# LECTURES ON RECONSTRUCTION ALGEBRAS IV

MICHAEL WEMYSS

## 1. Introduction

Last lecture Osamu introduced the notion of a special CM module and gave some of their properties. Here I briefly recap some of his lecture and also add the geometric part of the definition.

Let $X = \text{Spec}R$ be an affine complete rational surface singularity, denote the minimal resolution by $f : \widetilde{X} \to \text{Spec}R$ and the exceptional curves by $\{E_i\}$. Also, for a given CM module $M$ of $R$, denote by $\mathcal{M} := \pi^*M/\text{torsion}$ the corresponding vector bundle on $\widetilde{X}$.

**Definition-Proposition 1.1.** *A CM module $M$ is called special if one of the following equivalent conditions hold*
*1.* $H^1(\mathcal{M}^\vee) \cong \text{Ext}^1_{\widetilde{X}}(\mathcal{M}, \mathscr{O}) = 0$
*2.* $M \otimes_R \omega_R/\text{torsion}$ *is CM*
*3.* $\text{Ext}^1_R(M, R) = 0$
*4.* $\text{Hom}_R(M, R)$ *is the first syzygy of some CM module.*
*5.* $\Omega M \cong \text{Hom}_R(M, R)$.

How to interpret this: $1 \iff 2$ is due to Wunram, and links the geometric notion of 1 (which involves the minimal resolution) to the more algebraic notion of 2 (which does not involve the minimal resolution). Condition 3 says that we can deduce the vanishing of the ext group upstairs (i.e. 1) by deducing the vanishing of the ext group downstairs on the singularity. This is *very* useful, but note that such a phenomenon is very rare! It is still not clear from conditions 1,2 or 3 how to obtain special CM modules - it is 4 which now helps since just taking the syzygy of your favorite CM module (and then taking the dual) gives you a special CM module. Condition 5 is a refinement of condition 4 (for example when $G \leq \text{SL}(2, \mathbb{C})$ is gives an alternative proof that $\Omega^2 = id$) and is useful in proving homological statements.

We arrive at the definition:

**Definition 1.2.** *The ring* $\text{End}_R(\oplus M)$*, where the sum is over all indecomposable special CM modules, is called the reconstruction algebra.*

A long time ago I said that instead of viewing the minimal resolution as $G$-Hilb (which you can do), the new idea is to instead view the minimal resolution as being very similar to a space we already understand. It is the reconstruction algebra which tells us which space to compare to, and it is the reconstruction algebra which encodes the difference. This is related to why I call $\text{End}_R(\oplus M)$ the reconstruction algebra, which I shall now explain in the next section.

## 2. The Correspondence

Recall that given the data of a dual graph, simple combinatorics give us Artin's fundamental cycle $Z_f$. In the case of finite subgroups of $\text{SL}(2, \mathbb{C})$ these numbers are what you expect. I need one further piece of combinatorial data, since now the canonical sheaf need not be trivial and so we need to encode this combinatorially. It is already known how to do this: use the canonical cycle $Z_K$. It is the rational cycle defined by the condition

$$Z_K \cdot E_i = -K_{\widetilde{X}} \cdot E_i$$

for all $i$. By adjunction this means that

$$Z_K \cdot E_i = E_i^2 + 2$$

for all $i$. Note that on the minimal resolution the self-intersection number of every curve is $\leq -2$ and so consequently $Z_K \cdot E_i \leq 0$ for all $i$.

The canonical cycle appears in the theorem below since at some point in the proof Serre duality is envoked.

**Theorem 2.1.** *Let $\widetilde{X} \to \operatorname{Spec} R$ be the minimal resolution of some affine complete rational surface singularity. Then $\operatorname{End}_R(\oplus M)$ can be written as a quiver with relations as follows: for every exceptional curve $E_i$ associate a vertex labelled $i$, and also associate a vertex $\star$ corresponding to the free module. Then the number of arrows and relations between the vertices is given as follows:*

| | Number of arrows | Number of relations |
|---|---|---|
| $i \to j$ | $(E_i \cdot E_j)_+$ | $(-1 - E_i \cdot E_j)_+$ |
| $\star \to \star$ | $0$ | $-Z_K \cdot Z_f + 1 = -1 - Z_f \cdot Z_f$ |
| $i \to \star$ | $-E_i \cdot Z_f$ | $0$ |
| $\star \to i$ | $((Z_K - Z_f) \cdot E_i)_+$ | $((Z_K - Z_f) \cdot E_i)_-$ |

From this I should make some remarks

- We call $\operatorname{End}_R(\oplus M)$ the reconstruction algebra since it can be reconstructed from the dual graph of the minimal resolution. Although it looks quite complicated, the combinatorics are actually very easy (see lemma below).
- If you already know the dual graph (e.g. through the Brieskorn classification for quotient singularities) to obtain the quiver is very quick. If you don't know the dual graph then at least in the case of quotient singularities there is another way to build the reconstruction algebra, using the AR quiver. If you like, you can view this AR quiver method as another (but not so good) proof of the Brieskorn classification.
- Some version of the above theorem holds for non-minimal resolutions too, but the quiver and relations are sightly different.

Ideally we don't want to compute all the combinatorics in all examples, so the next lemma is useful since it reduces the calculation of the quiver to simply adding arrows to a certain base quiver[1]. This also tells you which space to compare to!

**Key Lemma 2.2.** *Suppose two curve systems $E = \{E_i\}$ and $F = \{F_i\}$ have the same dual graph **and** fundamental cycle, such that $-F_i^2 \leq -E_i^2$ for all $i$. Then the quiver for the curve system $E$ is obtained from the quiver of the curve system $F$ by adding $-E_i^2 + F_i^2$ extra arrows $i \to \star$ for every curve $E_i$.*

Thus if you have a dual graph and you want to compute the corresponding quiver, just reduce the self-intersction numbers (i.e. make them closer to $-2$) in such a way that the fundamental cycle does not change. Calculate this base quiver. Then just add extra arrows as in the Lemma. This will make more sense after some examples.

## 3. Some examples

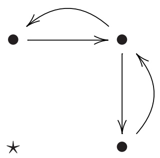
I'll start with type $A$, i.e. cyclic groups. Since its hard to draw $n$ vertices, consider only the case of $A_3$.

**Example 3.1.** Consider the group $\frac{1}{4}(1,3)$. For this example the dual graph is

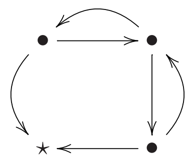
$$\underset{-2}{\bullet} \text{——} \underset{-2}{\bullet} \text{——} \underset{-2}{\bullet}$$

[1] If you were in my talk last week in Kyoto and were wondering why everything didn't make sense after some point, it is because I forgot to say the Key Lemma. Oops.

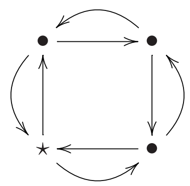
After the $i \to j$ and $\star \to \star$ steps in the theorem, we have the following picture



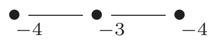
Now to calculate how to connect $\star$, we need to know the fundamental cycle. But here $Z_f = \begin{smallmatrix} 1 & 1 & 1 \end{smallmatrix}$ and so in matrix from $(-E_i \cdot Z_f)_{i \in I} = \begin{smallmatrix} 1 & 0 & 1 \end{smallmatrix}$. Thus after the $i \to \star$ step:



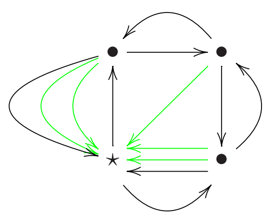
For the $\star \to i$ step notice that since all curves are $(-2)$-curves the canonical cycle is trivial, thus the number of arrows $\star \to i$ is equal to the number of arrows $i \to \star$. Consequently the quiver of the reconstruction algebra is



**Example 3.2.** Consider now the dual graph

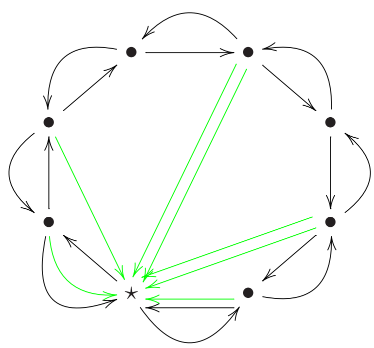



corresponding to the group $\frac{1}{40}(1, 11)$. Now the $Z_f$ is the same as the previous example, so by Lemma 2.2 we just have to add extra arrows to the above; we thus deduce that the reconstruction algebra is

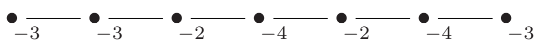


All other cyclic group cases are identical, and follow easily. For example

**Example 3.3.** For the group $\frac{1}{693}(1, 256)$, the reconstruction algebra is

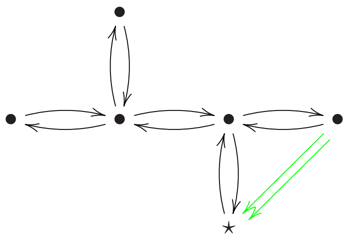
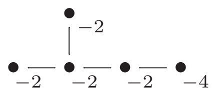
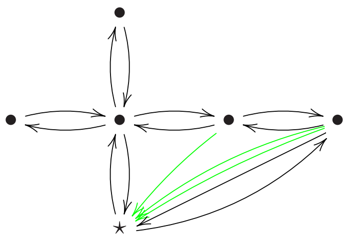
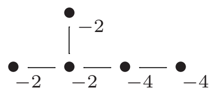
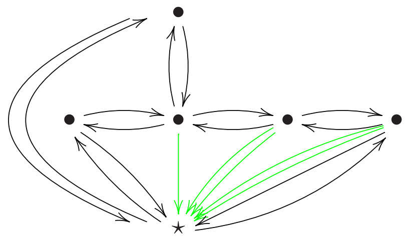
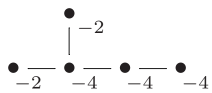


corresponding to the dual graph



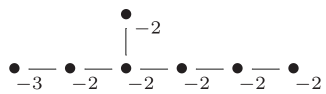
We now venture into some non-abelian groups. Right at the end I'll do some crazy non-quotient singularities.

**Example 3.4.** Some dihedral groups. I restrict to only 3 examples; notice that we have already seen the group $\mathbb{D}_{56,15}$ in Lecture 3.

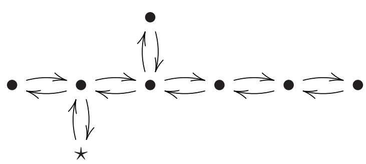
| Reconstruction Algebra | dual graph | $Z_f$ | group |
|---|---|---|---|
|  |   | $\begin{smallmatrix} & 1 & & \\ 1 & 2 & 2 & 1 \end{smallmatrix}$ | $\mathbb{D}_{10,7}$ |
|  |   | $\begin{smallmatrix} & 1 & & \\ 1 & 2 & 1 & 1 \end{smallmatrix}$ | $\mathbb{D}_{26,15}$ |
|  |   | $\begin{smallmatrix} & 1 & & \\ 1 & 1 & 1 & 1 \end{smallmatrix}$ | $\mathbb{D}_{56,15}$ |

Now for some examples of non-quotient singularities:

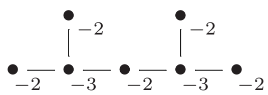
**Example 3.5.** Consider the dual graph




This is one of Artin's rational triple points; there are combinatorics which tell us that this corresponds to some rational singularity. It is not a quotient singularity by Brieskorn's classification, but it does look quite similar to the dual graph corresponding to the group $E_7$. Now here the fundamental cycle $Z_f = \begin{smallmatrix} & & 2 & & & \\ 1 & 3 & 4 & 3 & 2 & 1 \end{smallmatrix}$ (compare to $\begin{smallmatrix} & & 2 & & & \\ 2 & 3 & 4 & 3 & 2 & 1 \end{smallmatrix}$ for $E_7$) which makes the reconstruction algebra in this case



**Example 3.6.** Consider the dual graph



The fundamental cycle is reduced, i.e. $Z_f = \begin{smallmatrix} & 1 & & 1 & \\ 1 & 1 & 1 & 1 & 1 \end{smallmatrix}$ Hence the reconstruction algebra is