Tits Cone Intersections and Applications.

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Abstract

The first part of this memoir assigns hyperplane arrangements to any choice of vertices in a Coxeter graph, by taking an intersection arrangement inside a Tits cone. For each such arrangement obtained in this way, a combinatorial labelling of the chambers is given, and under mild assumptions a combinatorial description of wall crossing is also described. As a special case, given any choice $\mathcal{J}^c$ of n nodes in any Dynkin diagram, a tiling of $\mathbb{R}^n$ is produced. In the case $n = 2$ there are are precisely sixteen tilings, where three are the standard affine Lie algebra arrangements, and thirteen are new. The special case when $\mathcal{J}^c$ equals all nodes gives the full affine arrangement, however the flexibility of choice allows for the construction of affine-type structures, even in $\mathbb{R}^2$ and $\mathbb{R}^3$, when the non-affine situation does not classically have an affine analogue.

The second part is representation-theoretic, and is based around tilting theory. It is shown that the above combinatorial tilings control the tilting theory of contracted preprojective algebras, namely those algebras $e\Pi e$, where Π is the preprojective algebra associated to some Dynkin, or extended Dynkin quiver. This is used to give a derived classification of such algebras in both cases.

The third part is also algebraic, and focuses on cDV singularities, from the viewpoint of noncommutative resolutions and their variants. A bijection is established between the building blocks of noncommutative resolutions, namely modifying modules, and various (higher codimension) walls of the arrangement, thus classifying for the first time all such modules. Furthermore, it is shown that mutation corresponds to wall crossing, and from this, many strong properties are extracted, such as mutation being an involution on arbitrary modifying modules, at arbitrary summands. As a consequence, the Auslander–McKay Correspondence for cDV singularities [W2] is strengthened, and is furthermore extended into the affine setting by using the new infinite arrangements.

The final part contains all the geometric corollaries. The new combinatorial structure, together with the previous algebraic results, are combined to put the first affine-type actions both on the derived categories of 3-folds that admit flopping contractions, and on singular surfaces arising from partial resolutions of Kleinian singularities. A derived classification of partial crepant resolutions of Kleinian singularities is given. The fibre twist of [DW3] is extended to cover non-$\mathbb{Q}$-factorial singularities, and the non-affine actions from [HW1] is extended to cover flops with at worst Gorenstein terminal singularities. The maximum length of the braid relation for 3-fold flops is described, and various 'finite-type' group actions are shown to be faithful.

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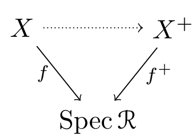
Introduction

The purpose of this work is to extend and generalise hyperplane arrangements from Coxeter theory, and to use these new structures as the fundamental ingredient that then establishes results in various algebraic geometric situations, such as surfaces and 3-folds, and also in various representation theoretic contexts, such as in contracted preprojective algebras, in noncommutative resolutions, and in all their variants. In essence, we first provide the correct combinatorial structure and describe its wall-crossing rules, then use this information to derive results in homological algebra, commutative algebra through reflexive modules, tilting theory of preprojective algebras, group actions on derived categories, and stability conditions.

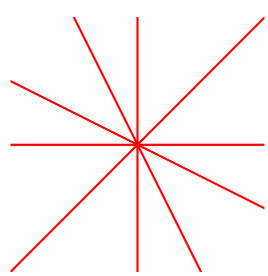
There are many further consequences, all underpinned by the same new combinatorial rules and structure. The hyperplane arrangements and tilings that are obtained, whilst not forming part of classical Coxeter theory, turn out to be surprisingly rich and beautiful, much like their classical Coxeter cousins.

When and What is Affine? Our original motivation stems from the following simple problem. Classical Coxeter theory asserts that various braid and Coxeter groups associated to the Coxeter graph I_5 do *not* admit an affine version. However, there is a situation arising in algebraic geometry, namely 3-fold flopping contractions, that suggests *something* 'affine' exists. This memoir grew out of first trying to uncover this structure, and as such, it is instructive to first review this motivational setting in slightly more detail.

Three dimensional flops are perhaps the most elementary higher-dimensional birational surgery, however many of their properties remain mysterious. It was observed in [**W2**, 7.2] that there exists a 3-fold flop



where both f and f^+ contract two intersecting curves to a point, and for which the movable cone of f is described, in suitable co-ordinates, by the following hyperplane arrangement

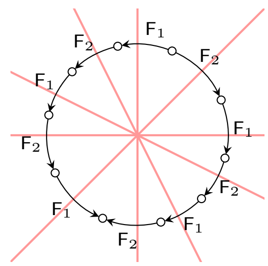


inside $\mathbb{R}^2$. As is well known, the fundamental group of the complexified complement $\mathbb{C}^2\backslash\mathcal{H}_\mathbb{C}$ is equal to the *pure braid group* $\mathsf{PBr}(I_5)$ associated to the Coxeter graph I_5. Starting from X, iteratively flopping the two individual curves (that is, performing the birational surgery of flop on the two curves individually and repeatedly), gives ten different flopping contractions $X_i \to \operatorname{Spec} R$. These correspond to the chambers of the movable cone, and so to the chambers of the above hyperplane arrangement.

All the X_i, whilst being non-isomorphic as R-schemes, are derived equivalent via the Bridgeland–Chen *flop functors*. It was shown in [**DW3**] that, in this example, the length five braid relation holds for the flops functors, namely

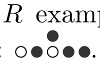
$$\mathsf{F}_1 \circ \mathsf{F}_2 \circ \mathsf{F}_1 \circ \mathsf{F}_2 \circ \mathsf{F}_1 \cong \mathsf{F}_2 \circ \mathsf{F}_1 \circ \mathsf{F}_2 \circ \mathsf{F}_1 \circ \mathsf{F}_2 .$$

Visually, this can be viewed as

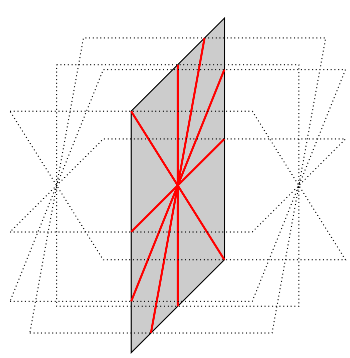


The existence of a group homomorphism $\mathsf{PBr}(I_5) \to \operatorname{Auteq} \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$ follows immediately. This homomorphism turns out to be quite important: it is injective [**HW1**], and its image is the Galois group of the universal cover $\operatorname{Stab}^\circ \mathcal{C} \to \mathbb{C}^2 \backslash \mathcal{H}_{\mathbb{C}}$, where $\mathcal{C}$ is the null category inside $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$, see [**HW2**].

However, this is not the whole picture. It was furthermore shown in [**DW3**] that, provided X is smooth (or more generally, $\mathbb{Q}$-factorial), there exists an *additional* autoequivalence, called the fibre twist, which, motivated by happens for smooth surfaces [**B6**], should be some *affine* element in some naturally occurring 'affine' version of $\mathsf{PBr}(I_5)$. The question is: what is this affine structure? In classical Coxeter theory, there is no affine pure braid group associated to I_5.

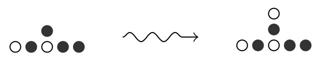
Solution: Finite and Affine. Of course, the question is not well posed, as the algebraic geometric setup contains more information. Whilst the finite arrangement with 10 chambers can be identified with the root system I_5, it turns out that it is much more natural to view it as the intersection arrangement (or localisation arrangement) inside a E_6 root system. Indeed, via Reid's general elephant conjecture and McKay correspondence, it is more natural to identify the two curves with a *subset* of vertices of an E_6 Dynkin diagram. For the $f\colon X \to \operatorname{Spec} R$ example above, the two curves correspond to the unshaded vertices in the following: .

In turn, the two unshaded vertices correspond to linearly independent vectors in the E_6 root system, taking so their span gives $\mathbb{R}^2$. Intersecting all the reflecting hyperplanes inside the root system (which is $\mathbb{R}^6$) with this $\mathbb{R}^2$ cuts out a hyperplane arrangement on the $\mathbb{R}^2$, which gives precisely the 10 chamber example above. A cartoon description of this intersection in $\mathbb{R}^6$ is depicted by the following diagram.

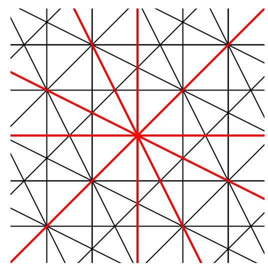


In particular, before even extending into the affine case, this on its own motivates us to develop a full theory for intersection arrangements inside *all* Coxeter root systems.

Our construction of the 'affine' version is verified in Example 2.10. It involves first adding the extended vertex



Then, inside the full Tits cone of the *affine* E_6 root system sits $\mathbb{R}^3$, based by the three unshaded vertices. Intersecting all the hyperplanes in the Tits cone with this $\mathbb{R}^3$ gives a cone, and after taking a suitable 'level' (for details, see below), the corresponding infinite hyperplane arrangement in $\mathbb{R}^2$ is precisely:



In general, taking intersection arrangements leaves the world of Coxeter arrangements, and so the language adopted is not one of global rules, governed by the Coxeter matrix, but rather a language of local rules, governing local wall crossing behaviour. Explicitly describing the affine arrangements produced in principle remains easy, as it is possible to start anywhere and iterate well-described local rules. In practice, these calculations are quite involved: even in the case of two curves, which produces a tiling of the plane, Theorem 0.5 below demonstrates that the tilings produced exhibit quite exceptional behaviour, and they take much longer to repeat than might naively be expected.

Nonetheless, Parts 2, 3 and 4 provide justification for calling these new infinite arrangements 'affine', since they achieve our geometric motivation, and much more.

Forward. With this motivation in hand, we start at the beginning, and develop a general theory for intersection arrangements in both finite and affine cases. We then use this new theory to help uncover and prove results in various algebraic and geometric settings. To achieve this, the memoir naturally splits into four parts.

- Part 1, which is entirely combinatorial and logically independent of all the other Parts, develops the general theory of intersection arrangements inside Coxeter arrangements. It constructs the arrangements and labels the chambers using Coxeter data, describes the local wall crossing rules, and classifies the arrangements, finite and affine, that arise in low dimension.
- Part 2 considers, for the first time, *contracted preprojective algebras*, which are $e\Pi e$ where Π is the usual preprojective algebra and e is some idempotent. Under various assumptions on the underlying quiver, one of the main results is that the tilting theory of these algebras is controlled by the affine arrangements constructed in Part 1. A derived classification is given.
- Part 3 lifts the technical heart of Part 2 into the world of 3-folds, and completely describes *all* noncommutative resolutions and their variants for compound Du Val (=cDV) singularities, again in terms of the affine hyperplane arrangements of Part 1. The finite arrangements correspond to a certain natural subset. In all cases, mutation corresponds to wall crossing, and crucially the topology of the arrangements is used to strengthen and extend many results in mutation to cover arbitrary rigid reflexive modules.
- The techniques in all of the Parts 1, 2 and 3 then allows, in Part 4, a return to the original algebraic geometry motivation, where the geometric corollaries are

spelled out in some detail in dimensions two and three. The advances in Part 3 allow many of the assumptions in the literature to be swept away, generalising many results from smooth flops to terminal flops, including braiding, existence of the fibre twists, affine actions, and various results on faithful actions.

We now describe the content of each of the four Parts in more detail.

Part 1. Intersection Arrangements. The construction requires two pieces of input data. The first is an $n \times n$ Coxeter matrix $M = (m_{ij})$, with entries in the set $\{1, 2, \ldots, \infty\}$. As is standard, M can alternatively be described by a Coxeter graph Δ with n nodes, where we draw an edge between i and j if and only if $m_{ij} \geq 3$. There is a naturally associated Coxeter group, denoted W_Δ.

Let V be the $\mathbb{R}$-vector space with basis $\{\alpha_i \mid i \in \Delta\}$, and B the symmetric bilinear form on V defined by $B(\alpha_i, \alpha_j) = -\cos(\pi/m_{ij})$. The Coxeter group W_Δ acts on V by $s_i(v) := v - 2\,B(\alpha_i, v)\alpha_i$. Set $\Theta := V^*$ to be the dual space of V, which has basis $\{\alpha_i^* \mid i \in \Delta\}$. The *Tits cone* $\mathsf{Cone}(\Delta)$ is defined to be

$$\mathsf{Cone}(\Delta) := \bigcup_{x \in W_\Delta} x(\overline{C}),$$

where $\overline{C} := \{\vartheta \in \Theta \mid \vartheta_i \geq 0 \text{ for all } i \in \Delta\}$.

Our second piece of input data is a choice $\mathcal{J}$ of a subset of vertices of Δ. Given this input pair $(\Delta, \mathcal{J})$, consider the vector space

$$\Theta_{\mathcal{J}} := \{\vartheta \in \Theta \mid \vartheta_i = 0 \text{ if } i \in \mathcal{J}\},$$

which has as basis $\{\alpha_i^* \mid i \notin \mathcal{J}\}$. The main object of our study, called the $\mathcal{J}$-cone, is the intersection

$$\mathsf{Cone}(\Delta, \mathcal{J}) := \mathsf{Cone}(\Delta) \cap \Theta_{\mathcal{J}}.$$

In order to describe this object, we first label its chambers. For $\mathcal{J} \subseteq \Delta$, let $\mathsf{Cham}(\Delta, \mathcal{J})$ be the set of those pairs (x, J), where x is an element of the Weyl group W_Δ, J is a subset of Δ, satisfying the two properties that $\ell(x) = \min\{\ell(y) \mid y \in xW_J\}$, and $W_{\mathcal{J}}x = xW_J$.

Theorem 0.1 (1.12). *Let Δ be a Coxeter graph. Then there is a bijection from the set $\mathsf{Cham}(\Delta, \mathcal{J})$ to the set of chambers in $\mathsf{Cone}(\mathcal{J})$, given by*

$$\mathsf{Cham}(\Delta, \mathcal{J}) \ni (x, J) \mapsto x(C_J).$$

Now for any $(x, J) \in \mathsf{Cham}(\Delta, \mathcal{J})$, by the above theorem topologically we land in some chamber of $\mathsf{Cone}(\Delta, \mathcal{J})$. From this, we try to *wall cross* into an adjacent chamber. Walls of the chamber $x(C_J)$ correspond to $i \in J^{\mathrm{c}}$. In particular, there are $|J^{\mathrm{c}}|$ of them.

To describe this, suppose we are in a situation where $i \in J^{\mathrm{c}}$ is such that W_{J+i} is finite. This happens often, and for example holds automatically for all extended ADE Dynkin diagrams. In this case, we define *simple wall crossing* by

$$\omega_i(x, J) := (xw_J w_{J+i}, J + i - \iota_{J+i}(i)),$$

where w_J is the longest element in W_J, w_{J+i} is the longest element in W_{J+i}, and ι_{J+i} is the involution on the graph $J + i$ from 1.2(2).

The name *simple wall crossing* in 1.16 is justified by the following theorem, which is the main result of Chapter 1.

Theorem 0.2 (1.20). *Let Δ be a Coxeter graph, and $\mathcal{J}$ a subset of Δ.*

(1) *For any $(x, J) \in \mathsf{Cham}(\Delta, \mathcal{J})$ and $i \in J^{\mathrm{c}}$ such that $J + i$ is Dynkin, the following assertions hold.*
   (a) *$\omega_i(x, J)$ belongs to $\mathsf{Cham}(\Delta, \mathcal{J})$ for any $(x, J) \in \mathsf{Cham}(\Delta, J)$ and $i \in J^{\mathrm{c}}$.*
   (b) *$x < xs_i \iff (x, J) < \omega_i(x, J)$, and $x > xs_i \iff (x, J) > \omega_i(x, J)$.*
   (c) *Wall crossing is involutive, that is,*

$$\omega_{i'}\omega_i(x, J) = (x, J)$$

   for $i' := \iota_{J+i}(i)$.

(d) *Let* $(y, J') := \omega_i(x, J)$. *Then the* $\mathcal{J}$*-chambers* $x(C_J)$ *and* $y(C_{J'})$ *are adjacent via the wall* $x(C_{J+i})$.

(2) *If* $\mathcal{J}$ *is strongly Dynkin, then the following assertions hold.*

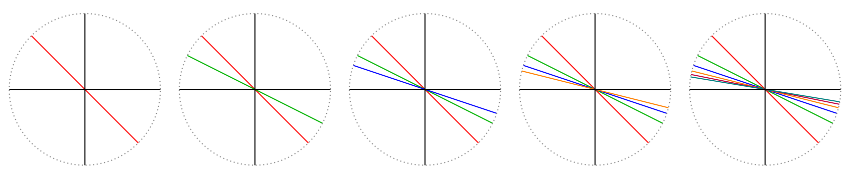
(a) *Any two elements in* $\mathsf{Cham}(\Delta, \mathcal{J})$ *are connected by a finite sequence of wall crossings.*

(b) *Two elements in* $\mathsf{Cham}(\Delta, \mathcal{J})$ *are related by a simple wall crossing if and only if the corresponding* $\mathcal{J}$*-chambers are adjacent.*

(3) *If* W_Δ *is finite, then* $\mathsf{Cham}(\Delta, \mathcal{J})$ *has the minimum element* $(1, \mathcal{J})$ *and the maximum element* $(w_{\mathcal{J}} w_\Delta, \iota_\Delta(\mathcal{J}))$.

It is the ADE Dynkin, and extended Dynkin cases that interest us the most. Suppose that Δ is ADE Dynkin, $\mathcal{J} \subseteq \Delta$, then the simplest case is when $\mathcal{J}^{\mathrm{c}} = \Delta \backslash \mathcal{J}$ is small. The case when $|\mathcal{J}^{\mathrm{c}}| = 1$ is degenerate, always having precisely two chambers (Lemma 3.1). The case $|\mathcal{J}^{\mathrm{c}}| = 2$ is much more surprising.

Theorem 0.3 (3.11). *Suppose that* Δ *is ADE Dynkin, and* $\mathcal{J} \subseteq \Delta$ *is such that* $|\mathcal{J}^{\mathrm{c}}| = 2$. *Then, up to changing the slopes of some of the hyperplanes,* $\mathsf{Cone}(\Delta, \mathcal{J})$ *is one of the following five hyperplane arrangements.*



In each case, the number of chambers is $6, 8, 10, 12$ *and* 16 *respectively.*

If we take into account the precise slopes of the hyperplanes, then more arrangements can occur, but for the vast majority of our applications the slopes of the hyperplanes do not matter. The slopes, and also an associated weighting of each hyperplane, do give a method of computing the infinite arrangement below, but this data is not strictly necessary; the infinite arrangement can be computed without knowledge of the slopes.

Consider next the extended Dynkin case Δ_{aff}. Given a subset $\mathcal{J}$ of vertices of the Dynkin diagram Δ, we can consider $\mathcal{J}$ as a subset of Δ_{aff}. We then call

$$\mathsf{Cone}(\mathcal{J}_{\mathsf{aff}}) := \mathsf{Cone}(\Delta_{\mathsf{aff}}, \mathcal{J}) \subseteq \mathbb{R}^{|\Delta_{\mathsf{aff}}| - |\mathcal{J}|}$$

the $\mathcal{J}$*-affine Tits cone*. As for usual Tits cones, there is redundancy as $\mathsf{Cone}(\mathcal{J}_{\mathsf{aff}})$ does not fill $\mathbb{R}^{|\Delta_{\mathsf{aff}}| - |\mathcal{J}|}$. Suppose that Δ_{aff} has corresponding $\Theta = V^*$, then for any $\mathcal{K} \subsetneq \Delta_{\mathsf{aff}}$, the *level* is defined to be

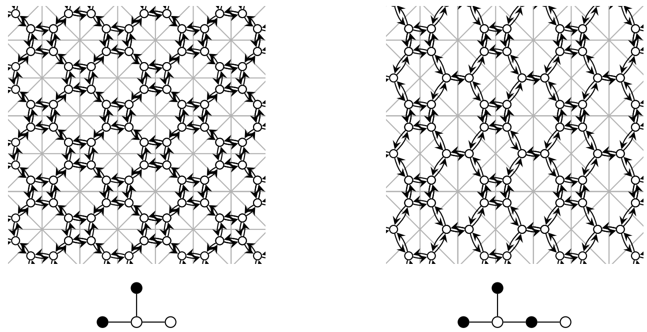
$$\mathsf{Level}(\mathcal{K}) := \{\vartheta \in \mathsf{Cone}(\Delta_{\mathsf{aff}}, \mathcal{K}) \mid \sum_{k \notin \mathcal{K}} \delta_k \vartheta_k = 1\}.$$

Thus for $\mathcal{J} \subseteq \Delta$, the level $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$ inside $\mathsf{Cone}(\Delta_{\mathsf{aff}}, \mathcal{J})$ is an infinite hyperplane arrangement in $\mathbb{R}^{|\mathcal{J}^c|}$, where $\mathcal{J}^c = \Delta \backslash \mathcal{J}$. The chambers of $\mathsf{Cone}(\mathcal{J}_{\mathsf{aff}})$ partition $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$ into *alcoves*. More details are given in Section 2.2, and all these concepts are illustrated in Example 2.9.

One remarkable feature is that different choices in different Dynkin diagrams can lead to the same finite hyperplane arrangement, but different affine arrangements.

Example 0.4. Consider the following Dynkin diagrams D_4 and D_5, where the shaded vertices denote the elements of a subset $\mathcal{J}$.

In both cases, $\mathsf{Cone}(\Delta, \mathcal{J})$ gives the second-left hyperplane arrangement in Theorem 0.3, with eight chambers. However, the levels $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$ differ, and they are both illustrated below. On top of these arrangements, we have drawn the dual groupoid, to illustrate the difference.

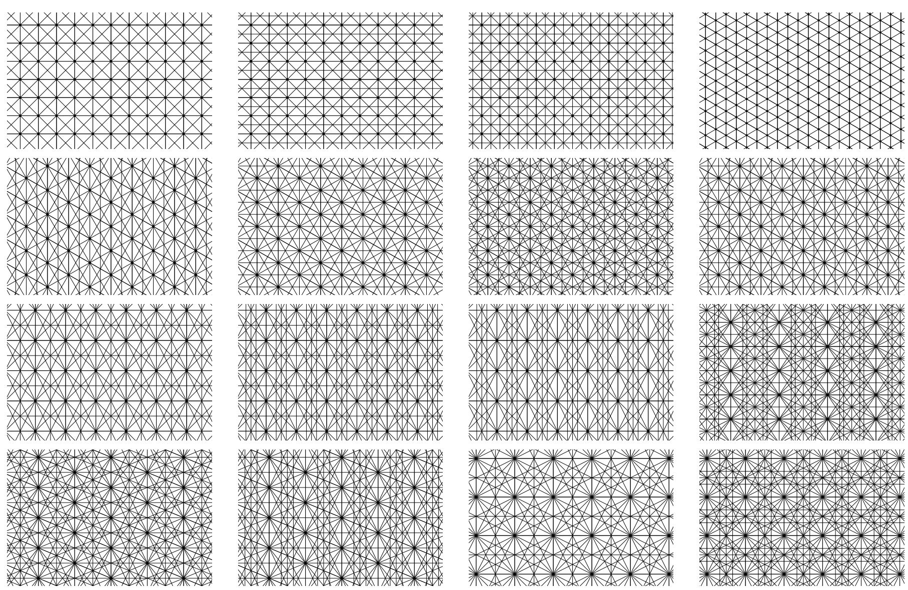


The left arrangement is the traditional affine B_2 arrangement, and the right arrangement is obtained from the left by removing hyperplanes.

The level $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$ then becomes the fundamental new object. In turn, this motivates the investigation of its basic properties, especially when $|\mathcal{J}^c|$ is small. The case $|\mathcal{J}^c| = 1$ is described in [**DW5, HW2**], which finds precisely six infinite hyperplane arrangements (equipped with $\mathbb{Z}$-action) in $\mathbb{R}$. Far from being the trivial case, these have already uncovered surprising new phenomena in derived autoequivalence groups.

The next case is $|\mathcal{J}^c| = 2$, which is treated in Chapter 4, where a full classification is obtained. Again, this unveils surprising structure.

THEOREM 0.5 (Section 4.2). *Suppose that Δ_{aff} is extended ADE Dynkin, and $\mathcal{K} \subseteq \Delta_{\mathsf{aff}}$ satisfies $|\mathcal{K}| = 3$. Then, up to changing the slopes of some of the hyperplanes,* $\mathsf{Level}(\mathcal{K})$ *is one of following sixteen hyperplane arrangements:*



In addition, each of the sixteen arrangements appears as $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$ *for some subset of the ADE Dynkin $\mathcal{J} \subseteq \Delta$ satisfying $|\mathcal{J}^c| = 2$.*

The first and third tilings in the top row are the same as abstract hyperplane arrangements, but they have different $\mathbb{Z}^2$ actions, illustrated by black dots. More details are given in Section 4.2. What is perhaps the most striking about the above is the sheer complexity of some of the tilings. This was unexpected, from the viewpoint of both the algebraic and the geometric applications below.

Part 2. Contracted Preprojective Algebras. To the input of a Coxeter graph Δ and a choice of vertices $\mathcal{J} \subseteq \Delta$, Part 2 investigates the first, and most basic, representation theory questions.

Let $Q = (Q_0, Q_1)$ be any quiver with underlying graph Δ, and let $\overline{Q}$ be the double quiver of Q. The preprojective algebra Π associated to this data is the (complete) preprojective algebra of Δ, that is the complete path algebra of $\overline{Q}$, modulo the closure of the ideal generated by the element

$$\sum_{a \in Q_1} (aa^* - a^*a).$$

For each vertex $i \in \Delta$, there is a corresponding idempotent e_i of Π. Given the subset $\mathcal{J} \subseteq \Delta$, consider the idempotent

$$e_{\mathcal{J}} := 1 - \sum_{j \in \mathcal{J}} e_j.$$

The following is our fundamental new object of study.

Definition 0.6. For any $\mathcal{J} \subseteq \Delta$, we call $\Gamma_{\mathcal{J}} := e_{\mathcal{J}} \Pi e_{\mathcal{J}}$ the *contracted preprojective algebra* associated to $\mathcal{J}$.

It turns out that tilting theory for $\Gamma_{\mathcal{J}}$ is controlled by the chambers $\mathsf{Cham}(\Delta, \mathcal{J})$. Set $\operatorname{tilt} \Gamma_{\mathcal{J}}$ to be the the set of isomorphism classes of basic tilting E-modules of projective dimension one. As notation, for $i \in \Delta$, let I_i be the two-sided ideal of Π generated by $1 - e_i$. For $w \in W$ with reduced expression $w = s_{i_1} \dots s_{i_\ell}$, recall that the ideal I_w of Π is defined

$$I_w := I_{i_1} \dots I_{i_\ell}.$$

This is independent of a choice of reduced expression [BIRS]. By convention, $I_1 = \Pi$.

Theorem 0.7 (5.2). *Let Δ be a non-Dynkin graph without loops, and Π the preprojective algebra of Δ. Let $\mathcal{J}$ be a strongly Dynkin subset of Δ.*

(1) *There is a map*

$$\mathsf{Cham}(\Delta, \mathcal{J}) \to \operatorname{tilt} \Gamma_{\mathcal{J}}$$

given by $(x, J) \mapsto e_{\mathcal{J}} I_x e_J$.

(2) *Wall crossing is compatible with mutation, that is, if $\omega_i(x, J) = (y, J')$, then $\nu_i(e_{\mathcal{J}} I_x e_J) = e_{\mathcal{J}} I_y e_{J'}$.*

(3) *If Δ is extended Dynkin, then the above map $\mathsf{Cham}(\Delta, \mathcal{J}) \to \operatorname{tilt} \Gamma_{\mathcal{J}}$ is a bijection.*

An immediate corollary of the above theorem is that the set of algebras $\Gamma_{\mathcal{J}}$, as $\mathcal{J}$ runs over $\mathcal{J} \subseteq \Delta$, is split into derived equivalence classes. Indeed, $\Gamma_{\mathcal{J}}$ and $\Gamma_{\mathcal{J}'}$ are derived equivalent provided that $\mathcal{J}$ and $\mathcal{J}'$ can be linked through a sequence of iterated combinatorial wall-crossing moves. In the extended Dynkin setting, this gives the first known derived equivalences between *partial* crepant resolutions of Kleinian singularities (see 0.23 below).

In Chapter 6 we consider the case when Δ is ADE Dynkin, and $\mathcal{J} \subseteq \Delta$. In this setting, both Π and $\Gamma_{\mathcal{J}}$ are finite dimensional algebras, but since Π is self-injective it has no classical tilting modules. The algebras Π and $\Gamma_{\mathcal{J}}$ do, however, have both silting and tilting complexes. Our main result in this context is the following, where $2\operatorname{silt} \Gamma_{\mathcal{J}}$ denotes the two-term silting complexes, and $2\operatorname{tilt} \Gamma_{\mathcal{J}}$ denotes the two-term silting complexes. The assumption $\iota(\mathcal{J}) = \mathcal{J}$ is necessary to ensure that $\Gamma_{\mathcal{J}}$ is also self-injective (see 6.2).

Theorem 0.8 (6.4). *Let Δ be ADE, and $\mathcal{J} \subseteq \Delta$ with $\iota(\mathcal{J}) = \mathcal{J}$.*

(1) *There are bijections*

$$\begin{array}{ccc} \mathsf{Cham}(\Delta, \mathcal{J}) & \longleftrightarrow & 2\operatorname{silt} \Gamma_{\mathcal{J}} \\ \cup & & \cup \\ \mathsf{Cham}(\Delta, \mathcal{J})^{\iota} & \longleftrightarrow & 2\operatorname{tilt} \Gamma_{\mathcal{J}}. \end{array}$$

(2) *The endomorphism algebra of any irreducible left tilting mutation of* $\Gamma_{\mathcal{J}}$ *is isomorphic to* Γ_J *for some* $J \subseteq \Delta$ *such that there exists* $(x, J) \in \mathsf{Cham}(\Delta, \mathcal{J})^\iota$*. In particular,* $\mathrm{K}^{\mathrm{b}}(\mathrm{proj}\, \Gamma_{\mathcal{J}})$ *is tilting-discrete.*

(3) *The derived and Morita equivalence classes of* $\Gamma_{\mathcal{J}}$ *coincide. The basic algebras in this class are precisely* $\{\Gamma_J \mid J \subseteq \Delta, \exists\, (x, J) \in \mathsf{Cham}(\Delta, \mathcal{J})^\iota\}$.

The focus of Chapter 7 is the case when Δ_{aff} is extended ADE, and $\mathcal{J} \subseteq \Delta_{\mathsf{aff}}$. The main idea is that, in this setting, the derived equivalence classification of *all* contracted preprojective algebras (not only those derived equivalent to partial resolutions) is always combinatorially determined, and furthermore the derived equivalence class does not contain anything unexpected.

Conjecture 0.9 (7.1). *Suppose that* $\mathcal{I} \subseteq \Delta_{\mathsf{aff}}$ *where* Δ_{aff} *is extended ADE Dynkin, and let* A *be a basic ring. Then* A *is derived equivalent to* $\Gamma_{\mathcal{I}}$ *if and only if there exists* $\mathcal{I}' \subseteq \Delta_{\mathsf{aff}}$ *such that* $A \cong \Gamma_{\mathcal{I}'}$*, and furthermore* $\mathcal{I}$ *and* $\mathcal{I}'$ *are iterated combinatorial mutation of each other, up to symmetries of* Δ_{aff}*.*

The direction ($\Leftarrow$) is clear follows using 0.7, since wall crossing gives derived equivalences (§5.6), as do isomorphisms. The content in the conjecture is the ($\Rightarrow$) direction. In Chapter 7 we prove the conjecture in all cases, except when $\Delta = D_n$ with $n \geq 8$, due to its combinatorial complexity. To this end, we introduce the following four invariants: the type, the cotype, the Grothendieck group, and the subgroup $H_{\mathcal{I}} + K_{\mathcal{I}}$; for definitions see §7.1. Our main result is the following.

Theorem 0.10 (7.21). *Suppose that* $\mathcal{I} \subseteq \Delta_{\mathsf{aff}}$ *and* $\mathcal{I}' \subseteq \Delta'_{\mathsf{aff}}$ *where* Δ *and* Δ' *are ADE Dynkin. Consider the following conditions.*

(1) $\Gamma_{\mathcal{I}}$ *is derived equivalent to* $\Gamma_{\mathcal{I}'}$*.*

(2) *The types match (namely* $\Delta = \Delta'$*), and* $\mathcal{I} \sim \mathcal{I}'$*.*

(3) *The types match, the cotypes match,* $G_0(\Gamma_{\mathcal{I}}) \cong G_0(\Gamma_{\mathcal{I}'})$*, and* $H_{\mathcal{I}} + K_{\mathcal{I}} \cong H_{\mathcal{I}'} + K_{\mathcal{I}'}$*.*

Then (2)$\Rightarrow$(1)$\Rightarrow$(3)*. If* $\Delta \in \{A_n, D_4, D_5, D_6, D_7, E_6, E_7, E_8\}$*, then* (1)$\Leftrightarrow$(2)$\Leftrightarrow$(3)*.*

In most cases it is possible to get by using less than the four invariants in (3) above; see 7.21 for a more precise statement.

Although slightly technical, the above 0.7 together with the natural partial order on the set tilt $\Gamma_{\mathcal{J}}$ is the key to much of what follows. It allows us to describe a large portion of the autoequivalence group of $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, \Gamma_{\mathcal{J}})$, to deduce many of the homological properties of three-dimensional cDV singularities below, to classifying noncommutative resolutions and their variants, and to verify that the wall crossing functors satisfy the relations of the Deligne groupoid, and hence give affine group actions in geometric settings.

Part 3. cDV Singularities. Compound du Val (=cDV) singularities are fundamental objects in birational geometry. Those cDV singularities with only isolated singularities are precisely the Gorenstein terminal singularities in dimension three, and these form the base of flopping contractions [R]. More generally, cDV singularities that are not isolated form the base of crepant divisor-to-curve contractions.

The distinction between isolated and non-isolated cDV singularities is in many ways artificial. The unifying feature is that each any such $\mathcal{R}$ admits a crepant birational morphism $X \to \operatorname{Spec} \mathcal{R}$, with only one-dimensional fibres, such that X has only $\mathbb{Q}$-factorial terminal singularities. The variety X is called a *minimal model* for $\operatorname{Spec} \mathcal{R}$; there are finitely many such minimal models, and they are all linked by flops. When one such X is smooth, all minimal models are smooth.

Our motivation behind Part 3 is to understand the birational geometry of $\operatorname{Spec} \mathcal{R}$ from a derived and homological perspective. We achieve this through understanding the representation theory of cDV singularities, namely through those modules which give rise to noncommutative minimal models of $\mathcal{R}$, and their variants.

Given a Gorenstein ring R, recall that a reflexive R-module M is called *modifying* if $\mathrm{End}_R(M)$ is (maximal) Cohen–Macaulay, as an R-module. Further, M is called maximal modifying provided it is modifying and maximal with respect to this property, and in such a case we call $\mathrm{End}_R(M)$ a maximal modification algebra (=MMA). In analogy with the paragraph above, if one MMA has finite global dimension, all MMAs have finite global dimension.

Henceforth, let $\mathcal{R}$ be a complete local cDV singularity. We first give a purely algebraic proof of the following result, first obtained in [**W2**]. The proof here is much shorter, and does not rely on tilting and on the existence of minimal models.

PROPOSITION 0.11 (9.4). *Let $\mathcal{R}$ be a cDV singularity, then $\mathcal{R}$ admits an MMA.*

Much of, in fact almost all of, the homological aspects of the MMAs of $\mathcal{R}$ turn out to be controlled by a factor of $\mathcal{R}$. Given an MMA $\mathrm{End}_{\mathcal{R}}(M)$, for generic $g \in \mathcal{R}$ there are isomorphisms

$$\mathrm{End}_{\mathcal{R}}(M)/g \cong \mathrm{End}_{\mathcal{R}/g}(M/g) \cong e\Pi e$$

for some idempotent e, linking the setting to the previous parts of this memoir. The following is then key, since it relates properties of maximal modifying $\mathcal{R}$-modules to tilting modules on $e\Pi e$, and hence to our previous affine hyperplane arrangements.

We remark that the following holds more generally; see 8.17 for full details. As notation, write $\mathrm{MM}\mathcal{R}$ for the isomorphism classes of basic maximal modifying modules, and $\mathrm{MMG}\mathcal{R}$ for the subset for those which have $\mathcal{R}$ as a direct summand.

PROPOSITION 0.12 (8.17). *With notation as above, set $\Lambda := \mathrm{End}_{\mathcal{R}}(M)$. Then for any $0 \neq g \in \mathcal{R}$, there is an injective map $\mathrm{MM}\mathcal{R} \hookrightarrow \mathrm{tilt}(\Lambda/g)$. If further that $\mathcal{R}$ is an isolated singularity, then the following statements hold.*

1. *The map is compatible with mutation.*
2. *If the exchange graph of $\mathrm{tilt}(\Lambda/g)$ is connected, then the map is bijective.*

Our main result is the following, which gives a full classification of maximal modification $\mathcal{R}$-modules. The result is quite unexpected, and is very specific to the cDV setting; usually there is no hope in being able to classify maximal modification modules in this way. The following can also be viewed as an extension of the Auslander–McKay correspondence in [**W2**] into the affine setting, which justifies the name.

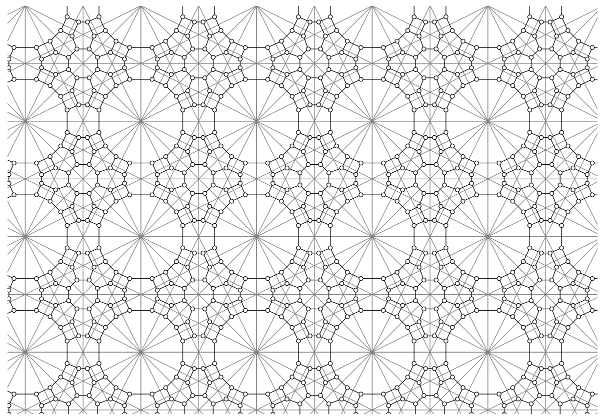
THEOREM 0.13 (9.8, Affine Auslander–McKay Correspondence). *Let $\mathcal{R}$ be a complete normal cDV singularity of type Δ and Δ_{aff} the corresponding extended Dynkin graph. Then the following assertions hold.*

1. *There exists a subset $\mathcal{J} \subseteq \Delta$ and an injective map*
$$\mathrm{MM}\mathcal{R} \hookrightarrow \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J}).$$
This induces an injective map $\mathrm{MMG}\mathcal{R} \hookrightarrow \mathsf{Cham}(\Delta, \mathcal{J})$.
2. *Wall crossing corresponds to mutation.*
3. *There exist only finitely many maximal modifying generators of $\mathcal{R}$, and finitely many indecomposable modifying Cohen-Macaulay $\mathcal{R}$-modules.*
4. *If furthermore $\mathcal{R}$ is an isolated singularity, then the following diagram commutes, where the horizontal arrows are bijections.*

$$\begin{array}{ccc} \mathrm{MM}\mathcal{R} & \xrightarrow{\sim} & \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J}) \\ \uparrow & & \uparrow \\ \mathrm{MMG}\mathcal{R} & \xrightarrow{\sim} & \mathsf{Cham}(\Delta, \mathcal{J}) \end{array}$$

COROLLARY 0.14. *Let $\mathcal{R}$ be a complete local cDV singularity. Then the exchange graphs of both $\mathrm{MM}\mathcal{R}$ and $\mathrm{MMG}\mathcal{R}$ are connected.*

In particular, it follows that the exchange graph of $\mathrm{MM}\mathcal{R}$ has a highly regular structure. An example of such an exchange graph, drawn on top of the associated hyperplane arrangement (see Example 4.29), is illustrated below in one case where the maximal modifying modules have three indecomposable summands.



One of our new observations is that the sets $\mathrm{MM}\mathcal{R}$ and $\mathrm{MMG}\mathcal{R}$, and by extension their geometric counterparts, do not have a natural order. However, for any fixed $M \in \mathrm{MM}\mathcal{R}$, the equivalence

$$\mathrm{Hom}_{\mathcal{R}}(M,-)\colon\ \mathrm{MM}\mathcal{R} \to \operatorname{tilt}\mathrm{End}_{\mathcal{R}}(M)$$

transfers information from $\mathrm{MM}\mathcal{R}$ to the category of reflexive tilting modules, which does have a partial order. It is by exploiting this partial order that we are able to obtain, rather easily, many of our results.

Perhaps the main content of Chapter 9 though, is that we then extend the Affine Auslander–McKay Correspondence in 0.13 to also cover the case when the modifying modules are not maximal. The extension of the representation theory to cover this case is much harder, since usually the set $\operatorname{modif}\mathcal{R}$ of modifying modules is not well behaved. For cDV singularities however, the payoff is significant: it turns out that the mutation class of *any* $N \in \operatorname{modif}\mathcal{R}$ also exhibits highly regular behaviour, and is again controlled topologically by some intersection arrangement. The extension of the theory to cover the non-maximal case, and the fact that mutation is still highly regular, is crucial later in order to understand the special case of a flopping contractions $X \to \operatorname{Spec}\mathcal{R}$ where X has only terminal singularities.

For $N \in \operatorname{modif}\mathcal{R}$, write $\operatorname{modif}^N\mathcal{R}$ for those modifying reflexive $\mathcal{R}$-modules than have a two-term approximation by $\operatorname{add}N$, and write $\mathrm{MM}^N R$ for those $L \in \operatorname{modif}^N R$ which have the same number of indecomposable summands as N.

Theorem 0.15 (9.25). *Let $\mathcal{R}$ be a cDV singularity, and fix $N = N_1 \oplus \ldots \oplus N_t \in \operatorname{modif}\mathcal{R}$ with indecomposable. For any $X \in \operatorname{modif}\mathcal{R}$, the following holds.*

(1) $X \in \operatorname{modif}^N\mathcal{R}$ *if and only if* $\mathrm{Hom}_{\mathcal{R}}(N,X) \in \text{ref-ptilt}\,\mathrm{End}_{\mathcal{R}}(N)$.
(2) $X \in \mathrm{MM}^N\mathcal{R}$ *if and only if* $\mathrm{Hom}_{\mathcal{R}}(N,X) \in \text{ref-tilt}\,\mathrm{End}_{\mathcal{R}}(N)$.
(3) *There are bijections*

$$\mathrm{Hom}_{\mathcal{R}}(N,-)\colon\ \operatorname{modif}^N\mathcal{R} \xrightarrow{\sim} \text{ref-ptilt}\,\mathrm{End}_{\mathcal{R}}(N)$$

$$\mathrm{Hom}_{\mathcal{R}}(N,-)\colon\ \mathrm{MM}^N\mathcal{R} \xrightarrow{\sim} \text{ref-tilt}\,\mathrm{End}_{\mathcal{R}}(N)$$

where ref-tilt *are those classical tilting modules that are reflexive with respect to $\mathcal{R}$, and* ref-ptilt *is the partial version, where we do not require generation.*

Given a summand X of $N \in \operatorname{modif}\mathcal{R}$, there always is a well-defined left and right mutation ν_X and μ_X of the module M. Over general Gorenstein rings, it is very rare that these operations coincide, even when X is indecomposable. The following result is thus very remarkable, even more so since we can use *any* summand. To ease the exposition we state the following in the case when $\mathcal{R}$ is isolated; the more general statement can be found in 9.28.

COROLLARY 0.16 (9.28). *Suppose that $\mathcal{R}$ is isolated cDV, $N \in \operatorname{modif}\mathcal{R}$, and X is an arbitrary summand of N.*

(1) $\nu_X(N) \cong \mu_X(N)$.
(2) $\nu_X\nu_X(N) \cong N$.

Since left mutation equals right mutation, we henceforth just refer to this process as *mutation*, and denote it ν_X. This means that in the exchange sequences, the same module appears on both the right and on the left. The power of 0.16 comes since it holds for *any* summand of N, where N is any modifying module; the proof boils down to mutation being a topological property of the hyperplane arrangement.

In fact, the statement $\nu_Y\nu_Y N \cong N$ can be strengthened further, as there is an even more remarkable symmetry in the exchange sequences. Again, there is a more general version of the following, but the case when $\mathcal{R}$ is isolated is easiest to state.

THEOREM 0.17 (9.29, 9.28). *Suppose that $\mathcal{R}$ is isolated cDV, and let $N \in \operatorname{modif}\mathcal{R}$.*

(1) $\mathrm{MM}^N\mathcal{R}$ *coincides with the mutation classes of N.*
(2) *For any direct summand $X = N_I$ of N, consider the exchange sequences*

$$0 \to N_I \to U_I \to \nu(N_I) \quad \text{and} \quad 0 \to \nu(N_I) \to V_I \to N_I.$$

Then there is an isomorphism $U_I \cong V_I$.

The first part is a strong version of the fact that for a maximal rigid object N in a 2-CY Krull–Schmidt triangulated categories, $N * N[1]$ coincides with the mutation class of N. Again, this is not typical behaviour for rigid objects, which makes the first part of 0.17 all the more remarkable. The second part, namely $U_I \cong V_I$, should be viewed as a strong form of Ext vanishing (see 9.30), and this has many consequences in the study of twist autoequivalences, developed further in Part 4.

The above combines to give the following general form of the Affine Auslander–McKay Correspondence. Again, for ease of exposition, here we restrict to the case where $\mathcal{R}$ has only isolated singularities.

THEOREM 0.18 (Affine Auslander–McKay Correspondence, general version). *Let $\mathcal{R}$ be cDV singularity of type Δ, with isolated singularities. Fix $N \in \operatorname{modif}\mathcal{R}$.*

(1) *There exist a subset $\mathcal{J} \subseteq \Delta$ and horizontal bijective maps*

$$\begin{array}{ccc} \mathrm{MM}^N\mathcal{R} & \xrightarrow{\sim} & \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J}) \\ \uparrow & & \uparrow \\ \mathrm{MMG}^N\mathcal{R} & \xrightarrow{\sim} & \mathsf{Cham}(\Delta, \mathcal{J}) \end{array}$$

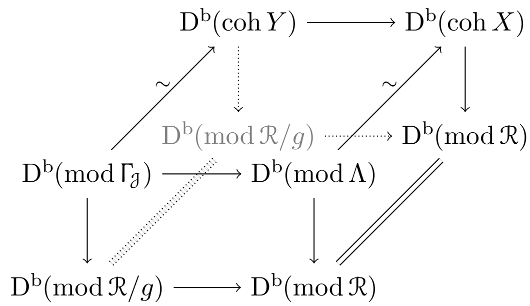
such that the diagram commutes. Furthermore, $\mathrm{MM}^N\mathcal{R}$ *coincides with the full mutation classes of N, and* $\mathrm{MMG}^N\mathcal{R}$ *coincides with the Cohen–Macaulay mutation class of N.*

(2) *Wall crossing corresponds to mutation.*

Part 4. Applications to Birational Geometry. Given a cDV singularity $\mathcal{R}$, and crepant birational map $X \to \operatorname{Spec}\mathcal{R}$ where X has only Gorenstein terminal singularities (e.g. a minimal model), then for generic $g \in \mathcal{R}$ consider the pullback diagram

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \operatorname{Spec}\mathcal{R}/g & \longrightarrow & \operatorname{Spec}\mathcal{R} \end{array} \tag{0.0.A}$$

By Reid's general elephant, the ring $\mathcal{R}/g$ is a Kleinian singularity, and the left hand morphism is a *partial* crepant resolution of singularities. Given the fibre dimension is at most one, there are canonical tilting bundles on X and Y. The endomorphism ring of the tilting bundle on X will be denoted Λ, and it is well-known that $\Lambda \cong \operatorname{End}_{\mathcal{R}}(M)$ for some $M \in \operatorname{modif} \mathcal{R}$. The endomorphism ring of the tilting bundle on Y is isomorphic to $\Gamma_{\mathcal{J}} = e_{\mathcal{J}} \Pi e_{\mathcal{J}}$, where Π is the preprojective algebra of extended Dynkin type, and $\mathcal{J} \subseteq \Delta$ is some subset of the vertices of the non-extended Dynkin diagram. This is summarised in the following commutative diagram.



Problems on the geometry of the back square can thus be transferred to the front square, involving $\Gamma_{\mathcal{J}}$ and Λ, where the techniques of the previous Parts come to the fore. Being derived equivalent to Y, the homological algebra of $\Gamma_{\mathcal{J}}$, developed in Part 2, thus controls partial crepant resolutions of Kleinian singularities, which are surfaces. In contrast, being derived equivalent to X, the homological algebra of Λ developed in Part 3 controls the 3-folds $X \to \operatorname{Spec} \mathcal{R}$, where $\mathcal{R}$ is cDV. Thus, as a special case, it controls all terminal 3-fold flopping contractions.

Our applications now split into two, depending on the dimension.

Surfaces. For simplicity, consider first $g\colon Y \to \mathbb{C}^2/\mathbb{Z}_3$, the minimal resolution of the $\mathbb{Z}_3$-Kleinian surface singularity, although all the arguments do work generally. It is well known that in this case the fibre above the origin, with reduced scheme structure, is $C_1 \cup C_2$, with both $C_i \cong \mathbb{P}^1$. To each of these curves we can associate the sheaf $E_i := \mathcal{O}_{C_i}(-1) \in \operatorname{coh} Y$, and these are examples of *spherical objects*, namely they satisfy

$$\operatorname{Ext}^t_Y(E_i, E_i) \cong \begin{cases} \mathbb{C} & \text{if } t = 0, 2, \\ 0 & \text{else,} \end{cases}$$

and $E_i \otimes_Y \omega_Y \cong E_i$. Thus, by [ST], we obtain two derived autoequivalences T_1 and T_2. It is not difficult to show that the relation $T_1 \circ T_2 \circ T_1 \cong T_2 \circ T_1 \circ T_2$ holds, and so there is an induced group homomorphism

$$\varphi\colon \mathsf{Br}_2 \to \operatorname{Auteq} \mathrm{D^b}(\operatorname{coh} Y), \tag{0.0.B}$$

where Br_2 is the braid group. With some more work, φ is even injective [BT].

On the other hand, the whole scheme-theoretic fibre $C := g^{-1}(0)$ gives a sheaf $\mathcal{O}_C$, and this also turns out to be spherical, and thus give another derived autoequivalence T_C. Adding this to the existing group above, it is easy to check that T_C, T_1, T_2 satisfy the relations of the affine braid group, and so φ lifts to a group homomorphism

$$\widetilde{\varphi}\colon \widetilde{\mathsf{Br}}_2 \to \operatorname{Auteq} \mathrm{D^b}(\operatorname{coh} Y), \tag{0.0.C}$$

which we will refer to as the *affine action* on the derived category.

Our results on contracted preprojective algebras generalise this to any partial crepant resolution of any Kleinian singularity. The caveat is that, for general partial resolutions, the best we can hope for is a *pure* braid-type action, due to the presence of other partial resolutions in the same derived equivalence class. This manifests itself as the fundamental group in the following result.

As notation, consider a partial crepant resolution $Y \to \mathbb{C}^2/G$ for some finite subgroup $G \leq \mathrm{SL}(2,\mathbb{C})$. As Y is dominated by the minimal resolution, it can be obtained by blowing down a subset $\mathcal{J}$ of curves in the minimal resolution, and thus by McKay correspondence a subset $\mathcal{J}$ of an ADE Dynkin configuration. From Part 1, consider the associated finite hyperplane arrangement $\mathsf{Cone}(\Delta,\mathcal{J})$ and infinite hyperplane arrangement $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$, both of which are hyperplane arrangements inside $\mathbb{R}^{|\mathcal{J}^c|}$. Write $\mathcal{X}$ for the complexification of $\mathsf{Cone}(\Delta,\mathcal{J})$, and $\mathcal{X}_{\mathsf{aff}}$ for the complexification of $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$.

THEOREM 0.19 (10.4). *Consider a partial crepant resolution* $Y \to \mathbb{C}^2/G$ *for some finite subgroup* $G \leq \mathrm{SL}(2,\mathbb{C})$*, with associated* $\mathcal{X}$ *and* $\mathcal{X}_{\mathsf{aff}}$ *as above. Then there exist group homomorphisms such that the following diagram commutes.*

$$\begin{array}{ccc} \pi_1(\mathcal{X}) & \xrightarrow{\varphi} & \operatorname{Auteq} \mathrm{D}^{\mathrm{b}}(\operatorname{coh} Y) \\ \downarrow & \nearrow_{\widetilde{\varphi}} & \\ \pi_1(\mathcal{X}_{\mathsf{aff}}) & & \end{array}$$

The above is in fact a direct consequence of a more general statement about the existence of a functor from the Deligne groupoid; we refer the reader to 5.35 for more details. The groupoid viewpoint illustrates one key difference between the classical case of the minimal resolution (which is $\mathcal{J} = \emptyset$) and here. Namely, in the formula for simple wall crossings, $J \mapsto J + i - \iota_{J+i}(i)$, and so J changes, in general. This translates into the categories in the groupoid not being equal, and thus we *must* monodromy in order to guarantee autoequivalences.

Thus, in general, the π_1 actions in 0.19 are the best that we can hope for. The homomorphism φ should be thought of as generalising and extending the *pure* braid group action on the minimal resolution in (0.0.B) to partial resolutions. The homomorphism $\widetilde{\varphi}$ generalises and extends the *pure* affine braid group action in (0.0.C). Also, as usual, the arrangements $\mathsf{Cone}(\Delta,\mathcal{J})$ and $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$ need not be Coxeter, and so there is no braid or affine braid group to aim for.

However, in some cases, most notably when $\mathcal{J} = \emptyset$ but also in many others examples, there will be some wall crossings in which the categories in each side of the wall are equal. In this case, it is reasonable to expect that wall crossing is given by some twist, over some possibly noncommutative base. Indeed, this is the case, and more surprisingly some new phenomena appear.

As notation, suppose that our partial crepant resolution $Y \to \mathbb{C}^2/G$ corresponds to $\mathcal{J} \subset \Delta$. For $i \in \Delta_{\mathsf{aff}} \backslash \mathcal{J}$, set $\mathcal{S}_i = \mathcal{O}_{\mathrm{C}_i}(-1)$ if i is not the extended vertex, and $\mathcal{S}_i = \omega_{\mathrm{C}}[1]$ if i is the extended vertex. In all cases, consider the noncommutative deformation theory (see e.g. [**DW2**]) of $\mathcal{S}_i$, which in this setting is representable. Write $\mathcal{E}_i$ for the universal sheaf, with endomorphism ring $\Gamma_{\mathcal{J},i}$.

PROPOSITION 0.20 (10.5). *Suppose that in a simple wall crossing,* $\omega_i(x,\mathcal{J}) = (xx_0,\mathcal{J})$*, i.e. the second term* $\mathcal{J}$ *does not change. Then the following hold.*

(1) $\mathcal{E}_i$ *is perfect, as a complex in* $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} Y)$.
(2) *There is an autoequivalence* Twist_i *fitting into a functorial triangle*

$$\mathbf{R}\mathrm{Hom}_Y(\mathcal{E}_i,-) \otimes^{\mathbf{L}}_{\Gamma_{\mathcal{J},i}} \mathcal{E}_i \to (-) \to \mathsf{Twist}_i(-) \to$$

The algebras $\Gamma_{\mathcal{J},i}$ are finite dimensional and self-injective, but they are not symmetric in general. For example, in 10.3 we obtain a spherical twist over the exterior algebra in two variables. This is a new example of a natural geometric autoequivalence over a noncommutative base, and is the first where the base is not a symmetric algebra. In particular, the cotwist is not the identity.

REMARK 0.21. In some cases, it may be the case that $\mathcal{J} \neq \emptyset$ (so, Y is not the minimal resolution), but yet all wall crossing rules satisfy the conditions in 0.20. In this case, the

π_1 actions in 0.19 can be improved substantially, as we no longer require to monodromy to obtain autoequivalences. Using this observation, many partial resolutions of ADE singularities admit braid and affine braid group actions, but crucially these braid actions need *not* be of ADE type. Example 10.3 constructs an action of type B_2 on a certain partial resolution of the D_4 surface singularity. By 4.20 (see also Example 10.3) there also exists partial resolutions of the E_6, E_7 and E_8 surface singularities that admit braid actions of type G_2.

Whilst 0.20 gives an intrinsic description of wall crossing, in some cases, in terms of twist functors and noncommutative deformation theory, we remark that we do not give an intrinsic geometric description of monodromy. There is an algebraic description, via tensoring by compositions of the ideals in 0.7, but an intrinsic twist-functor characterisation requires *derived* noncommutative deformation theory, since Y has canonical singularities. Booth describes the image of some monodromy under the finite action φ in [**B1**]; the general case remains open.

A small extension of the techniques in [**HW1**] then gives the following, which asserts that the finite action is faithful.

Theorem 0.22 (10.19). *The homomorphism φ in 0.19 is injective.*

We conjecture that our affine action is also faithful. However, even in the case of the classical affine braid group action on minimal resolutions, this is still not known in general. The papers [**IU, IUU**] establish this for minimal resolutions of cyclic groups (Type A), and are still the state-of-the-art.

The other main corollary of the results in the previous sections, and in particular of 0.7, is the implication (2)⇒(1) in the following. As above, by McKay correspondence we can identify partial crepant resolutions with subsets $\mathcal{J}$ of nodes of the associated ADE Dynkin diagram.

Corollary 0.23 (10.8). *Suppose that $Y \to \mathbb{C}^2/G$ and $Y' \to \mathbb{C}^2/G'$ are crepant partial resolutions, with associated $\mathcal{J} \subset \Delta_{\mathsf{aff}}$ and $\mathcal{J}' \subset \Delta'_{\mathsf{aff}}$. Consider the following conditions.*

1. *Y is derived equivalent to Y'.*
2. *$G \cong G'$ (equivalently $\Delta = \Delta'$) and up to symmetries of the extended ADE graph, $\mathcal{J}$ and $\mathcal{J}'$ can be linked through a sequence of iterated wall-crossing moves.*

Then (2)⇒(1). Further, if either $\Delta, \Delta' \in \{A_n, D_4, D_5, D_6, D_7, E_6, E_7, E_8\}$ then (1)⇒(2).

We conjecture that (1)⇒(2) is always true, and indeed this would follow from the stronger algebraic Conjecture 0.9.

Threefolds. The main applications of this memoir are to 3-folds. For the ease of exposition in this introduction, we restrict to the special case when $X \to \operatorname{Spec} \mathcal{R}$ is a flopping contraction, where X has only Gorenstein terminal singularities; necessarily $\mathcal{R}$ is isolated cDV. Many of the results below generalise to crepant partial resolutions of arbitrary cDV singularities.

The partial order on the tilting theory, and using 0.15 above, first allows us to elegantly recover, via an independent proof, the braiding of flop functors in [**DW3**, 1.1]. The following is stated globally, but it follows from the complete local case. In that setting, the technical assumption that the curves are independently floppable automatically holds.

Corollary 0.24 (10.11). *Suppose that $X \to X_{\mathrm{con}}$ is a flopping contraction between quasi-projective 3-folds, contracting precisely two independently floppable irreducible curves. If X has at worst Gorenstein terminal singularities, then*

$$\underbrace{\mathsf{F}_1 \circ \mathsf{F}_2 \circ \mathsf{F}_1 \circ \cdots}_{d} \cong \underbrace{\mathsf{F}_2 \circ \mathsf{F}_1 \circ \mathsf{F}_2 \circ \cdots}_{d}$$

where d is the number of hyperplanes in $\mathsf{Cone}(\Delta, \mathcal{J})$, where $\mathcal{J} \subseteq \Delta$ is the Dynkin type of the flopping contraction.

The number d is called the *length* of the braid relation. Whilst 0.24 above recovers a known result, from here onwards the results are new. Theorem 0.3 now gives very precise information on the possible braid relation lengths for 3-fold flops.

COROLLARY 0.25 (10.12). *Suppose that $X \to X_{\mathrm{con}}$ is a flopping contraction between quasi-projective 3-folds, as in 0.24. Then the length of the braid relation is either* $2, 3, 4, 5, 6$*, or* 8*. The first case, namely $d = 2$, holds if and only if the curves are disjoint.*

We are next able to extend previously known constructions. As one example, the following was established in [DW3] with an additional technical assumption that X is $\mathbb{Q}$-factorial. We can now drop this assumption, treating all simples same, and putting them on the same footing. There is a much more general version of the following result stated in 10.13; here we highlight only the second part.

THEOREM 0.26 (10.13). *Suppose that $f\colon X \to X_{\mathrm{con}}$ is a flopping contraction of quasi-projective 3-folds, where X has only Gorenstein terminal singularities. Let $\mathcal{E}_{\mathrm{fib}}$ be the universal object of the noncommutative deformation theory of $\mathcal{O}_{\mathrm{C}}$, and set $\mathrm{A}_{\mathrm{fib}} = \mathrm{End}_X(\mathcal{E}_{\mathrm{fib}})$. Then there is a fibre twist autoequivalence* FTwist, *together with a functorial triangle*

$$\mathbf{R}\mathrm{Hom}_X(\mathcal{E}_{\mathrm{fib}}, x) \otimes^{\mathbf{L}}_{\mathrm{A}_{\mathrm{fib}}} \mathcal{E}_{\mathrm{fib}} \to x \to \mathsf{FTwist}(x) \to$$

A consequence of our earlier tilting results in Part 2 is that there are a *lot* more autoequivalences than this. The following is one of main results, and is new even in the case when X is smooth. For convenience, we state the following locally. Recall that to $X \to \operatorname{Spec} \mathcal{R}$ we can associate a partial crepant resolution of a Kleinian singularity by (0.19), and thus associate a subset $\mathcal{J}$ of an ADE Dynkin diagram Δ exactly as above 0.19, with complexifications $\mathcal{X}$ and $\mathcal{X}_{\mathsf{aff}}$.

THEOREM 0.27 (10.14). *Let $X \to \operatorname{Spec} \mathcal{R}$ denote a 3-fold flopping contraction, where X has only Gorenstein terminal singularities. Then there are group homomorphisms such that the following diagram commutes.*

$$\begin{array}{ccc} \pi_1(\mathcal{X}) & \xrightarrow{\varphi} & \operatorname{Auteq} \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) \\ \downarrow & \nearrow_{\widetilde{\varphi}} & \\ \pi_1(\mathcal{X}_{\mathsf{aff}}) & & \end{array}$$

The above is a consequence of a more general result (explained in 9.36) that the mutation functors between $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} \operatorname{End}_R(M))$, where $M \in \mathrm{MM}^N R$, form a representation of the corresponding Deligne groupoid. A small extension of the techniques in [HW1] then gives the following, which asserts that the finite action is faithful.

THEOREM 0.28 (10.19). *The homomorphism φ in 0.27 is injective.*

Summary Theorem. We end this introduction with a summary theorem to illustrates the commonalities amongst all the four parts of the memoir. Given a subset $\mathcal{J}$ of nodes in an ADE Dynkin diagram Δ, we can associate:

- A partial crepant resolution $Y \to \mathbb{C}^2/G$, given by blowing down the curves in $\mathcal{J}$ from the minimal resolution.
- A contracted preprojective algebra $\Gamma_{\mathcal{J}} = e_{\mathcal{J}} \Pi e_{\mathcal{J}}$, where Π is the preprojective algebra of extended type Δ_{aff}.
- A flopping contraction $X \to \operatorname{Spec} \mathcal{R}$, where X has only terminal singularities, which slices to Y under generic $g \in \mathcal{R}$.
- To this flopping contraction, via Auslander–McKay, is a corresponding $N \in \operatorname{modif} \mathcal{R}$ such that $\mathcal{R} \in \operatorname{add} N$.
- A finite hyperplane arrangement $\mathsf{Cone}(\mathcal{J})$ inside $\mathbb{R}^{|\mathcal{J}^c|}$.
- The $\mathcal{J}$-Tits cone $\mathsf{Cone}(\Delta_{\mathsf{aff}}, \mathcal{J})$, and its level $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$ which is an infinite hyperplane arrangement inside $\mathbb{R}^{|\mathcal{J}^c|}$.

Alas, it is not possible in general to construct such an $X \to \operatorname{Spec} \mathcal{R}$ above where X is also smooth [KM]. Indeed, this is why the techniques in Part 3 are necessary.

The following summary theorem links all these notions, through a series of bijections.

THEOREM 0.29. *Let Δ be ADE Dynkin, and $\mathcal{J} \subset \Delta$ be arbitrary. With notation as above, there exist bijections between the following sets.*

(1) *Chambers of* $\mathsf{Cone}(\Delta_{\mathsf{aff}}, \mathcal{J})$.
(2) *Alcoves in* $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$.
(3) *Classical tilting modules for the contracted preprojective algebra* $\Gamma_{\mathcal{J}} = e_{\mathcal{J}} \Pi e_{\mathcal{J}}$.
(4) *Reflexive classical tilting modules for* $\operatorname{End}_{\mathcal{R}}(N)$.
(5) *Elements in the mutation class of* N.
(6) *Reflexive $\mathcal{R}$-modules with the same number of indecomposable summands as N, which furthermore admit a two-term approximation by* $\operatorname{add} N$.

If further $X \to \operatorname{Spec} \mathcal{R}$ is minimal model (equivalently, N is a maximal rigid module), the above sets are further in bijection with:

(7) *Maximal rigid $\mathcal{R}$-modules.*

If further X is smooth (equivalently, N is a cluster tilting object), the above sets are further in bijection with:

(8) *Cluster tilting $\mathcal{R}$-modules.*

In all cases, mutation corresponds to wall crossing.

The above bijections are set-theoretic, with an action by mutation/wall crossing. One level up, we have already seen above that these lift to the next categorical level, namely the corresponding categories and mutation functors form a representations of the corresponding Deligne groupoid. One further level up, summarised in the next subsection, all this information glues together to describe the stability manifold associated to X.

The finite analogue of 0.29 is the following, which may be of independent interest.

THEOREM 0.30. *If $\mathcal{J} \subset \Delta$ with Δ ADE Dynkin, then there are bijections between the following sets.*

(1) *Chambers of the finite arrangement* $\mathsf{Cone}(\Delta, \mathcal{J}) \subseteq \mathbb{R}^{|\mathcal{J}^c|}$.
(2) *Classical tilting modules for $\Gamma_{\mathcal{J}}$ which contain $e_0 \Pi e_0$ as a summand.*
(3) *Reflexive classical tilting modules for* $\operatorname{End}_{\mathcal{R}}(N)$ *containing summand* $\operatorname{Hom}_{\mathcal{R}}(N, \mathcal{R})$.
(4) *Elements in the mutation class of N containing $\mathcal{R}$ as a summand.*
(5) *Elements in the Cohen–Macaulay mutation class of* N.
(6) $L \in \operatorname{CM} \mathcal{R}$ *with the same number of summands of N, which have a two-term approximation by* $\operatorname{add} N$.

Again the above can be categorified, and lifted to the Deligne groupoid and Bridgeland stability manifold levels.

Further Uses. There are many further consequences of the results above, many of which appear elsewhere. As a quick summary, given a flopping contraction $f \colon X \to \operatorname{Spec} \mathcal{R}$, with scheme fibre C, consider the subcategories

$$\mathcal{C} := \{\mathcal{F} \in \mathrm{D^b}(\operatorname{coh} X) \mid \mathbf{R} f_* \mathcal{F} = 0\}$$
$$\mathcal{D} := \{\mathcal{F} \in \mathrm{D^b}(\operatorname{coh} X) \mid \operatorname{Supp} \mathcal{F} \subseteq \mathrm{C}\}.$$

Further consequences of this memoir include the following.

(1) The $\mathcal{J}$-Tits cones $\mathsf{Cone}(\Delta_{\mathsf{aff}}, \mathcal{J})$ and their levels give a description of Bridgeland stability conditions on $\mathcal{D}$ [HW2]. Indeed, there is a component of normalised stability conditions $\operatorname{Stab}_n^{\circ} \mathcal{D}$ such that the forgetful map

$$\operatorname{Stab}_n^{\circ} \mathcal{D} \to \mathcal{X}_{\mathsf{aff}}$$

is a regular covering map, with Galois group given by the image of the homomorphism $\widetilde{\varphi}$ from 0.27. Even in the case when X is smooth, the hyperplane technology in this memoir is heavily required to prove this result.

(2) The finite $\mathsf{Cone}(\Delta, \mathcal{J})$ describe stability conditions on $\mathcal{C}$, again in [**HW2**]. There is a component of stability conditions such that the forgetful map
$$\mathrm{Stab}^{\circ}\mathcal{C} \to \mathcal{X}$$
is the universal covering map, with Galois group given by the image of the homomorphism φ from 0.27. By 0.22, this group is isomorphic to $\pi_1(\mathcal{X})$.

(3) Autoequivalence groups. Let $\mathrm{PBr}\,\mathcal{C}$ denote the image of φ, and $\mathrm{APBr}\,\mathcal{D}$ denote the image of $\widetilde{\varphi}$. It turns out that for suitably defined autoequivalences preserving the stability manifold, $\mathrm{Aut}^{\circ}\mathcal{D} \cong \mathrm{APBr}\,\mathcal{D} \rtimes \mathrm{Pic}\,X$, and $\mathrm{Aut}^{\circ}\mathcal{C} \cong \mathrm{PBr}\,\mathcal{C}$ [**HW2**].

(4) The local wall-crossing rules developed here, together with the stability condition results above, allow for the first full computation of the Stingy Kähler Moduli Space (SKMS) in the flops setting [**DW5**]. Furthermore, the mutation results, most notably 0.17, allow for a full geometric description of monodromy on the SKMS to be realised, in terms of twist functors.

(5) Via Aulander–McKay, $f\colon X \to \mathrm{Spec}\,\mathcal{R}$ corresponds to some $N \in \mathrm{modif}\,\mathcal{R}$. The algebra $\mathrm{A}_{\mathrm{con}} := \underline{\mathrm{End}}_{\mathcal{R}}(N)$ is called the *contraction algebra*; it is a finite dimensional symmetric algebra. The strong form of mutation in 0.17 allows for a description of stability conditions on an arbitrary contraction algebra $\mathrm{A}_{\mathrm{con}}$, not just those from smooth minimal models, via $\mathsf{Cone}(\Delta, \mathcal{J})$ [**AW**]. It turns out that the forgetful map from the full space of Bridgeland stability conditions
$$\mathrm{Stab}\,(\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\mathrm{A}_{\mathrm{con}})) \to \mathcal{X}$$
is the universal cover. In turn, since $\mathrm{A}_{\mathrm{con}}$ is silting discrete, this establishes a homological proof of $K(\pi, 1)$ for all intersection arrangements inside ADE root systems [**AW**].

Conventions. All modules are left modules. If A is a ring, $\mathrm{mod}\,A$ is the category of finitely generated A-modules, and $\mathrm{proj}\,A$ is the subcategory of those projective modules. For $M \in \mathrm{mod}\,A$, we write $\mathrm{add}\,M$ to denote all summands of finite sums of M, and say that M is a *generator* if $A \in \mathrm{add}\,M$.

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Part 1

The Combinatorics of Tits Cone Intersections

CHAPTER 1

$\mathcal{J}$-Cones, $\mathcal{J}$-Chambers and Wall Crossing

This chapter investigates intersection arrangements inside Tits cones, in various levels of generality, and is fundamental to all what follows.

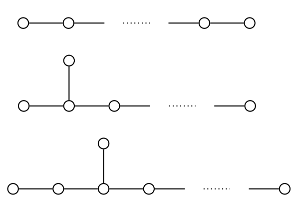
1.1. Coxeter Preliminaries

Recall that a symmetric $n \times n$ matrix $M = (m_{ij})$, with entries in the set $\{1, 2, \ldots, \infty\}$, is called *Coxeter* if $m_{ij} = 1 \iff i = j$. As is standard, M can be represented by a *Coxeter graph* $\Delta = (\Delta, \Delta_1)$ whose nodes are $\Delta := \{1, \ldots, n\}$ and where we draw an edge between i and j if and only if $m_{ij} \geq 3$. The edges with $m_{ij} \geq 4$ are labelled by that number. Typical examples arise from the following simply laced Dynkin graphs

A_n $(n \geq 1)$

D_n $(n \geq 4)$

E_n $(n = 6, 7, 8)$



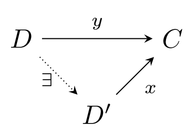
however, throughout when we refer to *Dynkin diagrams*, we will also allow for the non-simply laced cases $B_n = C_n$, F_4, G_2, $H_{2,3,4}$, and I_n.

The matrix $M = (m_{ij})$ determines the *Coxeter group* W_Δ, which is given abstractly as the group generated by $\{s_i \mid i \in \Delta\}$, subject to the relations

(1) $s_i^2 = 1$ for any $i \in \Delta$.
(2) $\underbrace{\ldots \circ s_j \circ s_i}_{m_{ij}} = \underbrace{\ldots \circ s_i \circ s_j}_{m_{ij}}$ for all $i, j \in \Delta$ with $i \neq j$.

For $x \in W_\Delta$, we say that an expression $x = s_{i_1} s_{i_2} \cdots s_{i_k}$ is *reduced* if k is smallest possible. In this case we write $k = \ell(x)$. We denote by $\leq$ the *right order* (=*weak order*) on the Coxeter group, that is, we write $x \leq y$ if $\ell(y) = \ell(x) + \ell(x^{-1}y)$ holds. Topologically, viewing elements of W_Δ as paths in the associated hyperplane arrangement ending at a fixed chamber C,

$x \leq y \iff$



Notation 1.1. To set notation, for a subset $J \subseteq \Delta$,

(1) If $j \in J$ and $i \in \Delta \backslash J$, we will write
$$J + i := J \sqcup \{i\},$$
$$J - j := J \backslash \{j\}.$$
(2) Write $J^c = \Delta \backslash J$, and note that this corresponds to the full subgraph obtained from Δ by removing the vertices in J.
(3) Consider the subgroup $W_J := \langle s_i \mid i \in J \rangle$ of W_Δ, which is called a *parabolic subgroup*. This is isomorphic to the Coxeter group associated with the full subgraph of Δ with vertices J, see e.g. [**BB**, 2.4.1(i)].

1.2. The Finite Case

Although we will not always be considering the case when W_Δ is finite, later we will sometimes have finite, or at least locally finite, assumptions, so we briefly recall some known facts here.

When W_Δ is finite we will write w_Δ for the *longest element* in W_Δ. For reference later, the length of the w_Δ for ADE Δ is summarised in the following table.

(1.2.A)

Δ	A_n	D_n	E_6	E_7	E_8
$\ell(w_\Delta)$	$\frac{n(n+1)}{2}$	$n(n-1)$	36	63	120

The following observation is basic, as is well-known. Throughout, we regard W_Δ as a poset with respect to the right order $\leq$.

LEMMA 1.2. *Let $W = W_\Delta$ be a finite Coxeter group, and w_Δ be the longest element.*

(1) *$(-)w_\Delta \colon W \to W$ and $w_\Delta(-) \colon W \to W$ are anti-automorphisms, and*

$$w_\Delta(-)w_\Delta \colon W \to W$$

is an automorphism of the poset W.

(2) *There exists an automorphism $\iota = \iota_\Delta \colon \Delta \to \Delta$ of the graph Δ such that for all $i \in \Delta$,*

$$w_\Delta s_i w_\Delta = s_{\iota(i)}.$$

(3) *For all $i \in \Delta$,*
   (a) $w_{\Delta - i} w_\Delta w_{\Delta-\iota(i)} w_\Delta = 1.$
   (b) $w_{\Delta - i} w_\Delta = w_\Delta w_{\Delta-\iota(i)}.$

PROOF. (1) is [BB, 3.1.5(i)(ii)], and (2) and (3) are consequences of (1). □

By 1.2, when W_Δ is finite, certainly w_Δ satisfies $w_\Delta w_\Delta = 1$, and further it induces an involution ι_Δ of the graph Δ. Again, although we do not need this until much later, for reference the involution ι_Δ for ADE Δ is summarised in the following table:

(1.2.B)

Δ	A_n	D_{2n}	D_{2n+1}	E_6	E_7	E_8
ι_Δ	!	*id*	!	!	*id*	*id*

where *id* is the identity, and ! is the unique non-trivial involution.

1.3. The Tits Cone

Let $\Delta = (\Delta, \Delta_1)$ be the Coxeter graph arising from a Coxeter matrix $M_\Delta = (m_{ij})$. Following [H, §5.13], we now recall the Tits cone associated with W_Δ. Let V be the $\mathbb{R}$-vector space with basis $\{\alpha_i \mid i \in \Delta\}$, and B the symmetric bilinear form on V defined by

$$B(\alpha_i, \alpha_j) = -\cos(\pi/m_{ij}).$$

Then the Coxeter group W_Δ acts on V by

$$s_i(v) := v - 2\,B(\alpha_i, v)\alpha_i,$$

and let $\Phi := \{x(\alpha_i) \mid i \in \Delta,\ x \in W_\Delta\}$ denote the set of *roots*.

The V^* be the dual space of V. To ease notation with duals, and so as to match the notation in [W2, §5], write $V^* = \Theta$. Note that Θ has basis $\{\alpha_i^* \mid i \in \Delta\}$. Throughout, elements of Θ will be written with respect to this basis, so $\vartheta \in \Theta$ will mean

$$\vartheta = (\vartheta_i)_{i \in \Delta} = \sum_{i \in \Delta} \vartheta_i \alpha_i^*.$$

The group W_Δ acts on Θ by $(x\vartheta)(v) = \vartheta(x^{-1}v)$ for all $x \in W_\Delta$, $\vartheta \in \Theta$ and $v \in V$. Each $i \in \Delta$ induces a hyperplane in Θ, namely

$$H_i := \{\vartheta \in \Theta \mid \vartheta_i = 0\},$$

and a decomposition

$$\Theta = H_i^- \sqcup H_i \sqcup H_i^+,$$

where $H_i := H_{\alpha_i}$, and H_i^+ and H_i^- are the half-spaces

$$H_i^+ = \{\vartheta \in \Theta \mid \vartheta_i > 0\}$$
$$H_i^- = \{\vartheta \in \Theta \mid \vartheta_i < 0\}.$$

Below, it will be convenient to consider the upper orthant $\overline{C}$ in Θ, defined as

$$\overline{C} := \{\vartheta \in \Theta \mid \vartheta_i \geq 0 \text{ for all } i \in \Delta\} = \bigcap_{i \in \Delta} (H_i \sqcup H_i^+),$$

and its open interior $C := \{\vartheta \in \Theta \mid \vartheta_i > 0 \text{ for all } i \in \Delta\} = \bigcap_{i\in\Delta} H_i^+$.

Definition 1.3. Suppose that Δ is a Coxeter graph.

(1) The *Tits cone* $\mathsf{Cone}(\Delta)$ is defined to be

$$\mathsf{Cone}(\Delta) := \bigcup_{x \in W_\Delta} x(\overline{C}).$$

(2) A *Weyl chamber* is an open subset of $\mathsf{Cone}(\Delta)$ of the form $x(C)$ for some $x \in W_\Delta$.

Weyl chambers are the connected components of $\mathsf{Cone}(\Delta) \backslash \bigcup_{\alpha\in\Phi} H_\alpha$, where

$$H_\alpha := \{\vartheta \in \Theta \mid \vartheta(\alpha) = 0\}.$$

It is well known that the Tits cone spans the whole of Θ (that is, $\mathsf{Cone}(\Delta) = \Theta$) if and only if W_Δ is finite (see e.g. [**Wan**, Prop 4.3A]).

1.4. $\mathcal{J}$-cones and $\mathcal{J}$-chambers

This subsection considers intersections of the Tits cone $\mathsf{Cone}(\Delta)$ with certain subspaces, and produces the combinatorial objects that will be needed later. These intersections need not give Coxeter arrangements; however, they still exhibit somewhat remarkable behaviour. Some limited examples are given throughout, with the understanding that many more can be found in Chapter 3.

Definition 1.4. For a subset J of Δ, we set

$$C_J := \left\{ \vartheta \in \Theta \;\middle|\; \begin{array}{ll} \vartheta_i = 0 & \text{if } i \in J, \\ \vartheta_i > 0 & \text{if } i \notin J \end{array} \right\},$$

that is $C_J = \left(\bigcap_{i\in J} H_i\right) \cap \left(\bigcap_{i \in J^c} H_i^+\right)$.

The degenerate case is $J = \emptyset$, when $C_\emptyset = C$. Every element in C_J has stabilizer W_J, and the Tits cone decomposes, although not into chambers, as

$$\mathsf{Cone}(\Delta) = \bigsqcup_{J \subseteq \Delta} \bigsqcup_{x \in W/W_J} x(C_J). \tag{1.4.A}$$

The following is our key new definition, which is motivated by finding an affine version of [**W2**, §5] and [**DW3**].

Definition 1.5. For every subset $\mathcal{J} \subseteq \Delta$, consider the following.

(1) The subspace $\Theta_\mathcal{J}$ of Θ, defined as

$$\Theta_\mathcal{J} := \{\vartheta \in \Theta \mid \vartheta_i = 0 \text{ if } i \in \mathcal{J}\}.$$

That is, $\Theta_\mathcal{J}$ is the subspace with basis $\{\alpha_i^* \mid i \notin \mathcal{J}\}$. Note that $\Theta_\mathcal{J} := \bigcap_{i\in\mathcal{J}} H_i$.

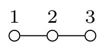
(2) We define the $\mathcal{J}$-cone to be the intersection
$$\mathsf{Cone}(\Delta, \mathcal{J}) := \mathsf{Cone}(\Delta) \cap \Theta_{\mathcal{J}}.$$
(3) We call $\Phi_{\mathcal{J}} := \{\alpha \in \Phi \mid \Theta_{\mathcal{J}} \not\subseteq H_\alpha\}$ the set of $\mathcal{J}$-*roots*.

Once $\mathcal{J}$ has been fixed, we can consider $x \in W_\Delta$ and another $J \subseteq \Delta$. Note that either $x(C_J) \subseteq \mathsf{Cone}(\Delta, \mathcal{J})$, or $x(C_J) \cap \mathsf{Cone}(\Delta, \mathcal{J}) = \emptyset$ holds, therefore it follows from (1.4.A) that $\mathsf{Cone}(\Delta, \mathcal{J})$ decomposes, although again not into chambers, as
$$\mathsf{Cone}(\Delta, \mathcal{J}) = \bigsqcup_{J \subseteq \Delta} \bigsqcup_{\substack{x \in W/W_J, \\ x(C_J) \subseteq \Theta_{\mathcal{J}}}} x(C_J).$$
We next consider the chamber structure of $\mathsf{Cone}(\Delta, \mathcal{J})$.

Definition 1.6. Let $\Delta = (\Delta, \Delta_1)$ be a graph without loops, and $W = W_\Delta$ the Coxeter group. Fix a subset $\mathcal{J}$ of Δ.

(1) For $J \subseteq \Delta$ and $x \in W/W_J$, we say that $x(C_J)$ is an $\mathcal{J}$-*chamber* if $x(C_J) \subseteq \Theta_{\mathcal{J}}$ and $|J| = |\mathcal{J}|$.
(2) A *wall* of a $\mathcal{J}$-chamber $x(C_J)$ is the intersection of the closure of $x(C_J)$ with H_α for some $\alpha \in \Phi_{\mathcal{J}}$.

The $\mathcal{J}$-chambers are the connected components of $\mathsf{Cone}(\Delta, \mathcal{J}) \setminus \bigcup_{\alpha \in \Phi_{\mathcal{J}}} H_\alpha$.

Example 1.7. Let $\Delta = A_3 =$ 

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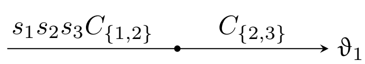
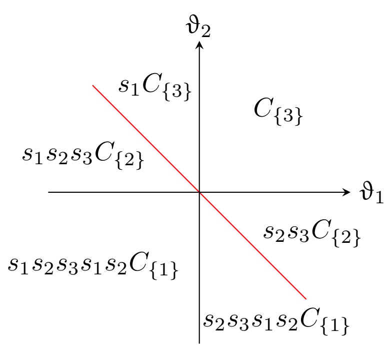
(1) If $\mathcal{J} = \{2, 3\}$, then $\mathsf{Cone}(A_3, \mathcal{J})$ is



(2) If $\mathcal{J} = \{3\}$, then $\mathsf{Cone}(A_3, \mathcal{J})$ is



$\alpha \in \Phi_{\mathcal{J}}$	Hyperplane in $\Theta_{\mathcal{J}}$
010, 011	$\vartheta_1 = 0$
100	$\vartheta_2 = 0$
110, 111	$\vartheta_1 + \vartheta_2 = 0$

We refer the reader to 2.10 and Chapter 3 for many more examples of $\mathsf{Cone}(\Delta, \mathcal{J})$. One of the key points is that $\mathsf{Cone}(\Delta, \mathcal{J})$ need not be Coxeter.

1.5. Labelling the $\mathcal{J}$-chambers

It will be convenient, especially with respect to wall crossing later, to be able to label the $\mathcal{J}$-chambers in a slightly different way.

Definition 1.8. Let $\Delta = (\Delta, \Delta_1)$ be a Coxeter graph, then for a fixed subset $\mathcal{J} \subseteq \Delta$, let $\mathsf{Cham}(\Delta, \mathcal{J})$ be the set of pairs (x, J) of elements $x \in W_\Delta$ and subsets $J \subseteq \Delta$ satisfying the following two conditions.

(1) $\ell(x) = \min\{\ell(y) \mid y \in xW_J\}$.
(2) $W_{\mathcal{J}} x = x W_J$.

By 1.14, we can identify $\mathsf{Cham}(\Delta, \mathcal{J})$ with the set of all left cosets C in $W_{\mathcal{J}} \backslash W_\Delta$ satisfying the following conditions.

(1) There exists a subset $J \subseteq \Delta$ such that C is a right coset in W_Δ / W_J.

We remark that the extreme case is $\mathcal{J} = \emptyset$, where since $W_\emptyset = \{1\}$,

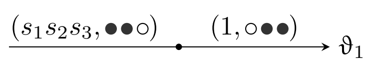
$$\mathsf{Cham}(\Delta, \emptyset) = \{(x, \emptyset) \mid x \in W_\Delta\}.$$

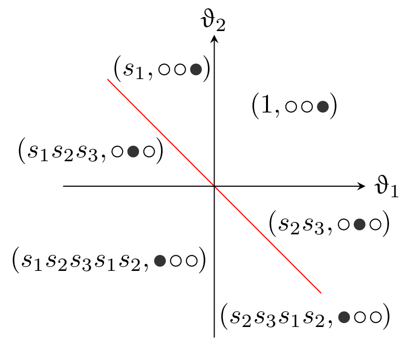
Thus in this case we can identify the set $\mathsf{Cham}(\Delta, \emptyset)$ with W_Δ.

Notation 1.9. We will depict a subset $\mathcal{J} \subseteq \Delta$ as a shading of the vertices of the graph Δ. The vertices j for which $j \in \mathcal{J}$ will be shaded (i.e. drawn $\bullet$), whilst the vertices j for which $j \notin \mathcal{J}$ will be unshaded (i.e. drawn $\circ$).

Remark 1.10. As calibration in our algebraic geometric flops setup in later chapters, $\mathcal{J}$ will correspond to the choice of curves that get contracted from the minimal resolution to describe the generic slice. Thus, the vertices not in $\mathcal{J}$, those drawn $\circ$, will correspond to the flopping curves. See Remark 10.9 for more information.

Example 1.11. Continuing the examples in 1.7, for $\mathcal{J} = \{2, 3\}$ and $\mathcal{J} = \{3\}$, the labels in $\mathsf{Cham}(\Delta, \mathcal{J})$ are, respectively





The benefit of the above approach is that wall crossing will become easier to describe visually, and we do this in the next subsection. The main result of this subsection is the following theorem, which verifies that the labelling is correct.

Theorem 1.12. *Let Δ be a Coxeter graph. Then there is a bijection from the set $\mathsf{Cham}(\Delta, \mathcal{J})$ to the set of chambers in $\mathsf{Cone}(\Delta, \mathcal{J})$, given by*

$$\mathsf{Cham}(\Delta, \mathcal{J}) \ni (x, J) \mapsto x(C_J).$$

We prove the theorem by first preparing a series of lemmas. As before, regard the Coxeter group W_Δ as a poset with respect to the right (weak) order $\leq$, and for $J \subseteq \Delta$, consider the parabolic subgroup $W_J = \langle s_i \mid i \in J \rangle$ of W_Δ.

By [BB, 2.4.4], for any element $x \in W_\Delta$, there exist unique $x^J \in xW_J$ and ${}^J x \in W_J x$ such that

$$\ell(x^J \cdot y) = \ell(x^J) + \ell(y) \quad \text{and} \quad \ell(y \cdot {}^J x) = \ell(y) + \ell({}^J x) \tag{1.5.A}$$

for all $y \in W_J$. The following observations follow immediately.

Lemma 1.13. *Let Δ be a Coxeter graph. Fix $x \in W_\Delta$ and $J \subseteq \Delta$.*

(1) *The subposet xW_J of W_Δ is isomorphic to W_J.*
(2) *If W_J is finite, then the map $(\cdot w_J)\colon xW_J \to xW_J$ is an anti-automorphism of posets. Thus xW_J has minimum element x^J and maximum element $x^J w_J$.*

Proof. For (1), by (1.5.A), the map $(x^J \cdot)\colon W_J \to x^J W_J = xW_J$ is an isomorphism of posets. Part (2) then follows immediately from (1), using 1.2(1). $\square$

The special case when a right coset coincides with a left coset is of particular interest.

Lemma 1.14. *Let Δ be a Coxeter graph. Consider subsets J and $\mathcal{J}$ of Δ, and $x \in W_\Delta$.*

(1) *The following conditions are equivalent.*

(a) $W_{\mathcal{J}}x = xW_J$.
(b) *There exists a bijection* $a\colon J \to \mathcal{J}$ *such that* $x^J s_j = s_{a(j)}x^J$ *for all* $j \in J$.

If these conditions are satisfied, then the following assertion holds.

(2) $W_{\mathcal{J}}x = xW_J$ *has minimum element* $x^J = {}^{\mathcal{J}}x$ *and maximum element* $w_{\mathcal{J}}x^J = x^J w_J$ *with respect to both of the left order and the right order.*

Proof. (1) It suffices to show that the first condition implies the second one. Since both of x^J and ${}^{\mathcal{J}}x$ are the unique elements with the minimal length in $W_{\mathcal{J}}x = xW_J$, we have $x^J = {}^{\mathcal{J}}x$. For any $i \in J$, using (1.5.A) twice, we have

$$\ell(x^J s_i (x^J)^{-1}) + \ell(x^J) = \ell(x^J s_i) = \ell(x^J) + 1$$

and so $\ell(x^J s_i (x^J)^{-1}) = 1$. Thus $x^J s_i = s_{a(i)}x^J$ holds for some $a(i) \in \Delta$. Then a has to be a bijection $J \to \mathcal{J}$ by [BB, 2.4.1(v)].
(2) We have shown $x^J = {}^{\mathcal{J}}x$. By (1.5.A), both of $x^J w_J$ and $w_{\mathcal{J}}\,{}^{\mathcal{J}}x$ are the unique elements with the maximal length in $W_{\mathcal{J}}x = xW_J$, thus they must coincide. ☐

It follows from 1.14(1) that a pair $(x, J) \in \mathsf{Cham}(\Delta, \mathcal{J})$ is uniquely determined by x, thus we can regard $\mathsf{Cham}(\Delta, \mathcal{J})$ as a subset of W_Δ. This need not be a subgroup, but nevertheless the right order on W_Δ induces a *right order* on $\mathsf{Cham}(\Delta, \mathcal{J})$, via

$$(x, J) \le (y, J') \iff x \le y.$$

Also, by 1.14(2), the following is immediate.

Corollary 1.15. *If* $W_{\mathcal{J}}x = xW_J$*, then the following are equivalent.*

(1) $(x, J) \in \mathsf{Cham}(\Delta, \mathcal{J})$.
(2) $\ell(x) = \min\{\ell(y) \mid y \in xW_J\}$.
(3) x *is the minimum element in* xW_J *with respect to the right order.*
(4) x *is the minimum element in* xW_J *with respect to the left order.*

Now we are ready to prove 1.12.

Proof of 1.12. Let $(x, J) \in \mathsf{Cham}(\Delta, \mathcal{J})$. Then $W_{\mathcal{J}}x = xW_J$ holds. Thus there exists a bijection $a\colon \mathcal{J} \to J$ such that $s_i x = x s_{a(i)}$ by 1.14(1). Comparing fixed points of $s_i = x s_{a(i)} x^{-1}$, we have $x(H_{a(i)}) = H_i$. Therefore

$$x(C_J) \subseteq \bigcap_{i \in J} x(H_i) = \bigcap_{i \in \mathcal{J}} H_i = \Theta_{\mathcal{J}}$$

holds, and $x(C_J)$ is a $\mathcal{J}$-chamber in $\mathsf{Cone}(\Delta, \mathcal{J})$.

Conversely, assume that $x(C_J)$ is a $\mathcal{J}$-chamber in $\mathsf{Cone}(\Delta, \mathcal{J})$. Then any $p \in C_J$ has a stabilizer W_J. Since $xp \in \Theta_{\mathcal{J}} = \bigcap_{i \in \mathcal{J}} H_i$, it is stablized by $W_{\mathcal{J}}$. Therefore $xW_Jx^{-1} \subseteq W_{\mathcal{J}}$ holds. Since $|J| = |\mathcal{J}|$ by definition of $\mathcal{J}$-chamber, the equality holds and so $W_{\mathcal{J}}x = xW_J$. Replacing x by the minimum element x^J in xW_J, it follows that $(x, J) \in \mathsf{Cham}(\Delta, \mathcal{J})$. ☐

1.6. Simple Wall Crossing

In this subsection we use the above labelling give a combinatorial model of simple wall crossing in the set of chambers $\mathsf{Cham}(\Delta, \mathcal{J})$. This holds under a suitable assumption on the label of the chamber, and on its wall. We remark that this assumption is local, and does not require any finiteness of the global W_Δ.

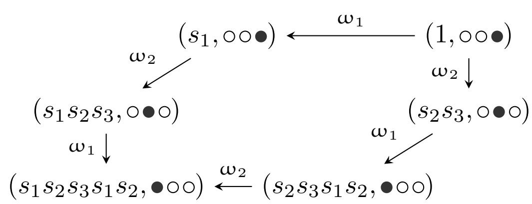
Definition 1.16. Let $\Delta = (\Delta, \Delta_1)$ be a Coxeter graph, and $\mathcal{J} \subseteq \Delta$. For any $(x, J) \in \mathsf{Cham}(\Delta, \mathcal{J})$ and $i \in J^c$ such that W_{J+i} is finite, we define *simple wall crossing* by

$$\omega_i(x, J) := (x w_J w_{J+i}, J + i - \iota_{J+i}(i)),$$

where w_J is the longest element in W_J, w_{J+i} is the longest element in W_{J+i}, and ι_{J+i} is the involution on the graph $J + i$ from 1.2(2).

REMARK 1.17. When Δ is ADE, the element $w_{\Delta - i} w_\Delta \in W_\Delta$ appears naturally in study of the Grassmannian of type Δ associated with the vertex i (e.g. [GLS]).

EXAMPLE 1.18. In the running example 1.11 with $\mathfrak{J} = \{3\}$, simple wall crossing in the set $\mathsf{Cham}(A_3, \mathfrak{J})$ is given as follows



Again we refer the reader to Chapter 3 for more substantial examples. Note that when W_Δ is infinite, it is still possible that the local assumption in 1.16 can hold at all chambers and all walls of $\mathsf{Cone}(\Delta, \mathfrak{J})$.

DEFINITION 1.19. We say that a subset $\mathfrak{J} \subsetneq \Delta$ is *strongly Dynkin* if $W_{\mathfrak{J}+i}$ is finite for all $i \in \mathfrak{J}^c$, that is the full subgraph $\mathfrak{J} + i$ of Δ is a disjoint union of Dynkin graphs for all $i \in \mathfrak{J}^c$.

For example, if Δ is Dynkin, then every subset $\mathfrak{J} \subsetneq \Delta$ is strongly Dynkin. Furthermore, if Δ is extended Dynkin, then every subset $\mathfrak{J}$ of Δ with $|\mathfrak{J}^c| \geq 2$ is strongly Dynkin.

The name *simple wall crossing* in 1.16 is justified by the following theorem, which is the main result of this subsection. Note that the assumptions become gradually stronger, as we move from (1) to (3).

THEOREM 1.20. *Let Δ be a Coxeter graph, and $\mathfrak{J}$ a subset of Δ.*

(1) *For any $(x, J) \in \mathsf{Cham}(\Delta, \mathfrak{J})$ and $i \in J^c$ such that $J + i$ is Dynkin, the following assertions hold.*
   (a) *$\omega_i(x, J)$ belongs to $\mathsf{Cham}(\Delta, \mathfrak{J})$ for any $(x, J) \in \mathsf{Cham}(\Delta, J)$ and $i \in J^c$.*
   (b) *$x < xs_i \iff (x, J) < \omega_i(x, J)$, and $x > xs_i \iff (x, J) > \omega_i(x, J)$.*
   (c) *Wall crossing is involutive, that is,*

$$\omega_{i'} \omega_i(x, J) = (x, J)$$

   for $i' := \iota_{J+i}(i)$.
   (d) *Let $(y, J') := \omega_i(x, J)$. Then the $\mathfrak{J}$-chambers $x(C_J)$ and $y(C_{J'})$ are adjacent via the wall $x(C_{J+i})$.*
(2) *If $\mathfrak{J}$ is strongly Dynkin, then the following assertions hold.*
   (a) *Any two elements in $\mathsf{Cham}(\Delta, \mathfrak{J})$ are connected by a finite sequence of wall crossings.*
   (b) *Two elements in $\mathsf{Cham}(\Delta, \mathfrak{J})$ are related by a simple wall crossing if and only if the corresponding $\mathfrak{J}$-chambers are adjacent.*
(3) *If W_Δ is finite, then $\mathsf{Cham}(\Delta, \mathfrak{J})$ has the minimum element $(1, \mathfrak{J})$ and the maximum element $(w_{\mathfrak{J}} w_\Delta, \iota_\Delta(\mathfrak{J}))$.*

The proof will be split into three propositions, and in the process we prove more than that stated above. We begin with the following lemma.

LEMMA 1.21. *Let W be a Coxeter group of a graph Δ, and $\mathfrak{J}$ a subset of Δ. For any $(x, J) \in \mathsf{Cham}(\Delta, \mathfrak{J})$, the following assertions hold.*

(1) *$W_{\mathfrak{J}} x = x W_J$ has minimum element x and maximum element $w_{\mathfrak{J}} x = x w_J$.*

(2) *If $i \in J^c$, then either $x < xs_i$ or $x > xs_i$ holds. Furthermore, the minimum and maximum elements of xW_{J+i} are*

	min element	*max element*
$x < xs_i$	x	xw_{J+i}
$x > xs_i$	$xw_J w_{J+i}$	xw_J

PROOF. By 1.13(1), we know that xW_{J+i} is isomorphic to the Coxeter group W_{J+i}.
(1) This is 1.14(2).
(2) The first assertion is a basic fact about the weak order. Suppose that $x < xs_i$, then by 1.2(2), we only have to show that x is the minimum element in xW_{J+i}. It suffices to show $x < xs_k$ holds for all $k \in J+i$. If $k \in J$, then this follows from (1). Otherwise $k = i$ holds, and we have $x < xs_i$ by our assumption.

Suppose that $x > xs_i$, then by 1.2(2), we only have to show that xw_J is the maximum element in xW_{J+i}. It suffices to show that $xw_J > xw_J s_k$ holds for any $k \in J+i$. If $k \in J$, then this follows from (1). Otherwise $k = i$ holds. Since $xs_i < x$ holds by our assumption,

$$\begin{aligned}\ell(xw_J s_i) = \ell(w_{\mathfrak{J}} x s_i) &\le \ell(w_{\mathfrak{J}}) + \ell(xs_i)\\ &< \ell(w_{\mathfrak{J}}) + \ell(x) = \ell(x) + \ell(w_J) = \ell(xw_J).\end{aligned}$$

Thus $xw_J s_i < xw_J$ holds. □

We are ready to prove 1.20(1).

PROPOSITION 1.22. *Let Δ be a graph, and $\mathfrak{J}$ a subset of Δ. For any $(x, J) \in$ $\mathsf{Cham}(\Delta, \mathfrak{J})$ and $i \in J^c$ such that $J + i$ is Dynkin, we have the following assertions.*

(1) *$\omega_i(x, J)$ belongs to $\mathsf{Cham}(\Delta, \mathfrak{J})$.*
(2) *$x < xs_i \iff (x, J) < \omega_i(x, J)$, and $x > xs_i \iff (x, J) > \omega_i(x, J)$.*
(3) *Mutation is involutive, that is, $\omega_{i'}\omega_i(x, J) = (x, J)$ holds for $i' := \iota_{J+i}(i)$.*
(4) *Let $(y, J') := \omega_i(x, J)$. Then the $\mathfrak{J}$-chambers $x(C_J)$ and $y(C_{J'})$ are adjacent through the wall $x(C_{J+i})$.*

PROOF. (1) Let $(y, J') := \omega_i(x, J) = (xw_J w_{J+i}, (J + i) - j)$. Then $y = xw_{J+i} w_{J'}$. We first show that $W_{\mathfrak{J}} y = yW_{J'}$. Since $W_{\mathfrak{J}} x = xW_J$ holds, we have

$$\begin{aligned}W_{\mathfrak{J}} y &= W_{\mathfrak{J}} x w_J w_{J+i}\\ &= xW_J w_J w_{J+i}\\ &= yw_{J+i} w_J W_J w_J w_{J+i} = yw_{J+i} W_J w_{J+i}\\ &= yW_{\iota_{J+i}(J)} = yW_{J'}. \quad \text{(by 1.2(2))}\end{aligned}$$

We next show that y is the minimum element in $yW_{J'}$, and to do this we divide into two cases. If $x > xs_i$, then $y = xw_J w_{J+i}$ is the minimum element in $xW_{J+i} \supset yW_{J'}$ by 1.21(2). If $x < xs_i$, then xw_{J+i} is the maximum element in $xW_{J+i} \supset yW_{J'}$ by 1.21(2), and so $y = xw_{J+i} w_{J'}$ is the minimum element in $yW_{J'}$ by 1.2(2). In either case, the assertion follows.
(2) Again by 1.21(2), the assertions follow.
(3) This is immediate from 1.2(3).
(4) Since C_{J+i} is a wall of C_J, it follows that $x(C_{J+i})$ is a wall of $x(C_J)$. Similarly $y(C_{J+i})$ is a wall of $y(C_{J'})$. Moreover $x^{-1}y = w_J w_{J+i} \in W_{J+i}$ holds. Since C_{J+i} is stabilised by W_{J+i}, we have $y(C_{J+i}) = x(C_{J+i})$, and so the assertion follows. □

Next we prove 1.20(2).

PROPOSITION 1.23. *Let Δ be a graph, and $\mathfrak{J}$ a strongly Dynkin subset of Δ.*

(1) *Any two elements in $\mathsf{Cham}(\Delta, \mathfrak{J})$ can be connected by a finite sequence of simple wall crossings.*

(2) *For any* $(x, J) \in \mathsf{Cham}(\Delta, \mathcal{J})$*, there exists a finite sequence* (x^t, J^t) *for* $t = 0, \ldots, s$ *satisfying the following conditions.*
  (a) $(x^0, \mathcal{J}) = (1, \mathcal{J})$ *and* $(x^s, J^s) = (x, J)$*.*
  (b) (x^{t+1}, J^{t+1}) *is a mutation of* (x^t, J^t) *for any* $t = 0, \ldots, s-1$*.*
  (c) $1 = x^0 < x^1 < \cdots < x^t = x$*.*

(3) *Two elements in* $\mathsf{Cham}(\Delta, \mathcal{J})$ *are related by a simple wall crossing if and only if the corresponding* $\mathcal{J}$*-chambers are adjacent.*

Proof. (2) Fix $(x, J) \in \mathsf{Cham}(\Delta, \mathcal{J})$. If $x > xs_i$ holds for some $i \in J^c$, then $(x, J) > \omega_i(x, J)$ by 1.22(2). Thus an inductive argument on $\ell(x)$ proves the assertion.

Now assume that $x < xs_i$ holds for all $i \in J^c$. Then $(x, J) \in \mathsf{Cham}(\Delta, \mathcal{J})$ implies that $x < xs_i$ holds for all $i \in \Delta$, from which $x = 1$ follows. The equality $W_{\mathcal{J}} = W_J$ implies $J = \mathcal{J}$ by [BB, 2.4.1(v)]. The assertion is clear in this case $(x, J) = (1, \mathcal{J})$.
(1) This follows immediately from (2).
(3)($\Rightarrow$) is 1.22(4). For ($\Leftarrow$), since all the walls of the $\mathcal{J}$-chamber $x(C_J)$ are given by $x(C_{J+i})$ for $i \in J^c$, and the chamber at the other side of these walls are given by the simple wall crossing formula by 1.22(4), the assertion follows. □

Finally we prove 1.20(3).

Proposition 1.24. *Suppose that* W_Δ *is finite, and* $\mathcal{J}$ *is a subset of* Δ*.*

(1) *If* $(x, J) \in \mathsf{Cham}(\Delta, \mathcal{J})$*, then* $(w_{\mathcal{J}} x w_\Delta, \iota_\Delta(J)) \in \mathsf{Cham}(\Delta, \mathcal{J})$*.*
(2) $\mathsf{Cham}(\Delta, \mathcal{J})$ *has an anti-automorphism* $(x, J) \mapsto (w_{\mathcal{J}} x w_\Delta, \iota_\Delta(J))$*.*
(3) $\mathsf{Cham}(\Delta, \mathcal{J})$ *has minimum element* $(1, \mathcal{J})$ *and maximum element* $(w_{\mathcal{J}} w_\Delta, \iota_\Delta(\mathcal{J}))$*.*
(4) $(x, J) \in \mathsf{Cham}(\Delta, \mathcal{J})$ *satisfies* $(x, J) < \omega_i(x, J)$ *(respectively,* $(x, J) > \omega_i(x, J)$*) for all* $i \in J^c$ *if and only if* $(x, J) = (1, \mathcal{J})$ *(respectively,* $(x, J) = (w_{\mathcal{J}} w_\Delta, \iota_\Delta(\mathcal{J}))$*).*

Proof. (1) We have

$$w_{\mathcal{J}} x w_\Delta W_{\iota_\Delta(J)} = w_{\mathcal{J}} x W_J w_\Delta = w_{\mathcal{J}} W_{\mathcal{J}} x w_\Delta = W_{\mathcal{J}} w_{\mathcal{J}} x w_\Delta.$$

Since x is the minimum element in $W_{\mathcal{J}} x$, the element $w_{\mathcal{J}} x$ is the maximum element in $W_{\mathcal{J}} x$ by 1.2(1). Thus the element $w_{\mathcal{J}} x w_\Delta$ is the minimum in $W_{\mathcal{J}} x w_\Delta = W_{\mathcal{J}} w_{\mathcal{J}} x w_\Delta$ again by 1.2(1). Thus $(w_{\mathcal{J}} x w_\Delta, \iota_\Delta(J))$ belongs to $\mathsf{Cham}(\Delta, \mathcal{J})$.
(2) Assume that $(x, J), (y, J')$ satisfy $x \leq y$, that is, $\ell(y) = \ell(x) + \ell(x^{-1}y)$. Then

$$\begin{aligned}\ell(w_{\mathcal{J}} y) &= \ell(w_{\mathcal{J}}) + \ell(y) = \ell(w_{\mathcal{J}}) + \ell(x) + \ell(x^{-1}y)\\ &= \ell(w_{\mathcal{J}} x) + \ell((w_{\mathcal{J}} x)^{-1} w_{\mathcal{J}} y),\end{aligned}$$

which implies that $w_{\mathcal{J}} x \leq w_{\mathcal{J}} y$. Thus $w_{\mathcal{J}} x w_\Delta \geq w_{\mathcal{J}} y w_\Delta$ holds by 1.2(1).
(3) Clearly $(1, \mathcal{J})$ is the minimum element in $\mathsf{Cham}(\Delta, \mathcal{J})$. By (2), $(w_{\mathcal{J}} w_\Delta, \iota_\Delta(\mathcal{J}))$ is the maximum element in $\mathsf{Cham}(\Delta, \mathcal{J})$.
(4) We only have to prove "if" part. It follows from 1.23(2) that $(x, J) < \omega_i(x, J)$ for any $i \in J^c$ implies $(x, J) = (1, \mathcal{J})$. Dually, by (2), $(x, J) > \omega_i(x, J)$ for any $i \in J^c$ implies $(x, J) = (w_{\mathcal{J}} w_\Delta, \iota_\Delta(\mathcal{J}))$. □

CHAPTER 2

Affine Arrangements, Levels and Groupoids

In this chapter we apply the theory in the last chapter to the case of affine, or extended, Dynkin diagrams. Given a subset $\mathcal{J}$ of an ADE Dynkin Δ, we may also view $\mathcal{J}$ as a subset of the extended diagram Δ_{aff}. Then, applying 1.5 to $\mathcal{J} \subseteq \Delta$ constructs $\mathsf{Cone}(\Delta, \mathcal{J})$, and applying 1.5 to $\mathcal{J} \subseteq \Delta_{\mathsf{aff}}$ constructs $\mathsf{Cone}(\Delta_{\mathsf{aff}}, \mathcal{J})$.

The arrangement $\mathsf{Cone}(\Delta, \mathcal{J})$ is finite and fills $\mathbb{R}^{|\Delta \backslash \mathcal{J}|}$. However, just as for the usual Tits cone (see 2.1), $\mathsf{Cone}(\Delta_{\mathsf{aff}}, \mathcal{J})$ does not fill the vector space $\mathbb{R}^{|\Delta_{\mathsf{aff}} \backslash \mathcal{J}|}$. As such, it is convenient to take the *level* in (2.1.B) below, which we will denote $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$. This exists in our more general setting, and we thus obtain an infinite hyperplane arrangement in one dimension lower, back in $\mathbb{R}^{|\Delta \backslash \mathcal{J}|}$. Taking the level make the picture easier to draw, without losing information, and also overlays the infinite arrangement on top of the finite arrangement, making the comparison easier.

Topologically, both the finite and infinite hyperplane arrangement is largely controlled by the Deligne, or arrangement, groupoid. We also introduce these here, mainly to set notation, as they will be heavily used in later chapters.

2.1. Θ_{aff} for extended Dynkin case

We mostly follow the setting Humphreys [H], but sometimes using different notation. Let Δ be an ADE Dynkin diagram, and Δ_{aff} the corresponding extended Dynkin diagram. Let Θ_{aff} be an $\mathbb{R}$-vector space with basis α_i^* for $i \in \Delta_{\mathsf{aff}}$, and let L be the lattice in Θ_{aff} generated by α_i^*. Thus

$$\Theta_{\mathsf{aff}} = \bigoplus_{i \in \Delta_{\mathsf{aff}}} \mathbb{R}\alpha_i^* \supset L = \bigoplus_{i \in \Delta_{\mathsf{aff}}} \mathbb{Z}\alpha_i^*.$$

We denote the dual vector space with dual basis α_i by

$$V_{\mathsf{aff}} = \bigoplus_{i \in \Delta_{\mathsf{aff}}} \mathbb{R}\alpha_i.$$

Write $\langle -, - \rangle \colon V_{\mathsf{aff}} \times \Theta_{\mathsf{aff}} \to \mathbb{R}$ for the natural paring, and for simplicity write $\cdot\alpha$ and $\vartheta\cdot$ for the maps $\langle -, \alpha \rangle \colon V_{\mathsf{aff}} \to \mathbb{R}$ and $\langle \vartheta, - \rangle \colon \Theta_{\mathsf{aff}} \to \mathbb{R}$, respectively.

Furthermore, write W_{aff} for the affine Weyl group, and

$$\Phi := \{ w\alpha_i \mid i \in \Delta_{\mathsf{aff}},\ w \in W_{\mathsf{aff}} \} \subset V_{\mathsf{aff}}$$

for the set of real roots. Consider also the null root

$$\delta := \sum_{i \in \Delta_{\mathsf{aff}}} \delta_i \alpha_i \in V_{\mathsf{aff}} \tag{2.1.A}$$

Any element $\alpha \in \Phi \sqcup \{\delta\}$ gives rise to a hyperplane

$$H_\alpha := \mathrm{Ker}(\cdot\alpha) \subset \Theta_{\mathsf{aff}}$$

and a decomposition

$$\Theta_{\mathsf{aff}} = H_\alpha^- \sqcup H_\alpha \sqcup H_\alpha^+,$$

where $H_\alpha^- = \{\vartheta \in \Theta_{\mathsf{aff}} \mid \vartheta \cdot \alpha < 0\}$ and $H_\alpha^+ = \{\vartheta \in \Theta_{\mathsf{aff}} \mid \vartheta \cdot \alpha > 0\}$.

Recall that the Tits cone $\mathsf{Cone}(\Delta_{\mathsf{aff}})$ is defined in 1.3 as

$$\mathsf{Cone}(\Delta_{\mathsf{aff}}) := \bigcup_{x \in W_{\mathsf{aff}}} x(\overline{C}).$$

and by definition is subset of Θ_{aff}. As is standard, in the Tits cone does not fill Θ_{aff}.

PROPOSITION 2.1. [H, §6.5] *If Δ is ADE, then* $\mathsf{Cone}(\Delta_{\mathsf{aff}}) = H_\delta^+ \sqcup \{0\}$.

By definition, a *chamber* is a connected component of $\mathsf{Cone}(\Delta_{\mathsf{aff}}) \setminus \bigcup_{\alpha\in\Phi} H_\alpha$, and we write $\mathsf{Cham}(\Delta_{\mathsf{aff}})$ for the set of all chambers. As usual, we describe $\mathsf{Cham}(\Delta_{\mathsf{aff}})$ by projecting to the hyperplane

$$\mathsf{Level} := \delta^{-1}(1) \subset \Theta_{\mathsf{aff}}.$$

For each $\alpha \in \Phi$, we consider the intersection hyperplane

$$\mathsf{H}_\alpha := \mathsf{Level} \cap H_\alpha. \tag{2.1.B}$$

An *alcove* is a connected component of $\mathsf{Level} \setminus \bigcup_{\alpha\in\Phi} \mathsf{H}_\alpha$, and we write $\mathsf{Alcove}(\Delta_{\mathsf{aff}})$ for the set of all alcoves. Since there is a homeomorphism

$$\mathbb{R}_{>0} \times \mathsf{Level} \xrightarrow{\sim} H_\delta^+$$

given by $(r, \vartheta) \mapsto r\vartheta$, the following is immediate.

PROPOSITION 2.2. [H, §6.5] *The maps $C \mapsto C \cap E$ and $A \mapsto \mathbb{R}_{>0}A$ give a bijection*

$$\mathsf{Cham}(\Delta_{\mathsf{aff}}) \xrightarrow{\sim} \mathsf{Alcove}(\Delta_{\mathsf{aff}}).$$

Next, consider the $\mathbb{R}$-vector space

$$V := \bigoplus_{i\in\Delta} \mathbb{R}\alpha_i \subset V_{\mathsf{aff}}.$$

Since $V_{\mathsf{aff}} = V \oplus \mathbb{R}\delta$, the pairing $\langle -,-\rangle\colon \Theta_{\mathsf{aff}} \times V_{\mathsf{aff}} \to \mathbb{R}$ restricts to a non-degenerate pairing

$$\langle -,-\rangle\colon H_\delta \times V \to \mathbb{R}.$$

Let $\{\varpi_i\}_{i\in\Delta} \subset H_\delta$ be the dual basis of $\{\alpha_i\}_{i\in\Delta} \subset V$. Let

$$\mathsf{CoWt} = \bigoplus_{i\in\Delta} \mathbb{Z}\varpi_i \subset H_\delta.$$

be the coweight lattice. In what follows, we write the roots $\alpha_0, \ldots, \alpha_n$, where 0 corresponds to the extended vertex.

LEMMA 2.3. *With notation as above, the following statements hold.*

(1) $\alpha_i^* = \delta_i\alpha_0^* + \varpi_i$ *for all* $i \in \Delta$.
(2) $L = \mathbb{Z}\alpha_0^* \oplus \mathsf{CoWt}$ *and* $L \cap \mathsf{Level} = \alpha_0^* + \mathsf{CoWt}$.

PROOF. (1) We compare both sides by evaluating elements in the basis $\{\alpha_j, \delta \mid j \in \Delta\}$ of V_{aff}. For all $j \in \Delta$, we have $\langle \alpha_i^*, \alpha_j\rangle = \delta_{ij} = \langle \delta_i\alpha_0^* + \varpi_i, \alpha_j\rangle$. Further $\langle \alpha_i^*, \delta\rangle = \delta_i = \langle \delta_i\alpha_0^* + \varpi_i, \delta\rangle$, thus the assertion follows.

(2) Immediate from (1). □

2.1.1. $\Theta_{\mathcal{J}_{\mathsf{aff}}}$ for Extended Dynkin Diagrams. Throughout Δ is ADE Dynkin diagram, with extended diagram Δ_{aff}. For a subset $\mathcal{J}$ of Δ, we may view $\mathcal{J}$ as a subset of Δ_{aff}, and consider

$$L_{\mathcal{J}} := \bigoplus_{i\in\Delta_{\mathsf{aff}}\setminus\mathcal{J}} \mathbb{Z}\alpha_i^* \quad\text{and}\quad \mathsf{CoWt}_{\mathcal{J}} := \bigoplus_{i\in\Delta\setminus\mathcal{J}} \mathbb{Z}\varpi_i.$$

The following is immediate from 2.3.

PROPOSITION 2.4. *For any subset $\mathcal{J}$ of Δ, we have*

$$L_{\mathcal{J}} = \mathbb{Z}\alpha_0^* \oplus \mathsf{CoWt}_{\mathcal{J}} \quad \textit{and} \quad L_{\mathcal{J}} \cap \mathsf{Level} = \alpha_0^* + \mathsf{CoWt}_{\mathcal{J}}.$$

In particular, there is a bijection

$$L_{\mathcal{J}} \cap \mathsf{Level} \simeq \mathsf{CoWt}_{\mathcal{J}} \quad \text{given by} \quad x \mapsto x - \alpha_0^*. \tag{2.1.A}$$

Regarding $\mathcal{J}$ as a subset of Δ, we can form $\Theta_{\mathcal{J}}$ as as subspace of Θ, as in 1.5. On the other hand, regarding $\mathcal{J}$ as a subset of Θ_{aff}, we can also form the corresponding subspace of Δ_{aff}. To avoid confusion, we will denote this by $\Theta_{\mathcal{J}_{\mathsf{aff}}}$, so that

$$\Theta_{\mathcal{J}_{\mathsf{aff}}} := \{\vartheta \in \Theta_{\mathsf{aff}} \mid \vartheta_i = 0 \text{ if } i \in \mathcal{J}\}.$$

Thus, $\Theta_{\mathcal{J}_{\mathsf{aff}}}$ is the subspace of Θ_{aff} with basis $\{\alpha_i^* \mid i \in \Delta_{\mathsf{aff}} \backslash \mathcal{J}\}$.

Similarly, using the coefficients of the null root in (2.1.A), consider

$$\delta_{\mathcal{J}_{\mathsf{aff}}} := \sum_{i \in \Delta_{\mathsf{aff}} \backslash \mathcal{J}} \delta_i \alpha_i,$$

which via the natural pairing $\langle -, - \rangle \colon \Theta^*_{\mathcal{J}_{\mathsf{aff}}} \times \Theta_{\mathcal{J}_{\mathsf{aff}}} \to \mathbb{R}$ can be viewed as a linear map $\Theta_{\mathcal{J}_{\mathsf{aff}}} \to \mathbb{R}$. Thus we define

$$\mathsf{Level}(\mathcal{J}_{\mathsf{aff}}) := \delta^{-1}_{\mathcal{J}_{\mathsf{aff}}}(1) \subset \Theta_{\mathcal{J}_{\mathsf{aff}}}. \tag{2.1.B}$$

It is clear that $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}}) = \mathsf{Level} \cap \Theta_{\mathcal{J}_{\mathsf{aff}}}$, since intersection is associative.

Definition 2.5. Suppose that Δ is ADE Dynkin, with extended diagram Δ_{aff}. Then for any $\mathcal{J} \subseteq \Delta$,

(1) We call $\mathsf{Cone}(\Delta, \mathcal{J}) \subseteq \mathbb{R}^{|\Delta \backslash \mathcal{J}|}$ the *$\mathcal{J}$-finite hyperplane arrangement.*
(2) We call $\mathsf{Cone}(\Delta_{\mathsf{aff}}, \mathcal{J}) \subseteq \mathbb{R}^{|\Delta_{\mathsf{aff}} \backslash \mathcal{J}|}$ the *$\mathcal{J}$-affine Tits cone.*
(3) We call $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}}) \subseteq \mathbb{R}^{|\Delta \backslash \mathcal{J}|}$ the $\mathcal{J}$-level.

The above constructions allow us to produce an infinite hyperplane arrangement for any subset of nodes $\mathcal{J}$ in any ADE Dynkin diagram. Indeed, continuing the notation and setting from above, specifically (2.1.B), let

$$\Phi_{\mathcal{J}_{\mathsf{aff}}} := \{\alpha \in \Phi \mid \Theta_{\mathcal{J}_{\mathsf{aff}}} \not\subseteq \mathsf{H}_\alpha\} = \Phi \setminus \bigoplus_{i \in \mathcal{J}_{\mathsf{aff}}} \mathbb{R}\alpha_i.$$

Thus inside $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$ is the infinite collection of hyperplanes

$$\mathcal{H}_{\mathcal{J}_{\mathsf{aff}}} := \{\mathsf{H}_\alpha \cap \Theta_{\mathcal{J}_{\mathsf{aff}}} \mid \alpha \in \Phi_{\mathcal{J}_{\mathsf{aff}}}\}.$$

Definition 2.6. A *$\mathcal{J}$-alcove* is a connected component of

$$\mathsf{Level}(\mathcal{J}_{\mathsf{aff}}) \setminus \bigcup_{\mathsf{H} \in \mathcal{H}_{\mathcal{J}_{\mathsf{aff}}}} \mathsf{H}.$$

We write $\mathsf{Alcove}(\mathcal{J}_{\mathsf{aff}})$ for the set of all $\mathcal{J}$-alcoves.

The following is clear, and generalises 2.2.

Proposition 2.7. *The maps $C \mapsto C \cap E$ and $A \mapsto \mathbb{R}_{>0}A$ give a bijection*

$$\mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J}) \xrightarrow{\sim} \mathsf{Alcove}(\mathcal{J}_{\mathsf{aff}}).$$

The labelling of the chambers of $\mathsf{Cone}(\Delta_{\mathsf{aff}}, \mathcal{J})$ in 1.12 allows for a more precise description and labelling of the alcoves, which we will need in Chapter 4. Consider again the open chamber decomposition

$$\bigsqcup_{(x,J) \in \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J})} x(C_J) \subset \mathsf{Cone}(\Delta_{\mathsf{aff}}, \mathcal{J}).$$

This induces an open decomposition of the level, and thus a labelling of the $\mathcal{J}$-alcoves. Indeed, for $(x, J) \in \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J}_{\mathsf{aff}})$, set

$$\mathsf{Alcove}_{(x,J)} := x(C_J) \cap \mathsf{Level}(\mathcal{J}_{\mathsf{aff}}).$$

From the induced open decomposition

$$\mathcal{A} := \bigsqcup_{(x,J)\in \mathsf{Cham}(\Delta_{\mathsf{aff}},\mathcal{J})} \mathsf{Alcove}_{(x,J)} \quad \subset \mathsf{Level}(\mathcal{J}_{\mathsf{aff}}),$$

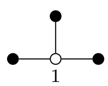
we recover the infinite hyperplane arrangement $\mathcal{H}_{\mathcal{J}_{\mathsf{aff}}}$ above as $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})\setminus \mathcal{A}$.

For Dynkin Δ and $\mathcal{J} \subseteq \Delta$, it is clear that $\mathcal{H}_{\mathcal{J}_{\mathsf{aff}}}$ is an infinite hyperplane arrangement in $\mathbb{R}^{|\Delta\setminus\mathcal{J}|}$. For extended Dynkin case, the $\mathcal{J}$-Tits cone does not fill $\Delta_{\mathcal{J}_{\mathsf{aff}}}$, and so sometimes the following is convenient.

NOTATION 2.8. For $\mathcal{I} \subset \Delta_{\mathsf{aff}}$, consider $\mathcal{W}_{\mathcal{I}}$, the set of full hyperplanes that separate the open chambers of $\mathsf{Cone}(\Delta_{\mathsf{aff}}, \mathcal{I})$. This is an infinite hyperplane arrangement in $\mathbb{R}^{|\Delta_{\mathsf{aff}}\setminus\mathcal{I}|}$.

These concepts are illustrated in the example below.

EXAMPLE 2.9. When $\Delta = D_4$ and $\mathcal{J}$ is



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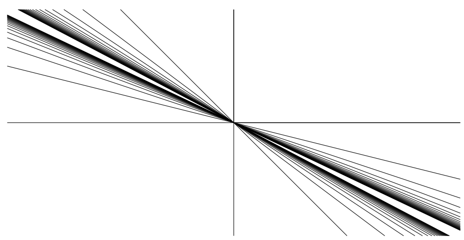
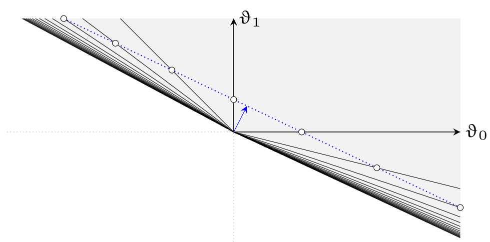
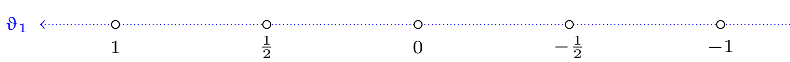
Viewing $\mathcal{J}$ as a subset of Δ_{aff}, then $\mathcal{W}_{\mathcal{J}}$ is the infinite hyperplane arrangement



The hyperplanes converge on the line $\vartheta_0 + 2\vartheta_1 = 0$, but $\mathcal{W}_{\mathcal{J}}$ does not contain this line. In contrast, $\mathsf{Cone}(\mathcal{J}_{\mathsf{aff}})$ is the shaded region in the following picture, and $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$ is illustrated by the dotted blue line $\vartheta_0 + 2\vartheta_1 = 1$.

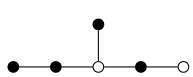


The circles on the blue line are, reading top left to bottom right, at $\vartheta_1 = \frac{3}{2}, 1, \frac{1}{2}, 0, -\frac{1}{2}, -1$. Thus basing $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$ by α_1^* (see §1.3), the level is the infinite hyperplane arrangement



The $\mathcal{J}$-alcoves are the open intervals on the blue line between two adjacent dots, and $\mathcal{H}_{\mathcal{J}_{\mathsf{aff}}}$ is the infinite collection of dots.

EXAMPLE 2.10. When $\Delta = E_6$ and $\mathcal{J}$ is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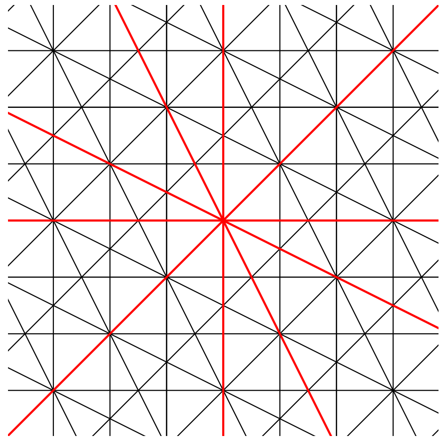
then $\mathsf{Cone}(\mathcal{J}_{\mathsf{aff}}) \subseteq \mathbb{R}^3$ is harder to draw, but $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}}) \subseteq \mathbb{R}^2$ is easier. Indeed, in suitable coordinates Example 4.19 shows that $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$ is the arrangement given by all lines



Note that, as in the introduction, $\mathsf{Cone}(\Delta, \mathcal{J})$ does not traditionally have an affine version.

Drawing $\mathsf{Cone}(\Delta, \mathcal{J})$ and $\mathsf{Cone}(\mathcal{J}_{\mathsf{aff}})$ in a canonical way, with all the correct angles, is possible, but it is long and tedious to calculate all the angles in all cases. We refrain from doing this since from the viewpoint of the applications in later chapters, the precise angles in the arrangements are not important. Even if we did calculate all the angles, coding the resulting pictures accurately is a non-trivial task.

2.2. $\mathcal{K}$-affine Arrangements and Levels

In the affine case, as already remarked in 2.1, the full Tits cone does not fill $\Theta_{\mathsf{aff}} = \mathbb{R}^{|\Delta_{\mathsf{aff}}|}$, and indeed $\mathsf{Cone}(\Delta_{\mathsf{aff}})$ is the region

$$\{\vartheta \in \Theta_{\mathsf{aff}} \mid \sum_{i \in \Delta_{\mathsf{aff}}} \delta_i \vartheta_i > 0\}.$$

The previous subsection required the assumption that $\mathcal{J} \subseteq \Delta$. From this, we then viewed $\mathcal{J}$ as a subset of Δ_{aff}, and from this constructed objects such as $\mathsf{Cone}(\Delta_{\mathsf{aff}}, \mathcal{J})$ and $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$. However, for much of the previous subsection (except notably 2.4), the assumption that $\mathcal{J} \subseteq \Delta$ is not required. We may in fact consider any subset $\mathcal{K} \subseteq \Delta_{\mathsf{aff}}$, and form a corresponding $\mathsf{Cone}(\Delta_{\mathsf{aff}}, \mathcal{K})$ and $\mathsf{Level}(\Delta_{\mathsf{aff}}, \mathcal{K})$

Definition 2.11. Suppose that Δ_{aff} is extended Dynkin, then for any $\mathcal{K} \subseteq \Delta_{\mathsf{aff}}$, the *level* is defined to be

$$\mathsf{Level}(\Delta_{\mathsf{aff}}, \mathcal{K}) := \{\vartheta \in \mathsf{Cone}(\Delta_{\mathsf{aff}}, \mathcal{K}) \mid \sum_{k \notin \mathcal{K}} \delta_k \vartheta_k = 1\}.$$

For most of our applications, this level of generality is not required. However, in Chapter 4 when we are classifying the possible arrangements in low dimension, we will work in this more general setting.

2.3. The affine $\mathcal{J}$-pure braid group

Let Δ be ADE Dynkin, and consider the extended diagram Δ_{aff}. For $\mathcal{J} \subseteq \Delta$, consider the finite $\mathsf{Cone}(\Delta, \mathcal{J})$ and the infinite $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$, both of which are inside $\mathbb{R}^{|\mathcal{J}^c|}$, where $\mathcal{J}^c := \Delta \backslash \mathcal{J}$. Both these arrangements are *locally finite*, i.e. every point of $\mathbb{R}^{|\mathcal{J}^c|}$ is contained in at most finitely many hyperplanes, and *essential*, i.e. the minimal intersections of hyperplanes are points.

2.3.1. Arrangements groupoids. In this subsection we briefly recall the basics of the arrangement (=Deligne) groupoids, mainly to set notation. In order to later apply this to both $\mathsf{Cone}(\Delta, \mathcal{J})$ and $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$, throughout this subsection let $\mathcal{H}$ be any essential, locally finite arrangement inside $\mathbb{R}^n$.

The graph $\Gamma_{\mathcal{H}}$ of oriented arrows is defined as follows. The vertices of $\Gamma_{\mathcal{H}}$ are the chambers (i.e. the connected components) of $\mathbb{R}^n \backslash \mathcal{H}$. There is a unique arrow $a\colon v_1 \to v_2$ from chamber v_1 to chamber v_2 if the chambers are adjacent, otherwise there is no arrow. For an arrow $a\colon v_1 \to v_2$, we set $s(a) := v_1$ and $t(a) := v_2$. By definition, if there is an arrow $a\colon v_1 \to v_2$, there is a unique arrow $b\colon v_2 \to v_1$ with the opposite direction of a.

A *positive path of length* n in $\Gamma_{\mathcal{H}}$ is defined to be a formal symbol

$$p = a_n \circ \ldots \circ a_2 \circ a_1,$$

whenever there exists a sequence of vertices $v_0, \ldots, v_n$ of $\Gamma_{\mathcal{H}}$ and exist arrows $a_i\colon v_{i-1} \to v_i$ in $\Gamma_{\mathcal{H}}$. Set $s(p) := v_0$, $t(p) := v_n$, and $\ell(p) := n$, and write $p\colon s(p) \to t(p)$. The notation $\circ$ should remind us of composition, but we will often drop the $\circ$'s in future. If $q = b_m \circ \ldots \circ b_2 \circ b_1$ is another positive path with $t(p) = s(q)$, we consider the formal symbol

$$q \circ p := b_m \circ \ldots \circ b_2 \circ b_1 \circ a_n \circ \ldots \circ a_2 \circ a_1,$$

and call it the *composition* of p and q.

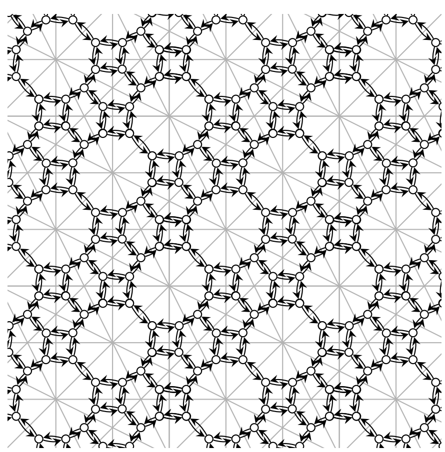
A positive path is called *reduced* if it does not cross any hyperplane twice. In this setting where $\mathcal{H}$ is locally-finite, reduced positive paths coincide with shortest positive paths; see e.g. [**S**, Lemma 2].

Following [**D2**, p7], let $\sim$ denote the smallest equivalence relation, compatible with morphism composition, that identifies all morphisms that arise as positive reduced paths with same source and target. Then consider the free category $\mathrm{Free}(\Gamma_{\mathcal{H}})$ on the graph $\Gamma_{\mathcal{H}}$, where morphisms are directed paths, and the quotient category

$$\mathcal{G}_{\mathcal{H}}^{+} := \mathrm{Free}(\Gamma_{\mathcal{H}})/\sim,$$

called the category of positive paths.

Example 2.12. For the infinite hyperplane arrangement in 2.10, $\mathcal{G}_{\mathcal{H}}^{+}$ is generated by the following arrows:



The relations in this example are generated by the polygon face relations, namely the two shortest paths around any 4-gon, 6-gon or 10-gon are identified.

Definition 2.13. The *arrangement (=Deligne) groupoid* $\mathcal{G}_{\mathcal{H}}$ is defined to be the groupoid completion of $\mathcal{G}_{\mathcal{H}}^{+}$, that is, a formal inverse is added for every morphism in $\mathcal{G}_{\mathcal{H}}^{+}$.

Warning 2.14. In this level of generality, it is not known whether the natural morphism $\mathcal{G}_{\mathcal{H}}^{+} \to \mathcal{G}_{\mathcal{H}}$ is injective. When $\mathcal{H}$ is a finite simplicial arrangement, the morphism is injective [**D**].

The following is well-known [**D, P1, P2, S**]. The statement below in our possibly infinite setting can be found for example in [**D2**, p9]. Recall that throughout this subsection, $\mathcal{H}$ is a locally finite, essential, arrangement inside $\mathbb{R}^n$.

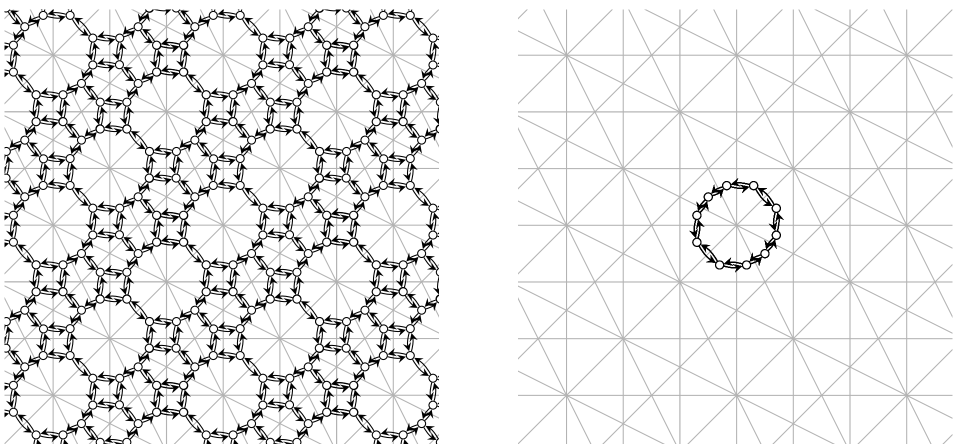
THEOREM 2.15. *If $v \in \mathcal{G}_{\mathcal{H}}$ is any vertex, then* $\operatorname{End}_{\mathcal{G}_{\mathcal{H}}}(v) \cong \pi_1(\mathbb{C}^n \backslash \mathcal{H}_{\mathbb{C}})$.

2.3.2. Groupoids for Intersection Arrangements. We revert to the setting of intersection arrangements. In this subsection we apply the above to these special cases, mainly to set notation for future sections.

NOTATION 2.16. Suppose that Δ is ADE Dynkin, and $\mathcal{J} \subseteq \Delta$.

(1) For $\mathcal{H} = \mathsf{Cone}(\Delta, \mathcal{J})$, which is a finite simplicial arrangement inside $\mathbb{R}^{|\mathcal{J}^c|}$, the resulting Deligne groupoid will be written $\mathcal{G}_{\mathcal{J}}$.
(2) For $\mathcal{H} = \mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$, which is an infinite arrangement inside $\mathbb{R}^{|\mathcal{J}^c|}$, the resulting Deligne groupoid will be written $\mathcal{G}_{\mathcal{J}_{\mathsf{aff}}}$.

EXAMPLE 2.17. For the $\mathcal{J}$ the subset of E_6 in 2.10, then $\mathcal{G}_{\mathcal{J}_{\mathsf{aff}}}$ and $\mathcal{G}_{\mathcal{J}}$ are respectively:



The following definition solves one of our main motivations, namely it constructs what we think of as the *affine pure* braid group, even when the affine braid group need not exist.

DEFINITION 2.18. Suppose that Δ is Dynkin, and $\mathcal{J} \subseteq \Delta$.

(1) Write $\pi_1(\mathcal{J})$ for any vertex group of $\mathcal{G}_{\mathcal{J}}$, and call it the $\mathcal{J}$-pure braid group.
(2) Write $\pi_1(\mathcal{J}_{\mathsf{aff}})$ for any vertex group of $\mathcal{G}_{\mathcal{J}_{\mathsf{aff}}}$, and call it the affine $\mathcal{J}$-pure braid group.

By 2.15 both $\pi_1(\mathcal{J})$ and $\pi_1(\mathcal{J}_{\mathsf{aff}})$ are fundamental groups of the complexified complements of the associated hyperplane arrangements. It will be through controlling the groupoids $\mathcal{G}_{\mathcal{J}}$ and $\mathcal{G}_{\mathcal{J}_{\mathsf{aff}}}$ that we will be able, in later sections, to produce π_1-group actions on derived categories in various algebraic and geometric settings.

CHAPTER 3

Finite Type Examples and Classification

In this chapter, when Δ is an ADE Dynkin diagram, we classify all the possible $\mathsf{Cham}(\Delta, \mathcal{J})$ that arise when $|\mathcal{J}^c| = 1, 2$, and illustrate the complexity in the general case. Furthermore, the techniques developed (specifically 3.4) significantly simplify the general theory of Chapter 1, which in turn simplify the calculation of the infinite arrangements in Chapter 4.

3.1. Degenerate Cases

When $\mathcal{J} = \emptyset$, as already remarked, $\mathsf{Cham}(\Delta, \emptyset) = \{(x, \emptyset) \mid x \in W_\Delta\}$, which can be identified with the usual root system for Δ. The other extreme is $\mathcal{J} = \Delta$, in which case $\mathsf{Cham}(\Delta, \Delta) = \{(1, \Delta)\}$.

The other main degenerate case is when $|\mathcal{J}^c| = 1$, which we will refer to as rank one.

Lemma 3.1. *If Δ is Dynkin and $\mathcal{J} = \Delta - i$ for some $i \in \Delta$, (equivalently, $|\mathcal{J}^c| = 1$), then wall crossing in $\mathsf{Cham}(\Delta, \mathcal{J})$ is described by*

$$(1, \Delta - i) \underset{j}{\overset{i}{\rightleftarrows}} (w_{\Delta - i} w_\Delta, \Delta - j)$$

where $j = \iota_\Delta(i)$. In particular, there are only ever two chambers, and one wall.

Proof. This is a direct consequence of 1.20. □

The other slightly degenerate case is when $\Delta = A_n$ and $\mathcal{J}$ is arbitrary. As is typical in this setting, see e.g. [**W2**, 6.5], constructions in type-A only ever give type-A phenomena. This is made precise in the following (see also 4.5).

Proposition 3.2. *If $\Delta = A_n$, and $\mathcal{J} \subseteq \Delta$, then the arrangement $\mathsf{Cone}(\Delta, \mathcal{J})$ is the finite root system of type $A_{n-|\mathcal{J}|}$.*

Proof. There are many ways to see this. Consider $\mathcal{J}^c$, and order its elements in increasing order $j_1 < \ldots < j_m$. It is easy to check, in a similar way to 3.4 below, that the vertices j_i and j_{i+1} braid under iterated wall crossing, with a length three braid relation. It is also easy to check that wall crossing under j_s and j_t commute provided that $|s-t| > 1$, since there is a vertex in between them, which splits the Dynkin diagram into two disjoint pieces. Thus, although the local labels change, the arrangement is controlled by the same global rules as the finite root system of type $A_{n-|\mathcal{J}|}$, and so it is $A_{n-|\mathcal{J}|}$.

The other way to see this is from the explicit description of roots of A_n in terms of connected chains of 1s on the Dynkin diagram. This immediately implies that the set of restricted roots on $\{j_1, \ldots, j_m\}$ is also given in terms of connected chains of 1s. From this, it follows that the arrangement is $A_{n-|\mathcal{J}|}$. □

3.2. Rank Two Examples and General Techniques

This section illustrates how to use the general theory to calculate any given $\mathsf{Cham}(\Delta, \mathcal{J})$ with $|\mathcal{J}^c| = 2$, which we will refer to as rank two. Example 3.3 explains this in full detail, however keeping track of the full Weyl group elements quickly becomes cumbersome. As such, group elements are then replaced by their length, leading to slimlined Examples 3.5, 3.6 and 3.7. These illustrate some phenomena appearing in the cases of E_6, E_7 and E_8

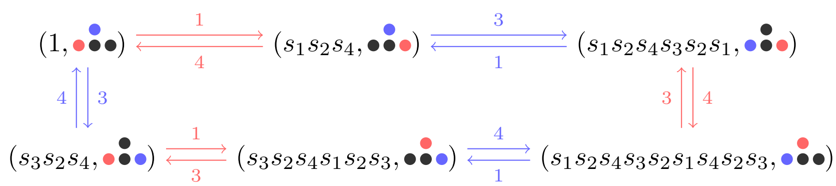
respectively, such as the slopes of the hyperplanes changing. These turn out to be a representative of the full rank two classification, which appears in the next section.

As before, in what follows we depict vertices in $\mathcal{J}$ by $\bullet$. It will be convenient to colour the other vertices, those in $\mathcal{J}^c$, so for example the picture

depicts the case Δ is E_8, with $\mathcal{J}$ equal to the set of black vertices.

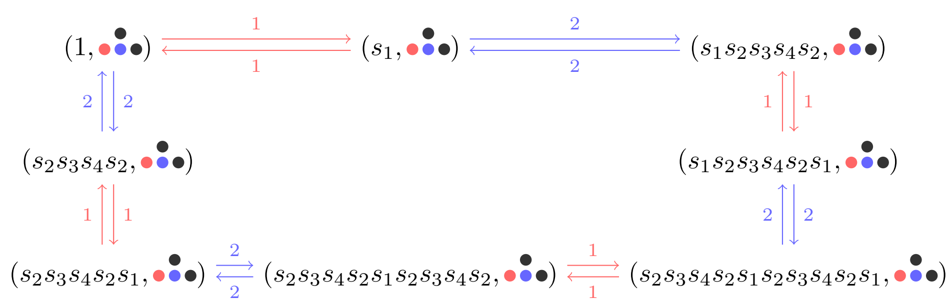
EXAMPLE 3.3. Let $\Delta = D_4$.

(1) For $\mathcal{J} =$, $\mathsf{Cham}(D_4, \mathcal{J})$ is

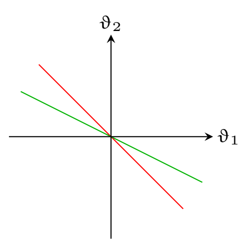


and $\mathsf{Cone}(D_4, \mathcal{J})$ is the hyperplane arrangement as in 1.7(2), with the same restricted roots 10, 01 and 11.

(2) For $\mathcal{J} =$, $\mathsf{Cham}(D_4, \mathcal{J})$ is given by the following calculation



Furthermore, $\mathsf{Cone}(D_4,$ $)$ is the following hyperplane arrangement



Hyperplane	Restricted Roots
$\vartheta_1 = 0$	10
$\vartheta_2 = 0$	01
$\vartheta_1 + \vartheta_2 = 0$	11
$\vartheta_1 + 2\vartheta_2 = 0$	12

In this example the equality $\nu_1\nu_2\nu_1\nu_2(1, \mathcal{J}) = \nu_2\nu_1\nu_2\nu_1(1, \mathcal{J})$ implies that

$$s_2s_3s_4s_2s_1s_2s_3s_4s_2s_1 = s_1s_2s_3s_4s_2s_1s_2s_3s_4s_2,$$

which takes some time to be checked directly by hand.

REMARK 3.4. In the case $|\mathcal{J}^c| = 2$, in fact we do not have to check such kind of equality since the chamber structure of $\mathsf{Cham}(\Delta, \mathcal{J})$ is given by positive roots. Indeed, consider first the table below (found using (1.2.A)), which records $\ell(w_{\Delta-i}w_\Delta) = \ell(w_\Delta) - \ell(w_{\Delta-i})$ for each Dynkin diagram Δ and each vertex $i \in \Delta$.

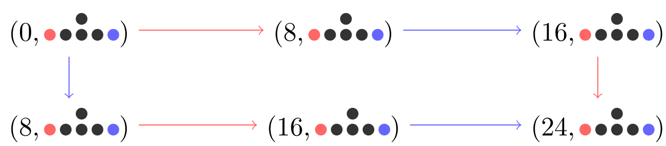
It follows from 1.20(2) that alternating wall crossings gives all elements in $\mathsf{Cham}(\Delta, \mathcal{J})$. Moreover, thanks to 1.20(3), we only have to calculate $\ell(x)$ for each $(x, J) \in \mathsf{Cham}(\Delta, \mathcal{J})$ until it reaches $\ell(w_{\mathcal{J}}w_\Delta)$. This can be done inductively, using the table below.

Δ	$\ell(w_{\Delta-i}w_\Delta) = \ell(w_\Delta) - \ell(w_{\Delta-i})$
A_n	$n \quad 2(n-1) \quad 3(n-2) \quad \cdots \quad (n-2)3 \quad (n-1)2 \quad n$
D_n	$\frac{n(n-1)}{2}$ (above the second entry)
	$\frac{n(n-1)}{2} \quad \frac{(n+5)(n-2)}{2} \quad \frac{(n+8)(n-3)}{2} \quad \frac{(n+11)(n-4)}{2} \quad \cdots \quad \frac{(4n-7)2}{2} \quad \frac{(4n-4)1}{2}$
E_6	21 (above the third entry)
	16 25 29 25 16
E_7	42 (above the third entry)
	33 47 53 50 42 27
E_8	92 (above the third entry)
	78 98 106 104 97 83 57

In the examples that follow, we thus write $(\ell(x), J)$ instead of (x, J).

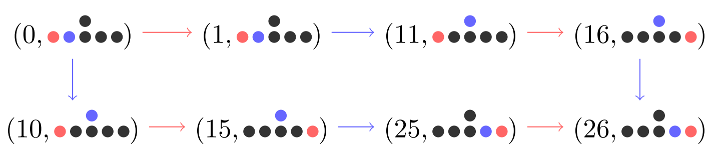
EXAMPLE 3.5. Let $\Delta = E_6$.

(1) When $\mathcal{J} =$ , then $\ell(w_{\Delta-\mathcal{J}}w_\Delta) = 36 - 12 = 24$ and $\mathsf{Cham}(E_6, \mathcal{J})$ is

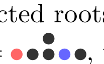


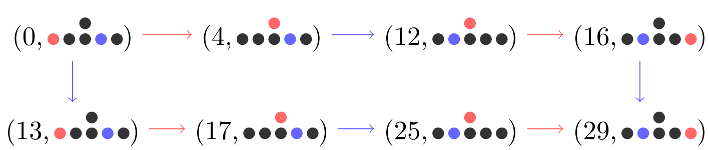
Moreover $\mathsf{Cone}(E_6, \mathcal{J})$ is the 6-chamber example in 1.7(2) and 3.3(1), with the same three restricted roots.

(2) When $\mathcal{J} =$ , then $\ell(w_{\Delta-\mathcal{J}}w_\Delta) = 36 - 10 = 26$ and $\mathsf{Cham}(E_6, \mathcal{J})$ is

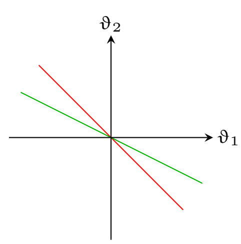


Further, $\mathsf{Cone}(E_6, \mathcal{J})$ is the 8-chamber example in 3.3(2), with the same four restricted roots.

(3) If $\mathcal{J} =$ , then $\ell(w_{\Delta-\mathcal{J}}w_\Delta) = 36 - (6+1) = 29$ and $\mathsf{Cham}(E_6, \mathcal{J})$ is



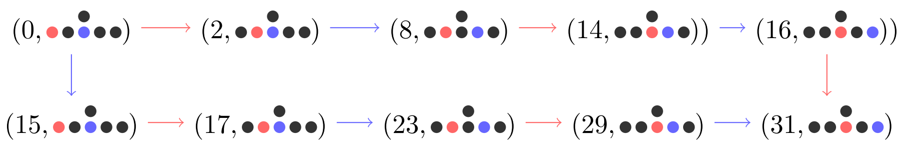
Again, $\mathsf{Cone}(E_6, \mathcal{J})$ is the 8-chamber arrangement in 3.3(2), but now the restricted roots have multiplicity.



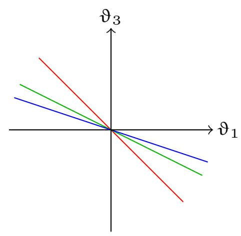
Hyperplane	Restricted Roots
$\vartheta_1 = 0$	10
$\vartheta_2 = 0$	01, 02
$\vartheta_1 + \vartheta_2 = 0$	11
$\vartheta_1 + 2\vartheta_2 = 0$	12

The restricted root 02 gives the hyperplane $2\vartheta_2 = 0$, which is the same as the hyperplane $\vartheta_2 = 0$. However, when later in the next chapter we translate by the integers (see 4.4), this multiplicity effects the affine arrangement that is obtained. Compare 4.15 versus 4.16 later.

(4) If $\mathcal{J} =$ , then $\ell(w_{\Delta-\mathcal{J}}w_{\Delta}) = 36 - (3+1+1) = 31$ and $\mathsf{Cham}(E_6, \mathcal{J})$ is



In this case, $\mathsf{Cone}(E_6, \mathcal{J})$ is the following hyperplane arrangement.



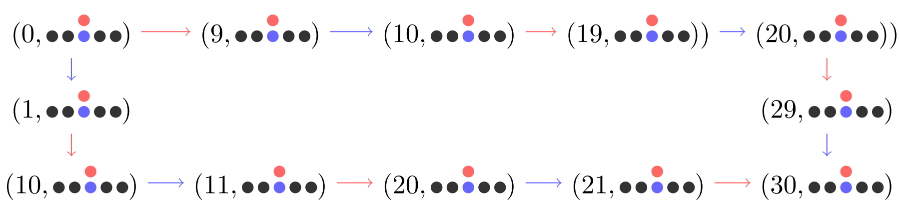
Hyperplane	Restricted Roots
$\vartheta_1 = 0$	10
$\vartheta_3 = 0$	01, 02
$\vartheta_1 + \vartheta_3 = 0$	11
$\vartheta_1 + 2\vartheta_3 = 0$	12
$\vartheta_1 + 3\vartheta_3 = 0$	13

Note that these simple calculations imply the non-obvious equality

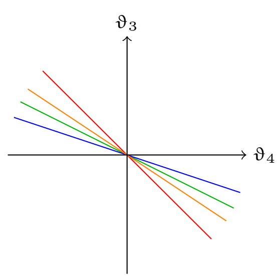
$$\begin{aligned} &s_3s_2s_4s_3s_5s_6s_3s_2s_4s_3s_5s_3s_2s_4s_3s_1s_2s_3s_5s_6s_4s_3s_5s_2s_3s_4s_1s_2s_3s_5s_6 \\ &= s_1s_2s_3s_5s_6s_4s_3s_5s_2s_3s_4s_1s_2s_3s_5s_6s_3s_4s_5s_3s_2s_1s_3s_4s_5s_3s_2s_3s_4s_5s_3 \end{aligned}$$

in the Weyl group W_{E_6}.

(5) Let $\mathcal{J} =$ . Then $\ell(w_{\Delta-\mathcal{J}}w_{\Delta}) = 36 - 6 = 30$ and $\mathsf{Cham}(E_6, \mathcal{J})$ is



Moreover, $\mathsf{Cone}(E_6, \mathcal{J})$ is the following hyperplane arrangement.

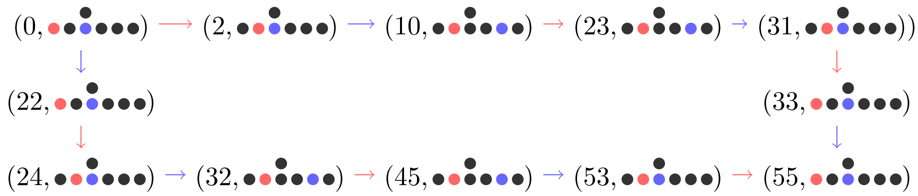


Hyperplane	Restricted Roots
$\vartheta_4 = 0$	10
$\vartheta_3 = 0$	01
$\vartheta_4 + \vartheta_3 = 0$	11
$\vartheta_4 + \frac{3}{2}\vartheta_3 = 0$	23
$\vartheta_4 + 2\vartheta_3 = 0$	12
$\vartheta_4 + 3\vartheta_3 = 0$	13

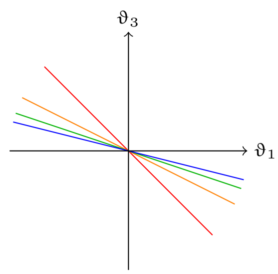
The next example, which takes place in E_7, illustrates that whilst from a topological (and π_1) perspective the hyperplane arrangements can be considered the same, the slopes of some of the hyperplanes may vary.

EXAMPLE 3.6. Consider $\Delta = E_7$.

(1) Let $\mathcal{J} =$. Then $\ell(w_{\Delta-\mathcal{J}}w_\Delta) = 63 - (6+1+1) = 55$ and $\mathsf{Cham}(E_7, \mathcal{J})$ is



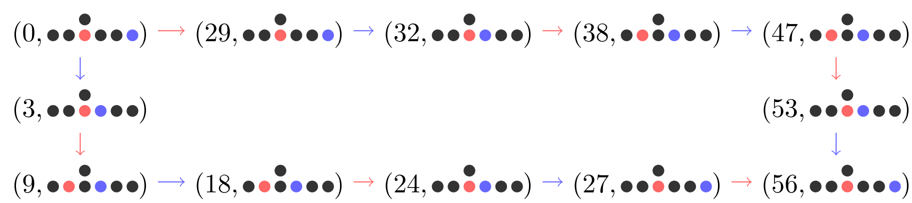
In this case $\mathsf{Cone}(E_7, \mathcal{J})$ is the following hyperplane arrangement.



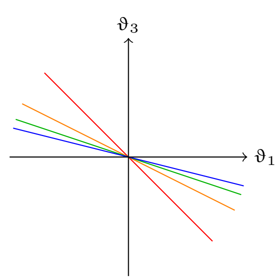
Hyperplane	Restricted Roots
$\vartheta_1 = 0$	10
$\vartheta_3 = 0$	01, 02
$\vartheta_1 + \vartheta_3 = 0$	11
$\vartheta_1 + 2\vartheta_3 = 0$	12, 24
$\vartheta_1 + 3\vartheta_3 = 0$	13
$\vartheta_1 + 4\vartheta_3 = 0$	14

Note that, up to gradients, this is the same as Example 3.5(5) above.

(2) Let $\mathcal{J} =$. Then $\ell(w_{\Delta-\mathcal{J}}w_\Delta) = 63 - (3+3+1) = 56$ and $\mathsf{Cham}(E_7, \mathcal{J})$ is



Moreover $\mathsf{Cone}(E_7, \mathcal{J})$ is the same hyperplane arrangement as in (1) above, but the multiplicities of the restricted roots differ.

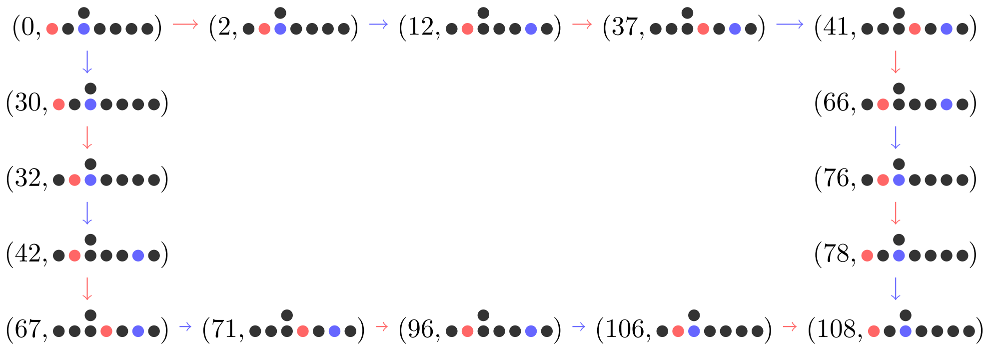


Hyperplane	Restricted Roots
$\vartheta_1 = 0$	10
$\vartheta_3 = 0$	01, 02, 03
$\vartheta_1 + \vartheta_3 = 0$	11
$\vartheta_1 + 2\vartheta_3 = 0$	12
$\vartheta_1 + 3\vartheta_3 = 0$	13
$\vartheta_1 + 4\vartheta_3 = 0$	14

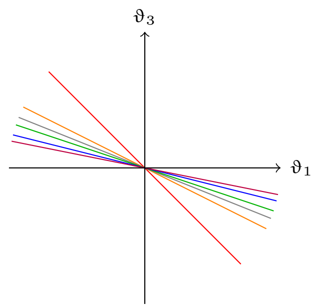
The multiplicities effect the affine arrangements later; compare 4.23 and 4.25.

Example 3.7. Let $\Delta = E_8$.

(1) $\mathcal{J} =$ . Then $\ell(w_{\Delta-\mathcal{J}}w_\Delta) = 120 - (10+1+1) = 108$ and $\mathsf{Cham}(E_8, \mathcal{J})$ is

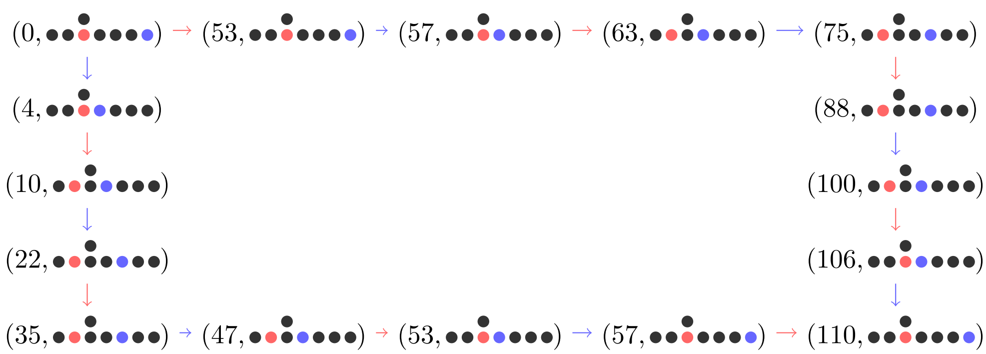


In this case $\mathsf{Cone}(E_8, \mathcal{J})$ is the following hyperplane arrangement.

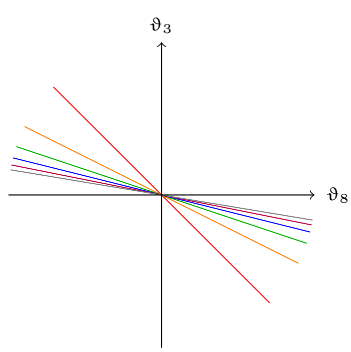


Hyperplane	Restricted Roots
$\vartheta_1 = 0$	10
$\vartheta_3 = 0$	01, 02
$\vartheta_1 + \vartheta_3 = 0$	11
$\vartheta_1 + 2\vartheta_3 = 0$	12, 24
$\vartheta_1 + \frac{5}{2}\vartheta_3 = 0$	25
$\vartheta_1 + 3\vartheta_3 = 0$	13, 26
$\vartheta_1 + 4\vartheta_3 = 0$	14
$\vartheta_1 + 5\vartheta_3 = 0$	15

(2) $\mathcal{J} =$ . Then $\ell(w_{\Delta-\mathcal{J}}w_\Delta) = 120 - (6+3+1) = 110$ and $\mathsf{Cham}(E_8, \mathcal{J})$ is



In this case $\mathsf{Cone}(E_8, \mathcal{J})$ is the following hyperplane arrangement



Hyperplane	Restricted Roots
$\vartheta_8 = 0$	10
$\vartheta_3 = 0$	01, 02, 03, 04
$\vartheta_8 + \vartheta_3 = 0$	11
$\vartheta_8 + 2\vartheta_3 = 0$	12
$\vartheta_8 + 3\vartheta_3 = 0$	13, 26
$\vartheta_8 + 4\vartheta_3 = 0$	14
$\vartheta_8 + 5\vartheta_3 = 0$	15
$\vartheta_8 + 6\vartheta_3 = 0$	16

Up to gradients, this is the same hyperplane arrangement as in (1) above. The gradients, and their multiplicities, effect the corresponding affine arrangements; compare 4.25 and 4.26.

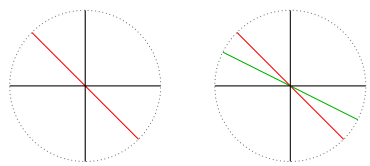
3.3. Rank Two Classification

In this section we consider $\mathsf{Cone}(\Delta, \mathcal{J})$ when Δ is ADE Dynkin and $|\mathcal{J}^c| = 2$, and we classify the hyperplane arrangements that can arise. The methods of the previous section allows for a full calculation of the arrangement, including the slopes of the hyperplanes, but for most purposes of this section (and for our applications in finite type), we will ignore the slopes.

The following results asserts that for $\Delta = A_n, D_n$, and $|\mathcal{J}^c| = 2$, few possibilities occur.

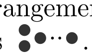
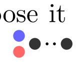
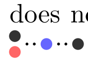
Proposition 3.8. *Consider $\mathcal{J} \subset \Delta$ with $|\mathcal{J}^c| = 2$.*

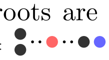
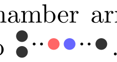
(1) *$\mathsf{Cone}(A_n, \mathcal{J})$ is the finite root system A_2, which has 6 chambers.*
(2) *Up to permutation of the co-ordinates, $\mathsf{Cone}(D_n, \mathcal{J})$ is one of the following:*

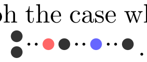


where the red line is $x + y = 0$ and the green line $2x + y = 0$.

Proof. (1) This follows immediately from 3.2.
(2) The case of D_4 is slightly degenerate. Up to mutation, and symmetries of the graph any $\mathcal{J}$ with $|\mathcal{J}^c| = 2$ is either the example in 3.3(1) or 3.3(2). These have have either six chambers, or eight chambers respectively, and are precisely the arrangements claimed.

Hence we can consider D_n with $n \geq 5$, and we draw this as . Consider the leftmost vertex in $\mathcal{J}^c$, and suppose it contains at least one of the left rank-one vertices. If it contains both, namely $\mathcal{J} =$ , then the restricted roots are $10, 01, 11$ and so we obtain the six chamber arrangement claimed. Note that this $\mathcal{J}$ is mutation equivalent to  and so this too gives the six chamber arrangement. Hence, we can assume that the second vertex in $\mathcal{J}^c$ does not have rank one, which up to symmetry of the left rank-one vertices means $\mathcal{J} =$ . In this case, the restricted roots are $10, 01, 11, 12$, which up to permutation of the co-ordinates is the eight chamber arrangement claimed.

Hence we can assume that the leftmost vertex in $\mathcal{J}^c$ does not have rank one. If the rightmost element of $\mathcal{J}^c$ has rank one, there are two cases. Firstly, if $\mathcal{J} =$ , then the restricted roots are $10, 01, 11, 21$, which is the eight chamber arrangement claimed. Secondly, if $\mathcal{J} =$ , then this is mutation equivalent to . This has restricted roots $10, 01, 11, 22, 21$, which gives the eight chamber arrangement claimed.

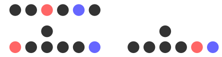
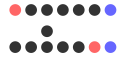
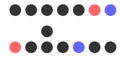
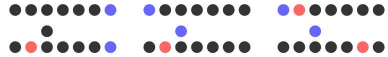
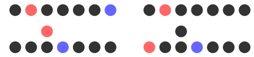
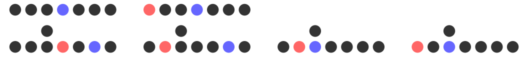
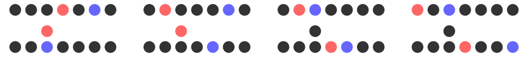
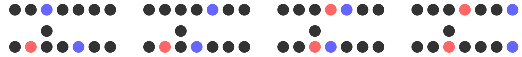
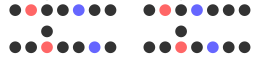
Hence we can assume that both vertices in $\mathcal{J}^c$ have rank two, and further by the above paragraph the case when both vertices are adjacent has already been covered. This means that $\mathcal{J} =$ . In this case, the restricted roots are $01, 10, 20, 11, 22, 21$, and again we obtain the eight chamber arrangement claimed. □

Remark 3.9. Another way to approach 3.8(2) is to classify the mutation classes in D_n with $|\mathcal{J}^c| = 2$, then run the techniques as in 3.3, with the simplification in 3.4. This method is slightly more time-consuming to write, but in practice is often more useful. It turns out that D_4 has four mutation classes, D_{2n+1} with $n \geq 2$ has n^2 mutation classes, and D_{2n} with $n \geq 3$ has $n^2 - 1$ mutation classes. Up to symmetries of the graph, D_4 has two classes, D_{2n+1} has n^2 classes, and D_{2n} with $n \geq 3$ has $n^2 - (n-1)$ classes.

The following classifies $\mathsf{Cone}(\Delta, \mathcal{J})$ when $\Delta = E_6, E_7, E_8$, and $|\mathcal{J}^c| = 2$, by first classifying the mutation classes, then performing a single calculation for each one. This reduces the number of cases substantially, whilst giving very precise information in each case.

Remarkably, all phenomena can be found using only E_6, E_7, and E_8. The hyperplane arrangements from Type A and D repeat here, but more arrangements are obtained. Later in Chapter 4 the same phenomena occurs: all our affine tilings in $\mathbb{R}^2$ can also be realised using only type E.

THEOREM 3.10. *For $\Delta = E_6, E_7, E_8$, and $|\mathcal{J}^c| = 2$, the mutation classes, and in each case the number of chambers in* $\mathsf{Cone}(\Delta, \mathcal{J})$*, are as follows.*

Family	Mutation Class	|Chambers|
$E_{6,1}$		10
$E_{6,2}$		6
$E_{6,3}$		12
$E_{6,4}$		8
$E_{6,5}$		8
$E_{7,1}$		8
$E_{7,2}$		10
$E_{7,3}$		12
$E_{7,4}$		12
$E_{7,5}$		12
$E_{7,6}$		12
$E_{7,7}$		8
$E_{7,8}$		12
$E_{7,9}$		8
$E_{8,1}$		8
$E_{8,2}$		12
$E_{8,3}$		12
$E_{8,4}$		12
$E_{8,5}$		12
$E_{8,6}$		12
$E_{8,7}$		16
$E_{8,8}$		16
$E_{8,9}$		16
$E_{8,10}$		16
$E_{8,11}$		16

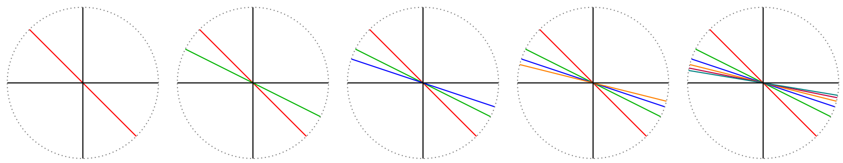
PROOF. In each line, it is easy to verify that the diagrams listed are all linked via the wall crossing rule 1.16, and further that the diagrams listed exhaust all possibilities of wall crossing at the coloured vertices. In particular, each line is a full mutation class.

Since $\binom{6}{2} = 15$, and the total number of diagrams in the five stated E_6 families is also 15, we have exhausted all possible choices of two nodes in an E_6 Dynkin diagram. Consequently, there can be no further E_6 families. Similarly, since $\binom{7}{2} = 21$, and $\binom{8}{2} = 28$, and there are 21 diagrams in the E_7 families above, and 28 diagrams in the E_8 families above, there are no further E_7 or E_8 mutation families. It follows that the above is a complete classification of mutation classes for E_6, E_7 and E_8.

The last column is a case-by-case analysis, using the method in 3.5, 3.6 and 3.7, or by looking at the restricted roots. □

The following is the main result in this section.

COROLLARY 3.11. *Suppose that* Δ *is Dynkin, and* $\mathcal{J} \subseteq \Delta$ *with* $|\mathcal{J}^c| = 2$*. Then, up to changing the slopes of some of the hyperplanes,* $\mathsf{Cone}(\Delta, \mathcal{J})$ *is one of the following five hyperplane arrangements.*



In each case, the number of chambers is $6, 8, 10, 12$ *and* 16 *respectively.*

PROOF. Simplicial hyperplane arrangements in $\mathbb{R}^2$ are determined, up to the slopes of the hyperplanes, by the number of chambers. Since we are ignoring slops, the result follows from the either 3.8 or the last column in 3.10, which is always $6, 8, 10, 12$ or 16. ☐

3.4. Rank Three Phenomena

The last section classified the $\mathsf{Cone}(\Delta, \mathcal{J})$ inside $\mathbb{R}^2$, when Δ is ADE Dynkin. It is clear that all simplicial arrangements inside $\mathbb{R}^2$ are, up to moving the slopes of the hyperplanes, Coxeter arrangements. In this section we will consider those $\mathsf{Cone}(\Delta, \mathcal{J})$ inside $\mathbb{R}^3$ when Δ is ADE Dynkin, and demonstrate both the non-Coxeter nature, and also the surprising complexity that occurs.

By 3.2, any choice of three vertices in Type A only ever gives Type A_3, which has 6 hyperplanes and 24 chambers. Thus, to find new phenomena, we must consider types D and E. The following is the most elementary example.

EXAMPLE 3.12. Consider $\mathcal{J} =$ . Then $\mathsf{Cone}(D_4, \mathcal{J})$ is the following non-Coxeter arrangement.

$$\begin{aligned} \vartheta_1 &= 0 \\ \vartheta_2 &= 0 \\ \vartheta_3 &= 0 \\ \vartheta_1 + \vartheta_2 &= 0 \\ \vartheta_1 + \vartheta_3 &= 0 \\ \vartheta_2 + \vartheta_3 &= 0 \\ \vartheta_1 + \vartheta_2 + \vartheta_3 &= 0 \end{aligned}$$

It turns out that there are many more possible arrangements that can occur. Both the number of hyperplanes and the number of chambers becomes surprisingly large.

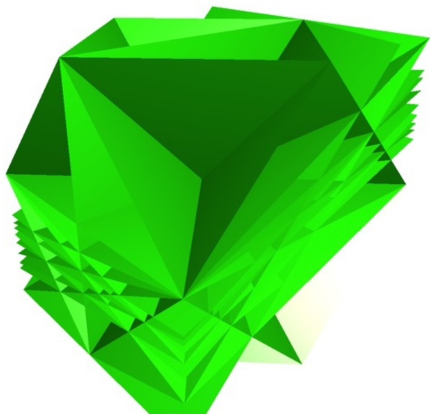
THEOREM 3.13. *Let* Δ *be ADE, and consider* $\mathcal{J} \subset \Delta$ *with* $|\mathcal{J}^c| = 3$*. Then* $\mathsf{Cone}(\Delta, \mathcal{J})$ *has either* $6, 7, 8, 9, 10, 11, 13, 16, 17$ *or* 19 *hyperplanes.*

PROOF. This is a case-by-case analysis over all types. As above, by 3.2, any choice of three vertices in Type A only ever gives Type A_3, which has 6 hyperplanes and 24 chambers. Type D is mildly harder, as in 3.8(2), however, any choice of three vertices gives 7, 8 or 9 hyperplanes. For E_6, via an exhaustive calculation aided by magma, any choice of three vertices gives either 8 or 10 hyperplanes. Similarly, for E_7, any choice of three vertices gives either 9, 10, 11 or 13 hyperplanes, and for E_8, any choice of three vertices gives 13, 16, 17 or 19 hyperplanes. ☐

REMARK 3.14. At this stage, it is not clear whether the number of hyperplanes in 3.13 determine the arrangement. It would seem that there are always 24, 32, 40, 48, 60, 72, 96, 144, 160, or 192 chambers respectively, but it is not clear whether the 1-skeleta remain constant within a given class.

The following is one example of a $\mathsf{Cone}(\Delta, \mathcal{J})$ inside $\mathbb{R}^3$ with 19 hyperplanes. This ignores multiplicity; with multiplicity there are 23. We explicitly describe the arrangement here, given it is our largest example inside $\mathbb{R}^3$.

EXAMPLE 3.15. Consider $\mathcal{J} =$ . Then $\mathsf{Cone}(E_8, \mathcal{J})$ is the following hyperplane arrangement, which has 19 hyperplanes and 192 chambers.



$$\begin{aligned} \vartheta_1 &= 0 \\ \vartheta_2 &= 0 \\ \vartheta_3 &= 0 \\ \vartheta_1 + \vartheta_2 &= 0 \\ \vartheta_2 + \vartheta_3 &= 0 \\ 2\vartheta_2 + 3\vartheta_3 &= 0 \\ \vartheta_2 + 2\vartheta_3 &= 0 \\ \vartheta_2 + 3\vartheta_3 &= 0 \end{aligned} \qquad \begin{aligned} \vartheta_1 + \vartheta_2 + \vartheta_3 &= 0 \\ \vartheta_1 + \vartheta_2 + 2\vartheta_3 &= 0 \\ \vartheta_1 + \vartheta_2 + 3\vartheta_3 &= 0 \\ \vartheta_1 + 2\vartheta_2 + 2\vartheta_3 &= 0 \\ \vartheta_1 + 2\vartheta_2 + 3\vartheta_3 &= 0 \\ \vartheta_1 + 2\vartheta_2 + 4\vartheta_3 &= 0 \\ \vartheta_1 + 3\vartheta_2 + 3\vartheta_3 &= 0 \\ \vartheta_1 + 3\vartheta_2 + 4\vartheta_3 &= 0 \\ \vartheta_1 + 3\vartheta_2 + 5\vartheta_3 &= 0 \\ \vartheta_1 + 3\vartheta_2 + 6\vartheta_3 &= 0 \\ \vartheta_1 + 4\vartheta_2 + 6\vartheta_3 &= 0 \end{aligned}$$

CHAPTER 4

Affine Tilings in $\mathbb{R}^2$

Given a subset $\mathcal{J}$ of vertices of the Dynkin diagram Δ, Section 2.2 constructs an infinite hyperplane arrangement $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$ inside $\mathbb{R}^{|\mathcal{J}^c|}$. The case $|\mathcal{J}^c| = 2$ is then particularly important: given any choice of two vertices $\mathcal{J}^c$ in an ADE Dynkin diagram, there is a corresponding tiling of the plane $\mathbb{R}^2$. This chapter classifies the tilings that are constructed in this way. The main result is that there are precisely sixteen tilings, counted with $\mathbb{Z}^2$ action, or fifteen counted without the $\mathbb{Z}^2$ action. Only three of the fifteen are Coxeter.

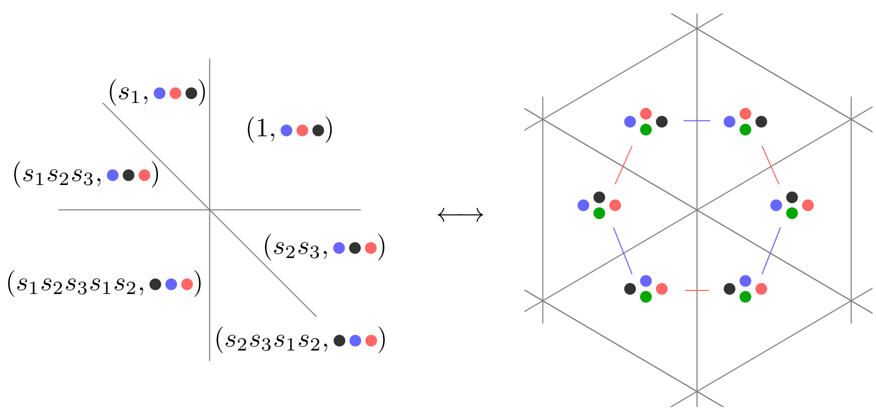
We view these new tilings as fundamental building blocks, and so in this chapter we draw each of them in some detail. They turn out to exhibit remarkable phenomena, which have further applications later in this memoir, and also elsewhere in the literature.

4.1. The Main Techniques

In this section we show how to calculate the associated tiling of the plane in one extended example. In later sections we use this technique implicitly, and for brevity we will mainly just present the results for the exceptional types, without outlining the underlying calculations.

Since wall crossing is combinatorial, we will begin in one chamber, and wall-cross repeatedly. The first step is always to view $\mathsf{Cone}(\Delta, \mathcal{J})$ inside $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$, as illustrated in the following example.

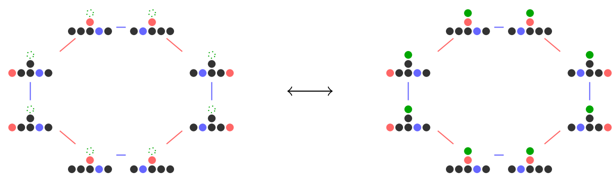
Example 4.1. Consider $\mathcal{J}$ = ●●●, which is $\mathcal{J}$ = ●●●● when viewed inside the affine diagram. Since $\mathsf{Cone}(\Delta, \mathcal{J}) \subset \mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$, we begin by fixing the extended vertex, and mutating at the two other colours. Since the $w_J w_{J+i}$ wall crossing rules are not effected by the addition of the extended vertex, the calculation 1.11 can be transferred to describe part of the affine tiling (see also 5.27). This is illustrated below, where in the right hand picture we drop the labelling coming from the Weyl group for convenience.



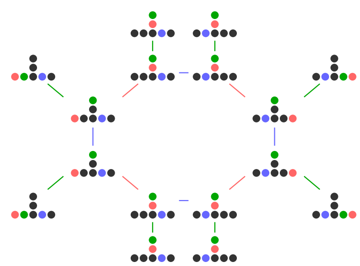
All the tilings below will be calculated by taking the first step $\mathsf{Cone}(\Delta, \mathcal{J}) \subset \mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$, and for each of those chambers, calculating the wall crossing for the extended vertex. From here, the calculations in the rank two ADE setting in §3.2, and especially 3.4, will be used to determine the size of the new tiles that are glued on. The calculation repeats, stopping once it becomes clear that the same tiles are being repeated.

We illustrate this main argument in one extended example.

EXAMPLE 4.2. As in 3.5(3) consider $\mathcal{J}$ = , which is  when viewed inside the affine diagram. As in 4.1, we begin with $\mathsf{Cone}(\Delta, \mathcal{J}) \subset \mathsf{Level}\,\mathcal{J}_{\mathsf{aff}}$, obtained by fixing the extended vertex, and mutating at the others. As above, since the $w_J w_{J+i}$ wall crossing rules are not effected by the addition of the extended vertex, the extended vertex can be ignored for the purposes of this calculation. So, 3.5(3) gives the left hand side of the following, namely an 8-gon with appropriate labels. The right hand side views this inside the affine picture.



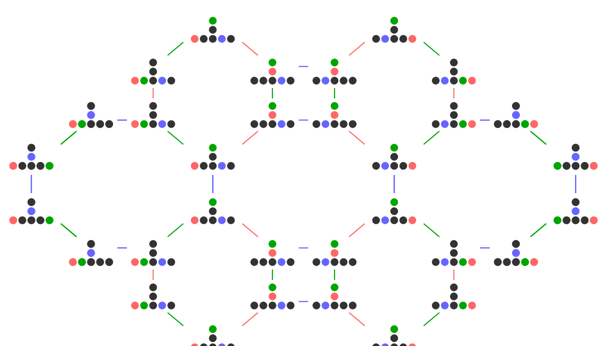
The calculation continues by crashing through the remaining wall in each of these chambers. In each case, this is the wall crossing given by the extended vertex. We thus extend the calculation out, to cover these green wall crossings, and obtain the following.



We next complete the outer edges into full n-gons. The value of n varies. For example, consider the very bottom of the above picture, where red is being fixed, and we are mutating green and blue. Since removing the green vertex leaves a disjoint union, in which the red and blue vertices are in different pieces, the wall-crossing rules in §3.2 imply a length two braid relation, and so we glue on a 4-gon to the bottom of the picture.

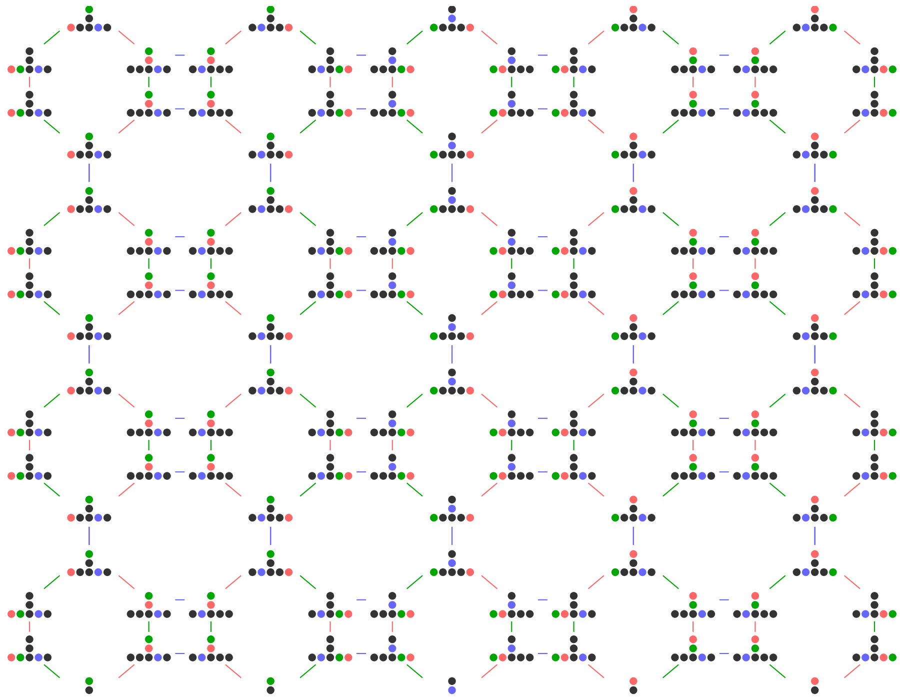
In contrast, consider the right hand side of the above, where red is fixed, and we are mutating green and blue. As in 4.1, the $w_J w_{J+i}$ wall crossing rule is not effected if we delete the fixed red vertex. Doing this we obtain the situation in 3.5(3), and thus we glue an 8-gon onto the right hand side.

Continuing, completing the outer edges into full n-gons gives the following.



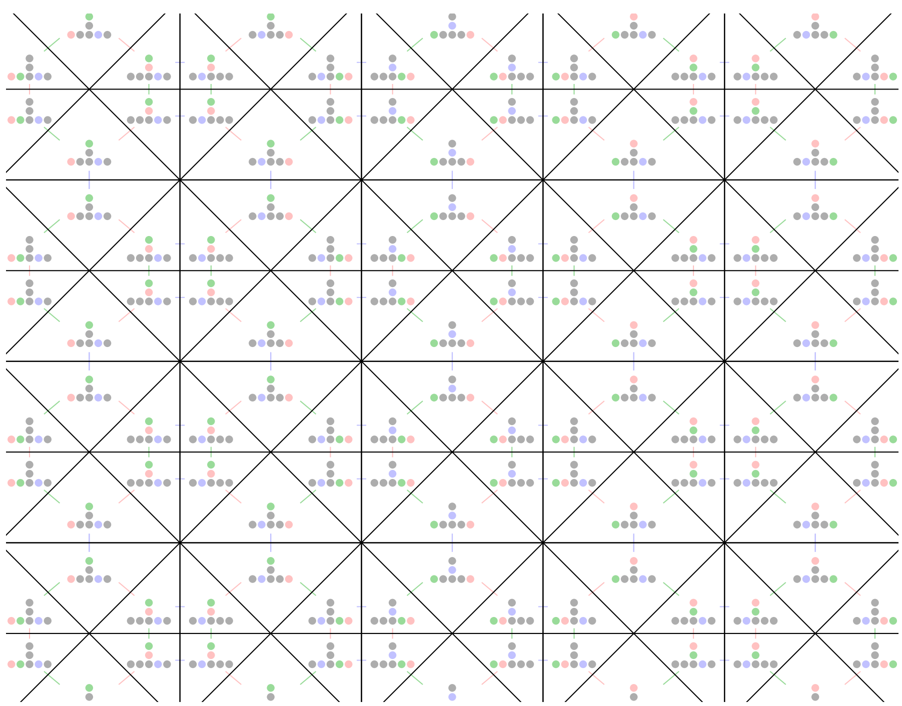
We next continue by completing the new outer edges into full n-gons. Again, in each case, 6-gons and 8-gons arise by 3.5(3) or 3.5(2), and sometimes 6-gons arise by the Type A situation (see 3.2). The 4-gons always arise whenever removing the fixed vertex leaves a disjoint union, in which the two remaining vertices are in different pieces.

Continuing in this way, over and over, we obtain the following.



There is a clear line of reflection in a central vertical line, which fixes blue but swaps green and red. The is also a line of reflection in a central horizontal line, which fixes all colours. This determines the rest of the tiling.

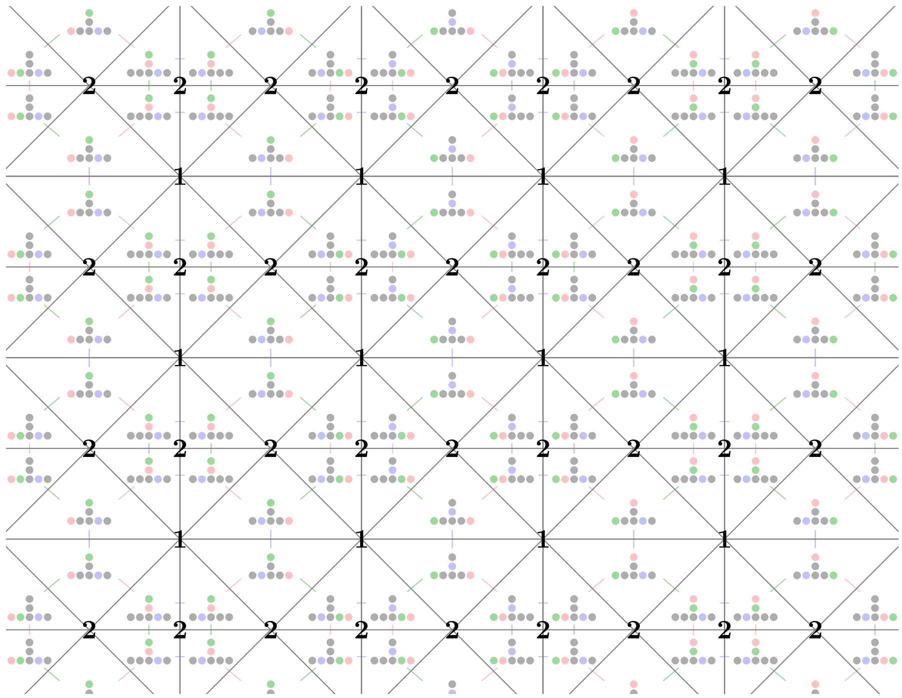
Since in the above picture, at every label there are three wall crossings, it follows that there are no further chambers, and so the above is the 1-skeleton of the hyperplane arrangement. As such, the infinite hyperplane arrangement is the following.



This is example $\mathcal{E}_{6,5}$ in 4.16 later.

All the hyperplane arrangements in §4.2 below are calculated using the above method. In each case, after deleting the fixed vertex for any potential n-gon, the value of n is calculated using 3.4.

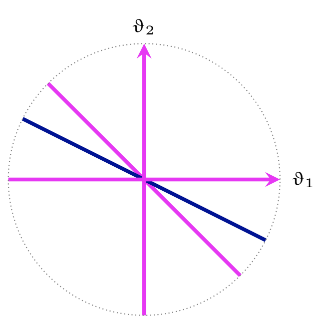
Remark 4.3. The above calculation enriches the hyerplane arrangement with the information of the null root $\delta := \sum_{i\in\Delta_{\mathsf{aff}}} \delta_i\alpha_i$. Indeed, if at the centre of each n-gon we record the value of δ_i for the vertex in $\mathcal{J}^c$ which is being fixed, we obtain the following.



The 1's form a $\mathbb{Z}^2$ lattice, which we will later refer to as the action of the *class group*. This action becomes very important in Part 3.

Remark 4.4. It is possible to calculate the hyperplane arrangements by translating the finite arrangements and their multiples (see below). However, the above gives us more information, as it links different choices through wall crossing, and thus makes classification significantly easier. The above method gives the class group action, and the other numbers in 4.3 also become important in Part 3 later.

We also illustrate the translation method here. Consider $\mathcal{J} =$ , which exactly as 3.5(3) has the following finite arrangement:

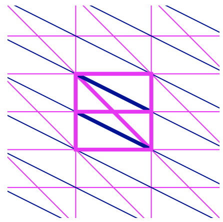


Hyperplane	Restricted Roots
$\vartheta_2 = 0$	$01, 02$
$\vartheta_1 + 2\vartheta_2 = 0$	12
$\vartheta_1 + \vartheta_2 = 0$	11
$\vartheta_1 = 0$	10

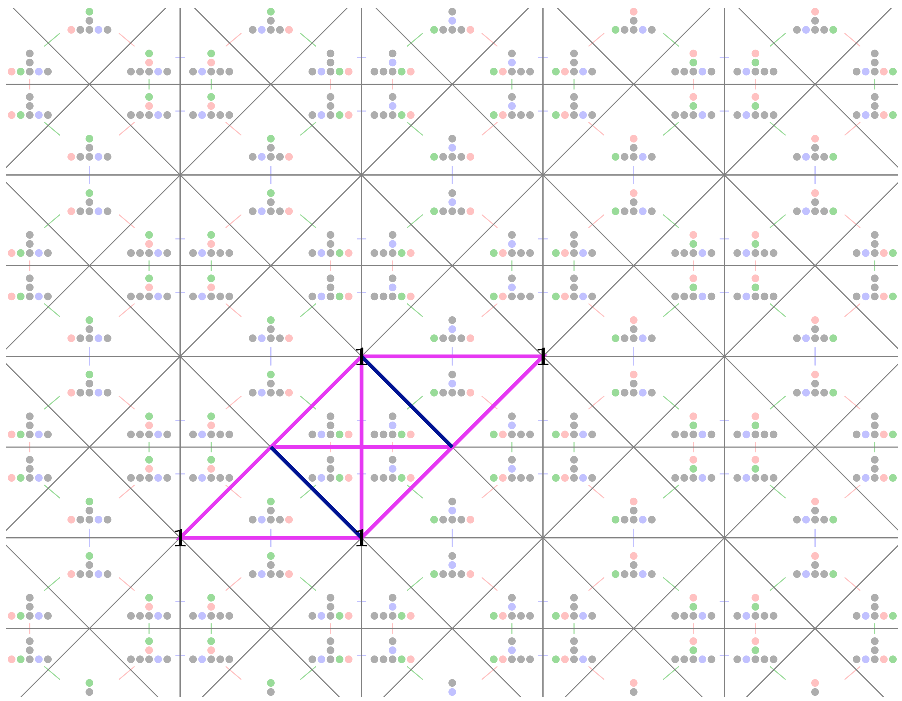
We then translate all hyperplanes given by the restricted roots, and so consider

$$\vartheta_1 = z, \qquad \begin{matrix}\vartheta_2 = z,\\ 2\vartheta_2 = z,\end{matrix} \qquad \vartheta_1 + \vartheta_2 = z, \quad \vartheta_1 + 2\vartheta_2 = z$$

for all $z \in \mathbb{Z}$. The set where all $z = 0$ is precisely the finite arrangement illustrated, whilst translating all by the integers give the following infinite arrangement, extended to infinity in all directions:



The area in bold bounds a rectangle from $(0,0)$ to $(1,1)$, and is the fundamental region. It terms of the arrangement in 4.3, this bold region corresponds to the following:



4.2. Summary of Classification

Given a subset $\mathcal{J}$ of vertices of a Dynkin diagram Δ, such that $|\Delta \backslash \mathcal{J}| = 2$, this section classifies the infinite hyperplane arrangements $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$ that can occur. Equivalently, in the notation of §2.2, it classifies those $\mathsf{Level}(\Delta_{\mathsf{aff}}, \mathcal{K})$ such that $|\Delta_{\mathsf{aff}} \backslash \mathcal{K}| = 3$ and furthermore $\mathcal{K}$ is mutation equivalent to $\mathcal{J}$ such that $\Delta_{\mathsf{aff}} \backslash \mathcal{J}$ contains the extended vertex.

For our applications later, we are mostly interested in the hyperplane arrangements $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$ equipped with their class group $\mathbb{Z}^2$-action explained in 4.3. Consequently, we classify not the abstract arrangements (of which there are 14), but instead the arrangements together with the $\mathbb{Z}^2$-action (of which there are 16). Of course, by the wall crossing rules, the only possible $\mathsf{Level}(\Delta_{\mathsf{aff}}, \mathcal{K})$ that have a class group action are those $\mathcal{K}$ which are, up to symmetries of the graph, mutation equivalent to $\mathcal{J}$ such that $\Delta_{\mathsf{aff}} \backslash \mathcal{J}$ contains the extended vertex. This is why we restrict our attention to $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$. However, many of our arguments are more general, and we will make various remarks on the more general case $\mathsf{Level}(\Delta_{\mathsf{aff}}, \mathcal{K})$ throughout.

4.2.1. Extended Type A. Exactly as above 3.2 in the finite type A setting, it turns out that constructions in extended Type A also only ever give extended Type A

phenomena, as follows. This implies that no new phenomena arise in extended Type A, and so for classification purposes it can largely be ignored.

PROPOSITION 4.5. *Let $\Delta = A_n$. Then for any $\mathcal{K} \subsetneq \Delta_{\mathsf{aff}}$, the cone $\mathsf{Cone}(\Delta_{\mathsf{aff}}, \mathcal{K})$ is precisely the Tits cone $\mathsf{Cone}(\Gamma_{\mathsf{aff}})$ for $\Gamma = A_{n-|\mathcal{K}|}$.*

PROOF. If $\mathcal{K} = \emptyset$ then $\mathsf{Cone}(\Delta_{\mathsf{aff}}, \mathcal{K}) = \mathsf{Cone}(\Delta_{\mathsf{aff}})$ and there is nothing to prove, hence we can assume that $\mathcal{K} \neq \emptyset$. This being the case, label the vertices of Δ_{aff} by $0, 1, \dots, n$, reading around the circle clockwise, where 0 is the extended vertex, and suppose that the elements of $\Delta_{\mathsf{aff}} \backslash \mathcal{K}$ are $i_1 < \dots < i_t$. By rotating if necessary, up to symmetries of the graph we can assume that $i_1 = 0$.

The fact that the vertices $i_2, \dots, i_t$ satisfy the Type A braid rules is 3.2, using 4.1 to see that adding the extended vertex does not effect the braiding. It is easy to check that the extended vertex braids with i_2 and i_t, and commutes with the others. Thus, although the local labels change, $\mathsf{Cone}(\Delta_{\mathsf{aff}}, \mathcal{K})$ is controlled by the same global rules as the Tits cone for affine $A_{n-|\mathcal{K}|}$, and hence it is the Tits cone $\mathsf{Cone}(\Gamma_{\mathsf{aff}})$ for $\Gamma = A_{n-|\mathcal{K}|}$.

The other way to see the result is that, since we can assume $i_1 = 0$ by rotating if necessary, as in 4.4 the level can be obtained from the finite root system by translating the restricted roots. By 3.2 these are connected chains of 1s, and hence translating them gives the standard level of affine $A_{n-|\mathcal{K}|}$. Since by 2.2 the cones can be recovered from their levels, the result follows. ☐

REMARK 4.6. The rotational symmetry used in the proof of 4.5 above shows that, for $\Delta = A_n$, the more general setting of $\mathsf{Level}(\Delta_{\mathsf{aff}}, \mathcal{K})$ from §2.2 yields no new arrangements than simply considering those $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$ with $\mathcal{J} \subseteq \Delta$, irrespective of $|\mathcal{J}|$

4.2.2. Extended Type D. In this subsection we consider the case $\Delta = D_n$. By 3.8(2), up to slopes, $\mathsf{Cone}(\Delta, \mathcal{J})$ with $|\mathcal{J}^c| = 2$ can be one of two options. However, both the slopes and the multiplicities of the hyperplanes are required in order to translate these arrangements, as in 4.4, to obtain the affine versions.

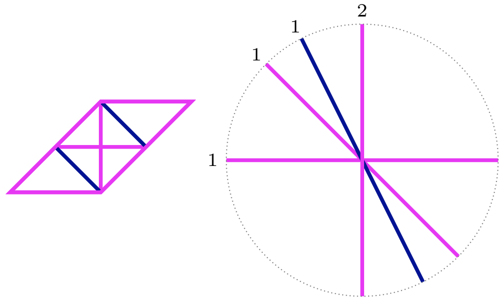
The following show that, in total, if $\Delta = D_n$ and $\mathcal{J} \subseteq \Delta$ with $|\mathcal{J}^c| = 2$, then $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$ is one of four affine arrangements. One is the classical affine A_2 arrangement, and two (4.15 and 4.17) are the classical affine B_2 arrangement, albeit with different $\mathbb{Z}^2$-lattices. The other, namely 4.16, is similar to affine B_2, but is mildly different.

PROPOSITION 4.7. *Let $\Delta = D_n$, and $\mathcal{J} \subseteq \Delta$ with $|\mathcal{J}^c| = 2$. Then $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$, together with its class group action, is one of the four arrangements on the right hand side of the following table. The middle column shows the restricted roots that translate to give the affine arrangement, and thus $\mathcal{J}$ is mutation equivalent to some $\mathcal{I}$ for which $\mathsf{Cone}(\Delta, \mathcal{I})$ has these restricted roots.*

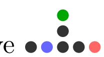
Family Name	Finite Arrangement	Affine Arrangement
$\mathcal{D}_1$	$\{10, 01, 11\}$	4.18
$\mathcal{D}_2$	$\{10, 01, 11, 21\}$	4.15
$\mathcal{D}_3$	$\{10, 01, 11, 22, 21\}$	4.16
$\mathcal{D}_4$	$\{01, 10, 20, 11, 22, 21\}$	4.17

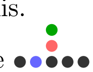
PROOF. The key point is that, inspecting the proof of 3.8(2), we see that $\mathcal{J}$ is mutation equivalent to some $\mathcal{I}$ for which $\mathsf{Cone}(\Delta, \mathcal{I})$ is given by one of the four sets of restricted roots listed. Our method of calculating the affine arrangement via wall crossing in 4.2 shows that $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}}) = \mathsf{Level}(\mathcal{I}_{\mathsf{aff}})$, i.e. the affine arrangements are the same. As in 4.4, we can then calculate $\mathsf{Level}(\mathcal{I}_{\mathsf{aff}})$ by translating the finite hyperplanes listed, together with multiplicity. This results in 4.18, 4.15, 4.16, and 4.17 respectively, where in each case the translation is visually illustrated. ☐

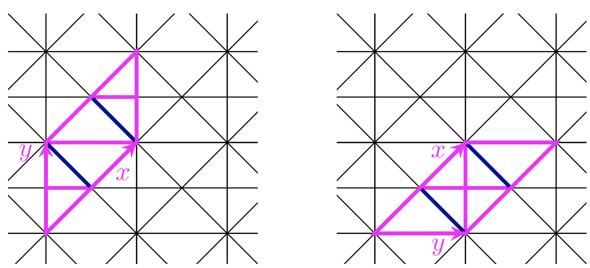
REMARK 4.8. The 'up to mutation' in the statement and proof of 4.7 is important, but subtle. Indeed, $\mathsf{Cone}(\Delta, \mathcal{J})$ may be given by a different set of restricted roots than those listed, but after translation these still give the affine arrangement listed. Such a phenomena already appears in 4.4, where given the representative $\mathcal{J} =$  of the mutation class, both the affine fundamental region and the finite hyperplane arrangement are illustrated below, with the fundamental region being exactly as in 4.4.



Hyperplanes	
01	—
11	\
21	—
10, 20	|

Note that, in 4.4, the affine representative  sits in the bottom left triangle of this fundamental region. Also, we remark that the bottom horizontal line on the left hand side corresponds to the vertical y-axis in the above arrangement, and the leftmost diagonal on the left hand side corresponds to the x-axis.

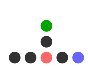
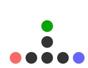
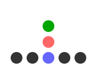
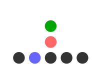
On the other hand, the representative  can be obtained as a single wall crossing. In contrast, its fundamental region and finite hyperplane arrangement (ignoring the extended vertex) are illustrated in 4.16. Both of these are mildly different to the fundamental region and the finite hyperplane arrangement above, however, being mutation equivalent, they give the same tiling. Visually, the two fundamental regions are as follows.



4.2.3. Extended Type E_6. In the previous subsections, $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$ with $\mathcal{J} \subseteq A_n, D_n$ and $|\mathcal{J}^c| = 2$ has yielded only four affine arrangements. Many more arise from extended type E, and in this subsection we consider the case $\Delta = E_6$, and $\mathcal{K} \subseteq \Delta_{\mathsf{aff}}$ with $|\Delta_{\mathsf{aff}} \backslash \mathcal{K}| = 3$. In this case, it turns out to be easiest to first classify the affine mutation classes, then in each case to calculate the associated affine arrangement.

As symmetries of the graph do not effect the arrangements, in fact it suffices to classify the mutation classes, up to symmetries.

Proposition 4.9. *Up to symmetries, there are five mutation classes for $\mathcal{K} \subseteq \Delta_{\mathsf{aff}}$, when $\Delta = E_6$ and $|\Delta_{\mathsf{aff}} \backslash \mathcal{K}| = 3$. Four of these can be realised via $\mathcal{J} \subseteq \Delta$, and one cannot. The table below summarises a representative of each mutation class, the number in each, and in the case when the class arises from Δ, where the associated affine arrangement may be found.*

Rep.						
Label	$\mathcal{E}_{6,1}$	$\mathcal{E}_{6,2}$	$\mathcal{E}_{6,3}$	$\mathcal{E}_{6,4}$	$\mathcal{E}_{6,5}$	$\mathcal{E}_{6,6}$
|Class|	15	1	1*	6	9	1
Arr.	4.19	4.18	4.20	4.15	4.16	x

The star denotes a mutation class that is not closed under symmetries. Thus $\mathcal{E}_{6,3}$ represents any of three singleton mutation classes obtained from the representative above, and its image under the action of $\mathbb{Z}_3$.

Proof. Via the wall-crossing rule 1.16, it is very easy to check, in each case, that the number in the mutation class is as claimed. Since there are three versions of $\mathcal{E}_{6,3}$, in total there are $15+1+3+6+9+1=35$ diagrams in the above mutation classes. Since $\binom{7}{3}=35$, the above classes must be them all. In each case, the affine arrangement can be calculated by using either the method in 4.2, or the translation method in 4.4. This is briefly summarised in each of the corresponding linked examples. □

Remark 4.10. The method in 4.2 can be used to calculate $\mathsf{Level}(\Delta_{\mathsf{aff}},\mathcal{K})$ for the family $\mathcal{E}_{6,6}$. This gives the honeycomb tiling, but where the numbers on the intersection points (as in 4.3) are always 2. Hence, ignoring the $\mathbb{Z}^2$ lattice, this is 4.18.

4.2.4. Extended Type E_7. This subsection considers the case $\Delta = E_7$, and $\mathcal{K} \subseteq \Delta_{\mathsf{aff}}$ with $|\Delta_{\mathsf{aff}}\backslash\mathcal{K}| = 3$. Again, it turns out to be easiest to first classify the affine mutation classes, then in each case to calculate the associated affine arrangement. The symmetry group is now $\mathbb{Z}_2$.

Proposition 4.11. *Let $\Delta = E_7$, and $\mathcal{K} \subseteq \Delta_{\mathsf{aff}}$ with $|\Delta_{\mathsf{aff}}\backslash\mathcal{K}| = 3$. Up to symmetries of the graph, the table below summarises the mutation classes, a representative of each, the number in each, and in the case when the class arises from Δ, where the associated affine arrangement may be found.*

Rep.						
Label	$\mathcal{E}_{7,1}$	$\mathcal{E}_{7,2}$	$\mathcal{E}_{7,3}$	$\mathcal{E}_{7,4}$	$\mathcal{E}_{7,5}$	$\mathcal{E}_{7,6}$
|Class|	3	10	1*	8	5*	11
Arr.	4.16	4.19	4.20	4.22	4.23	4.25

Rep.					
Label	$\mathcal{E}_{7,7}$	$\mathcal{E}_{7,8}$	$\mathcal{E}_{7,9}$	$\mathcal{E}_{7,10}$	$\mathcal{E}_{7,11}$
|Class|	1*	2*	2	3	1
Arr.	4.17	4.21	4.15	x	x

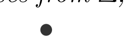
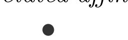
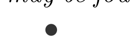
Again, $*$ denotes a mutation class that is not closed under symmetries, so each class starred represents two classes, of the size stated.

Proof. The proof is the same as 4.9: via the wall-crossing rule 1.16, it is very easy to check, in each case, that the number in the mutation class is as claimed. As each starred class is doubled, since it represents two classes, the total number in the classes above is 56, which equals $\binom{8}{3}$. Hence these are the only classes. Again, in each case, the affine arrangement can be calculated either by using 4.2 or 4.4. □

Remark 4.12. As in 4.10, ignoring the $\mathbb{Z}^2$-lattice, the family $\mathcal{E}_{7,11}$ gives the affine B_2 arrangement 4.15, and the family $\mathcal{E}_{7,10}$ gives the arrangement 4.16.

4.2.5. Extended Type E_8. This subsection considers the case $\Delta = E_8$, and $\mathcal{K} \subseteq \Delta_{\mathsf{aff}}$ with $|\Delta_{\mathsf{aff}}\backslash\mathcal{K}| = 3$. As above, we first classify the affine mutation classes, then in each case to calculate the associated affine arrangement. The symmetry group is now trivial.

Proposition 4.13. *Let $\Delta = E_8$, and $\mathcal{K} \subseteq \Delta_{\mathsf{aff}}$ with $|\Delta_{\mathsf{aff}}\backslash\mathcal{K}| = 3$. The table below summarises the mutation classes, a representative of each, the number in each, and in the case where the class arises from Δ, where the associated affine arrangement may be found.*

Rep.						
Label	$\mathcal{E}_{8,1}$	$\mathcal{E}_{8,2}$	$\mathcal{E}_{8,3}$	$\mathcal{E}_{8,4}$	$\mathcal{E}_{8,5}$	$\mathcal{E}_{8,6}$
|Class|	1	1	2	5	4	7
Arr.	4.17	4.20	4.21	4.23	4.22	4.24

Rep.					
Label	$\mathcal{E}_{8,7}$	$\mathcal{E}_{8,8}$	$\mathcal{E}_{8,9}$	$\mathcal{E}_{8,10}$	$\mathcal{E}_{8,11}$
|Class|	5	12	11	15	10
Arr.	4.29	4.27	4.28	4.26	4.30

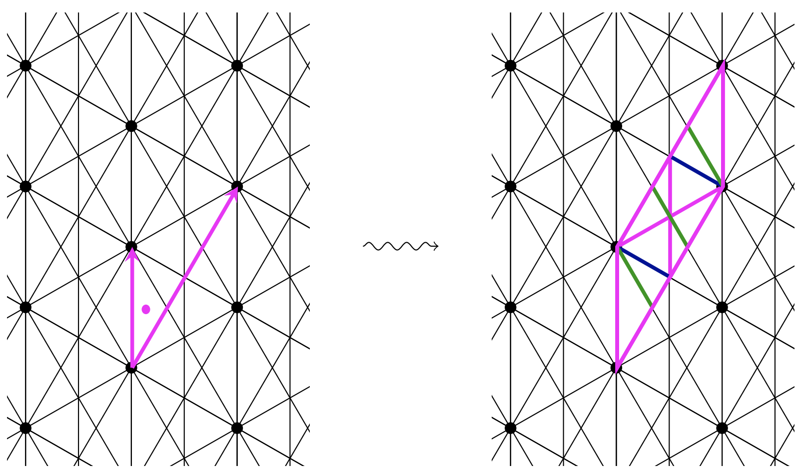
Rep.						
Label	$\mathcal{E}_{8,12}$	$\mathcal{E}_{8,13}$	$\mathcal{E}_{8,14}$	$\mathcal{E}_{8,15}$	$\mathcal{E}_{8,16}$	$\mathcal{E}_{8,17}$
|Class|	2	1	1	1	1	5
Rep.	x	x	x	x	x	x

Proof. The proof is the same as 4.11: via the wall-crossing rule 1.16, it is very easy to check, in each case, that the number in the mutation class is as claimed. The total number in the classes stated above is 84, which equals $\binom{9}{3}$, hence these are the only mutation classes. Again, in each case, the affine arrangement can be calculated either by using 4.2 or 4.4. □

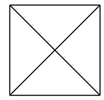
Remark 4.14. As in 4.10 and 4.12, using the method in 4.2 if we ignore the $\mathbb{Z}^2$-lattices, the families $\mathcal{E}_{8,12}$, $\mathcal{E}_{8,13}$, $\mathcal{E}_{8,14}$, $\mathcal{E}_{8,15}$, $\mathcal{E}_{8,16}$, and $\mathcal{E}_{8,17}$ give the arrangements 4.21, 4.15, 4.15, 4.20, 4.20 and 4.23 respectively.

4.2.6. Summary of Notation and Tables. Using the above subsections, the following sixteen pages summarise the classification of arrangements in $\mathbb{R}^2$ obtained as $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$ for some $\mathcal{J} \subseteq \Delta$, with Δ ADE Dynkin.

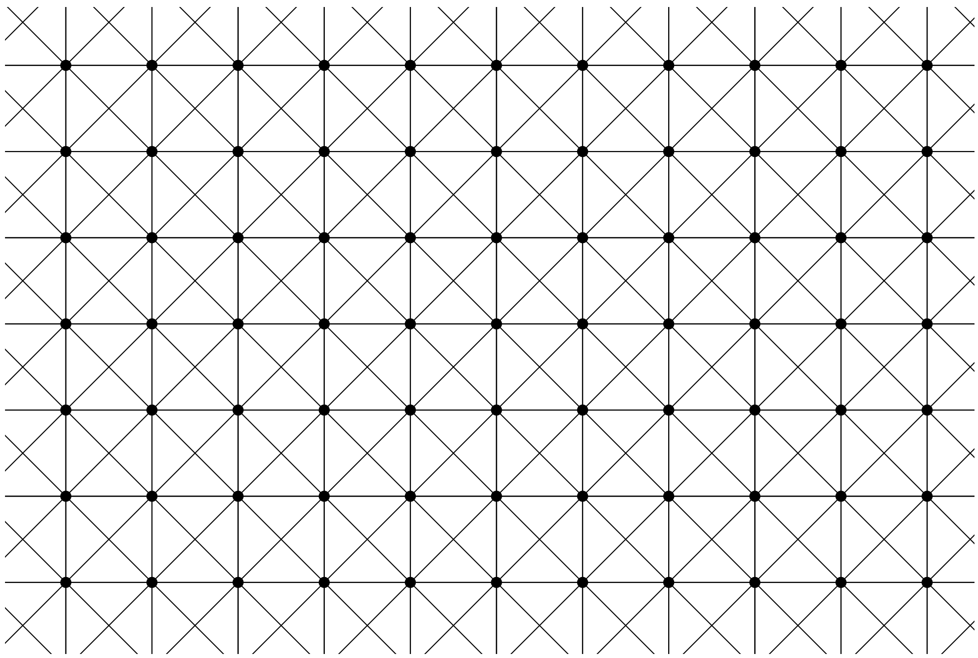
In each case, we draw the arrangement equipped with the $\mathbb{Z}^2$-lattice explained in 4.3. We will also draw a fundamental domain in each case. As in 4.8, the fundamental domain depends on the choice of representative, and so in each case we also state the representative for the fundamental domain we choose. This representative will always correspond to the bottom left triangle, in the chamber marked with a dot in the following picture. From that chamber, we extend out arrows until they reach lattice points. The fundamental region, shown on the right, is the rectangle enclosing this.



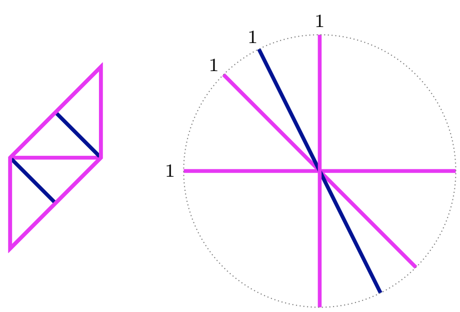
EXAMPLE 4.15. The families $\mathcal{D}_2$, $\mathcal{E}_{6,4}$, and $\mathcal{E}_{7,9}$.



Affine arrangement:



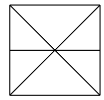
Made of: 8-gons, 4-gons. As an abstract arrangement, this is affine B_2.



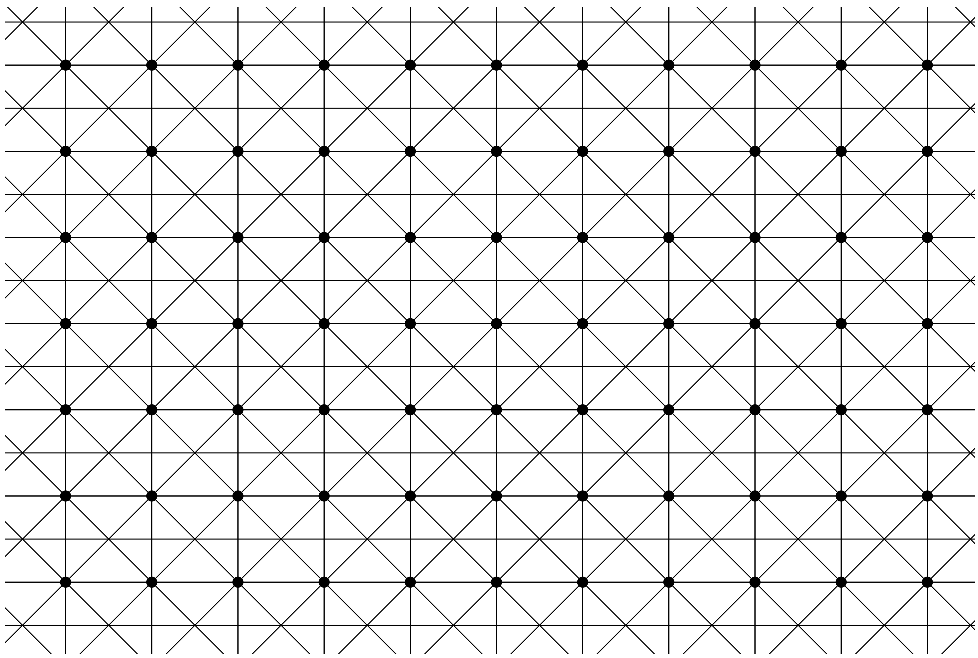
Hyperplanes
01
11
21
10

Fundamental domain representatives: ,  respectively.

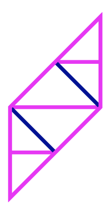
Example 4.16. The families $\mathcal{D}_3$, $\mathcal{E}_{6,5}$, and $\mathcal{E}_{7,1}$.

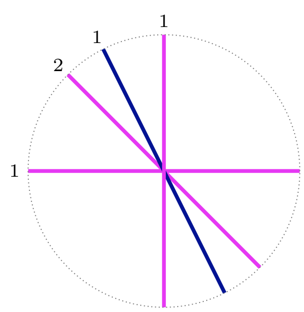


Affine arrangement:

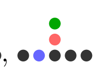


Made of: 8-gons, 6-gons, 4-gons.

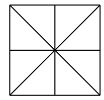




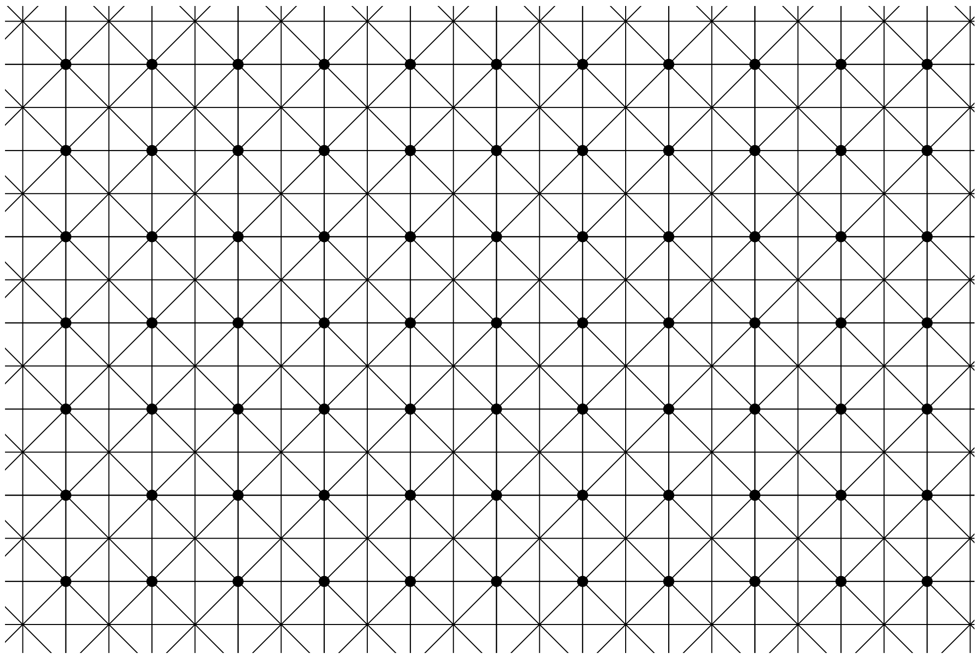
Hyperplanes	
01	
11, 22	
21	
10	

Fundamental domain representatives: ,  respectively.

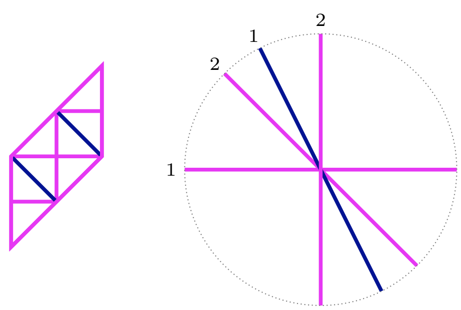
EXAMPLE 4.17. The families $\mathcal{D}_4$, $\mathcal{E}_{7,7}$, and $\mathcal{E}_{8,1}$.



Affine arrangement:



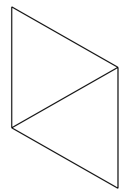
Made of: 8-gons, 4-gons. Same arrangement as 4.15, but the $\mathbb{Z}^2$-lattice is different.



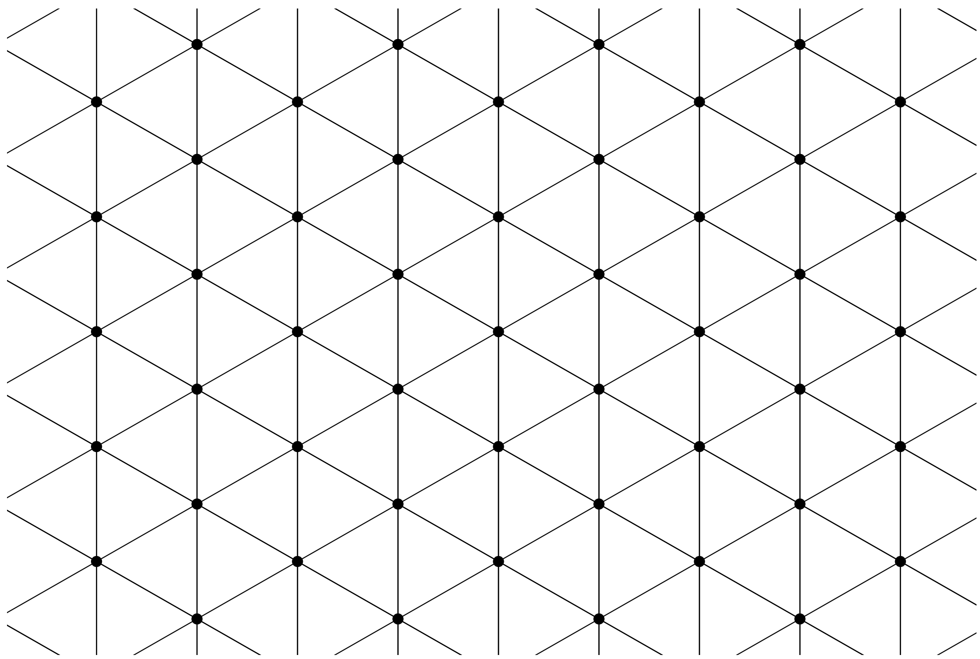
Hyperplanes	
01	
11, 22	
21	
10, 20	

Fundamental domain representatives: ,  respectively.

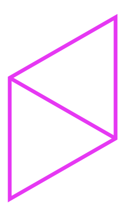
EXAMPLE 4.18. The families $\mathcal{A}$, $\mathcal{D}_1$, and $\mathcal{E}_{6,2}$.

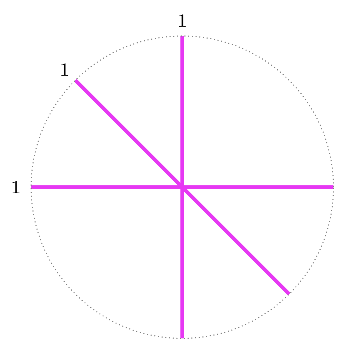


Affine arrangement:

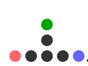


Made of: 6-gons. This is affine A_2.

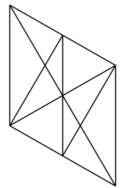




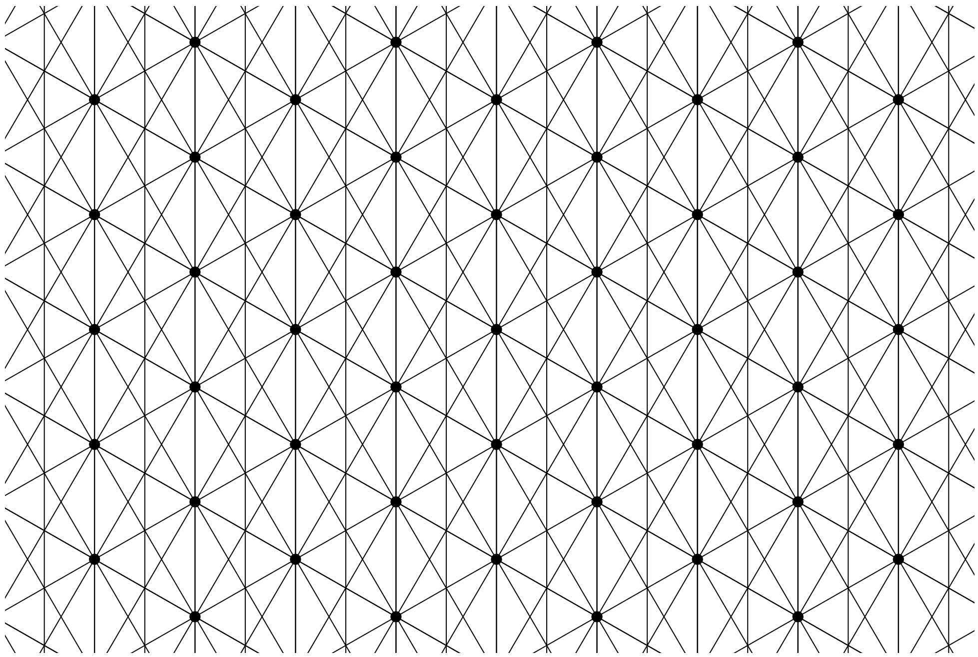
Hyperplanes	
01	
11	
10	

Fundamental domain representative: .

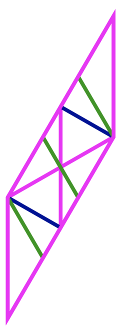
Example 4.19. The families $\mathcal{E}_{6,1}$ and $\mathcal{E}_{7,2}$.

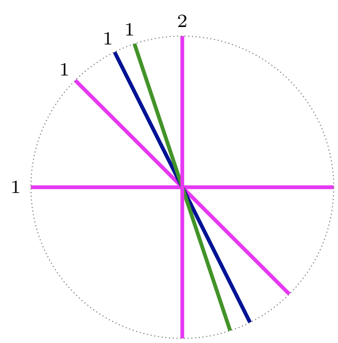


Affine arrangement:



Made of: 10-gons, 6-gons, 4-gons.

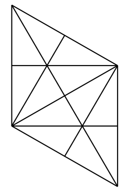




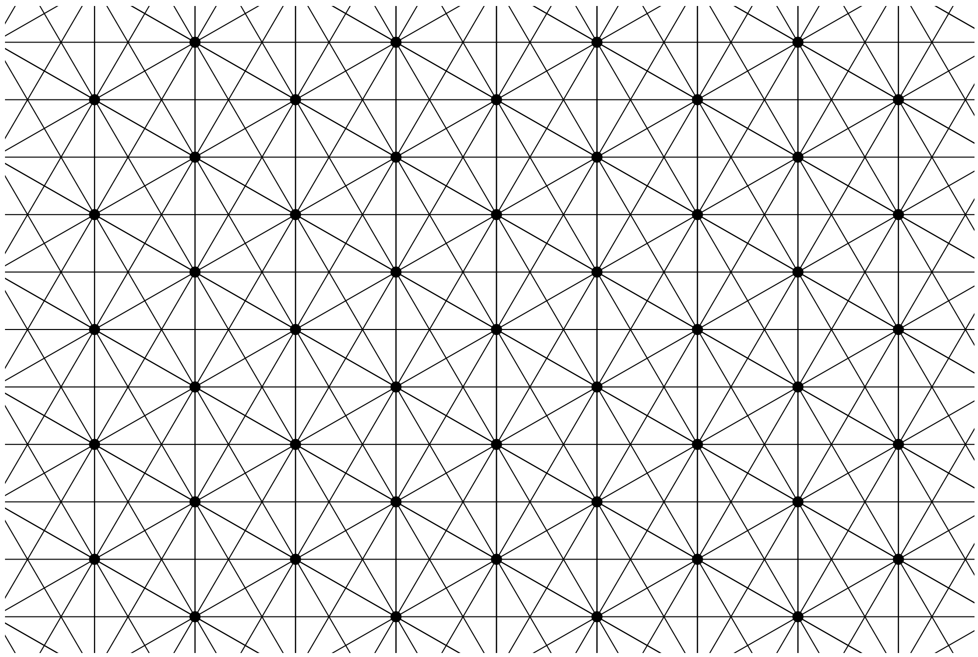
Hyperplanes	
01	
11	
21	
31	
10, 20	

Fundamental domain representatives:  and  respectively.

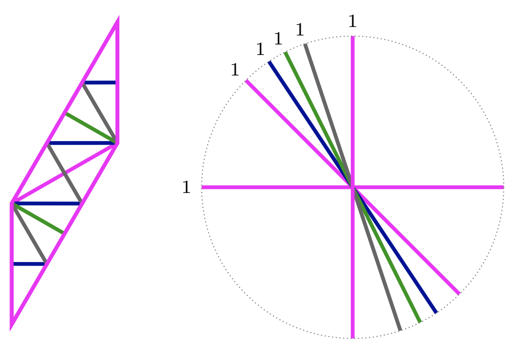
EXAMPLE 4.20. The families $\mathcal{E}_{6,3}$, $\mathcal{E}_{7,3}$, and $\mathcal{E}_{8,2}$.



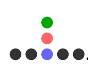
Affine arrangement:



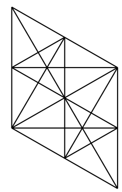
Made of: 12-gons, 6-gons, 4-gons. This is affine G_2.



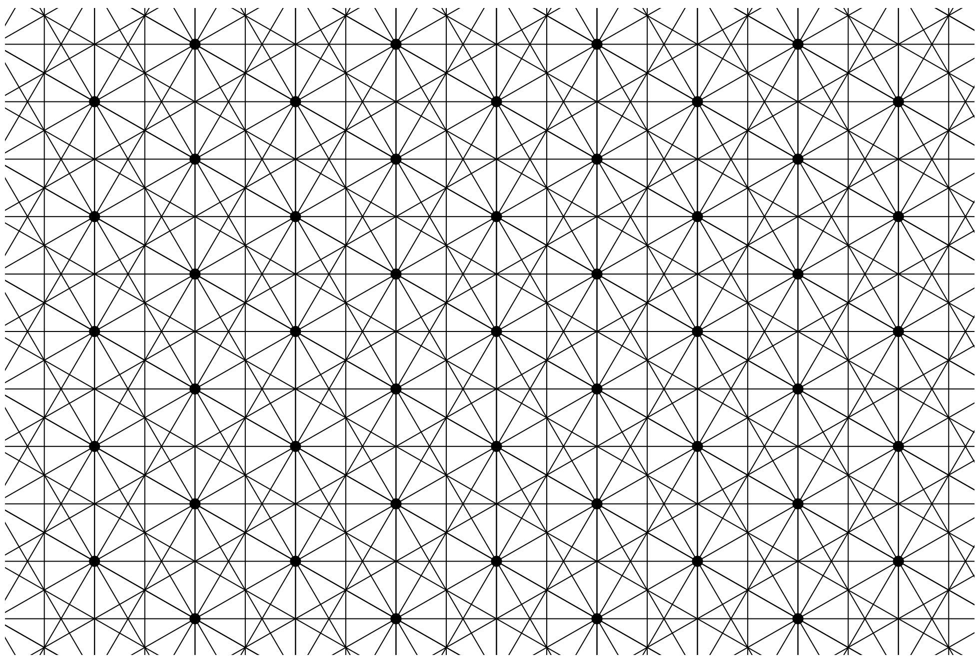
Hyperplanes
01
11
32
21
31
10

Fundamental domain representatives: , , and  respectively.

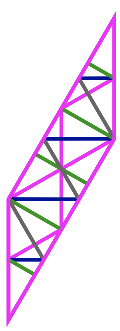
EXAMPLE 4.21. The families $\mathcal{E}_{7,8}$ and $\mathcal{E}_{8,3}$.

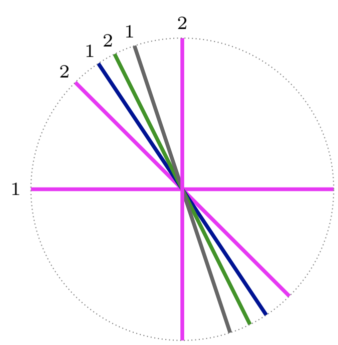


Affine arrangement:



Made of: 12-gons, 8-gons, 6-gons, 4-gons.

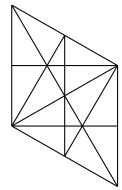




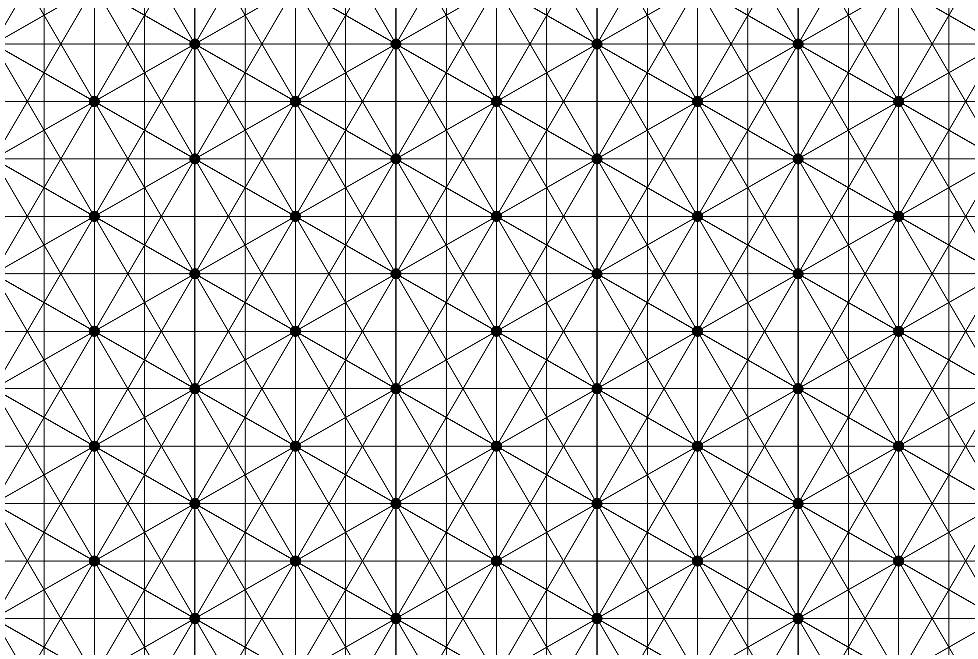
Hyperplanes
01
11, 22
32
21, 42
31
10, 20

Fundamental domain representatives: , and  respectively.

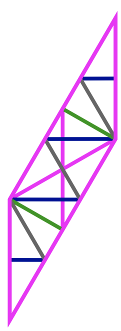
Example 4.22. The families $\mathcal{E}_{7,4}$ and $\mathcal{E}_{8,5}$.

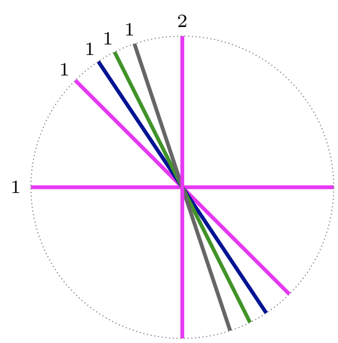


Affine arrangement:



Made of: 12-gons, 6-gons, 4-gons.

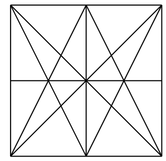




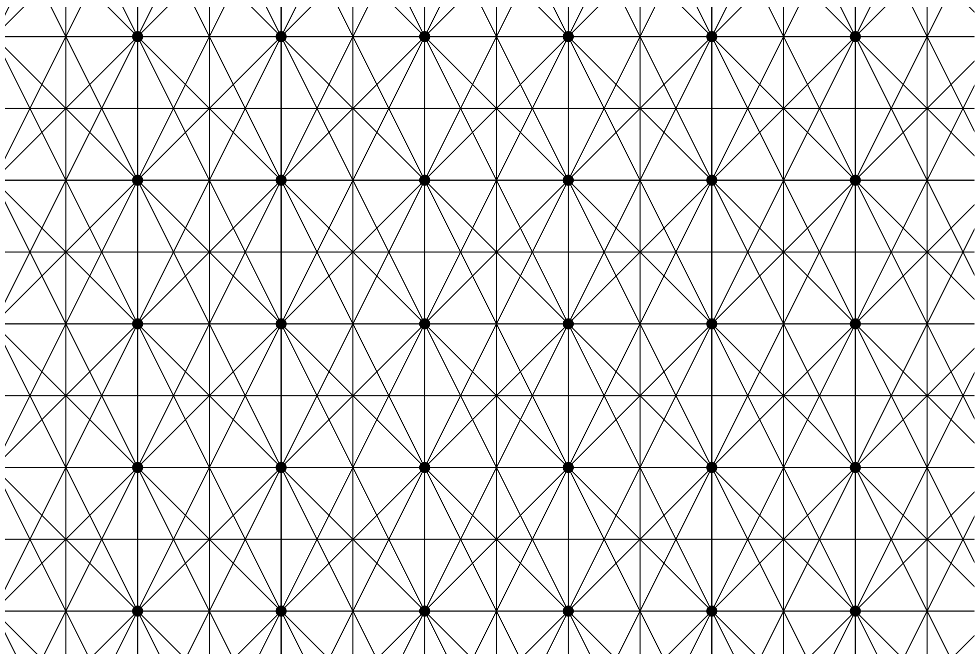
Hyperplanes	
01	
11	
32	
21	
31	
10, 20	

Fundamental domain representatives: , and  respectively.

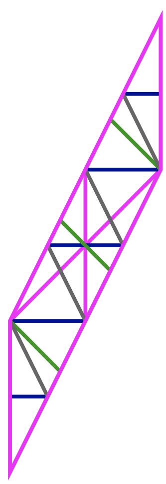
EXAMPLE 4.23. The families $\mathcal{E}_{7,5}$ and $\mathcal{E}_{8,4}$.

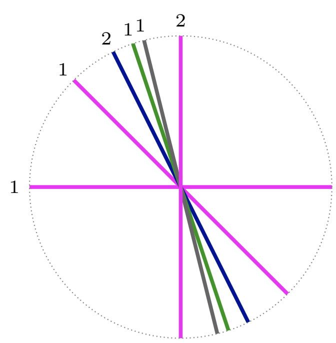


Affine arrangement:



Made of: 12-gons, 8-gons, 6-gons, 4-gons.

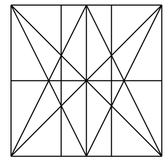




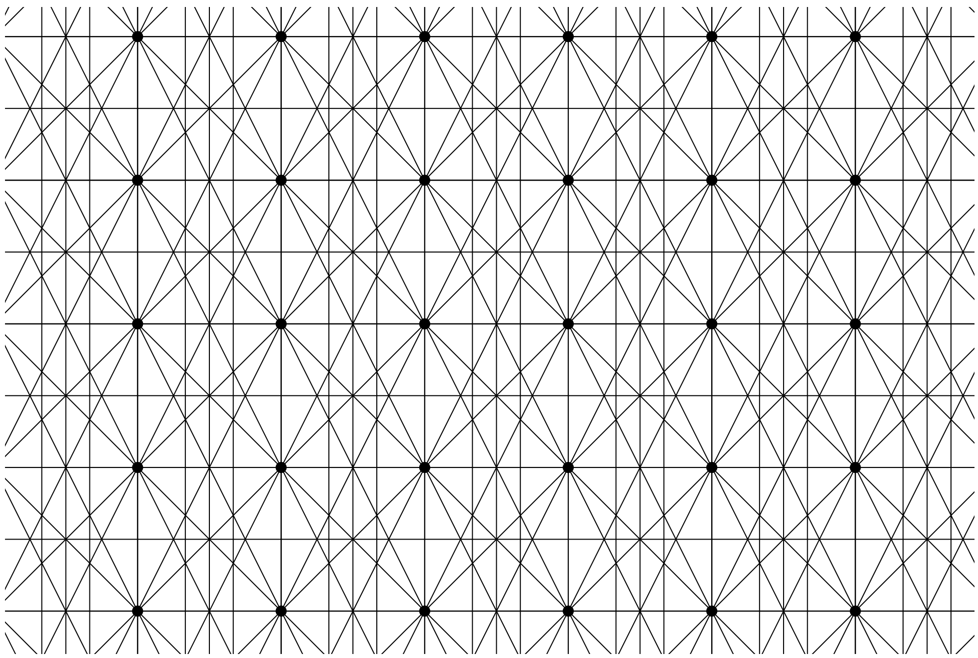
Hyperplanes
01
11
21, 42
31
41
10, 20

Fundamental domain representatives: , and .

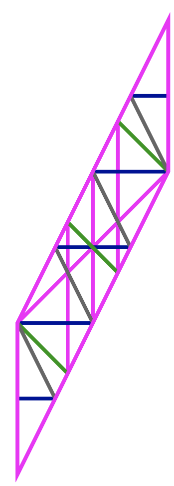
Example 4.24. The family $\mathcal{E}_{8,6}$.

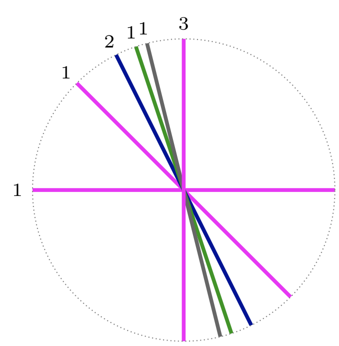


Affine arrangement:



Made of: 12-gons, 8-gons, 6-gons, 4-gons.

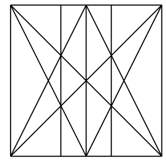




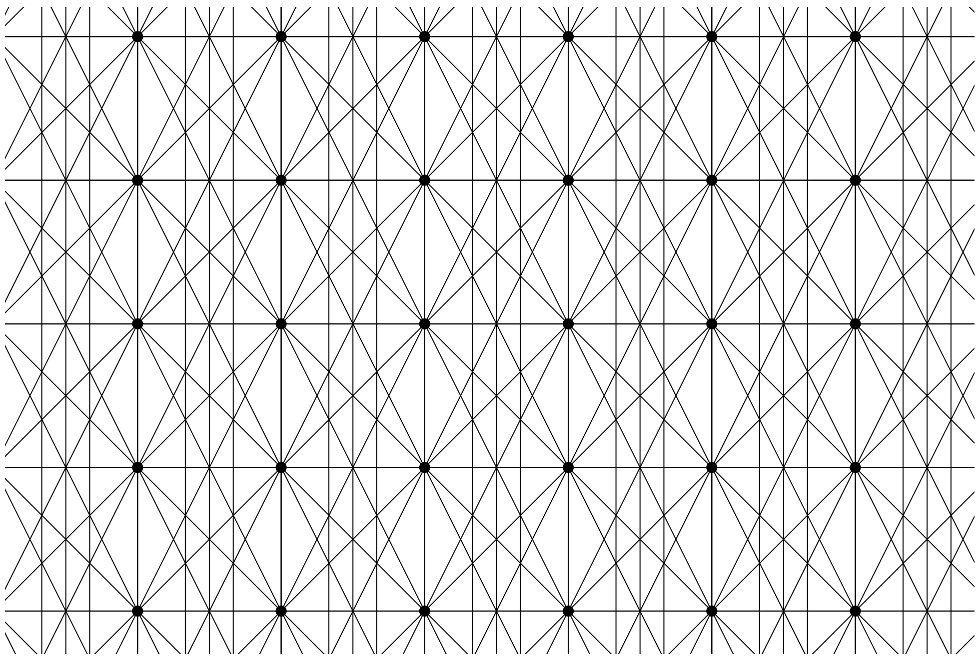
Hyperplanes	
01	
11	
21, 42	
31	
41	
10, 20, 30	

Fundamental domain representative: .

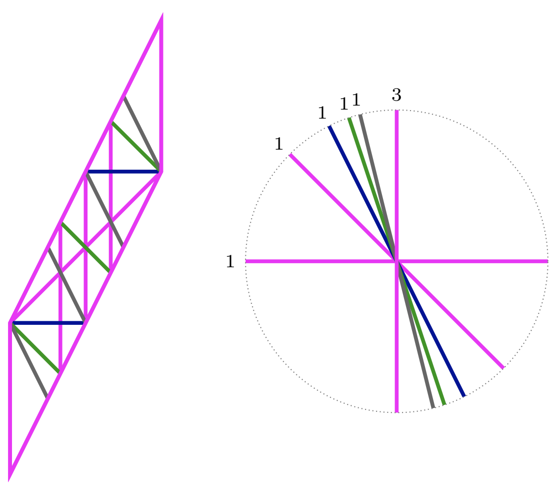
Example 4.25. The family $\mathcal{E}_{7,6}$.



Affine arrangement:



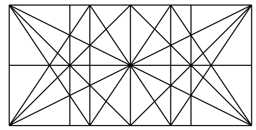
Made of: 12-gons, 8-gons, 6-gons, 4-gons.



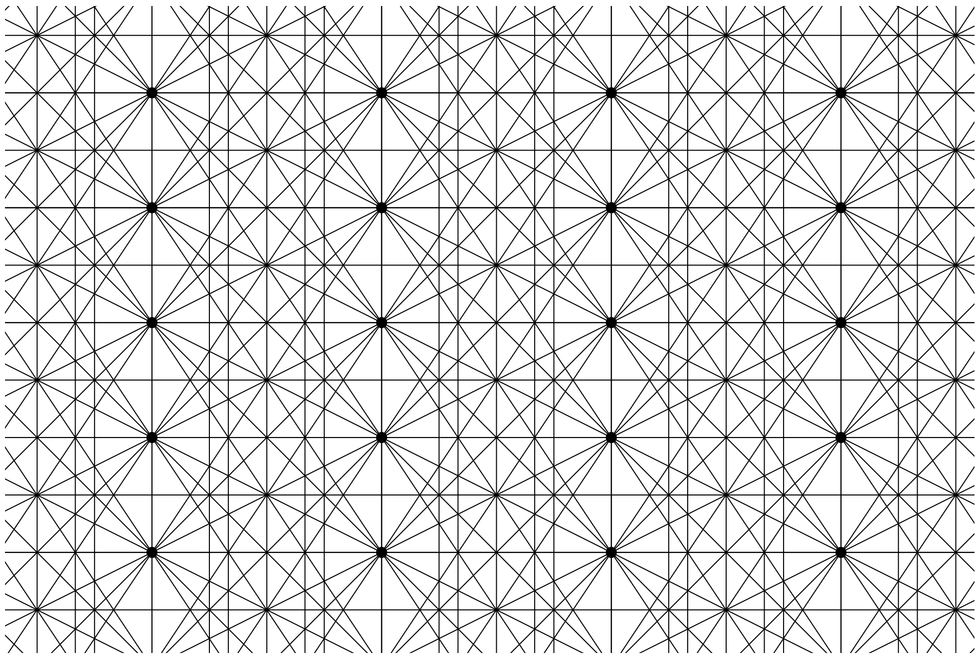
Hyperplanes
01
11
21
31
41
10, 20, 30

Fundamental domain representative: .

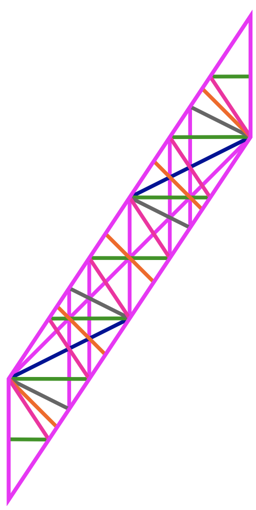
EXAMPLE 4.26. The family $\mathcal{E}_{8,10}$.

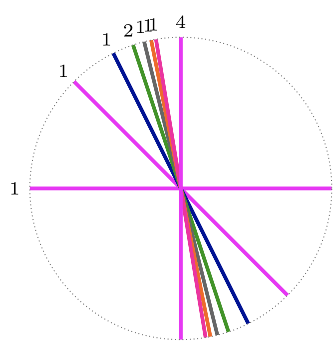


Affine arrangement:



Made of: 16-gons, 12-gons, 8-gons, 6-gons, 4-gons.

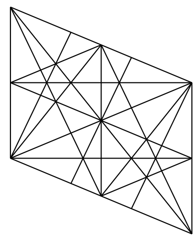




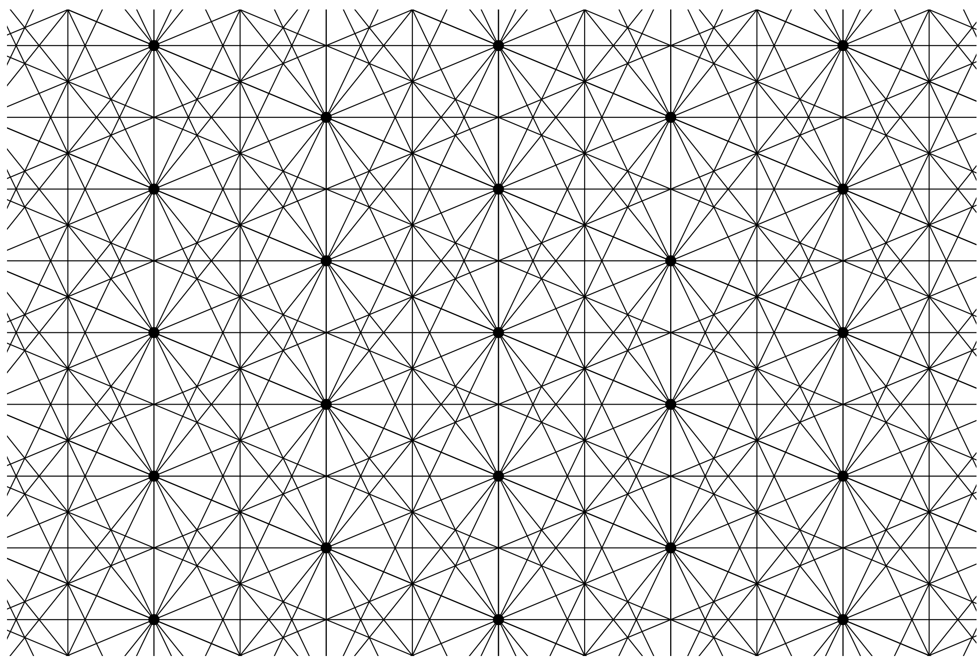
Hyperplanes
01
11
21
31, 62
41
51
61
10, 20, 30, 40

Fundamental domain representative: .

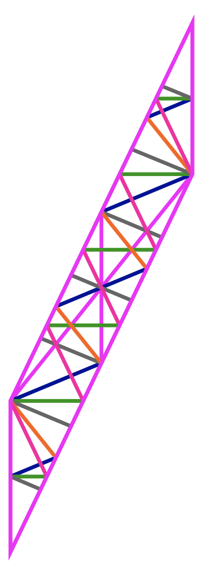
EXAMPLE 4.27. The family $\mathcal{E}_{8,8}$.

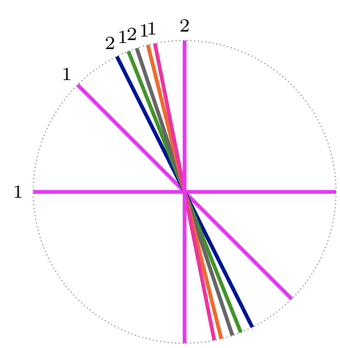


Affine arrangement:

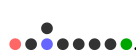


Made of: 16-gons, 10-gons, 8-gons, 6-gons, 4-gons.

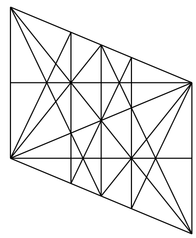




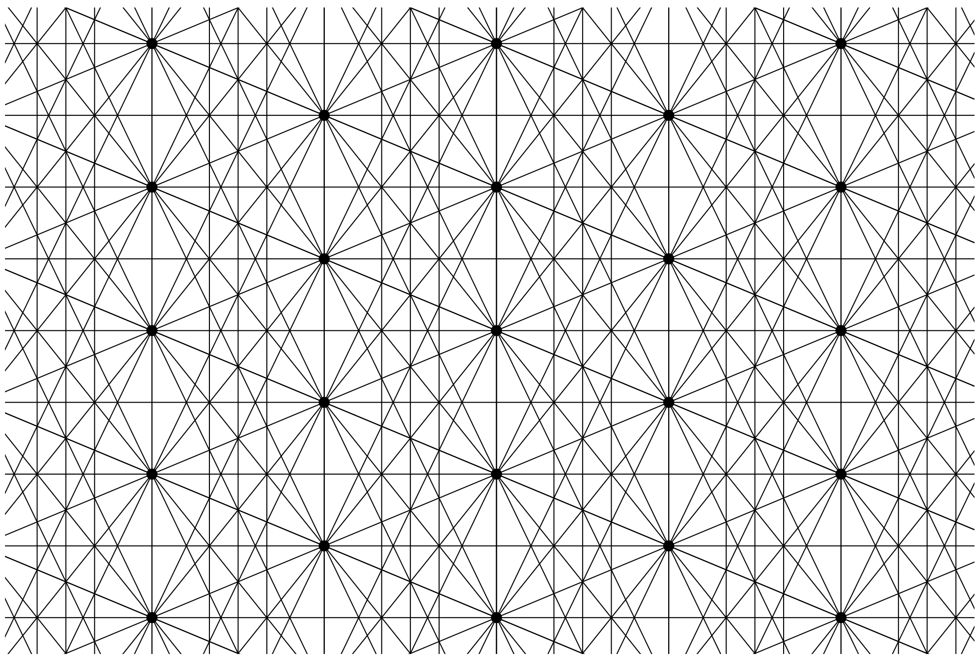
Hyperplanes
01
11
21, 42
52
31, 62
41
51
10, 20

Fundamental domain representative: .

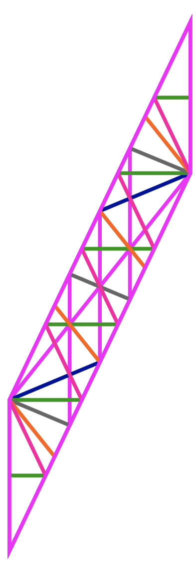
EXAMPLE 4.28. The family $\mathcal{E}_{8,9}$.

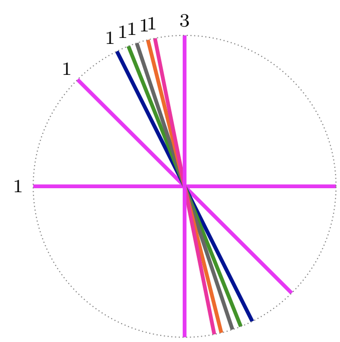


Affine arrangement:



Made of: 16-gons, 8-gons, 6-gons, 4-gons.

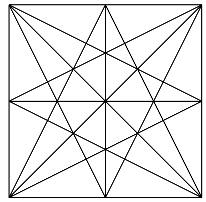




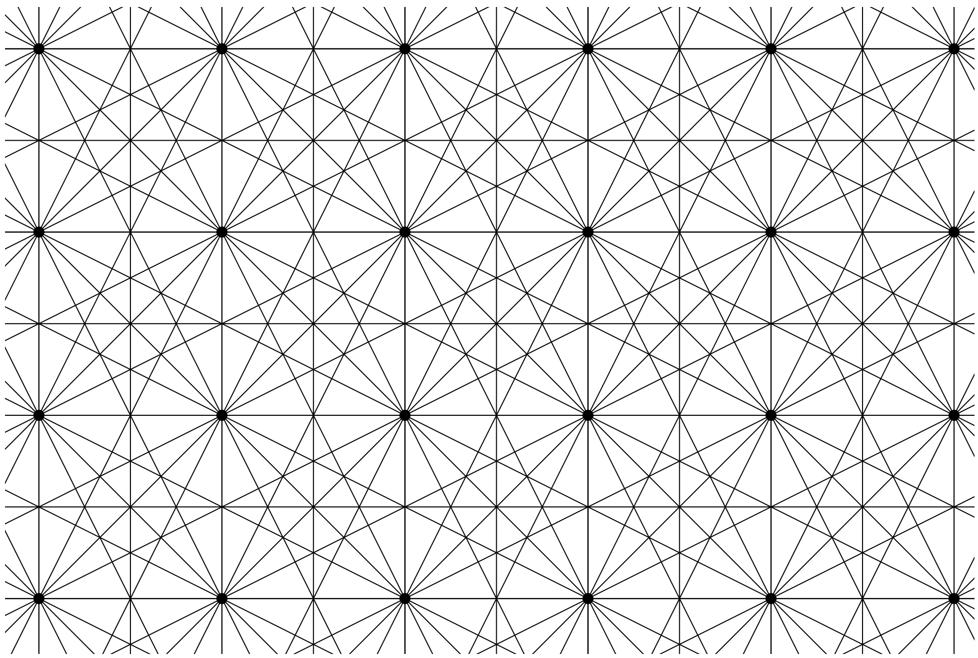
Hyperplanes
01
11
21
52
31
41
51
10, 20, 30

Fundamental domain representative: .

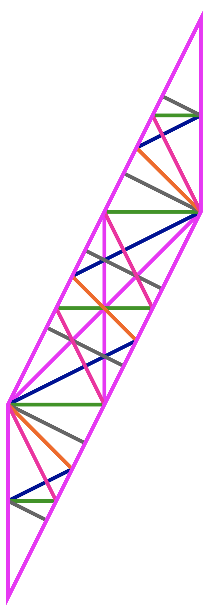
Example 4.29. The family $\mathcal{E}_{8,7}$.

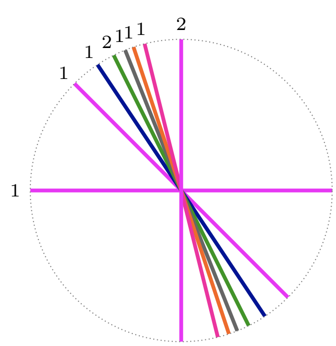


Affine arrangement:



Made of: 16-gons, 8-gons, 6-gons, 4-gons.

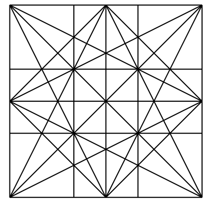




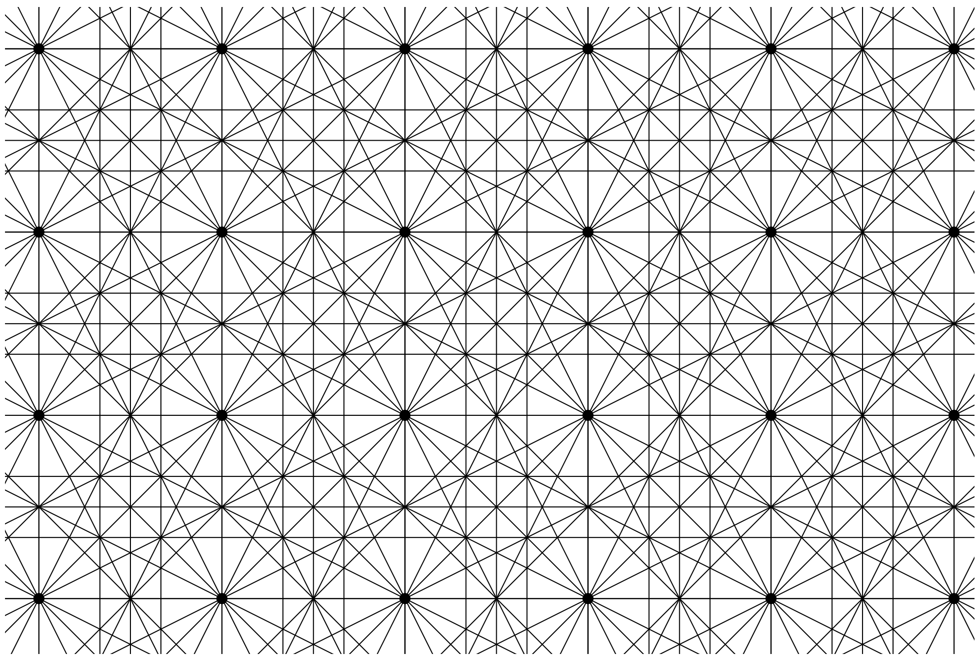
Hyperplanes	
01	
11	
32	
21, 42	
52	
31	
41	
10, 20	

Fundamental domain representative: .

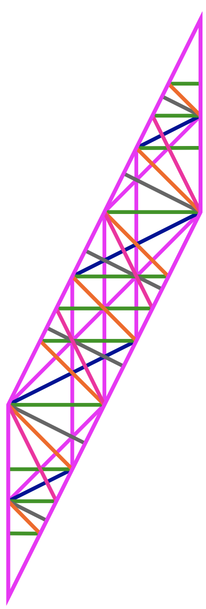
Example 4.30. The family $\mathcal{E}_{8,11}$.

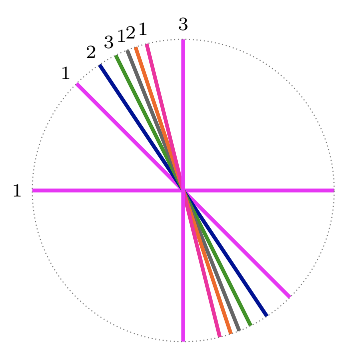


Affine arrangement:



Made of: 16-gons, 12-gons, 10-gons, 8-gons, 6-gons, 4-gons.





Hyperplanes	
01	
$11, 22$	
32	
$21, 42, 63$	
52	
$31, 62$	
41	
$10, 20, 30$	

Fundamental domain representative: .

Part 2

Contracted Preprojective Algebras

CHAPTER 5

Classical Tilting Modules of Contracted Preprojective Algebras

In this chapter we introduce contracted preprojective algebras $\Gamma_{\mathcal{J}}$, and then describe their classical tilting modules using the combinatorics of $\mathcal{J}$-chambers from the previous sections. These classical tilting modules have a partial order, which can be described using the Coxeter combinatorics of Part 1. We exploit this in §5.7 to prove that $\Gamma_{\mathcal{J}}$ carries the action of the fundamental group $\pi_1(\mathcal{J}_{\mathsf{aff}})$, which generalises the braiding of spherical twists of Seidel–Thomas. More generally, we construct a representation of the corresponding infinite groupoid.

5.1. Contracted Preprojective Algebras

Let $Q = (Q_0, Q_1)$ be a quiver with underlying graph Δ, and $\overline{Q}$ the double quiver of Q. Let Π be the (complete) preprojective algebra of Δ, that is the complete path algebra of $\overline{Q}$, modulo the closure of the ideal generated by the element

$$\sum_{a \in Q_1} (aa^* - a^*a).$$

For each vertex $i \in \Delta$, we write e_i for the corresponding idempotent of Π. Subsets $\mathcal{J} \subseteq \Delta$ can be identified with idempotents of Π which are sums of $e_1, \ldots, e_n$, and we use the convention that

$$e_{\mathcal{J}} := 1 - \sum_{j \in \mathcal{J}} e_j.$$

Definition 5.1. For any subset $\mathcal{J} \subseteq \Delta$, we call $\Gamma_{\mathcal{J}} := e_{\mathcal{J}} \Pi e_{\mathcal{J}}$ the *contracted preprojective algebra* associated to $\mathcal{J}$.

The aim of this section is to understand, for every $\mathcal{J} \subseteq \Delta$, classical tilting $\Gamma_{\mathcal{J}}$-modules in terms of elements of $\mathsf{Cham}(\Delta, \mathcal{J})$. For $i \in \Delta$, let $I_i := \langle 1 - e_i \rangle$ be the two-sided maximal ideal of Π generated by $1 - e_i$. For $w \in W_\Delta$ with reduced expression $w = s_{i_1} \ldots s_{i_\ell}$, recall that the ideal I_w of Π is defined by

$$I_w := I_{i_1} \ldots I_{i_\ell}.$$

This is independent of a choice of reduced expression [**BIRS**]. By convention $I_1 = \Pi$.

The aim of this chapter is to prove the following result.

Theorem 5.2. *Let Δ be a non-Dynkin graph without loops, and Π the preprojective algebra of Δ. Let $\mathcal{J}$ be a strongly Dynkin subset of Δ.*

(1) *There is a map*
$$\mathsf{Cham}(\Delta, \mathcal{J}) \to \operatorname{tilt} \Gamma_{\mathcal{J}}$$
given by $(x, J) \mapsto e_{\mathcal{J}} I_x e_J$.

(2) *Wall crossing is compatible with mutation, that is, if $\omega_i(x, J) = (y, J')$, then $\nu_i(e_{\mathcal{J}} I_x e_J) = e_{\mathcal{J}} I_y e_{J'}$.*

(3) *The tilting order is the reflection of the weak order. Namely, if $\omega_i(x, J) = (y, J')$, then $e_{\mathcal{J}} I_x e_J > e_{\mathcal{J}} I_y e_{J'}$ if and only if $x < y$.*

(4) *If Δ is extended Dynkin, then the above map $\mathsf{Cham}(\Delta, \mathcal{J}) \to \operatorname{tilt} \Gamma_{\mathcal{J}}$ is a bijection, and the exchange graph of $\operatorname{tilt} \Pi_{\mathcal{J}}$ is connected.*

Corollary 5.3. *Let Δ be a Dynkin graph, $\mathcal{J} \subset \Delta_{\mathsf{aff}}$, with contracted preprojective algebra $\Gamma_{\mathcal{J}}$. Then the following assertions hold.*

(1) *There exist bijective maps*

$$\operatorname{ptilt}\Gamma_{\mathcal{J}} \xrightarrow{\sim} L_{\mathcal{J}}^{+} \quad \text{and} \quad \mathrm{C}\colon \operatorname{tilt}\Gamma_{\mathcal{J}} \xrightarrow{\sim} \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J}).$$

Moreover, if $T = T_1 \oplus \cdots \oplus T_n \in \operatorname{tilt}\Gamma_{\mathcal{J}}$ is basic with indecomposable T_i, then $T_1, \ldots, T_n$ is a basis of $\overline{\mathrm{C}(T)} \cap L_{\mathcal{J}}$.

(2) *If $0 \notin \mathcal{J}$, then the maps above restrict to bijective maps*

$$\operatorname{ptilt}(\Gamma_{\mathcal{J}}, \Gamma_{\mathcal{J}} e_0) \xrightarrow{\sim} L_{\Delta,\mathcal{J}}^{+} \quad \text{and} \quad \mathrm{C}\colon \operatorname{tilt}(\Gamma_{\mathcal{J}}, \Gamma_{\mathcal{J}} e_0) \xrightarrow{\sim} \mathsf{Cham}(\Delta, \mathcal{J}).$$

In particular, $\#\operatorname{tilt}(\Gamma_{\mathcal{J}}, \Gamma_{\mathcal{J}} e_0) < \infty$.

(3) *Let $T, U \in \operatorname{tilt}\Gamma_{\mathcal{J}}$. Then T and U are mutation of each other if and only if $\mathrm{C}(T)$ and $\mathrm{C}(U)$ are wall crossing of each other.*

The proof will be split into a series of lemmas; parts (1), (2) and (3) will be proved at the end of §5.4, and part (4) will be proved in §5.5. All results need properties of the preprojective algebras Π, and the ideals I_x, which we briefly review in the next subsection.

5.2. Reminder on Tilting Modules

Let A be a ring. For an A-module X, we write $\operatorname{add} X$ for the category of A-modules which are direct summands of finite direct sums of copies of X.

Definition 5.4. $T \in \operatorname{mod} A$ is called *tilting* if the following are satisfied.

(1) There exists an exact sequence $0 \to P_1 \to P_0 \to T \to 0$ with $P_0, P_1 \in \operatorname{add} A$
(2) There exists an exact sequence $0 \to A \to T^0 \to T^1 \to 0$ and $T^0, T^1 \in \operatorname{add} T$.
(3) $\operatorname{Ext}_A^1(T, T) = 0$.

T is called *partial tilting* if it satisfies (1) and (3). Write $\operatorname{ptilt} A$ for the set of isomorphism classes of (not necessarily basic) partial tilting A-modules, and $\operatorname{tilt} A$ for the set of isomorphism classes of basic tilting A-modules.

If $T \in \operatorname{tilt} A$, then it is very well-known that A and $\operatorname{End}_A(T)$ is derived equivalent, via the functor $\mathbf{R}\mathrm{Hom}_A(T, -)$.

Let Λ be a noetherian ring, and $X, Y \in \operatorname{mod}\Lambda$. A morphism $f : X' \to Y$ is called a *right* $\operatorname{add} X$*-approximation of* Y if $X' \in \operatorname{add} X$ and the map

$$f\colon \operatorname{Hom}_\Lambda(X, X') \to \operatorname{Hom}_\Lambda(X, Y)$$

is surjective. It is called *right minimal* if each morphism $g : X' \to X'$ satisfying $f = fg$ is an automorphism. A right $\operatorname{add} X$-approximation which is right minimal is called a *minimal right* $\operatorname{add} X$*-approximation*. Dually, we define a *(minimal) left* $\operatorname{add} X$*-approximation*.[1]

It is basic that Y has a right (respectively, left) $\operatorname{add} X$-approximation if and only if $\operatorname{Hom}_\Lambda(X, Y)$ is finitely generated as an $\operatorname{End}_\Lambda(X)$-module (respectively, $\operatorname{End}_\Lambda(Y)^{\mathrm{op}}$-module). This condition is automatically satisfied if Λ is a module-finite algebra over a commutative noetherian ring. More strongly, if Λ is a module-finite algebra over a commutative noetherian complete local ring, then each $Y \in \operatorname{mod}\Lambda$ has a minimal right (respectively, left) $\operatorname{add} X$-approximation.

The following result is basic (see e.g. [**IR**, 5.2]).

Proposition 5.5. *Let A be a ring and $T = X \oplus U$ a tilting A-module.*

(1) *Suppose $0 \to X \xrightarrow{f} U' \to Y \to 0$ is an exact sequence where f is a left $\operatorname{add} U$-approximation of X, and $\operatorname{proj.dim}_A Y \leq 1$. Then $Y \oplus U$ is a tilting A-module.*

(2) *Suppose $0 \to Y \to U' \xrightarrow{f} X \to 0$ is an exact sequence where f is a right $\operatorname{add} U$-approximation of X. Then $Y \oplus U$ is a tilting A-module.*

[1] A right (respectively, left) $\operatorname{add} X$-approximation is often called an $\operatorname{add} X$*-precover* (respectively, $\operatorname{add} X$*-preenvelope*).

If either case occurs, we write $\nu_X(T) := Y \oplus U$, and call it the *tilting mutation* of T at X. Now recall that A-modules X and Y are called *additively equivalent* if $\operatorname{add} X = \operatorname{add} Y$ holds; we write $\operatorname{tilt} A$ for the set of additive equivalence classes of tilting A-modules. For a ring A and $T \in \operatorname{tilt} A$, we assume that $\operatorname{End}_A(T)$ is semiperfect and that $T = T_1 \oplus \cdots \oplus T_n$ for non-isomorphic indecomposable A-modules T_i. In this case, we write

$$\nu_i(T) := \nu_{T_i}(T).$$

One of the key properties of $\operatorname{tilt} A$ is that it has a partial order, defined by

$$T \geq U \Longrightarrow \operatorname{Fac} T \supset \operatorname{Fac} U.$$

PROPOSITION 5.6. [**AI**, 2.25] *Let Λ be a semiperfect ring. For $T \in \operatorname{tilt} \Lambda$, take a minimal projective resolution $0 \to P_1 \to P_0 \to T \to 0$. Then P_0 and P_1 do not have non-zero common direct summands.*

The following is due to Riedtmann–Schofield [**RS**] in the context of finite dimensional algebras.

PROPOSITION 5.7. [**IR**, 4.2] *Suppose that Λ is a module-finite algebra over a commutative noetherian complete local ring. If $T \in \operatorname{tilt} \Lambda$, with indecomposable summand T_i, then there is at most one $T_i' \ncong T_i$ such that $(T/T_i) \oplus T_i' \in \operatorname{tilt} \Lambda$.*

We recall basic properties on tilting mutation which will be used later.

PROPOSITION 5.8. *If $T \geq U$, then there exists exact sequences*

$$0 \to T \to U^0 \to U^1 \to 0 \quad \text{and } 0 \to T_1 \to T_0 \to U \to 0$$

with $U^i \in \operatorname{add} U$ and $T_i \in \operatorname{add} T$.

PROPOSITION 5.9. *Let A be a ring A and $T, U \in \operatorname{tilt} A$. Assume that $\operatorname{End}_A(T)$ is semiperfect and that $T = T_1 \oplus \cdots \oplus T_n$ for non-isomorphic indecomposable A-modules T_i. If $T > U$, then there exists $i \in \{1, \ldots, n\}$ such that $\nu_i(T) \geq U$.*

PROOF. This follows easily from [**AI**, 2.36] (see also [**IW2**, 4.4]). □

5.3. Recap on Tilting on Preprojective Algebras

Throughout this subsection Δ will be a non-Dynkin graph without loops, and Π the preprojective algebra of Δ. For an idempotent $e \in \Pi$, set $\Pi_e := \Pi/\langle e \rangle$.

The following singular 2-Calabi-Yau property of $e\Pi e$ is known. Recall that if A is a k-algebra, we write $A^{\mathrm{en}} = A \otimes_k A^{\mathrm{op}}$.

PROPOSITION 5.10. *Suppose that Δ is a non-Dynkin graph without loops, and let e be an idempotent of Π such that $\dim_k(\Pi_e) < \infty$. There is a functorial isomorphism*

$$\mathbf{R}\mathrm{Hom}_{e\Pi e}(X, Y) \cong D\, \mathbf{R}\mathrm{Hom}_{e\Pi e}(Y, X[2])$$

for all $X \in \mathsf{K}^{\mathrm{b}}(\operatorname{proj} e\Pi e)$ and $Y \in \mathsf{D}^{\mathrm{b}}_{\mathrm{fd}\, e\Pi e}(\operatorname{Mod} e\Pi e)$.

PROOF. It is well-known (e.g. [**K2**]) that Π is a 2-Calabi-Yau algebra, that is, $\Pi \in \operatorname{per} \Pi^{\mathrm{en}}$ and $\mathbf{R}\mathrm{Hom}_{\Pi^{\mathrm{en}}}(\Pi, \Pi^{\mathrm{en}}) \cong \Pi[-2]$ in $\mathsf{D}(\operatorname{Mod} \Pi^{\mathrm{en}})$. Our assumption $\dim_k(\Pi_e) < \infty$ impies that $e\Pi e$ is a singular 2-Calabi-Yau algebra, that is,

$$\mathbf{R}\mathrm{Hom}_{(e\Pi e)^{\mathrm{en}}}(e\Pi e, (e\Pi e)^{\mathrm{en}}) \cong \Pi[-2] \tag{5.3.A}$$

in $\mathsf{D}(\operatorname{Mod}(e\Pi e)^{\mathrm{en}})$ by [**AIR**, Remark 2.7], noting that the assumption (A2) there is not used in the proof.

As is then standard, the assertion follows from (5.3.A) using an identical argument as in the proof of [**K**, 4.1], where our assumption $X \in \mathsf{K}^{\mathrm{b}}(\operatorname{proj} e\Pi e)$ replaces the smoothness assumption of [**K**]. □

LEMMA 5.11. *Suppose that Δ is a non-Dynkin graph without loops. For a subset $\mathcal{J} \subseteq \Delta$, let $e = \sum_{j \in \mathcal{J}} e_j = 1 - e_{\mathcal{J}}$. Then for any $i \in \mathcal{J}$, the following statements hold.*

(1) $\Pi_{e-e_i}e_i \cong \Pi e \otimes_{e\Pi e} S_i$ *as* Π*-modules.*
(2) $e_i\Pi_{e-e_i} \cong S_i^{\mathrm{op}} \otimes_{e\Pi e} e\Pi$ *as* Π^{op}*-modules.*

PROOF. We only prove (1), since (2) is the dual. Since $S_i = e\Pi_{e-e_i}e_i$ as $(e\Pi e)$-modules, it follows that

$$\Pi e \otimes_{e\Pi e} S_i = \Pi e \otimes_{e\Pi e} e\Pi_{e-e_i}e_i = \frac{\Pi e_i}{\Pi e\langle e-e_i\rangle e_i} = \Pi_{e-e_i}e_i. \qquad \square$$

LEMMA 5.12. *Suppose that Δ is a non-Dynkin graph without loops, and let e be an idempotent of Π such that* $\dim_k(\Pi_e) < \infty$.

(1) *If X is a $(e\Pi e)$-submodule of $e\Pi$ with* $\mathrm{proj.dim}_{e\Pi e}X < \infty$, *then necessarily* $\mathrm{proj.dim}_{e\Pi e}X \leq 1$.
(2) *If Y is a finite dimensional $(e\Pi e)$-module, then*
   (a) $\dim_k(\Pi e \otimes_{e\Pi e} Y) < \infty$.
   (b) $\mathrm{Ext}^t_{e\Pi e}(Y, e\Pi) = 0$ *for* $t = 0, 1$.

PROOF. (2) For any simple $(e\Pi e)$-module S_i, by 5.11 $\Pi e \otimes_{e\Pi e} S_i = \Pi_{e-e_i}e_i$, which is finite dimensional since Π_e is by assumption. Therefore, if Y is a finite dimensional $(e\Pi e)$-module, then so is $\Pi e \otimes_{e\Pi e} Y$. Thus

$$\begin{aligned} \mathbf{R}\mathrm{Hom}_{e\Pi e}(Y, e\Pi) &= \mathbf{R}\mathrm{Hom}_{\Pi}(\Pi e \overset{\mathbf{L}}{\otimes}_{e\Pi e} Y, \Pi) \\ \text{(by 5.10)} \qquad &= D\mathbf{R}\mathrm{Hom}_{\Pi}(\Pi, \Pi e \overset{\mathbf{L}}{\otimes}_{e\Pi e} Y[2]) \\ &= D(\Pi e \overset{\mathbf{L}}{\otimes}_{e\Pi e} Y)[-2]. \end{aligned}$$

Taking cohomology, it follows that for $t = 0, 1$

$$\mathrm{Ext}^t_{e\Pi e}(Y, e\Pi) = H^{t-2}(D(\Pi e \overset{\mathbf{L}}{\otimes}_{e\Pi e} Y)) = DH^{2-t}(\Pi e \overset{\mathbf{L}}{\otimes}_{e\Pi e} Y) = 0,$$

since $H^i(\Pi e \overset{\mathbf{L}}{\otimes}_{e\Pi e} Y) = 0$ for all $i > 0$.
(1) Since $\mathrm{proj.dim}_{e\Pi e}X < \infty$, we have

$$\mathrm{Ext}^2_{(e\Pi e)^{\mathrm{op}}}(X, S_i) \overset{5.10}{=} D\,\mathrm{Hom}_{(e\Pi e)^{\mathrm{op}}}(S_i, X) = 0,$$

where the last equality holds by (2) since X is a submodule of $e\Pi$. By the existence of a minimal projective resolution of X, the result follows. $\square$

We next recall some basic properties of the ideals I_x of Π.

LEMMA 5.13. *Suppose that Δ is a non-Dynkin graph without loops, and let $x, y \in W_\Delta$.*

(1) *If $\ell(xy) = \ell(x) + \ell(y)$ holds, then we have $I_{xy} = I_xI_y$. Moreover*
$$I_y \cong \mathrm{Hom}_\Pi(I_x, I_{xy}) \quad \textit{and} \quad I_x \cong \mathrm{Hom}_{\Pi^{\mathrm{op}}}(I_y, I_{xy}).$$
via $a \mapsto (b \mapsto ba)$ and $a \mapsto (b \mapsto ab)$ respectively.
(2) $\ell(s_ix) = \ell(x) + 1$ *if and only if* $I_iI_x \subsetneq I_x$ *if and only if* $\mathrm{Hom}_\Pi(I_x, S_i) \neq 0$ *if and only if* $\mathrm{Ext}^1_\Pi(I_x, S_i) = 0$ *if and only if* $\mathrm{Ext}^1_\Pi(S_i, I_x) = 0$.
(3) $\ell(s_ix) = \ell(x) - 1$ *if and only if* $I_iI_x = I_x$ *if and only if* $\mathrm{Hom}_\Pi(I_x, S_i) = 0$ *if and only if* $\mathrm{Ext}^1_\Pi(I_x, S_i) \neq 0$ *if and only if* $\mathrm{Ext}^1_\Pi(S_i, I_x) \neq 0$.
(4) *If $\ell(s_ix) = \ell(x) - 1$, then I_{s_ix} is maximum among left ideals I of Π satisfying the following condition.*
   - *$I \supset I_x$ and any composition factor of the Π-module I/I_x is S_i.*

PROOF. (1) This is shown in [BIRS, Part II].
(2)(3) The first equivalence is [BIRS, III.1.10]. The second one is clear. The third is basic in tilting theory. The fourth follows from 5.10.
(4) Since $\ell(s_i(s_ix)) = \ell(x) = \ell(s_ix) + \ell(s_i)$ holds, there is an isomorphism $I_{s_ix} \cong \mathrm{Hom}_\Pi(I_i, I_x)$, $a \mapsto (b \mapsto ba)$ by (1). Thus $I_{s_ix} = \{a \in \Pi \mid I_ia \in I_x\}$ holds, the assertion follows. $\square$

The following properties of $I_{w_J w_{J+i}}$ play a key role. Recall $e_J = 1-\sum_{j\in J} e_j = \sum_{i\notin J} e_i$.

PROPOSITION 5.14. *Suppose that Δ is a non-Dynkin graph without loops, and let $J \subseteq \Delta$. For $i \in J^{\mathrm{c}}$ such that $J+i$ is Dynkin, let $j := \iota_{J+i}(i)$ and $x_0 := w_J w_{J+i}$.*

(1) *We have $\langle e_J\rangle = I_{w_J}$ and $\langle e_J - e_i\rangle = I_{w_{J+i}}$.*
(2) *$I_{x_0} \supset \langle e_J - e_i\rangle$ and $\langle e_J\rangle I_{x_0} = \langle e_J - e_i\rangle$.*
(3) *I_{x_0} is maximum among left ideals I of Π satisfying the following condition.*
   - *$I \supset \langle e_J - e_i\rangle$ and S_i is not a composition factor of the Π-module $I/\langle e_J - e_i\rangle$.*
(4) *I_{x_0} is maximum among right ideals I of Π satisfying the following condition.*
   - *$I \supset \langle e_J - e_i\rangle$, and furthermore S_j is not a composition factor of the Π^{op}-module $I/\langle e_J - e_i\rangle$.*
(5) *We have $I_{x_0}(e_J - e_i) = \Pi(e_J - e_i) = \langle e_J - e_i\rangle(e_J - e_i)$ and $I_{x_0}e_j = \langle e_J - e_i\rangle e_j$.*

PROOF. (1) This is shown in [BIRS, III.3.5].
(2) Since $\ell(w_J) + \ell(x_0) = \ell(w_{J+i})$,

$$\langle e_J\rangle I_{x_0} \overset{(1)}{=} I_{w_J} I_{x_0} \overset{5.13(1)}{=} I_{w_{J+i}} \overset{(1)}{=} \langle e_J - e_i\rangle.$$

(3) Applying 5.13(4) repeatedly, we know from (2) that any composition factor of the Π-module $I_{x_0}/\langle e_J - e_i\rangle$ has a form S_k for some $k \in J$. In particular, S_i is not a composition factor of $I_{x_0}/\langle e_J - e_i\rangle$. Now for all $k \in J$, there is an inequality

$$\ell(s_k x_0) = \ell(w_{J+i}) - \ell(s_k w_J) > \ell(w_{J+i}) - \ell(w_J) = \ell(x_0).$$

Thus $\mathrm{Ext}^1_\Pi(I_{x_0}, S_k) = 0$ holds by 5.13(2). This implies the desired maximality.
(4) Since $I_{x_0} = I_{w_{J'+j} w_{J'}}$ holds by 1.2(3), the assertion is dual to (3).
(5) Since $\Pi \supset I_{x_0} \supset \langle e_J - e_i\rangle$, it follows that

$$\Pi(e_J - e_i) \supset I_{x_0}(e_J - e_i) \supset \langle e_J - e_i\rangle(e_J - e_i) = \Pi(e_J - e_i).$$

Thus the first assertion follows. Further by (4), $(I_{x_0}/\langle e_J - e_i\rangle)e_j = 0$ and so the second assertion follows. $\square$

5.4. Tilting Mutation for Contracted Preprojective Algebras

Again, throughout this section let Δ be a non-Dynkin graph without loops, and Π the preprojective algebra of Δ. We require the following basic result.

PROPOSITION 5.15. *Let e be an idempotent of Π with $\dim_k(\Pi_e) < \infty$, and $x, y \in W_\Delta$.*

(1) *If $x \leq y$, then there is an isomorphism*

$$\mathrm{Hom}_\Pi(I_x, I_y) \to \mathrm{End}_{e\Pi e}(eI_x, eI_y), \quad f \mapsto ef.$$

(2) *There is an isomorphism of k-algebras*

$$\Pi \to \mathrm{End}_{e\Pi e}(eI_x), \quad a \mapsto (\cdot a).$$

(3) *There are equivalences*

$$eI_x \otimes_\Pi - \colon \ \mathrm{add}_\Pi \Pi \to \mathrm{add}_{e\Pi e}(eI_x),$$
$$\mathrm{Hom}_{\Pi^{\mathrm{op}}}(-, eI_x) \colon \ \mathrm{add}\, \Pi_\Pi \to \mathrm{add}_{e\Pi e}(eI_x).$$

PROOF. (1) For injectivity, suppose that $f \in \mathrm{Hom}_\Pi(I_x, I_y)$ satisfies $ef = 0$. Then $\mathrm{Im}\, f$ is a finitely generated Π_e-module, and thus $\dim_k(\mathrm{Im}\, f) < \infty$ by our assumption. It follows that

$$\mathrm{Hom}_\Pi(\mathrm{Im}\, f, \Pi) = D\,\mathrm{Ext}^2_\Pi(\Pi, \mathrm{Im}\, f) = 0$$

by 5.10, so $\mathrm{Hom}_\Pi(\mathrm{Im}\, f, I_y) = 0$ holds. Thus $f = 0$.

For surjectivity, let $g \in \mathrm{End}_{e\Pi e}(eI_x, eI_y)$. Consider the obvious multiplication map $m_x \colon \Pi e \otimes_{e\Pi e} eI_x \to I_x$, then there is an exact sequence

$$0 \to \mathrm{Ker}\, m_x \to \Pi e \otimes_{e\Pi e} eI_x \xrightarrow{m_x} I_x \to \mathrm{Cok}\, m_x \to 0.$$

Since $em_x\colon e\Pi e \otimes_{e\Pi e} eI_x \to eI_x$ is an isomorphism, both $\operatorname{Ker} m_x$ and $\operatorname{Cok} m_x$ are Π_e-modules, and hence by our assumption are finite dimensional. Thus $\operatorname{Hom}_\Pi(\operatorname{Ker} m_x, \Pi) = 0$ again by 5.10, which implies that $\operatorname{Hom}_\Pi(\operatorname{Ker} m_x, I_y) = 0$ and so $\operatorname{Hom}_\Pi(\operatorname{Ker} m_x, \operatorname{Im} m_y) = 0$.

Therefore $1 \otimes g \in \Pi e \otimes_{e\Pi e} \operatorname{Hom}_\Pi(eI_x, eI_y)$ gives a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ker} m_x & \longrightarrow & \Pi e \otimes_{e\Pi e} eI_x & \longrightarrow & \operatorname{Im} m_x & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle 1\otimes g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & \operatorname{Ker} m_y & \longrightarrow & \Pi e \otimes_{e\Pi e} eI_y & \longrightarrow & \operatorname{Im} m_y & \longrightarrow & 0 \end{array}$$

for some $h \in \operatorname{Hom}_\Pi(\operatorname{Im} m_x, \operatorname{Im} m_y)$. Since $\operatorname{Cok} m_x$ is a factor of I_x, and $\operatorname{Ext}^1_\Pi(I_x, I_y) = 0$ by [**BIRS**, II.1.13] since $x \le y$, it follows that $\operatorname{Ext}^1_\Pi(I_y, \operatorname{Cok} m_x) = 0$. Hence by 5.10,

$$\operatorname{Ext}^1_\Pi(\operatorname{Cok} m_x, I_y) \cong D \operatorname{Ext}^1_\Pi(I_y, \operatorname{Cok} m_x) = 0.$$

We can thus lift h to a morphism $f \in \operatorname{Hom}_\Pi(I_x, I_y)$ fitting into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Im} m_x & \longrightarrow & I_x & \longrightarrow & \operatorname{Cok} m_x & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} & & & & \\ 0 & \longrightarrow & \operatorname{Im} m_y & \longrightarrow & I_y & \longrightarrow & \operatorname{Cok} m_y & \longrightarrow & 0 \end{array}$$

Now $ef = eh = e(1 \otimes g) = g$, as required.
(2) Since Π is non-Dynkin, by [**BIRS**, III.1.6] there is an isomorphism of k-algebras $\Pi \to \operatorname{End}_\Pi(I_x)$ given by $a \mapsto (\cdot a)$. This gives an isomorphism of k-algebras

$$\Pi \to \operatorname{End}_\Pi(I_x), \quad a \mapsto (\cdot a).$$

Thus it is enough to show that the map $\operatorname{End}_\Pi(I_x) \to \operatorname{End}_{e\Pi e}(eI_x)$ given by $f \mapsto ef$ is an isomorphism, which follows by (1).
(3) This is immediate from (2). $\square$

Corollary 5.16. *Let Δ be a non-Dynkin graph without loops, and suppose $\mathfrak{J} \subseteq \Delta$ such that $e_{\mathfrak{J}} = 1 - \sum_{j \in \mathfrak{J}} e_j$ satisfies $\dim_k(\Pi_{e_{\mathfrak{J}}}) < \infty$. Then for any $J \subseteq \Delta$ and for any $x \in W_\Delta$, there is an isomorphism of k-algebras*

$$\Gamma_J \to \operatorname{End}_{\Gamma_{\mathfrak{J}}}(e_{\mathfrak{J}} I_x e_J), \quad a \mapsto (\cdot a).$$

Proof. By 5.15(2) there is an isomorphism $\Pi \to \operatorname{End}_{e_{\mathfrak{J}}\Pi e_{\mathfrak{J}}}(e_{\mathfrak{J}} I_x)$ given by $a \mapsto (\cdot a)$. Applying $e_J(-)e_J$ gives the result. $\square$

As in the case $e = 1$, the following result plays an important role.

Proposition 5.17. *Suppose that Δ is a non-Dynkin graph without loops. For a subset $J \subseteq \Delta$ and $i \in J^{\mathrm{c}}$, assume that $J + i$ is Dynkin, and let $e = e_J$ and $j := \iota_{J+i}(i)$. Then there exists an exact sequence*

$$0 \to \Pi e_i \xrightarrow{f} P \to \langle e - e_i \rangle e_j \to 0$$

of Π-modules with a left $\operatorname{add} \Pi(e - e_i)$-approximation f.

Proof. Since Π is complete, let $0 \to P' \xrightarrow{f} P \to \langle e - e_i \rangle e_j \to 0$ be a minimal projective resolution of the Π-module $\langle e - e_i \rangle e_j$. Then $P \in \operatorname{add} \Pi(e - e_i)$ holds. Applying $\operatorname{Hom}_\Pi(-, \Pi)$ to the exact sequence

$$0 \to P' \xrightarrow{f} P \to \Pi e_j \to \Pi_{e-e_i} e_j \to 0$$

gives an exact sequence

$$\operatorname{Hom}_\Pi(P, \Pi) \xrightarrow{f} \operatorname{Hom}_\Pi(P', \Pi) \to \operatorname{Ext}^1_\Pi(\langle e - e_i \rangle e_j, \Pi) \to 0 \tag{5.4.A}$$

of $\Pi^{\rm op}$-modules. Since $f\colon \operatorname{Hom}_\Pi(P,\Pi) \to \operatorname{Hom}_\Pi(P',\Pi)$ belongs to the radical and

$$\operatorname{Ext}^1_\Pi(\langle e-e_i\rangle e_j,\Pi) \cong \operatorname{Ext}^2_\Pi(\Pi_{e-e_i}e_j,\Pi) \overset{5.10}{\cong} D(\Pi_{e-e_i}e_j) \cong e_i\Pi_{e-e_i} \tag{5.4.B}$$

holds, it follows that $P' \cong \Pi e_i$. On the other hand, multiplying (5.4.B) by $e-e_i$ on the right,

$$\operatorname{Ext}^1_\Pi(\langle e-e_i\rangle e_j,\Pi(e-e_i)) \cong e_i\Pi_{e-e_i}(e-e_i) = 0.$$

Hence multiplying (5.4.A) by $e-e_i$ on the right gives an exact sequence

$$\operatorname{Hom}_\Pi(P,\Pi(e-e_i)) \xrightarrow{f} \operatorname{Hom}_\Pi(P',\Pi(e-e_i)) \to 0,$$

showing that f is a left $\operatorname{add}\Pi(e-e_i)$-approximation. The assertions follow. □

We need the following vanishing property.

LEMMA 5.18. *If $x < xs_i$, then $\operatorname{Tor}^\Pi_1(I_x,\Pi_{e-e_i}) = 0$ holds.*

PROOF. Our assumptions $(x,J) \in \mathsf{Cham}(\Delta,\mathcal{J})$ and $x < xs_i$ imply $x < xs_k$ for any $k \in J+i$. By 5.13(2), we have $\operatorname{Ext}^1_{\Pi^{\rm op}}(I_x,S_k) = 0$ for any $k \in J+i$ and hence

$$\operatorname{Tor}^\Pi_1(I_x,S_k) = D\operatorname{Ext}^1_{\Pi^{\rm op}}(I_x,S_k) = 0.$$

Since any composition factor of the Π-module Π_{e-e_i} has the form S_k for some $k \in J+i$, we have $\operatorname{Tor}_1^\Pi(I_x,\Pi_{e-e_i}) = 0$. □

In the rest of this subsection, we will consider the following setup.

SETUP 5.19. Suppose that $\mathcal{J} \subset \Delta$, and $(x,J) \in \mathsf{Cham}(\Delta,\mathcal{J})$ with J strongly Dynkin. For $i \in J^c$, set

$$i' := \iota_{J+i}(i),\ \ J' := (J+i)-i' \ \text{ and } \ (y,J') := (xw_Jw_{J+i},J') = \omega_i(x,J).$$

The following is our crucial observation.

PROPOSITION 5.20. *Under Setup 5.19, assume that $x < xs_i$ and $e_\mathcal{J} I_x e_J \in \operatorname{tilt} e_\mathcal{J}\Pi e_\mathcal{J}$. Then $\nu_i(e_\mathcal{J} I_x e_J) = e_\mathcal{J} I_y e_{J'}$, and so in particular $e_\mathcal{J} I_y e_{J'} \in \operatorname{tilt} e_\mathcal{J}\Pi e_\mathcal{J}$.*

PROOF. Recall that $e_J = 1-\sum_{j\in J} e_j = \sum_{k\in J^c} e_k$. Note that Π_{e_J} is the preprojective algebra of type J, where J is Dynkin by assumption, and thus Π_{e_J} is finite dimensional. Hence we can appeal to 5.17, under which applying $e_\mathcal{J} I_x \otimes_\Pi (-)$ to the exact sequence

$$0 \to \Pi e_i \xrightarrow{f} P \to \langle e_J - e_i\rangle e_j \to 0$$

and using $\operatorname{Tor}^\Pi_1(e_\mathcal{J} I_x,\langle e_J-e_i\rangle e_j) = \operatorname{Tor}^\Pi_3(e_\mathcal{J}(\Pi/I_x),\Pi_{e_J-e_i}e_j) = 0$ by dimension shifting (twice), gives an exact sequence

$$0 \to e_\mathcal{J} I_x e_i \xrightarrow{1\otimes f} e_\mathcal{J} I_x \otimes_\Pi P \to e_\mathcal{J} I_x \otimes_\Pi \langle e_J - e_i\rangle e_j \to 0. \tag{5.4.C}$$

Moreover the map $1\otimes f$ is a left $\operatorname{add}(e_\mathcal{J}\Pi(e_J-e_i))$-approximation by 5.15(3).

On the other hand, applying $e_\mathcal{J} I_x \otimes_\Pi (-)$ to the exact sequence

$$0 \to \langle e_J - e_i\rangle e_j \xrightarrow{g} \Pi e_j \to \Pi_{e_J-e_i}e_j \to 0$$

gives an exact sequence

$$0 \overset{5.18}{=} \operatorname{Tor}^\Pi_1(e_\mathcal{J} I_x,\Pi_{e_J-e_i}e_j) \to e_\mathcal{J} I_x \otimes_\Pi \langle e_J - e_i\rangle e_j \xrightarrow{1\otimes g} e_\mathcal{J} I_x e_j,$$

where $\operatorname{Im}(1\otimes g) = e_\mathcal{J} I_x\langle e_J-e_i\rangle e_j$ holds. Therefore $e_\mathcal{J} I_x \otimes_\Pi \langle e_J - e_i\rangle e_j \cong e_\mathcal{J} I_x\langle e_J-e_i\rangle e_j$.

Hence $\operatorname{proj.dim}_{e_\mathcal{J}\Pi e_\mathcal{J}}(e_\mathcal{J} I_x\langle e_J-e_i\rangle e_j)$ is finite by the sequence (5.4.C), so it is at most one by 5.12(1). By 5.5(1), it follows that

$$\begin{aligned} \nu_i(e_\mathcal{J} I_x e) &= e_\mathcal{J} I_x(e_J-e_i) \oplus e_\mathcal{J} I_x\langle e_J-e_i\rangle e_j \\ \text{(by 5.14(5))} \qquad &= e_\mathcal{J} I_x I_{w_Jw_{J+i}}(e_J-e_i+e_j) \\ &= e_\mathcal{J} I_x I_{w_Jw_{J+i}} e_{J'}. \end{aligned}$$

Since our assumption $x < xs_i$ implies that $x < y = xw_J w_{J+i}$ by 1.22(2), by 5.13(1) we see that $e_{\mathcal{J}} I_x I_{w_J w_{J+i}} e_{J'} = e_{\mathcal{J}} I_y e_{J'}$. The assertion follows. ☐

With the above, we can now prove 5.2(1)–(3).

Proof. (1) Since $I_1 = \Pi$, and $e_{\mathcal{J}} I_1 e_{\mathcal{J}} = e_{\mathcal{J}} \Pi e_{\mathcal{J}}$ is clearly a tilting $e_{\mathcal{J}} \Pi e_{\mathcal{J}}$ module, the assertion follows inductively by using 5.20 and the wall crossing sequence given in 1.23(2).
(2) Certainly either $x < y$ or $x > y$ by 1.22(2). Replacing (x, J) by (y, J') as necessary, we can assume that $x < y$. In this case $x < xs_i$, again by 1.22(2). The assertion then follows from 5.20.
(3)(⇐) If $x < y$, then $x < xs_i$ by 1.22(2), and so by the proof of 5.20 there is an isomorphism $e_{\mathcal{J}} I_x \otimes_\Pi \langle e - e_i \rangle e_j \cong e_{\mathcal{J}} I_x \langle e - e_i \rangle e_j$. Consequently, (5.4.C) becomes

$$0 \to e_{\mathcal{J}} I_x e_i \xrightarrow{1 \otimes f} e_{\mathcal{J}} I_x \otimes_\Pi P \to e_{\mathcal{J}} I_x \langle e - e_i \rangle e_j \to 0.$$

In passing from $e_{\mathcal{J}} I_x e_J$ to $\nu_i(e_{\mathcal{J}} I_x e_J)$, we replace the summand $e_{\mathcal{J}} I_x e_i$ with $e_{\mathcal{J}} I_x \langle e - e_i \rangle e_j$. As is standard, the approximation sequence above implies that $e_{\mathcal{J}} I_x e_J > \nu_i(e_{\mathcal{J}} I_x e)$.
(⇒) By contrapositive, suppose that $x \not< y$. As in the proof of (2) above, since either $x < y$ or $y < x$ holds, we deduce that $y < x$. Exactly the same reasoning as above, now using 5.20 applied to $y < ys_i$, gives $e_{\mathcal{J}} I_y e_{J'} > \nu_i(e_{\mathcal{J}} I_y e_{J'}) = e_{\mathcal{J}} I_x e_J$. It follows that $e_{\mathcal{J}} I_x e_J \not> e_{\mathcal{J}} I_y e_J$, as required. ☐

5.5. The Extended Dynkin Setting

The proof of the last part of 5.2, namely 5.2(4), we involve a localization argument, and this requires Π to have a large centre; this is why we will restrict to the extended Dynkin setting. However, most of the preparatory results in this subsection hold more generally.

The following is a mild generalization of 5.15(2).

Lemma 5.21. *Suppose that Δ is a non-Dynkin graph without loops, and let e be an idempotent such that $\dim_k \Pi_e < \infty$. Then for any $x \in W$, the map $\Pi \to \operatorname{Hom}_{e\Pi e}(eI_x, e\Pi)$ given by $a \mapsto (\cdot a)$ is an isomorphism.*

Proof. Applying $\operatorname{Hom}_{e\Pi e}(-, e\Pi)$ to $0 \to eI_x \to e\Pi \to e(\Pi/I_x) \to 0$, gives an exact sequence

$$\operatorname{Hom}_{e\Pi e}(e(\Pi/I_x), e\Pi) \to \operatorname{Hom}_{e\Pi e}(e\Pi, e\Pi) \to \operatorname{Hom}_{e\Pi e}(eI_x, e\Pi) \to \operatorname{Ext}^1_{e\Pi e}(e(\Pi/I_x), e\Pi)$$

where the two outer spaces are zero by 5.12(2). Therefore there are isomorphisms

$$\Pi \overset{5.15(2)}{\cong} \operatorname{Hom}_{e\Pi e}(e\Pi, e\Pi) \cong \operatorname{Hom}_{e\Pi e}(eI_x, e\Pi)$$

and it is easy to check that the composition is given by $a \mapsto (\cdot a)$. ☐

Lemma 5.22. *Suppose that Δ is a non-Dynkin graph without loops, and let e be an idempotent such that $\dim_k \Pi_e < \infty$. Then for any chain $x_1 > x_2 > \dots$ in W_Δ,*

(1) $\lim_i \dim_k(\Pi/I_{x_i}) = \infty$,
(2) $\lim_i \dim_k(e(\Pi/I_{x_i})e) = \infty$.

Proof. (1) By the dual of 5.13(2) there is a strictly descending chain $I_{x_1} \supsetneq I_{x_2} \supsetneq \dots$, proving the assertion.
(2) Assume that $\lim_i \dim_k(e(\Pi/I_{x_i})e) < \infty$. Then $\mathfrak{n} := \bigcap_{i \geq 0}(eI_{x_i}e)$ satisfies $\mathfrak{n} = eI_{x_i}e$ for $i \gg 0$ and hence $\dim_k(e\Pi e/\mathfrak{n}) < \infty$. Using 5.12 twice,

$$\dim_k(\Pi e \otimes_{e\Pi e} (e\Pi e/\mathfrak{n})) < \infty \text{ and } \dim_k(\Pi e \otimes_{e\Pi e} (e\Pi e/\mathfrak{n}) \otimes_{e\Pi e} e\Pi) < \infty.$$

But since there is a surjective map

$$\Pi e \otimes_{e\Pi e} (e\Pi e/\mathfrak{n}) \otimes_{e\Pi e} e\Pi \twoheadrightarrow \Pi e\Pi/\Pi\mathfrak{n}\Pi,$$

it follows that $\dim_k(\Pi e\Pi/\Pi\mathfrak{n}\Pi) < \infty$ and hence $\dim_k(\Pi/\Pi\mathfrak{n}\Pi) < \infty$. But then since

$$I_{x_i} \supset \Pi e I_{x_i} e\Pi \supset \Pi\mathfrak{n}\Pi,$$

it follows that

$$\dim_k(\Pi/I_{x_i}) < \dim_k(\Pi/\Pi\mathfrak{n}\Pi) < \infty$$

for all $i > 0$, which contradicts $\lim_i \dim_k(\Pi/I_{x_i}) = \infty$. ☐

The following is the key technical result of this subsection; we show below that all the assumptions hold in the extended Dynkin case.

Proposition 5.23. *Let $\Gamma_{\mathfrak{J}} = e_{\mathfrak{J}}\Pi e_{\mathfrak{J}}$ be a complete partial preprojective algebra of non-Dynkin type, and $\mathfrak{J}$ a strongly Dynkin subset of Δ. If a tilting $\Gamma_{\mathfrak{J}}$ module T satisfies*

$$\dim_k\left(\frac{\Gamma_{\mathfrak{J}}}{\sum_{f\in\mathrm{Hom}_{\Gamma_{\mathfrak{J}}}(T,\Gamma_{\mathfrak{J}})}\mathrm{Im}\,f}\right) < \infty, \tag{5.5.A}$$

then there exists $(x, J) \in \mathsf{Cham}(\Delta, \mathfrak{J})$ such that $\mathrm{add}\,T = \mathrm{add}(e_{\mathfrak{J}}I_x e_J)$.

Proof. To ease notation, set $\Gamma := \Gamma_{\mathfrak{J}}$. Assume that T does not satisfy the desired condition. Using 5.9 repeatedly, there is an infinite sequence

$$\Gamma > T^1 > T^2 > \dots$$

of tilting mutation such that $T^i > T$ for all $i \geq 0$. By 5.2(2), there exists $(x_i, J^i) \in \mathsf{Cham}(\Delta, \mathfrak{J})$ such that $T^i = e_{\mathfrak{J}}I_{x_i}e_{J^i}$. Then $T^i > T$ implies that

$$\sum_{f\in\mathrm{Hom}_\Gamma(T,\Gamma)}\mathrm{Im}\,f \subseteq \sum_{f\in\mathrm{Hom}_\Gamma(T^i,\Gamma)}\mathrm{Im}\,f \overset{5.21}{=} (e_{\mathfrak{J}}I_{x_i}e_{J^i})(e_{J^i}\Pi e_{\mathfrak{J}}) \subseteq e_{\mathfrak{J}}I_{x_i}e_{\mathfrak{J}}$$

for all i. Hence

$$\dim_k\left(\frac{\Gamma}{\sum_{f\in\mathrm{Hom}_\Gamma(T,\Gamma)}\mathrm{Im}\,f}\right) > \dim_k(e_{\mathfrak{J}}(\Pi/I_{x_i})e_{\mathfrak{J}})$$

for any $i > 0$, which contradicts the fact $\lim_i \dim_k(e_{\mathfrak{J}}(\Pi/I_{x_i})e_{\mathfrak{J}}) = \infty$ in 5.22. ☐

As final preparation before 5.2(4) we require the following, which is very well-known.

Lemma 5.24. *Let Π be a preprojective algebra of extended Dynkin type.*

1. *The centre R of Π is a simple singularity in dimension two.*
2. $\Pi \cong \mathrm{End}_R(M)$ *for some Cohen-Macaulay R-module M.*
3. $\Pi_{\mathfrak{p}}$ *is Morita equivalent to the local ring $R_{\mathfrak{p}}$ for all non-maximal primes $\mathfrak{p}$ of R.*

Proof. (1) and (2) are well-known [CBH], and (3) follows since $\Pi_{\mathfrak{p}} \cong \mathrm{End}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ where $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module for all non-maximal primes. ☐

With the above, we now prove 5.2(4).

Proof. Again set $\Gamma := e_{\mathfrak{J}}\Pi e_{\mathfrak{J}}$, and consider $C := \frac{\Gamma}{\sum_{f\in\mathrm{Hom}_\Gamma(T,\Gamma)}\mathrm{Im}\,f}$. By 5.23, we only have to check that $\dim_k C < \infty$.

To prove this, it suffices to show that $C_{\mathfrak{p}} = 0$ holds for any non-maximal prime ideal $\mathfrak{p}$ of R. Since $\Gamma_{\mathfrak{p}} \cong e_{\mathfrak{J}}\Pi_{\mathfrak{p}}e_{\mathfrak{J}}$ is Morita equivalent to the local ring $R_{\mathfrak{p}}$ by 5.24(3), any tilting $\Gamma_{\mathfrak{p}}$-module is a progenerator. Since tilting modules are preserved by localization, it follows that $T_{\mathfrak{p}}$ a progenerator, and so certainly

$$\Gamma_{\mathfrak{p}} = \sum_{f\in\mathrm{Hom}_{\Gamma_{\mathfrak{p}}}(T_{\mathfrak{p}},\Gamma_{\mathfrak{p}})}\mathrm{Im}\,f$$

holds. Since $\mathrm{Hom}_\Gamma(T,\Gamma)_{\mathfrak{p}} \cong \mathrm{Hom}_{\Gamma_{\mathfrak{p}}}(T_{\mathfrak{p}},\Gamma_{\mathfrak{p}})$, this implies that $C_{\mathfrak{p}} = 0$. ☐

We can now use 5.2(4), in the extended Dynkin setting, to link tilting to chambers via K-theory. As such, let Δ be ADE Dynkin, with affine version Δ_{aff}, and recall that Θ is a $\mathbb{R}$-vector space with basis α_i^* with $i \in \Delta_{\mathsf{aff}}$, and that L is the lattice in Θ generated by α_i^* with $i \in \Delta_{\mathsf{aff}}$. There is a natural identification

$$\beta\colon \mathrm{K}_0(\operatorname{proj}\Pi) \xrightarrow{\sim} L,$$

given by $\Pi e_i \mapsto \alpha_i^*$. The following is known.

THEOREM 5.25. [IR] *There are natural bijections*

(1) $\operatorname{ptilt}\Pi \xrightarrow{\sim} L^+$ *given by* $T \mapsto \beta[T]$.
(2) $\operatorname{tilt}\Pi \xrightarrow{\sim} \mathsf{Cham}(\Delta_{\mathsf{aff}})$ *given by* $T = T_1 \oplus \ldots \oplus T_n \mapsto \sum_{i=1}^n \mathbb{R}_{>0}(\beta[T_i])$.

For any subset $\mathcal{J}$ of Δ_{aff}, recall that $\Theta_{\mathcal{J}}$ is the subspace of Θ spanned by α_i^* with $i \notin \mathcal{J}$, and that $L_{\mathcal{J}}$ is the lattice in $\Theta_{\mathcal{J}}$ generated by α_i^* with $i \notin \mathcal{J}$. Set

$$L_{\mathcal{J}}^+ := L_{\mathcal{J}} \cap \mathsf{Cone}(\Delta_{\mathsf{aff}}, \mathcal{J}).$$

As above, set $e_{\mathcal{J}} = 1 - \sum_{j \in \mathcal{J}} e_j$, and $\Gamma_{\mathcal{J}} = e_{\mathcal{J}}\Pi e_{\mathcal{J}}$, where Π is the preprojective algebra of type Δ_{aff}. There is a natural identification

$$\beta_{\mathcal{J}}\colon \mathrm{K}_0(\operatorname{proj}\Gamma_{\mathcal{J}}) \xrightarrow{\sim} L_{\mathcal{J}},$$

given by $\Gamma_{\mathcal{J}} e_i \mapsto \alpha_i^*$. The following extends 5.25, and will be used later.

THEOREM 5.26. *For any subset $\mathcal{J}$ of Δ_{aff}, there are natural bijections*

(1) $\operatorname{ptilt}\Gamma_{\mathcal{J}} \xrightarrow{\sim} L_{\mathcal{J}}^+$ *given by* $T \mapsto \beta_{\mathcal{J}}[T]$.
(2) $\operatorname{tilt}\Gamma_{\mathcal{J}} \xrightarrow{\sim} \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J})$ *given by* $T = T_1 \oplus \ldots \oplus T_n \mapsto \sum_{i=1}^n \mathbb{R}_{>0}(\beta_{\mathcal{J}}[T_i])$.

PROOF. By 5.25, there is an isomorphism $\beta\colon \mathrm{K}_0(\operatorname{proj}\Pi) \xrightarrow{\sim} L$ given by $\Pi e_i \mapsto \alpha_i^*$. For any $x \in W$ and $i \in \Delta_{\mathsf{aff}}$, we have $\beta[I_x e_i] = x\alpha_i^*$.

Every element of $\operatorname{tilt}\Gamma_{\mathcal{J}}$ is isomorphic to $e_{\mathcal{J}} I_x e_K$ for some $(x, K) \in \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J})$, using 5.2(4). Since $e_{\mathcal{J}} I_x e_K = \bigoplus_{i \notin K} e_{\mathcal{J}} I_x e_i$, and $\beta_{\mathcal{J}}[e_{\mathcal{J}} I_x e_i] = x\alpha_i^*$ holds via the commutative diagram

$$\begin{array}{ccc} \mathrm{K}_0(\operatorname{proj}\Pi) & \longrightarrow & L \\ \uparrow & & \uparrow \\ \mathrm{K}_0(\operatorname{proj}\Gamma_{\mathcal{J}}) & \longrightarrow & L_{\mathcal{J}} \end{array}$$

we see that $[-] \circ \beta$ takes elements of $\operatorname{tilt}\Gamma_{\mathcal{J}}$ to the primitive vectors defining the chambers $\mathsf{Cham}(\Delta, J)$. Thus (2) holds. Part (1) follows immediately, since by Bongartz completion, every partial tilting module is the summand of a (not necessarily basic) tilting module. $\square$

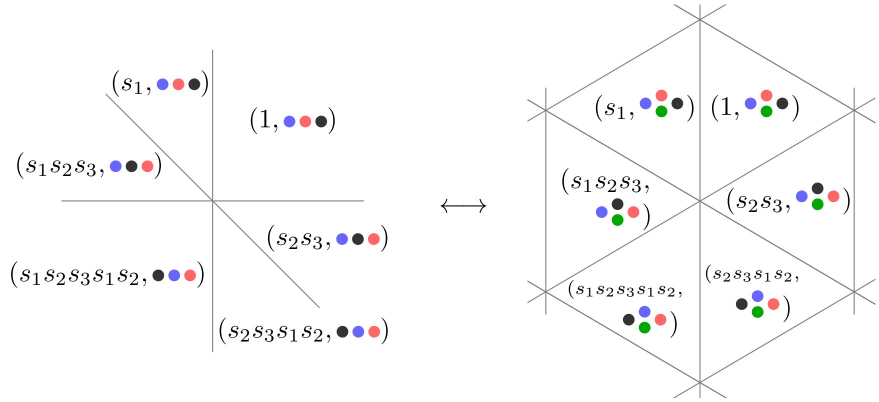
5.6. Orders, Paths and Basepoints

As notation, recall for a fixed $\mathcal{J} \subseteq \Delta_{\mathsf{aff}}$, by 5.2 there is a bijection

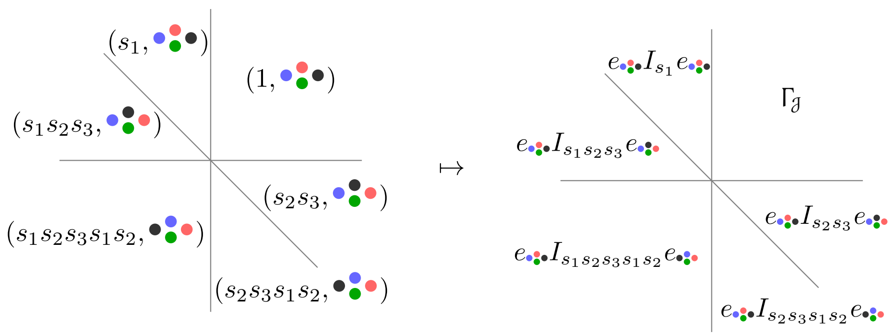
$$\mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J}) \to \operatorname{tilt}\Gamma_{\mathcal{J}}$$

given by $(x, J) \mapsto e_{\mathcal{J}} I_x e_J$, under which wall crossing corresponds to mutation.

EXAMPLE 5.27. Continuing Example 1.11, for the A_3 Dynkin diagram consider the choice $\mathcal{J} = \bullet\bullet\bullet$, viewed in affine A_3 as $\mathcal{J} = \bullet\bullet\bullet\bullet$, where green is the extended vertex. As in 4.1, provided that we do not mutate at the extended vertex, then the $w_J w_{J+i}$ wall crossing rules are not effected by this additional vertex, so the calculation 1.11 can be transferred to describe part of the affine tiling. This is illustrated below.



It is easier to draw the left hand side, but to emphasise that we are working with the preprojective algebra of the extended Dynkin quiver, we will always label the chambers there with the extra green extended vertex. Doing this, the map $\mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J}) \to \operatorname{tilt} \Gamma_{\mathcal{J}}$ restricted to $\mathsf{Cham}(\Delta, \mathcal{J})$ then sends



We now decompose the elements of $\operatorname{tilt} \Gamma_{\mathcal{J}}$ into smaller pieces. For the case $\mathcal{J} = \emptyset$, in which case $\Gamma_{\mathcal{J}} = \Pi$, this was achieved in [BIRS]; see also [SY, 2.13(2)]. Indeed, when $\mathcal{J} = \emptyset$ then every tilting module has the form I_w for some $w \in W$, and any choice of reduced expression $w = s_{i_n} \circ \ldots \circ s_{i_1}$ induces isomorphisms

$$I_w \cong I_{s_{i_n}} \ldots I_{s_{i_1}} \cong I_{s_{i_n}} \otimes^{\mathbf{L}}_{\Pi} \circ \ldots \circ \otimes^{\mathbf{L}}_{\Pi} I_{s_{i_1}}.$$

The purpose of this section is to replicate the above decomposition in the setting of tilting modules for an arbitrary $\Gamma_{\mathcal{J}}$.

Recall that, for any hyperplane arrangement, the *length* of a positive path is the number of simple wall crossings that it traverses.

LEMMA 5.28. *If* $\beta \colon (x_1, \mathcal{J}_1) \to (x_2, \mathcal{J}_2)$ *is a positive path in* $\mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J})$, *then the following are equivalent.*

(1) β *is reduced, that is, it does not cross any hyperplane twice.*
(2) β *is minimal, that is, there is no path* $(x_1, \mathcal{J}_1) \to (x_2, \mathcal{J}_2)$ *of shorter length.*

PROOF. This is a property of locally finite arrangements; see e.g. [S, Lemma 2]. □

LEMMA 5.29. *Suppose that* $\beta \colon (1, \mathcal{J}) \to (x, J)$ *is a reduced path, and consider a simple wall crossing* $\omega_i \colon (x, J) \to (y, J')$. *The following are equivalent*

(1) $\omega_i \circ \beta$ *is reduced.*
(2) $\omega_i \circ \beta$ *is minimal.*
(3) $x < y$.

PROOF. Write H for the hyperplane separating (x, J) and (y, J'). Since β is reduced, $\omega_i \circ \beta$ is not reduced if and only if β also crosses H.
(1)⇔(2) is 5.28.
(3)⇔(1) By 5.2(3), $y < x$ if and only if $e_{\mathcal{J}} I_y e_J > e_{\mathcal{J}} I_x e_{J'}$. Passing to K-theory classes using 5.26, an argument similar to [**HW1**, B.4] shows that $e_{\mathcal{J}} I_y e_J > e_{\mathcal{J}} I_x e_{J'}$ if and only if the chamber (y, J') is on the same side of H as $[\Gamma_{\mathcal{J}}] \in (1, \mathcal{J})$. Clearly this holds if and only if $\omega_i \circ \beta$ crosses H at least twice, which holds if and only if β crosses H. By the top paragraph of the proof, this holds if and only $\omega_i \circ \beta$ is not reduced. □

Consider a positive reduced path $\alpha\colon (1, \mathcal{J}) \to (x, J)$, and then decompose α into simple wall crossings

$$\alpha\colon \quad (1, \mathcal{J}) = (x_1, \mathcal{J}_1) \xrightarrow{\omega_{i_1}} (x_2, \mathcal{J}_2) \xrightarrow{\omega_{i_2}} \dots \xrightarrow{\omega_{i_n}} (x_{n+1}, \mathcal{J}_{n+1}) = (x, J) \quad \text{in } \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J})$$

By 5.29, necessarily $x_1 < x_2 < \dots < x_{n+1}$. Recall from 1.16 that every simple wall crossing is of the form

$$\omega_{i_t}(x_t, \mathcal{J}_t) := (x_t w_{\mathcal{J}_t} w_{\mathcal{J}_t + i_t}, \mathcal{J}_t + i_t - \iota_{\mathcal{J}_t + i_t}(i_t)).$$

Thus, under each wall crossing $(x_t, \mathcal{J}_t) \to (x_{t+1}, \mathcal{J}_{t+1})$, we can obtain x_{t+1} from x_t by post multiplying by $w_{\mathcal{J}_t} w_{\mathcal{J}_t + i_t}$.

Consider now instead $\mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J}_t)$, which is in bijection with tilt $\Gamma_{\mathcal{J}_t}$. From $(1, \mathcal{J}_t) \in \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J}_t)$, we can still wall cross ω_{i_t}, which now becomes

$$\omega_{i_t}\colon (1, \mathcal{J}_t) \to (w_{\mathcal{J}_t} w_{\mathcal{J}_t + i_t}, \mathcal{J}_{t+1}) \quad \text{in } \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J}).$$

Applying 5.2 to $\mathcal{J}_t$, we obtain $e_{\mathcal{J}_t} I_{w_{\mathcal{J}_t} w_{\mathcal{J}_t + i_t}} e_{\mathcal{J}_{t+1}} \in \text{tilt}\, \Gamma_{\mathcal{J}_t}$ for all $t = 1, \dots, n$. For calibration, when $\mathcal{J} = \emptyset$, all $\mathcal{J}_t = \emptyset$, and these tilting modules are precisely $I_{s_{i_t}}$.

Summarising, for every decomposition of a reduced positive path α into simple wall crossings,

$$\alpha\colon \quad (1, \mathcal{J}) = (x_1, \mathcal{J}_1) \xrightarrow{\omega_{i_1}} (x_2, \mathcal{J}_2) \xrightarrow{\omega_{i_2}} \dots \xrightarrow{\omega_{i_n}} (x_{n+1}, \mathcal{J}_{n+1})$$

we can form $e_{\mathcal{J}} I_{w_{\mathcal{J}} w_{\mathcal{J}+i_1}} e_{\mathcal{J}_2} \otimes^{\mathbf{L}}_{\Gamma_{\mathcal{J}_2}} e_{\mathcal{J}_2} I_{w_{\mathcal{J}_2} w_{\mathcal{J}_2+i_2}} e_{\mathcal{J}_3} \otimes^{\mathbf{L}}_{\Gamma_{\mathcal{J}_3}} \dots \otimes^{\mathbf{L}}_{\Gamma_{\mathcal{J}_n}} e_{\mathcal{J}_n} I_{w_{\mathcal{J}_n} w_{\mathcal{J}_n + i_n}} e_{\mathcal{J}_{n+1}}$. As the derived tensor of tilting modules, this is a tilting complex. The following is the main result of this section, which in particular asserts that this tilting complex is in fact a tilting module, and it is independent of the decomposition of α.

THEOREM 5.30. *Suppose that Δ_{aff} is extended Dynkin, and $\mathcal{J}$ is a subset of vertices. Then for any decomposition of a positive reduced path $\alpha\colon (1, \mathcal{J}) \to (x, J)$ as above, there are isomorphisms of bimodules*

$$\begin{aligned} & e_{\mathcal{J}} I_{w_{\mathcal{J}} w_{\mathcal{J}+i_1}} e_{\mathcal{J}_2} \otimes^{\mathbf{L}}_{\Gamma_{\mathcal{J}_2}} e_{\mathcal{J}_2} I_{w_{\mathcal{J}_2} w_{\mathcal{J}_2+i_2}} e_{\mathcal{J}_3} \otimes^{\mathbf{L}}_{\Gamma_{\mathcal{J}_3}} \dots \otimes^{\mathbf{L}}_{\Gamma_{\mathcal{J}_n}} e_{\mathcal{J}_n} I_{w_{\mathcal{J}_n} w_{\mathcal{J}_n+i_n}} e_{\mathcal{J}_{n+1}} \\ \cong\ & e_{\mathcal{J}} I_{w_{\mathcal{J}} w_{\mathcal{J}+i_1}} e_{\mathcal{J}_2} \otimes_{\Gamma_{\mathcal{J}_2}} e_{\mathcal{J}_2} I_{w_{\mathcal{J}_2} w_{\mathcal{J}_2+i_2}} e_{\mathcal{J}_3} \otimes_{\Gamma_{\mathcal{J}_3}} \dots \otimes_{\Gamma_{\mathcal{J}_n}} e_{\mathcal{J}_n} I_{w_{\mathcal{J}_n} w_{\mathcal{J}_n+i_n}} e_{\mathcal{J}_{n+1}} \\ \cong\ & e_{\mathcal{J}} I_{x_{n+1}} e_{\mathcal{J}_{n+1}} \end{aligned}$$

PROOF. We proceed by induction on the length of the path α, where in the case of length one there is nothing to prove. Hence we can assume that the result is true for paths of smaller length, and so it suffices to prove that there are bimodule isomorphisms

$$e_{\mathcal{J}} I_{x_n} e_{\mathcal{J}_n} \otimes^{\mathbf{L}}_{\Gamma_{\mathcal{J}_n}} e_{\mathcal{J}_n} I_{w_{\mathcal{J}_n} w_{\mathcal{J}_n+i_n}} e_{\mathcal{J}_{n+1}} \cong e_{\mathcal{J}} I_{x_n} e_{\mathcal{J}_n} \otimes_{\Gamma_{\mathcal{J}_n}} e_{\mathcal{J}_n} I_{w_{\mathcal{J}_n} w_{\mathcal{J}_n+i_n}} e_{\mathcal{J}_{n+1}} \cong e_{\mathcal{J}} I_{x_{n+1}} e_{\mathcal{J}_{n+1}}.$$

To ease notation, set $A = e_{\mathcal{J}} I_{x_n} e_{\mathcal{J}_n}$, $B = e_{\mathcal{J}_n} I_{w_{\mathcal{J}_n} w_{\mathcal{J}_n+i_n}} e_{\mathcal{J}_{n+1}}$, so that we need to prove there are bimodule isomorphisms

$$A \otimes^{\mathbf{L}}_{\Gamma_{\mathcal{J}_n}} B \cong A \otimes_{\Gamma_{\mathcal{J}_n}} B \cong e_{\mathcal{J}} I_{x_{n+1}} e_{\mathcal{J}_{n+1}}. \tag{5.6.A}$$

As above, since α is reduced, by 5.29 necessarily $x_n < x_{n+1}$, and thus by 5.2(3) $A > \nu_{i_n} A$. Given this last fact, as is standard (see e.g. [**HW1**, B.1]), it follows that

$$\nu_{i_n} A \cong A \otimes^{\mathbf{L}}_{\operatorname{End}_{\Gamma_{\mathcal{J}}}(A)} \nu_{i_n} \operatorname{End}_{\Gamma_{\mathcal{J}}}(A)$$

as left $\Gamma_{\mathcal{J}}$-modules, and further $\mathbf{R}\mathrm{Hom}_{\Gamma_{\mathcal{J}}}(A, \nu_{i_n} A) \cong \mathrm{Hom}_{\Gamma_{\mathcal{J}}}(A, \nu_{i_n} A)$.

By 5.16, $\mathrm{End}_{\Gamma_{\mathcal{J}}}(A) \cong \Gamma_{\mathcal{J}}$, and by definition $\nu_{i_n} \mathrm{End}_{\Gamma_{\mathcal{J}}}(A) \cong B$. Note that under the isomorphism in 5.16, the natural right action of $\Gamma_{\mathcal{J}_n}$ on A by multiplication coincides with the natural right action of $\mathrm{End}_{\Gamma_{\mathcal{J}}}(A)$ on A, and thus

$$\nu_{i_n} A \cong A \otimes^{\mathbf{L}}_{\Gamma_{\mathcal{J}_n}} B$$

as left $\Gamma_{\mathcal{J}}$-modules. In particular, the right hand side is only concentrated in degree zero, so truncating in the category of bimodules establishes the first isomorphism in (5.6.A).

For the second bimodule isomorphism in (5.6.A), note first that

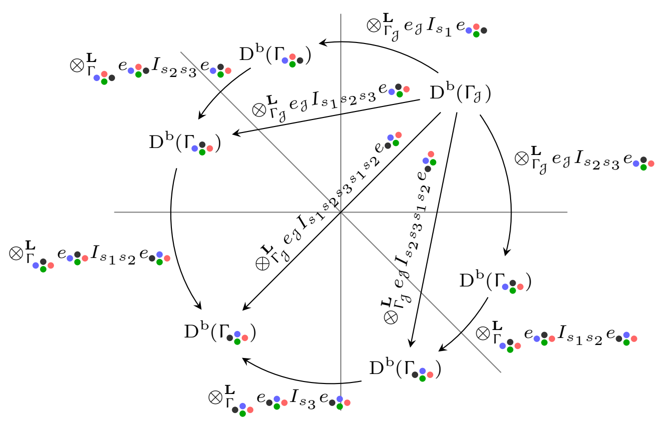
$$\begin{aligned}
B &= e_{\mathcal{J}_n} I_{w_{\mathcal{J}_n} w_{\mathcal{J}_{n+i_n}}} e_{\mathcal{J}_{n+1}} \\
\text{(by 5.13(1))} \quad &\cong e_{\mathcal{J}_n} \mathrm{Hom}_{\Pi}(I_{x_n}, I_{x_{n+1}}) e_{\mathcal{J}_{n+1}} \\
\text{(by 5.15(1))} \quad &\cong e_{\mathcal{J}_n} \mathrm{Hom}_{\Gamma_{\mathcal{J}}}(e_{\mathcal{J}} I_{x_n}, e_{\mathcal{J}} I_{x_{n+1}}) e_{\mathcal{J}_{n+1}} \\
&\cong \mathrm{Hom}_{\Gamma_{\mathcal{J}}}(e_{\mathcal{J}} I_{x_n} e_{\mathcal{J}_n}, e_{\mathcal{J}} I_{x_{n+1}} e_{\mathcal{J}_{n+1}}) \\
&= \mathrm{Hom}_{\Gamma_{\mathcal{J}}}(A, \nu_{i_n} A),
\end{aligned}$$

and so $B \cong \mathrm{Hom}_{\Gamma_{\mathcal{J}}}(A, \nu_{i_n} A)$ via $b \mapsto (\cdot b)$. Then, consider the composition of isomorphisms

$$A \otimes_{\Gamma_{\mathcal{J}_n}} B \xrightarrow{\sim} A \otimes_{\Gamma_{\mathcal{J}_n}} \mathrm{Hom}_{\Gamma_{\mathcal{J}}}(A, \nu_{i_n} A) \xrightarrow{\sim} \nu_{i_n} A$$

where the first given by $a \otimes b \mapsto a \otimes (\cdot b)$ above, and the second is the derived adjunction (after noting $\mathbf{R}\mathrm{Hom}_{\Gamma_{\mathcal{J}}}(A, \nu_{i_n} A) \cong \mathrm{Hom}_{\Gamma_{\mathcal{J}}}(A, \nu_{i_n} A)$ above), which is $a \otimes f \mapsto f(a)$. The composition is the isomorphism $a \otimes b \mapsto ab$, which is clearly a bimodule isomorphism. □

Example 5.31. Continuing the running Example 5.27, as a consequence of 5.30, the following diagram commutes.



5.7. The $\mathcal{J}$-cone Groupoid

In this section we will re-interpret the above results in terms of Deligne groupoid $\mathcal{G}_{\mathcal{J}_{\mathsf{aff}}}$ from §2.3, and show that iterated tilts form a representation of the groupoid. As a corollary, we obtain an action of both the finite and affine $\mathcal{J}$-pure braid group on the derived category of contracted preprojective algebras.

We observed above that under each wall crossing $\omega_i(x, \mathcal{I}_1) \to (y, \mathcal{I}_2)$, we can obtain y from x_t by post-multiplying by $w_{\mathcal{I}_1} w_{\mathcal{I}_1+i}$.

Definition 5.32. Let Δ be ADE Dynkin, and consider $\mathcal{J} \subseteq \Delta$.

(1) The groupoid $\mathbb{G}_{\mathcal{J}_{\mathsf{aff}}}$ is defined as follows. As objects, for every chamber $(x, \mathcal{I}) \in \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J})$, associate a vertex labelled $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, \Gamma_{\mathcal{I}})$. The morphisms are generated by the simple wall crossings, where to $\omega_i(x, \mathcal{I}_1) = (y, \mathcal{I}_2)$ we associate the equivalence $\mathbf{R}\mathrm{Hom}_{\Gamma_{\mathcal{I}_1}}(e_{\mathcal{I}_1} I_{w_{\mathcal{I}_1} w_{\mathcal{I}_1+i}} e_{\mathcal{I}_2}, -)$.

(2) The groupoid $\mathbb{G}_{\mathcal{J}}$ is defined as follows. As objects, for every chamber $(x, \mathcal{I}) \in \mathsf{Cham}(\Delta, \mathcal{J})$, associate a vertex labelled $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Gamma_{\mathcal{I}})$. The morphisms are generated by the simple wall crossings, where to $\omega_i(x, \mathcal{I}_1) = (y, \mathcal{I}_2)$ we associate the equivalence $\mathbf{R}\mathrm{Hom}_{\Gamma_{\mathcal{I}_1}}(e_{\mathcal{I}_1} I_{w_{\mathcal{I}_1} w_{\mathcal{I}_1+i}} e_{\mathcal{I}_2}, -)$.

Example 5.33. Continuing the example 2.17, the groupoids $\mathbb{G}_{\mathcal{J}_{\mathsf{aff}}}$ and $\mathbb{G}_{\mathcal{J}}$ are obtained from those pictures by replacing each dot with an appropriate derived category of a contracted preprojective algebra, and each arrow by the equivalence described in 5.32.

Remark 5.34. Even although by 1.20 simple wall crossing is an involution, the functorial version is not an involution. Indeed, even in the case $\mathcal{J} = \emptyset$, the functor $\mathbf{R}\mathrm{Hom}_{\Pi}(I_i, -) \circ \mathbf{R}\mathrm{Hom}_{\Pi}(I_i, -) \cong \mathbf{R}\mathrm{Hom}_{\Pi}(I_i \otimes^{\mathbf{L}}_{\Pi} I_i, -)$ is not the identity, and is instead a spherical twist. A generalisation of this, in special cases, is given in 10.2 below, however the general description of monodromy in terms of spherical twists needs derived noncommutative deformation theory in this setting; see [B1].

The following is the main result of this section.

Theorem 5.35. *Suppose that Δ is ADE Dynkin, and $\mathcal{J} \subseteq \Delta$. Then there are functors:*

$$\mathcal{G}_{\mathcal{J}} \to \mathbb{G}_{\mathcal{J}}$$
$$\mathcal{G}_{\mathcal{J}_{\mathsf{aff}}} \to \mathbb{G}_{\mathcal{J}_{\mathsf{aff}}}$$

given, in both cases, by sending a vertex corresponding to a chamber labelled $(x, \mathcal{I})$ to $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Gamma_{\mathcal{I}})$, and to $\omega_i(x, \mathcal{I}_1) = (y, \mathcal{I}_2)$ the equivalence $\mathbf{R}\mathrm{Hom}_{\Gamma_{\mathcal{I}_1}}(e_{\mathcal{I}_1} I_{w_{\mathcal{I}_1} w_{\mathcal{I}_1+i}} e_{\mathcal{I}_2}, -)$

Proof. All the work has already been done. In either case, denote the functor above by F. It suffices to show that the relations on $\mathcal{G}_{\mathcal{J}}$ and $\mathcal{G}_{\mathcal{J}_{\mathsf{aff}}}$ in 2.16 are satisfied functorially in $\mathbb{G}_{\mathcal{J}}$ and $\mathbb{G}_{\mathcal{J}_{\mathsf{aff}}}$. By definition, in 2.13, it suffices to show that any positive two reduced paths

$$\alpha, \beta \colon (x, \mathcal{I}_1) \to (y, \mathcal{I}_2)$$

give rise to isomorphic functors $F(\alpha) \cong F(\beta)$. This is just a relabelling trick. We can change the labels of the chambers, indexing instead by $\mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{I}_1)$, such that both α and β start at $(1, \mathcal{I}_1)$, and end at $(y', \mathcal{I}_2)$ say. This reindexing does not effect the wall crossing functors, and thus does not effect the functors $F(\alpha)$ or $F(\beta)$, which are compositions of these. The result then follows immediately from 5.30, since both $F(\alpha)$ and $F(\beta)$ are isomorphic to the direct functor given by $\mathbf{R}\mathrm{Hom}_{\Gamma_{\mathcal{I}_1}}(e_{\mathcal{I}_1} I_{y'} e_{\mathcal{I}_2}, -)$. $\square$

Recall the notation $\pi_1(\mathcal{J})$ and $\pi_1(\mathcal{J}_{\mathsf{aff}})$ from 2.18. By passing to vertex groups, the following is then immediate from 5.35.

Corollary 5.36. *Suppose that Δ is ADE Dynkin, and $\mathcal{J} \subseteq \Delta$. Then there are group homomorphisms such that the following diagram commutes.*

$$\begin{array}{ccc} \pi_1(\mathcal{J}) & \xrightarrow{\varphi} & \operatorname{Auteq} \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Gamma_{\mathcal{J}}) \\ \downarrow & \nearrow_{\widetilde{\varphi}} & \\ \pi_1(\mathcal{J}_{\mathsf{aff}}) & & \end{array}$$

We will show in Part 4 that φ is faithful.

CHAPTER 6

Derived Classification: Dynkin Type

Throughout this chapter, let Π be the preprojective algebra of an ADE Dynkin quiver, and for a fixed subset $\mathcal{J} \subseteq \Delta$, consider the corresponding contracted preprojective algebra $\Gamma_{\mathcal{J}} := e_{\mathcal{J}}\Pi e_{\mathcal{J}}$. In this setting, both Π and $\Gamma_{\mathcal{J}}$ are finite dimensional algebras.

In this setting, since Π is self-injective, the only modules of finite projective dimension are free. Thus Π has no classical tilting modules, and the results of the previous chapter do not apply. The algebra Π does, however, have both silting and tilting complexes, and its derived equivalence class is understood [**AM**].

In this chapter we describe two-term silting and tilting complexes for $\Gamma_{\mathcal{J}}$, under the assumption that $\iota(\mathcal{J}) = \mathcal{J}$. This assumption is needed to ensure that $\Gamma_{\mathcal{J}}$ is also self-injective (see 6.2 below). We establish that various silting and tilting complexes for $\Gamma_{\mathcal{J}}$ can be described in terms of the intersection arrangements from Chapter 1 and in the process, intersection arrangements from non-ADE Dynkin diagrams naturally arise. This gives some justification to the level of generality developed in Part 1. One of the main consequences of this chapter is that in the case $\iota(\mathcal{J}) = \mathcal{J}$, the algebra $\Gamma_{\mathcal{J}}$ is tilting-discrete, its derived equivalence class is finite, and we give a complete classification of all basic members of this class.

6.1. Silting, Tilting and Folding

This chapter, and Chapter 7, will be concerned with properties of silting and tilting complexes. We recall the following, mainly to set notation.

DEFINITION 6.1. Let A be a ring, $P = \ldots \to P_{-1} \to P_0 \to P_1 \to \ldots \in \mathrm{K^b}(\operatorname{proj} A)$.

(1) P is called *two-term* if $P_i = 0$ for all $i \neq -1, 0$.
(2) P is called silting if $\operatorname{Hom}_{\mathrm{K^b}(\operatorname{proj} A)}(P, P[i]) = 0$ for all $i > 0$.
(3) P is called tilting if $\operatorname{Hom}_{\mathrm{K^b}(\operatorname{proj} A)}(P, P[i]) = 0$ for all $i \neq 0$.

When $\operatorname{proj} A$ is Krull–Schmidt, We write $\operatorname{tilt}^{\boldsymbol{\cdot}} A$ for the set of basic tilting complexes, $2\operatorname{silt} A$ for the set of basic two-term silting complexes, and $2\operatorname{tilt} A$ for the set of basic two-term tilting complexes.

It is a classical fact that the preprojective algebra Π of Dynkin type is self-injective finite dimensional algebra; this corresponds to the $\mathcal{J} = \emptyset$ case of the following. Recall the notation $\iota_\Delta = \iota$ from 1.2, which denotes the Dynkin involution.

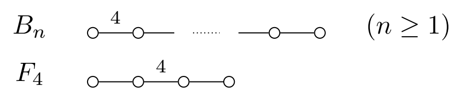
LEMMA 6.2. *Let Δ be ADE Dynkin and $\mathcal{J} \subseteq \Delta$. Then $\Gamma_{\mathcal{J}}$ is self-injective if and only if $\iota_\Delta(\mathcal{J}) = \mathcal{J}$.*

PROOF. $\Gamma_{\mathcal{J}}$ is self-injective if and only if the Nakayama functor $\mathcal{N} = D\operatorname{Hom}_{\Gamma_{\mathcal{J}}}(-, \Gamma_{\mathcal{J}})$ preserves projectives. These are precisely the $e_{\mathcal{J}}\Pi e_i$ with $i \in \Delta \backslash \mathcal{J}$. Now

$$\mathcal{N}(e_{\mathcal{J}}\Pi e_i) = D\operatorname{Hom}_{e_{\mathcal{J}}\Pi e_{\mathcal{J}}}(e_{\mathcal{J}}\Pi e_i, e_{\mathcal{J}}\Pi e_{\mathcal{J}}) \cong D\operatorname{Hom}_\Pi(\Pi e_i, \Pi e_{\mathcal{J}}) = e_{\mathcal{J}} D\operatorname{Hom}_\Pi(\Pi e_i, \Pi).$$

Since $D\operatorname{Hom}_\Pi(-, \Pi)$ is the Nakayama functor on Π, and ι_Δ is the Nakayama permutation, $D\operatorname{Hom}_\Pi(\Pi e_i, \Pi) \cong \Pi e_{\iota(i)}$. Hence $\mathcal{N}(e_{\mathcal{J}}\Pi e_i) \cong e_{\mathcal{J}}\Pi e_{\iota(i)}$, so $\mathcal{N}$ preserves projectives if and only if $\iota(i) \in \Delta \backslash \mathcal{J}$ for all $i \in \Delta \backslash \mathcal{J}$. Clearly this is equivalent to $\iota(\mathcal{J}) = \mathcal{J}$. $\square$

In what follows, we will largely restrict to the case $\iota(\mathcal{J}) = \mathcal{J}$. Now recall that the non-ADE Dynkin diagrams B_n and F_4 are defined to be:



Given any ADE Dynkin digram Δ, we now fold the diagram under the action of the Dynkin involution to obtain the *folded graph* Δ_{f}, defined using the following table.

Δ	A_{2n-1}	A_{2n}	D_{2n}	D_{2n+1}	E_6	E_7	E_8
Δ_{f}	B_n	B_n	D_{2n}	B_{2n}	F_4	E_7	E_8

There is a natural map $\Delta \to \Delta_{\mathrm{f}}$ which induces a natural map

$$\{\mathcal{J} \subseteq \Delta \mid \iota(\mathcal{J}) = \mathcal{J}\} \to \{\mathcal{K} \subseteq \Delta_{\mathrm{f}}\},$$

which, to set notation, sends $\mathcal{J} \mapsto \mathcal{J}_{\mathrm{f}}$.

Example 6.3. If $\mathcal{J} =$ , then $\mathcal{J}_{\mathrm{f}} =$, viewed as a subset of $\Delta_{\mathrm{f}} = B_4$.

These foldings arise naturally via the subgroup $(W_\Delta)^\iota = \{w \in W_\Delta \mid \iota(w) = w\}$, where recall from 1.2 that $\iota(w) = w_\Delta w w_\Delta$. Indeed, it is easy to show that $(W_\Delta)^\iota = \langle t_i \mid i \in \Delta_{\mathrm{f}} \rangle$, where

$$t_i = \begin{cases} s_i & \text{if } i = \iota(i) \\ s_i s_{\iota(i)} s_i & \text{if there is an edge } i - \iota(i) \\ s_i s_{\iota(i)} & \text{if there is no edge } i - \iota(i), \end{cases}$$

and that $\phi\colon W_{\Delta_{\mathrm{f}}} \xrightarrow{\sim} (W_\Delta)^\iota$ as groups, via $\phi(\mathsf{s}_i) = t_i$ (see e.g. [AM, 3.1]). Furthermore, given any $\mathcal{J} \subseteq \Delta$ such that $\iota(\mathcal{J}) = \mathcal{J}$, the following diagram commutes

$$\begin{array}{ccc} W_{\Delta_{\mathrm{f}}} & \xrightarrow[\sim]{\phi} & (W_\Delta)^\iota \\ \cup & & \cup \\ W_{\mathcal{J}_{\mathrm{f}}} & \xrightarrow[\sim]{\phi} & (W_{\mathcal{J}})^\iota \end{array} \tag{6.1.A}$$

To apply this to intersection arrangements, consider the fixed subset

$$\mathsf{Cham}(\Delta, \mathcal{J})^\iota := \{(x, J) \in \mathsf{Cham}(\Delta, \mathcal{J}) \mid (x, J) = (\iota(w), \iota(J))\}.$$

6.2. Main Results, and Derived Classification

The aim of this section is to prove the following result. The first part generalises [M, AM], and the other parts [AM], who all considered the case $\mathcal{J} = \emptyset$.

Theorem 6.4. *Let Δ be ADE, and $\mathcal{J} \subseteq \Delta$ with $\iota(\mathcal{J}) = \mathcal{J}$. Then the following hold.*

(1) *There are bijections*

$$\begin{array}{ccc} \mathsf{Cham}(\Delta, \mathcal{J}) & \longleftrightarrow & 2\,\mathrm{silt}\,\Gamma_{\mathcal{J}} \\ \cup & & \cup \\ \mathsf{Cham}(\Delta, \mathcal{J})^\iota & \longleftrightarrow & 2\,\mathrm{tilt}\,\Gamma_{\mathcal{J}}. \end{array}$$

(2) *The endomorphism algebra of any irreducible left tilting mutation of $\Gamma_{\mathcal{J}}$ is isomorphic to Γ_J for some $J \subseteq \Delta$ such that there exists $(x, J) \in \mathsf{Cham}(\Delta, \mathcal{J})^\iota$. In particular, $\mathrm{K}^{\mathrm{b}}(\mathrm{proj}\,\Gamma_{\mathcal{J}})$ is tilting-discrete.*

(3) *The derived and Morita equivalence classes of $\Gamma_{\mathcal{J}}$ coincide. The basic algebras in this class are precisely $\{\Gamma_J \mid J \subseteq \Delta,\ \exists\,(x, J) \in \mathsf{Cham}(\Delta, \mathcal{J})^\iota\}$.*

Part (1) follows from 6.7 and 6.8 below. Parts (2) and (3) are 6.12, 6.13 and 6.14 respectively.

To approach these problems, our main main new insight is to leverage the fact that the natural restriction of scalars functor

$$F\colon \mathrm{D^b}(\operatorname{mod}\Gamma_{\mathcal{J}}) \to \mathrm{D^b}(\operatorname{mod}\Lambda_{\mathcal{J}})$$

is spherical, where $\Lambda_{\mathcal{J}}$ is the contracted preprojective algebra of extended Dynkin type (see below). Mapping the tilting theory for $\Lambda_{\mathcal{J}}$, established in 5.2, under the left adjoint of F will rather easily establish the properties for the silting and tilting modules for $\Gamma_{\mathcal{J}}$. Even in the case $\mathcal{J} = \emptyset$, this is a considerable simplification.

Since we will be passing to the extended Dynkin diagram, to avoid a proliferation of tildes, or affs, we now ease notation.

SETUP 6.5. For the remainder of this chapter, let Δ be ADE Dynkin, and consider a subset $\mathcal{J} \subset \Delta$, satisfying $\iota(\mathcal{J}) = \mathcal{J}$. The preprojective algebra associated to Δ will be written Π, and the preprojective algebra associated to Δ_{aff} will be written Λ. Viewing $\mathcal{J} \subset \Delta$, we will write

$$\Gamma_{\mathcal{J}} = e_{\mathcal{J}}\Pi e_{\mathcal{J}},$$

which is a finite dimensional algebra. Viewing $\mathcal{J}$ as a subset of Δ_{aff}, we will write

$$\Lambda_{\mathcal{J}} = e_{\mathcal{J}}\Lambda e_{\mathcal{J}},$$

which is 2-sCY. Denote the extended vertex by 0, so that $\Gamma_{\mathcal{J}} = \Lambda_{\mathcal{J}}/\langle e_0\rangle$.

6.2.1. Two-term silting and tilting. By 5.2, all tiling $\Lambda_{\mathcal{J}}$-modules are of the form $e_{\mathcal{J}}I_x e_J$ for $(x, J) \in \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J})$. Consider again the natural inclusion

$$\Psi\colon \mathsf{Cham}(\Delta, \mathcal{J}) \hookrightarrow \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J}).$$

We will be interested in those tilting $\Lambda_{\mathcal{J}}$-modules that arise from the image of Ψ, which we then push down to $\Gamma_{\mathcal{J}}$ as follows.

DEFINITION 6.6. Under Setup 6.5, for $(x, J) \in \operatorname{Im}\Psi$ define $S_{x,J}$ to be

$$S_{x,J} := \Gamma_{\mathcal{J}} \otimes^{\mathbf{L}}_{\Lambda_{\mathcal{J}}} e_{\mathcal{J}}I_x e_J.$$

Since $\operatorname{proj.dim}_{\Lambda_{\mathcal{J}}} e_{\mathcal{J}}I_x e_J \leq 1$, the $S_{x,J}$ are clearly two-term complexes of projective $\Gamma_{\mathcal{J}}$-modules. We will show in 6.7 below, using the crucial assumption $\iota(\mathcal{J}) = \mathcal{J}$, that the $S_{x,J}$ are precisely the two-term silting complexes for $\Gamma_{\mathcal{J}}$.

Since $\iota(\mathcal{J}) = \mathcal{J}$, the kernel of the natural homomorphism $\Lambda_{\mathcal{J}} \to \Gamma_{\mathcal{J}}$ is given by $e_{\mathcal{J}}I_{w_{\mathcal{J}}w_{\Delta}}e_{\mathcal{J}}$, by 1.20(3). Hence, it follows that

$$0 \to e_{\mathcal{J}}I_{w_{\mathcal{J}}w_{\Delta}}e_{\mathcal{J}} \to \Lambda_{\mathcal{J}} \to \Gamma_{\mathcal{J}} \to 0 \tag{6.2.A}$$

is a short exact sequence of $\Lambda_{\mathcal{J}}$-bimodules.

LEMMA 6.7. *Under Setup 6.5, the map*

$$\mathsf{Cham}(\Delta, \mathcal{J}) \to 2\operatorname{silt}\Gamma_{\mathcal{J}}$$

sending $(x, J) \mapsto S_{x,J}$ is a bijection.

PROOF. We first claim that $S_{x,J} \in 2\operatorname{silt}\Gamma_{\mathcal{J}}$ for $(x, J) \in \operatorname{Im}\Psi$. Since they are clearly two-term complexes of projectives, we just need to show that $\operatorname{Hom}_{\Gamma_{\mathcal{J}}}(S_{x,J}, S_{x,J}[1]) = 0$.

To ease notation write $I = e_{\mathcal{J}}I_{w_{\mathcal{J}}w_{\Delta}}e_{\mathcal{J}}$ from (6.2.A), and $a = e_{\mathcal{J}}I_x e_J$. Applying $-\otimes^{\mathbf{L}}_{\Lambda_{\mathcal{J}}} a$ to (6.2.A) gives a triangle

$$I \otimes^{\mathbf{L}}_{\Lambda_{\mathcal{J}}} a \to a \to \Gamma_{\mathcal{J}} \otimes^{\mathbf{L}}_{\Lambda_{\mathcal{J}}} a$$

of (left) $\Lambda_{\mathcal{J}}$-modules. Applying $\operatorname{Hom}_{\Lambda_{\mathcal{J}}}(a, -)$ then gives an exact sequence

$$0 \to \operatorname{Hom}_{\Lambda_{\mathcal{J}}}(a, \Gamma_{\mathcal{J}} \otimes^{\mathbf{L}}_{\Lambda_{\mathcal{J}}} a[1]) \to \operatorname{Hom}_{\Lambda_{\mathcal{J}}}(a, I \otimes^{\mathbf{L}}_{\Lambda_{\mathcal{J}}} a[2]) \to 0,$$

where we have used $\mathrm{Ext}^i_{\Lambda_{\mathcal{J}}}(a,a)=0$ for $i>0$ since $a\in\operatorname{tilt}\Lambda_{\mathcal{J}}$. Hence

$$\begin{aligned} \mathrm{Hom}_{\Gamma_{\mathcal{J}}}(S_{x,J},S_{x,J}[1]) &= \mathrm{Hom}_{\Gamma_{\mathcal{J}}}(\Gamma_{\mathcal{J}}\otimes^{\mathbf{L}}_{\Lambda_{\mathcal{J}}} a, \Gamma_{\mathcal{J}}\otimes^{\mathbf{L}}_{\Lambda_{\mathcal{J}}} a[1]) \\ \text{(ext/res of scalars)} \qquad &\cong \mathrm{Hom}_{\Lambda_{\mathcal{J}}}(a, \Gamma_{\mathcal{J}}\otimes^{\mathbf{L}}_{\Lambda_{\mathcal{J}}} a[1]) \\ &\cong \mathrm{Hom}_{\Lambda_{\mathcal{J}}}(a, I\otimes^{\mathbf{L}}_{\Lambda_{\mathcal{J}}} a[2]). \end{aligned}$$

We claim that this last group is zero. Indeed, we can compute this group by replacing a by its projective resolution (which is a complex in degrees -1 and 0), and computing the Hom space in the homotopy category. However, $I\otimes^{\mathbf{L}}_{\Lambda_{\mathcal{J}}} a[2]$ is given by applying $I\otimes_{\Lambda_{\mathcal{J}}}-$ to the projective resolution of a and shifting, hence is a complex in degrees -3 and -2. Thus there can be no morphisms in the homotopy category, and so $\mathrm{Hom}_{\Gamma_{\mathcal{J}}}(S_{x,J},S_{x,J}[1])\cong 0$. It follows that $S_{x,J}\in 2\operatorname{silt}\Gamma_{\mathcal{J}}$.

To prove the bijection, we use g-vectors. By 5.3(2), for $(x,J)\in\operatorname{Im}\Psi$, each $e_{\mathcal{J}}I_xe_J$ has P_0 as a summand, and the g-vectors of the other summands describe the intersection arrangement $\mathsf{Cone}(\Delta,\mathcal{J})$. Since tensoring $\Gamma_{\mathcal{J}}\otimes_{\Lambda_{\mathcal{J}}}-$ sends $P_0\mapsto 0$, but maps the other projective $\Lambda_{\mathcal{J}}$-modules to projective $\Gamma_{\mathcal{J}}$-modules, it is clear that the g-vectors of the $S_{x,J}$ still describe the (finite) intersection arrangement $\mathsf{Cone}(\Delta,\mathcal{J})$. In particular, the open chambers of the associated $S_{x,J}$ are the chambers of the finite hyperplane arrangement, so we deduce the following.

(1) The open chambers for the varying $S_{x,J}$ do not overlap. Hence the $S_{x,J}$ are mutually non-isomorphic, and so $(x,J)\mapsto S_{x,J}$ is injective.
(2) The closure of the union of the all the open chambers is the whole vector space. As is now standard [**DIJ**], this implies that $S_{x,J}$ are all elements of $2\operatorname{silt}\Gamma_{\mathcal{J}}$, and thus $(x,J)\mapsto S_{x,J}$ is surjective. □

It is also possible to describe explicitly which of the $S_{x,J}$ are tilting. Recall that $\mathcal{N}$ is the Nakayama functor on $\Gamma_{\mathcal{J}}$.

Proposition 6.8. *Under Setup 6.5, for $(x,J)\in\operatorname{Im}\Psi$, the following are equivalent.*
(1) $S_{x,J}\in 2\operatorname{tilt}\Gamma_{\mathcal{J}}$.
(2) $\mathcal{N}(S_{x,J})\cong S_{x,J}$.
(3) $(x,J)\in\mathsf{Cham}(\Delta,\mathcal{J})^{\iota}\cong\mathsf{Cham}(\Delta_{\mathrm{f}},\mathcal{J}_{\mathrm{f}})$.
(4) $\iota(x)=x$.

Proof. (1)⇔(2) It is a general fact that a basic silting object is tilting iff it is fixed by the Nakayama functor, see e.g. [**A**, A.4].
(2)⇔(3). As already observed in 6.2, $\mathcal{N}(e_{\mathcal{J}}\Pi e_i)\cong e_{\mathcal{J}}\Pi e_{\iota(i)}$, and so $\mathcal{N}$ acts on $K_0(\operatorname{proj}\Gamma_{\mathcal{J}})$ via ι. Since two-term silting complexes are determined by their g-vectors,

$$\mathcal{N}(S_{x,J})\cong S_{x,J}\Leftrightarrow\mathcal{N}(C(S_{x,J}))\cong C(S_{x,J})\Leftrightarrow\iota(x(C_J))=x(C_J)\Leftrightarrow\iota(x,J)=(x,J).$$

This holds iff $(x,J)\in\mathsf{Cham}(\Delta,\mathcal{J})^{\iota}$.
(3)⇒(4) is clear, and (4)⇒(3) holds since both (x,J) and $(x,\iota(J))=(\iota(x),\iota(J))$ belong to $\mathsf{Cham}(\Delta,\mathcal{J})$, and by 1.14(1) any chamber (y,K) is determined by y. Hence $J=\iota(J)$, and so $(x,J)\in\mathsf{Cham}(\Delta,\mathcal{J})^{\iota}$. □

6.2.2. Endomorphism rings of two-term tilting complexes. Leading up to a proof of 6.4(2), we need to control the endomorphism rings of the two-term tilting complexes established in 6.8. This requires the following two results. The first is a general fact, since given any ring homomorphism $\rho\colon A\to B$ we can consider the following four functors $\operatorname{Mod}A\to\operatorname{Mod}A$, and natural transformations between them

$$\begin{array}{ccc} 1_A & \xrightarrow{\ \eta\ } & \mathrm{Hom}_B(B,B\otimes_A-) \\ \sim\downarrow & & \downarrow\sim \\ A\otimes_A- & \xrightarrow{\ \rho\otimes-\ } & B\otimes_A,- \end{array} \tag{6.2.A}$$

All morphisms are given by the obvious maps, and by inspection of these maps the diagram commutes. Furthermore, the top map η is the unit of the restriction and extension of scalars adjunction. The leftmost functors are exact, whilst the rightmost functors are right exact, so we can form their left derived functors. The following is standard.

LEMMA 6.9. *Given a ring homomorphism $A \to B$, consider the derived restriction and extension of scalars adjunction, with unit η. Then the following diagram commutes*

$$\begin{array}{ccc} 1_A & \xrightarrow{\eta} & \mathrm{Hom}_B(B, B \otimes^{\mathbf{L}}_A -) \\ \sim\big\downarrow & & \big\downarrow\sim \\ A \otimes_A - & \xrightarrow{\rho\otimes -} & B\otimes^{\mathbf{L}}_{A}, - \end{array}$$

PROOF. The commutativity follows formally from the commutativity of (6.2.A), together with standard properties of derived functors. □

Now, under Setup 6.5 consider a two-term tilting complex $S_{x,J}$. By 6.8, necessarily $(x, J) \in \mathsf{Cham}(\Delta, \mathcal{J})^\iota$, and so $\iota(J) = J$. Thus, on one hand we have $I := e_{\mathcal{J}} I_{w_{\mathcal{J}} w_\Delta} e_{\mathcal{J}}$ and the short exact sequence of $\Lambda_{\mathcal{J}}$-bimodules (6.2.A), which we now write as

$$0 \to I \xrightarrow{\mathrm{i}} \Lambda_{\mathcal{J}} \xrightarrow{\rho} \Gamma_{\mathcal{J}} \to 0 \tag{6.2.B}$$

On the other hand, we have $I' := e_J I_{w_J w_\Delta} e_J$ and a short exact sequence of Λ_J-bimodules

$$0 \to I' \xrightarrow{\mathrm{i}} \Lambda_J \to \Gamma_J \to 0. \tag{6.2.C}$$

In both cases, i denotes the inclusion map. In the following, to again ease notation, for any $\mathcal{K} \subseteq \Delta$, set $\otimes_{\mathcal{K}} := \otimes_{\Lambda_{\mathcal{K}}}$.

PROPOSITION 6.10. *Under Setup 6.5, for $(x, J) \in \mathsf{Cham}(\Delta, \mathcal{J})^\iota$ consider $a = e_{\mathcal{J}} I_x e_J$. Then there is an isomorphism of $\Lambda_{\mathcal{J}}$-Λ_J-bimodules $I \otimes_{\mathcal{J}} a \xrightarrow{\sim} a \otimes_J I'$ such that*

$$\begin{array}{ccc} I \otimes_{\mathcal{J}} a & \xrightarrow{\mathrm{i}\otimes 1} & \Lambda_{\mathcal{J}} \otimes_{\mathcal{J}} a \\ \sim\big\downarrow & & \big\downarrow\sim\ \mathrm{sw} \\ a \otimes_J I' & \xrightarrow{1\otimes \mathrm{i}} & a \otimes_J \Lambda_J \end{array}$$

commutes, where the right hand map sw *sends $1 \otimes a \mapsto a \otimes 1$.*

PROOF. This is just a repeated use of 5.30. Set $y := x^{-1} w_{\mathcal{J}} w_\Delta = w_J w_\Delta x^{-1}$, and consider $b := e_J I_y e_{\mathcal{J}} \in \operatorname{tilt} \Gamma_J$. Since $(x, J) \in \mathsf{Cham}(\Delta, \mathcal{J})$, and the longest positive minimal path $(1, \mathcal{J}) \to (w_{\mathcal{J}} w_\Delta, \mathcal{J})$ factors into positive minimal paths

$$(1, \mathcal{J}) \to (x, J) \to (w_{\mathcal{J}} w_\Delta, \mathcal{J}),$$

it follows from the proof of 5.30 that there is a bimodule isomorphism

$$a \otimes_J b \xrightarrow{\sim} I$$

which sends $f \otimes g \mapsto fg$. Write μ for this multiplication map.

Similarly, since $(y, \mathcal{J}) = (w_J w_\Delta x^{-1}, \mathcal{J}) \in \mathsf{Cham}(\Delta, J)$, and the longest positive minimal path $(1, J) \to (y, w_J w_\Delta, J)$ factors into positive minimal paths

$$(1, J) \to (y, \mathcal{J}) \to (w_J w_\Delta, J),$$

it follows from the proof of 5.30 that there is a bimodule isomorphism

$$b \otimes_{\mathcal{J}} a \xrightarrow{\sim} I'$$

which sends $f \otimes g \mapsto fg$. Again, write μ for this multiplication map.

Consider also the multiplication maps $\upmu\colon b\otimes_{\mathcal{J}} a\to\Lambda_J$ and $\upmu\colon a\otimes_J b\to\Lambda_{\mathcal{J}}$, then it is clear by inspection that the top and the bottom squares in the following diagram commute:

$$\begin{array}{ccc}
I\otimes_{\mathcal{J}} a & \xrightarrow{\;\mathrm{i}\otimes 1\;} & \Lambda_{\mathcal{J}}\otimes_{\mathcal{J}} a\\
{\scriptstyle \upmu\otimes 1}\uparrow\wr & & \|\\
a\otimes_J b\otimes_{\mathcal{J}} a & \xrightarrow{\;\upmu\otimes 1\;} & \Lambda_{\mathcal{J}}\otimes_{\mathcal{J}} a\\
\| & & \wr\downarrow{\scriptstyle \mathrm{sw}}\\
a\otimes_J b\otimes_{\mathcal{J}} a & \xrightarrow{\;1\otimes\upmu\;} & a\otimes_J \Lambda_J\\
{\scriptstyle 1\otimes\upmu}\downarrow\wr & & \|\\
a\otimes_J I' & \xrightarrow{\;1\otimes\mathrm{i}\;} & a\otimes_J\Lambda_J
\end{array}$$

The middle square also commutes, by inspection. Composing horizontal maps from top to bottom gives the required statement. □

THEOREM 6.11. *Under Setup 6.5, if $(x,J)\in\mathsf{Cham}(\Delta,\mathcal{J})^\iota$ then there is an isomorphism of rings* $\mathrm{End}_{\Gamma_{\mathcal{J}}}(S_{x,J})\cong\Gamma_J$.

PROOF. As in 6.10, set $a=e_{\mathcal{J}}I_xe_J$. We first claim that $\mathbf{R}\mathrm{Hom}_{\Gamma_{\mathcal{J}}}(a,I\otimes_{\mathcal{J}}a)\cong I'$. This follows since

(by 6.10)	$\mathbf{R}\mathrm{Hom}_{\Gamma_{\mathcal{J}}}(a,I\otimes_{\mathcal{J}}a)\cong\mathbf{R}\mathrm{Hom}_{\Gamma_{\mathcal{J}}}(a,a\otimes_J I')$
(see e.g. [IR, 2.10(2)])	$\cong\mathbf{R}\mathrm{Hom}_{\Lambda_{\mathcal{J}}}(a,a)\otimes_J I'$
(since $a\in\operatorname{tilt}\Lambda_{\mathcal{J}}$)	$\cong I'$

In particular, $\mathrm{Hom}_{\Gamma_{\mathcal{J}}}(a,I\otimes_{\mathcal{J}}a[1])=0$. Further, since $S_{x,J}$ is tilting by 6.8, by extension and restriction of scalars we have

$$\mathrm{Hom}_{\Gamma_{\mathcal{J}}}(a,\Gamma_{\mathcal{J}}\otimes^{\mathbf{L}}_{\mathcal{J}}a[-1])\cong\mathrm{Hom}_{\Gamma_{\mathcal{J}}}(S_{x,J},S_{x,J}[-1])=0.$$

Using these facts, first applying $-\otimes^{\mathbf{L}}_{\mathcal{J}}a$ to (6.2.B), then applying $\mathrm{Hom}_{\Gamma_{\mathcal{J}}}(a,-)$ gives a short exact sequence

$$0\to\mathrm{Hom}_{\Gamma_{\mathcal{J}}}(a,I\otimes_{\mathcal{J}}a)\to\mathrm{Hom}_{\Gamma_{\mathcal{J}}}(a,\Lambda_{\mathcal{J}}\otimes_{\mathcal{J}}a)\to\mathrm{Hom}_{\Gamma_{\mathcal{J}}}(a,\Gamma_{\mathcal{J}}\otimes^{\mathbf{L}}_{\mathcal{J}}a)\to 0.$$

Now consider the ring homomorphism $\uprho$ in (6.2.B). Set $F=\mathrm{Hom}_{\Gamma_{\mathcal{J}}}(\Gamma_{\mathcal{J}},-)$ to be restriction of scalars, with left adjoint $F^{\mathrm{LA}}=\Gamma_{\mathcal{J}}\otimes^{\mathbf{L}}_{\Lambda_{\mathcal{J}}}-$. Dropping $\mathrm{Hom}_{\Gamma_{\mathcal{J}}}$ from the notation, we claim that the following diagram commutes. Indeed, the top square follows from standard properties of adjunctions, and the second square follows from 6.9. The third square is 6.10, and the bottom square is clear; again $\mathbf{R}\mathrm{Hom}_{\Lambda_{\mathcal{J}}}(a,a\otimes d)\cong\mathbf{R}\mathrm{Hom}_{\Lambda_{\mathcal{J}}}(a,a)\otimes d\cong d$ since a is tilting.

$$
\begin{array}{ccccccccc}
 & & & & (a,a) & \xrightarrow{F^{\mathrm{LA}}} & (F^{\mathrm{LA}}a, F^{\mathrm{LA}}a) & & \\
 & & & & \Vert & & \uparrow{\scriptstyle \mathrm{adj}} & & \\
 & & & & (a,a) & \xrightarrow{\eta\circ} & (a, FF^{\mathrm{LA}}a) & & \\
 & & & & {\scriptstyle\sim}\uparrow & & \uparrow{\scriptstyle\sim} & & \\
0 & \longrightarrow & (a, I\otimes_{\mathcal{J}} a) & \xrightarrow{(\mathrm{i}\otimes 1)\circ} & (a, \Lambda_{\mathcal{J}}\otimes_{\mathcal{J}} a) & \xrightarrow{(\rho\otimes 1)\circ} & (a, \Gamma_{\mathcal{J}}\otimes^{\mathbf{L}}_{\mathcal{J}} a) & \longrightarrow & 0 \\
 & & {\scriptstyle\sim}\uparrow & & {\scriptstyle\sim}\uparrow{\scriptstyle \mathrm{sw}\circ} & & & & \\
 & & (a, a\otimes_J I') & \xrightarrow{(1\otimes \mathrm{i})\circ} & (a, a\otimes_J \Lambda_J) & & & & \\
 & & {\scriptstyle\sim}\uparrow & & {\scriptstyle\sim}\uparrow & & & & \\
 & & I' & \xrightarrow{\qquad \mathrm{i} \qquad} & \Lambda_J & & & &
\end{array}
$$

Since unadorned maps are the obvious ones, the middle vertical maps compose to give a map $\Lambda_J \to (a,a)$ sending

$$\lambda \mapsto (a \mapsto a\otimes\lambda = a\lambda\otimes 1) \mapsto (a\mapsto 1\otimes a\lambda) \mapsto (a\mapsto a\lambda),$$

which is clearly a ring homomorphism. Hence, composing with the top right ring homomorphism given by F^{LA}, we obtain a surjective ring homomorphism $\Lambda_J \to (F^{\mathrm{LA}}a, F^{\mathrm{LA}}a)$, with kernel I'. It follows that $\mathrm{End}_{\Gamma_{\mathcal{J}}}(F^{\mathrm{LA}}a) \cong \Lambda_J/I' \cong \Gamma_J$, using (6.2.C). $\square$

6.2.3. Tilting discrete and derived classification. In this subsection we finally prove 6.4(2) and 6.4(3). In particular, we show that under Setup 6.5, the number of basic algebras in the derived equivalence class of $\Gamma_{\mathcal{J}}$ is finite, and has a very precise description.

LEMMA 6.12. *Under Setup 6.5, the endomorphism algebra of any irreducible left tilting mutation of* $\Gamma_{\mathcal{J}}$ *is isomorphic to* Γ_J *for some* $J\subseteq\Delta$ *such that* $\exists\,(x,J)\in\mathsf{Cham}(\Delta,\mathcal{J})^{\iota}$.

PROOF. This is a simple induction on the length of $T = \mu_{(t)}\dots\mu_{(1)}\Gamma_{\mathcal{J}}$, where each $\mu_{(i)}$ is an irreducible tilting mutation, with the case $t=1$ being 6.11. Hence we can assume that $T' = \mu_{(t-1)}\dots\mu_{(1)}\Gamma_{\mathcal{J}}$ satisfies $\mathrm{End}_{\Gamma_{\mathcal{J}}}(T')\cong\Gamma_K$ for some $K\subseteq\Delta$ such that $\exists\,(z,K)\in\mathsf{Cham}(\Delta,\mathcal{J})^{\iota}$. But since T' is tilting, there exists an equivalence

$$\mathrm{K}^{\mathrm{b}}(\mathrm{proj}\,\Gamma_{\mathcal{J}}) \xrightarrow{\sim} \mathrm{K}^{\mathrm{b}}(\mathrm{proj}\,\Gamma_K)$$

sending $T'\mapsto\Gamma_K$. Since equivalences preserve mutation, necessarily

$$\mathrm{End}_{\Gamma_{\mathcal{J}}}(T) = \mathrm{End}_{\Gamma_{\mathcal{J}}}(\mu_{(t)}T') \cong \mathrm{End}_{\Gamma_K}(\mu_{(t)}\Gamma_K).$$

Applying 6.11 to Γ_K, it follows that $\mathrm{End}_{\Gamma_K}(\mu_{(t)}\Gamma_K)\cong\Gamma_J$ for some J such that $\exists\,(y,J)\in\mathsf{Cham}(\Delta,K)^{\iota}$. Now $W_{\mathcal{J}}zy = zW_Ky = zyW_J$, and the element $zy(C_J)\in\mathsf{Cone}(\Delta,\mathcal{J})$ is fixed by ι. Hence, under the bijection in 1.12, the corresponding $(x,J)\in\mathsf{Cham}(\Delta,\mathcal{J})$ is fixed by ι, and the statement follows. $\square$

PROPOSITION 6.13. *Under Setup 6.5,* $\mathrm{K}^{\mathrm{b}}(\mathrm{proj}\,\Gamma_{\mathcal{J}})$ *is tilting-discrete.*

PROOF. With 6.12 in hand, this is quite elementary. We just need to check [AM, 2.11], namely $\{U\in\mathrm{tilt}^{\cdot}\,\Gamma_{\mathcal{J}} \mid T\geq U\geq T[1]\}$ is a finite set for all T obtained from $\Gamma_{\mathcal{J}}$ by iterated irreducible tilting mutation. Choose such a T, then by 6.12 $\mathrm{End}_{\Gamma_{\mathcal{J}}}(T)\cong\Gamma_J$ say. Since T is tilting, there exists an equivalence

$$\mathrm{K}^{\mathrm{b}}(\mathrm{proj}\,\Gamma_{\mathcal{J}}) \xrightarrow{\sim} \mathrm{K}^{\mathrm{b}}(\mathrm{proj}\,\Gamma_J)$$

sending $T\mapsto\Gamma_J$. Thus

$$\{U\in\mathrm{tilt}^{\cdot}\,\Gamma_{\mathcal{J}} \mid T\geq U\geq T[1]\} = \{V\in\mathrm{tilt}^{\cdot}\,\Gamma_J \mid \Gamma_J\geq V\geq\Gamma_J[1]\}.$$

But this is precisely the set of two-term tilting complexes for Γ_J, which is finite by 6.8. □

The following is the main result of this chapter.

COROLLARY 6.14. *Under Setup 6.5, there are only finitely many basic algebras derived equivalent to* $\Gamma_{\mathcal{J}}$*, and these are precisely* $\{\Gamma_J \mid J \subseteq \Delta, \exists\,(x, J) \in \mathsf{Cham}(\Delta, \mathcal{J})^{\iota}\}$.

PROOF. Let $T \in \operatorname{tilt}^{\cdot}\Gamma_{\mathcal{J}}$. By [**AI**, 2.4] there exists $\ell \geq 0$ such that $\Gamma_{\mathcal{J}} \geq T[\ell]$. Thus, since $\Gamma_{\mathcal{J}}$ is tilting discrete by 6.13, $T[\ell]$ can be obtained from $\Gamma_{\mathcal{J}}$ by iterated irreducible left mutation [**A**, 3.5]. Now $\operatorname{End}_{\Gamma_{\mathcal{J}}}(T) \cong \operatorname{End}_{\Gamma_{\mathcal{J}}}(T[\ell])$, and by 6.12 this is isomorphic to Γ_J for some $J \subseteq \Delta$ such that $\exists\,(x, J) \in \mathsf{Cham}(\Delta, \mathcal{J})^{\iota}$. That all such Γ_J arise in the derived equivalence class follows from 6.8 and 6.11. □

CHAPTER 7

Derived Classification: Extended Dynkin Type

Given an extended ADE Dynkin diagram Δ_{aff}, write Π for the associated preprojective algebra. For any subset $\mathcal{J} \subseteq \Delta_{\mathsf{aff}}$, consider the corresponding contracted preprojective algebra $\Gamma_{\mathcal{J}} = e_{\mathcal{J}}\Pi e_{\mathcal{J}}$, where recall that $e_{\mathcal{J}} := 1 - \sum_{i \in \mathcal{J}} e_i$. For our geometric applications, we will be most interested in the case when $\mathscr{J} \subseteq \Delta$, which we then view as a subset of Δ_{aff} and form $\Gamma_{\mathscr{J}} = e_{\mathscr{J}}\Pi e_{\mathscr{J}}$.

The following asserts that the derived equivalence classification of contracted preprojective algebras is entirely combinatorial, and also that the derived equivalence class does not contain anything unexpected.

Conjecture 7.1. *Suppose that $\mathcal{J} \subseteq \Delta_{\mathsf{aff}}$ where Δ_{aff} is extended ADE Dynkin, and let A be a basic ring. Then A is derived equivalent to $\Gamma_{\mathcal{J}}$ if and only if there exists $\mathcal{J}' \subseteq \Delta_{\mathsf{aff}}$ such that $A \cong \Gamma_{\mathcal{J}'}$, and furthermore $\mathcal{J}$ and $\mathcal{J}'$ are iterated combinatorial mutation of each other, up to symmetries of Δ_{aff}.*

The direction ($\Leftarrow$) is clear, since wall crossing gives derived equivalences (§5.6), as do isomorphisms. The content in the conjecture is the ($\Rightarrow$) direction, where amongst other things we need to produce invariants that distinguish between different mutation classes.

In this chapter we prove the conjecture in all cases, except when $\Delta = D_n$ with $n \geq 8$, due to its combinatorial complexity. Our main result is 7.21, which also describes which invariants are needed in order to distinguish the derived equivalence classes; this varies, according to Dynkin type. Geometric applications are given later, in 10.8.

The above shows that the endomorphism rings of all tilting complexes for $\Gamma_{\mathcal{J}}$ are well behaved. For applications to stability manifolds and autoequivalences, it is in fact more important to show that the tilting complexes themselves are controlled, and behave well. We then make partial progress towards this in §7.4, where in 7.24 we show that the 2-term tilting complexes for $\Gamma_{\mathcal{J}}$ are also controlled by Coxeter-style data, in the form of the infinite hyperplane arrangement $\mathcal{W}_{\mathcal{J}}$ from 2.8.

7.1. Derived Invariants

As in Chapter 3, in what follows we depict vertices in $\mathcal{J}$ by $\bullet$. It will be convenient to colour the other vertices red, which are precisely the vertices in $\Delta_{\mathsf{aff}} \backslash \mathcal{J}$. The 'up to symmetries' part of 7.1 is important: the two choices of $\mathcal{J}$ given by

 and 

are not in the same mutation class, but one can be obtained from the other via a symmetry. The corresponding contracted preprojective algebras are isomorphic, so in particular they are derived equivalent.

Remark 7.2. As calibration, the case where all vertices are red corresponds to $\Gamma_{\mathcal{J}} = \Pi$, and thus the minimal resolution. The case where only the extended vertex is red corresponds to $\Gamma_{\mathcal{J}} = \mathfrak{R}/g$, and thus the singularity $\mathbb{C}^2/G$.

Definition 7.3. Define $\sim$ by $\mathcal{J}_1 \sim \mathcal{J}_2$ if and only if $\mathcal{J}_1$ can be obtained from $\mathcal{J}_2$ by a finite sequence of mutation moves and symmetries of the graph.

(1) The resulting equivalence classes are called the *symmetric mutation classes*.

(2) A symmetric mutation class is called *geometric* if there is some element of the class in which the extended node is red.

We approach 7.1 by first associating a triple of invariants to each contracted preprojective algebra $\Gamma_{\mathcal{J}}$. This will consist of the *type* of the extended Dynkin diagram to which $\mathcal{J}$ is a subset, the *cotype* defined below, and the Grothendieck group $G_0(\Gamma_{\mathcal{J}}) := K_0(\mathrm{D^b}(\operatorname{mod}\Gamma_{\mathcal{J}}))$. We will see below that all three of these invariants are preserved under derived equivalence, and furthermore the triple distinguishes the derived equivalence classes when $\Delta \in \{A_n, D_4, D_5, D_6, E_6, E_8\}$. In other cases, we will need slightly finer invariants.

Definition 7.4. We say that the partial preprojective algebra $\Gamma_{\mathcal{J}} = e_{\mathcal{J}}\Pi e_{\mathcal{J}}$ has *cotype* $\Delta^1 \ldots \Delta^m$ if the full subgraph given by the vertices in $\mathcal{J}$ is a disjoint union of $\Delta^1, \ldots, \Delta^m$.

Example 7.5. $\mathcal{J} =$  has cotype A_1A_5.

It is clear that the type and cotype are constant over all members of a given symmetric mutation class. The Grothendieck group $G_0(\Gamma_{\mathcal{J}})$ is also an invariant of the symmetric mutation class, since it is invariant under derived equivalences. Our next result will show that in fact all three invariants can be extracted from just the derived equivalence class. This requires the following preparation, where $\mathrm{D_{sg}}(\Gamma_{\mathcal{J}}) = \mathrm{D^b}(\operatorname{mod}\Gamma_{\mathcal{J}})/\mathrm{K^b}(\operatorname{proj}\Gamma_{\mathcal{J}})$.

Proposition 7.6. *For a subset $\mathcal{J} \subseteq \Delta_{\mathsf{aff}}$, the following conditions hold.*

(1) *The centre $Z(\Gamma_{\mathcal{J}})$ is isomorphic to a Kleinian singularity, and the type of $\mathcal{J}$ equals the type of this Kleinian singularity.*
(2) *The AR quiver of $\mathrm{D_{sg}}(\Gamma_{\mathcal{J}})$ is the double quiver of the cotype graph.*

In particular, both the type and cotype can be obtained from the derived equivalence class.

Proof. (1) It is clear that $Z(\Gamma_{\mathcal{J}}) = Z(\Pi)$, and hence by 5.24 is isomorphic to a Kleinian singularity of the given type.
(2) By 5.24, we can write $\Pi \cong \operatorname{End}_R(M)$ with $M = \bigoplus_{i\in\Delta_{\mathsf{aff}}} M_i$. Since $e_{\mathcal{J}} := 1 - \sum_{i\in\mathcal{J}} e_i$, setting $M_{\mathcal{J}} = \bigoplus_{i\in\Delta_{\mathsf{aff}}\setminus\mathcal{J}} M_i$, it follows that

$$\Gamma_{\mathcal{J}} = e_{\mathcal{J}}\Pi e_{\mathcal{J}} \cong \operatorname{End}_R(M_{\mathcal{J}}).$$

Since $\dim R = 2$, reflexive equivalence is $\operatorname{Hom}_R(M_{\mathcal{J}}, -)\colon \operatorname{CM} R \xrightarrow{\sim} \operatorname{CM}\Gamma_{\mathcal{J}}$, where by $\operatorname{CM}\Gamma_{\mathcal{J}}$ we mean those $\Gamma_{\mathcal{J}}$-modules that are maximal Cohen–Macaulay as an R-module. Clearly, this equivalence sends $\operatorname{add} M_{\mathcal{J}}$ to $\operatorname{proj}\Gamma_{\mathcal{J}}$, and so induces an equivalence

$$(\operatorname{CM} R)/[M_{\mathcal{J}}] \simeq \underline{\operatorname{CM}}\,\Gamma_{\mathcal{J}}$$

where the Hom-spaces on the left hand side are modulo those morphisms that factor through $\operatorname{add} M_{\mathcal{J}}$, and the Hom-spaces on the right hand side are modulo those morphisms that factor through $\operatorname{proj}\Gamma_{\mathcal{J}}$. Furthermore, since $\Gamma_{\mathcal{J}} \cong \operatorname{End}_R(M_{\mathcal{J}})$ is 2-sCY by [IW, 2.22], it follows that CM modules in the sense above are precisely the Gorenstein projective modules. Consequently, $(\operatorname{CM} R)/[M_{\mathcal{J}}] \simeq \underline{\operatorname{CM}}\,\Gamma_{\mathcal{J}} \simeq \mathrm{D_{sg}}(\Gamma_{\mathcal{J}})$, and so the result follows. □

Corollary 7.7. *The type, cotype and Grothendieck groups are derived invariants.*

Proof. The type is a derived invariant by 7.6(1), since derived equivalences preserve the centre [R89, 9.2]. Cotype is a derived invariant by 7.6(2), since derived equivalences induce equivalences of singularity categories. The fact that the Grothendieck group is a derived invariant is clear. □

The following will be used to calculate the Grothendieck group $G_0(\Gamma_{\mathcal{J}})$, which since $\Gamma_{\mathcal{J}}$ has infinite projective dimension, is a priori difficult. The point is that Π has finite global dimension, and all its simples have prescribed projective resolutions.

Proposition 7.8. *For $\mathcal{J} \subseteq \Delta_{\mathsf{aff}}$, the Grothendieck group $G_0(\Gamma_{\mathcal{J}})$ is isomorphic to the free abelian group with basis $\{\mathcal{P}_i \mid i \in \Delta_{\mathsf{aff}}\}$, modulo the subgroup generated by the projective resolution of the simple Π-modules $\mathcal{S}_i$ with $i \in \mathcal{J}$.*

PROOF. For a partial preprojective algebra $\Gamma_{\mathcal{J}} = e_{\mathcal{J}}\Pi e_{\mathcal{J}}$, the standard idempotent recollement induces equivalences

$$\begin{aligned}\mathrm{D^b}(\mathrm{mod}\,\Gamma_{\mathcal{J}}) &\cong \mathrm{D^b}(\mathrm{mod}\,\Pi)/\mathrm{D^b_{mod(\Pi/\langle 1-e_{\mathcal{J}}\rangle)}}(\mathrm{mod}\,\Pi)\\ &\cong \mathrm{K^b}(\mathrm{proj}\,\Pi)/\mathrm{D^b_{mod(\Pi/\langle 1-e_{\mathcal{J}}\rangle)}}(\mathrm{mod}\,\Pi).\end{aligned}$$

Since $1 - e_{\mathcal{J}} = \sum_{j\in\mathcal{J}} e_j$, it follows that the Grothendieck group $G_0(\Gamma_{\mathcal{J}}) = K_0(\mathrm{D^b}(\mathrm{mod}\,\Gamma_{\mathcal{J}}))$ is the quotient of the free abelian group with basis $\{\mathcal{P}_i \mid i \in \Delta_{\mathsf{aff}}\}$ modulo the subgroup generated by the projective resolution of the simple Π-modules $\mathcal{S}_i$ with $i \in \mathcal{J}$. $\square$

The following illustrates how to use 7.8, and also demonstrates that the type and cotype alone are not enough to distinguish symmetric mutation classes.

COROLLARY 7.9. *Consider* $\mathcal{J}_1 =$  *and* $\mathcal{J}_2 =$ . *Both have type extended* D_5, *and cotype* A_3. *However,* $\Gamma_{\mathcal{J}_1}$ *and* $\Gamma_{\mathcal{J}_2}$ *are not derived equivalent, since* $G_0(\Gamma_{\mathcal{J}_1}) \ncong G_0(\Gamma_{\mathcal{J}_1})$.

PROOF. Label the vertices $\begin{smallmatrix}2&&&5\\&3&4&\\1&&&6\end{smallmatrix}$ so that $\mathcal{J}_1 = \{4, 5, 6\}$ and $\mathcal{J}_2 = \{3, 4, 5\}$. By 7.8,

$$G_0(\Gamma_{\mathcal{J}_1}) \cong \mathbb{Z}^6/\langle S_4, S_5, S_6\rangle = \mathbb{Z}^6/\left(\begin{smallmatrix}0&0&-1&2&-1&-1\\0&0&0&-1&2&0\\0&0&0&-1&0&2\end{smallmatrix}\right) \cong \mathbb{Z}^3 \oplus (\mathbb{Z}/2\mathbb{Z}).$$

On the other hand, again by 7.8,

$$G_0(\Gamma_{\mathcal{J}_2}) \cong \mathbb{Z}^6/\langle S_3, S_4, S_5\rangle = \mathbb{Z}^6/\left(\begin{smallmatrix}-1&-1&2&-1&0&0\\0&0&-1&2&-1&-1\\0&0&0&-1&2&0\end{smallmatrix}\right) \cong \mathbb{Z}^3.$$

Since the Grothendieck groups differ, $\Gamma_{\mathcal{J}_1}$ and $\Gamma_{\mathcal{J}_2}$ are not derived equivalent. $\square$

It turns out that the triple (type, cotype, G_0) also does not distinguish symmetric mutation classes, as 7.11 below demonstrates. In this case, we will use the following more refined invariants. Any derived equivalence $\mathrm{D^b}(\mathrm{mod}\,\Gamma_{\mathcal{J}}) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{mod}\,\Gamma_{\mathcal{J}})$ preserves the subcategory of perfect complexes, and also the subcategory of compactly supported objects, so restricts to give the following commutative diagrams

$$\begin{array}{ccc}\mathrm{D^b}(\mathrm{mod}\,\Gamma_{\mathcal{J}}) & \longrightarrow & \mathrm{D^b}(\mathrm{mod}\,\Gamma_{\mathcal{J}})\\ \uparrow & & \uparrow\\ \mathrm{K^b}(\mathrm{proj}\,\Gamma_{\mathcal{J}}) & \xrightarrow{\sim} & \mathrm{K^b}(\mathrm{proj}\,\Gamma_{\mathcal{J}})\end{array} \qquad \begin{array}{ccc}\mathrm{D^b}(\mathrm{mod}\,\Gamma_{\mathcal{J}}) & \longrightarrow & \mathrm{D^b}(\mathrm{mod}\,\Gamma_{\mathcal{J}})\\ \uparrow & & \uparrow\\ \mathrm{D^b_{fd}}(\mathrm{mod}\,\Gamma_{\mathcal{J}}) & \xrightarrow{\sim} & \mathrm{D^b_{fd}}(\mathrm{mod}\,\Gamma_{\mathcal{J}})\end{array}$$

The inclusion $\mathrm{D^b_{fd}}(\mathrm{mod}\,\Gamma_{\mathcal{J}}) \hookrightarrow \mathrm{D^b}(\mathrm{mod}\,\Gamma_{\mathcal{J}})$ gives rise to a homomorphism

$$K_0(\mathrm{D^b_{fd}}(\mathrm{mod}\,\Gamma_{\mathcal{J}})) \to G_0(\Gamma_{\mathcal{J}})$$

sending $[\mathcal{S}] \mapsto [\mathcal{S}]$, and let $H_{\mathcal{J}}$ denote the image. Writing $G_0(\Gamma_{\mathcal{J}}) = (\bigoplus_{i\in\Delta_{\mathsf{aff}}} \mathbb{Z}[\mathcal{P}_i])/\mathcal{R}$ by 7.8, then $H_{\mathcal{J}}$ is the subgroup generated by $\{[\mathcal{S}_i] + \mathcal{R} \mid i \in \Delta_{\mathsf{aff}}\backslash\mathcal{J}\}$.

Similarly, the inclusion $\mathrm{K^b}(\mathrm{proj}\,\Gamma_{\mathcal{J}}) \hookrightarrow \mathrm{D^b}(\mathrm{mod}\,\Gamma_{\mathcal{J}})$ induces a homomorphism

$$K_0(\mathrm{K^b}(\mathrm{proj}\,\Gamma_{\mathcal{J}})) \to G_0(\Gamma_{\mathcal{J}})$$

sending $[\mathcal{P}] \mapsto [\mathcal{P}]$. Let $K_{\mathcal{J}}$ denote the image, so that $K_{\mathcal{J}}$ is the subgroup of $G_0(\Gamma_{\mathcal{J}})$ generated by $\{[\mathcal{P}_i] + \mathcal{R} \mid i \in \Delta_{\mathsf{aff}}\backslash\mathcal{J}\}$.

LEMMA 7.10. *The subgroups* $H_{\mathcal{J}}$, $K_{\mathcal{J}}$ *and* $H_{\mathcal{J}} + K_{\mathcal{J}}$ *of* $G_0(\Gamma_{\mathcal{J}})$ *are all derived invariants.*

PROOF. The first two statements are a consequence of the above commutative diagrams. The final statement follows from the first two. $\square$

COROLLARY 7.11. *Consider* $\mathcal{J} =$  *and* $\mathcal{J}' =$ . *Both have type extended* D_6, *cotype* $(A_1)^3$, *and Grothendieck group* $\mathbb{Z}^4 \oplus \mathbb{Z}/2\mathbb{Z}$. *However,* $\Gamma_{\mathcal{J}}$ *and* $\Gamma_{\mathcal{J}'}$ *are not derived equivalent, since* $H_{\mathcal{J}} + K_{\mathcal{J}} \cong \mathbb{Z}^4$ *whilst* $H_{\mathcal{J}'} + K_{\mathcal{J}'} \cong \mathbb{Z}^4 \oplus \mathbb{Z}/2\mathbb{Z}$, *so* $H_{\mathcal{J}} + K_{\mathcal{J}} \ncong H_{\mathcal{J}'} + K_{\mathcal{J}'}$.

PROOF. Label the vertices $\substack{1\\7}\,2\,3\,4\,\substack{5\\6}$ so that $\mathcal{J} = \{1, 3, 6\}$ and $\mathcal{J}' = \{2, 5, 6\}$. The type is extended D_6, and cotype of both $\mathcal{J}$ and $\mathcal{J}'$ is clearly $(A_1)^3$. Further, using 7.8

$$G_0(\Gamma_{\mathcal{J}}) \cong \mathbb{Z}^7/\langle S_1, S_3, S_6\rangle = \mathbb{Z}^7/\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & 0 \end{pmatrix} \cong \mathbb{Z}^4 \oplus (\mathbb{Z}/2\mathbb{Z}).$$

Inside this group,

$$H_{\mathcal{J}} = \left\langle \begin{matrix} [\,-1 & 2 & -1 & 0 & 0 & 0 & -1\,] \\ [\,0 & 0 & -1 & 2 & -1 & -1 & 0\,] \\ [\,0 & 0 & 0 & -1 & 2 & 0 & 0\,] \\ [\,0 & -1 & 0 & 0 & 0 & 0 & 2\,] \end{matrix} \right\rangle \quad \text{and} \quad K_{\mathcal{J}} = \left\langle \begin{matrix} [\,0 & 1 & 0 & 0 & 0 & 0 & 0\,] \\ [\,0 & 0 & 0 & 1 & 0 & 0 & 0\,] \\ [\,0 & 0 & 0 & 0 & 1 & 0 & 0\,] \\ [\,0 & 0 & 0 & 0 & 0 & 0 & 1\,] \end{matrix} \right\rangle$$

By elementary transformations of integer matrices, $H_{\mathcal{J}} + K_{\mathcal{J}} \cong \mathbb{Z}^4$. The verifications that $H_{\mathcal{J}'} + K_{\mathcal{J}'} \cong \mathbb{Z}^4 \oplus \mathbb{Z}/2\mathbb{Z}$ is very similar. ☐

7.2. Symmetric Mutation Classes

In this section we determine the symmetric equivalence classes for all extended ADE Dynkin diagrams (except D_n with $n \geq 8$), since this will be required for the derived equivalence classification in the next section. We keep the notation that elements in $\mathcal{J} \subset (\Delta)_{\mathsf{aff}}$ are drawn black, the other vertices are drawn red, and recall that we mutate at the red vertices.

We begin with Type A, which is elementary. By convention, $A_0 = \emptyset$.

LEMMA 7.12. *Let $\Delta = A_n$ with $n \geq 1$, and consider Δ_{aff}.*

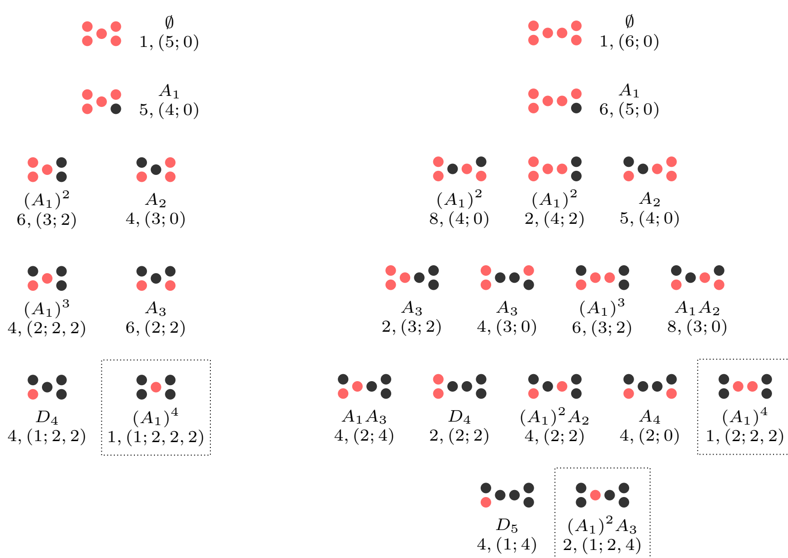
(1) *If $\mathcal{J}_1$ and $\mathcal{J}_2$ are subsets of Δ_{aff} with the same cotype, then they can connected by a finite sequence of mutations, up to symmetries of the graph.*
(2) *For each i such that $0 \leq i \leq n$, Δ_{aff} has a unique symmetric mutation class with cotype A_i, and this class contains $\binom{n+1}{i}$ elements.*

In particular, symmetric mutation classes are indexed by cotype.

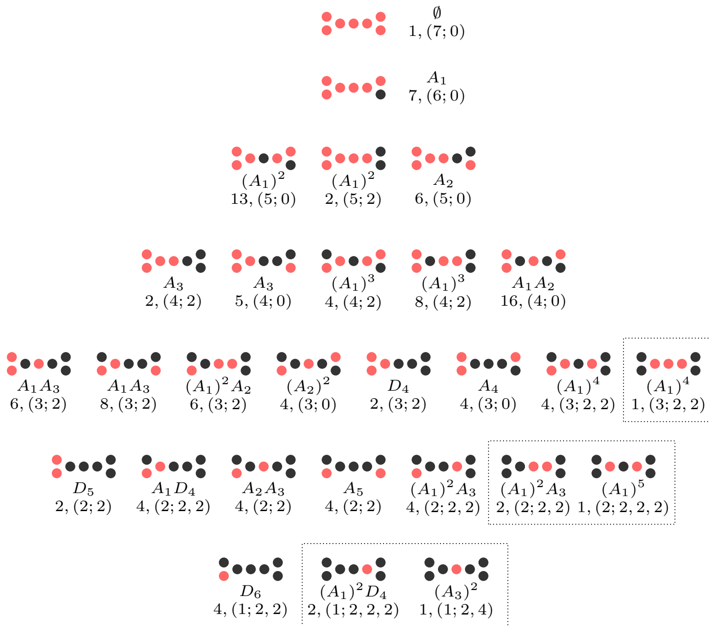
PROOF. (1) is a direct verification, using the wall-crossing rule, and (2) follows. ☐

NOTATION 7.13. In the notation $(a; b)$ below, a is the rank, and b is the torsion. So, for example $(1; 3, 3) = \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, and $(6; 0) = \mathbb{Z}^6$. Also, in what follows we will discount the case $\mathcal{J} = \Delta_{\mathsf{aff}}$, since in that case $\Gamma_{\mathcal{J}} = 0$.

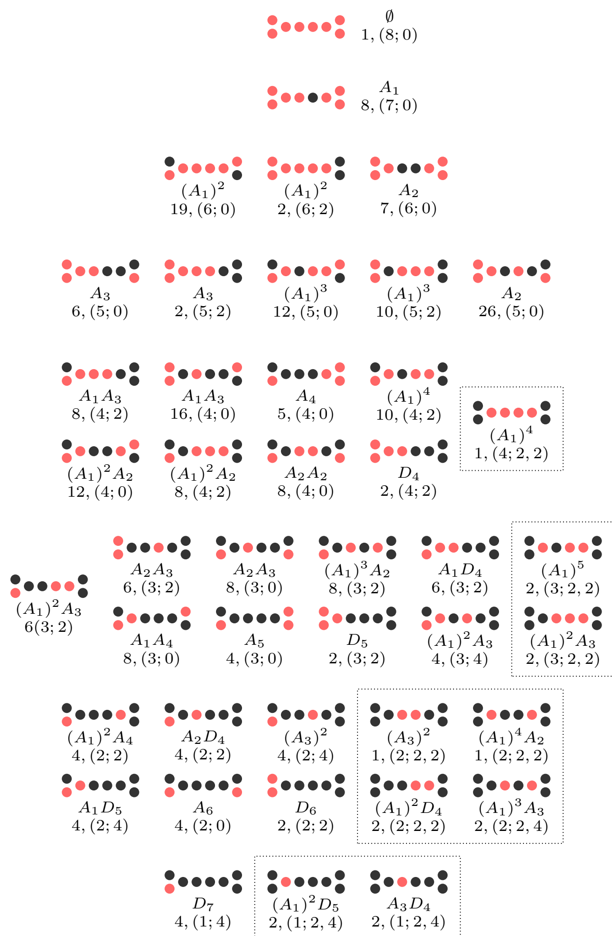
PROPOSITION 7.14. *The symmetric mutation classes for extended D_4 and D_5 are as follows. In each case, a representative of the class, and the number in each class is listed. For convenience later, we also list the cotype and $G_0(\Gamma_{\mathcal{J}})$.*



Furthermore, the symmetric equivalence classes for extended D_6 are as follows:

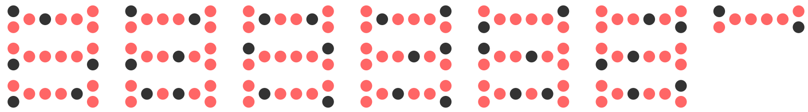


and the symmetric equivalence classes for extended D_7 are:



The classes boxed are precisely those symmetric equivalence classes that are not geometric.

Proof. Each row in each case is a direct verification. Here, we prove the third row for $\Delta = D_7$, to illustrate the method. For this, consider the set $\mathcal{C} = \{\mathcal{J} \subset \Delta_{\mathsf{aff}} : |\mathcal{J}| = 2\}$, which has $\binom{8}{2} = 28$ elements, and begin by choosing an element in $\mathcal{C}$, say $\mathcal{J} =$ . It is easy to verify that the set



is closed under the wall crossing, and symmetries of the graph. Hence it is precisely the symmetric mutation class containing $\mathcal{J}$.

Next, choose an element of $\mathcal{C}$ which is not in the above class. There are $28 - 19 = 9$ to choose from, say $\mathcal{J} =$ . In this case, the set

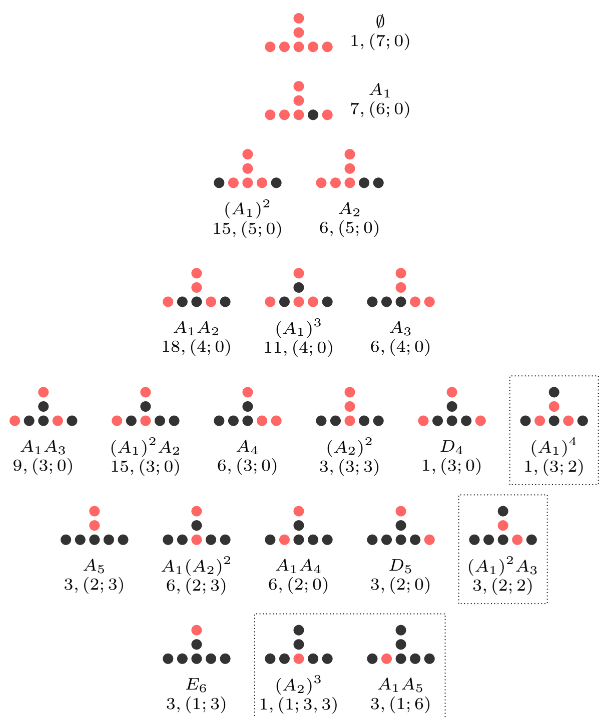
is closed under wall crossing and symmetries, so is the second symmetric mutation class.

Finally, choose an element in $\mathcal{C}$ which is not in the above two classes. There are $28 - (19 + 2) = 7$ to choose from, say . In this case, the set

is closed under wall crossing and symmetries, so is the third symmetric mutation class.

The above three classes total 28 elements, so exhaust all elements of the $\mathcal{C}$. As such, there can be no more symmetric mutation classes with $|\mathcal{J}| = 2$. The number of elements in each of the three classes is 19, 2, 7 respectively, and the cotype is $(A_1)^2, (A_1)^2$ and A_2 respectively. In each class, the Grothendieck group $\Gamma_{\mathcal{J}}$ is calculated using 7.8, and is easily seen to be $\mathbb{Z}^6$, $\mathbb{Z}^6 \oplus (\mathbb{Z}/2\mathbb{Z})$ and $\mathbb{Z}^6$ respectively. ☐

Proposition 7.15. *Let $\Delta = E_6$. Then Δ_{aff} has the following 21 symmetric equivalence classes. In each case, a representative of the class, the number in each class, the cotype and $G_0(\Gamma_{\mathcal{J}})$ is listed.*



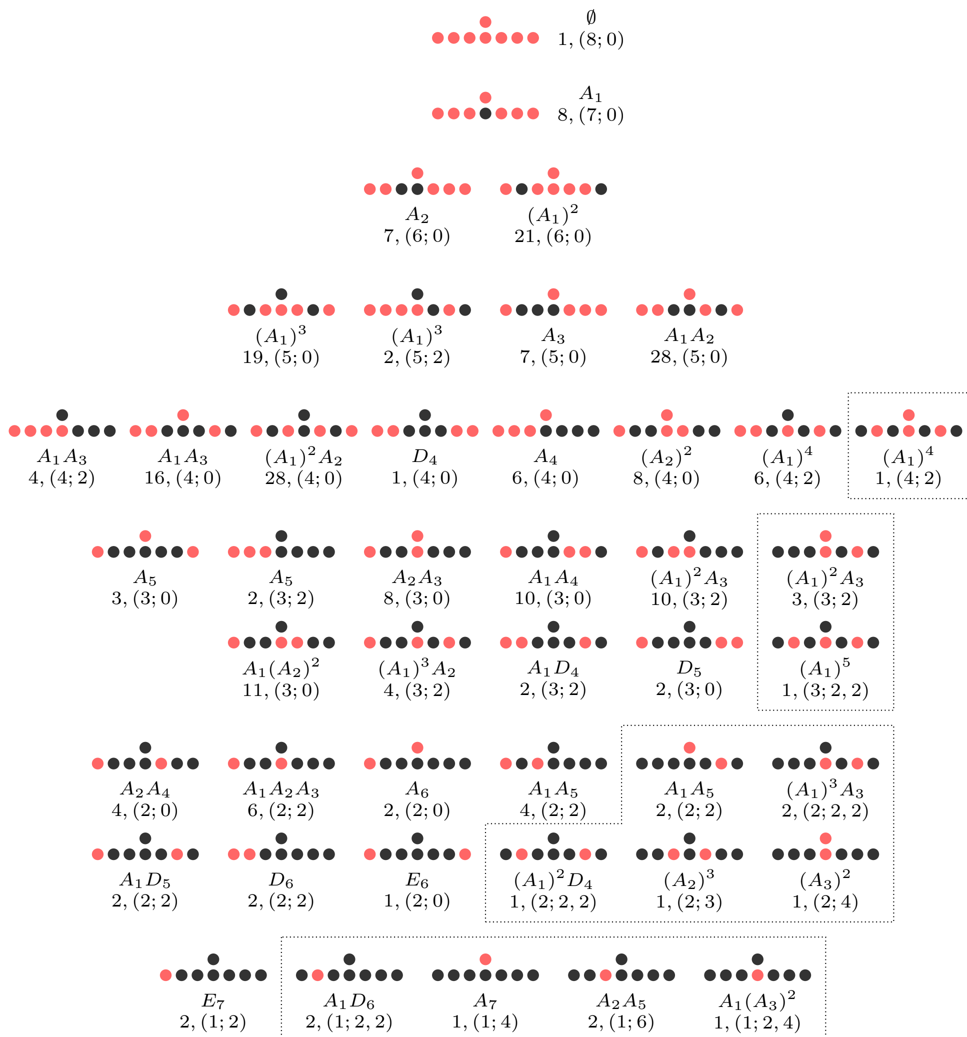
The classes boxed are precisely those symmetric equivalence classes that are not geometric.

Proof. This is also a direct verification, exactly as in 7.14. Note that here, the case $|\mathcal{J}| = 4$ is 4.9. ☐

Remark 7.16. Direct inspection of 7.12, 7.14 and 7.15 reveals the following.

1. For $\Delta \in \{A_n, D_4, E_6\}$, cotype distinguishes all symmetric equivalence classes.
2. For $\Delta \in \{D_5, D_7\}$, the pair (cotype, G_0) distinguishes the symmetric equivalence classes. For example, when $\Delta = D_5$, everything is distinguished just using cotype, with the exception of $(A_1)^2$ and A_3, each of which is the cotype for two different classes. However, in these two, cases G_0 distinguishes.
3. For $\Delta = D_6$, the pair (cotype, G_0) does not distinguish the classes, but later the triple (cotype, G_0, $H + K$) will.

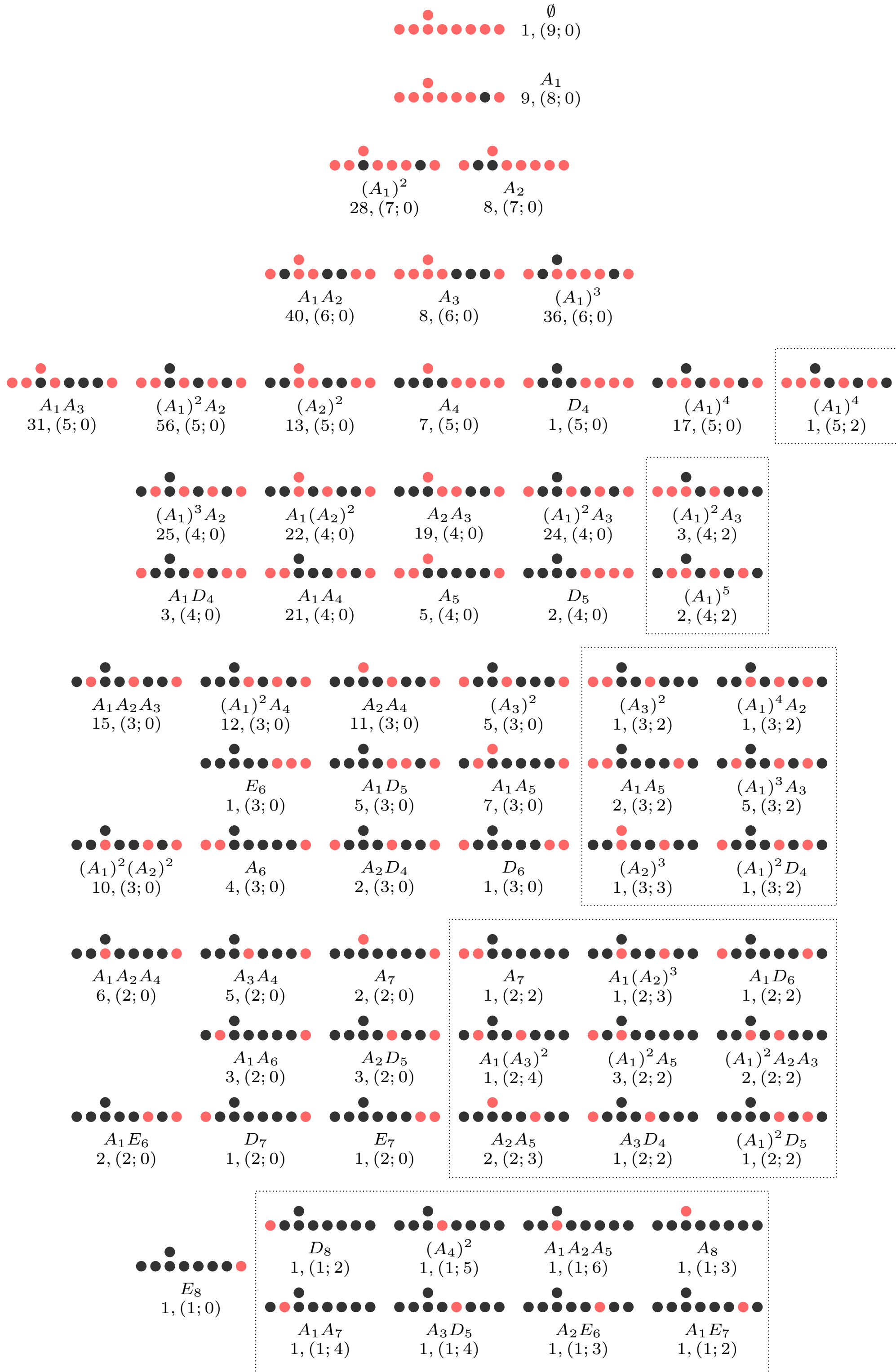
Proposition 7.17. *Let $\Delta = E_7$ and consider Δ_{aff}. Then Δ_{aff} has the following 44 symmetric equivalence classes. In each case, a representative of the class, the number in each class, the cotype and $G_0(\Gamma_{\mathcal{J}})$ is listed.*



The classes boxed are precisely those symmetric equivalence classes that are not geometric.

Proof. This is again a direct verification, exactly as in 7.14. The case $|\mathcal{J}| = 5$ has already been verified in 4.11, but the classes here are permuted. ☐

PROPOSITION 7.18. *Let $\Delta = E_8$ and consider Δ_{aff}. Then Δ_{aff} has the following 67 symmetric equivalence classes. In each case, a representative of the class, the number in each class, the cotype and $G_0(\Gamma_{\mathfrak{J}})$ is listed.*



The classes boxed are precisely those symmetric equivalence classes that are not geometric.

Proof. Direct verification, exactly as in 7.14. The case $|\mathcal{J}| = 6$ is 4.13. □

Remark 7.19. Direct inspection of 7.18 reveals that when $\Delta = E_8$, the pair (cotype, G_0) distinguishes the symmetric mutation classes for Δ_{aff}.

7.3. The Derived Classification

The main result in this section, 7.21, gives a full derived equivalence classification of contracted preprojective algebras $\Gamma_{\mathcal{J}}$ of extended Dynkin type, in particular confirming 7.1, when $\Delta \in \{A_n, D_4, D_5, D_6, D_7, E_6, E_7, E_8\}$. Partial results are obtained in all cases, including the ($\Leftarrow$) direction of 7.1.

The following also verifies that the first part of ($\Rightarrow$) in 7.1 holds in all cases.

Lemma 7.20. *Suppose that $\mathcal{J} \subseteq \Delta_{\mathsf{aff}}$ where Δ is ADE Dynkin, and suppose that A is a basic ring that is derived equivalent to $\Gamma_{\mathcal{J}}$. Then there exists $\mathcal{J}' \subseteq \Delta_{\mathsf{aff}}$ such that $A \cong \Gamma_{\mathcal{J}'}$.*

Proof. Using the notation from the proof of 7.6, say $\Gamma_{\mathcal{J}} \cong \operatorname{End}_R(M_{\mathcal{J}})$. Since $\dim R = 2$, it is automatic that $\Gamma_{\mathcal{J}} \in \operatorname{CM} R$, so $\Gamma_{\mathcal{J}}$ is a modifying R-algebra. By [IW, 4.6(1)], since A is derived equivalent to $\Gamma_{\mathcal{J}}$, necessarily $A \cong \operatorname{End}_R(N)$ for some modifying R-module N. Since R has finite CM type, necessarily $N \cong M_{\mathcal{J}'}$ for some subset $\mathcal{J}'$. □

Thus, to verify 7.1, we can restrict to considering derived equivalences between contracted preprojective algebras, of the same type.

Theorem 7.21. *Suppose that $\mathcal{J} \subseteq \Delta_{\mathsf{aff}}$ and $\mathcal{J}' \subseteq \Delta'_{\mathsf{aff}}$ where Δ and Δ' are ADE Dynkin. Consider the following conditions.*

(1) *$\Gamma_{\mathcal{J}}$ is derived equivalent to $\Gamma_{\mathcal{J}'}$.*
(2) *The types match (namely $\Delta = \Delta'$), and $\mathcal{J} \sim \mathcal{J}'$.*
(3) *The types match, and the cotypes match.*
(4) *The types match, the cotypes match, and $G_0(\Gamma_{\mathcal{J}}) \cong G_0(\Gamma_{\mathcal{J}'})$.*
(5) *The types match, the cotypes match, $G_0(\Gamma_{\mathcal{J}}) \cong G_0(\Gamma_{\mathcal{J}'})$, and $H_{\mathcal{J}} + K_{\mathcal{J}} \cong H_{\mathcal{J}'} + K_{\mathcal{J}'}$.*

Then (2)⇒(1). If $\Delta \in \{A_n, D_4, D_5, D_6, D_7, E_6, E_7, E_8\}$, then (1)⇒(2). Furthermore:

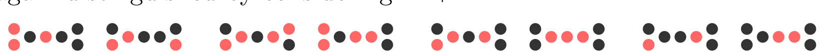
- *If $\Delta \in \{A_n, D_4, E_6\}$ then (1)⇔(2)⇔(3).*
- *If $\Delta \in \{D_5, D_7, E_8\}$ then (1)⇔(2)⇔(4).*
- *If $\Delta \in \{D_6, E_7\}$ then (1)⇔(2)⇔(5).*

Proof. The implication (2)⇒(1) always holds, since wall crossing gives derived equivalences (§5.6), as do isomorphisms. The statement regarding (1)⇒(2) follows from the bulleted statements, which we prove now.

- If $\Delta \in \{A_n, D_4, E_6\}$, then (1)⇒(3) by 7.7, and 7.16(1) shows that (3)⇒(2).
- If $\Delta \in \{D_5, D_7, E_8\}$, then (1)⇒(4) by 7.7, and 7.16(2) and 7.19 show that (4)⇒(2).
- If $\Delta \in \{D_6, E_7\}$, then (1)⇒(5) by 7.7 and 7.10. Hence it suffices to show (5)⇒(2). For E_7, by 7.17 the pair (cotype, G_0) distinguishes all classes except the following. These are distinguished by considering $H + K$, which are calculated using 7.11.

cotype	$(A_1)^4$	$(A_1)^4$	$(A_1)^2A_3$	$(A_1)^2A_3$	A_1A_5	A_1A_5
G_0	$(4;2)$	$(4;2)$	$(3;2)$	$(3;2)$	$(2;2)$	$(2;2)$
$H+K$	$(4;0)$	$(4;2)$	$(3;0)$	$(3;2)$	$(2;0)$	$(2;2)$

For D_6, by 7.14 the pair (cotype, G_0) distinguishes all classes except the following, which are again distinguished by considering $H + K$



cotype	A_1A_3	A_1A_3	$(A_1)^3$	$(A_1)^3$	$(A_1)^4$	$(A_1)^4$	$(A_1)^2A_3$	$(A_1)^2A_3$
G_0	$(3;2)$	$(3;2)$	$(4;2)$	$(4;2)$	$(3;2,2)$	$(3;2,2)$	$(2;2,2)$	$(2;2,2)$
$H+K$	$(3;2)$	$(3;0)$	$(4;0)$	$(4;2)$	$(3;0)$	$(3;2,2)$	$(2;0)$	$(2;2,2)$

The result follows. □

Corollary 7.22. *Conjecture 7.1 is true if* $\Delta \in \{A_n, D_4, D_5, D_6, D_7, E_6, E_7, E_8\}$.

Proof. ($\Rightarrow$) holds by combining 7.20 with 7.21(1)$\Rightarrow$(2). ($\Leftarrow$) is 7.21(2)$\Rightarrow$(1). $\square$

Remark 7.23. Consider $\mathcal{J}$ =  and $\mathcal{J}'$ = . These are in different symmetric mutation classes, but both have type extended D_8, and both have cotype $(A_1)^4$. Furthermore, in both cases the Grothendieck group G_0 is isomorphic to $\mathbb{Z}^5 \oplus \mathbb{Z}/2\mathbb{Z}$, and in both cases $H + K \cong \mathbb{Z}^5 \oplus \mathbb{Z}/2\mathbb{Z}$. This is the first case where the invariants in 7.21 do not distinguish between $\Gamma_{\mathcal{J}}$ and $\Gamma_{\mathcal{J}'}$. Consequently, to extend 7.21 to cover all of type D_n will require finer invariants.

7.4. Two-term Tilting Complexes

This section classifies all 2-term tilting complexes for $\Gamma_{\mathcal{J}}$, for any $\mathcal{J} \subseteq \Delta_{\mathsf{aff}}$. The main result 7.24 is that the 2-tilting complexes of $\Gamma_{\mathcal{J}}$ are in bijection with chambers of the infinite arrangement $\mathcal{W}_{\mathcal{J}}$ from 2.8, which is the 'double' of the associated Tits cone (see e.g. 2.10). Just as $\Theta_{\mathcal{J}}$ splits into two halves, it turns out that there are two types of 2-term tilting complexes, and their mutation graphs never meet. This whole section should be seen as generalisation of [**KiM**], which described the case $\mathcal{J} = \emptyset$.

7.4.1. Construction. For $(x, J) \in \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J})$, set $I_{x,J} = e_{\mathcal{J}} I_x e_J$. By 5.2, each $I_{x,J}$ is a tilting $\Gamma_{\mathcal{J}}$-module with projective dimension one, and so in particular their projective resolutions give two-term tilting complexes. Below, these will give half of the two-term tilting complexes.

In order to obtain the other half, recycling the notation from the proof of 7.6, say $\Gamma_{\mathcal{J}} \cong \mathrm{End}_R(M_{\mathcal{J}})$. Then $\Gamma_{\mathcal{J}}^{\mathrm{op}} \cong \mathrm{End}_R(M_{\mathcal{J}}^*)$, where $()^* = \mathrm{Hom}_R(-, R)$. But under McKay Correspondence, the dual $(-)^*$ fixes the extended vertex, and acts as the Dynkin involution on the other vertices. Write ι_{aff} for this symmetry. Thus $\mathrm{End}_R(M_{\mathcal{J}}^*) \cong \Gamma_{\mathcal{J}'}$, where $\mathcal{J}' = \iota_{\mathsf{aff}}(\mathcal{J})$. Since $\mathcal{J}$ and $\mathcal{J}'$ differ by the symmetry of the graph ι_{aff}, it is clear that they induce exactly the same Tits cone, and so have exactly the same tilting theory.

Now, since $\Gamma_{\mathcal{J}}$ is a symmetric R-order, by [**IR**, p1103], there is a duality

$$\mathrm{D}^-(\mathrm{mod}\,\Gamma_{\mathcal{J}}) \xrightarrow{\mathbf{R}\mathrm{Hom}_{\Gamma_{\mathcal{J}}}(-,\Gamma_{\mathcal{J}})} \mathrm{D}^+(\mathrm{mod}\,\Gamma_{\mathcal{J}}^{\mathrm{op}}) \xrightarrow{\mathbf{R}\mathrm{Hom}_{\Gamma_{\mathcal{J}}^{\mathrm{op}}}(-,\Gamma_{\mathcal{J}})} \mathrm{D}^-(\mathrm{mod}\,\Gamma_{\mathcal{J}}).$$

The strategy is to apply the functor $\mathbf{R}\mathrm{Hom}_{\Gamma_{\mathcal{J}}^{\mathrm{op}}}(-, \Gamma_{\mathcal{J}}[1])$ to tilt $\Gamma_{\mathcal{J}'} = \text{tilt}\,\Gamma_{\mathcal{J}}^{\mathrm{op}}$ to obtain the 'other half' of the two-term tilting complexes. As notation, set

$$R_{y,K} = \mathbf{R}\mathrm{Hom}_{\Gamma_{\mathcal{J}}^{\mathrm{op}}}(e_{\mathcal{J}'} I_y e_K, \Gamma_{\mathcal{J}}[1])$$

for $(y, K) \in \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J}')$. Replacing $e_{\mathcal{J}'} I_y e_K$ by its two-term projective resolution, it is clear that $R_{y,K}$ is a two-term tilting complex for $\Gamma_{\mathcal{J}}$.

7.4.2. Two-Term Tilting. Our previous 7.21 showed which algebras are in the derived equivalence class of $\Gamma_{\mathcal{J}}$, and so demonstrated that the endomorphism rings of general tilting complexes behave well. In comparison, the following, which is a generalisation of [**KiM**, 2.7, 3.1], takes this further, and gives evidence that the set of *all* tilting complexes could in fact be very well-behaved, not just their endomorphism rings.

Theorem 7.24. *Suppose that Δ is ADE, $\mathcal{J} \subset \Delta_{\mathsf{aff}}$, and $\mathcal{J}' = \iota_{\mathsf{aff}}(\mathcal{J})$.*

(1) $\{I_{x,J} \mid (x, J) \in \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J})\} \cap \{R_{y,K} \mid (y, K) \in \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J}')\} = \emptyset$. *In particular, there exist two different families of tilting complexes for $\Gamma_{\mathcal{J}}$.*

(2) $2\,\text{tilt}\,\Gamma_{\mathcal{J}} = 2\,\text{silt}\,\Gamma_{\mathcal{J}}$.

(3) *Under the isomorphism $\Theta_{\mathcal{J}} \cong K_0(\mathrm{proj}\,\Gamma_{\mathcal{J}})$,*

$$\Theta_{\mathcal{J}} \backslash \mathcal{W}_{\mathcal{J}} = \left(\bigcup_{(x,J) \in \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J})} C(I_{x,J}) \right) \bigcup \left(\bigcup_{(y,K) \in \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J}')} C(R_{y,K}) \right).$$

Furthermore, the closure equals $\Theta_{\mathcal{J}}$.

(4) $2\,\mathrm{tilt}\,\Gamma_{\mathcal{J}} = \{I_{x,J} \mid (x,J) \in \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J})\} \cup \{R_{y,K} \mid (y,K) \in \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J}')\}$.

Proof. (1) By 5.2 the $I_{x,J} = e_{\mathcal{J}} I_x e_J$ are classical tilting modules, and so give 2-term tilting complexes with homology only in degree zero. In contrast, since $e_{\mathcal{J}'} I_y e_K \in \mathrm{tilt}\,\Gamma_{\mathcal{J}'}$ and so give two-term tilting complexes for $\Gamma_{\mathcal{J}}^{\mathrm{op}}$, by construction the $R_{y,K}$ are two-term tilting complexes for $\Gamma_{\mathcal{J}}$. They certainly have homology in degree -1. For these homology reasons, the intersection is thus empty.
(2) Since $\Gamma_{\mathcal{J}}$ is a symmetric R-order, this is clear (see e.g. [KiM, A.2]).
(3) Under the isomorphism, we already know that elements in $\mathrm{tilt}\,\Gamma_{\mathcal{J}}$, namely

$$\bigcup_{(x,J)\in\mathsf{Cham}(\Delta_{\mathsf{aff}},\mathcal{J})} C(I_{x,J})$$

gives $\mathsf{Cone}(\Delta_{\mathsf{aff}}, \mathcal{J})$. This fills precisely half of $K_0(\mathrm{proj}\,\Gamma_{\mathcal{J}})$, and so half of $\Theta_{\mathcal{J}}$.

Applying the same logic to $\mathrm{tilt}\,\Gamma_{\mathcal{J}'}$,

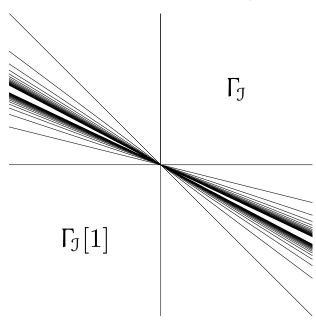
$$\bigcup_{(y,K)\in\mathsf{Cham}(\Delta_{\mathsf{aff}},\mathcal{J}')} C(I_{y,K})$$

gives precisely half of $K_0(\mathrm{proj}\,\Gamma_{\mathcal{J}'})$. Since the dividing half-plane for both $\mathcal{J}$ and $\mathcal{J}'$ is the same (dual respects the rank), mapping this across the duality, it is clear that

$$\bigcup_{(y,K)\in\mathsf{Cham}(\Delta_{\mathsf{aff}},\mathcal{J}')} C(R_{y,K})$$

fills the other half of $\Theta_{\mathcal{J}}$. Furthermore, since $\mathcal{J}$ and $\mathcal{J}'$ differ by the symmetry ι_{aff}, it is clear that their Tits cones are identical. Consequently, the full hyperplanes $\mathcal{W}_{\mathcal{J}}$ describe the walls on both halves of the dividing half-plane, and so the statement about $\Theta_{\mathcal{J}} \backslash \mathcal{W}_{\mathcal{J}}$ follows. That the closure is everything is clear.
(4) As is standard, this follows from (3) by a [DIJ]-type argument (see e.g. [KiM, 3.8(2)]). $\square$

Example 7.25. Continuing 2.9, consider $\mathcal{J} =$ . Then by 7.24, the g-vector fan of the two-term tilting complexes for $\Gamma_{\mathcal{J}}$ is the following.



The hyperplanes converge on the line $[\mathcal{P}_0] + 2[\mathcal{P}_1] = 0$, but $\mathcal{W}_{\mathcal{J}}$ does not contain this line.

Part 3

The Representation Theory of cDV Singularities

CHAPTER 8

Tilting Modules and Modifying Modules

This chapter is concerned with general properties of tilting, and tilting modulo a regular element. These results obtained here are general; they are then specialised, and strengthened, in the cDV setting in Chapter 9.

The first section describes tilting modules modulo a regular element for module-finite R-algebras. Section 8.2 recalls the setting of 3-dimensional Gorenstein rings, recaps the notion of modifying modules and mutation, and recalls known theorems. The section after, §8.3, generalises some of these, and gives three new general results, the most notable that K-theory can be used to detect partial tilting modules. All these results, new and old, are combined in Section 8.4 to give the most general results for 3-dimensional Gorenstein rings, and their associated modification algebras.

8.1. Tilting Modules Modulo a Regular Element

When R is a commutative noetherian ring and $g \in R$, write $\overline{R} := R/gR$ and

$$\overline{(-)} := \overline{R} \otimes_R - \colon \operatorname{mod} R \to \operatorname{mod} \overline{R}.$$

In particular, if Λ is an R-algebra, then we can consider the $\overline{R}$-algebra $\overline{\Lambda}$.

Notation 8.1. Given an R-algebra Λ, and $g \in R$, write $\operatorname{tilt}_g \Lambda$ for the subset of $\operatorname{tilt} \Lambda$ consisting of those T on which g acts a non-zerodivisor.

We need the following elementary observation.

Lemma 8.2. *If g is a non-zerodivisor on R, then for any $X \in \operatorname{mod} R$,*

$$\operatorname{Tor}_1^R(\overline{R}, X) = \{x \in X \mid gx = 0\}.$$

Proof. The short exact sequence $0 \to R \xrightarrow{g} R \to \overline{R} \to 0$ gives a projective resolution of $\overline{R}$. Applying $- \otimes_R X$ gives an exact sequence

$$0 \to \operatorname{Tor}_1^R(\overline{R}, X) \to X \xrightarrow{g} X,$$

which proves the assertion. □

The following general observation is the main result of this section.

Theorem 8.3. *Let R be a commutative noetherian ring and Λ a module-finite R-algebra. Assume that g is a non-zerodivisor on R.*

(1) *There is a map*

$$\overline{(-)} \colon \operatorname{tilt}_g \Lambda \to \operatorname{tilt} \overline{\Lambda}$$

making the following diagram commute.

(8.1.A)
$$\begin{array}{ccc} \operatorname{tilt}_g \Lambda & \xrightarrow{\overline{(-)}} & \operatorname{tilt} \overline{\Lambda} \\ \downarrow & & \downarrow \\ \mathrm{K}_0(\operatorname{proj} \Lambda) & \xrightarrow{\overline{(-)}} & \mathrm{K}_0(\operatorname{proj} \overline{\Lambda}) \end{array}$$

(2) *If g is contained in all maximal ideals of R, then the map in (1) is injective.*
(3) *If R is complete local, then the map in (1) is compatible with mutation.*

PROOF. (1) Fix $T \in \operatorname{tilt}_g \Lambda$, and let

$$0 \to P_1 \to P_0 \to T \to 0 \tag{8.1.B}$$

be a projective resolution of the Λ-module T. Applying $\overline{R} \otimes_R -$ gives an exact sequence

$$0 = \operatorname{Tor}_1^R(\overline{R}, T) \to \overline{P}_1 \to \overline{P}_0 \to \overline{T} \to 0, \tag{8.1.C}$$

where the first term is zero by 8.2, since $T \in \operatorname{tilt}_g \Lambda$. Thus $\overline{T}$ has projective dimension at most one, as a $\overline{\Lambda}$-module.

Applying $\operatorname{Hom}_\Lambda(-, T)$ to (8.1.B), and $\operatorname{Hom}_{\overline{\Lambda}}(-, \overline{T})$ to (8.1.C), gives a commutative diagram of exact sequences

$$\begin{array}{ccccc} \operatorname{Hom}_\Lambda(P_1, T) & \longrightarrow & \operatorname{Hom}_\Lambda(P_0, T) & \longrightarrow & \operatorname{Ext}^1_\Lambda(T, T) = 0 \\ \downarrow & & \downarrow & & \\ \operatorname{Hom}_{\overline{\Lambda}}(\overline{P}_1, \overline{T}) & \longrightarrow & \operatorname{Hom}_{\overline{\Lambda}}(\overline{P}_0, \overline{T}) & \longrightarrow & \operatorname{Ext}^1_{\overline{\Lambda}}(\overline{T}, \overline{T}) \longrightarrow 0 \end{array}$$

where the vertical maps are clearly surjective. It follows that $\operatorname{Ext}^1_{\overline{\Lambda}}(\overline{T}, \overline{T}) = 0$.

Lastly, take an exact sequence $0 \to \Lambda \to T_0 \to T_1 \to 0$ with $T_i \in \operatorname{add} T$. Applying $\overline{R} \otimes_R -$ gives an exact sequence

$$0 = \operatorname{Tor}_1^R(\overline{R}, T_1) \to \overline{\Lambda} \to \overline{T}_0 \to \overline{T}_1 \to 0,$$

where the first term is zero by 8.2, since $T \in \operatorname{tilt}_g \Lambda$. Consequently, $\overline{T} \in \operatorname{tilt} \overline{\Lambda}$. Using (8.1.B) and (8.1.C), $[\overline{T}] = [\overline{P}_0] - [\overline{P}_1] = \overline{[P_0] - [P_1]} = \overline{[T]}$ and so the diagram commutes.
(2) Without loss of generality, we can assume that R is complete local. Indeed, by [IW, 2.26], if $\operatorname{add}(\widehat{R}_\mathfrak{m} \otimes_R T) = \operatorname{add}(\widehat{R}_\mathfrak{m} \otimes_R U)$ for all maximal ideals $\mathfrak{m}$, then $\operatorname{add} T = \operatorname{add} U$.

For any $T \in \operatorname{tilt} \Lambda$, write $\operatorname{Fac} T$ for the full subcategory of $\operatorname{mod} \Lambda$ consisting of the factor modules of an object in $\operatorname{add} T$. It is well-known in tilting theory that

$$\operatorname{Fac} T = \{X \in \operatorname{mod} \Lambda \mid \operatorname{Ext}^1_\Lambda(T, X) = 0\}$$

holds, and we can recover $\operatorname{add} T$ from $\operatorname{Fac} T$ as Ext-projective objects, that is,

$$\operatorname{add} T = \{X \in \operatorname{Fac} T \mid \operatorname{Ext}^1_\Lambda(X, \operatorname{Fac} T) = 0\}. \tag{8.1.D}$$

Assume that $T, U \in \operatorname{tilt}_g \Lambda$ satisfies $\operatorname{add} \overline{T} = \operatorname{add} \overline{U}$. To prove $\operatorname{add} T = \operatorname{add} U$, it suffices to show $\operatorname{Fac} T = \operatorname{Fac} U$ by (8.1.D), and by symmetry, we only have to show $U \in \operatorname{Fac} T$. For any $\ell > 0$, consider the short exact sequence

$$0 \to U/gU \xrightarrow{g^{\ell-1}} U/g^\ell U \xrightarrow{h_\ell} U/g^{\ell-1}U \to 0. \tag{8.1.E}$$

Since $\overline{U} = U/gU$ belongs to $\operatorname{add} \overline{T} \subseteq \operatorname{Fac} T$ and $\operatorname{Fac} T$ is closed under extensions, it follows that $U/g^\ell U \in \operatorname{Fac} T$ for all $\ell > 0$ by using (8.1.E) repeatedly. Therefore $\operatorname{Ext}^1_\Lambda(T, U/g^\ell U) = 0$ holds for all $\ell > 0$.

Now fix an epimorphism $f_1 \colon T' \to \overline{U} = U/gU$ for $T' \in \operatorname{add} T$. Using the sequence (8.1.E) repeatedly, there is a morphism $f_\ell \colon T' \to U/g^\ell U$ satisfying $h_\ell \circ f_\ell = f_{\ell-1}$. We lift $(f_\ell)_\ell$ to a morphism $f \colon T' \to \varprojlim_\ell U/g^\ell U = U$, where the equality holds since R is complete. By Nakayama's Lemma, f is surjective, and thus $U \in \operatorname{Fac} T$ holds.
(3) This is clear, since if $T, U \in \operatorname{tilt}_g \Lambda$ share all summands except one, then $\overline{T}, \overline{U} \in \operatorname{tilt}_g \Lambda$ share all summands except one. □

8.2. Modifying Modules, Mutation and Tilting

In this section we recall various concepts and results, mainly to set notation. Let R be a local Gorenstein normal ring, write $\operatorname{ref} R$ for the category of reflexive R-modules, and $\operatorname{CM} R$ for the category of Cohen-Macaulay R-modules. For a module-finite R-algebra Λ, we consider the category

$$\operatorname{ref} \Lambda := \{X \in \operatorname{mod} \Lambda \mid X \in \operatorname{ref} R\}.$$

For non-zero $M \in \operatorname{ref} R$, there is an equivalence [RV, 1.2] (see also [IR, 2.4(2)(i)])

$$\operatorname{Hom}_R(M,-)\colon \operatorname{ref} R \xrightarrow{\sim} \operatorname{ref} \operatorname{End}_R(M). \tag{8.2.A}$$

which we will refer to as *reflexive equivalence*.

Recall from [IW] that $M \in \operatorname{ref} R$ is called *modifying* if $\operatorname{End}_R(M) \in \operatorname{CM} R$. An R-module M is called *maximal modifying* if it is modifying, and maximal with respect to this property; equivalently

$$\operatorname{add} M = \{X \in \operatorname{ref} R \mid \operatorname{End}_R(M \oplus X) \in \operatorname{CM} R\}.$$

We write $\operatorname{MM} R$ (respectively, $\operatorname{MMG} R$) for the set of additive equivalence classes of maximal modifying R-modules (respectively, maximal modifying generators of R).

The following properties are elementary.

Lemma 8.4. [IW, 2.7, 5.12] *With notation as above,*

1. *If $M \in \operatorname{CM} R$ is a modifying R-module, then so is $R \oplus M$. Therefore a maximal modifying R-module M is Cohen-Macaulay if and only if $R \in \operatorname{add} M$.*
2. *If $\dim R = 3$, then $M \in \operatorname{CM} R$ is modifying if and only if $\operatorname{Ext}^1_R(M,M)$ has positive depth.*
3. *If $\dim R = 3$ and R is isolated, then $M \in \operatorname{CM} R$ is modifying if and only if M is* rigid *(that is, $\operatorname{Ext}^1_R(M,M) = 0$).*

As for tilting modules, there is an operation on $\operatorname{MM} R$ called *mutation* [IW, §6]. We describe this for the case when R is complete local. Let $M \in \operatorname{modif} R$ be basic, and let $M = N \oplus L$ be a direct sum decomposition. There exists an exact sequence

$$0 \to K \xrightarrow{g} V \xrightarrow{f} L \tag{8.2.B}$$

where f is a minimal right $(\operatorname{add} N)$-approximation. We call (8.2.B) an *exchange sequence*, and set

$$\mu_L(M) := N \oplus \operatorname{Ker} f,$$

which is called the *right mutation* of M at L. Furthermore, we say $\mu_L(M)$ is an *artinian mutation* if $\operatorname{End}_R(M)/[N]$ is an artinian ring. Later, when we work over a field, this condition is equivalent to $\operatorname{End}_R(M)/[N]$ being a finite dimensional algebra.

Dually, there exists an exact sequence

$$0 \to L \xrightarrow{f} U \xrightarrow{g} C \tag{8.2.C}$$

where f is a minimal left $(\operatorname{add} N)$-approximation, such that

$$0 \to C^* \xrightarrow{g^*} U^* \xrightarrow{f^*} L^*$$

is exact. We call (8.2.C) an *exchange sequence*. Set

$$\nu_L(M) := N \oplus C,$$

and call $\nu_L(M)$ the *left mutation* of M at L. Again, we call $\nu_L(M)$ an *artinian mutation* if $\operatorname{End}_R(M)/[N]$ is an artinian ring. Clearly this is equivalent to $\mu_L(M)$ being an artinian mutation.

Proposition 8.5. *The following assertions hold.*

1. [IW, 6.10] $\mu_L(M), \nu_L(M) \in \operatorname{modif} R$.
2. [IW, 6.4] *g in (8.2.B) is a minimal left $(\operatorname{add} N)$-approximation, and g in (8.2.C) is a minimal right $(\operatorname{add} N)$-approximation.*
3. [IW, 6.5] *μ and ν are inverse operations of each other, that is,*

$$\nu_{L'} \circ \mu_L(M) \cong M \quad \text{and} \quad \mu_{L''} \circ \nu_L(M) \cong M$$

hold for $L' := \operatorname{Ker} f$ and $L'' := (\operatorname{Ker} g)^*$.

(4) [**IW**, 6.14] *If* $\mu_L(M)$ *and* $\nu_L(M)$ *are artinian mutations, then*

$$\mathrm{Hom}_R(\mu_L(M), M) \in \text{ref-tilt}\,\Lambda^{\mathrm{op}} \quad \textit{and} \quad \mathrm{Hom}_R(M, \nu_L(M)) \in \text{ref-tilt}\,\Lambda.$$

hold for $\Lambda = \mathrm{End}_R(M)$*, and the following sequences are exact.*

$$0 \to \mathrm{Hom}_R(L, M) \to \mathrm{Hom}_R(V, M) \to \mathrm{Hom}_R(K, M) \to 0,$$
$$0 \to \mathrm{Hom}_R(M, L) \to \mathrm{Hom}_R(M, U) \to \mathrm{Hom}_R(M, C) \to 0.$$

Now let $M = M_1 \oplus \ldots \oplus M_n \in \mathrm{modif}\, R$ be a basic modifying R-module such that each M_i is indecomposable. For simplicity, set

$$\mu_i(M) := \mu_{M_i}(M) \ \text{ and } \ \nu_i(M) := \nu_{M_i}(M),$$

and call them *simple mutation*. In the following case, simple mutation behaves nicely.

Proposition 8.6. [**IW**, 6.25] *Assume* $\dim R = 3$ *and* $M \in \mathrm{MM}R$*. Let* $M = M_1 \oplus \ldots \oplus M_n$ *with indecomposable* M_i*. For each* $1 \le i \le n$*, the following assertions hold.*

(1) $\nu_i(M) \cong \mu_i(M)$.
(2) $\nu_i(M)$ *and* $\mu_i(M)$ *are finite dimensional mutations if and only if* $\nu_i(M) \not\cong M$ *if and only if* $\mu_i(M) \not\cong M$*.*

In the setting of the above proposition, we define the exchange graph as follows.

Notation 8.7. The *exchange graph* of $\mathrm{MM}R$ is the graph where:

- The vertices are the elements of $\mathrm{MM}R$.
- For $M, N \in \mathrm{MM}R$, we draw an edge between M and N if they are can be obtained from each other by a simple mutation.

In the rest of this section, we assume $\dim R = 3$ in addition to the first assumptions that R is a local Gorenstein normal ring.

Proposition 8.8. [**IW**, 4.12] *Let* R *be a local Gorenstein normal ring with* $\dim R = 3$*. For each* $M \in \mathrm{MM}R$ *and* $N \in \mathrm{modif}\, R$*, there exists exact sequences*

$$0 \to M_1 \to M_0 \xrightarrow{f} N \quad \textit{and} \quad 0 \to N \xrightarrow{g} M^0 \to M^1 \tag{8.2.D}$$

with $M_i, M^i \in \mathrm{add}\, M$ *such that* f *is right minimal,* g *is left minimal and the following sequences are exact.*

$$0 \to \mathrm{Hom}_R(M, M_1) \to \mathrm{Hom}_R(M, M_0) \to \mathrm{Hom}_R(M, N) \to 0,$$
$$0 \to \mathrm{Hom}_R(M^1, M) \to \mathrm{Hom}_R(M^0, M) \to \mathrm{Hom}_R(N, M) \to 0.$$

Moreover, M_0 *and* M_1 *do not have non-zero common direct summands, and* M^0 *and* M^1 *do not have non-zero common direct summands.*

Proof. In [**IW**, 4.12], clearly we can choose the sequences such that f is right minimal and g is left minimal. The last assertion follows from 5.6. □

Notation 8.9. For a module-finite R-algebra Λ, write

$$\text{ref-ptilt}\,\Lambda := \{T \in \text{ptilt}\,\Lambda \mid T \in \text{ref}\, R\},$$
$$\text{ref-tilt}\,\Lambda := \{T \in \text{tilt}\,\Lambda \mid T \in \text{ref}\, R\}.$$

The following bijections are fundamental, and will be heavily used.

Proposition 8.10. [**IW**, 4.17] *Let* R *be a Gorenstein normal local domain such that* $\dim R = 3$*. For each* $M \in \mathrm{MM}R$*, there are bijections*

$$\mathrm{Hom}_R(M, -)\colon \ \mathrm{modif}\, R \xrightarrow{\sim} \text{ref-ptilt}\,\mathrm{End}_R(M) \tag{8.2.E}$$
$$\mathrm{Hom}_R(M, -)\colon \ \mathrm{MM}R \xrightarrow{\sim} \text{ref-tilt}\,\mathrm{End}_R(M). \tag{8.2.F}$$

8.3. Three General Results

This section contains three new general results, which for the most part are just mild extensions of known techniques. All are needed for our applications in Chapter 9.

The following result gives a characterization of artinian mutation. The first part justifies our abuse of notation of writing ν_i for simple mutation on modifying modules, and also writing ν_i for mutation on tilting modules.

THEOREM 8.11. *Let R be a complete local Gorenstein normal domain with $\dim R = 3$. Let $N = N_1 \oplus \ldots \oplus N_n \in \operatorname{modif} R$ be basic with indecomposable N_i, $\Lambda = \operatorname{End}_R(N)$, and let i be such that $1 \le i \le n$.*

(1) *The following conditions are equivalent.*
   (a) *$\nu_i(N)$ is an artinian mutation*
   (b) *$\nu_i(\Lambda) \in \operatorname{ref-tilt} \Lambda$.*
   (c) *$\nu_i(\Lambda) \cong \operatorname{Hom}_R(N, \nu_i(N))$.*

Assuming any of the equivalent conditions in (1), then the following holds.

(2) *Let $0 \to N_i \to U \xrightarrow{f} L$ be the exchange sequence. For any $M \in \operatorname{modif} R$ such that $N \in \operatorname{add} M$, the map f is a right $(\operatorname{add} M)$-approximation.*

We need the following preparation.

LEMMA 8.12. *If R is a complete local Gorenstein normal ring with $\dim R = 2$, and*

$$0 \to X \xrightarrow{a} Y \xrightarrow{b} Z \tag{8.3.A}$$

be an exact sequence with terms in $\operatorname{ref} R$ such that $b\colon \operatorname{Hom}_R(X,Y) \to \operatorname{Hom}_R(X,Z)$ is surjective. Then a is a split monomorphism.

PROOF. Write $X = X' \oplus P$, where P is the maximal projective direct summand of X. By our assumptions on R, Auslander–Reiten translation τ is the identity, and there is a functorial isomorphism $\operatorname{Ext}^1_R(L,M) \cong D\overline{\operatorname{Hom}}_R(M,L)$ for $L, M \in \operatorname{ref} R$, called *Auslander–Reiten duality*.

First, consider the case $P = 0$. For $Z' := \operatorname{Im} b$, there is an exact sequence

$$\operatorname{Hom}_R(X,Y) \to \operatorname{Hom}_R(X,Z') \to \operatorname{Ext}^1_R(X,X) \to \operatorname{Ext}^1_R(X,Y).$$

By our assumption, the left map is surjective and hence the right map is injective. By Auslander–Reiten duality, the map $\overline{\operatorname{Hom}}_R(Y,X) \to \overline{\operatorname{Hom}}_R(X,X)$ is surjective. Since X does not have a non-zero projective direct summand, the map $\operatorname{Hom}_R(Y,X) \to \operatorname{Hom}_R(X,X)$ is also surjective. Thus a is a split monomorphism.

Next, we consider the case $P \neq 0$. Then b must be surjective. Further, since $Z \in \operatorname{ref} R$ implies $\operatorname{Ext}^1_R(Z,R) = 0$, the sequence (8.3.A) is a direct sum of $0 \to P \xrightarrow{1} P \to 0$ and $0 \to X' \xrightarrow{a'} Y' \to Z$, where the latter sequence satisfies the same condition. By the first case, a' is a split monomorphism. Thus so is a. $\square$

With this preparation, we now prove 8.11.

PROOF OF 8.11. (1)(a)⇒(b) If $\nu_i(N)$ is an artinian mutation, then by 8.5(4), we obtain $\nu_i(\Lambda) \cong \operatorname{Hom}_R(N, \nu_i(N)) \in \operatorname{ref} \Lambda$.

(b)⇒(c) Assume $\nu_i(\Lambda) \in \operatorname{ref} \Lambda$. By construction [IW, 6.8], applying $\operatorname{Hom}_R(N,-)$ to the exchange sequence

$$0 \to N_i \xrightarrow{a} U \to L, \tag{8.3.B}$$

gives an exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow \Lambda e_i & \xrightarrow{f} & \operatorname{Hom}_R(N,U) & & \longrightarrow & & \operatorname{Hom}_R(N,L) \\ & & & \searrow & & \nearrow_{\alpha} & \\ & & & & C & & \end{array} \tag{8.3.C}$$

where f is a right $\operatorname{add}\Lambda(1-e_i)$-approximation, $\nu_i(\Lambda) = \Lambda(1-e_i) \oplus C$, and α is a height one isomorphism. Since $C \in \operatorname{ref} R$ by our assumption, α is thus a height one isomorphism between reflexive modules, and thus is an isomorphism.

(c)⇒(a) Assume $\nu_i(\Lambda) \cong \operatorname{Hom}_R(N, \nu_i(N))$, then in particular $\nu_i(\Lambda) \in \operatorname{ref} R$. Hence, by the above, applying $\operatorname{Hom}_R(N, -)$ to the exchange sequence gives an exact sequence

$$0 \to \operatorname{Hom}_R(N, N_i) \to \operatorname{Hom}_R(N, U) \to \operatorname{Hom}_R(N, L) \to 0. \tag{8.3.D}$$

Now we fix $\mathfrak{p} \in \operatorname{Spec} R$ with $\dim R_\mathfrak{p} = 2$. Localising (8.3.B) at $\mathfrak{p}$ gives an exact sequence

$$0 \to N_{i\mathfrak{p}} \xrightarrow{a_\mathfrak{p}} U_\mathfrak{p} \to L_\mathfrak{p}$$

with terms in $\operatorname{ref} R_\mathfrak{p}$, such that $\operatorname{Hom}_{R_\mathfrak{p}}(N_\mathfrak{p}, U_\mathfrak{p}) \twoheadrightarrow \operatorname{Hom}_{R_\mathfrak{p}}(N_\mathfrak{p}, L_\mathfrak{p})$. By 8.12, the map $a_\mathfrak{p}$ is a split monomorphism.

The sequence obtained by applying $\operatorname{Hom}_R(-, N_i)$ to (8.3.B) is isomorphic to the sequence obtained by applying $\operatorname{Hom}_\Lambda(-, \operatorname{Hom}_R(N, N_i))$ to (8.3.D), hence

$$\operatorname{Hom}_R(U, N_i) \xrightarrow{\operatorname{Hom}_R(a, N_i)} \operatorname{Hom}_R(N_i, N_i) \to \Lambda/(1-e_i) \to 0. \tag{8.3.E}$$

is exact. Since $a_\mathfrak{p}$ is a split monomorphism, $\operatorname{Hom}_{R_\mathfrak{p}}(a_\mathfrak{p}, (N_i)_\mathfrak{p})$ is a split epimorphism. Localising the sequence (8.3.E) at $\mathfrak{p}$ then implies that $(\Lambda/(1-e_i))_\mathfrak{p} = 0$, and so the support of the R-module $\Lambda/(1-e_i)$ contains only the maximal ideal of R. Consequently, it is an artinian ring.

(2) Applying $\operatorname{Hom}_\Lambda(\mathbb{F}M, -)$ to (8.3.D), where $\mathbb{F} = \operatorname{Hom}_R(N, -)$, then applying reflexive equivalence (8.2.A) gives an an exact sequence

$$0 \to \operatorname{Hom}_R(M, N_i) \to \operatorname{Hom}_R(M, U) \to \operatorname{Hom}_R(M, L) \to \operatorname{Ext}^1_\Lambda(\mathbb{F}M, \mathbb{F}N_i).$$

The last term vanishes since $\mathbb{F}M \in \operatorname{CM}\Lambda$ and $\mathbb{F}N_i \in \operatorname{proj}\Lambda$, so the assertion follows. □

LEMMA 8.13. *For any $N \in \operatorname{modif} R$ and $T \in \operatorname{ref\text{-}ptilt} \operatorname{End}_R(N)$, there exists $L \in \operatorname{modif} R$ such that $T = \operatorname{Hom}_R(N, L)$.*

PROOF. The assertion is clear for $N = 0$, so assume $N \neq 0$ and set $\Lambda = \operatorname{End}_R(N)$. Take $T \in \operatorname{ref\text{-}ptilt}\Lambda$. By Bongartz completion [**IR**, 2.8], there exists U such that $T \oplus U \in \operatorname{ref\text{-}tilt}\Lambda$. By reflexive equivalence (8.2.A), $\operatorname{Hom}_R(N, -)\colon \operatorname{ref} R \xrightarrow{\sim} \operatorname{ref}\Lambda$, so there exists $L' \in \operatorname{ref} R$ such that $T \oplus U = \operatorname{Hom}_R(N, L')$. Thus $\operatorname{End}_R(L') \cong \operatorname{End}_\Lambda(T \oplus U)$ is derived equivalent to Λ. Since modifying algebras are closed under derived equivalences [**IW**, 4.6(1)], we obtain $L' \in \operatorname{modif} R$. Again by reflexive equivalence, L' has a direct summand L such that $T = \operatorname{Hom}_R(N, L)$. □

Approximations behave well with respect to rank.

LEMMA 8.14. *Let R be a commutative normal ring, $M, N \in \operatorname{ref} R$ such that $f\colon M' \to N$ is a right $\operatorname{add} M$-approximation. Write $K := \operatorname{Ker} f$, then the following hold.*

1. *$0 \to K_\mathfrak{p} \to M'_\mathfrak{p} \to N_\mathfrak{p} \to 0$ is exact for all height one primes $\mathfrak{p} \in \operatorname{Spec} R$.*
2. *$\operatorname{rank}_R M' = \operatorname{rank}_R K + \operatorname{rank}_R N$.*

PROOF. (1) By definition there is an exact sequence

$$0 \to \operatorname{Hom}_R(M, K) \to \operatorname{Hom}_R(M, M') \to \operatorname{Hom}_R(M, N) \to 0 \tag{8.3.F}$$

Localising at a height one prime $\mathfrak{p}$ gives the following commutative diagram, where the top row is exact

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Hom}_R(M, K)_\mathfrak{p} & \longrightarrow & \operatorname{Hom}_R(M, M')_\mathfrak{p} & \longrightarrow & \operatorname{Hom}_R(M, N)_\mathfrak{p} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, K_\mathfrak{p}) & \longrightarrow & \operatorname{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, M'_\mathfrak{p}) & \longrightarrow & \operatorname{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) & \longrightarrow & 0
\end{array}$$

Hence the bottom row is exact. Since R is normal, $M_\mathfrak{p} \in \operatorname{add} R_\mathfrak{p}$, so the assertion follows.

(2) This is immediate from (1), after further localising to the zero ideal. □

K-theory can be used to detect partial tilting modules.

THEOREM 8.15. *Let R be a complete local Gorenstein normal domain with* $\dim R = 3$*, and* $M \in \operatorname{MM} R$*. Let* $N = N_1 \oplus \ldots \oplus N_t \in \operatorname{add} M$ *with indecomposable* N_i*, and* $\Gamma = \operatorname{End}_R(N)$*. Then for any* $X \in \operatorname{modif} R$*, the following conditions are equivalent.*

(1) $\operatorname{Hom}_R(N, X)$ *is a partial tilting* Γ*-module.*
(2) $[\operatorname{Hom}_R(M, X)]$ *belongs to the subgroup of* $\mathrm{K}_0(\operatorname{proj}\Gamma)$ *generated by the elements* $[\operatorname{Hom}_R(M, N_i)]$ *with* $1 \le i \le t$*.*

PROOF. (2)⇒(1) By (8.2.E) we know that $\operatorname{Hom}_R(M, X)$ is a partial tilting Γ-module. Thus there exists a projective resolution

$$0 \to P_1 \to P_0 \to \operatorname{Hom}_R(M, X) \to 0$$

such that P_0 and P_1 do not have common direct summands. Then $[\operatorname{Hom}_S(M, X)] = [P_0] - [P_1]$ holds in $\mathrm{K}_0(\operatorname{End}_S(M))$. Since $[\operatorname{Hom}_S(M, X)]$ belongs to the subgroup generated by $[\operatorname{Hom}_R(M, N_i)]$ with $1 \le i \le t$, and P_0 and P_1 do not share any common summands, it follows that both $[P_0]$ and $[P_1]$ also belong to the subgroup.

By projectivisation, for $i = 0, 1$, write $P_i := \operatorname{Hom}_R(M, M_i)$ with $M_i \in \operatorname{add} M$. Since $[P_0]$ and $[P_1]$ belong to the subgroup, M_0 and M_1 belong to $\operatorname{add} N$. Since the above projective resolution induces an exact sequence

$$0 \to \operatorname{Hom}_R(N, M_1) \to \operatorname{Hom}_R(N, M_0) \to \operatorname{Hom}_R(N, X) \to 0,$$

it follows again by projectivisation that $\operatorname{Hom}_R(N, X)$ has projective dimension at most one, as a Γ-module. As in [IW, §4], it is routine to check that it has no self $\operatorname{Ext}^1_\Gamma$'s.
(1)⇒(2) Write $\mathbb{F} = \operatorname{Hom}_R(N, -)$. By assumption $\mathbb{F}X$ is a partial tilting Γ-module, so in particular it has projective dimension at most one. Thus there is an exact sequence

$$0 \to \mathbb{F}M_1 \to \mathbb{F}M_0 \to \mathbb{F}X \to 0$$

for some $M_1, M_0 \in \operatorname{add} N$. Applying $\operatorname{Hom}_\Gamma(\mathbb{F}M, -)$, and dropping Hom from the notation, we obtain a commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow {}_\Gamma(\mathbb{F}M, \mathbb{F}M_1) & \longrightarrow & {}_\Gamma(\mathbb{F}M, \mathbb{F}M_0) & \longrightarrow & {}_\Gamma(\mathbb{F}M, \mathbb{F}X) & \longrightarrow & \operatorname{Ext}^1_\Gamma(\mathbb{F}M, \mathbb{F}M_1) \\
\cong \uparrow & & \cong \uparrow & & \cong \uparrow & & \\
0 \longrightarrow {}_R(M, M_1) & \longrightarrow & {}_R(M, M_0) & \longrightarrow & {}_R(M, X) & &
\end{array}$$

where the vertical isomorphisms are reflexive equivalence. Since N is a summand of $M \in \operatorname{MM} R$, we have $\mathbb{F}M \in \operatorname{CM} R$. Thus by [IR, 3.4(5)], since Γ is 3-sCY, $\operatorname{Ext}^1_\Gamma(\mathbb{F}M, \Gamma) \cong \operatorname{Ext}^1_R(\mathbb{F}M, R) = 0$, from which since $M_1 \in \operatorname{add} N$, $\operatorname{Ext}^1_\Gamma(\mathbb{F}M, \mathbb{F}M_1) = 0$ follows. Hence the top row of the above diagram is a short exact sequence, and hence so too is the bottom row. This clearly implies that $[\operatorname{Hom}_R(M, X)]$ belongs to the stated subgroup. ☐

8.4. Modifying Modules Modulo a Regular Element

We now apply the general 8.3 to the setting in §8.2 of modifying modules over Gorenstein local normal domains in dimension three, and obtain some general corollaries. These can, and will, be strengthened in the cDV setting in Chapter 9. Both here and later, restricting to subsets of $\operatorname{tilt}\Lambda$ that share common summands will be useful.

NOTATION 8.16. If A is a ring, and $P \in \operatorname{mod} A$, write $\operatorname{tilt}(A, P)$ for the subset of $\operatorname{tilt} A$ consisting of those T which satisfy $P \in \operatorname{add} T$.

THEOREM 8.17. *Let R be a complete local Gorenstein normal domain with* $\dim R = 3$*,* $M \in \operatorname{MM} R$*, and set* $\Lambda := \operatorname{End}_R(M)$*. Then for any* $0 \ne g \in R$*, the following hold.*

(1) *There is an injective map*

$$\overline{F} := \overline{\mathrm{Hom}_R(M,-)}\colon\ \mathrm{MM}R \to \mathrm{tilt}\,\overline{\Lambda},$$

which induces an injective map $\overline{F}\colon\ \mathrm{MMG}R \to \mathrm{tilt}(\overline{\Lambda}, \overline{FR})$.

(2) *If* $N \in \mathrm{MM}R$ *satisfies* $\nu_i(N) \not\cong N$, *then* $\overline{F(\nu_i(N))} \cong \nu_i(\overline{FN})$.

(3) *If* R *is an isolated singularity, then the map in* (1) *is compatible with mutation.*

PROOF. (1) By (8.2.F), there is a bijection

$$F := \mathrm{Hom}_R(M,-)\colon\ \mathrm{MM}R \xrightarrow{\sim} \text{ref-tilt}\,\Lambda.$$

Since g is a non-zero element in the domain R, it is a non-zerodivisor on any $N \in \mathrm{ref}\,R$ and also on $\mathrm{Hom}_R(M,N)$. Therefore ref-tilt $\Lambda \subseteq \mathrm{tilt}_g\,\Lambda$, so composing with the injective map

$$\overline{(-)}\colon\ \mathrm{tilt}_g\,\Lambda \hookrightarrow \mathrm{tilt}\,\overline{\Lambda}$$

in 8.3 gives an injective map $\overline{F}\colon\ \mathrm{MM}R \to \mathrm{tilt}\,\overline{\Lambda}$. This clearly induces an injective map $\overline{F}\colon\ \mathrm{MMG}R \to \mathrm{tilt}(\overline{\Lambda}, \overline{FR})$.
(2) Since both $\overline{F(\nu_i(N))}$ and $\nu_i(\overline{FN})$ are tilting modules, distinct from $\overline{FN}$, and they have the same indecomposable direct summands except one, they must be isomorphic by 5.7.
(3) If R has an isolated singularity, then $\nu_i(N) \not\cong N$ holds by [**IW**, 6.22]. Thus (3) follows immediately from (2). □

Immediately we obtain the following corollary, which reduce many questions regarding the representation theory of CM and reflexive modules to the tilting theory of an algebra one dimension lower.

COROLLARY 8.18. *In 8.17, assume further that* R *is an isolated singularity.*

(1) *If the exchange graph of* $\mathrm{tilt}\,\overline{\Lambda}$ *is connected, then* $\overline{F}\colon\ \mathrm{MM}R \to \mathrm{tilt}\,\overline{\Lambda}$ *is bijective.*
(2) *If the exchange graph* $\mathrm{tilt}(\overline{\Lambda}, \overline{FR})$ *is connected, then* $\overline{F}\colon\ \mathrm{MMG}R \to \mathrm{tilt}(\overline{\Lambda}, \overline{FR})$ *is bijective.*

PROOF. (1) By 8.17(1) it suffices to prove surjectivity. Let $T \in \mathrm{tilt}\,\overline{\Lambda}$. By our assumption, there is a finite sequence of mutations such that $T = \nu_{i_\ell} \dots \nu_{i_1}(\overline{\Lambda})$. Then $N := \nu_{i_\ell} \dots \nu_{i_1}(M)$ satisfies $\overline{FN} = T$ by 8.17(2)(3).
(2) The proof is very similar to (1). □

The following criterion for connectedness of the exchange graphs plays a key role in the next chapter.

PROPOSITION 8.19. *In 8.17, assume that there is no* $T \in \mathrm{tilt}\,\overline{\Lambda}$ *and an infinite sequence* $\overline{\Lambda} = T_0 > T_1 > T_2 > \dots$ *such that* $T_i \geq T$ *for all* $i \geq 0$. *Then the exchange graph of* $\mathrm{MM}R$ *is connected.*

PROOF. Fix $M, N \in \mathrm{MM}R$. We will find a sequence of mutation from M to N. Let $\Lambda = \mathrm{End}_R(M)$ and $F := \mathrm{Hom}_R(M,-)\colon\ \mathrm{MM}R \to \mathrm{tilt}\,\Lambda$. Then $FN \in$ ref-tilt Λ. Applying 5.9 to $\Lambda \geq FN$ repeatedly, we obtain a sequence of mutations

$$\Lambda = U_0 > U_1 > U_2 > \dots \tag{8.4.A}$$

such that

(1) $U_i \geq FN$ for each $i \geq 0$.
(2) Either there exists $\ell \geq 0$ such that $U_\ell = FN$, or the sequence is infinite.

By 5.8, the first condition implies that there exists an exact sequence

$$0 \to U_i \to V^0 \to V^1 \to 0$$

with $V^0, V^1 \in \operatorname{add} FN$. Since $FN \in \text{ref-tilt}\,\Lambda$, and reflexive modules are closed under kernels, it follows that $U_i \in \text{ref-tilt}\,\Lambda \subset \text{tilt}_g\,\Lambda$. By 8.3(3), applying $\overline{(-)}$ gives a sequence of mutations

$$\overline{\Lambda} = \overline{U}_0 > \overline{U}_1 > \overline{U}_2 > \dots$$

such that $\overline{T}_i \geq \overline{FN}$ for each $i \geq 0$. By our assumption, the sequence has to be finite, and hence $U_\ell = FN$ holds for some $\ell \geq 0$. Since $U_i \in \text{ref-tilt}\,\Lambda$ for each $i \geq 0$, by 8.17(2) the sequence (8.4.A) corresponds to a sequence of mutations in $\operatorname{MM} R$ from M to N. ☐

For a commutative ring R and $g \in R$, recall $\overline{(-)} = \overline{R} \otimes_R -$. Consider the functor

$$(-)^\star := \operatorname{Hom}_{\overline{R}}(-, \overline{R})\colon \operatorname{mod} \overline{R} \to \operatorname{mod} \overline{R}$$

and the evaluation map $\varepsilon_X\colon X \to X^{\star\star}$ for each $X \in \operatorname{mod} \overline{R}$. The following result will play a key role in the next charter.

Proposition 8.20. *Let R be a noetherian local ring with $\dim R = 3$, and $g \in R$ a non-zerodivisor on R such that $\overline{R}$ is a normal domain. For $M \in \operatorname{ref} R$, the following hold.*

(1) $\overline{M}^{\star\star} \in \operatorname{CM} \overline{R}$.
(2) $\varepsilon_{\overline{M}}$ *is injective.*
(3) $(\varepsilon_{\overline{M}})_{\mathfrak{p}}$ *is an isomorphism for all non-maximal prime ideals* $\mathfrak{p} \in \operatorname{Spec} \overline{R}$.
(4) *If* $M \in \operatorname{modif} R$, *then there is a canonical isomorphism* $\overline{\operatorname{End}_R(M)} \cong \operatorname{End}_{\overline{R}}(\overline{M}^{\star\star})$ *of* $\overline{R}$*-algebras.*

Proof. (1) Since $\overline{R}$ is normal, $\overline{M}^{\star\star} \in \operatorname{ref} \overline{R}$. Further, since g is a non-zerodivisor on R, clearly $\dim \overline{R} = 2$. By Serre's (S_2) criteria, normal surfaces are automatically CM, and so $\operatorname{ref} \overline{R} = \operatorname{CM} \overline{R}$.
(2) Since $M \in \operatorname{ref} R$, by applying $(-)^* = \operatorname{Hom}_R(-, R)$ to a projective presentation of M^*, and splicing, gives exact sequences

$$0 \to M \to F_0 \to N \to 0 \quad \text{and} \quad 0 \to N \to F_1$$

with $F_i \in \operatorname{proj} R$. Since g is a non-zerodivisor on R, the second sequence and 8.2 implies that $\operatorname{Tor}_1^R(\overline{R}, N) = 0$. Applying $\overline{(-)}$ to the first sequence then gives an exact sequence

$$0 = \operatorname{Tor}_1^R(\overline{R}, N) \to \overline{M} \xrightarrow{a} \overline{F}_0 \to \overline{N} \to 0. \tag{8.4.B}$$

Since a is injective, and the evaluation gives the commutative diagram

$$\begin{array}{ccc} \overline{M} & \xrightarrow{\quad a \quad} & \overline{F}_0 \\ {\scriptstyle \varepsilon_{\overline{M}}}\downarrow & & \downarrow{\scriptstyle \varepsilon_{\overline{F}_0}}\ \wr \\ \overline{M}^{\star\star} & \xrightarrow{\quad a^{\star\star} \quad} & \overline{F}_0^{\star\star}. \end{array}$$

it follows that $\varepsilon_{\overline{M}}$ is injective.
(3) For each non-maximal prime ideal $\mathfrak{p}$ of $\overline{R}$, since $\overline{R}_{\mathfrak{p}}$ is regular, the sequence (8.4.B) shows that $\overline{M}_{\mathfrak{p}} \in \operatorname{proj} \overline{R}_{\mathfrak{p}}$ and hence $(\varepsilon_{\overline{M}})_{\mathfrak{p}}$ is an isomorphism. .
(4) Since g is a non-zerodivisor on R, it is also a non-zerodivisor on $M \in \operatorname{ref} R$. Thus $0 \to M \xrightarrow{g} M \to \overline{M} \to 0$ is exact. Applying $\operatorname{Hom}_R(M, -)$ and using $\operatorname{Hom}_R(M, \overline{M}) \cong \operatorname{End}_R(\overline{M}) \cong \operatorname{End}_{\overline{R}}(\overline{M})$ gives an exact sequence

$$0 \to \operatorname{End}_R(M) \xrightarrow{g} \operatorname{End}_R(M) \to \operatorname{End}_{\overline{R}}(\overline{M})$$

and thus an injective morphism of rings

$$\phi\colon \overline{\operatorname{End}_R(M)} \to \operatorname{End}_{\overline{R}}(\overline{M}).$$

We now show that $\phi_{\mathfrak{p}}$ is an isomorphism for each non-maximal $\mathfrak{p} \in \operatorname{Spec} \overline{R}$. Take $\mathfrak{q} \in \operatorname{Spec} R$ such that $\mathfrak{p} = \mathfrak{q}/gR$. Then $R_{\mathfrak{q}}/gR_{\mathfrak{q}} = \overline{R}_{\mathfrak{p}}$ is regular since $\overline{R}$ is normal and $\dim \overline{R} = 2$. Since g is a non-zerodivisor on $R_{\mathfrak{q}}$, we have $\dim R_{\mathfrak{q}}/gR_{\mathfrak{q}} = \dim R_{\mathfrak{q}} - 1$. Thus

$R_{\mathfrak{q}}$ is also regular, by inspection of the minimal number of generators of the maximal ideal $\mathfrak{q}R_{\mathfrak{q}}$. In particular, $M_{\mathfrak{q}} \in \operatorname{ref} R_{\mathfrak{q}} = \operatorname{proj} R_{\mathfrak{q}}$. Thus

$$\overline{\operatorname{End}_R(M)}_{\mathfrak{p}} = \operatorname{End}_{R_{\mathfrak{q}}}(M_{\mathfrak{q}})/g\operatorname{End}_{R_{\mathfrak{q}}}(M_{\mathfrak{q}}) \cong \operatorname{End}_{R_{\mathfrak{q}}/gR_{\mathfrak{q}}}(M_{\mathfrak{q}}/gM_{\mathfrak{q}}) = \operatorname{End}_{\overline{R}}(\overline{M})_{\mathfrak{p}},$$

where the middle isomorphism holds since $M_{\mathfrak{q}} \in \operatorname{proj} R_{\mathfrak{q}}$.

On the other hand, consider the canonical morphism of rings

$$\psi\colon \operatorname{End}_{\overline{R}}(\overline{M}) \to \operatorname{End}_{\overline{R}}(\overline{M}^{\star\star}).$$

Each $b \in \operatorname{End}_{\overline{R}}(\overline{M})$ gives a commutative diagram

$$\begin{array}{ccc} \overline{M} & \xrightarrow{\quad b \quad} & \overline{M} \\ {\scriptstyle \varepsilon_{\overline{M}}}\downarrow & & \downarrow{\scriptstyle \varepsilon_{\overline{M}}} \\ \overline{M}^{\star\star} & \xrightarrow{\quad b^{\star\star} \quad} & \overline{M}^{\star\star}. \end{array}$$

Since $\varepsilon_{\overline{M}}$ is injective by (2), $b^{\star\star} = 0$ implies $b = 0$, so ψ is injective. For each non-maximal $\mathfrak{p} \in \operatorname{Spec} \overline{R}$, $(\varepsilon_{\overline{M}})_{\mathfrak{p}}$ is an isomorphism by (1), and hence $\psi_{\mathfrak{p}}$ is also an isomorphism.

Consequently, the composition

$$\psi \circ \phi\colon \overline{\operatorname{End}_R(M)} \to \operatorname{End}_{\overline{R}}(\overline{M}^{\star\star})$$

is injective and $(\psi \circ \phi)_{\mathfrak{p}}$ is an isomorphism for each non-maximal $\mathfrak{p} \in \operatorname{Spec} \overline{R}$. Thus $\operatorname{Cok}(\psi \circ \phi) \in \operatorname{fd} \overline{R}$. Since $M \in \operatorname{modif} R$, it follows that $\overline{\operatorname{End}_R(M)} \in \operatorname{CM} \overline{R}$. Further, since $\overline{M}^{\star\star} \in \operatorname{CM} \overline{R}$ by (1), and $\overline{R}$ is a normal surface, $\operatorname{End}_{\overline{R}}(\overline{M}^{\star\star}) \in \operatorname{CM} \overline{R}$. In particular, both $\overline{\operatorname{End}_R(M)}$ and $\operatorname{End}_{\overline{R}}(\overline{M}^{\star\star})$ have depth two. Thus $\operatorname{Cok}(\psi \circ \phi) = 0$ holds, as desired. □

8.5. Summary of Notation

In what follows, all modules are finitely generated, Λ is a module-finite R-algebra, and whenever *basic* is mentioned, implicitly R will be complete local.

Notation	Reference	Meaning
$\operatorname{CM} R$	§8.2	(Maximal) Cohen-Macaulay R-modules
$\operatorname{ref} \Lambda$	§8.2	Λ-modules which are reflexive as R-modules.
$\operatorname{modif} R$	§8.2	Modifying reflexive R-modules.
$\operatorname{MM} R$	§8.2	Maximal modifying reflexive R-modules.
$\operatorname{MMG} R$	§8.2	Those $X \in \operatorname{MM} R$ such that $R \in \operatorname{add} X$.
$\nu_L(M)$	§8.2	Left mutation of M at L.
$\mu_L(M)$	§8.2	Right mutation of M at L.
$\operatorname{tilt} \Lambda$	5.4	Basic tilting Λ-modules of projective dimension one.
$\operatorname{ptilt} \Lambda$	5.4	Not-necessarily-basic partial tilting Λ-modules.
$\operatorname{tilt}(\Lambda, P)$	8.16	Basic tilting Λ-modules containing fixed projective P.
$\operatorname{tilt}_g \Lambda$	8.1	Those elements of $\operatorname{tilt} \Lambda$ where g acts a non-zerodivisor
$2\operatorname{tilt} \Lambda$	6.1	Two-term tilting complexes for Λ
$\operatorname{ref-ptilt} \Lambda$	8.9	Elements in $(\operatorname{ref} \Lambda) \cap (\operatorname{ptilt} \Lambda)$. These need not be basic.
$\operatorname{ref-tilt} \Lambda$	8.9	Elements in $(\operatorname{ref} \Lambda) \cap (\operatorname{tilt} \Lambda)$. These are basic.

CHAPTER 9

Modifying Modules on cDV Singularities

Throughout this chapter, let $\mathcal{R}$ be a complete *cDV singularity*, that is,

$$\mathcal{R} = \Bbbk[[x,y,z,t]]/(f(x,y,z) + tg(x,y,z,t))$$

for some simple surface singularity f and arbitrary g. Note that $\mathcal{R}$ is normal if $\Bbbk$ is perfect; see 9.2 below. We write $\operatorname{modif}\mathcal{R}$ for the set of isomorphism classes of not-necessarily-basic modifying $\mathcal{R}$-modules, and $\operatorname{MM}\mathcal{R}$ (respectively, $\operatorname{MMG}\mathcal{R}$) for the set of isomorphism classes of basic maximal modifying $\mathcal{R}$-modules (respectively, generators of $\mathcal{R}$). By [B2, VII.4.7], there is an isomorphism

$$\mathbb{Z} \oplus \operatorname{Cl}(\mathcal{R}) = \operatorname{K}_0(\operatorname{mod}\mathcal{R})/\langle [X] \mid \dim_{\mathcal{R}} X \le 1\rangle. \tag{9.0.A}$$

In particular, there is a natural map

$$\operatorname{mod}\mathcal{R} \to \mathbb{Z} \oplus \operatorname{Cl}(\mathcal{R}). \tag{9.0.B}$$

Below we will introduce the *index* as the composition

$$\operatorname{ind}\colon \operatorname{mod}\mathcal{R} \xrightarrow{[-]} \operatorname{K}_0(\operatorname{mod}\mathcal{R}) \xrightarrow{(9.0.B)} \mathbb{Z} \oplus \operatorname{Cl}(\mathcal{R}), \tag{9.0.C}$$

which sends M to $(\operatorname{rank}_{\mathcal{R}} M, \det M)$, and consider the submonoid of $\mathbb{Z}\oplus\operatorname{Cl}(\mathcal{R})$ defined by

$$(\mathbb{Z} \oplus \operatorname{Cl}(\mathcal{R}))_+ := (\mathbb{Z}_{>0} \oplus \operatorname{Cl}(\mathcal{R})) \cup \{0\}.$$

The purpose of this chapter is to prove the following result, in the setting of isolated cDV singularities.

THEOREM 9.1. *Let $\mathcal{R}$ be an isolated cDV singularity.*

(1) *There exists $N \in \operatorname{MM}\mathcal{R}$, say $N = \bigoplus_{i=1}^n N_i$.*
(2) *There is an isomorphism $\operatorname{Cl}(\mathcal{R}) \cong \mathbb{Z}^{n-1}$.*
(3) *The map (9.0.C) restricts to a bijection $\operatorname{modif}\mathcal{R} \cong (\mathbb{Z}\oplus\operatorname{Cl}(\mathcal{R}))_+$.*
(4) *There exists an extended Dynkin diagram Δ_{aff}, a subset $\mathcal{J}$, and bijections*

$$\operatorname{modif}\mathcal{R} \cong L_{\mathcal{J}}^+ \quad \text{and} \quad \operatorname{MM}\mathcal{R} \cong \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J}).$$

(5) *$\#(\operatorname{modif}\mathcal{R} \cap \operatorname{ind}\operatorname{CM}\mathcal{R}) < \infty$ and $\#\operatorname{MMG}\mathcal{R} < \infty$.*

The proof is split across the following subsections. Many of the preparatory results do not require the assumption that $\mathcal{R}$ is not isolated, and so in what follows, we do *not* assume that $\mathcal{R}$ has an isolated singularity unless explicitly stated otherwise. For example, in the case $g = 0$, $\mathcal{R} = \Bbbk[[x,y,z,t]]/(f)$ is a non-isolated cDV singularity.

9.1. Existence of Maximal modifying modules

Throughout this subsection, let $\mathcal{R}$ be an arbitrary cDV singularity.

PROPOSITION 9.2. *When k is a perfect field, $\mathcal{R}$ is a normal domain.*

PROOF. Let $\mathcal{R} = S/hS$ be a cDV singularity for $S = \Bbbk[[x,y,z,t]]$. Since $\mathcal{R}$ is Cohen-Macaulay, by Serre's normality criterion, it suffices to show $\dim(\operatorname{Sing}\mathcal{R}) \le 1$. By the Jacobian criterion [E, 16.20], $\operatorname{Sing}\mathcal{R} = V(J)\cap\operatorname{Spec}\mathcal{R} = \operatorname{Spec} A$ where J is the Jacobi ideal and $A := S/(J+hS)$. Again by the Jacobian criterion, $\operatorname{Sing}(\mathcal{R}/t\mathcal{R}) = V(J)\cap\operatorname{Spec}(\mathcal{R}/t\mathcal{R}) = \operatorname{Spec}(A/tA)$ and thus

$$\dim(\operatorname{Sing}\mathcal{R}) - \dim(\operatorname{Sing}(\mathcal{R}/t\mathcal{R})) = \dim A - \dim(A/tA) \le 1.$$

Since $\mathcal{R}/t\mathcal{R}$ is a simple singularity, $\dim(\mathrm{Sing}(\mathcal{R}/t\mathcal{R})) = 0$, which proves the assertion. □

For a cDV singularity $\mathcal{R}$, we take a hyperplane $g \in \mathcal{R}$ such that

$$\overline{\mathcal{R}} := \mathcal{R}/g\mathcal{R}$$

is a simple singularity of type Δ. Let Δ_{aff} be the extended Dynkin graph as above, and $\delta = \sum_{i\in\Delta_{\mathsf{aff}}} \delta_i\alpha_i$ the imaginary root, where $\{\alpha_i^* \mid i \in \Delta_{\mathsf{aff}}\}$ and $\{\alpha_i \mid i \in \Delta_{\mathsf{aff}}\}$ are the dual basis. We regard δ as a map $\Theta \to \mathbb{R}$. The following is well-known by McKay Correspondence, where $\mathsf{indec}\,\mathcal{C}$ denotes the set of indecomposable objects of a category $\mathcal{C}$.

PROPOSITION 9.3. *The following assertions hold.*

(1) *There exists a bijection* $\Delta_{\mathsf{aff}} \xrightarrow{\sim} \mathsf{indec}(\mathrm{CM}\,\overline{\mathcal{R}})$ *given by* $i \mapsto N_i$, *such that* $\mathrm{rank}_{\overline{\mathcal{R}}} N_i = \delta_i$ *for all* $i \in \Delta_{\mathsf{aff}}$.
(2) *There is an algebra isomorphism* $\Pi \cong \mathrm{End}_{\overline{\mathcal{R}}}(\bigoplus_{i\in\Delta_{\mathsf{aff}}} N_i)$, *where* Π *is the preprojective algebra of type* Δ_{aff}.

PROOF. See e.g. [AV, 1.11], [LW, 6.31]. □

Consider $\overline{(-)} := \overline{\mathcal{R}} \otimes_{\mathcal{R}} -$. The following result is the starting point of our study of $\mathcal{R}$.

PROPOSITION 9.4. *Under the above setting, let* $M \in \mathrm{modif}\,\mathcal{R}$. *Then the following statements hold.*

(1) *There exist a subset* $\mathcal{J}_M \subseteq \Delta_{\mathsf{aff}}$ *and an isomorphism of algebras:*
$$\overline{\mathrm{End}_{\mathcal{R}}(M)} \cong \Gamma_{\mathcal{J}_M}.$$
(2) *There exist a natural bijection from* $\Delta_{\mathsf{aff}} \setminus \mathcal{J}_M$ *to the isomorphism classes of indecomposable direct summands of* M.
(3) *Let* M_i *be an indecomposable direct summand of* M *corresponding to* $i \in \Delta_{\mathsf{aff}} \setminus \mathcal{J}_M$ *via the bijection in* (2). *Then* $\mathrm{rank}_{\mathcal{R}} M_i = \delta_i$.

PROOF. (1) Let $(-)^{\star} = \mathrm{Hom}_{\overline{\mathcal{R}}}(-, \overline{\mathcal{R}})$. By 8.20(4), $\overline{\mathrm{End}_{\mathcal{R}}(M)} \cong \mathrm{End}_{\overline{\mathcal{R}}}(\overline{M}^{\star\star})$. By 8.20(1), $\overline{M}^{\star\star} \in \mathrm{CM}\,\overline{\mathcal{R}}$ holds. Thus the assertion follows from 9.3.
(2) Let $E := \mathrm{End}_{\mathcal{R}}(M)$. By projectivisation there is a bijection

$$\mathsf{indec}(\mathrm{add}\,M) \simeq \mathsf{indec}(\mathrm{proj}\,E).$$

On the other hand, any maximal left ideal I of E contains gE by Nakayama's Lemma for $\mathcal{R}$-modules. Thus gE belongs to the radical of E, and there is a bijection

$$\mathsf{indec}(\mathrm{proj}\,E) \simeq \mathsf{indec}(\mathrm{proj}\,\overline{E}).$$

The righthand side corresponds bijectively with $\Delta_{\mathsf{aff}} \setminus \mathcal{J}_M$. Composing the above bijections gives the desired one.
(3) This is immediate since $\mathrm{rank}_{\mathcal{R}} M_i = \mathrm{rank}_{\overline{\mathcal{R}}} \overline{M_i}$, and this equals δ_i by 9.3. □

The following result follows immediately from 9.4(2). This was shown in [W2, 1.12] by using minimal models of $\mathrm{Spec}\,\mathcal{R}$.

THEOREM 9.5. *Let* $\mathcal{R}$ *be a cDV singularity of type* Δ. *Then any modifying* $\mathcal{R}$*-module has at most* $1 + |\Delta|$ *non-isomorphic indecomposable direct summands. In particular,* $\mathcal{R}$ *has a maximal modifying generator.*

9.2. Modifying Modules and Chambers

Let $\mathcal{R}$ be an arbitrary complete local cDV singularity, and let $M \in \mathrm{MM}\mathcal{R}$, which exists by 9.5. Since $\mathcal{R}$ is a three-dimensional Gorenstein normal domain, recall from (8.2.E) and (8.2.F) that there are bijections

$$\mathrm{Hom}_{\mathcal{R}}(M, -)\colon \mathrm{modif}\,\mathcal{R} \xrightarrow{\sim} \text{ref-ptilt}\,\mathrm{End}_{\mathcal{R}}(M) \tag{9.2.A}$$

$$\mathrm{Hom}_{\mathcal{R}}(M, -)\colon \mathrm{MM}\mathcal{R} \xrightarrow{\sim} \text{ref-tilt}\,\mathrm{End}_{\mathcal{R}}(M). \tag{9.2.B}$$

These bijections hold for general three dimensional Gorenstein normal domains admitting a maximal modifying module. The point here is that cDV singularities $\mathcal{R}$ have much more structure, and we now extend these bijections, and describe them using Coxeter combinatorics.

As above, choose a hyperplane $g \in \Bbbk[[x,y,z,t]]$ such that $\mathcal{R}/g\mathcal{R}$ is a simple singularity of type Δ, and write $\overline{\mathcal{R}} := \mathcal{R}/(g)$ and $\overline{(-)} := \overline{\mathcal{R}} \otimes_{\mathcal{R}} -$. From 9.4 recall that:

- There is a bijection $\mathsf{indec}(\mathrm{CM}\,\overline{\mathcal{R}}) \xrightarrow{\sim} \Delta_{\mathsf{aff}}$ for some extended Dynkin Δ,
- There exists a subset $\mathcal{J}_M$ of Δ_{aff} such that $\overline{\mathrm{End}_{\mathcal{R}}(M)} \cong \Gamma_{\mathcal{J}_M}$.

Then, using 8.3, it follows that there are injections:

$$\overline{(-)}\colon \text{ref-ptilt}\,\mathrm{End}_{\mathcal{R}}(M) \to \text{ptilt}\,\Gamma_{\mathcal{J}_M} \tag{9.2.C}$$

$$\overline{(-)}\colon \text{ref-tilt}\,\mathrm{End}_{\mathcal{R}}(M) \to \text{tilt}\,\Gamma_{\mathcal{J}_M}. \tag{9.2.D}$$

Combining these facts, we immediately obtain the following crucial result.

THEOREM 9.6. *Let $\mathcal{R}$ be cDV, $M \in \mathrm{MM}\mathcal{R}$ and set $\overline{\mathbb{F}}_M = \overline{\mathrm{Hom}_{\mathcal{R}}(M,-)}$.*

(1) *For $P := \overline{\mathbb{F}}_M R$, there are injections*

$$\overline{\mathbb{F}}_M\colon \text{modif}\,\mathcal{R} \to \text{ptilt}\,\Gamma_{\mathcal{J}_M} \tag{9.2.E}$$

$$\overline{\mathbb{F}}_M\colon \mathrm{MM}\mathcal{R} \to \text{tilt}\,\Gamma_{\mathcal{J}_M} \tag{9.2.F}$$

$$\overline{\mathbb{F}}_M\colon \text{modif}\,\mathcal{R} \cap \mathrm{CM}\,\mathcal{R} \to \text{ptilt}(\Gamma_{\mathcal{J}_M}, P) \tag{9.2.G}$$

$$\overline{\mathbb{F}}_M\colon \mathrm{MMG}\mathcal{R} \to \text{tilt}(\Gamma_{\mathcal{J}_M}, P). \tag{9.2.H}$$

(2) *Let $L, N \in \mathrm{MM}R$ with $L \ncong N$. Then L and N are related by a simple mutation if and only if the tilting modules $\overline{\mathbb{F}}_M L$ and $\overline{\mathbb{F}}_M N$ are related by a simple tilting mutation.*

(3) *The exchange graphs of $\mathrm{MM}\mathcal{R}$ and $\mathrm{MMG}\mathcal{R}$ are connected.*

(4) *If $\mathcal{R}$ is an isolated singularity, then all the maps in (1) are bijective.*

PROOF. (1) This is now immediate, by composing (9.2.A) and (9.2.C), respectively (9.2.B) and (9.2.D).
(2)(⇒) follows using 8.17(2). (⇐) follows by injectivity of the map in (1). Indeed, suppose that $\overline{\mathbb{F}}_M L$ and $\overline{\mathbb{F}}_M N$ share all summands except one. Since (9.2.D) is injective, $\mathrm{Hom}_{\mathcal{R}}(M,L)$ and $\mathrm{Hom}_{\mathcal{R}}(M,N)$ share all summands except one. Given this, the fact that L and N are linked by a simple mutation is [**IW2**, 4.5(2)].
(3) We apply the criterion 8.19 for connectedness, namely there is no $T \in \text{tilt}\,\Gamma_{\mathcal{J}_M}$ and an infinite sequence $\Gamma_{\mathcal{J}_M} = T_0 > T_1 > T_2 > \dots$ such that $T_i \geq T$ for all $i \geq 0$. Since each T_i corresponds to a chamber, this follows using exactly the same argument as in 5.23.
(4) The exchange graph of tilt $\Gamma_{\mathcal{J}_M}$ is connected by 5.2. Thus all the assertion follow from 8.17(1). □

To apply our results in tilting theory of contracted preprojective algebras, recall from 5.26 that there is a natural bijection

$$\beta_{\mathcal{J}_M}\colon \mathrm{K}_0(\text{proj}\,\Gamma_{\mathcal{J}_M}) \xrightarrow{\sim} L_{\mathcal{J}_M}.$$

The following *index* allows us to assign to every modifying $\mathcal{R}$-module a point in a combinatorial lattice described using the Coxeter combinatorics of Part 1.

DEFINITION 9.7. For $M \in \mathrm{MM}\mathcal{R}$, write ind_M for the composition of bijections

$$\text{modif}\,\mathcal{R} \xrightarrow[(9.2.E)]{\overline{\mathrm{Hom}_{\mathcal{R}}(M,-)}} \text{ptilt}\,\Gamma_{\mathcal{J}_M} \xrightarrow[(5.26)]{\beta_{\mathcal{J}_M}\circ[-]} L^{+}_{\mathcal{J}_M}.$$

For $N = N_1 \oplus \ldots \oplus N_\ell \in \operatorname{modif} \mathcal{R}$, let

$$\mathrm{C}_M(N) := \sum_{i=1}^{\ell} \mathbb{R}_{>0}(\operatorname{ind}_M N_i) \subset \Theta_{\mathcal{J}_M}.$$

The following is the main result of this section. The injections in part (2) are known by [W2], but here we reprove it without using any birational geometry. All other parts are new, and should be viewed as the providing the *affine* version of the Auslander–McKay correspondence in [W2].

COROLLARY 9.8. *Let $\mathcal{R}$ be cDV of type Δ, and fix $M \in \operatorname{MM}\mathcal{R}$. Then the following assertions hold.*

(1) *There exists a subset $\mathcal{J}_M \subseteq \Delta$ and injective maps*

$$\operatorname{ind}_M\colon \operatorname{modif}\mathcal{R} \hookrightarrow L^+_{\mathcal{J}_M} \quad \text{and} \quad \mathrm{C}_M\colon \operatorname{MM}\mathcal{R} \hookrightarrow \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J}_M).$$

Furthermore, if $N = N_1 \oplus \ldots \oplus N_n \in \operatorname{MM}\mathcal{R}$ is basic with indecomposables N_i, then $\operatorname{ind}_M N_1, \ldots, \operatorname{ind}_M N_n$ is a basis of $\overline{\mathrm{C_M(N)}} \cap L_{\mathcal{J}_M}$.

(2) *If M has R as a direct summand, then the maps above restrict to injective maps*

$$\operatorname{ind}_M\colon \operatorname{modif}\mathcal{R} \cap \operatorname{CM}\mathcal{R} \hookrightarrow L^+_{\Delta,\mathcal{J}_M} \quad \text{and} \quad \mathrm{C}_M\colon \operatorname{MMG}\mathcal{R} \hookrightarrow \mathsf{Cham}(\Delta, \mathcal{J}_M).$$

In particular, $\#(\operatorname{modif}\mathcal{R} \cap \mathsf{indec}\operatorname{CM}\mathcal{R}) < \infty$ and $\#\operatorname{MMG}\mathcal{R} < \infty$.

(3) *Let $L, N \in \operatorname{MM}R$ with $L \ncong N$. Then L and N linked by a simple mutation if and only if $\mathrm{C}_M(L)$ and $\mathrm{C}_M(N)$ are linked by a simple wall crossing.*

(4) *If $\mathcal{R}$ has an isolated singularity, then the maps in (1) and (2) are bijective.*

PROOF. All the assertions follow from 9.6 and 5.3. □

For a fixed $M \in \operatorname{MM}\mathcal{R}$, the following property of the index ind_M is elementary.

PROPOSITION 9.9. *Let $N \in \operatorname{modif}\mathcal{R}$, and consider $0 \to M_1 \to M_0 \to N$, namely the exact sequence (8.2.D). Then*

$$\operatorname{ind}_M N = \operatorname{ind}_M M_0 - \operatorname{ind}_M M_1. \tag{9.2.I}$$

PROOF. Since $0 \to \operatorname{Hom}_{\mathcal{R}}(M, M_1) \to \operatorname{Hom}_{\mathcal{R}}(M, M_0) \to \operatorname{Hom}_{\mathcal{R}}(M, N) \to 0$ in 8.8 is exact, this is clear. □

Let $\delta = \sum_{i \in \Delta_{\mathsf{aff}}} \delta_i \alpha_i$ be the imaginary root for the extended Dynkin graph Δ_{aff}, where $\{\alpha_i^* \mid i \in \Delta_{\mathsf{aff}}\}$ and $\{\alpha_i \mid i \in \Delta_{\mathsf{aff}}\}$ are the dual basis. We regard δ as a map $\Theta_{\mathsf{aff}} \to \mathbb{R}$.

PROPOSITION 9.10. *For $M \in \operatorname{MM}\mathcal{R}$, the following statements hold.*

(1) *If $N \in \operatorname{modif}\mathcal{R}$, then $\operatorname{rank}_{\mathcal{R}} N = \delta(\operatorname{ind}_M N)$.*

(2) *For each positive integer r, there is an injection*

$$\operatorname{ind}_M\colon \{X \in \operatorname{modif}\mathcal{R} \mid \operatorname{rank} X = r\} \to L_{\mathcal{J}_M} \cap \delta^{-1}(r).$$

In particular, there is an injection $\operatorname{ind}_M\colon \operatorname{Cl}(\mathcal{R}) \to L_{\mathcal{J}_M} \cap \mathsf{Level}$.

(3) *Assume that $\mathcal{R}$ has an isolated singularity. Then the maps in (2) are bijective. Moreover, $L_{\mathcal{J}_M} = \operatorname{ind}_M(\operatorname{Cl}(\mathcal{R})) + \mathbb{Z}\operatorname{ind}_M \mathcal{R}$ holds.*

PROOF. (1) By 9.4(3), we have $M = \bigoplus_{i \in \Delta_0 \setminus \mathcal{J}_M} M^i$ and $\operatorname{rank}_{\mathcal{R}} M^i = \delta_i$. Thus

$$\operatorname{rank}_{\mathcal{R}} M^i = \delta_i = \delta(\alpha_i^*) = \delta(\operatorname{ind}_M M^i). \tag{9.2.J}$$

Now consider an arbitrary $N \in \operatorname{modif}\mathcal{R}$. For any $N \in \operatorname{modif}\mathcal{R}$, there exists an exact sequence $0 \to M_1 \to M_0 \to N$ with $M_i \in \operatorname{add} M$ where the last map is an $\operatorname{add} M$-approximation. By (9.2.I), we obtain $\operatorname{ind}_M N = \operatorname{ind}_M M_0 - \operatorname{ind}_M M_1$. By 8.14(2) we see that $\operatorname{rank}_{\mathcal{R}} N = \operatorname{rank}_{\mathcal{R}} M_0 - \operatorname{rank}_{\mathcal{R}} M_1$. Thus

$$\operatorname{rank}_{\mathcal{R}} N = \operatorname{rank}_{\mathcal{R}} M_0 - \operatorname{rank}_{\mathcal{R}} M_1 \overset{(9.2.J)}{=} \delta(\operatorname{ind}_M M_0) - \delta(\operatorname{ind}_M M_1) = \delta(\operatorname{ind}_M N).$$

(2) The first assertion is clear from (1) and 9.8(1). The second one is clear since $\mathrm{Cl}(\mathcal{R}) = \{N \in \operatorname{modif} \mathcal{R} \mid \operatorname{rank} N = 1\}$.
(3) The first assertion follows from (2) and 9.8(4). To show the second assertion, fix $x \in L_{\mathcal{J}_M}$. For $r := \delta(x)$ and $y := x - (r-1)\operatorname{ind}_M \mathcal{R}$, we have $x = y + (r-1)\operatorname{ind}_M \mathcal{R}$, where $y \in L_{\mathcal{J}_M} \cap \delta^{-1}(1) = \operatorname{ind}_M(\mathrm{Cl}(\mathcal{R}))$. □

9.3. Universality of index

The definition of the index ind_M given in the previous section depends on the choice of a fixed $M \in \mathrm{MM}\mathcal{R}$. We now show, in 9.11, that ind_M does not depend on this choice of M, up to a specific isomorphism, then in 9.17 show that it is left/right symmetric.

As such, consider $N \in \mathrm{MM}\mathcal{R}$. By 9.4 applied to N, there is an associated subset $\mathcal{J}_N$ of Δ_{aff}. Furthermore, it is clear that the image of N under the map

$$\mathrm{MM}\mathcal{R} \to \operatorname{tilt} \Gamma_{\mathcal{J}_M} \cong \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J}_M).$$

has the form $(x_{NM}, \mathcal{J}_N)$ for some element x_{NM} of the affine Weyl group. It is also clear that multiplication $x_{NM} \colon \Theta_{\mathsf{aff}} \to \Theta_{\mathsf{aff}}$ relabels chambers, and so restricts to an isomorphism $x_{NM} \colon L_{\mathcal{J}_N} \to L_{\mathcal{J}_M}$ via the labelling rules of 1.12.

Theorem 9.11. *For $M, N \in \mathrm{MM}\mathcal{R}$, the following diagram is commutative.*

$$\begin{array}{ccc} \operatorname{modif} \mathcal{R} & \xrightarrow{\operatorname{ind}_N} & L_{\mathcal{J}_N} \\ \| & & \downarrow{\scriptstyle x_{NM}} \\ \operatorname{modif} \mathcal{R} & \xrightarrow{\operatorname{ind}_M} & L_{\mathcal{J}_M} \end{array}$$

To prove this requires the following notion.

Definition 9.12. For $\mathcal{J} \subset \Delta_{\mathsf{aff}}$, we call a map

$$f \colon \operatorname{modif} \mathcal{R} \to L_{\mathcal{J}}^{+}$$

compatible if it is injective and, furthermore, the following two conditions hold for each $N = N_1 \oplus \ldots \oplus N_n \in \mathrm{MM}\mathcal{R}$ with indecomposable N_i.

(1) The elements $f(N_1), \ldots, f(N_n)$ form a $\mathbb{Z}$-basis of some $\mathrm{C}_f(N) \in \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J})$.
(2) For all $m_1, \ldots, m_n \geq 0$, $f(\bigoplus_{i=1}^n N_i^{\oplus m_i}) = \sum_{i=1}^n m_i f(N_i)$.

We also require the following easy observations.

Lemma 9.13. *Let $\mathcal{J}$ be a subset of Δ_{aff}.*

(1) *For each $M \in \mathrm{MM}\mathcal{R}$, the map $\operatorname{ind}_M \colon \operatorname{modif} \mathcal{R} \to L_{\mathcal{J}_M}$ is compatible.*
(2) *Let $(x, J) \in \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J})$ and $f \colon \operatorname{modif} \mathcal{R} \to L_J$ be a compatible map. Then $x \circ f \colon \operatorname{modif} \mathcal{R} \to L_{\mathcal{J}}$ is also a compatible map.*
(3) *Let $f \colon \operatorname{modif} \mathcal{R} \to L_{\mathcal{J}}$ be a compatible map and $M, N \in \mathrm{MM}\mathcal{R}$. If M and N are linked by a simple mutation, then $\mathrm{C}_f(M)$ and $\mathrm{C}_f(N)$ are related by a simple wall crossing.*
(4) *Let $f, g \colon \operatorname{modif} \mathcal{R} \to L_{\mathcal{J}}$ be compatible maps. Then $f = g$ holds if there exists basic $M = M_1 \oplus \ldots \oplus M_n \in \mathrm{MM}\mathcal{R}$ with indecomposable M_i such that $f(M_i) = g(M_i)$ for all i with $1 \leq i \leq n$.*

Proof. (1) This is an immediate consequence of the definition, using 9.8(1).
(2) This is clear since the isomorphism $x \colon L_J \to L_{\mathcal{J}}$ sends the chambers in $\mathsf{Cham}(\Delta_{\mathsf{aff}}, J)$ to those in $\mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J})$.
(3) This is clear, since M and N share all summands except one.
(4) By [IW, 4.18(1)], any element in $\operatorname{modif} \mathcal{R}$ belongs to $\operatorname{add} N$ for some $N \in \mathrm{MM}\mathcal{R}$. It suffices to show that $f = g$ holds on $\operatorname{add} N$ for each $N \in \mathrm{MM}\mathcal{R}$.

Let $N, L \in \mathrm{MM}\mathcal{R}$ be linked by a simple mutation. If $f = g$ holds on $\operatorname{add} N$, then $f = g$ holds on $\operatorname{add} L$ by (3). Since the exchange graph of $\mathrm{MM}\mathcal{R}$ is connected by 9.6(3), the existence of M implies that $f = g$ holds on $\operatorname{add} L$ for each $L \in \mathrm{MM}\mathcal{R}$. ☐

Now for $M, N \in \mathrm{MM}\mathcal{R}$ there are isomorphisms

$$\overline{\mathrm{End}_{\mathcal{R}}(M)} \cong \Gamma_{\mathcal{J}_M} \quad \text{and} \quad \overline{\mathrm{End}_{\mathcal{R}}(N)} \cong \Gamma_{\mathcal{J}_N}$$

of algebras and an isomorphism

$$\overline{\mathrm{Hom}_{\mathcal{R}}(M, N)} \cong e_{\mathcal{J}_M} I_{x_{NM}} e_{\mathcal{J}_N} \tag{9.3.A}$$

of $(\overline{\mathrm{End}_{\mathcal{R}}(M)}, \overline{\mathrm{End}_{\mathcal{R}}(N)})$-bimodules.

Proof of 9.11. By 9.13(1)(2), the maps ind_M and $x_{NM} \circ \mathrm{ind}_N$ are compatible. Let $N = N_1 \oplus \ldots \oplus N_n$ with indecomposable N_i. Then

$$\mathrm{ind}_M N_i = \beta_{\mathcal{J}_N}[\overline{\mathrm{Hom}_{\mathcal{R}}(M, N_i)}] \overset{(9.3.A)}{=} [e_{\mathcal{J}_M} I_{x_{NM}} e_i] \overset{5.26}{=} x_{NM}(\alpha_i^*) = x_{NM}(\mathrm{ind}_N N_i).$$

By 9.13(3), it follows that $\mathrm{ind}_M = x_{NM} \circ \mathrm{ind}_N$. ☐

Remark 9.14. It follows from the above that the following diagram commutes.

$$\begin{array}{ccc} \operatorname{modif} \mathcal{R} & \xrightarrow{\mathrm{Hom}_{\mathcal{R}}(N,-)} & \mathrm{K}_0(\operatorname{proj} \mathrm{End}_R(N)) \\ \| & & \uparrow{\scriptstyle [\mathbf{R}\mathrm{Hom}_{\Lambda_M}(\mathrm{Hom}_{\mathcal{R}}(M,N),-)]} \\ \operatorname{modif} \mathcal{R} & \xrightarrow{\mathrm{Hom}_{\mathcal{R}}(M,-)} & \mathrm{K}_0(\operatorname{proj} \mathrm{End}_R(M)) \end{array}$$

In what follows, a key role is played by the *swap involution*

$$\mathsf{sw} \colon \Pi \xrightarrow{\sim} \Pi^{\mathrm{op}}$$

of the preprojective algebra Π defined by

$$\mathsf{sw}(e_i) = e_i \quad \text{and} \quad \mathsf{sw}(a) = a^*.$$

Let $\mathcal{J}$ be a subset of Δ. Since sw fixes the idempotent $e_{\mathcal{J}} \in \Pi$, it induces an involution

$$\mathsf{sw} \colon \Gamma_{\mathcal{J}} \xrightarrow{\sim} \Gamma_{\mathcal{J}}^{\mathrm{op}}.$$

This induces an equivalence $\mathsf{sw}^* \colon \operatorname{mod} \Gamma_{\mathcal{J}} \cong \operatorname{mod} \Gamma_{\mathcal{J}}^{\mathrm{op}}$ which clearly gives rise to bijections

$$\mathsf{sw}^* \colon \operatorname{ptilt} \Gamma_{\mathcal{J}} \xrightarrow{\sim} \operatorname{ptilt} \Gamma_{\mathcal{J}}^{\mathrm{op}} \quad \text{and} \quad \mathsf{sw}^* \colon \operatorname{tilt} \Gamma_{\mathcal{J}} \xrightarrow{\sim} \operatorname{tilt} \Gamma_{\mathcal{J}}^{\mathrm{op}}.$$

Similarly, if $M \in \mathrm{MM}\mathcal{R}$, then $\mathcal{R}$-dual satisfies $M^* \in \mathrm{MM}\mathcal{R}$, and furthermore $\mathrm{End}_{\mathcal{R}}(M^*) \cong \mathrm{End}_{\mathcal{R}}(M)^{\mathrm{op}}$. Then the dual version of (9.2.A) is

$$\mathrm{Hom}_{\mathcal{R}}(M^*, -) \cong \mathrm{Hom}_{\mathcal{R}}(-, M) \colon \operatorname{modif} \mathcal{R} \xrightarrow{\sim} \text{ref-ptilt} \, \mathrm{End}_{\mathcal{R}}(M)^{\mathrm{op}}$$

The right module version of 8.3 asserts that

$$\overline{(-)} \colon \text{ref-tilt} \, \mathrm{End}_{\mathcal{R}}(M)^{\mathrm{op}} \hookrightarrow \operatorname{tilt} \Gamma_{\mathcal{J}_M}^{\mathrm{op}},$$

and so combining gives the dual version of (9.2.E), namely

$$\overline{\mathrm{Hom}_{\mathcal{R}}(-, M)} \colon \operatorname{modif} \mathcal{R} \hookrightarrow \operatorname{tilt} \Gamma_{\mathcal{J}_M}^{\mathrm{op}}. \tag{9.3.B}$$

Lemma 9.15. *If $M \in \mathrm{MM}\mathcal{R}$, then the following diagram is commutative.*

$$\begin{array}{ccccc} \operatorname{modif} \mathcal{R} & \xrightarrow{\mathrm{Hom}_{\mathcal{R}}(M,-)} & \text{ref-ptilt} \, \mathrm{End}_{\mathcal{R}}(M) & \xrightarrow{\overline{(-)}} & \operatorname{ptilt} \Gamma_{\mathcal{J}_M} \\ \| & & & & {\scriptstyle \sim}\downarrow{\scriptstyle \mathsf{sw}^*} \\ \operatorname{modif} \mathcal{R} & \xrightarrow{\mathrm{Hom}_{\mathcal{R}}(-,M)} & \text{ref-ptilt} \, \mathrm{End}_{\mathcal{R}}(M)^{\mathrm{op}} & \xrightarrow{\overline{(-)}} & \operatorname{ptilt} \Gamma_{\mathcal{J}_M}^{\mathrm{op}} \end{array}$$

Proof. By the same argument as in the proof of 9.13(4), it suffices to show that the diagram commutes for each indecomposable direct summand M_i of M. Let $e_i \in \Lambda$ be the idempotent corresponding to M_i. Then $\overline{\mathrm{Hom}_{\mathcal{R}}(M, M_i)} = \Gamma_{\mathcal{J}_M} e_i$ and $\overline{\mathrm{Hom}_{\mathcal{R}}(M_i, M)} = e_i \Gamma_{\mathcal{J}_M}$ hold. Thus

$$\mathsf{sw}^* \overline{\mathrm{Hom}_{\mathcal{R}}(M, M_i)} = \mathsf{sw}^*(\Gamma_{\mathcal{J}_M} e_i) = e_i \Gamma_{\mathcal{J}_M} = \overline{\mathrm{Hom}_{\mathcal{R}}(M_i, M)}.$$

Hence the diagram commutes for M_i, as desired. □

Next we show that the index is left-right symmetric in the following sense. Fix $M \in \mathrm{MM}\mathcal{R}$, then in a similar way to 5.26, there is a bijection

$$\beta^{\mathrm{op}}_{\mathcal{J}_M} \colon \mathrm{K}_0(\mathrm{proj}\, \Gamma^{\mathrm{op}}_{\mathcal{J}_M}) \xrightarrow{\sim} L_{\mathcal{J}_M}.$$

sending $e_i \Gamma_{\mathcal{J}_M} \mapsto \alpha_i^*$.

Definition 9.16. For $M \in \mathrm{MM}\mathcal{R}$, write ind^M for the composition

$$\mathrm{modif}\, \mathcal{R} \xhookrightarrow[(9.3.\mathrm{B})]{\overline{\mathrm{Hom}_{\mathcal{R}}(-,M)}} \mathrm{ptilt}\, \Gamma^{\mathrm{op}}_{\mathcal{J}_M} \xrightarrow[\sim]{\beta^{\mathrm{op}}_{\mathcal{J}_M} \circ [-]} L^+_{\mathcal{J}_M}.$$

Theorem 9.17. *For any $M \in \mathrm{MM}\mathcal{R}$, we have $\mathrm{ind}_M = \mathrm{ind}^M$.*

Proof. Consider the following diagram, where the top row is ind_M and the bottom row is ind^M.

$$\begin{array}{ccccc} \mathrm{modif}\, \mathcal{R} & \xrightarrow{\overline{\mathrm{Hom}_{\mathcal{R}}(M,-)}} & \mathrm{ptilt}\, \Gamma_{\mathcal{J}_M} & \xrightarrow{\beta_{\mathcal{J}_M} \circ [-]} & L_{\mathcal{J}_M} \\ \| & & \downarrow{\scriptstyle \mathsf{sw}^*} & & \| \\ \mathrm{modif}\, \mathcal{R} & \xrightarrow{\overline{\mathrm{Hom}_{\mathcal{R}}(-,M)}} & \mathrm{ptilt}\, \Gamma^{\mathrm{op}}_{\mathcal{J}_M} & \xrightarrow{\beta^{\mathrm{op}}_{\mathcal{J}_M} \circ [-]} & L_{\mathcal{J}_M} \end{array}$$

The left hand square commutes by 9.15, and the right hand square clearly commutes. □

A corollary of the above is the following remarkable symmetry property for cDV singularities, which does not usually hold for general 3-dimensional Gorenstein rings. We will extend the following in §9.5 later.

Corollary 9.18. *Let $M \in \mathrm{MM}\mathcal{R}$ and $L \in \mathrm{modif}\, \mathcal{R}$. Then the sequences* (8.2.D)

$$0 \to L \to U_0 \to U_1 \quad \textit{and} \quad 0 \to V_1 \to V_0 \to L$$

satisfy $U_0 \cong V_0$ and $U_1 \cong V_1$. Moreover, $\mathrm{ind}_M L = \mathrm{ind}_M U_0 - \mathrm{ind}_M U_1$ holds.

Proof. By 9.9 and its dual, we have $\mathrm{ind}_M L = \mathrm{ind}_M V_0 - \mathrm{ind}_M V_1$ and $\mathrm{ind}^M L = \mathrm{ind}^M U_0 - \mathrm{ind}^M U_1$. Hence, using 9.17, $\mathrm{ind}_M V_0 - \mathrm{ind}_M V_1 = \mathrm{ind}_M L = \mathrm{ind}_M U_0 - \mathrm{ind}_M U_1$. In particular

$$\mathrm{ind}_M(V_0 \oplus U_1) = \mathrm{ind}_M V_0 + \mathrm{ind}_M U_1 = \mathrm{ind}_M V_1 + \mathrm{ind}_M U_0 = \mathrm{ind}_M(V_1 \oplus U_0).$$

Hence, by injectivity of ind_M, $V_1 \oplus U_0 \cong V_0 \oplus U_1$. Since V_0 and V_1 (respectively, U_0 and U_1) do not share non-zero common direct summands by 8.8, the assertion follows. □

As a consequence, we obtain the following remarkable property of exchange sequences.

Proposition 9.19. *Let $M = M_1 \oplus \ldots \oplus M_n \in \mathrm{MM}\mathcal{R}$ with indecomposable M_i. For all i such that $1 \le i \le n$, consider the exchange sequences*

$$0 \to N_i \to V_i \to M_i \quad \textit{and} \quad 0 \to M_i \to U_i \to N_i.$$

Then $U_i \cong V_i$. Moreover, $\mathrm{ind}_M M_i + \mathrm{ind}_M N_i = \mathrm{ind}_M U_i = \mathrm{ind}_M V_i$.

PROOF. Let $\Lambda_i = \operatorname{End}_{\mathcal{R}}(M)/(1-e_i)$. The proof divides into two cases. If Λ_i is not artinian, then $M_i \cong N_i$ holds by 8.6(2). Thus the two exchange sequences are isomorphic by 8.5(2), and, in particular, $U_i \cong V_i$. On the other hand, if Λ_i is artinian, then by 8.5(4), $U_i \to N_i$ is a right $(\operatorname{add} M)$-approximation, and $N_i \to V_i$ is a left $(\operatorname{add} M)$-approximation. By 9.18, $U_i \cong V_i$ holds.

In either case, using 9.9, $\operatorname{ind}_M N_i = \operatorname{ind}_M U_i - \operatorname{ind}_M M_i$ holds. □

9.4. Class Group and Global Index

Throughout this section, let $\mathcal{R}$ be a cDV singularity. We regard $\mathbb{Z}\oplus\operatorname{Cl}(\mathcal{R})$ has a factor group of $\mathrm{K}_0(\mathcal{R})$ by the map (9.0.B), and as in (9.0.C) we define the *global index* to be the composition

$$\operatorname{ind}\colon \operatorname{mod}\mathcal{R} \xrightarrow{[-]} \mathrm{K}_0(\operatorname{mod}\mathcal{R}) \xrightarrow{(9.0.\mathrm{B})} \mathbb{Z}\oplus\operatorname{Cl}(\mathcal{R}).$$

This sends M to $(\operatorname{rank}_{\mathcal{R}} M, \det M)$. In this section, for each $M \in \operatorname{MM}\mathcal{R}$, we will show in 9.21 that ind and ind_M are related by a specific isomorphism.

We begin by constructing a morphism from $L_{\mathcal{J}_M}$ to $\mathbb{Z}\oplus\operatorname{Cl}(\mathcal{R})$. Consider the triangle functor

$$M \overset{\mathbf{L}}{\otimes}_{\operatorname{End}_R(M)} - \colon \mathrm{K}^{\mathrm{b}}(\operatorname{proj}\operatorname{End}_{\mathcal{R}}(M)) \to \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\mathcal{R}),$$

which induces a group homomorphism

$$[M \overset{\mathbf{L}}{\otimes}_{\operatorname{End}_{\mathcal{R}}(M)} -]\colon \mathrm{K}_0(\operatorname{proj}\operatorname{End}_{\mathcal{R}}(M)) \to \mathrm{K}_0(\operatorname{mod}\mathcal{R}).$$

We define the group homomorphism $\gamma_M\colon L_{\mathcal{J}_M} \to \mathbb{Z}\oplus\operatorname{Cl}(\mathcal{R})$ via the following commutative diagram.

$$\begin{array}{ccc} \mathrm{K}_0(\operatorname{proj}\operatorname{End}_{\mathcal{R}}(M)) & \xrightarrow[\sim]{\beta_{\mathcal{J}_M}\overline{(-)}} & L_{\mathcal{J}_M} \\ {\scriptstyle [M \overset{\mathbf{L}}{\otimes}_{\operatorname{End}_{\mathcal{R}}(M)} -]}\downarrow & & \downarrow{\scriptstyle \gamma_M} \\ \mathrm{K}_0(\operatorname{mod}\mathcal{R}) & \xrightarrow{(9.0.\mathrm{B})} & \mathbb{Z}\oplus\operatorname{Cl}(\mathcal{R}) \end{array} \tag{9.4.A}$$

Leading up to our next result, we need the following general observations. Note that all are clear for the case $M \in \operatorname{MMG}\mathcal{R}$, since then M is a projective Λ-module.

LEMMA 9.20. *Let $M \in \operatorname{ref}\mathcal{R}$ and $\Lambda = \operatorname{End}_{\mathcal{R}}(M)$.*

(1) *For each $X \in \operatorname{mod}\Lambda$, we have $[M \overset{\mathbf{L}}{\otimes}_\Lambda X] = [M \otimes_\Lambda X]$ in $\mathbb{Z}\oplus\operatorname{Cl}(\mathcal{R})$.*
(2) *For each $N \in \operatorname{ref}\mathcal{R}$, we have*

$$[M \overset{\mathbf{L}}{\otimes}_\Lambda \operatorname{Hom}_{\mathcal{R}}(M,N)] = [M \otimes_\Lambda \operatorname{Hom}_{\mathcal{R}}(M,N)] = [N] \quad \textit{in} \quad \mathbb{Z}\oplus\operatorname{Cl}(\mathcal{R}).$$

PROOF. (1) Let $\Lambda = \operatorname{End}_{\mathcal{R}}(M)$. For all height one prime ideal $\mathfrak{p} \in \operatorname{Spec} R$, the $\mathcal{R}_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is free. Hence $M_{\mathfrak{p}}$ is projective as a module over $\operatorname{End}_{\mathcal{R}_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \Lambda_{\mathfrak{p}}$. In particular,

$$\operatorname{Tor}_i^\Lambda(M,N)_{\mathfrak{p}} = \operatorname{Tor}_i^{\Lambda_{\mathfrak{p}}}(M_{\mathfrak{p}},N_{\mathfrak{p}}) = 0$$

holds for all $i > 0$. Thus $\dim_R \operatorname{Tor}_i^\Lambda(M,N) \le 1$ for all $i > 0$. By (9.0.A), we obtain the assertion.
(2) The first equality is (1). For the second equality, by the same argument as above, the canonical map $M\otimes_\Lambda \operatorname{Hom}_{\mathcal{R}}(M,N) \to N$ becomes an isomorphism after localising at each height one prime ideal of $\mathcal{R}$. □

THEOREM 9.21. *Let $\mathcal{R}$ be a cDV singularity and $M \in \operatorname{MM}\mathcal{R}$.*

(1) *The following diagram commutes.*

$$\begin{array}{ccc} \operatorname{modif} \mathcal{R} & \xrightarrow{\operatorname{ind}_M} & L_{\mathcal{J}_M} \\ \big\downarrow & & \big\downarrow{\scriptstyle \gamma_M} \\ \operatorname{mod} \mathcal{R} & \xrightarrow{\operatorname{ind}} & \mathbb{Z} \oplus \operatorname{Cl}(\mathcal{R}) \end{array}$$

(2) γ_M *is surjective, and satisfies* $\gamma_M(\operatorname{ind}_M I) = (1, I)$ *for all* $I \in \operatorname{Cl}(\mathcal{R})$.
(3) *If* $\mathcal{R}$ *is isolated, then the map* $\delta \colon L_{\mathcal{J}_M} \to \mathbb{Z}$ *coincides with the composition of* $\gamma_M \colon L_{\mathcal{J}_M} \to \mathbb{Z} \oplus \operatorname{Cl}(\mathcal{R})$ *and the first projection* $\mathbb{Z} \oplus \operatorname{Cl}(\mathcal{R}) \to \mathbb{Z}$.

PROOF. (1) Let $N \in \operatorname{modif} \mathcal{R}$. Then

$$\gamma_M(\operatorname{ind}_M N) = \gamma_M \circ \beta_{\mathcal{J}_M}([\overline{\operatorname{Hom}_{\mathcal{R}}(M, N)}]) \overset{(9.4.A)}{=} [M \overset{\mathbf{L}}{\otimes}_{\Lambda} \operatorname{Hom}_{\mathcal{R}}(M, N)] \overset{9.20(2)}{=} [N].$$

(2) By (1), we have $\gamma_M(\operatorname{ind}_M I) = [I] = (1, I)$. Since $\gamma_M(\operatorname{ind}_M R) = (1, 0)$, the image of γ_M contains both $\mathbb{Z}$ and $\operatorname{Cl}(\mathcal{R})$. Thus γ_M is surjective.
(3) It suffices to show the assertion for each element $x \in L^+_{\mathcal{J}_M}$. Since $\mathcal{R}$ is isolated, by 9.8(4) we can write $x = \operatorname{ind}_M N$ by some $N \in \operatorname{modif} \mathcal{R}$. The assertion then follows since

$$\gamma_M(x) \overset{(1)}{=} \operatorname{ind} N = (\operatorname{rank}_{\mathcal{R}} N, \det N) \overset{9.10(1)}{=} (\delta(x), \det N). \qquad \square$$

As a consequence, we obtain the following explicit description of $\operatorname{Cl}(\mathcal{R})$. The first part can also be obtained using the theory of minimal models, but the following description falls out of the Coxeter combinatorics.

COROLLARY 9.22. *Let* $\mathcal{R}$ *be isolated cDV, and* $M \in \operatorname{MM}\mathcal{R}$.

(1) $\gamma_M \colon L_{\mathcal{J}_M} \to \mathbb{Z} \oplus \operatorname{Cl}(\mathcal{R})$ *is a group isomorphism. Therefore* $\operatorname{Cl}(\mathcal{R}) \cong \mathbb{Z}^{n-1}$, *where* n *is the number of indecomposable direct summands of* M.
(2) *If* $M \in \operatorname{MMG}\mathcal{R}$, *then* γ_M *restricts to a group isomorphism* $\mathsf{CoWt}_{\mathcal{J}_M} \xrightarrow{\sim} \operatorname{Cl}(\mathcal{R})$.
(3) *There are bijections*

$$\operatorname{modif} \mathcal{R} \xrightarrow[\sim]{\operatorname{ind}_M} L^+_{\mathcal{J}_M} \xrightarrow[\sim]{\gamma_M} (\mathbb{Z} \oplus \operatorname{Cl}(\mathcal{R}))^+$$

PROOF. (1) The map $\gamma_M \colon L_{\mathcal{J}_M} \to \mathbb{Z} \oplus \operatorname{Cl}(\mathcal{R})$ is surjective by 9.21(2), so it suffices to prove injectivity. Let $r = \operatorname{ind}_M \mathcal{R} \in L_{\mathcal{J}_M}$. Since $\mathcal{R}$ has an isolated singularity, 9.10(3) shows that each $x \in L_{\mathcal{J}_M}$ can be written as $x = \operatorname{ind}_M I + \ell \operatorname{ind}_M \mathcal{R}$ for some $I \in \operatorname{Cl}(\mathcal{R})$ and $\ell \in \mathbb{Z}$. If $\gamma_M(x) = 0$, then

$$0 = \gamma_M(x) \overset{9.21(1)}{=} \operatorname{ind} I + \ell \operatorname{ind} R = (1 + \ell, I).$$

Thus $\ell = -1$ and $I = R$ hold, and hence $x = 0$. This proves the first assertion. The second assertion follows, since $L_{\mathcal{J}_M} \cong \mathbb{Z}^n$.
(2) Since $M \in \operatorname{MMG}\mathcal{R}$, the subset $\mathcal{J}_M \subseteq \Delta_{\mathsf{aff}}$ can be viewed as a subset of Δ. Hence, as in §2.1, it is possible to consider $\mathsf{CoWt}_{\mathcal{J}_M}$. Since $\mathsf{CoWt}_{\mathcal{J}_M} = L_{\mathcal{J}_M} \cap \delta^{-1}(0)$, the result follows by combining (1) and 9.21(3).
(3) The first bijection is 9.8(4), and the second bijection is a direct consequence of (1). $\square$

The following observation is elementary.

PROPOSITION 9.23. *Let* $X \in \operatorname{modif} \mathcal{R}$, *and* $I \in \operatorname{Cl}(\mathcal{R})$.

(1) $\operatorname{ind} X + \operatorname{ind} X^* = (2, 0)$.
(2) $\operatorname{ind}((I \otimes_{\mathcal{R}} X)^{**}) = \operatorname{ind} X + \operatorname{rank} X (\operatorname{ind} I - \operatorname{ind} \mathcal{R})$.

Proof. (1) This is clear.
(2) We have $\det(I \otimes_{\mathcal{R}} X) = \det X + \operatorname{rank} X \cdot \det I$ (e.g. [DITW, 3.1]). Thus

$$\begin{aligned}\operatorname{ind}(I \otimes_{\mathcal{R}} X) &= (\operatorname{rank}(I \otimes_{\mathcal{R}} X), \det(I \otimes_{\mathcal{R}} X)) \\ &= (\operatorname{rank} X, \det X + \operatorname{rank} X \cdot \det I) \\ &= \operatorname{ind} X + \operatorname{rank} X(\operatorname{ind} I - \operatorname{ind} \mathcal{R}).\end{aligned}$$ □

One of the advantage of the global index is that it simplifies some arguments.

Lemma 9.24. *Let $A, B, C \in \operatorname{ref} \mathcal{R}$, and let*

$$0 \to A \to B \xrightarrow{f} C$$

be exact such that either

(1) $\dim_{\mathcal{R}} \operatorname{Cok} f \leq 1$, *or*
(2) *there exists non-zero $X \in \operatorname{ref} \mathcal{R}$ such that $\operatorname{Hom}_{\mathcal{R}}(X, f)$ is surjective.*

Then $\operatorname{ind} B = \operatorname{ind} A + \operatorname{ind} C$.

Proof. (1) This is clear since $\operatorname{Cok} f$ is zero in $\mathbb{Z} \oplus \operatorname{Cl}(\mathcal{R})$.
(2) Let $\mathfrak{p} \in \operatorname{Spec} \mathcal{R}$ be a height one prime ideal. Then $X_{\mathfrak{p}}$ is a non-zero free $R_{\mathfrak{p}}$-module. Since $\operatorname{Hom}_{\mathcal{R}_{\mathfrak{p}}}(X_{\mathfrak{p}}, f_{\mathfrak{p}})\colon \operatorname{Hom}_{\mathcal{R}_{\mathfrak{p}}}(X_{\mathfrak{p}}, B_{\mathfrak{p}}) \to \operatorname{Hom}_{\mathcal{R}_{\mathfrak{p}}}(X_{\mathfrak{p}}, C_{\mathfrak{p}})$ is surjective, $f_{\mathfrak{p}}\colon B_{\mathfrak{p}} \to C_{\mathfrak{p}}$ is surjective. It follows that $\dim_{\mathcal{R}} \operatorname{Cok} f \leq 1$, so the assertion follows from (1). □

As an application, we give a more direct proof of 9.18 for isolated cDV singularities. Another application will be in 9.29 below.

Proof of 9.18 for isolated case. Since the sequences

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{R}}(M, V_1) \longrightarrow \operatorname{Hom}_{\mathcal{R}}(M, V_0) \longrightarrow \operatorname{Hom}_{\mathcal{R}}(M, L) \longrightarrow 0$$

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{R}}(M^*, U_1^*) \longrightarrow \operatorname{Hom}_{\mathcal{R}}(M^*, U_0^*) \longrightarrow \operatorname{Hom}_{\mathcal{R}}(M^*, L^*) \longrightarrow 0$$

are exact, by 9.24 we have $\operatorname{ind} L = \operatorname{ind} V_0 - \operatorname{ind} V_1$ and $\operatorname{ind} L^* = \operatorname{ind} U_0^* - \operatorname{ind} U_1^*$. Hence

$$\begin{aligned}
&\text{(by 9.23(1))} & \operatorname{ind} L &= \operatorname{ind} L^* - 2(\operatorname{rank} L, 0) \\
&& &= \operatorname{ind} L^* - 2(\operatorname{rank} U_0, 0) + 2(\operatorname{rank} U_1, 0) \\
&& &= (\operatorname{ind} U_0^* - \operatorname{ind} U_1^*) - 2(\operatorname{rank} U_0) + 2(\operatorname{rank} U_1) \\
&\text{(by 9.23(1))} & &= \operatorname{ind} U_0 - \operatorname{ind} U_1.
\end{aligned}$$

Equating both expressions for $\operatorname{ind} L$, it follows that

$$\operatorname{ind}(V_0 \oplus U_1) = \operatorname{ind} V_0 + \operatorname{ind} U_1 = \operatorname{ind} U_0 + \operatorname{ind} V_1 = \operatorname{ind}(U_0 \oplus V_1),$$

and hence $V_0 \oplus U_1 \simeq U_0 \oplus V_1$ since ind is bijective on modifying modules by 9.21(1) and 9.22(3). Since V_0 and V_1 (respectively, U_0 and U_1) do not have non-zero common direct summands by 8.8, the assertion follows. □

9.5. Extension to Modifying Modules

In this subsection we extend the above symmetry properties to all modifying modules. Geometrically this is important; later, this is precisely what allows us to work on arbitrary crepant partial resolutions. The consequences are striking: we prove in 9.28 that when $\mathcal{R}$ is isolated, the mutation of any modifying module, at any summand, is an involution. This is not the typical behaviour for 3-dimensional Gorenstein rings.

Let $\mathcal{R}$ be a cDV singularity. For a fixed modifying $\mathcal{R}$-module

$$N = N_1 \oplus \ldots \oplus N_t \in \operatorname{modif} \mathcal{R}$$

with indecomposable summands N_i, choose some $M \in \operatorname{MM}\mathcal{R}$, and define

$$H_N := \langle \operatorname{ind}_M N_1, \dots, \operatorname{ind}_M N_t \rangle \subset L_{\mathcal{J}_M},$$
$$\operatorname{modif}^N \mathcal{R} := \{L \in \operatorname{modif}\mathcal{R} \mid \operatorname{ind}_M X \in H_N\},$$
$$\operatorname{MM}^N \mathcal{R} := \{L \in \operatorname{modif}^N \mathcal{R} \mid |L| = |N|\}.$$

where $|X|$ denotes the number of non-isomorphic indecomposable summands of X. By 9.11 (see also 9.25(3)), $\operatorname{modif}^N \mathcal{R}$ and $\operatorname{MM}^N\mathcal{R}$ are independent of the choice of $M \in \operatorname{MM}\mathcal{R}$. Note that $\operatorname{MM}^N\mathcal{R}$ is not a subset of $\operatorname{MM}\mathcal{R}$, unless $N \in \operatorname{MM}\mathcal{R}$.

In the following theorem, we remark that all parts are not typical behaviour of modifying modules on 3-dimensional Gorenstein rings. The key point is that it is our precise understanding of the K-theory of MMAs for cDV singularities in the previous sections, via Coxeter combinatorics, that allows us to extract these corollaries.

THEOREM 9.25. *Let $\mathcal{R}$ be a cDV singularity and $M \in \operatorname{MM}\mathcal{R}$. Fix $N = N_1 \oplus \dots \oplus N_t \in \operatorname{modif}\mathcal{R}$ with indecomposable. For any $X \in \operatorname{modif}\mathcal{R}$, the following holds.*

(1) *$X \in \operatorname{modif}^N\mathcal{R}$ if and only if $\operatorname{Hom}_{\mathcal{R}}(N, X) \in \text{ref-ptilt}\operatorname{End}_{\mathcal{R}}(N)$.*
(2) *$X \in \operatorname{MM}^N\mathcal{R}$ if and only if $\operatorname{Hom}_{\mathcal{R}}(N, X) \in \text{ref-tilt}\operatorname{End}_{\mathcal{R}}(N)$.*
(3) *There are bijections*

$$\operatorname{Hom}_{\mathcal{R}}(N, -)\colon \operatorname{modif}^N \mathcal{R} \xrightarrow{\sim} \text{ref-ptilt}\operatorname{End}_{\mathcal{R}}(N) \tag{9.5.A}$$

$$\operatorname{Hom}_{\mathcal{R}}(N, -)\colon \operatorname{MM}^N \mathcal{R} \xrightarrow{\sim} \text{ref-tilt}\operatorname{End}_{\mathcal{R}}(N). \tag{9.5.B}$$

PROOF. (1) The following diagram is clearly commutative, where the symbol $\sim$ denotes a bijection, and the symbol $\cong$ a group isomorphism.

$$\begin{array}{ccccc}
\text{ref-ptilt}\operatorname{End}_{\mathcal{R}}(M) & \xrightarrow{\overline{(-)}} & \operatorname{ptilt}\Gamma_{\mathcal{J}_M} & \xrightarrow[\sim]{\beta_{\mathcal{J}_M}\circ[-]} & L^+_{\mathcal{J}_M} \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{K}_0(\operatorname{proj}\operatorname{End}_{\mathcal{R}}(M)) & \xrightarrow[\cong]{\overline{(-)}} & \operatorname{K}_0(\operatorname{proj}\Gamma_{\mathcal{J}_M}) & \xrightarrow[\cong]{\beta_{\mathcal{J}_M}} & L_{\mathcal{J}_M}
\end{array}$$

Thus $X \in \operatorname{modif}^N\mathcal{R}$ holds if and only if $[\operatorname{Hom}_{\mathcal{R}}(M, X)] \in \operatorname{K}_0(\operatorname{proj}\operatorname{End}_{\mathcal{R}}(M))$ belongs to the subgroup generated by $[\operatorname{Hom}_{\mathcal{R}}(M, N_i)]$ with $1 \le i \le t$. By 8.15, this is equivalent to $\operatorname{Hom}_{\mathcal{R}}(N, X) \in \text{ref-ptilt}\operatorname{End}_{\mathcal{R}}(N)$.
(2) This is clear from (1).
(3) If $N = 0$, then the assertion is clear since $\operatorname{modif}^0\mathcal{R} = 0$. Assume $N \neq 0$. The maps are well-defined by (1) and (2). Since $N \neq 0$, there is a reflexive equivalence

$$\operatorname{Hom}_{\mathcal{R}}(N, -)\colon \operatorname{ref}\mathcal{R} \xrightarrow{\sim} \operatorname{ref}\operatorname{End}_R(N) \tag{9.5.C}$$

by (8.2.A). In particular, both maps in the statement are injective. To prove the surjectivity, fix $T \in \text{ref-ptilt}\operatorname{End}_{\mathcal{R}}(N)$. By 8.13, there exists $L \in \operatorname{modif}\mathcal{R}$ such that $T = \operatorname{Hom}_{\mathcal{R}}(N, L)$. By (1), $L \in \operatorname{modif}^N\mathcal{R}$. This shows that the first map is surjective. If $T \in \text{ref-tilt}\operatorname{End}_{\mathcal{R}}(N)$, then

$$|{}_R L| \overset{(9.5.C)}{=} |{}_\Lambda T| = |{}_\Lambda \Lambda| \overset{(9.5.C)}{=} |{}_R N|.$$

Thus $L \in \operatorname{MM}^N\mathcal{R}$ holds, and the second map is also surjective. ☐

We will now work towards showing that, when $\mathcal{R}$ is isolated, $\operatorname{MM}^N\mathcal{R}$ is the mutation class of N, and $\operatorname{modif}^N\mathcal{R}$ are all possible summands of these. Although many of the arguments below are more general, the non-isolated case remains more subtle; see 9.27.

The following is a generalisation of 9.19, and also generalises [W2, 5.22].

COROLLARY 9.26. *Let $N = N_1 \oplus \dots \oplus N_n \in \operatorname{modif}\mathcal{R}$ be basic with indecomposable N_i, and let $M \in \operatorname{MM}\mathcal{R}$ contain N as a summand. Then for each i such that $1 \le i \le n$, the following statement holds.*

(1) $\nu_i(N)$ *is an artinian mutation if and only if there exists* $N_i' \not\cong N_i$ *such that* $(N/N_i) \oplus N_i' \in \mathrm{MM}^N \mathcal{R}$.

Assuming further the equivalent conditions in (1), *then the following statements hold.*

(2) $\nu_i(N) \cong \mu_i(N) \cong (N/N_i) \oplus N_i' \in \mathrm{MM}^N \mathcal{R}$.

(3) *The exchange sequences*

$$0 \to N_i' \to V_i \to N_i \quad \text{and} \quad 0 \to N_i \to U_i \to N_i'$$

satisfy $U_i \cong V_i$ *and* $\mathrm{ind}_M N_i + \mathrm{ind}_M N_i' = \mathrm{ind}_M U_i = \mathrm{ind}_M V_i$.

PROOF. Let $\Lambda = \mathrm{End}_{\mathcal{R}}(N)$.

(1)($\Rightarrow$) By 8.11, $\nu_{\Lambda e_i}(\Lambda) \in \text{ref-tilt}\,\Lambda$. Using (9.5.B), there exists $N_i' \not\cong N_i$ such that $(N/N_i) \oplus N_i' \in \mathrm{MM}^N \mathcal{R}$ and $\nu_{\Lambda e_i}(\Lambda) = \mathrm{Hom}_{\mathcal{R}}(N, (N/N_i) \oplus N_i')$.

($\Leftarrow$) Again by (9.5.B), $\mathrm{Hom}_{\mathcal{R}}(N, (N/N_i) \oplus N_i') \in \text{ref-tilt}\,\Lambda$. Thus $\nu_i(\Lambda)$ belongs to ref-tilt Λ. By 8.11(1), $\nu_i(N)$ is artinian mutation.

(2) On one hand, by 8.11(1) $\nu_i(\Lambda) \cong \mathrm{Hom}_{\mathcal{R}}(N, \nu_i(N))$. On the other hand, by (1) above $\nu_i(\Lambda) \cong \mathrm{Hom}_{\mathcal{R}}(N, (N/N_i) \oplus N_i')$. Hence by reflexive equivalence $\nu_i(N) \cong (N/N_i) \oplus N_i'$. Since $H_N = \langle \mathrm{ind}^M N_1, \ldots, \mathrm{ind}^M N_n \rangle$ holds by 9.17, the dual argument tells us that $\mu_i(N) \cong (N/N_i) \oplus N_i'$.

(3) Applying $\mathrm{Hom}_{\mathcal{R}}(M, -)$ to the second sequence, we obtain an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{R}}(M, N_i) \to \mathrm{Hom}_{\mathcal{R}}(M, U_i) \to \mathrm{Hom}_{\mathcal{R}}(M, N_i') \to 0$$

by 8.11(2). Dually, applying $\mathrm{Hom}_{\mathcal{R}}(-, M)$ to the first sequence, we obtain an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{R}}(N_i, M) \to \mathrm{Hom}_{\mathcal{R}}(V_i, M) \to \mathrm{Hom}_{\mathcal{R}}(N_i', V) \to 0.$$

Now the assertions follow from 9.18. □

REMARK 9.27. If $\nu_i(N)$ is not an artinian mutation, then $\nu_i(N) \in \mathrm{MM}^N \mathcal{R}$ does not necessarily hold. For example, let $\mathcal{R} = k[[x, y, u, v]]/(x^3 - uv)$ and $N = \mathcal{R} \oplus (u, x)$. Then $\mu_2(N) = \mathcal{R} \oplus (u, x^2)$ does not belong to $\mathrm{MM}^N \mathcal{R}$. We do not know whether $\nu_i(N) \cong \mu_i(N)$ holds in general.

This has the following striking consequence.

COROLLARY 9.28. *Let* $N \in \mathrm{modif}\,\mathcal{R}$, *and* L *be an* arbitrary *direct summand of* N *such that* $\nu_L(N)$ *is an artinian mutation. Let* $M \in \mathrm{MM}\mathcal{R}$ *contain* N *as a summand.*

(1) *There is an isomorphism* $\nu_L(N) \cong \mu_L(N)$, *and so* $\nu_{L'}\nu_L(N) \cong N$.

(2) *The exchange sequences*

$$0 \to L' \to V \to L \quad \text{and} \quad 0 \to L \to U \to L'.$$

satisfy $U \cong V$, *and further* $\mathrm{ind}_M L + \mathrm{ind}_M L' = \mathrm{ind}_M U = \mathrm{ind}_M V$.

PROOF. Let $L = L_1 \oplus \ldots \oplus L_\ell$ with indecomposable L_i. For each $1 \le i \le \ell$, by 9.26 applied to $(N/L) \oplus L_i$, we have

$$(N/L) \oplus L_i' := \nu_{L_i}((N/L) \oplus L_i) \cong \mu_{L_i}((N/L) \oplus L_i). \tag{9.5.D}$$

However by definition of mutation,

$$\nu_L(N) = (N/L) \oplus L_1' \oplus \ldots \oplus L_\ell' \cong \mu_L(N).$$

This proves (1). For (2), the exchange sequences are the direct sum of the exchange sequences for the mutations (9.5.D), which satisfy the desired properties by 9.26. □

COROLLARY 9.29. *Let* $\mathcal{R}$ *be isolated cDV. If* $N \in \mathrm{modif}\,\mathcal{R}$, *then* $\mathrm{MM}^N \mathcal{R}$ *coincides with the mutation classes of* N.

PROOF. Let $X \in \mathrm{MM}^N\mathcal{R}$. First, consider the exchange sequence

$$0 \to X_i \to U_i \to X_i',$$

then by 9.24 $\operatorname{ind} U_i = \operatorname{ind} X_i + \operatorname{ind} X_i'$. Hence, combining 9.21(1) and 9.22(1), $\operatorname{ind}_M U_i = \operatorname{ind}_M X_i + \operatorname{ind}_M X_i'$. This shows that $\nu_i(X) \in \mathrm{MM}^N\mathcal{R}$, and so $\mathrm{MM}^N\mathcal{R}$ is closed under mutation. In particular $\mathrm{MM}^N\mathcal{R}$ contains the mutation class of N.

We next claim that $\mathrm{MM}^N\mathcal{R}$ is the full mutation class. Consider the composition

$$\mathrm{MM}^N\mathcal{R} \xrightarrow{\operatorname{Hom}_{\mathcal{R}}(N,-)} \text{ref-tilt}\operatorname{End}_{\mathcal{R}}(N) \hookrightarrow \operatorname{tilt}\Gamma_{\mathcal{J}_N} \xrightarrow{\sim} \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J}_N), \tag{9.5.E}$$

where the first bijection is (9.5.B), the second injection is 8.3(1), and the third is 5.26. Since $\mathrm{MM}^N\mathcal{R}$ is closed under mutation, the first bijection is clearly compatible with mutation; the second map is compatible with mutation by 8.3(3), and in the third, mutation corresponds to wall crossing by 5.2.

Consider an arbitrary chamber C in $\mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J}_N)$. By 1.20(2), this chamber can be connected to $(1, \mathcal{J}_N)$, the chamber corresponding to N, via a finite sequence of wall crossings. By induction starting at N, it follows that the chamber C is the image of some $Y \in \mathrm{MM}^N\mathcal{R}$, and furthermore Y can be obtained from N via a finite sequence of mutations. In particular, the composition (9.5.E) is a bijection. It also follows that $\mathrm{MM}^N\mathcal{R}$ is precisely the mutation class of N. $\square$

As notation for the next corollary, for $N \in \operatorname{modif}\mathcal{R}$ write $\mathrm{A} = \operatorname{End}_{\mathcal{R}}(N)$, and $S_0, \ldots, S_n$ for the simple A-modules. These correspond bijectively with the indecomposable summands $N_0, \ldots, N_n$ of N. For any subset $\mathcal{I} \subsetneq \{0, \ldots, n\}$, set $N_{\mathcal{I}} = \bigoplus_{i\in\mathcal{I}} N_i$, $\nu_{\mathcal{I}}(N) = \nu_{N_{\mathcal{I}}}(N)$, and write $\mathrm{A}_{\mathcal{I}} = \mathrm{A}/(1 - \sum_{i\in\mathcal{I}} e_i)$ for the associated contraction algebra.

In the case when $\mathcal{R}$ is isolated, all mutations are artinian, so the following generalises [**DW3**, 4.7, 4.9] by removing *all* restrictions on $\mathcal{I}$. It is also required for applications to twist autoequivalences in Part 4.

PROPOSITION 9.30. *Let $\mathcal{R}$ be cDV, suppose that $N \in \operatorname{modif}\mathcal{R}$ and $\mathcal{I} \subsetneq \{0, \ldots, n\}$ is such that $\nu_{\mathcal{I}}N$ is an artinian mutation. Then the minimal projective resolution of $\mathrm{A}_{\mathcal{I}}$ is*

$$0 \to \bigoplus_{i\in\mathcal{I}} \mathcal{P}_i \to \mathcal{Q} \to \mathcal{Q} \to \bigoplus_{i\in\mathcal{I}} \mathcal{P}_i \to \mathrm{A}_{\mathcal{I}} \to 0$$

where $\mathcal{P}_i \notin \operatorname{add}\mathcal{Q}$ for all $i \in \mathcal{I}$. In particular, $\operatorname{proj.dim}_{\mathrm{A}} \mathrm{A}_{\mathcal{I}} = 3$ and

$$\operatorname{Ext}^t_{\mathrm{A}}(\mathrm{A}_{\mathcal{I}}, S_i) = \begin{cases} \mathbb{C} & \text{if } t = 0, 3, \\ 0 & \text{else.} \end{cases}$$

PROOF. Since $\mathcal{R}$ is isolated, $\dim_{\mathbb{C}} \mathrm{A}_{\mathcal{I}} < \infty$. Further, we know $\nu_{\mathcal{I}}\nu_{\mathcal{I}}N \cong N$ by 9.28. These last two conditions are precisely the assumptions [**W2**, A.6(b)], hence the projective resolution of $\mathrm{A}_{\mathcal{I}}$ is now immediate from [**W2**, A.7]. Since the approximation sequences are minimal, the projective resolution is minimal, and so all other statements follow. $\square$

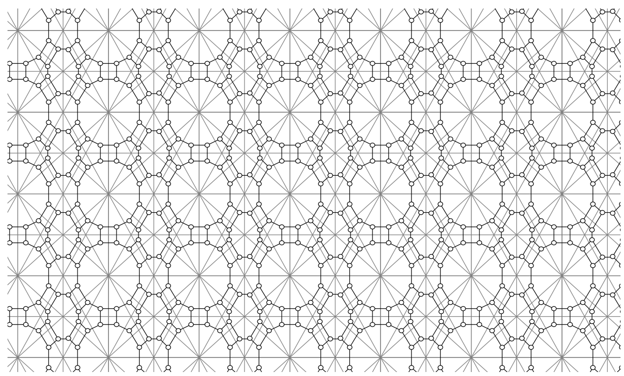
The following is largely a summary of some of the results so far, in the special case that $\mathcal{R}$ is isolated.

COROLLARY 9.31. *Let $\mathcal{R}$ be isolated cDV, and $N \in \operatorname{modif}\mathcal{R}$ with associated $\mathcal{J} = \mathcal{J}_N$. Then the following hold.*

(1) *The exchange graph of N equals the 1-skeleton of $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$. In particular, the exchange graph is connected.*
(2) *If further $\mathcal{R} \in \operatorname{add} N$, then*
 (a) *The elements of $\mathrm{MM}^N\mathcal{R} \cap \mathrm{CM}\,\mathcal{R}$ are in bijection with $\mathsf{Cham}(\Delta, \mathcal{J}_N)$. Wall crossing corresponds to mutation at non-free summands.*
 (b) *$\mathrm{MM}^N\mathcal{R} \cap \mathrm{CM}\,\mathcal{R}$ coincides with the mutation classes of N where we only mutate at non-free summands. In particular, this is a finite number, and this subgraph is connected.*

PROOF. (1) As already remarked in the proof of 9.29, the composition $\mathrm{MM}^N\mathcal{R} \to \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J}_N)$ in (9.5.E) is bijective, and mutation corresponds to wall crossing. Since by 9.29 $\mathrm{MM}^N\mathcal{R}$ is the mutation class of N, the result follows.
(2) Elements of $\mathrm{MM}^N\mathcal{R} \cap \mathrm{CM}\,\mathcal{R}$ are precisely those elements of $\mathrm{MM}^N\mathcal{R}$ which contain $\mathcal{R}$ as a summand. Across the bijection $\mathrm{MM}^N\mathcal{R} \to \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J}_N)$, this corresponds to all chambers in the image of $\mathsf{Cham}(\Delta, \mathcal{J}_N) \to \mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J}_N)$. Part (a) follows. For part (b), we can run the argument in 9.29, since we can connect any two chambers in $\mathsf{Cham}(\Delta, \mathcal{J}_N)$ by a finite sequence of wall crossings, at each step remaining in $\mathsf{Cham}(\Delta, \mathcal{J}_N)$. $\square$

EXAMPLE 9.32. Suppose that $\mathcal{R}$ is isolated of type cE_7, and that $N \in \operatorname{modif}\mathcal{R}$ has corresponding . Such an example exists. Combining 9.31 with 4.23, the exchange graph of N is illustrated as follows.



9.6. The $\mathcal{J}$-cone Groupoid

In this section we will re-interpret the above results in terms of the Deligne groupoid $\mathcal{G}_{\mathcal{J}_{\mathsf{aff}}}$ from §2.3, and we show that the mutation functors between $\operatorname{End}_{\mathcal{R}}(N)$ and $\operatorname{End}_{\mathcal{R}}(\nu_i N)$, ranging over the whole mutation class, form a representation of the groupoid. This subsection is the 3-fold version of §5.7.

To set notation, throughout this section $\mathcal{R}$ will be isolated cDV, $N \in \operatorname{modif}\mathcal{R}$, with associated $\mathcal{J} = \mathcal{J}_N$. By 9.31, we know the following.

- The mutation class of N is in bijection with the chambers $\mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J}_N)$. Write $X \mapsto \mathrm{C}_X$ for this bijection.
- If further $\mathcal{R} \in \operatorname{add} N$, then the Cohen–Macaulay mutation class of N is in bijection with the chambers $\mathsf{Cham}(\Delta, \mathcal{J}_N)$. We will abuse notation and still write $X \mapsto \mathrm{C}_X$ for this bijection.

DEFINITION 9.33. Let $\mathcal{R}$ be isolated cDV, and $N \in \operatorname{modif}\mathcal{R}$, with associated $\mathcal{J} = \mathcal{J}_N$.

1. The groupoid $\mathbb{H}_{\mathcal{J}_{\mathsf{aff}}}$ is defined as follows. As objects, to the chamber C_X associate a vertex labelled $\mathrm{D}^{\mathrm{b}}(\operatorname{mod}\operatorname{End}_{\mathcal{R}}(X))$. The morphisms are generated by the simple wall crossings, where to $\omega_i\colon X \to \nu_i X$ we associate the equivalence $\mathbf{R}\mathrm{Hom}_{\operatorname{End}_{\mathcal{R}}(X)}(\operatorname{Hom}_{\mathcal{R}}(X, \nu_i X), -)$.
2. If $\mathcal{R} \in \operatorname{add} N$, we also associate the groupoid $\mathbb{H}_{\mathcal{J}}$. This is defined in a similar way, where now the vertices are labelled $\mathrm{D}^{\mathrm{b}}(\operatorname{mod}\operatorname{End}_{\mathcal{R}}(X))$ for those X in the Cohen–Macaulay mutation class of N. Wall crossing is as above.

The following is the three-dimensional version of 5.30. It extends [HW1, 4.6] in two ways: firstly by constructing the affine version, and secondly by removing all smoothness assumptions.

PROPOSITION 9.34. *Let $\mathcal{R}$ be isolated cDV, and $N \in \operatorname{modif}\mathcal{R}$ with associated $\mathcal{J} = \mathcal{J}_N$. Consider a chain of simple mutations*

$$N = N_1 \to N_2 \to \ldots \to N_m \to N_{m+1}$$

then via 9.31*, this corresponds to a path in* $\mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J})$*. If this path is reduced, then there is a bimodule isomorphism*

$$\mathrm{Hom}_{\mathcal{R}}(N_m, N_{m+1}) \overset{\mathbf{L}}{\otimes} \dots \overset{\mathbf{L}}{\otimes} \mathrm{Hom}_{\mathcal{R}}(N_2, N_3) \overset{\mathbf{L}}{\otimes} \mathrm{Hom}_{\mathcal{R}}(N, N_2) \cong \mathrm{Hom}_{\mathcal{R}}(N, N_{m+1})$$

where, reading right to left, the tensors are over $\mathrm{End}_{\mathcal{R}}(M_i)$ *for* $i = 2, \dots, m$*.*

Proof. Exactly as in 5.30, we induct on the length of the path. Setting $\Lambda_m = \mathrm{End}_{\mathcal{R}}(M_m)$, it suffices to prove that there is an isomorphism

$$\mathrm{Hom}_{\mathcal{R}}(N_m, N_{m+1}) \otimes^{\mathbf{L}}_{\Lambda_m} \mathrm{Hom}_{\mathcal{R}}(N, N_m) \cong \mathrm{Hom}_{\mathcal{R}}(N, N_{m+1}) \tag{9.6.A}$$

in the category of bimodules. Since the path is reduced, $\mathrm{Hom}_{\mathcal{R}}(N, N_m) > \mathrm{Hom}_{\mathcal{R}}(N, N_{m+1})$. Given this last fact, as is standard (see e.g. [**HW1**, B.1]), it follows that the left hand side of (9.6.A) is concentrated in degree zero. Truncating in the category of bimodules, it follows that

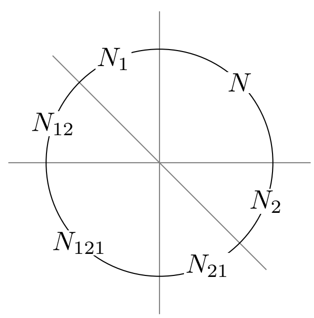
$$\mathrm{Hom}_{\mathcal{R}}(N_m, N_{m+1}) \otimes^{\mathbf{L}}_{\Lambda_m} \mathrm{Hom}_{\mathcal{R}}(N, N_m) \cong \mathrm{Hom}_{\mathcal{R}}(N_m, N_{m+1}) \otimes_{\Lambda_m} \mathrm{Hom}_{\mathcal{R}}(N, N_m)$$

as bimodules. From here, the proof is word-to-word identical to [**HW1**, 4.6]. Namely, set $\Lambda = \mathrm{End}_{\mathcal{R}}(N)$, $\mathbb{F} = \mathrm{Hom}_{\mathcal{R}}(N, -)$ and $T = \mathrm{Hom}_{\mathcal{R}}(N, N_n)$, then there is a chain of isomorphisms

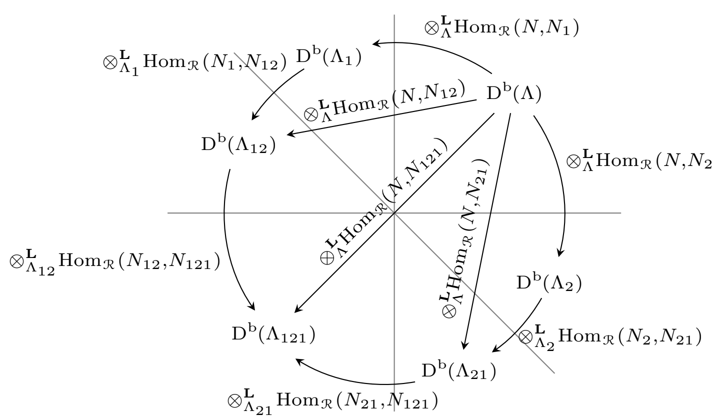
$$\mathrm{Hom}_{\mathfrak{R}}(N_m, N_{m+1}) \otimes \mathrm{Hom}_{\mathfrak{R}}(N, N_m) \xrightarrow{\sim} \mathrm{Hom}_{\Lambda}(T, \mathbb{F}N_{m+1}) \otimes T \xrightarrow{\sim} \mathbb{F}N_{m+1}$$

where the first is reflexive equivalence $g \otimes f \mapsto (g \circ -) \otimes f$, and the second is the adjunction from the derived equivalence (using the last statement in [**HW1**, B.1]), which takes $\varphi \otimes t \mapsto \varphi(t)$. The above composition takes $g \otimes f \mapsto g \circ f$, which by inspection this an isomorphism in the category of bimodules. □

Example 9.35. Suppose that $N \in \mathrm{modif}\,\mathcal{R}$ such that $\mathcal{J}_N =$ ••••. Such an example exists. Drawing the picture as in 5.27, consider the following part of the exchange graph:



Set $\Lambda_i = \mathrm{End}_{\mathcal{R}}(N_i)$, and $\Lambda = \mathrm{End}_{\mathcal{R}}(N)$. Then, as a direct consequence of 9.34, the following diagram commutes.



The above generalises to the following statement, which is the 3-fold version of 5.35.

THEOREM 9.36. *Let $\mathcal{R}$ be isolated cDV, and $N \in \operatorname{modif}\mathcal{R}$ with associated $\mathcal{J} = \mathcal{J}_N$. Then there are functors:*

$$\mathcal{G}_{\mathcal{J}} \to \mathbb{H}_{\mathcal{J}}$$
$$\mathcal{G}_{\mathcal{J}_{\mathsf{aff}}} \to \mathbb{H}_{\mathcal{J}_{\mathsf{aff}}}$$

given, in both cases, by sending a vertex corresponding to a chamber labelled C_X to $\mathrm{D^b}(\operatorname{mod}\operatorname{End}_{\mathcal{R}}(X))$. On morphisms, since wall crossing corresponds to mutation, the functor takes a wall crossing $X \to \nu_i X$ to $\mathbf{R}\mathrm{Hom}_{\mathrm{End}_{\mathcal{R}}(X)}(\mathrm{Hom}_{\mathcal{R}}(X, \nu_i X), -)$.

PROOF. In either case, denote the functor above by F. It suffices to show that the relations on $\mathcal{G}_{\mathcal{J}}$ and $\mathcal{G}_{\mathcal{J}_{\mathsf{aff}}}$ in 2.16 are satisfied functorially in $\mathbb{G}_{\mathcal{J}}$ and $\mathbb{G}_{\mathcal{J}_{\mathsf{aff}}}$. By definition, in 2.13, it suffices to show that any positive two reduced paths

$$\alpha, \beta \colon C_X \to C_Y$$

in $\mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J}_N)$ give rise to isomorphic functors $F(\alpha) \cong F(\beta)$.

We can view both paths as a sequence of mutations in $\mathsf{Cham}(\Delta_{\mathsf{aff}}, \mathcal{J}_X)$, starting at C_+. Clearly this reindexing does not effect whether the paths are reduced. The result then follows immediately from 9.34, since both $F(\alpha)$ and $F(\beta)$ are isomorphic to the direct functor given by $\mathbf{R}\mathrm{Hom}_{\mathrm{End}_{\mathcal{R}}(X)}(\mathrm{Hom}_{\mathcal{R}}(X, Y), -)$. □

Recall the notation $\pi_1(\mathcal{J})$ and $\pi_1(\mathcal{J}_{\mathsf{aff}})$ from 2.18. By passing to vertex groups, the following is then immediate from 9.36, and is the 3-fold version of 5.36.

COROLLARY 9.37. *Let $\mathcal{R}$ be isolated cDV, and $N \in \operatorname{modif}\mathcal{R}$ with associated $\mathcal{J} = \mathcal{J}_N$. Then there are group homomorphisms such that the following diagram commutes.*

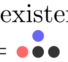
$$\begin{array}{ccc} \pi_1(\mathcal{J}) & \xrightarrow{\varphi} & \operatorname{Auteq}\mathrm{D^b}(\operatorname{mod}\operatorname{End}_{\mathcal{R}}(N)) \\ \downarrow & \nearrow_{\widetilde{\varphi}} & \\ \pi_1(\mathcal{J}_{\mathsf{aff}}) & & \end{array}$$

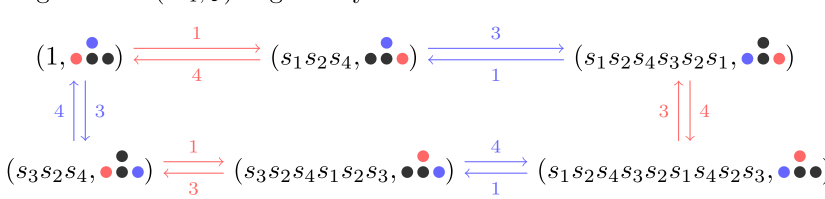
We will show in Part 4 that φ is faithful.

9.7. Global Ordering

One of the remarkable properties of Coxeter arrangements in $\mathbb{R}^n$ is that their 1-skeleta can be labelled by $s_1, \ldots, s_n$, globally, in such a way that:

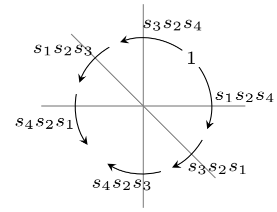
(1) Every vertex has precisely the labels $s_1, \ldots, s_n$ emerging from it.
(2) If s_i labels the simple wall crossing $C \to D$, then s_i also labels the simple wall crossing $D \to C$.

For our restriction arrangements, which need not be Coxeter, the existence of such a global ordering is not so clear. For example, consider the example $\mathcal{J} =$  in 3.3(1), in which wall crossing in $\mathsf{Cham}(D_4, \mathcal{J})$ is given by



Even although this is itself a Coxeter arrangement, the 'natural' numbers on the wall are written on the arrows, are induced from the number of the vertex being mutated.

These are constantly changing. The other 'natural' labelling on the walls is induced by multiplying by the relevant $w_{\mathcal{J}} w_{\mathcal{J}+i}$, namely:



However, whilst this satisfies property (2) above, it does not satisfy property (1).

To rectify this, we can always find some isolated cDV singularity $\mathcal{R}$ admitting a crepant resolution $X \to \operatorname{Spec} \mathcal{R}$ which slices to the minimal resolution of $\mathcal{R}/g\mathcal{R}$ (see e.g. [**T**]). By HomMMP [**W2**], the associated basic modifying module M, which is a generator, has associated $\mathcal{J}_M = \emptyset$. Thus, the following hold.

- For any choice $\mathcal{K} \subseteq \Delta_{\mathsf{aff}}$, we may find a summand of N of M such that $\mathcal{J}_N = \mathcal{K}$.
- For any choice $\mathcal{J} \subseteq \Delta$, we may find a summand L of M, which contains $\mathcal{R}$ as a summand, such that $\mathcal{J}_L = \mathcal{J}$, thought of as a subset of Δ_{aff}.

The purpose of this section to use the Krull–Schmidt decomposition of L and N to put a global ordering on the 1-skeleton of the arrangements $\mathsf{Cone}(\Delta, \mathcal{J})$ and $\mathsf{Cone}(\Delta_{\mathsf{aff}}, \mathcal{K})$, satisfying conditions (1) and (2).

We do this for $(\Delta_{\mathsf{aff}}, \mathcal{K})$, with the finite type case being similar. Fix a Krull–Schmidt decomposition

$$N \cong N_1 \oplus \ldots \oplus N_n.$$

The summands N_i give rise to mutations $\nu_i(N) = \nu_{N_i}(N)$ of N, which by 9.31 correspond to wall crossings out of the chamber $(1, \mathcal{K})$. Label these n wall crossings $(1, \mathcal{K}) \to C_{\nu_i(N)}$ by s_i, and in chamber $C_{\nu_i(N)}$ fix the ordering

$$\nu_i(N) \cong N_1 \oplus \ldots \oplus N_{i-1} \oplus N_i' \oplus N_{i+1} \oplus \ldots \oplus N_n.$$

Reading from left to right, the n indecomposable summands give rise to n mutations of $C_{\nu_i(N)}$, which we write $\nu_j \nu_i(N)$, as j runs from 1 to n. Label these n wall crossings $C_{\nu_i(N)} \to C_{\nu_j \nu_i(N)}$ by s_j, and in the chamber $C_{\nu_j \nu_i(N)}$ fix the ordering on $\nu_j \nu_i(N)$ from the fixed ordering on $\nu_i(N)$, replacing the jth summand.

Continue in this way. Given an arbitrary chamber C, find a positive minimal path $\alpha\colon (1, \mathcal{K}) \to C$. Repeating the above steps at each simple wall crossing of α gives a fixed ordering on the module L corresponding to C. Reading left to right of this fixed ordering of the summands of L gives mutations $\nu_1, \ldots, \nu_n$, and we use these to label the wall-crossings out of C by $s_1, \ldots, s_n$.

THEOREM 9.38. *For any pair* $(\Delta_{\mathsf{aff}}, \mathcal{K})$ *or* $(\Delta, \mathcal{J})$*, where* Δ *is ADE Dynkin, the above construction gives a well-defined global ordering on* $\mathsf{Cone}(\Delta, \mathcal{J})$ *and* $\mathsf{Cone}(\Delta_{\mathsf{aff}}, \mathcal{K})$*, satisfying properties* (1) *and* (2) *above.*

PROOF. The main problem is to show that the second part of the construction is well-defined. Suppose that $\alpha, \beta\colon (1, \mathcal{K}) \to C_X$ are both positive minimal paths. Then they induce the same Krull–Schmidt ordering on the summands of X if and only if

$$\beta^{-1} \circ \alpha\colon (1, \mathcal{K}) \to (1, \mathcal{K})$$

induces the fixed Krull–Schmidt order $N_1 \oplus \ldots \oplus N_n$ of N. But we can write the cycle of mutations $N \to N$ corresponding to $\beta^{-1} \circ \alpha$ as a product of conjugate rank two cycles (i.e. cycles around a codimension two wall crossings). Since by inspection in the rank two setting, which is in effect a finite hyperplane arrangement in $\mathbb{R}^2$, the mutation cycle induces the same fixed ordering, the above procedure is well defined.

Given it is well-defined, property (1) is clear by construction. Property (2) is just the statement that $\nu_i \nu_i(X) \cong X$. □

Part 4

Applications to Birational Geometry

CHAPTER 10

Autoequivalences and Faithful Actions

Given a subset $\mathcal{J} \subseteq \Delta$, where Δ is an ADE Dynkin diagram, we can associate:

(1) By Part 1, the following combinatorial data:
  (a) A finite hyperplane arrangement $\mathsf{Cone}(\mathcal{J})$ inside $\mathbb{R}^{|\mathcal{J}^c|}$.
  (b) The $\mathcal{J}$-Tits cone $\mathsf{Cone}(\Delta_{\mathsf{aff}}, \mathcal{J})$, and its level $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$ which is an infinite hyperplane arrangement inside $\mathbb{R}^{|\mathcal{J}^c|}$.
  (c) The arrangement groupoids $\mathcal{G}_{\mathcal{J}}$ and $\mathcal{G}_{\mathcal{J}_{\mathsf{aff}}}$ of 2.16, and the vertex groups $\pi_1(\mathcal{J})$ and $\pi_1(\mathcal{J}_{\mathsf{aff}})$ of 2.18.

(2) The following surfaces data:
  (a) A Kleinian singularity $\mathbb{C}^2/G$, where G corresponds to Δ by McKay correspondence.
  (b) A partial crepant resolution $g\colon Y \to \mathbb{C}^2/G$, obtained from the minimal resolution by blowing down the curves in $\mathcal{J}$.
  (c) A canonical tilting bundle $\mathcal{V}_Y$ on Y, such that its dual is generated by global sections [**VdB**]. It is well known [**KIWY**] that $\mathrm{End}_Y(\mathcal{V}_Y) \cong e_{\mathcal{J}}\Pi e_{\mathcal{J}} = \Gamma_{\mathcal{J}}$, where Π is the preprojective algebra of type Δ_{aff}.
  (d) The derived equivalence

$$\text{(10.0.A)} \qquad \mathbf{R}\mathrm{Hom}_Y(\mathcal{V}_Y, -)\colon\ \mathrm{D^b}(\mathrm{coh}\, Y) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{mod}\, \Gamma_{\mathcal{J}}).$$

(3) The following 3-fold data:
  (a) A (in fact many) flopping contraction $f\colon X \to \mathrm{Spec}\, \mathcal{R}$, where X has only terminal singularities, which slices to Y under generic $g \in \mathcal{R}$. Namely, there is a pullback diagram

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}\, \mathcal{R}/g & \longrightarrow & \mathrm{Spec}\, \mathcal{R} \end{array}$$

  where by Reid's general elephant $\mathrm{Spec}\, \mathcal{R}/g = \mathbb{C}^2/G$, and furthermore the left hand vertical map is precisely that appearing in (2) above.
  (b) The derived equivalence

$$\text{(10.0.B)} \qquad \mathbf{R}\mathrm{Hom}_X(\mathcal{V}_X, -)\colon\ \mathrm{D^b}(\mathrm{coh}\, X) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{mod}\, \mathrm{End}_{\mathcal{R}}(N)).$$

  where $\mathcal{V}_X$ is the canonical tilting bundle on X such that its dual is generated by global sections [**VdB**].
  (c) The $\mathcal{R}$-module $N := f_*\mathcal{V}_X$. It is well-known that $N \in \mathrm{modif}\, \mathcal{R}$, and since by construction $\mathcal{V}_X$ has summand $\mathcal{O}_X$, necessarily $\mathcal{R} \in \mathrm{add}\, N$.

Conversely, given either a crepant partial resolution $Y \to \mathbb{C}^2/G$, or a flopping contraction $X \to \mathrm{Spec}\, \mathcal{R}$ where X has only Gorenstein terminal singularities, we can associate such a $\mathcal{J}$ via slicing if necessary, then using McKay correspondence. Thus, throughout this chapter, the question of what is the input is really just a function of viewpoint.

The purpose of this chapter is to transfer the results of the previous parts across the derived equivalences above, and to spell out all the geometric corollaries. In §10.1 we will use the tilting theory of $\Gamma_{\mathcal{J}}$ from Part 2 to give new results about crepant partial resolutions $\mathbb{C}^2/G$, and in §10.2 we use the results about modifying modules in Part 3 to give new results about flopping contractions. In §10.3 we prove that the 'finite' group actions constructed in both settings are faithful generalising [**BT**, **HW1**].

10.1. Surfaces

As notation, consider a partial crepant resolution $Y \to \mathbb{C}^2/G$ for some finite subgroup $G \leq \mathrm{SL}(2,\mathbb{C})$. As Y is dominated by the minimal resolution, it can be obtained by blowing down a subset $\mathcal{J}$ of curves in the minimal resolution, and thus by McKay correspondence a subset $\mathcal{J}$ of an ADE Dynkin configuration.

Remark 10.1. Recall that the nodes in $\mathcal{J}$ are shaded $\bullet$, and the curves not in $\mathcal{J}$ are coloured. Thus, the coloured vertices correspond to curves in the partial resolution.

10.1.1. Twists for Wall Crossings. Here we gives an intrinsic description of certain wall crossing in terms of twist functors, and in the process produce new phenomena of spherical twists. Simple wall crossing can only possibly give an autoequivalence if the categories on both sides of the wall are equal; this translates into the condition that the label $\mathcal{J}$ does not change under the wall crossing. In the case $\mathcal{J} = \emptyset$ below, we recover the Seidel–Thomas twists on the minimal resolution.

As notation, for a fixed $\Gamma_{\mathcal{J}}$, which has unit $e_{\mathcal{J}}$, for each $i \in \mathcal{J}^c$ consider the factor $\Gamma_{\mathcal{J},i} = \Gamma_{\mathcal{J}}/\Gamma_{\mathcal{J}}(e_{\mathcal{J}} - e_i)\Gamma_{\mathcal{J}}$. This is the algebra that represents the noncommutative deformations of the simple $\Gamma_{\mathcal{J}}$-module $\mathcal{S}_i$ at the vertex i of $\Gamma_{\mathcal{J}}$.

Proposition 10.2. *Suppose that in the simple wall crossing formula, $\omega_i(x,\mathcal{J}) = (xx_0,\mathcal{J})$, i.e. the second term $\mathcal{J}$ does not change. Set $\mathsf{Twist}_i = \mathbf{R}\mathrm{Hom}_{\Gamma_{\mathcal{J}}}(e_{\mathcal{J}}I_{x_0}e_{\mathcal{J}}, -)$, which is the wall crossing equivalence from 5.32. Then the following hold.*

(1) *$\Gamma_{\mathcal{J},i}$ has finite projective dimension, as a $\Gamma_{\mathcal{J}}$-module.*
(2) *There is a functorial triangle*

$$\mathbf{R}\mathrm{Hom}_{\Gamma_{\mathcal{J}}}(\Gamma_{\mathcal{J},i}, -) \otimes^{\mathbf{L}}_{\Gamma_{\mathcal{J},i}} \Gamma_{\mathcal{J},i} \to (-) \to \mathsf{Twist}_i(-) \to$$

Proof. Set $j = \iota_{J+i}(i)$, then by assumption $j = i$. It follows that

$$\begin{aligned}
&\text{(since } i=j) && e_{\mathcal{J}}I_{x_0}e_{\mathcal{J}} = e_{\mathcal{J}}(I_{x_0}(e_{\mathcal{J}} - e_i) \oplus I_{x_0}e_j) \\
&\text{(by 5.14(5))} && \qquad = e_{\mathcal{J}}(\langle e_{\mathcal{J}} - e_i\rangle(e_{\mathcal{J}} - e_i) \oplus \langle e_{\mathcal{J}} - e_i\rangle e_j) \\
& && \qquad = e_{\mathcal{J}}\langle e_{\mathcal{J}} - e_i\rangle e_{\mathcal{J}} \\
& && \qquad = e_{\mathcal{J}}\Pi(e_{\mathcal{J}} - e_i)\Pi e_{\mathcal{J}} \\
&\text{(since } e_{\mathcal{J}}e_i = e_i = e_ie_{\mathcal{J}}) && \qquad = e_{\mathcal{J}}\Pi e_{\mathcal{J}}(e_{\mathcal{J}} - e_i)e_{\mathcal{J}}\Pi e_{\mathcal{J}} \\
& && \qquad = \Gamma_{\mathcal{J}}(e_{\mathcal{J}} - e_i)\Gamma_{\mathcal{J}}.
\end{aligned}$$

It follows that $\Gamma_{\mathcal{J},i} = \Gamma_{\mathcal{J}}/e_{\mathcal{J}}I_{x_0}e_{\mathcal{J}}$, and so there is a short exact sequence of bimodules

$$0 \to e_{\mathcal{J}}I_{x_0}e_{\mathcal{J}} \to \Gamma_{\mathcal{J}} \to \Gamma_{\mathcal{J},i} \to 0.$$

Since $e_{\mathcal{J}}I_{x_0}e_{\mathcal{J}}$ is tilting, and thus has projective dimension one, part (1) follows. Part (2) follows from the above bimodule exact sequence, exactly as in [**DW1**, 6.10]. ☐

Example 10.3. Consider $\mathcal{J} =$  . This is fixed under all wall crossing rules. In this example, a knitting calculation [**DW1**, 3.16, 3.17] confirms that $\Gamma_{\mathcal{J},\bullet} \cong \mathbb{C}$, and $\Gamma_{\mathcal{J},\bullet} \cong \mathbb{C}\langle x,y\rangle/(x^2, xy+yx, y^2)$. The green wall crossing thus gives rise to a spherical twist over $\mathbb{C}$, which is a classical spherical twist, and the red wall crossing gives a spherical twist over the exterior algebra in two variables.

10.1.2. Finite and Affine Actions. Keeping the notation that $Y \to \mathbb{C}^2/G$ is a crepant partial resolution corresponding to $\mathcal{J}$, from Part 1, consider the associated finite hyperplane arrangement $\mathsf{Cone}(\Delta, \mathcal{J})$, and infinite hyperplane arrangement $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$, both of which are hyperplane arrangements inside $\mathbb{R}^{|\Delta \setminus \mathcal{J}|}$. Write $\mathcal{X}$ for the complexification of $\mathsf{Cone}(\Delta, \mathcal{J})$, and $\mathcal{X}_{\mathsf{aff}}$ for the complexification of $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$.

THEOREM 10.4. *Consider a partial crepant resolution $Y \to \mathbb{C}^2/G$ for some finite subgroup $G \leq \mathrm{SL}(2, \mathbb{C})$, with associated $\mathcal{X}$ and $\mathcal{X}_{\mathsf{aff}}$ as above. Then there exist group homomorphisms such that the following diagram commutes.*

$$\begin{array}{ccc} \pi_1(\mathcal{X}) & \xrightarrow{\varphi} & \operatorname{Auteq} \mathrm{D}^{\mathrm{b}}(\operatorname{coh} Y) \\ \big\downarrow & \nearrow_{\widetilde{\varphi}} & \\ \pi_1(\mathcal{X}_{\mathsf{aff}}) & & \end{array}$$

PROOF. This follows from 2.15 and 5.36, passing through (10.0.A). $\square$

In general, since labels $\mathcal{J}$ change under wall crossing, and so the spaces on each side of the wall need not be isomorphic, the fundamental group action is the best we can hope for. However, when $\mathcal{J}$ is always fixed, below we improve the above to a braid group action (possibly of a non-ADE braid group!). When $\mathcal{J}$ is sometimes fixed, we improve the above to a mixed braid group action.

We first translate the spherical twists above into geometric notation. Under the equivalence (10.0.A), suppose that $\mathcal{E}_i$ corresponds to $\Gamma_{\mathcal{J},i}$. It turns out that $\mathcal{E}_i$ is the universal sheaf from the noncommutative deformation theory of $\mathcal{O}_{\mathrm{C}_i}(-1)$ [**DW1**, §3.2].

PROPOSITION 10.5. *Suppose that in a simple wall crossing, $\omega_i(x, \mathcal{J}) = (xx_0, \mathcal{J})$, i.e. the second term $\mathcal{J}$ does not change. Then the following hold.*

(1) *$\mathcal{E}_i$ is perfect, as a complex in $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} Y)$.*
(2) *There is an autoequivalence Twist_i fitting into a functorial triangle*

$$\mathbf{R}\mathrm{Hom}_Y(\mathcal{E}_i, -) \otimes^{\mathbf{L}}_{\Gamma_{\mathcal{J},i}} \mathcal{E}_i \to (-) \to \mathsf{Twist}_i(-) \to$$

PROOF. Part (1) follows immediately from 10.2(1), since derived equivalences preserve perfect complexes. Part (2) is a translation of 10.2(2), exactly as in [**DW1**, §6.4]. $\square$

WARNING 10.6. The assumption in 10.5 that the second term $\mathcal{J}$ does not change under wall crossing cannot be removed. In general, the sheaf $\mathcal{E}_i$ is not perfect. This should be contrasted with the 3-fold terminal flops setting, where the universal sheaf of the noncommutative deformation theory is automatically perfect [**DW1**, 5.6]. Homologically, canonical singularities behave much worse than terminal singularities.

In the case when the label never changes under wall crossing, we can identify all chambers and obtain a group action by an appropriate braid group. We now illustrate this via examples, which illustrate the key features.

REMARK 10.7. In the following examples, the label never changes under wall crossing.

(1) Consider $\mathcal{J} =$  and the corresponding partial resolution

$$Y \to \mathbb{C}^2/G$$

where G is the binary dihedral group of order eight. In this case, the finite hyperplane arrangement is 3.3(2), which is B_2. The infinite hyperplane arrangement is 4.15, which is affine B_2.

Since the label $\mathcal{J}$ is always fixed under all wall crossing rules, by 10.5 each wall crossing is in fact a twist autoequivalence. The length two and four braid relations have already been verified in 5.30, and so it follows that $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} Y)$

carries the action of the braid group, and of the affine braid group, of type B_2. The actions in 10.4 are the *pure* braid group, and the pure affine braid group, respectively.

(2) Consider $\mathcal{J} =$ , and the corresponding partial crepant resolution Y. In this case, the finite hyperplane arrangement is 3.5(5), which is G_2. The infinite hyperplane arrangement is 4.20, which is affine G_2.

Again, since the label $\mathcal{J}$ is always fixed under all wall crossing rules, by 10.5 each wall crossing is in fact a twist autoequivalence, and so $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} Y)$ carries the action of the braid group, and of the affine braid group, of type G_2.

In the above cases, and also in many others, the braid and affine braid group actions of non-ADE type appear, and are quite unexpected. It is still not clear precisely which non-ADE braid groups appear, and this problem has a similar flavour to 3.11, where perhaps classification of all the possible intersection arrangements is required.

10.1.3. Derived Classification of Partial Resolutions. The algebraic results in Chapter 7 have immediate geometric corollaries. The following is a derived equivalence classification of all partial crepant resolutions of Kleinian singularities. This problem is quite fundamental, and it is surprising it has not been investigated before.

Corollary 10.8. *Suppose that $Y \to \mathbb{C}^2/G$ and $Y' \to \mathbb{C}^2/G'$ are two crepant partial resolutions, with associated $\mathcal{J} \subset \Delta_{\mathsf{aff}}$ and $\mathcal{J}' \subset \Delta'_{\mathsf{aff}}$. Consider the following two conditions.*

(1) *Y is derived equivalent to Y'.*
(2) *$\Delta = \Delta'$ and $\mathcal{J} \sim \mathcal{J}'$, namely up to symmetries of the extended ADE graph, $\mathcal{J}$ and $\mathcal{J}'$ can be linked through a sequence of iterated wall-crossing moves.*

Then (2)⇒(1). Further, if either $\Delta, \Delta' \in \{A_n, D_4, D_5, D_6, D_7, E_6, E_7, E_8\}$ then (1)⇒(2).

Proof. This follows from 7.21, after passing through the equivalences (10.0.A). □

10.2. Threefolds

We next consider the case of a flopping contraction $f : X \to \operatorname{Spec} \mathcal{R}$, where X has only Gorenstein terminal singularities. As before, this slices under generic $g \in \mathcal{R}$ to give a partial crepant resolution of a Kleinian singularity. The partial crepant resolution is obtained from the minimal resolution by blowing down the curves, and the subset $\mathcal{J}$ of vertices of the associated ADE Δ records which curves are blown down to produce it.

Remark 10.9. Recall from 10.1 that the nodes in $\mathcal{J} \subseteq \Delta$ are shaded $\bullet$, whereas the nodes not in $\mathcal{J}$ (which thus correspond to the flopping curves) are coloured.

The following result is then immediate from Part 3. The first proof of this theorem, in the Gorenstein terminal setting, was given in [**DW3**] using moduli tracking and a case-by-case analysis. This was simplified somewhat via tilting in [**HW1**], at the cost of assuming that X is smooth. Here we can use the tilting results of Chapter 9 to give a simplified proof in all cases.

Corollary 10.10. *Suppose that $X \to \operatorname{Spec} \mathcal{R}$ is a flopping contraction, contracting precisely two intersecting irreducible curves, where X has at worst Gorenstein terminal singularities. Then*

$$\underbrace{\mathsf{F}_1 \circ \mathsf{F}_2 \circ \mathsf{F}_1 \circ \cdots}_{d} \cong \underbrace{\mathsf{F}_2 \circ \mathsf{F}_1 \circ \mathsf{F}_2 \circ \cdots}_{d}$$

where d is the number of hyperplanes in $\mathsf{Cone}(\Delta, \mathcal{J})$, where $\mathcal{J} \subseteq \Delta$ is the Dynkin type of the flopping contraction.

Proof. The assumptions imply that $\mathcal{R}$ is isolated. By [**W2**, 4.2], the flop functor is isomorphic to the inverse of the mutation functor. Hence it suffices to show that the mutation functors braid in the stated manner. But this is just 9.36. □

The now standard local-to-global techniques of [**DW1**, **DW3**], specifically [**DW3**, 3.13, 3.15], then immediately lift the above complete local result 10.10 to give the following global consequence.

COROLLARY 10.11. *Suppose that $X \to X_{\text{con}}$ is a flopping contraction between quasi-projective 3-folds, contracting precisely two intersecting independently floppable irreducible curves. If X has at worst Gorenstein terminal singularities, then*

$$\underbrace{\mathsf{F}_1 \circ \mathsf{F}_2 \circ \mathsf{F}_1 \circ \cdots}_{d} \cong \underbrace{\mathsf{F}_2 \circ \mathsf{F}_1 \circ \mathsf{F}_2 \circ \cdots}_{d}$$

where d is the number of hyperplanes in $\mathsf{Cone}(\Delta, \mathcal{J})$, where $\mathcal{J} \subseteq \Delta$ is the Dynkin type of the flopping contraction.

In addition to vastly simplifying the proof of 10.10, and avoiding case-by-case analysis, we are for the first time able to precisely determine d, the length of the braid relation.

COROLLARY 10.12. *Suppose that $X \to X_{\text{con}}$ is a flopping contraction between quasi-projective 3-folds, as in 10.11. Then the length of the braid relation is either $2, 3, 4, 5, 6$, or 8. The first case, namely $d = 2$, holds if and only if the curves are disjoint.*

PROOF. If the curves are disjoint, it is very well-known that the flop functors commute (see e.g. [**DW3**]), so the braid relation has length two. If the curves intersect, by 10.11 the flops functors braid where d is the number of hyperplanes in $\mathsf{Cone}(\Delta, \mathcal{J})$. By 3.11, this number is $3, 4, 5, 6$, or 8. ☐

The real power of this memoir is that there are *many* more autoequivalences than those given as compositions of flop functors. Before stating the full affine action results, we restrict first to the following special cases.

As preparation, consider $\mathcal{S}_0, \ldots, \mathcal{S}_n$, where $\mathcal{S}_0 = \omega_{\mathrm{C}}[1]$, and $\mathcal{S}_i = \mathcal{O}_{\mathrm{C}_i}(-1)$ for all $i > 0$. By [**VdB**, 3.5.8], these are precisely the simple $\mathrm{A} = \mathrm{End}_{\mathcal{R}}(N)$-modules under the equivalence (10.0.B). With 9.30 now established, Part (1) of the following result removes all restrictions on [**DW3**, §4.1]. Part (2) was the main motivation behind this memoir.

THEOREM 10.13. *Suppose that $f\colon X \to X_{\text{con}}$ is a flopping contraction of quasi-projective 3-folds, where X has only Gorenstein terminal singularities.*

(1) *For any $\mathcal{I} \subsetneq \{0, \ldots, n\}$, let $\mathcal{E}_{\mathcal{I}}$ be the universal object of the noncommutative deformation theory of $\{\mathcal{S}_i\}_{i \in \mathcal{I}}$, and set $\mathrm{A}_{\mathcal{I}} = \mathrm{End}_X(\mathcal{E}_{\mathcal{I}})$. Then there is a twist autoequivalence, together with a functorial triangle*

$$\mathbf{R}\mathrm{Hom}_X(\mathcal{E}_{\mathcal{I}}, x) \otimes^{\mathbf{L}}_{\mathrm{A}_{\mathcal{I}}} \mathcal{E}_{\mathcal{I}} \to x \to \mathsf{Twist}_{\mathcal{I}}(x) \to$$

(2) *In particular, for $\mathcal{I} = \{0\}$, Let $\mathcal{E}_{\text{fib}}$ be the universal sheaf of the noncommutative deformation theory of ω_{C}, and set $\mathrm{A}_{\text{fib}} = \mathrm{End}_X(\mathcal{E}_{\text{fib}})$. Then there is a fibre twist autoequivalence FTwist, together with a functorial triangle*

$$\mathbf{R}\mathrm{Hom}_X(\mathcal{E}_{\text{fib}}, x) \otimes^{\mathbf{L}}_{\mathrm{A}_{\text{fib}}} \mathcal{E}_{\text{fib}} \to x \to \mathsf{FTwist}(x) \to$$

PROOF. (1) This follows word-for-word [**DW3**, §5], where 9.30 replaces either [**DW3**, 4.7] or [**DW3**, 4.9] appropriately.
(2) Is an immediate special case of (1). ☐

The above 10.13 gives many more twist autoequivalences than just compositions of flop functors. To give the most general result, the full affine action, as in Part 3 we pass to the Deligne groupoid, but only from an algebraic perspective. As notation, recall that to $X \to \operatorname{Spec} \mathcal{R}$ we can associate a subset $\mathcal{J}$ of an ADE Dynkin Δ. As in §10.1.2, consider the associated finite hyperplane arrangement $\mathsf{Cone}(\Delta, \mathcal{J})$, and infinite hyperplane arrangement $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$, both of which are hyperplane arrangements inside $\mathbb{R}^{|\Delta \setminus \mathcal{J}|}$.

Given $X \to \operatorname{Spec} \mathcal{R}$, we can associate $N \in \operatorname{modif} \mathcal{R}$, and by 9.36:

(1) The mutation functors between $\mathrm{D^b}(\operatorname{mod}\operatorname{End}_{\mathcal{R}}(M))$, where $M \in \mathrm{MM}^N\mathcal{R} \cap \mathrm{CM}\,\mathcal{R}$, form a representation of the corresponding Deligne groupoid $\mathcal{G}_{\mathcal{J}}$.
(2) The mutation functors between $\mathrm{D^b}(\operatorname{mod}\operatorname{End}_{\mathcal{R}}(M))$, where $M \in \mathrm{MM}^N\mathcal{R}$, form a representation of the corresponding Deligne groupoid $\mathcal{G}_{\mathcal{J}_{\mathsf{aff}}}$.

It is possible to give a strictly geometric version of (1), by replacing it with the flop functors between $\mathrm{D^b}(\operatorname{coh} Y)$, where Y is obtained from X via iterated flop. However, there is no good geometric replacement of (2), and so we must rely on the noncommutative resolutions (and their variants) to construct it. Regardless of this lack of a birational-geometric version of the affine Deligne groupoid, we still have the following. As before, write $\mathcal{X}$ for the complexification of $\mathsf{Cone}(\Delta, \mathcal{J})$, and $\mathcal{X}_{\mathsf{aff}}$ for the complexification of $\mathsf{Level}(\mathcal{J}_{\mathsf{aff}})$.

Corollary 10.14. *Let $X \to \operatorname{Spec}\mathcal{R}$ denote a 3-fold flopping contraction, where X has only Gorenstein terminal singularities. Then there are group homomorphisms such that the following diagram commutes.*

$$\begin{array}{ccc} \pi_1(\mathcal{X}) & \xrightarrow{\varphi} & \operatorname{Auteq}\mathrm{D^b}(\operatorname{coh} X) \\ \downarrow & \nearrow_{\widetilde{\varphi}} & \\ \pi_1(\mathcal{X}_{\mathsf{aff}}) & & \end{array}$$

Proof. This follows from 2.15 and 9.37, passing through (10.0.B). □

10.3. Faithful Actions

In this section, which is a mild extension of the techniques in [HW1], we prove that the 'finite actions' φ on surfaces and 3-folds in 10.4 and 10.14 are faithful. The action in the surfaces case is new, when $\mathcal{J} \neq \emptyset$, as is the faithfulness of it. The action in the 3-fold setting was proved in [HW1] under the additional assumption that X was smooth; here we again use the tilting advances in Part 3 to remove this.

Setup 10.15. We consider one of the following two settings.

(1) A partial crepant resolution $Y \to \mathbb{C}^2/G$, for some finite $G \leq \mathrm{SL}(2,\mathbb{C})$. There is an associated subset $\mathcal{J}$ of an affine ADE Dynkin configuration, which does not contain the extended vertex. As in (10.0.A), Y is derived equivalent to $\Gamma_{\mathcal{J}}$.
(2) A 3-fold flopping contraction $X \to \operatorname{Spec}\mathcal{R}$ where X has at worst Gorenstein terminal singularities. Necessarily X is derived equivalent to some $\operatorname{End}_{\mathcal{R}}(N)$ with $N \in \operatorname{modif}\mathcal{R}$ and $\mathcal{R} \in \operatorname{add} N$.

In case (1), below set $A = \Gamma_{\mathcal{J}}$ and $d = 2$. In case (2), below set $A = \operatorname{End}_{\mathcal{R}}(N)$ and $d = 3$. In either case, write $\mathcal{S}_0, \ldots, \mathcal{S}_n$ for the simple A-modules, and $\mathcal{S} = \bigoplus_{i=0}^n \mathcal{S}_n$. By convention, in case (1) $\mathcal{S}_0$ corresponds to the extended vertex, and in case (2) $\mathcal{S}_0$ corresponds to the summand $\mathcal{R}$.

Throughout, for a triangulated category $\mathcal{C}$ and $a, b \in \mathcal{C}$, we match the notation in [BT] and write
$$[a, b]_t := \operatorname{Hom}_{\mathcal{C}}(a, b[t]).$$
The following is elementary.

Lemma 10.16 ([HW1, 6.2]). *In Setup 10.15, suppose that N is a non-zero A-module of finite length.*

(1) *If $y \in \mathrm{D^b}(\operatorname{mod} A)$ is such that $[\mathcal{S}, y]_{\geq p} = 0$, then $[N, y]_{\geq p} = 0$.*
(2) $[N, A]_d \neq 0$ *and* $[N, A]_{\geq d+1} = 0$.

For $\mathcal{H} = \mathsf{Cone}(\Delta, \mathcal{J})$, recall from §2.3 that the category of positive paths is defined $\mathcal{G}_{\mathcal{J}}^+ := \operatorname{Free}(\Gamma_{\mathcal{H}})/\sim$, and the Deligne groupoid is obtained as the groupoid completion of this. Since $\mathcal{H}$ is a finite simplicial hyperplane arrangement, by 2.14 the natural functor

$\mathcal{G}_{\mathcal{J}}^{+} \to \mathcal{G}_{\mathcal{J}}$ is faithful. This fact, together with the existence of Deligne normal form for finite simplicial arrangements, is crucial for the proofs below.

As in [**HW1**], for any α, define

$$t_\alpha = F(\alpha),$$

where F is the functor in either 5.35 or 9.36. The key point, exactly as in [**HW1**, §5], is that 5.30 and 9.34 allow us to give direct functors in the case of reduced paths, and thus we can use the torsion pairs arguments from [**HW1**]. The following is a variation of the main technical lemma from [**BT**] and [**HW1**], which follows in our general setting here via the previous sections.

PROPOSITION 10.17. *In Setup 10.15, let $\alpha \in \mathcal{G}_{\mathcal{J}}^{+}$ have Deligne normal form $\alpha = \alpha_k \circ \ldots \circ \alpha_1$. Then the following statements hold.*

1. $[\mathcal{S}, t_\alpha A]_{\geq k+d+1} = 0$.
2. $[\mathcal{S}_i, t_\alpha A]_{k+d} \neq 0$ *if and only if $i \neq 0$, and the atom α_k ends (up to the relations in $\mathcal{G}_{\mathcal{J}}^{+}$) by passing through wall i. In particular $[\mathcal{S}, t_\alpha A]_{k+d} \neq 0$.*
3. *The maximum p such that $[\mathcal{S}, t_\alpha A]_{k+d} \neq 0$ is precisely $p = k + d$.*

PROOF. With 5.30 and 9.34 already established, this follows word-for-word as in [**HW1**, 6.3], using $b = \mathrm{A}$. □

COROLLARY 10.18. *The functors F in 5.35 and 9.36 are faithful.*

PROOF. Given 10.17, this is now word-for-word identical to [**HW1**, 6.5] □

Passing to vertex groups, and using 2.15, the following is then immediate.

COROLLARY 10.19. *Under the Setup 10.15, the following statements hold.*

1. *The homomorphism $\pi_1(\mathcal{X}) \to \operatorname{Auteq} \mathrm{D}^{\mathrm{b}}(\operatorname{coh} Y)$ in 10.4 is injective.*
2. *The homomorphism $\pi_1(\mathcal{X}) \to \operatorname{Auteq} \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$ in 10.14 is injective*

REMARK 10.20. By exactly the same proof of 10.17, and exactly as in [**HW1**, p21], it follows that the non-ADE actions of B_2 and G_2 in 10.7 are also faithful. These are the first known examples of non-ADE Coxeter braid group actions in an algebraic-geometric context.

REMARK 10.21. Even in the case of the classical case $\mathcal{J} = \emptyset$, with the affine braid group acting on the minimal resolution, it is still not known if the affine action is faithful. The papers [**IU, IUU**] establish this for the minimal resolutions of cyclic groups.

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