S_0 \in \mathcal{Y}_\alpha$. $\qquad\square$

Corollary 5.6. *Under the assumptions 4.1, suppose that $\alpha\colon C \to D$ is an atom. If $N \in \mathcal{F}_\alpha$ is nonzero, then there exists some $j \neq 0$ such that α starts with s_j, and further $\operatorname{Hom}_{\Lambda_C}(S_j, N) \neq 0$.*

Proof. Certainly N is filtered by simples, so there exists some $0 \leq j \leq n$ with $S_j \hookrightarrow N$. In particular $\operatorname{Hom}(S_j, N) \neq 0$. Since $\mathcal{F}_\alpha$ is closed under submodules $S_j \in \mathcal{F}_\alpha$, and so by 5.5 necessarily $j \neq 0$. The result then follows from 5.4(1). $\qquad\square$

6. Proof of Faithfulness

Keeping the notation in the previous sections, under the flops setup of 4.1, recall from 5.2 that every chamber D has an associated algebra Λ_D and simple modules $S_0, S_1, \ldots, S_n$, and we set $\mathcal{S} := \bigoplus_{i=0}^n S_i$. As in the Conventions, we write $[a, b]_t = \operatorname{Hom}_{\mathrm{D^b}(\Lambda_D)}(a, b[t])$. Although the D is suppressed in this notation, it will be clear from the context in which category to view $\mathcal{S}$.

We will reduce to a key technical lemma in 6.3, which is an analogue of [BT, Prop. 3.1]. The key point in Brav–Thomas is to first find an object b such that

$$[\mathcal{S}, b]_{\geq d+1} = 0, \tag{6.A}$$

where $d = \dim \mathfrak{R}$. For this there are many choices. To ensure that the method below can be used in future papers to cover situations where Λ has infinite global dimension (or flopping contractions $U \to \operatorname{Spec} \mathfrak{R}$ where U need not be smooth), throughout we choose $b = \Lambda$, as is justified in the following lemma.

Lemma 6.1. *Suppose that $\mathfrak{R}$ is a d-dimensional complete local Gorenstein ring, and that $\Lambda \cong \operatorname{End}_{\mathfrak{R}}(M)$ for some $M \in \operatorname{ref}\mathfrak{R}$, with $\Lambda \in \operatorname{CM}\mathfrak{R}$ (that is, Λ is a modifying $\mathfrak{R}$-algebra). Then $b := \Lambda$ satisfies*

$$[S_i, b]_d \neq 0 \text{ for all } 0 \leq i \leq n, \quad \text{and} \quad [\mathcal{S}, b]_{\geq d+1} = 0. \tag{6.B}$$

Proof. We know that $\operatorname{Ext}_{\Lambda}^t(S_i, b) := \operatorname{Ext}_{\Lambda}^t(S_i, \Lambda) \cong \operatorname{Ext}_{\mathfrak{R}}^t(S_i, \mathfrak{R})$, where the last isomorphism is [IR, 3.4(5)]. Hence by local duality

$$\operatorname{depth}_{\mathfrak{R}} S_i = d - \sup\{t \geq 0 \mid [S_i, b]_t \neq 0\}.$$

Clearly, being finite length, $\operatorname{depth}_{\mathfrak{R}} S_i = 0$, so we deduce that (6.B) holds. □

6.1. **The Main Result.** Throughout this subsection we will work under the setting of 4.1, and write $b := \Lambda$. The initial step requires the following elementary lemma.

Lemma 6.2. *Suppose that $0 \neq N \in \operatorname{fl}\Lambda_D$.*
  (1) *If $y \in \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda_D)$ is such that $[\mathcal{S}, y]_{\geq p} = 0$, then $[N, y]_{\geq p} = 0$.*
  (2) *$[N, b]_d \neq 0$ and $[N, b]_{\geq d+1} = 0$.*

Proof. (1) is an easy induction on the length of the filtration of N, using the long exact sequence from $[-, y]$.
(2) By 6.1 $[\mathcal{S}, b]_{\geq d+1} = 0$, so the second statement is a consequence of (1). The first also follows by an induction on the length of the filtration of N, using $[S_i, b]_d \neq 0$ and $[S_i, b]_{d+1} = 0$ for all $0 \leq i \leq n$. □

Now for $\alpha \in \mathbb{G}_{\mathcal{H}}^+$, we can decompose α into length one atoms $\alpha = s_{i_n} \ldots s_{i_1}$ and define $t_\alpha := t_{i_n} \circ \ldots \circ t_{i_1}$ (where the t_{i_t} are defined in 4.3), or alternatively we can decompose α into Deligne normal form $\alpha = \alpha_k \ldots \alpha_1$ and define $t_\alpha := t_{\alpha_k} \circ \ldots \circ t_{\alpha_1}$ (where the t_{α_i} are also defined in 4.3). The crucial point in the proof of faithfulness is that by 4.6(2) these yield the same functor.

The following is our analogue of the main technical lemma of Brav–Thomas [BT, Prop. 3.1]. Using torsion pairs, the proof only needs to induct on the number of Deligne factors, whereas Brav–Thomas use a more complicated double induction.

Proposition 6.3. *Let $1 \neq \alpha \in \mathbb{G}_{\mathcal{H}}^+$ have Deligne normal form $\alpha = \alpha_k \circ \ldots \circ \alpha_1$. Then*
  (1) *$[\mathcal{S}, t_\alpha b]_{\geq k+d+1} = 0$.*
  (2) *$[S_i, t_\alpha b]_{k+d} \neq 0$ if and only if $i \neq 0$ and the atom α_k ends (up to the relations in $\mathbb{G}_{\mathcal{H}}^+$) by passing through wall i. In particular $[\mathcal{S}, t_\alpha b]_{k+d} \neq 0$.*
  (3) *The maximal p such that $[\mathcal{S}, t_\alpha b]_p \neq 0$ is precisely $p = k + d$.*

Proof. Statement (3) follows immediately from (1) and (2), so we prove both (1) and (2) together using induction on the number of Deligne factors.

Base Case: $k = 1$, i.e. α is an atom. Since S_i is simple, there are only two cases, namely $S_i \in \mathcal{Y}_\alpha$ or $S_i \in \mathcal{X}_\alpha$, and using 5.4 and 5.5 we can characterise these:

(a) $S_i \in \mathcal{Y}_\alpha$ (equivalently, $i = 0$, or $i \neq 0$ and α does not end with s_i). By (5.B) $t_\alpha^{-1}(S_i) \cong N$ for some finite length module N. Hence by 6.2(2)

$$[S_i, t_\alpha b]_{\geq d+1} = [N, b]_{\geq d+1} = 0.$$

(b) $S_i \in \mathcal{X}_\alpha$ (equivalently, $i \neq 0$ and α ends with s_i). By (5.B) $t_\alpha^{-1}(S_i) \cong N[1]$ for some finite length module N. Hence again by 6.2(2)

$$[S_i, t_\alpha b]_{\geq d+2} = [N[1], b]_{\geq d+2} = [N, b]_{\geq d+1} = 0,$$

and

$$[S_i, t_\alpha b]_{d+1} = [N, b]_d \neq 0.$$

Combining (a) and (b) proves (1)(2) in the case $k = 1$.

Induction Step. We assume that the result is true for all paths with less than or equal to $k - 1$ Deligne factors. Write $\alpha = \alpha_k \circ \beta$ where $\beta := \alpha_{k-1} \circ \ldots \circ \alpha_1$. By induction

$$[\mathcal{S}, t_\beta b]_{\geq k+d} = 0$$

and $[S_j, t_\beta b]_{k+d-1} \neq 0$ if and only if $j \neq 0$ and α_{k-1} ends with s_j. Again there are only two cases:

(a) $S_i \in \mathcal{Y}_{\alpha_k}$ (equivalently, $i = 0$, or $i \neq 0$ and α_k does not end with s_i). By (5.B) $t_{\alpha_k}^{-1}(S_i) \cong N$ for some finite length module N. Hence

$$[S_i, t_\alpha b]_{\geq k+d} = [t_{\alpha_k}^{-1} S_i, t_\beta b]_{\geq k+d} = [N, t_\beta b]_{\geq k+d} \overset{6.2(1)}{=} 0.$$

(b) $S_i \in \mathcal{X}_{\alpha_k}$ (equivalently, $i \neq 0$ and α_k ends with s_i). By (5.B) $t_{\alpha_k}^{-1}(S_i) \cong N[1]$ for some finite length module N. Thus

$$[S_i, t_\alpha b]_{\geq k+d+1} = [N[1], t_\beta b]_{\geq k+d+1} = [N, t_\beta b]_{\geq k+d} \overset{6.2(1)}{=} 0.$$

Similarly

$$[S_i, t_\alpha b]_{k+d} = [N[1], t_\beta b]_{k+d} = [N, t_\beta b]_{k+d-1}$$

so it remains to show that $[N, t_\beta b]_{k+d-1} \neq 0$. But by 5.6, there exists $j \neq 0$ such that α_k starts with s_j, and $S_j \hookrightarrow N$. Write C for the cokernel, which necessarily has finite length, and consider the long exact sequence

$$\ldots \to [C, t_\beta b]_{k+d-1} \to [N, t_\beta b]_{k+d-1} \to [S_j, t_\beta b]_{k+d-1} \to [C, t_\beta b]_{k+d} = 0.$$

Since α_k starts with s_j, necessarily α_{k-1} ends with s_j, else $s_j \circ \alpha_{k-1}$ is an atom by 5.1, which would contradict the fact that $\alpha_k \circ \alpha_{k-1} \circ \ldots \circ \alpha_1$ is in Deligne normal form. Thus $[S_j, t_\beta b]_{k+d-1} \neq 0$ by the inductive hypothesis. It follows that $[N, t_\beta b]_{k+d-1} \neq 0$.

Combining (a) and (b) proves (1)(2) in the case of k factors, so by induction the result follows. $\square$

The remainder of the proof of faithfulness is straightforward.

Definition 6.4. *Define the groupoid $\mathcal{G}_\Lambda$ as follows:*
   (1) *The vertices are $\mathrm{D}^\mathrm{b}(\mathrm{mod}\,\Lambda_C)$, for chambers C of $\mathcal{H}$.*
   (2) *The morphisms between any two vertices are all triangle equivalences between the corresponding derived categories.*

By 4.6(2) and 4.7, there is a natural functor

$$F_\Lambda \colon \mathbb{G}_\mathcal{H} \to \mathcal{G}_\Lambda$$

which sends a simple wall crossing s_i to the corresponding equivalence t_i.

Theorem 6.5. *The functor F_Λ is faithful.*

Proof. This is an easy induction. We use 2.11, so suppose that

$$t_\alpha = t_\beta \colon \mathrm{D}^\mathrm{b}(\mathrm{mod}\,\Lambda_C) \to \mathrm{D}^\mathrm{b}(\mathrm{mod}\,\Lambda_D)$$

for some $\alpha, \beta \in \mathbb{G}_\mathcal{H}^+$. Since $t_\alpha = t_\beta$, we deduce from 6.3(3) that α and β have the same number of Deligne factors, so write

$$\alpha = \alpha_k \ldots \alpha_1 \quad \text{and} \quad \beta = \beta_k \ldots \beta_1$$

in Deligne normal form. By induction, it is enough to show that $\alpha_k = \beta_k$ and $t_{\alpha_{k-1}\ldots\alpha_1} = t_{\beta_{k-1}\ldots\beta_1}$. We may assume that $\ell := \ell(\alpha_k) \leq \ell(\beta_k)$. By 6.3(2), since $t_\alpha = t_\beta$, both α_k and β_k end with the same simple wall crossing, say s_{i_1}, so we can write $\alpha_k = s_{i_1}\widetilde{\alpha}_k$ and $\beta_k = s_{i_1}\widetilde{\beta}_k$. Hence applying $t_{i_1}^{-1}$ to $t_\alpha = t_\beta$ we deduce that $t_{\widetilde{\alpha}_k \alpha_{k-1}\ldots\alpha_1} = t_{\widetilde{\beta}_k \beta_{k-1}\ldots\beta_1}$.

Repeating the above argument, we can write $\alpha_k = s_{i_1}\ldots s_{i_\ell}$ and $\beta_k = s_{i_1}\ldots s_{i_\ell}\gamma$ for some $\gamma \in \mathbb{G}_\mathcal{H}^+$, and so we have $t_{\alpha_{k-1}\ldots\alpha_1} = t_{\gamma\beta_{k-1}\ldots\beta_1}$. But again by 6.3(3), γ must be a length zero path. Hence we have $\alpha_k = s_{i_1}\ldots s_{i_\ell} = \beta_k$ and $t_{\alpha_{k-1}\ldots\alpha_1} = t_{\beta_{k-1}\ldots\beta_1}$, as required. $\square$

Corollary 6.6. *For every chamber C, the induced map*

$$\pi_1(\mathbb{C}^n\backslash\mathcal{H}_\mathbb{C}) \to \operatorname{Auteq} D^b(\operatorname{mod}\Lambda_C)$$

is an injective group homomorphism

Proof. By 2.12, this follows immediately from 6.5.   □

6.2. **Geometric Corollaries.** Although the above results were stated in the formal fibre setting, they easily imply the following global results.

Corollary 6.7. *Suppose that $f\colon X \to X_{\operatorname{con}}$ is a flopping contraction between 3-folds, where X is smooth, and all curves in the contraction f are individually floppable. Then there is an injective group homomorphism*

$$\varphi\colon \pi_1(\mathbb{C}^n\backslash\mathcal{H}_\mathbb{C}) \to \operatorname{Auteq} D^b(\operatorname{coh} X).$$

Proof. As in [DW3, 6.2], the functors in the image of φ fix the skyscraper sheaves away from the flopping curves. Hence the relations can be detected on the formal fibre, where the result is 6.6.   □

In the case when the n curves are not individually floppable, there is still a group action, but only by a subgroup S of $\pi_1(\mathbb{C}^n\backslash\mathcal{H}_\mathbb{C})$ defined to be the subgroup generated by the J-twists of [DW3], where J runs through all subsets of $\{1,\ldots,n\}$. The proof of faithfulness extends to this case too.

Corollary 6.8. *Suppose that $f\colon X \to X_{\operatorname{con}}$ is a flopping contraction between 3-folds, where X is smooth. Then there is an injective group homomorphism*

$$S \to \operatorname{Auteq} D^b(\operatorname{coh} X).$$

Proof. Again, by [DW3, 6.2], the functors in the image of the above homomorphism fix the skyscraper sheaves away from the flopping curves. Hence the relations can be detected on the formal fibre. Since there $\pi_1(\mathbb{C}^n\backslash\mathcal{H}_\mathbb{C})$ acts faithfully by 6.6, so does any subgroup.   □

Recall that if $\mathcal{A}$ is the heart of a bounded t-structure on a triangulated category $\mathcal{D}$, and $\mathcal{A}$ admits a torsion pair $(\mathcal{T},\mathcal{F})$, then the tilt of $\mathcal{A}$ with respect to this torsion pair is defined to be

$$\mathcal{A}^\sharp := \{E \in \mathcal{D} \mid H^i(E) = 0 \text{ for } i \notin \{-1,0\}, H^{-1}(E) \in \mathcal{F} \text{ and } H^0(E) \in \mathcal{T}\}.$$

By [HRS, 2.1], $\mathcal{A}^\sharp$ is also the heart of a bounded t-structure on $\mathcal{D}$.

Now for a 3-fold flopping contraction $f\colon X \to X_{\operatorname{con}}$, consider the full subcategories

$$\mathcal{T}_0 := \{T \in \operatorname{coh} X \mid \mathbf{R}^1 f_*(T) = 0\}$$
$$\mathcal{F}_0 := \{F \in \operatorname{coh} X \mid f_*(F) = 0, \operatorname{Hom}(\mathcal{C}, F) = 0\},$$

where $\mathcal{C} \subset \operatorname{coh} X$ is the full subcategory consisting of objects E such that $\mathbf{R}f_*(E) = 0$. Then $(\mathcal{T}_0, \mathcal{F}_0)$ is a torsion pair by [V, Lemma 3.1.2], and the category of perverse sheaves relative to f is defined to be

$$^0\!\operatorname{Per}(X, X_{\operatorname{con}}) := (\operatorname{coh} X)^\sharp,$$

namely the tilt of the standard heart $\operatorname{coh} X \subset D^b(\operatorname{coh} X)$ with respect to the torsion pair $(\mathcal{T}_0, \mathcal{F}_0)$.

The following is a further consequence of the results in this paper, and may be of independent interest. The first part is implicit in [DW3], the second part is new.

Theorem 6.9. *Consider two crepant resolutions*

$$X \qquad\qquad Y$$
$$\searrow \qquad \swarrow$$
$$\operatorname{Spec} R$$

of $\operatorname{Spec} R$, where R is an isolated cDV singularity.

(1) *Given two minimal chains of flops connecting X and Y, the composition of flop functors associated to each chain are functorially isomorphic.*
(2) *Perverse sheaves on Y, namely $^0\mathrm{Per}(Y, R)$, can be obtained from perverse sheaves on X, namely $^0\mathrm{Per}(X, R)$, by a single tilt at a torsion pair.*

Proof. (1) By [D, 1.10, 1.12], any two minimal paths can be identified provided that in the Deligne groupoid the codimension two relations hold. By [DW3, 3.20] the codimension two relations are precisely correspond to the braiding of the 2-curve flop functors, which is proved in [DW3, 3.9, 3.20].
(2) Consider a minimal path of flops

$$\mathrm{D^b}(\mathrm{coh}\, X) \xrightarrow{\mathsf{F}_{i_1}} \mathrm{D^b}(\mathrm{coh}\, X_{i_1}) \xrightarrow{\mathsf{F}_{i_2}} \ldots \xrightarrow{\mathsf{F}_{i_n}} \mathrm{D^b}(\mathrm{coh}\, Y)$$

connecting X and Y. By [V] X is derived equivalent to $\mathrm{End}_R(M)$ say, and Y is derived equivalent to $\mathrm{End}_R(N)$ say, and under this identification $^0\mathrm{Per}(X, R)$ corresponds to $\mathrm{mod}\,\mathrm{End}_R(M)$, and $^0\mathrm{Per}(Y, R)$ corresponds to $\mathrm{mod}\,\mathrm{End}_R(N)$. Hence it suffices to show that $\mathrm{mod}\,\mathrm{End}_R(N)$ can be obtained from $\mathrm{mod}\,\mathrm{End}_R(M)$ by a tilt at a torsion pair.

Consider $T := \mathrm{Hom}_R(M, N)$. This is a tilting $\mathrm{End}_R(M)$-module, by [IW, 4.17]. But since $\mathrm{End}_R(M)$ is noetherian,

$$\mathcal{T} := \{X \in \mathrm{mod}\,\mathrm{End}_R(M) \mid \mathrm{Ext}^1_{\mathrm{End}_R(M)}(T, X) = 0\}$$
$$\mathcal{F} := \{X \in \mathrm{mod}\,\mathrm{End}_R(M) \mid \mathrm{Hom}_{\mathrm{End}_R(M)}(T, X) = 0\}$$

gives a torsion pair $(\mathcal{T}, \mathcal{F})$ on $\mathrm{mod}\,\mathrm{End}_R(M)$; the proof is identical to [SY, 2.7(3)]. Using the finitely-generated version of the equivalences (5.A), it is then clear that $\mathrm{mod}\,\mathrm{End}_R(N)$ is obtained from $\mathrm{mod}\,\mathrm{End}_R(M)$ by tilting at $(\mathcal{T}, \mathcal{F})$.

□

APPENDIX A. BRAV–THOMAS REVISITED

In this appendix, which can be read independently of the previous sections, we give a direct proof of the faithfulness of the braid action on the minimal resolution of Kleinian singularities, just to demonstrate that our torsion pairs viewpoint simplifies the [BT] proof. Thus in this section we consider the minimal resolution $X \to \mathrm{Spec}\, R$ of a Kleinian singularity, let Λ denote the completion of the preprojective algebra of the corresponding extended Dynkin diagram, and set $b := \mathcal{S}$, where $\mathcal{S}$ is the direct sum of the vertex simples $S_0, S_1, \ldots, S_n$.

The initial step requires the following elementary lemma, which replaces 6.2.

Lemma A.1. *Suppose that $M \in \mathsf{fl}\,\Lambda$.*

(1) *If $y \in \mathrm{D^b}(\mathrm{mod}\,\Lambda)$ is such that $[\mathcal{S}, y]_{\geq p} = 0$, then $[M, y]_{\geq p} = 0$.*
(2) *$[M, \mathcal{S}]_2 \neq 0$ and $[M, \mathcal{S}]_{\geq 3} = 0$.*

Proof. (1) is an easy induction on the length of the filtration of M, using the long exact sequence from $[-, y]$.
(2) Since Λ is 2-CY, $[\mathcal{S}, \mathcal{S}]_{\geq 3} = 0$, so the second statement is a consequence of (1). The first also follows by an induction on the length of the filtration of M, using the fact that $[S_i, \mathcal{S}]_2 \neq 0$ for all $0 \leq i \leq n$.   □

For every primitive idempotent e_i corresponding to a vertex of the extended Dynkin diagram, following [IR, §6] we set

$$I_i := \Lambda(1 - e_i)\Lambda.$$

It is known by [DW1, Section 6] that $\mathbf{R}\mathrm{Hom}_\Lambda(I_i, -)$ is functorially isomorphic to the twist functor t_i. To control iterations, for any $\alpha \in W$ where W is the associated Weyl group, choose a reduced expression $\alpha = s_{i_n} \circ \ldots \circ s_{i_1}$ and define

$$I_\alpha := I_{i_n} \ldots I_{i_1}.$$

Since the expression is reduced,

$$I_\alpha \cong I_{i_n} \otimes^{\mathbf{L}}_\Lambda \ldots \otimes^{\mathbf{L}}_\Lambda I_{i_1}$$

by [SY, 2.21], so that

$$t_\alpha := \mathbf{R}\mathrm{Hom}_\Lambda(I_\alpha, -) \cong t_{i_n} \circ \ldots \circ t_{i_1}. \qquad (\mathrm{A.A})$$

By the usual torsion pair associated to a tilting module, as in [SY, 2.9] and §5, for any vertex simple S_i, either $S_i \in \mathcal{X}_\alpha$ or $S_i \in \mathcal{Y}_\alpha$, where

$$\mathcal{X}_\alpha := \{N \in \mathsf{fl}\,\Lambda \mid N \otimes_\Lambda I_\alpha = 0\}$$

$$\mathcal{Y}_\alpha := \{N \in \mathsf{fl}\,\Lambda \mid \mathrm{Tor}_1^\Lambda(N, I_\alpha) = 0\},$$

and furthermore the equivalence (A.A) forces

$$t_\alpha^{-1}(S_i) = \left\{ \begin{array}{ll} \mathrm{Tor}_1(S_i, I_\alpha)[1] & \text{if } S_i \in \mathcal{X}_\alpha \\ S_i \otimes I_\alpha & \text{if } S_i \in \mathcal{Y}_\alpha. \end{array} \right. \qquad (\mathrm{A.B})$$

There is a corresponding version of the results 5.4, 5.5 and 5.6, which we will use freely below, since these were already very well known [SY, 2.28, 5.4] in the preprojective algebra setting. With this, we can now prove the main technical lemma [BT, Prop. 3.1] in the setting of minimal resolutions of Kleinian singularities.

Proposition A.2. *Let $1 \neq \alpha \in \mathbb{G}_{\mathcal{H}}^+$ have Deligne normal form $\alpha = \alpha_k \circ \ldots \circ \alpha_1$. Then*

(1) $[\mathcal{S}, t_\alpha \mathcal{S}]_{\geq k+3} = 0$.
(2) $[S_i, t_\alpha \mathcal{S}]_{k+2} \neq 0$ *if and only if $i \neq 0$ and the atom α_k ends (up to the relations in $\mathbb{G}_{\mathcal{H}}^+$) by passing through wall i. In particular $[\mathcal{S}, t_\alpha \mathcal{S}]_{k+2} \neq 0$.*
(3) *The maximal p such that $[\mathcal{S}, t_\alpha \mathcal{S}]_p \neq 0$ is precisely $p = k + 2$.*

Proof. Statement (3) follows immediately from (1) and (2), so we prove both (1) and (2) together using induction on the number of Deligne factors.

Base Case: $k = 1$, i.e. α is an atom. Since S_i is simple, there are only two cases, namely $S_i \in \mathcal{Y}_\alpha$ or $S_i \in \mathcal{X}_\alpha$, and using 5.4 and 5.5 we can characterise these:

(a) $S_i \in \mathcal{Y}_\alpha$ (equivalently, $i = 0$, or $i \neq 0$ and α does not end with s_i). By (A.B) $t_\alpha^{-1}(S_i) \cong M$ for some finite length module M. Hence by A.1(2)

$$[S_i, t_\alpha \mathcal{S}]_{\geq 3} = [M, \mathcal{S}]_{\geq 3} = 0.$$

(b) $S_i \in \mathcal{X}_\alpha$ (equivalently, $i \neq 0$ and α ends with s_i). By (A.B) $t_\alpha^{-1}(S_i) \cong M[1]$ for some finite length module M. Hence again by A.1(2)

$$[S_i, t_\alpha \mathcal{S}]_{\geq 4} = [M[1], \mathcal{S}]_{\geq 4} = [M, \mathcal{S}]_{\geq 3} = 0,$$

and

$$[S_i, t_\alpha \mathcal{S}]_3 = [M, \mathcal{S}]_2 \neq 0.$$

Combining (a) and (b) proves (1)(2) in the case $k = 1$.

Induction Step. We assume that the result is true for all paths with less than or equal to $k - 1$ Deligne factors. Write $\alpha = \alpha_k \circ \beta$ where $\beta := \alpha_{k-1} \circ \ldots \circ \alpha_1$. By induction

$$[\mathcal{S}, t_\beta \mathcal{S}]_{\geq k+2} = 0$$

and $[S_j, t_\beta \mathcal{S}]_{k+1} \neq 0$ if and only if $j \neq 0$ and α_{k-1} ends with s_j. Again there are only two cases:

(a) $S_i \in \mathcal{Y}_{\alpha_k}$ (equivalently, $i = 0$, or $i \neq 0$ and α_k does not end with s_i). By (A.B) $t_{\alpha_k}^{-1}(S_i) \cong M$ for some finite length module M. Hence

$$[S_i, t_\alpha \mathcal{S}]_{\geq k+2} = [t_{\alpha_k}^{-1} S_i, t_\beta \mathcal{S}]_{\geq k+2} = [M, t_\beta \mathcal{S}]_{\geq k+2} \stackrel{\mathrm{A.1(1)}}{=} 0.$$

(b) $S_i \in \mathcal{X}_{\alpha_k}$ (equivalently, $i \neq 0$ and α_k ends with s_i). By (A.B) $t_{\alpha_k}^{-1}(S_i) \cong M[1]$ for some finite length module M. Thus

$$[S_i, t_\alpha \mathcal{S}]_{\geq k+3} = [M[1], t_\beta \mathcal{S}]_{\geq k+3} = [M, t_\beta \mathcal{S}]_{\geq k+2} \stackrel{\text{A.1(1)}}{=} 0.$$

Similarly

$$[S_i, t_\alpha \mathcal{S}]_{k+2} = [M[1], t_\beta \mathcal{S}]_{k+2} = [M, t_\beta \mathcal{S}]_{k+1}$$

so it remains to show that $[M, t_\beta \mathcal{S}]_{k+1} \neq 0$. But by 5.6, there exists $j \neq 0$ such that α_k starts at s_j, and $S_j \hookrightarrow M$. Write C for the cokernel, which necessarily has finite length, and consider the long exact sequence

$$\ldots \to [C, t_\beta \mathcal{S}]_{k+1} \to [M, t_\beta \mathcal{S}]_{k+1} \to [S_j, t_\beta \mathcal{S}]_{k+1} \to [C, t_\beta \mathcal{S}]_{k+2} = 0.$$

Since α_k starts with s_j, necessarily α_{k-1} ends with s_j, else $s_j \circ \alpha_{k-1}$ is an atom by 5.1, which would contradict the fact that $\alpha_k \circ \alpha_{k-1} \circ \ldots \circ \alpha_1$ is in Deligne normal form. Thus $[S_j, t_\beta \mathcal{S}]_{k+1} \neq 0$ by the inductive hypothesis. It follows that $[M, t_\beta \mathcal{S}]_{k+1} \neq 0$.

Combining (a) and (b) proves (1)(2) in the case of k factors, so by induction the result follows. $\qquad\square$

From here, the proof of faithfulness follows exactly as in [BT, Thm. 3.1]. Alternatively, we can use 2.11 as in 6.5 to deduce that the groupoid action is faithful. Since $B_\Gamma \cong \pi_1((\mathbb{C}^n \backslash \mathcal{H}_\mathbb{C})/W_\Gamma)$, and each vertex of $\mathcal{G}$ is by definition the same $D^b(\text{coh } X)$, as is standard by identifying all vertices we can simply re-interpret the faithful groupoid action as an injective group homomorphism $B_\Gamma \to \text{Auteq } D^b(\text{coh } X)$.

Appendix B. Tilting Background

In this appendix, which is logically independent of all other sections, we give some known tilting results that were used in the text, and we also prove 3.4 and 3.5.

Throughout Λ is a basic $\mathfrak{R}$-algebra, where $\mathfrak{R}$ is a complete local domain. Recall that if $T \in \text{tilt}_0 \Lambda$ and its mutation $\nu_i T$ at a direct summand T_i exists, either there is an exact sequence

$$0 \to T_i \xrightarrow{f} T' \to U_i \to 0$$

where f is a minimal left $\text{add}(T/T_i)$-approximation, or an exact sequence

$$0 \to U_i \to T' \xrightarrow{g} T_i \to 0$$

where g is a minimal right $\text{add}(T/T_i)$-approximation. By definition, $T > \nu_i T$ in the former case, and $T < \nu_i T$ in the latter case.

Suppose that $T \in \text{tilt } \Lambda$ with $\text{End}_\Lambda(T) \cong \Gamma$. By projectivization, the indecomposable summands of Γ correspond to the indecomposable summands of T. Hence we can try to mutate $T \in \text{tilt } \Lambda$ to form $\nu_i T$, and similarly we can try to mutate $\Gamma \in \text{tilt } \Gamma$ to form $\nu_i \Gamma$. Although the following is elementary and is known to experts, references to the literature only exist when $\text{mod } \Lambda$ is Hom-finite, so here we give the proof in full.

Proposition B.1. *Suppose that* $T \in \text{tilt } \Lambda$, *and set* $\Gamma := \text{End}_\Lambda(T)$. *If* $\nu_i T$ *exists and further* $T > \nu_i T$, *then* $\nu_i \Gamma \in \text{tilt } \Gamma$ *exists, there is an isomorphism* $\nu_i T \cong \nu_i \Gamma \otimes_\Gamma^\mathbf{L} T$ *in* $D^b(\text{mod } \Lambda)$, *and further* $\nu_i T \in \mathcal{T} := \{N \in \text{mod } \Lambda \mid \text{Ext}_\Lambda^1(T, N) = 0\}$.

Proof. To ease notation write $V := T/T_i$.
(1) Since $\nu_i T = V \oplus U_i$ exists and $T > \nu_i T$, as above there exists an exact sequence

$$0 \to T_i \xrightarrow{f} T' \to U_i \to 0$$

where f is a minimal left $\text{add } V$-approximation. Applying $\text{Hom}_\Lambda(T, -)$ gives an exact sequence

$$0 \to \text{Hom}_\Lambda(T, T_i) \xrightarrow{f \circ} \text{Hom}_\Lambda(T, T') \to \text{Hom}_\Lambda(T, U_i) \to 0 \qquad \text{(B.A)}$$

Write $\Gamma = \text{Hom}_\Lambda(T, T) = \text{Hom}_\Lambda(T, V) \oplus \text{Hom}_\Lambda(T, T_i) := \Gamma_V \oplus \Gamma_i$, then by projectivisation (B.A) is a projective resolution of $\text{Hom}_\Lambda(T, U_i)$. We claim that $(f \circ)$ is a minimal left

add Γ_V-approximation. To see this, simply apply $\mathrm{Hom}_\Gamma(-, \Gamma_V)$ to (B.A) to obtain a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_\Gamma(\Gamma_i, \Gamma_V) & \longleftarrow & \mathrm{Hom}_\Gamma(\mathrm{Hom}_\Lambda(T, T'), \Gamma_V) \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\
\mathrm{Hom}_\Lambda(T_i, V) & \xleftarrow{\ \circ f\ } & \mathrm{Hom}_\Lambda(T', V)
\end{array}
$$

where the vertical maps are isomorphisms by projectivisation, and the bottom map is surjective since f is an $\mathrm{add}\,V$-approximation. It follows that the top map is surjective, and hence $(f\circ)$ is a left $\mathrm{add}\,\Gamma_V$-approximation. The minimality of $(f\circ)$ follows since the left $\mathrm{add}\,V$-approximation f is minimal, and the functor $\mathrm{Hom}_\Lambda(T, -)\colon \mathrm{add}\,T \to \mathrm{proj}\,\Gamma$ is fully faithful.

As is standard [IW, 6.6], since $(f\circ)$ in (B.A) is injective and an approximation, it follows that $\Gamma_V \oplus \mathrm{Hom}_\Lambda(T, U_i) \in \mathsf{tilt}\,\Gamma$, and evidently $\nu_i\Gamma \cong \Gamma_V \oplus \mathrm{Hom}_\Lambda(T, U_i)$ since $\nu_i\Gamma$ and Γ differ at only one indecomposable summand.

Now, using (B.A) to compute the derived tensor in $\mathrm{D}^b(\mathrm{mod}\,\Lambda)$, observe first that

$$
\mathrm{Hom}_\Lambda(T, U_i) \otimes^{\mathbf{L}}_\Gamma T \cong \ldots \to 0 \to \mathrm{Hom}_\Lambda(T, T_i) \otimes_\Gamma T \to \mathrm{Hom}_\Lambda(T, T') \otimes_\Gamma T \to 0 \to \ldots
$$

$$
\cong \ldots \to 0 \to T_i \xrightarrow{f} T' \to 0 \to \ldots,
$$

which since f is injective, is clearly isomorphism to U_i (in degree zero). Hence

$$
\nu_i\Gamma \otimes^{\mathbf{L}}_\Gamma T \cong (\mathrm{Hom}_\Lambda(T, V) \otimes_\Gamma T) \oplus (\mathrm{Hom}_\Lambda(T, U_i) \otimes^{\mathbf{L}}_\Gamma T)
$$

$$
\cong V \oplus U_i,
$$

where $\mathrm{Hom}_\Lambda(T, V) \otimes_\Gamma T \cong V$ holds since T is tilting and V is projective. It follows that $\nu_i\Gamma \otimes^{\mathbf{L}}_\Gamma T \cong \nu_i T$ in $\mathrm{D}^b(\mathrm{mod}\,\Lambda)$. Applying $\mathbf{R}\mathrm{Hom}_\Lambda(T, -)$ gives the final statement. $\qquad\square$

Lemma B.2 (3.4). *Suppose that Λ is a basic $\mathfrak{R}$-algebra, where $\mathfrak{R}$ is a complete local domain. If $T, U \in \mathsf{tilt}_0\,\Lambda$ are related by a mutation at an indecomposable summand, then C_T and C_U do not overlap, and are separated by a codimension one wall.*

Proof. By assumption, there are indecomposable modules $T_0, \ldots, T_n$ and U_n such that $T = T_{<n} \oplus T_n$ and $U = T_{<n} \oplus U_n$, where $T_{<n} := \bigoplus_{i=0}^{n-1} T_i$. We may assume that $T > U$, and then there is an exact sequence

$$
0 \to T_n \to X_{<n} \to U_n \to 0
$$

where $X_{<n} \in \mathrm{add}\,T_{<n}$, say $X_{<n} := T_0^{\oplus a_0} \oplus T_1^{\oplus a_1} \oplus \ldots \oplus T_{n-1}^{\oplus a_{n-1}}$. Thus, recalling that the $[-]$ notation works modulo $\mathrm{Span}\{\mathbf{e}_0\}$, we see that

$$
[T_n] = -[U_n] + \sum_{i=1}^{n-1} a_i [T_i]
$$

in Θ_Λ, and so

$$
C_T := \{ \sum_{i=1}^{n} \vartheta_i [T_i] \mid \vartheta_i > 0 \text{ for all } 1 \le i \le n \}
$$

$$
= \{ \sum_{i=1}^{n-1} (\vartheta_i + a_i\vartheta_n)[T_i] - \vartheta_n[U_n] \mid \vartheta_i > 0 \text{ for all } 1 \le i \le n \}.
$$

Since U is tilting, the classes of indecomposable summands of U, namely $[T_0], \ldots, [T_{n-1}]$, and $[U_n]$, span $K_0 \otimes_\mathbb{Z} \mathbb{R} \cong \mathbb{R}^{n+1}$. Hence they form a basis of $K_0 \otimes_\mathbb{Z} \mathbb{R}$, and in particular the classes $[T_1], \ldots, [T_{n-1}], [U_n]$ in Θ_Λ form a basis of Θ_Λ.

Write $H \subset \Theta_\Lambda$ for the linear subspace spanned by $[T_1], \ldots, [T_{n-1}]$. Then H separates Θ_Λ into two half spaces $H_+ := \{ \sum_{i=1}^{n-1} b_i[T_i] + a[U_n] \mid a > 0 \}$ and $H_- := \{ \sum_{i=1}^{n-1} b_i[T_i] +$

$a[U_n] \mid a < 0\}$. Since $C_T \subset H_-$, $C_U \subset H_+$, and $H_+ \cap H_- = \emptyset$, we obtain $C_T \cap C_U = \emptyset$. It is clear that C_T and C_U are separated by a codimension one wall contained in H. $\square$

Lemma B.3. *Suppose that Λ is a basic $\mathfrak{R}$-algebra, where $\mathfrak{R}$ is a complete local domain. Suppose that $T, U \in \mathsf{tilt}_0 \Lambda$ are related by a mutation at an indecomposable summand T_n. If $T > U$, then there exists an exact sequence $0 \to \Lambda \to T' \to T'' \to 0$ such that $T_n \notin \mathrm{add}\, T''$.*

Proof. By the definition of tilting modules, there is an exact sequence $0 \to \Lambda \to T' \to T'' \to 0$ with $T', T'' \in \mathrm{add}\, T$, and this induces the following triangle in $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, \Lambda)$

$$\Lambda \to T' \to T'' \xrightarrow{f} \Lambda[1].$$

Since $\mathrm{Ext}^1(T, T) = 0$, we see that $f \colon T'' \to \Lambda[1]$ is a right $\mathrm{add}\, T$-approximation. Replacing T' and T'' if necessary, we may assume that the approximation f is right minimal, and we will show that for such a sequence $0 \to \Lambda \to T' \to T'' \to 0$, we have $T_n \notin \mathrm{add}\, T''$.

Suppose that $T_n \in \mathrm{add}\, T''$, and let Y be the summand of T'' such that $T_n \notin \mathrm{add}\, Y$ and $T'' = (T_n)^{\oplus a} \oplus Y$ for some $a > 0$. Let $f_n \colon (T_n)^{\oplus a} \to \Lambda[1]$ and $f_Y \colon Y \to \Lambda[1]$ be the components of f. By assumption, there are indecomposable modules T_n and U_n such that $T = X \oplus T_n$ and $U = X \oplus U_n$. Since $T > U$, there is an exact sequence

$$0 \to T_n \xrightarrow{g} X' \to U_n \to 0,$$

where $X' \in \mathrm{add}\, X$. Applying $\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, \Lambda)}(-, \Lambda[1])$ to the above gives an exact sequence

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, \Lambda)}(X', \Lambda[1]) \xrightarrow{\circ g} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, \Lambda)}(T_n, \Lambda[1]) \to \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, \Lambda)}(U_n[-1], \Lambda[1]) = 0,$$

since $\mathrm{pd}_\Lambda U_n \leq 1$. Hence there exists a morphism $h \colon X'^{\oplus a} \to \Lambda[1]$ such that $f_n = h \circ g^{\oplus a}$. But $h + f_Y \colon X'^{\oplus a} \oplus Y \to \Lambda[1]$ is a right $\mathrm{add}\, T$-approximation, with $X'^{\oplus a} \oplus Y \in \mathrm{add}\, X$, and so $T_n \notin \mathrm{add}(X'^{\oplus a} \oplus Y)$. This contradicts the minimality of $f \colon T'' \to \Lambda[1]$, since $T_n \in \mathrm{add}\, T''$. Hence $T_n \notin \mathrm{add}\, T''$. $\square$

Theorem B.4 (3.5)**.** *Suppose that Λ is a basic $\mathfrak{R}$-algebra, where $\mathfrak{R}$ is a complete local domain. Suppose that $T, U \in \mathsf{tilt}_0 \Lambda$ are related by a mutation at an indecomposable summand, so by B.2 C_T and C_U are separated by H. Suppose that $[\Lambda] \notin H$. Then $T > U$ iff C_T lies on the same side of H as $[\Lambda]$.*

Proof. $(\Rightarrow)$ Suppose that $T > U$. Since the summands of T (excluding $T_0 = P_0$) form a basis for Θ_Λ, we can write

$$[\Lambda] = b_1[T_1] + \ldots + b_{n-1}[T_{n-1}] + b_n[T_n]$$

Certainly $b_n \neq 0$, else $[\Lambda] \in H$, which is false by assumption. Since by B.3 there are objects $T', T'' \in \mathrm{add}\, T$ such that $[\Lambda] = [T'] - [T'']$ and $T_n \notin \mathrm{add}\, T''$, necessarily $b_n > 0$ given that it is non-zero. It follows that $[\Lambda]$ is on the same side of H as C_T.
$(\Leftarrow)$ If $\neg(T > U)$, then since by the assumption T and U are the mutation of each other at an indecomposable summand, necessarily $U > T$. Replicating the above proof word-for-word, we conclude that C_U is on the same side of H as $[\Lambda]$. Since C_T is on the other side of H than C_U, it follows that C_T is not on the same side of H as $[\Lambda]$. $\square$

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