

STRINGY KÄHLER MODULI, MUTATION AND MONODROMY

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ABSTRACT. This paper gives the first description of derived monodromy on the stringy Kähler moduli space (SKMS) for a general irreducible flopping curve C in a 3-fold X with mild singularities. We do this by constructing two new infinite helices: the first consists of sheaves supported on C , and the second comprises vector bundles in a tubular neighbourhood. We prove that these helices determine the simples and projectives in iterated tilts of the category of perverse sheaves, and that all objects in the first helix induce a twist autoequivalence for X . We show that these new derived symmetries, along with established ones, induce the full monodromy on the SKMS.

The helices have many further applications. We (1) prove representability of non-commutative deformations of the sheaves $\mathcal{O}_C, \dots, \mathcal{O}_{\ell C}$ associated to a length ℓ flopping curve, via tilting theory, (2) control the representing objects, characterise when they are not commutative, and their central quotients, and (3) give new and sharp theoretical lower bounds on Gopakumar–Vafa invariants for a curve of length ℓ . When X is smooth and resolves an affine base, we furthermore (4) prove that the second helix classifies all tilting reflexive sheaves on X , and thus that (5) all noncommutative crepant resolutions arise from tilting bundles on X .

1. INTRODUCTION

Describing the simples and projectives in categories of perverse sheaves, and their tilts, is a fundamental problem. In this paper, in the setting of 3-fold flopping contractions, we describe both, for iterated tilts of zero perverse sheaves ${}^0\text{Per}$ at simple objects.

Our main breakthrough is the construction of two new invariants of the flopping curve in the form of infinite helices of sheaves $\{\mathcal{S}_i\}$ and $\{\mathcal{V}_i\}$, which we show control the simples and projectives in iterated tilts of ${}^0\text{Per}$, respectively. We believe that the new helices are intrinsic and of wider importance: we show that they uncover new unexpected phenomena in the autoequivalence groups of 3-folds, where each \mathcal{S}_i gives rise to a certain generalised spherical (or Dehn) twist, and we show that the helices have further applications to deformation theory, noncommutative resolutions, and Gopakumar–Vafa invariants.

1.1. SKMS and Monodromy. The stringy Kähler moduli space (SKMS) associated to a variety X is central to the study of mirror symmetry: it is isomorphic to the complex structure moduli of the mirror manifold, and furthermore its fundamental group is conjectured to recover the derived symmetries of X . However, even in the crucial setting of Calabi–Yau 3-folds, this conjecture has only been verified rigorously in certain very restricted classes of examples.

In the simplest case, namely the Atiyah flop between resolutions of $uv = x^2 + y^2$, Aspinwall [A] explains how the associated derived symmetries may be recovered from an SKMS given by the sphere minus three points, as illustrated in Figure 1. The variety X and its flop X^+ correspond to ‘large radius limits’ near two of the removed points, shown at the poles of the sphere. Derived equivalences correspond to homotopy classes of paths between the two basepoints; monodromy around the central hole corresponds to a spherical, or Dehn, twist.

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Toda generalised this result to the flop between resolutions of $uv = x^2 + y^{2n}$, known as flops of length one [T2, T3]. In this case the SKMS, denoted $\mathcal{M}_{\mathcal{S}\mathcal{X}}$, can be considered as a certain factor of a Bridgeland stability manifold associated to X . Toda proves that in this class of examples, $\mathcal{M}_{\mathcal{S}\mathcal{X}}$ is still a sphere minus three points, and furthermore its fundamental group acts on the derived category of X via compositions of the functors shown in Figure 1, where F is the flop functor.

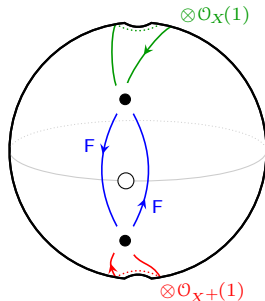


FIGURE 1. Monodromy on $\mathcal{M}_{\mathcal{S}\mathcal{X}}$ for length one flops.

Irreducible flops are fundamental building blocks of Calabi–Yau geometry, and are the most elementary of higher-dimensional birational surgeries. In this paper we give a geometric description of monodromy on $\mathcal{M}_{\mathcal{S}\mathcal{X}}$ when $X \rightarrow \text{Spec } R$ is an arbitrary irreducible 3-fold flop, of any type, and of any length, where X can even have mild singularities. In this setting, the defining equations of R are not even precisely known.

To do this, we generalise Figure 1, revealing a surprising and beautiful structure.

1.2. The Simple Helix. Much of the richness of a general flopping contraction arises because the exceptional locus need not be reduced. For an irreducible 3-fold flopping contraction $X \rightarrow X_{\text{con}}$ where X has only Gorenstein terminal singularities, write C for the exceptional locus with reduced scheme structure. Suppose that the contraction has length ℓ , as recalled in §2.1, which is necessarily a number between one and six. Then there exists a sequence of successive thickenings

$$\mathcal{O}_C, \mathcal{O}_{2C}, \dots, \mathcal{O}_{\ell C}.$$

Full definitions are given in §2.1, but for calibration, $\mathcal{O}_{\ell C}$ is the structure sheaf of the scheme-theoretic fibre of the contraction.

Our simple helix $\{\mathcal{S}_i\}_{i \in \mathbb{Z}}$ is defined in §4.4, first by specifying a finite region of size N , then by translating this region by tensoring by line bundles. We refer to N as the *helix period*, and it is defined as follows:

ℓ	1	2	3	4	5	6
N	1	2	4	6	10	12

The period N depends only on the length ℓ of the curve, but in contrast to the length, N does not yet have an obvious algebro-geometric interpretation.

When $\ell = 1$, the period $N = 1$ and the helix is simply $\mathcal{S}_i := \mathcal{O}_C(i - 1)$. For length $\ell > 1$, the period is always even, and the helix is determined by the following properties.

(1) We put

$$\mathcal{S}_0, \dots, \mathcal{S}_{N/2} = \begin{cases} \mathcal{O}_C(-1), \mathcal{O}_{\ell C}, \dots, \mathcal{O}_{2C} & \text{if } \ell \leq 4 \\ \mathcal{O}_C(-1), \mathcal{O}_{\ell C}, \dots, \mathcal{O}_{3C}, \mathcal{Z}, \mathcal{O}_{2C} & \text{if } \ell = 5, 6 \end{cases}$$

where \mathcal{Z} is explained below.

(2) $\mathcal{S}_{-n} = \mathbb{D}\mathcal{S}_n[-1]$ for all n , where \mathbb{D} is the dualizing functor.

(3) $\mathcal{S}_{i+N} \cong \mathcal{S}_i \otimes \mathcal{O}(1)$ for all i .

It follows that the simples helix contains, albeit not in order, the sheaves $\mathcal{O}_C, \mathcal{O}_{2C}, \dots, \mathcal{O}_{\ell C}$, the canonical bundles $\omega_C, \omega_{2C}, \dots, \omega_{\ell C}$, and also all line bundle twists of both. However, there is a surprise: when $\ell = 5, 6$ we require a further sheaf \mathcal{Z} above, which we construct in 4.14 as the unique non-split extension $0 \rightarrow \mathcal{O}_{3C} \rightarrow \mathcal{Z} \rightarrow \mathcal{O}_{2C} \rightarrow 0$.

Remark 1.1. We call $\{\mathcal{S}_i\}_{i \in \mathbb{Z}}$ a helix by analogy with the helices $\{E_i\}_{i \in \mathbb{Z}}$ for the total space of the canonical bundle ω_Z of a del Pezzo surface Z , see for instance [B2, BS, R2]. Here, the relation $E_{i+n} \cong E_i \otimes \omega_Z$ is replaced by $\mathcal{S}_{i+N} = \mathcal{S}_i \otimes \mathcal{O}(1)$, and the fact that the E_i induce spherical objects is replaced by the fact that whilst the \mathcal{S}_i do not determine spherical objects exactly, they do after noncommutative deformation.

1.3. Monodromy on the SKMS. The simples helix gives new derived autoequivalences. Indeed, our first result shows that all members \mathcal{S}_i induce an autoequivalence in the global quasi-projective setting. This answers a question of Kawamata [K2, 6.8] for the sheaves $\mathcal{O}_C, \mathcal{O}_{2C}, \dots, \mathcal{O}_{\ell C}$, but furthermore includes other sheaves like \mathcal{Z} . We prove in 7.9 that the noncommutative deformation functor for each \mathcal{S}_i is representable, and as a consequence obtain universal sheaves \mathcal{E}_i .

Theorem 1.2 (7.10). *Let $Y \rightarrow Y_{\text{con}}$ be a flopping contraction of quasi-projective 3-folds, where Y has at worst Gorenstein terminal singularities. For any contracted curve, consider the simples helix $\{\mathcal{S}_i\}_{i \in \mathbb{Z}}$. Then \mathcal{E}_i is perfect on Y , and there is an autoequivalence $\text{Twist}_{\mathcal{S}_i}$ of $D^b(\text{coh } Y)$ which fits into a functorial triangle*

$$\mathbf{R}\text{Hom}_Y(\mathcal{E}_i, -) \otimes_{\text{End}_Y(\mathcal{E}_i)}^{\mathbf{L}} \mathcal{E}_i \rightarrow \text{Id} \rightarrow \text{Twist}_{\mathcal{S}_i} \rightarrow .$$

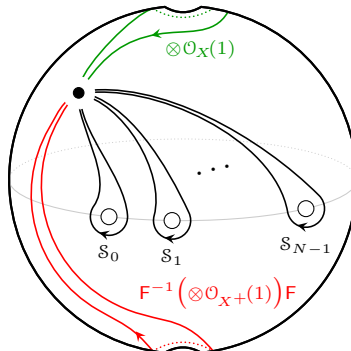
Motivated by mirror symmetry (§1.1), the hard work in this paper goes into showing how the above autoequivalences knit together, and describing monodromy on the stringy Kähler moduli space. In our 3-fold flop setting, the SKMS \mathcal{M}_{SK} is defined to be a certain quotient of normalised Bridgeland stability conditions [T2, HW2]. It follows easily using the techniques in [HW2], summarised in 2.7, that if $X \rightarrow \text{Spec } \mathfrak{R}$ is a local length ℓ Gorenstein terminal flop, then \mathcal{M}_{SK} is the 2-sphere minus $N + 2$ points, where N is the helix period of §1.2. Henceforth \mathcal{M}_{SK} is the 2-sphere, as above, but now with points removed from its poles, and N points removed from the equator.

Our main result is that the simples helix describes the derived monodromy on \mathcal{M}_{SK} .

Theorem 1.3 (6.5). *There is a group homomorphism*

$$\begin{aligned} \pi_1(\mathcal{M}_{\text{SK}}) &\rightarrow \text{Auteq } D^b(\text{coh } X) \\ q_i &\mapsto \text{Twist}_{\mathcal{S}_i} \\ q_- &\mapsto - \otimes_{\mathcal{O}_X}(-1) \\ q_+ &\mapsto F^{-1} \circ (- \otimes_{\mathcal{O}_X}(-1)) \circ F \end{aligned}$$

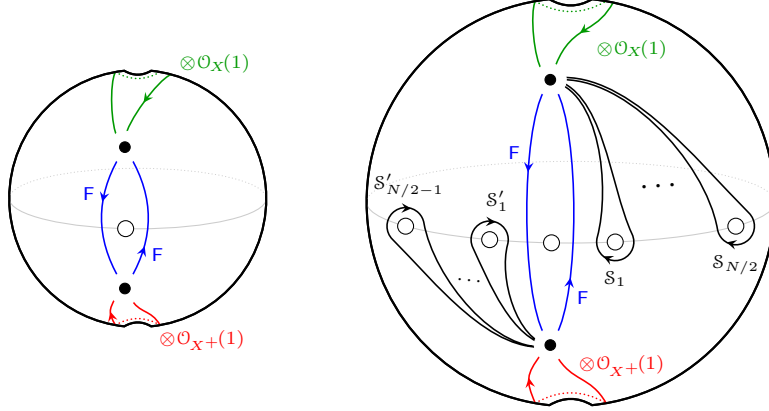
illustrated in the following diagram



where F is the flop functor, and black monodromies indicate twist functors Twist_{S_i} on X , defined in a similar way as above.

Given $X \rightarrow \text{Spec } \mathfrak{R}$, we obtain a symmetric version of the above result if we systematically account for the flop $X^+ \rightarrow \text{Spec } \mathfrak{R}$. Flopping is an involution, and this new version shows that this symmetry is reflected in the derived monodromy action.

Theorem 1.4 (6.8). *For a 3-fold flop $X \dashrightarrow X^+$, with length invariant ℓ , there is an action of a fundamental groupoid of $\mathcal{M}_{S^3 X}$ with two basepoints on $D^b(\text{coh } X)$ and $D^b(\text{coh } X^+)$ given, in the cases $\ell = 1$ and $\ell > 1$ respectively, as follows:*



Black monodromies indicate twist functors Twist_{S_i} on X from 1.2, along with functors $\text{Twist}_{S'_i}$ on X^+ defined in the same way.

The above result extends Figure 1 to higher length flops. It turns out that only the sheaves \mathcal{Z} and $\mathcal{O}_{\mathbb{C}}$ in the above theorem can be genuinely spherical (7.5), and for all others we need deformation theory to obtain the autoequivalences.

Our main new technique to construct the above actions is to geometrically describe the simples and projectives in iterated tilts of the category of perverse sheaves ${}^0\text{Per}$. The projectives control the noncommutative deformation theory, and construct the twists. The actions then follow since monodromy corresponds to twisting around the simples.

1.4. Tilts of Perverse Sheaves. In the following, we construct a family of sheaves $\{\mathcal{V}_i\}_{i \in \mathbb{Z}}$, which we then prove are vector bundles.

Proposition 1.5 (5.2, 5.8, 5.10). *For all $i \in \mathbb{Z}$, \mathcal{V}_i is a vector bundle on X , and furthermore the following hold.*

- (1) *There is a short exact sequence*

$$0 \rightarrow \mathcal{V}_{i-1} \rightarrow \mathcal{V}_i^{\oplus n_i} \rightarrow \mathcal{V}_{i+1} \rightarrow 0,$$

where the numbers n_i , and the ranks of the \mathcal{V}_i , are given in 2.6.

- (2) $\mathcal{V}_{i+N} \cong \mathcal{V}_i \otimes \mathcal{O}(1)$.

Motivated by 1.5(2), we refer to $\{\mathcal{V}_i\}_{i \in \mathbb{Z}}$ as the vector bundle helix.

Remark 1.6. For length one flops, this helix is simply $\mathcal{V}_i = \mathcal{O}(i)$, and the short exact sequences above are just pullbacks to X of the Euler sequence $0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O}^{\oplus 2} \rightarrow \mathcal{O}(1) \rightarrow 0$ on \mathbb{P}^1 , twisted by $\mathcal{O}(i)$. For higher length flops, we thus view the new exact sequences in 1.5(1) as generalised Euler sequences.

We construct, in 4.5 and 4.9, a \mathbb{Z} -indexed family of hearts $\mathcal{Tilt}_i({}^0\text{Per} X)$ obtained as successive tilts. Our main technical result is the following, which shows that our helices $\{\mathcal{S}_i\}_{i \in \mathbb{Z}}$ and $\{\mathcal{V}_i\}_{i \in \mathbb{Z}}$ completely describe the simples and projectives in iterated tilts of perverse sheaves.

Theorem 1.7. *For all $i \in \mathbb{Z}$, the abelian category $\mathcal{Tilt}_i({}^0\mathcal{P}erX)$ has:*

- (1) (5.9) *Progenerator $\mathcal{P}_i = \mathcal{V}_{i-1} \oplus \mathcal{V}_i$, which is a tilting bundle on X .*
- (2) (4.13) *Simples $\mathcal{S}_{i-1}[1]$ and \mathcal{S}_i .*

In the process of establishing the above, it turns out that there are two main actions on the family $\mathcal{Tilt}_i({}^0\mathcal{P}erX)$, namely tensoring and duality, as follows. We refer to the duality in (2) below as Perverse–Tilt duality.

Theorem 1.8. *With notation as above, for all $i, k \in \mathbb{Z}$, the following statements hold, where \mathfrak{fl} denotes the finite length subcategory.*

- (1) (4.10) $\mathcal{Tilt}_{i+kN}({}^0\mathcal{P}erX) = \mathcal{Tilt}_i({}^0\mathcal{P}erX) \otimes \mathcal{O}(k)$.
- (2) (4.11) $\mathbb{D}(\mathfrak{fl}\mathcal{Tilt}_i({}^0\mathcal{P}erX)) = \mathfrak{fl}\mathcal{Tilt}_{1-i}({}^0\mathcal{P}erX)$.

Combining 1.7 and 1.8 we obtain a helix of abelian categories, with prescribed simples and projectives. Figure 2 below illustrates this in the case $\ell = 5$, where $\mathcal{A}_i = \mathcal{Tilt}_i({}^0\mathcal{P}erX)$. The inner circle illustrates the projectives, and the outer circle the simples. Here \mathcal{Z}^ω denotes the sheaf, defined in §4.4, which is dual to \mathcal{Z} in an appropriate sense.

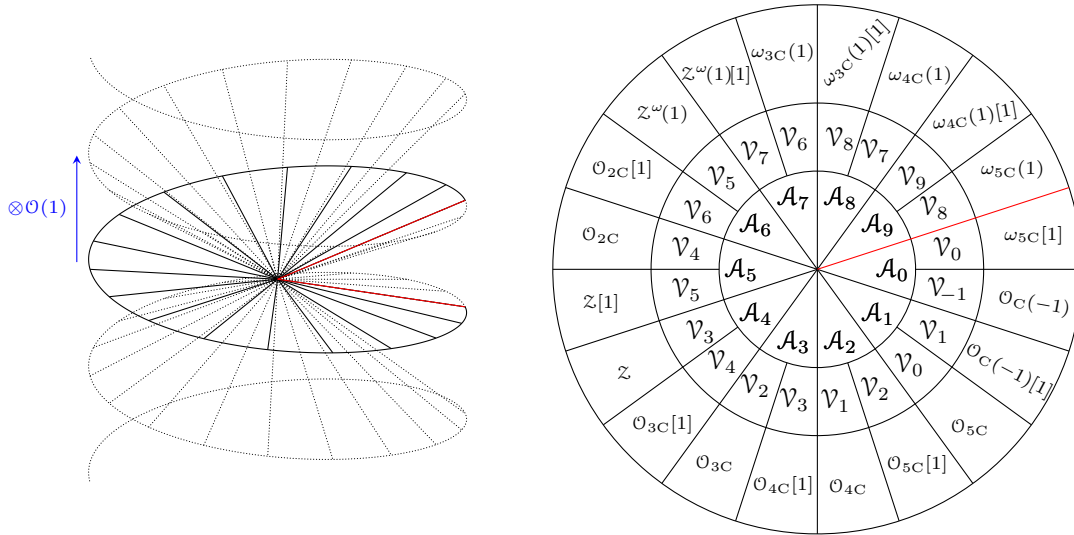


FIGURE 2. The simples and projectives helices for length five flops.

A surprising consequence, in the local setting $X \rightarrow \text{Spec}\mathfrak{R}$ where X is smooth, is that the helix $\{\mathcal{V}_i\}_{i \in \mathbb{Z}}$ classifies *all* tilting reflexive sheaves on X , as follows. In particular, all noncommutative crepant resolutions of \mathfrak{R} arise from tilting bundles on X .

Corollary 1.9 (5.12). *Suppose that \mathcal{P} is a basic reflexive tilting sheaf on X , and that X is smooth. Then $\mathcal{P} \cong \mathcal{P}_i$ for some $i \in \mathbb{Z}$. In particular, the set of all basic tilting bundles on X equals $\{\mathcal{P}_i = \mathcal{V}_{i-1} \oplus \mathcal{V}_i\}_{i \in \mathbb{Z}}$, and all reflexive tilting sheaves on X are vector bundles.*

1.5. Further Applications. As in our previous work [DW1, DW2] taking factors of local tilting bundles turns out to control the deformation theory of associated sheaves. With our new \mathbb{Z} -indexed family \mathcal{P}_i of such bundles from 1.7, consider $\text{End}_X(\mathcal{P}_i)$ and write $[\mathcal{V}_i]$ for the two-sided ideal of morphisms that factor through a summand of a finite sum of copies of \mathcal{V}_i . We thus obtain a \mathbb{Z} -indexed family of algebras

$$\Lambda_i^{\text{def}} = \text{End}_X(\mathcal{P}_i)/[\mathcal{V}_i].$$

Each is finite dimensional as a vector space, since \mathfrak{R} is isolated. It turns out that Λ_i^{def} represents the functor of noncommutative deformations of \mathcal{S}_i .

Theorem 1.10 (7.9). *In the quasi-projective setting $Y \rightarrow Y_{\text{con}}$ of 1.2, consider the formal fibre $X \rightarrow \text{Spec } \mathfrak{R}$. Then Λ_i^{def} represents noncommutative deformations of $\mathcal{S}_i \in \text{coh } Y$.*

In particular, the abelianisation of Λ_i^{def} represents commutative deformations of \mathcal{S}_i . Since Λ_i^{def} is a factor of a tilting algebra, the techniques of [DW1, DW3] apply, and this homological control allows us to extract very fine information about the dimension of the deformation spaces. In 7.4 we determine precisely when Λ_i^{def} is not commutative, and we give lower bounds on its dimension, and on the dimension of its abelianisation.

Recall that the *contraction algebra* A_{con} was defined in [DW1] via noncommutative deformations of $\mathcal{O}_C(-1)$. In the notation here, $A_{\text{con}} = \Lambda_0^{\text{def}}$.

Corollary 1.11 (7.7). *For smooth X , there exist lower bounds as follows, where GV are the Gopakumar–Vafa invariants.*

ℓ	GV lower bounds	$\dim A_{\text{con}}$ lower bound
1	(1)	1
2	(4, 1)	8
3	(5, 3, 1)	26
4	(6, 4, 2, 1)	56
5	(7, 6, 4, 2, 1)	124
6	(6, 6, 4, 3, 2, 1)	200

Our last corollary, which may be of independent interest, shows that noncommutative deformations detect higher multiples of the curve. When the representing object of noncommutative deformation theory is not commutative, we say that strictly noncommutative deformations exist.

Corollary 1.12 (7.5, 7.6). *For $1 \leq a \leq \ell$, higher multiples naC of aC exist (i.e. $2a \leq \ell$) if and only if there exist strictly noncommutative deformations of the sheaf \mathcal{O}_{aC} .*

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2. THICKENINGS, PERVERSITY AND DYNKIN COMBINATORICS

The setup is a flopping contraction $f: X \rightarrow \text{Spec } R$ of an irreducible rational curve, where (R, \mathfrak{m}) is a complete local \mathbb{C} -algebra, and X has Gorenstein terminal singularities.

2.1. Thickenings and Katz sequences. If g is a generic element of \mathfrak{m} , pulling back $X \rightarrow \text{Spec } R$ along the map $\text{Spec } R/g \rightarrow \text{Spec } R$ gives a morphism $S \rightarrow \text{Spec } R/g$, say. By Reid’s general elephant theorem [R1], R/g is an ADE surface singularity, and S is a partial crepant resolution. As such, by the McKay correspondence the exceptional curve in S corresponds to some vertex v in some ADE Dynkin diagram. Labelling the nodes of the Dynkin diagram by the rank of the highest root, the number attached to the node v is called the *length* ℓ of the curve.

Notation 2.1. Let C be the fibre $f^{-1}(\mathfrak{m})$ with *reduced* scheme structure.

As usual $C \cong \mathbb{P}^1$. Let $\mathcal{J} = \mathcal{J}_{C,S}$ be the ideal sheaf of C in S . For each $1 \leq a \leq \ell$, let \mathcal{J}_a be the saturation of $\mathcal{J} \cdots \mathcal{J}$ (a times), namely the smallest ideal sheaf containing $\mathcal{J} \cdots \mathcal{J}$ that defines a subscheme of S of pure dimension one. This necessarily has support C . Write $aC \subset X$ for the subscheme of X defined by \mathcal{J}_a .

Proposition 2.2. *When X has only Gorenstein terminal singularities, then for $1 \leq a \leq \ell$, the following statements hold.*

- (1) aC is a Cohen–Macaulay (CM) scheme of dimension one. Furthermore, ℓC is the scheme-theoretic fibre over \mathfrak{m} .

- (2) $H^0(\mathcal{O}_{aC}) = \mathbb{C}$ and $H^1(\mathcal{O}_{aC}) = 0$.
- (3) If $a \geq 2$, there is a non-split short exact sequence

$$0 \rightarrow \mathcal{O}_C(-1) \rightarrow \mathcal{O}_{aC} \rightarrow \mathcal{O}_{(a-1)C} \rightarrow 0. \tag{2.A}$$

Proof. This is essentially [K1, Lemma 3.2], but since Katz works under the assumption that X is smooth, and relies on a case-by-case argument that uses Katz–Morrison [KM] (which is false in our singular setting), we give a more general argument here.

(1) The first statement is word-for-word [K1, Lemma 3.2(i)]. For the second statement, exactly as in [K1, Lemma 3.2(ii)] it suffices to show that the scheme fibre has no embedded points: this is just [V, 3.4.2].

(2) The surjection $\mathcal{O} \rightarrow \mathcal{O}_{aC}$ together with the fact that $\mathbf{R}f_*\mathcal{O} = \mathcal{O}$ gives $H^1(\mathcal{O}_{aC}) = 0$. Since ℓC is the scheme fibre by (1), necessarily $H^0(\mathcal{O}_{\ell C}) = \mathbb{C}$ by [V, 3.4.2]. It is also clear that $H^0(\mathcal{O}_C) = \mathbb{C}$, since $C \cong \mathbb{P}^1$. Thus when $\ell = 1, 2$, there is nothing more to prove. When $\ell = 5, 6$, necessarily R/g is an E_8 surface singularity, and further S is the partial resolution already considered in [K1]. Thus when $\ell = 5, 6$ the result holds by [K1, Lemma 3.2(iii)].

It thus suffices to prove that $H^0(\mathcal{O}_{2C}) = \mathbb{C}$ when $\ell = 3$, and to prove that $H^0(\mathcal{O}_{2C}) = \mathbb{C} = H^0(\mathcal{O}_{3C})$ when $\ell = 4$. Since $\ell = 3$ can appear in various places in the longest root of an ADE Dynkin diagram, and likewise for $\ell = 4$, there are various cases, and each needs to be independently verified. We illustrate the technique in the hardest case below (namely when S has an E_6 singularity), with the technique being general, and all other cases being similar.

Consider the minimal resolution $Y \rightarrow \text{Spec } R/g$ where R/g is an E_8 singularity, and consider the length three curve obtained by contracting all curves in Y except for the shaded vertex indicated below (which has value 3 in the longest root), as follows.

$$Y \xrightarrow{h} S \longrightarrow \text{Spec } R/g$$

The singular surface S contains precisely one length three curve, and on this curve is one E_6 singularity and one A_1 singularity. We next construct a sheaf on Y which pushes down to give \mathcal{O}_{2C} on S . Let Δ denote the set of vertices of the E_8 Dynkin diagram. Then we consider the root

$$0 \ 1 \ \overset{1}{2} \ 2 \ 2 \ 2 \ 1 \ = (a_i)_{i \in \Delta}$$

of E_8 , chosen since it has value 2 on the shaded vertex. These numbers $(a_i)_{i \in \Delta}$ determine a divisor $D = \sum_{i \in \Delta} a_i E_i$, where E_i are the exceptional curves in Y , and we consider the exact sequence

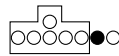
$$0 \rightarrow \mathcal{O}(-D) \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_D \rightarrow 0.$$

Since $(a_i)_{i \in \Delta}$ is a root, it is well known (and proved by induction) that $H^0(Y, \mathcal{O}_D) = \mathbb{C}$.

The line bundle $\mathcal{O}(-D)$ can easily be computed using intersection theory: for each curve E_i , the following diagram gives the intersection product $-D \cdot E_i$.

$$-1 \ 0 \ \overset{0}{0} \ 0 \ 0 \ 1 \ 0 \tag{2.B}$$

We next claim that $\mathbf{R}^1 h_* \mathcal{O}(-D) = 0$. By Grothendieck’s Theorem on Formal Functions, this can be computed locally on S . A neighbourhood of the E_6 singular point gives rise to a neighbourhood of Y containing only the six curves of E_6 illustrated below.



Against these six curves, (2.B) verifies that $\mathcal{O}(-D)$ restricts to a line bundle with intersection products as follows.

$$-1 \ 0 \ \overset{0}{0} \ 0 \ 0 \ 0$$

Since this is a summand of the Artin–Verdier tilting bundle on the minimal resolution of E_6 , it follows that $\mathbf{R}^1 h_* \mathcal{O}(-D)$ is zero near the E_6 singular point. (The construction of the summands of this tilting bundle is recalled in Section 2.2. Take f there to be the minimal resolution, and \mathcal{L} to instead be the line bundle which has degree one against a specific curve, and degree zero against the others. Then the corresponding summand of the Artin–Verdier bundle is the constructed \mathcal{N} .) Near the A_1 singular point, (2.B) verifies that restricting $\mathcal{O}(-D)$ to the single curve

$$\circ \circ \circ \circ \circ \bullet \square$$

gives the trivial bundle. Thus $\mathbf{R}^1 h_* \mathcal{O}(-D)$ is also zero near the A_1 singular point.

The claim is verified, thus $\mathbf{R}^1 h_* \mathcal{O}(-D) = 0$ and so

$$0 \rightarrow h_* \mathcal{O}(-D) \rightarrow \mathcal{O}_S \rightarrow h_* \mathcal{O}_D \rightarrow 0$$

is exact. This defines a scheme structure on the curve in S , which has multiplicity two at the smooth points, since this is true in Y , by construction of D . Since $H^0(h_* \mathcal{O}_D) = H^0(Y, \mathcal{O}_D) = \mathbb{C}$, it follows that this scheme structure gives a CM curve, since embedded points would give rise to additional sections. Thus we have produced a CM curve of multiplicity two in S , and so as observed in [K1, beginning of proof of Lemma 3.2] it follows from uniqueness that $h_* \mathcal{O}_D \cong \mathcal{O}_{2C}$. In particular $H^0(\mathcal{O}_{2C}) = \mathbb{C}$, as required.

All the other cases involving partial resolutions of Kleinian singularities with length three curves, respectively length four curves, follow using the same technique. In each case, we can lift aC (with $a = 2$, respectively $a = 2, 3$) to a root, then construct a sheaf on Y using this root. This sheaf is known to have only constant sections: intersection theory then allows us to push it down to obtain \mathcal{O}_{aC} , and thus \mathcal{O}_{aC} has only constant sections.

(3) There is an exact sequence

$$0 \rightarrow \mathcal{J}_{a-1}/\mathcal{J}_a \rightarrow \mathcal{O}_{aC} \rightarrow \mathcal{O}_{(a-1)C} \rightarrow 0.$$

Since $\mathcal{J}_{a-1}/\mathcal{J}_a$ is a torsion-free sheaf of rank 1 on $C \cong \mathbb{P}^1$, it is locally free. By (2) we know that $H^0(\mathcal{J}_{a-1}/\mathcal{J}_a) = 0 = H^1(\mathcal{J}_{a-1}/\mathcal{J}_a)$, hence it follows that $\mathcal{J}_{a-1}/\mathcal{J}_a \cong \mathcal{O}_C(-1)$, as required. The sequence does not split since applying $\mathrm{Hom}_X(-, \mathcal{O}_C(-1))$ to the surjection $\mathcal{O}_X \rightarrow \mathcal{O}_{aC}$ shows that $\mathrm{Hom}_X(\mathcal{O}_{aC}, \mathcal{O}_C(-1)) = 0$, and so there can be no splitting morphism. \square

The dualizing sheaves on the thickenings aC turn out to be important. As notation, consider the commutative diagram

$$\begin{array}{ccc} aC & \xhookrightarrow{\iota} & X \\ & \searrow q & \downarrow p \\ & & \mathrm{Spec} \mathbb{C} \end{array}$$

and recall that $\mathbb{D}_X = \mathbf{R}\mathcal{H}om_X(-, p^! \mathbb{C})$. The following is very well known.

Lemma 2.3. *For $1 \leq a \leq \ell$, there is an isomorphism $\mathbb{D}_X(\mathcal{O}_{aC}) \cong \omega_{aC}[1]$.*

Proof. By definition $\mathbb{D}_X(\mathcal{O}_{aC})$ is $\mathbb{D}_X(\iota_* \mathcal{O}_{aC})$. Since ι is a closed embedding, $\mathbf{R}\iota_* = \iota_*$. Furthermore, closed immersions are proper, thus sheafified Grothendieck duality gives

$$\mathbb{D}_X(\mathcal{O}_{aC}) = \mathbf{R}\mathcal{H}om_X(\iota_* \mathcal{O}_{aC}, p^! \mathbb{C}) = \iota_* \mathbf{R}\mathcal{H}om_{aC}(\mathcal{O}_{aC}, \iota^! p^! \mathbb{C}) = \iota_*(\iota^! p^! \mathbb{C}).$$

But $\iota^! p^! \mathbb{C} = q^! \mathbb{C} = \omega_{aC}[1]$ since aC is CM by 2.2, hence this is $\iota_* \omega_{aC}[1]$, as required. \square

2.2. Perverse Sheaves. Recall [B1] that *zero perverse sheaves* are defined

$${}^0\mathrm{Per}(X, R) = \left\{ \mathcal{F} \in \mathrm{D}^b(\mathrm{coh} X) \left| \begin{array}{l} H^i(\mathcal{F}) = 0 \text{ if } i \neq 0, -1 \\ f_* H^{-1}(\mathcal{F}) = 0, \mathbf{R}^1 f_* H^0(\mathcal{F}) = 0 \\ \mathrm{Hom}(c, H^{-1}(\mathcal{F})) = 0 \text{ for all } c \in \mathcal{C}_0 \end{array} \right. \right\},$$

where $\mathcal{C} := \{\mathcal{G} \in D^b(\text{coh } X) \mid \mathbf{R}f_*\mathcal{G} = 0\}$ and \mathcal{C}_0 denotes the full subcategory of \mathcal{C} whose objects have cohomology only in degree 0.

There is a bundle $\mathcal{O}_X(1)$ on X with degree 1 on \mathcal{C} . Writing $\mathcal{L} := \mathcal{O}_X(1)$, there is an exact sequence

$$0 \rightarrow \mathcal{O}_X^{\oplus(\ell-1)} \rightarrow \mathcal{M} \rightarrow \mathcal{L} \rightarrow 0$$

associated to a minimal set of $\ell - 1$ generators of $H^1(X, \mathcal{L}^*)$ [V, 3.5.4]. Write $\mathcal{N} := \mathcal{M}^*$, and $\Lambda := \text{End}_X(\mathcal{O} \oplus \mathcal{N})$. Then $\mathcal{O} \oplus \mathcal{N}$ gives a derived equivalence Ψ , which restricts to an equivalence on bounded hearts as illustrated in the following commutative diagram.

$$\begin{array}{ccc} D^b(\text{coh } X) & \xrightarrow[\sim]{\Psi := \mathbf{R}\text{Hom}_X(\mathcal{O} \oplus \mathcal{N}, -)} & D^b(\text{mod } \Lambda) \\ \uparrow & & \uparrow \\ {}^0\mathcal{P}\text{er}(X, R) & \xrightarrow[\sim]{\Psi} & \text{mod } \Lambda \end{array} \quad (2.C)$$

2.3. Summary of Dynkin Combinatorics. This subsection summarises some results from [IW2, §9, §10] and [HW2], mainly to set notation. Under our setup $f: X \rightarrow \text{Spec } R$, consider $R \oplus f_*\mathcal{M}$. This R -module is reflexive, and rigid in the sense that its self-extension group is zero.

Theorem 2.4. [IW2, 10.7] *The mutation class containing $R \oplus f_*\mathcal{M}$ is in bijection with the chambers of an infinite hyperplane arrangement in \mathbb{R}^1 .*

To set notation, we label the walls by rigid reflexive modules V_i with $i \in \mathbb{Z}$, and chambers are then labelled by their direct sum, as follows.

$$\cdots \circ \cdots V_{-2} \oplus V_{-1} \cdots \circ \cdots V_{-1} \oplus V_0 \cdots \circ \cdots V_0 \oplus V_1 \cdots \circ \cdots V_1 \oplus V_2 \cdots \circ \cdots \quad (2.D)$$

Crossing the wall labelled V_i from left to right means that we consider $V_{i-1} \oplus V_i$, keep V_i , and replace V_{i-1} by V_{i+1} . This mutation process is governed homologically by *exchange sequences*: in this context, for all $i \in \mathbb{Z}$ there is an exact sequence

$$0 \rightarrow V_{i-1} \rightarrow V_i^{\oplus n_i} \rightarrow V_{i+1} \quad (2.E)$$

such that applying $\text{Hom}_R(V_{i-1} \oplus V_i, -)$ gives a short exact sequence [IW1, (6.Q)]. Similarly, for wall crossing from right to left, there exists an exchange sequence

$$0 \rightarrow V_{i+1} \rightarrow V_i^{\oplus n'_i} \rightarrow V_{i-1}. \quad (2.F)$$

For isolated cDV singularities $n_i = n'_i$ [IW2, 10.4], and furthermore

$$\text{rank}_R V_{i+1} + \text{rank}_R V_{i-1} = n_i \cdot \text{rank}_R V_i. \quad (2.G)$$

We will use both these facts implicitly throughout.

By convention, since $R \oplus f_*\mathcal{M}$ must appear in its mutation class, we set $V_0 = R$ and $V_1 = f_*\mathcal{M}$. Under this convention, $L = f_*\mathcal{O}(1)$ generates a subgroup of the class group $\text{Cl}(R)$ which acts on the above hyperplane arrangement, by translating to the right.

Proposition 2.5. [IW2, 9.10, 10.7] *Set $L := f_*\mathcal{O}(1)$ and consider $\mathbb{Z} \cong \langle L \rangle \leq \text{Cl}(R)$. Then L acts on (2.D) by translation, taking the wall labelled $V_0 = R$ to the next wall to the right for which its label V_i has rank one.*

Thus there always exists a number N , which below turns out to depend only on ℓ , such that $V_{i+N} \cong V_i \cdot L := (V_i \otimes L)^{**}$ for all $i \in \mathbb{Z}$. This number, together with the ranks of the V_i , and the n_i , can be calculated combinatorially. When X is smooth, this was achieved in [HW2, 7.11] using a result of Katz–Morrison [KM] which fails in the more general singular setting here. The following completes the calculation in all cases.

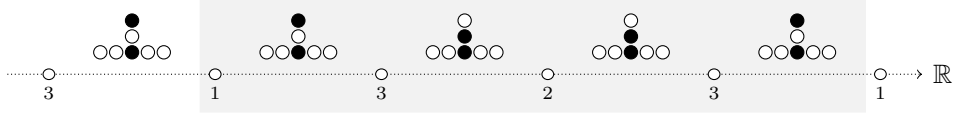
Proposition 2.6. *Suppose that $X \rightarrow \text{Spec } R$ is a length ℓ flop, where X has only Gorenstein terminal singularities. Then in the hyperplane arrangement the walls are numbered by the ranks of the V_0, \dots, V_{N-1} in the table below.*

ℓ	N	Ranks of V_0, \dots, V_{N-1}	n_0, \dots, n_{N-1}
1	1	1	2
2	2	1,2	4,1
3	4	1,3,2,3	6,1,3,1
4	6	1,4,3,2,3,4	8,1,2,3,2,1
5	10	1,5,4,3,5,2,5,3,4,5	10,1,2,3,1,5,1,3,2,1
6	12	1,6,5,4,3,5,2,5,3,4,5,6	12,1,2,2,3,1,5,1,3,2,2,1

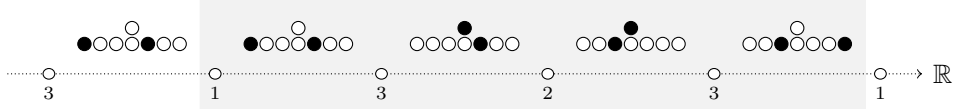
Proof. We do this for $\ell = 3$, with all other cases being similar. There are precisely five places where a vertex is labelled 3 in an ADE Dynkin diagram:



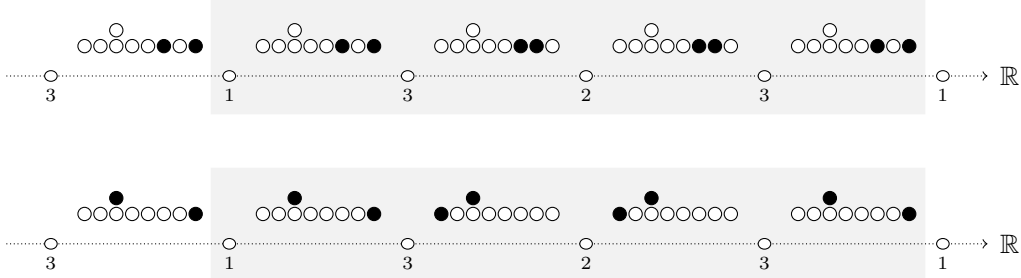
We analyse each individually using the combinatorics of wall crossing described in [IW2, 1.16]. The first is calculated in [W3, Ex. 1.1] (see also [HW2, 7.11]), and is as follows.



The next two cases are covered by the following calculation



and the last two cases are given below.



In all cases we see that $N = 4$, and the walls are numbered by 1, 3, 2, 3, then repeat. That these are the ranks of the V_i is [IW2, 9.10(2), 10.4]. The numbers n_i follow from these, using (2.G). \square

Consider the full subcategory $\mathcal{D} := \{\mathcal{F} \in \text{D}^b(\text{coh } X) \mid \text{Supp } \mathcal{F} \subseteq C\}$, and the manifold $\text{Stab}_n \mathcal{D}$ of locally finite Bridgeland stability conditions on \mathcal{D} satisfying the normalisation $Z([\mathcal{O}_x]) = i$. There is a connected component $\text{Stab}_n^\circ \mathcal{D}$ containing those normalised stability conditions with heart ${}^0\text{Per} \cap \mathcal{D}$. Fourier–Mukai autoequivalences of $\text{D}^b(\text{coh } X)$ restrict to \mathcal{D} , and we consider those that furthermore preserve $\text{Stab}_n^\circ \mathcal{D}$ and are R -linear. This group is denoted $\text{Aut}^\circ \mathcal{D}$.

Corollary 2.7. *Suppose that $X \rightarrow \text{Spec } R$ is a length ℓ flop, where X has only Gorenstein terminal singularities. Then $\mathcal{M}_{\text{Stab}} := \text{Stab}_n^\circ \mathcal{D} / \text{Aut}^\circ \mathcal{D}$ is homeomorphic to the 2-sphere with $N + 2$ points removed, where N is given in 2.6.*

Proof. In the smooth case, this is [HW2, 7.12]. Exactly the same argument works here, using that N is calculated in 2.6, and the dependence of N on ℓ is the same for the singular setting and for the smooth case. \square

3. TWISTING AND MUTATING ON THE STRIP

With notation as in (2.D), set $\Lambda_i := \text{End}_R(M_i)$, where $M_i := V_{i-1} \oplus V_i$. The ring Λ_i always has two projective modules, namely $\text{Hom}_R(M_i, V_{i-1})$ and $\text{Hom}_R(M_i, V_i)$.

3.1. Global Ordering on Projectives and Simples. Both for iteration purposes later, and to make theorems easier to state, it is convenient to now fix an ordering on the projectives. This ordering is induced by mutation. Concretely,

$$P_0 = \begin{cases} \text{Hom}_R(M_i, V_i) & \text{if } i \text{ is even} \\ \text{Hom}_R(M_i, V_{i-1}) & \text{if } i \text{ is odd,} \end{cases} \quad P_1 = \begin{cases} \text{Hom}_R(M_i, V_{i-1}) & \text{if } i \text{ is even} \\ \text{Hom}_R(M_i, V_i) & \text{if } i \text{ is odd.} \end{cases}$$

This is a slight abuse of notation, since P_0 and P_1 depend on which Λ_i is being considered. As a similar abuse, we will always write S_0 for the simple corresponding to P_0 , and S_1 for the simple corresponding to P_1 , regardless of Λ_i .

Via (2.C), this then fixes an ordering on the simples in ${}^0\mathcal{P}\text{er}$. Indeed, for $\Lambda = \Lambda_0$ across the equivalence (2.C),

$$\begin{array}{ccc} \mathcal{O} & \longleftrightarrow & P_0 \\ \mathcal{N} & \longleftrightarrow & P_1 \end{array} \quad \text{and} \quad \begin{array}{ccc} \omega_{\ell\mathbb{C}}[1] & \longleftrightarrow & S_0 \\ \mathcal{O}_{\mathbb{C}}(-1) & \longleftrightarrow & S_1. \end{array}$$

3.2. Mutation and Wall Crossing. Given $\text{End}_R(A)$ and $\text{End}_R(B)$, consider the functor

$$\mathbf{R}\text{Hom}_{\text{End}_R(A)}(T_{AB}, -): \text{D}^b(\text{mod } \text{End}_R(A)) \rightarrow \text{D}^b(\text{mod } \text{End}_R(B)) \quad (3.A)$$

where $T_{AB} = \text{Hom}_R(A, B)$. Since R is isolated cDV, if A and B are rigid reflexive R -modules that are connected by a finite sequence of mutations, then the above functor is an equivalence [IW2, 10.1, 10.5]. This then gives a chain of equivalences, which by e.g. [W2, 4.15(2)] and our choice of orderings send

$$\begin{array}{ccccccc} \cdots & \rightarrow & \text{D}^b(\text{mod } \Lambda_{-1}) & \rightarrow & \text{D}^b(\text{mod } \Lambda) & \rightarrow & \text{D}^b(\text{mod } \Lambda_1) \rightarrow \text{D}^b(\text{mod } \Lambda_2) \rightarrow \cdots \\ & & S_0 \longmapsto & S_0[-1] & & S_0 \longmapsto & S_0[-1] \\ & & P_1 \longmapsto & P_1 & & P_1 \longmapsto & P_1 \\ \longmapsto & S_1[-1] & & S_1 \longmapsto & S_1[-1] & & S_1 \longmapsto \\ \longmapsto & P_0 & & P_0 \longmapsto & P_0 & & P_0 \longmapsto \end{array} \quad (3.B)$$

We index these functors using their domain, and thus write the following.

$$\cdots \xrightarrow{\Phi_{-2}} \text{D}^b(\text{mod } \Lambda_{-1}) \xrightarrow{\Phi_{-1}} \text{D}^b(\text{mod } \Lambda) \xrightarrow{\Phi_0} \text{D}^b(\text{mod } \Lambda_1) \xrightarrow{\Phi_1} \text{D}^b(\text{mod } \Lambda_2) \xrightarrow{\Phi_2} \cdots$$

Considering T_{BA} instead of T_{AB} in (3.A) there are also equivalences in the reverse direction, which again by our choice of orderings send

$$\begin{array}{ccccccc} \cdots & \longleftarrow & \text{D}^b(\text{mod } \Lambda_{-1}) & \longleftarrow & \text{D}^b(\text{mod } \Lambda) & \longleftarrow & \text{D}^b(\text{mod } \Lambda_1) \longleftarrow \text{D}^b(\text{mod } \Lambda_2) \longleftarrow \cdots \\ & & S_0[-1] \longleftarrow & S_0 & & S_0[-1] \longleftarrow & S_0 \\ & & P_1 \longleftarrow & P_1 & & P_1 \longleftarrow & P_1 \\ \longleftarrow & S_1 & & S_1[-1] \longleftarrow & S_1 & & S_1[-1] \longleftarrow \\ \longleftarrow & P_0 & & P_0 \longleftarrow & P_0 & & P_0 \longleftarrow \end{array}$$

We index these functors using their codomain. Combining gives the following ‘strip’ of functors.

$$\begin{array}{ccccccc} \xleftrightarrow{\Phi_{-2}} & \text{D}^b(\text{mod } \Lambda_{-1}) & \xleftrightarrow{\Phi_{-1}} & \text{D}^b(\text{mod } \Lambda) & \xleftrightarrow{\Phi_0} & \text{D}^b(\text{mod } \Lambda_1) & \xleftrightarrow{\Phi_1} & \text{D}^b(\text{mod } \Lambda_2) & \xleftrightarrow{\Phi_2} \\ \xleftarrow{\Phi_{-2}} & & \xleftarrow{\Phi_{-1}} & & \xleftarrow{\Phi_0} & & \xleftarrow{\Phi_1} & & \xleftarrow{\Phi_2} \end{array} \quad (3.C)$$

The notation is deliberate: comparing to (2.D) we see that the categories above correspond to the chambers, and the functors correspond to wall-crossing. Indeed, the functor labels above are precisely induced from the label on the corresponding wall.

$$\cdots \overset{-2}{\circ} \cdots V_{-2} \oplus V_{-1} \cdots \overset{-1}{\circ} \cdots V_{-1} \oplus V_0 \cdots \overset{0}{\circ} \cdots V_0 \oplus V_1 \cdots \overset{1}{\circ} \cdots V_1 \oplus V_2 \cdots \overset{2}{\circ} \cdots$$

Later, we will see that monodromy in the complexified complement around wall i will correspond to the composition $\Phi_i \circ \Phi_i$.

For any fixed i , both functors labelled Φ_i are governed by the same numerics. These mutation functors are governed by the exchange sequences (2.E) and (2.F) respectively, and, as has already been noted, $n_i = n'_i$. The following will be used heavily later.

Lemma 3.1. *For a fixed i , consider either of the functors labelled Φ_i . Under the (P_0, P_1) and (S_0, S_1) ordering conventions above, the following hold.*

- (1) *If i is even, then $\Phi_i(S_1) = S_1[-1]$ and $\Phi_i(S_0)$ is a module of dim vector $(1, n_i)$.*
- (2) *If i is odd, then $\Phi_i(S_0) = S_0[-1]$ and $\Phi_i(S_1)$ is a module of dim vector $(n_i, 1)$.*

Proof. Consider $\Phi_i: D^b(\text{mod } \Lambda_i) \rightarrow D^b(\text{mod } \Lambda_{i+1})$. The proof for $\Phi_i: D^b(\text{mod } \Lambda_{i+1}) \rightarrow D^b(\text{mod } \Lambda_i)$ is similar.

Being the mutation functor, Φ_i is induced by the tilting bimodule $\text{Hom}_R(M_i, M_{i+1})$, which as a Λ_i -module decomposes as $\text{Hom}_R(M_i, V_i) \oplus \text{Hom}_R(M_i, V_{i+1})$. This is either $P_0 \oplus \text{Hom}_R(M_i, V_{i+1})$ if i is even, or $P_1 \oplus \text{Hom}_R(M_i, V_{i+1})$ if i is odd. The first statement regarding shifting S_1 , respectively S_0 , is already in (3.B). For the second statement, we assume that i is even: the same proof below works in the case i is odd, simply by swapping subscripts 1 and 0 throughout.

The exchange sequence induces an exact sequence

$$0 \rightarrow \text{Hom}_R(M_i, V_{i-1}) \rightarrow \text{Hom}_R(M_i, V_i)^{\oplus n_i} \rightarrow \text{Hom}_R(M_i, V_{i+1}) \rightarrow 0$$

which since i is even is

$$0 \rightarrow P_1 \rightarrow P_0^{\oplus n_i} \rightarrow \text{Hom}_R(M_i, V_{i+1}) \rightarrow 0.$$

Applying $\text{Hom}_{\Lambda_i}(-, S_0)$ we deduce that $\mathbf{R}\text{Hom}_{\Lambda_i}(\text{Hom}_R(M_i, V_{i+1}), S_0) = \mathbb{C}^{\oplus n_i}$. It is clear that $\mathbf{R}\text{Hom}_{\Lambda_i}(P_0, S_0) = \mathbb{C}$, and so the result follows. \square

3.3. Identifying Line Bundle Twists. The subgroup of the class group generated by $L = f_*\mathcal{O}(1)$ acts as in 2.5, so for all $i \in \mathbb{Z}$ there is an isomorphism

$$\Lambda_i = \text{End}_R(V_{i-1} \oplus V_i) \xrightarrow{(-\otimes L)^{**}} \text{End}_R(V_{i+N-1} \oplus V_{i+N}) = \Lambda_{i+N}.$$

This isomorphism relates the ordered projectives for Λ_i and Λ_{i+N} as follows: when N is odd (which occurs precisely when $\ell = 1$, in which case $N = 1$) it permutes the ordered projectives, whereas when N is even (which occurs precisely when $\ell \geq 2$) it preserves the ordering. For later reference, we summarize this as follows.

Notation 3.2. For any i , by abuse of notation we write β for the above isomorphism. In particular, β induces an isomorphism of categories $D^b(\text{mod } \Lambda_i) \rightarrow D^b(\text{mod } \Lambda_{i+N})$, which we will also denote by β .

Lemma 3.3. β is a ring isomorphism that sends $P_i \mapsto P_i$ and $S_i \mapsto S_i$ when $\ell > 1$, and interchanges $P_0 \leftrightarrow P_1$ and $S_0 \leftrightarrow S_1$ when $\ell = 1$.

It will be convenient to write $\kappa_0 = \text{Id}$, and for $i \geq 1$ to set

$$\kappa_i := \Phi_{i-1} \circ \dots \circ \Phi_0: D^b(\text{mod } \Lambda) \xrightarrow{\sim} D^b(\text{mod } \Lambda_i).$$

For $i < 0$, we set

$$\lambda_i := \Phi_{-1} \circ \dots \circ \Phi_i: D^b(\text{mod } \Lambda_i) \xrightarrow{\sim} D^b(\text{mod } \Lambda).$$

We also define a functor $D^b(\text{coh } X) \rightarrow D^b(\text{mod } \Lambda_i)$ as follows.

$$\Psi_i := \begin{cases} \kappa_i \circ \Psi & \text{if } i \geq 0 \\ \lambda_i^{-1} \circ \Psi & \text{if } i < 0 \end{cases}$$

It is obvious that if $i \in \mathbb{Z}$, and $j \geq 1$, then

$$\Phi_{i+j-1} \circ \dots \circ \Phi_i \circ \Psi_i \cong \Psi_{i+j}. \quad (3.D)$$

The following is an easy extension of [HW2, 7.4].

Proposition 3.4. *For all $i \in \mathbb{Z}$, and $k \geq 1$, the following diagram commutes.*

$$\begin{array}{ccc} D^b(\text{coh } X) & \xrightarrow{-\otimes \mathcal{O}(-k)} & D^b(\text{coh } X) \\ \Psi_i \downarrow & & \beta \circ \Psi_i \downarrow \\ D^b(\text{mod } \Lambda_i) & \xrightarrow{\Phi_{i+kN-1} \circ \dots \circ \Phi_i} & D^b(\text{mod } \Lambda_{i+kN}) \end{array}$$

Proof. Consider the diagram

$$\begin{array}{ccccc} D^b(\text{coh } X) & \xrightarrow{-\otimes \mathcal{O}(-1)} & D^b(\text{coh } X) & \xrightarrow{\Psi_1} & D^b(\text{mod } \Lambda_1) \\ \Psi_0 \downarrow & \searrow \Psi_1 & \Psi_0 \downarrow & \searrow \Phi_0 & \beta \downarrow \\ D^b(\text{mod } \Lambda_0) & \xrightarrow{\Phi_0} & D^b(\text{mod } \Lambda_0) & \xrightarrow{\Phi_0} & D^b(\text{mod } \Lambda_1) \\ & & \beta \downarrow & & \beta \downarrow \\ & & D^b(\text{mod } \Lambda_N) & \xrightarrow{\Phi_N} & D^b(\text{mod } \Lambda_{N+1}) \end{array}$$

The left hand square commutes by [HW2, 7.4]. We claim the right hand square also commutes. On one hand, $\beta \circ \Phi_0$ is given by the tilting bimodule $\text{Hom}_R(M_0, M_1)$, with standard action of $\text{End}_R(M_0)$ and the action of $\text{End}_R(M_{N+1})$ via β^{-1} . On the other hand, $\Phi_N \circ \beta$ is given by the tilting bimodule $\text{Hom}_R(M_N, M_{N+1})$, with the action of $\text{End}_R(M_0)$ via β , and the standard action by $\text{End}_R(M_{N+1})$. These tilting modules are isomorphic via tensor by L . This is clearly an isomorphism of bimodules since both functors β are induced by tensoring by L .

Since $\Psi_1 = \Phi_0 \circ \Psi$, from the above two commutative squares, the result follows for $i = 1$ and $k = 1$. In an identical way, composing suitable squares proves the result for all $i \geq 0$ and $k = 1$. Then, for any $i \geq 0$, all three squares in the following diagram commute

$$\begin{array}{ccccc} D^b(\text{coh } X) & \xrightarrow{-\otimes \mathcal{O}(-1)} & D^b(\text{coh } X) & \xrightarrow{-\otimes \mathcal{O}(-1)} & D^b(\text{coh } X) \\ \Psi_i \downarrow & & \Psi_i \downarrow & & \Psi_i \downarrow \\ D^b(\text{mod } \Lambda_i) & \xrightarrow{\dots \circ \Phi_i} & D^b(\text{mod } \Lambda_{i+N}) & \xrightarrow{\dots \circ \Phi_{i+N}} & D^b(\text{mod } \Lambda_{2i+N}) \end{array}$$

which proves the result for all $i \geq 0$ and all $k = 2$. In a similar way, the result follows for all $i \geq 0$ and all $k \geq 1$. The result for all $i < 0$ and $k \geq 1$ is proved in a very similar way,

starting with the following commutative diagram

$$\begin{array}{ccccc}
& & \mathrm{D}^b(\mathrm{coh} X) & \xrightarrow{-\otimes \mathcal{O}(-1)} & \mathrm{D}^b(\mathrm{coh} X) \\
& & \downarrow \Psi_0 & & \Psi_0 \downarrow \\
& \Psi_{-1} \swarrow & & & \mathrm{D}^b(\mathrm{mod} \Lambda_{-1}) \xrightarrow{\Phi_{-1}} \mathrm{D}^b(\mathrm{mod} \Lambda_0) \\
& & \mathrm{D}^b(\mathrm{mod} \Lambda_{-1}) & \xrightarrow{\Phi_{-1}} & \mathrm{D}^b(\mathrm{mod} \Lambda_0) \\
& & \downarrow \beta & & \downarrow \beta \\
\mathrm{D}^b(\mathrm{mod} \Lambda_{-1}) & \xrightarrow{\Phi_{-1}} & \mathrm{D}^b(\mathrm{mod} \Lambda_0) & \xrightarrow{\kappa_{N-1}} & \mathrm{D}^b(\mathrm{mod} \Lambda_{N-1}) \xrightarrow{\Phi_{N-1}} \mathrm{D}^b(\mathrm{mod} \Lambda_N). \quad \square
\end{array}$$

4. T-STRUCTURES ON THE STRIP, DUALITY AND THE SIMPLES HELIX

This section establishes the existence of the simples helix $\{\mathcal{S}_i\}_{i \in \mathbb{Z}}$, and shows that these describe the simples in iterated tilts of perverse sheaves. In later sections, this allows us to give an intrinsic description of monodromy on the SKMS.

4.1. Generalities on t-structures. Recall that a t-structure on a triangulated category \mathcal{D} is a full subcategory $\mathcal{F} \subset \mathcal{D}$, satisfying $\mathcal{F}[1] \subset \mathcal{F}$, such that setting

$$\mathcal{F}^\perp := \{d \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(f, d) = 0 \text{ for all } f \in \mathcal{F}\}$$

then for every $d \in \mathcal{D}$ there is a triangle $f \rightarrow d \rightarrow g \rightarrow$ with $f \in \mathcal{F}$ and $g \in \mathcal{F}^\perp$. The heart is defined to be

$$\mathcal{A} = \mathcal{F} \cap \mathcal{F}^\perp[1].$$

A t-structure $\mathcal{F} \subset \mathcal{D}$ is called bounded if $\mathcal{D} = \bigcup_{i, j \in \mathbb{Z}} \mathcal{F}[i] \cap \mathcal{F}^\perp[j]$. A bounded t-structure \mathcal{F} is determined by its heart \mathcal{A} . Indeed, \mathcal{F} is the extension-closed subcategory generated by the subcategories $\mathcal{A}[j]$ for integers $j \geq 0$.

In what follows, recall that the truncation functor $\tau_{\leq 0}^{\mathcal{A}}$ is defined to be the right adjoint to the inclusion $\mathcal{F} \subset \mathcal{D}$, and $\tau_{\geq 0}^{\mathcal{A}}$ as left adjoint to the inclusion $\mathcal{F}^\perp[1] \subset \mathcal{D}$.

Lemma 4.1. *Suppose that \mathcal{A}, \mathcal{B} are hearts of bounded t-structures in \mathcal{D} , such that $\mathbb{D}\mathcal{A} = \mathcal{B}$ for some exact duality $\mathbb{D}: \mathcal{D}^{\mathrm{op}} \rightarrow \mathcal{D}$, namely an exact anti-autoequivalence with $\mathbb{D}^2 = \mathrm{Id}$. Then $\mathbb{D}\tau_{\geq 0}^{\mathcal{A}} \cong \tau_{\leq 0}^{\mathcal{B}}\mathbb{D}$ and $\mathbb{D}\tau_{\leq 0}^{\mathcal{A}} \cong \tau_{\geq 0}^{\mathcal{B}}\mathbb{D}$.*

Proof. Suppose that \mathcal{F} and \mathcal{G} are the two bounded t-structures that give the hearts $\mathcal{A} = \mathcal{F} \cap \mathcal{F}^\perp[1]$ and $\mathcal{B} = \mathcal{G} \cap \mathcal{G}^\perp[1]$. Since \mathcal{G} is the extension-closed subcategory generated by $\mathcal{B}[j]$ for integers $j \geq 0$, it follows that $\mathbb{D}(\mathcal{G})$ is the extension-closed subcategory generated by $\mathcal{A}[j]$ for $j \leq 0$. This is $\mathcal{F}^\perp[1]$. Thus \mathbb{D} restricts to a duality

$$\mathcal{F}^\perp[1] \longleftrightarrow \mathcal{G}.$$

The chain of functorial isomorphisms

$$\mathrm{Hom}_{\mathcal{G}}(g, \tau_{\leq 0}^{\mathcal{B}}\mathbb{D}d) \cong \mathrm{Hom}_{\mathcal{D}}(d, \mathbb{D}g) \cong \mathrm{Hom}_{\mathcal{F}^\perp[1]}(\tau_{\geq 0}^{\mathcal{A}}d, \mathbb{D}g) \cong \mathrm{Hom}_{\mathcal{G}}(g, \mathbb{D}\tau_{\geq 0}^{\mathcal{A}}d)$$

then shows that $\mathbb{D}\tau_{\geq 0}^{\mathcal{A}} \cong \tau_{\leq 0}^{\mathcal{B}}\mathbb{D}$. Swapping \mathcal{A} and \mathcal{B} , we also see that $\tau_{\leq 0}^{\mathcal{A}}\mathbb{D} \cong \mathbb{D}\tau_{\geq 0}^{\mathcal{B}}$. Applying \mathbb{D} on both sides gives the the second isomorphism. \square

Now suppose that \mathcal{A} is the finite length heart of a bounded t-structure in a triangulated category \mathcal{D} . Each of the simple objects $S \in \mathcal{A}$ induces two torsion theories, $(\langle S \rangle, \mathcal{F})$ and $(\mathcal{T}, \langle S \rangle)$ on \mathcal{A} , where $\langle S \rangle$ is the full subcategory of objects whose simple factors are isomorphic to S , and the subcategories \mathcal{F} and \mathcal{T} are defined by

$$\begin{aligned}
\mathcal{F} &:= \{a \in \mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(S, a) = 0\} \\
\mathcal{T} &:= \{a \in \mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(a, S) = 0\}.
\end{aligned}$$

The corresponding tilted hearts are defined by

$$\begin{aligned}
\mathcal{L}_S(\mathcal{A}) &:= \{d \in \mathcal{D} \mid H_{\mathcal{A}}^i(d) = 0 \text{ for } i \notin \{0, 1\}, H_{\mathcal{A}}^0(d) \in \mathcal{F} \text{ and } H_{\mathcal{A}}^1(d) \in \langle S \rangle\} \\
\mathcal{R}_S(\mathcal{A}) &:= \{d \in \mathcal{D} \mid H_{\mathcal{A}}^i(d) = 0 \text{ for } i \notin \{-1, 0\}, H_{\mathcal{A}}^0(d) \in \mathcal{T} \text{ and } H_{\mathcal{A}}^{-1}(d) \in \langle S \rangle\},
\end{aligned}$$

where $H_{\mathcal{A}}(-)$ is the cohomological functor associated to the t-structure defining \mathcal{A} . The heart $\mathcal{R}_S(\mathcal{A})$ is called the *right tilt* of \mathcal{A} with respect to the simple S , and $\mathcal{L}_S(\mathcal{A})$ is called the *left tilt* at S .

Lemma 4.2. *Suppose that \mathcal{A} and \mathcal{B} are finite length hearts of bounded t-structures. If $\mathbb{D}\mathcal{A} = \mathcal{B}$, and S is a simple in \mathcal{A} , then $\mathcal{L}_{\mathbb{D}S}(\mathcal{B}) = \mathbb{D}(\mathcal{R}_S\mathcal{A})$, so that \mathbb{D} restricts to a duality*

$$\mathcal{L}_{\mathbb{D}S}(\mathcal{B}) \longleftrightarrow \mathcal{R}_S(\mathcal{A}).$$

Proof. We claim that for all $i \in \mathbb{Z}$

$$H_{\mathcal{A}}^{-i}(\mathbb{D}d) = \mathbb{D}(H_{\mathcal{B}}^i(d)). \quad (4.A)$$

The case $i = 0$ holds since by repeated use of 4.1

$$H_{\mathcal{A}}^0(\mathbb{D}d) = \tau_{\geq 0}^{\mathcal{A}} \tau_{\leq 0}^{\mathcal{A}} \mathbb{D}d = \tau_{\geq 0}^{\mathcal{A}} \mathbb{D} \tau_{\geq 0}^{\mathcal{B}} d = \mathbb{D} \tau_{\leq 0}^{\mathcal{B}} \tau_{\geq 0}^{\mathcal{B}} d = \mathbb{D}(H_{\mathcal{B}}^0(d)).$$

The general case of (4.A) then holds by the following chain of equalities

$$H_{\mathcal{A}}^{-i}(\mathbb{D}d) = H_{\mathcal{A}}^0(\mathbb{D}d[-i]) = H_{\mathcal{A}}^0(\mathbb{D}(d[i])) = \mathbb{D}(H_{\mathcal{B}}^0(d[i])) = \mathbb{D}(H_{\mathcal{B}}^i(d))$$

where the first and last equalities hold by definition, the second is obvious, and the third is the case $i = 0$ above. Now, by definition

$$\mathcal{L}_{\mathbb{D}S}(\mathcal{B}) = \left\{ d \in \mathcal{D} \left| \begin{array}{l} H_{\mathcal{B}}^i(d) = 0 \text{ for } i \notin \{0, 1\}, \\ \text{Hom}_{\mathcal{B}}(\mathbb{D}S, H_{\mathcal{B}}^0(d)) = 0, \\ H_{\mathcal{B}}^1(d) \in \langle \mathbb{D}S \rangle \end{array} \right. \right\}.$$

Consider the first condition, and note that

$$\begin{aligned} H_{\mathcal{B}}^i(d) = 0 \text{ for } i \notin \{0, 1\} &\iff \mathbb{D}(H_{\mathcal{B}}^i(d)) = 0 \text{ for } i \notin \{0, 1\} && \text{(by duality)} \\ &\iff H_{\mathcal{A}}^i(\mathbb{D}d) = 0 \text{ for } i \notin \{0, -1\}. && \text{(by (4.A))} \end{aligned}$$

Similarly, for the second condition

$$\begin{aligned} \text{Hom}_{\mathcal{B}}(\mathbb{D}S, H_{\mathcal{B}}^0(d)) = 0 &\iff \text{Hom}_{\mathcal{A}}(\mathbb{D}(H_{\mathcal{B}}^0(d)), S) = 0 && \text{(by duality)} \\ &\iff \text{Hom}_{\mathcal{A}}(H_{\mathcal{A}}^0(\mathbb{D}d), S) = 0. && \text{(by (4.A))} \end{aligned}$$

Finally, by duality it is clear that the third condition is equivalent to $\mathbb{D}(H_{\mathcal{B}}^1(d)) \in \langle S \rangle$, which again by (4.A) is equivalent to $H_{\mathcal{A}}^{-1}(\mathbb{D}d) \in \langle S \rangle$.

Summarising the above, it follows that $d \in \mathcal{L}_{\mathbb{D}S}(\mathcal{B})$ if and only if $\mathbb{D}d$ satisfies the three conditions defining $\mathcal{R}_S(\mathcal{A})$. Consequently, \mathbb{D} restricts to the desired duality. \square

4.2. Iterated Algebraic Tilts. Returning to the flops setting, with ordering as in §3.1, and notation as in §3.2, $\mathcal{A}_i = \text{mod } \Lambda_i$ is the heart of a bounded t-structure on $\text{D}^b(\text{mod } \Lambda_i)$, with projectives P_0 and P_1 , and simples S_0 and S_1 . For our applications later, we will need to tilt both \mathcal{A}_i and its subcategory of finite length modules, written $\text{fl } \mathcal{A}_i$.

The case of $\text{fl } \mathcal{A}_i$ is slightly easier. By [IR, 2.5] we have $\text{D}^b(\text{fl } \Lambda_i) = \text{D}_{\text{fl } \Lambda_i}^b(\text{mod } \Lambda_i)$, and so $\text{fl } \mathcal{A}_i = \text{fl } \Lambda_i$ is the finite length heart of a bounded t-structure in this subcategory. Appealing to the previous subsection, for each of the S_j , write the torsion theories as $(\langle S_j \rangle, \mathcal{F}_j)$ and $(\mathcal{T}_j, \langle S_j \rangle)$, with resulting tilted hearts $\mathcal{L}_j(\text{fl } \mathcal{A}_i)$ and $\mathcal{R}_j(\text{fl } \mathcal{A}_i)$.

Proposition 4.3 ([HW2, 5.5]). *The $\mathcal{L}_j(\text{fl } \mathcal{A}_i)$ and $\mathcal{R}_j(\text{fl } \mathcal{A}_i)$ are the image of a standard heart under the mutation functors and their inverses, as follows:*

$$\begin{array}{ccccc} \cdots & \text{D}^b(\text{mod } \Lambda_{-1}) & \begin{array}{c} \xrightarrow{\Phi_{-1}} \\ \xleftarrow{\Phi_{-1}} \end{array} & \text{D}^b(\text{mod } \Lambda) & \begin{array}{c} \xrightarrow{\Phi_0} \\ \xleftarrow{\Phi_0} \end{array} & \text{D}^b(\text{mod } \Lambda_1) & \cdots \\ & \mathcal{L}_1(\text{fl } \mathcal{A}_{-1}) & \swarrow & \text{fl } \mathcal{A} & \searrow & \mathcal{L}_1(\text{fl } \mathcal{A}_1) & \\ & \mathcal{R}_0(\text{fl } \mathcal{A}_{-1}) & \searrow & & \swarrow & \mathcal{R}_0(\text{fl } \mathcal{A}_1) & \end{array}$$

Definition 4.4. For $i \geq 1$, consider the iterated tilt

$$\mathfrak{Tilt}_i(\mathfrak{f}l\mathcal{A}) := \underbrace{\dots \mathcal{R}_1 \mathcal{R}_0 \mathcal{R}_1}_{i}(\mathfrak{f}l\mathcal{A}),$$

that is, we first tilt $\mathfrak{f}l\mathcal{A}$ at S_1 . The resulting t -structure $\mathcal{R}_1(\mathfrak{f}l\mathcal{A})$ has two simples, being equivalent to $\mathfrak{f}l\mathcal{A}_1$, one of which is $S_1[1]$. We then right tilt $\mathcal{R}_1(\mathfrak{f}l\mathcal{A})$ at the other, new, simple to obtain $\mathcal{R}_0\mathcal{R}_1(\mathfrak{f}l\mathcal{A})$. We then right tilt this t -structure at its new simple, and repeat. Similarly, for $i < 0$ consider the iterated tilt

$$\mathfrak{Tilt}_i(\mathfrak{f}l\mathcal{A}) := \underbrace{\dots \mathcal{L}_0 \mathcal{L}_1 \mathcal{L}_0}_{-i}(\mathfrak{f}l\mathcal{A}).$$

For all $i \in \mathbb{Z}$, we call $\mathfrak{Tilt}_i(\mathfrak{f}l\mathcal{A})$ the i th tilt of the heart $\mathfrak{f}l\mathcal{A}$.

In contrast, we tilt the full abelian category \mathcal{A} using the tilting modules and notation introduced in §3.3. This can also be achieved using iterated tilts at torsion theories (see e.g. [HW1, (5.A)]), but for our purposes here, the following definition suffices.

Definition 4.5. For $i \geq 1$, consider the iterated tilt

$$\mathfrak{Tilt}_i(\mathcal{A}) := (\Phi_{i-1} \circ \dots \circ \Phi_0)^{-1}(\mathcal{A}_i).$$

Similarly, for $i < 0$ consider the iterated tilt

$$\mathfrak{Tilt}_i(\mathcal{A}) := (\Phi_{-1} \circ \dots \circ \Phi_i)(\mathcal{A}_i).$$

For all $i \in \mathbb{Z}$, we call $\mathfrak{Tilt}_i(\mathcal{A})$ the i th tilt of the heart \mathcal{A} .

The following is an immediate consequence of the definition above, together with the notation Ψ_i from §3.3. Later it will be used to generalise (2.C).

Lemma 4.6. For all $i \in \mathbb{Z}$, there is a commutative diagram as follows.

$$\begin{array}{ccc} \mathrm{D}^b(\mathrm{mod} \Lambda) & \xrightarrow[\sim]{\Psi_i \circ \Psi_i^{-1}} & \mathrm{D}^b(\mathrm{mod} \Lambda_i) \\ \uparrow & & \uparrow \\ \mathfrak{Tilt}_i(\mathcal{A}) & \xrightarrow[\sim]{} & \mathcal{A}_i \end{array}$$

Definitions 4.4 and 4.5 are compatible, as follows.

Corollary 4.7. $\mathfrak{f}l \mathfrak{Tilt}_i(\mathcal{A}) = \mathfrak{Tilt}_i(\mathfrak{f}l\mathcal{A})$ for all $i \in \mathbb{Z}$.

Proof. The statement that $\mathfrak{f}l \mathfrak{Tilt}_i(\mathcal{A}) = \Psi \circ \Psi_i^{-1}(\mathfrak{f}l\mathcal{A}_i)$ holds tautologically, since the bottom map in 4.6 is an abelian equivalence. On the other hand, by 4.3, $\Psi \circ \Psi_i^{-1}(\mathfrak{f}l\mathcal{A}_i) = \mathfrak{Tilt}_i(\mathfrak{f}l\mathcal{A})$. \square

In what follows, set $\mathbb{D}_\Lambda := \Psi \circ \mathbb{D} \circ \Psi^{-1}$, where \mathbb{D} is Grothendieck duality.

Proposition 4.8. For all $i \in \mathbb{Z}$, the following hold.

- (1) $\mathbb{D}_\Lambda(\mathfrak{Tilt}_i(\mathfrak{f}l\mathcal{A})) = \mathfrak{Tilt}_{1-i}(\mathfrak{f}l\mathcal{A})$.
- (2) \mathbb{D}_Λ preserves the ordering on the simples.

Proof. Consider first the case $i = 1$. As observed by Bridgeland [B1, (4.8)], with corrected sign in [T4, (27)], the image of ${}^0\mathrm{Per}X^+$ under the flop functor is ${}^{-1}\mathrm{Per}X$. Since the flop functor is the inverse of the mutation functor Φ_0 [W2, 4.2], we see that ${}^{-1}\mathrm{Per}X = \Psi^{-1}(\mathfrak{Tilt}_1(\mathcal{A}))$. In particular $\mathfrak{f}l {}^{-1}\mathrm{Per}X = \Psi^{-1}(\mathfrak{f}l \mathfrak{Tilt}_1(\mathcal{A}))$, and so

$$\mathbb{D}_\Lambda(\mathfrak{Tilt}_1(\mathfrak{f}l\mathcal{A})) \stackrel{4.7}{=} \mathbb{D}_\Lambda(\mathfrak{f}l \mathfrak{Tilt}_1(\mathcal{A})) = \Psi \circ \mathbb{D}(\mathfrak{f}l {}^{-1}\mathrm{Per}X).$$

But by [V, 3.5.8] $\mathbb{D}(\mathfrak{f}l {}^{-1}\mathrm{Per}X) = \mathfrak{f}l {}^0\mathrm{Per}X$, and so the right hand side equals $\Psi(\mathfrak{f}l {}^0\mathrm{Per}X)$. In turn, this is just $\mathfrak{f}l\mathcal{A}$. This establishes the statement (1) in the case $i = 1$.

Applying \mathbb{D}_Λ gives $\mathbb{D}_\Lambda(\mathrm{fl}\mathcal{A}) = \mathcal{T}\mathrm{ilt}_1(\mathrm{fl}\mathcal{A})$. Now track $S_1 \in \mathrm{fl}\mathcal{A}$. Note first that $\mathbb{D}: \mathcal{O}_C(-1) \mapsto \mathcal{O}_C(-1)[1]$, hence $\mathbb{D}_\Lambda: S_1 \mapsto S_1[1]$, which is the first simple in $\mathcal{T}\mathrm{ilt}_1(\mathrm{fl}\mathcal{A})$. It follows that the zeroth simple must get sent to the zeroth simple. This then establishes (2) in the case $i = 1$.

When $i = 2$, apply 4.2 to hearts $\mathcal{R}_1(\mathrm{fl}\mathcal{A})$ and $\mathrm{fl}\mathcal{A}$, with S the zeroth simple of $\mathcal{R}_1(\mathrm{fl}\mathcal{A}) = \mathcal{T}\mathrm{ilt}_1(\mathrm{fl}\mathcal{A})$. Since by the above $\mathbb{D}_\Lambda(S)$ is the zeroth simple of $\mathrm{fl}\mathcal{A}$,

$$\mathcal{T}\mathrm{ilt}_{1-i}(\mathrm{fl}\mathcal{A}) = \mathcal{L}_0(\mathrm{fl}\mathcal{A}) \stackrel{4.2}{=} \mathbb{D}(\mathcal{R}_0\mathcal{R}_1(\mathrm{fl}\mathcal{A})) = \mathbb{D}(\mathcal{T}\mathrm{ilt}_2(\mathrm{fl}\mathcal{A})),$$

establishing the case $i = 2$. The fact that \mathbb{D}_Λ preserves the ordering on the simples can be seen this time by tracking the zeroth simple, which tracks to the zeroth simple. The first simple must thus do likewise. This establishes both (1) and (2) in the case $i = 2$. The proof then simply proceeds by induction, alternating the simples: for $i = 3$, apply 4.2 to $\mathcal{R}_0\mathcal{R}_1(\mathrm{fl}^0\mathcal{P}\mathrm{er})$ and $\mathcal{L}_0(\mathrm{fl}^0\mathcal{P}\mathrm{er})$, with S the first simple. \square

4.3. Tensors and Dualities on Perverse Tilts.

Definition 4.9. *The i th tilt of perverse sheaves, written $\mathcal{T}\mathrm{ilt}_i({}^0\mathcal{P}\mathrm{er})$, is defined in an identical way to $\mathcal{T}\mathrm{ilt}_i(\mathcal{A})$ in 4.5, using the ordering on its simples from §3.1.*

Equivalently, we can define $\mathcal{T}\mathrm{ilt}_i({}^0\mathcal{P}\mathrm{er}) := \Psi^{-1}\mathcal{T}\mathrm{ilt}_i(\mathcal{A})$. Combining with 4.6, for all $i \in \mathbb{Z}$ there is thus a commutative diagram as follows.

$$\begin{array}{ccccc} \mathrm{D}^b(\mathrm{coh} X) & \xrightarrow[\sim]{\Psi} & \mathrm{D}^b(\mathrm{mod} \Lambda) & \xrightarrow[\sim]{\Psi_i \circ \Psi^{-1}} & \mathrm{D}^b(\mathrm{mod} \Lambda_i) \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{T}\mathrm{ilt}_i({}^0\mathcal{P}\mathrm{er}) & \xrightarrow[\sim]{} & \mathcal{T}\mathrm{ilt}_i(\mathcal{A}) & \xrightarrow[\sim]{} & \mathcal{A}_i \end{array} \quad (4.B)$$

In the following two propositions, we prove that the categories $\mathcal{T}\mathrm{ilt}_i({}^0\mathcal{P}\mathrm{er})$ admit two key properties: they are closed under tensor, and their finite length subcategories are closed under duality.

Proposition 4.10. *For all $i \in \mathbb{Z}$, and $k \geq 0$,*

$$\mathcal{T}\mathrm{ilt}_{i+kN}({}^0\mathcal{P}\mathrm{er}) = \mathcal{T}\mathrm{ilt}_i({}^0\mathcal{P}\mathrm{er}) \otimes \mathcal{O}(k).$$

Furthermore, tensoring by $\mathcal{O}(k)$ relates the ordering on simples as follows.

- (1) *When $\ell = 1$, and so $N = 1$, the ordering on simples is swapped when k is odd, and preserved when k is even.*
- (2) *When $\ell > 1$, the ordering on simples is preserved.*

Proof. We chase the heart $\mathrm{mod} \Lambda_{i+kN}$ both ways from the bottom right to top left in 3.4. On one hand, traversing the inverses of the top and right takes

$$\mathrm{mod} \Lambda_{i+kN} \xrightarrow{\beta^{-1}} \mathrm{mod} \Lambda_i \xrightarrow{\Psi_i^{-1}} \mathcal{T}\mathrm{ilt}_i({}^0\mathcal{P}\mathrm{er}) \xrightarrow{-\otimes \mathcal{O}(k)} \mathcal{T}\mathrm{ilt}_i({}^0\mathcal{P}\mathrm{er}) \otimes \mathcal{O}(k),$$

where the middle equivalence is (4.B). On the other hand, by (3.D) the composition of the left and bottom functors is Ψ_{i+kN} , and so traversing the inverses of the bottom and left functors gives, again using (4.B), $\mathcal{T}\mathrm{ilt}_{i+kN}({}^0\mathcal{P}\mathrm{er})$. Since 3.4 is commutative, the first stated result follows. Both statements (1) and (2) follow, using 3.3. \square

Proposition 4.11 (Perverse–Tilt Duality). *For all $i \in \mathbb{Z}$, the following hold.*

- (1) $\mathbb{D}(\mathrm{fl} \mathcal{T}\mathrm{ilt}_i({}^0\mathcal{P}\mathrm{er})) = \mathrm{fl} \mathcal{T}\mathrm{ilt}_{1-i}({}^0\mathcal{P}\mathrm{er})$.
- (2) \mathbb{D} *preserves the ordering on the simples.*

Proof. Since $\mathbb{D}_\Lambda = \Psi \circ \mathbb{D} \circ \Psi^{-1}$, by 4.8 it follows that

$$\Psi \circ \mathbb{D} \circ \Psi^{-1}(\mathcal{T}\mathrm{ilt}_i(\mathrm{fl}\mathcal{A})) = \mathcal{T}\mathrm{ilt}_{1-i}(\mathrm{fl}\mathcal{A}).$$

But $\mathcal{T}\text{ilt}_n(\text{fl}\mathcal{A}) = \Psi \circ \Psi_n^{-1}(\text{fl}\mathcal{A}_n)$ for all n , by 4.7, thus

$$\Psi \circ \mathbb{D} \circ \Psi_i^{-1}(\text{fl}\mathcal{A}_i) = \Psi \circ \Psi_{1-i}^{-1}(\text{fl}\mathcal{A}_{1-i}).$$

Since the bottom line in (4.B) is an abelian equivalence, $\Psi_n^{-1}(\text{fl}\mathcal{A}_n) = \text{fl}\mathcal{T}\text{ilt}_n({}^0\mathcal{P}\text{er})$ for all n . Substituting and applying Ψ^{-1} to each side establishes (1).

For (2), the fact that the ordering is preserved is 4.8(2), given that the ordering on the simples in $\mathcal{T}\text{ilt}_i({}^0\mathcal{P}\text{er})$ is induced from the algebraic ordering in $\mathcal{T}\text{ilt}_i(\mathcal{A})$. \square

Combining with 4.10, the following will later allow us to compute the simples in $\mathcal{T}\text{ilt}_i({}^0\mathcal{P}\text{er})$ for all $i \in \mathbb{Z}$, by computing them only in a finite region.

Corollary 4.12. *With notation as above, the two simples of $\mathcal{T}\text{ilt}_i({}^0\mathcal{P}\text{er})$ are given as follows.*

- (1) *If $i \geq 0$ the simples are $\Psi_i^{-1}(S_0)$ and $\Psi_i^{-1}(S_1)$.*
- (2) *If $i < 0$ the simples are $\mathbb{D}\Psi_{1-i}^{-1}(S_0)$ and $\mathbb{D}\Psi_{1-i}^{-1}(S_1)$.*

Proof. The case $i \geq 0$ follows immediately from (4.B). The case $i < 0$ is then immediate, since $\text{fl}\mathcal{T}\text{ilt}_i({}^0\mathcal{P}\text{er}) = \mathbb{D}(\text{fl}\mathcal{T}\text{ilt}_{1-i}({}^0\mathcal{P}\text{er}))$ by Perverse-Tilt Duality 4.11. \square

4.4. The Simples Helix. We now define the sequence $\{\mathcal{S}_i\}_{i \in \mathbb{Z}}$. The sheaf $\mathcal{S}_0 := \mathcal{O}_C(-1)$, and the terms $\mathcal{S}_1, \dots, \mathcal{S}_{N/2}$ are defined as being either

$$\begin{cases} \mathcal{O}_{\ell C}, \dots, \mathcal{O}_{2C} & \text{if } \ell \leq 4 \\ \mathcal{O}_{\ell C}, \dots, \mathcal{O}_{3C}, \mathcal{Z}, \mathcal{O}_{2C} & \text{if } \ell = 5, 6. \end{cases}$$

The sheaf \mathcal{Z} will be constructed in the proof of 4.13 below, and it will be shown in 4.14 that \mathcal{Z} occurs as the unique non-split extension

$$0 \rightarrow \mathcal{O}_{3C} \rightarrow \mathcal{Z} \rightarrow \mathcal{O}_{2C} \rightarrow 0.$$

This extension group vanishes when $\ell \leq 4$ (also shown in 4.14 below), which explains why \mathcal{Z} only appears for high length E_8 flops.

The terms $\mathcal{S}_{N/2+1}, \dots, \mathcal{S}_{N-1}$ are then defined to be either

$$\begin{cases} \omega_{3C}(1), \dots, \omega_{\ell C}(1) & \text{if } \ell \leq 4 \\ \mathcal{Z}^\omega(1), \omega_{3C}(1), \dots, \omega_{\ell C}(1) & \text{if } \ell = 5, 6 \end{cases}$$

where $\mathcal{Z}^\omega := \mathbb{D}(\mathcal{Z})[-1]$, and \mathbb{D} is the Grothendieck dual. The full simples helix $\{\mathcal{S}_i\}_{i \in \mathbb{Z}}$ is then defined by translating the region $\mathcal{S}_0, \dots, \mathcal{S}_{N-1}$ via the rule $\mathcal{S}_{i+N} = \mathcal{S}_i \otimes \mathcal{O}(1)$ for all i . Note that, by construction, the terms $\mathcal{S}_{-N/2}, \dots, \mathcal{S}_{-1}$ satisfy the rule

$$\mathcal{S}_{-i} = \mathbb{D}(\mathcal{S}_i)[-1]. \tag{4.C}$$

In most cases these are just ω_{kC} , by 2.3.

Theorem 4.13. *Consider the simples helix $\{\mathcal{S}_i\}_{i \in \mathbb{Z}}$ above.*

- (1) *For all $i \geq 0$, the category $\mathcal{T}\text{ilt}_i({}^0\mathcal{P}\text{er})$ has simples $\mathcal{S}_{i-1}[1]$ and \mathcal{S}_i . Furthermore, with respect to the order in §3.1*

$$\Psi_i^{-1}S_0 = \begin{cases} \mathcal{S}_{i-1}[1] & \text{if } i \text{ even} \\ \mathcal{S}_i & \text{if } i \text{ odd} \end{cases} \quad \text{and} \quad \Psi_i^{-1}S_1 = \begin{cases} \mathcal{S}_i & \text{if } i \text{ even} \\ \mathcal{S}_{i-1}[1] & \text{if } i \text{ odd.} \end{cases}$$

- (2) *For all $i < 0$, the category $\mathcal{T}\text{ilt}_i({}^0\mathcal{P}\text{er})$ has simples $\mathcal{S}_{i-1}[1]$ and \mathcal{S}_i .*

Proof. Step 1: When $i = 0, 1$, both statements in (1) are of course already known. Indeed, when $i = 0$ the projectives are

$$P_0 = \text{Hom}_R(R \oplus N, R) \quad \text{and} \quad P_1 = \text{Hom}_R(R \oplus N, N),$$

and the corresponding simples are $\omega_{\ell C}[1]$ and $\mathcal{O}_C(-1)$ respectively [V, 3.5.8], which are $\mathcal{S}_{-1}[1]$ and \mathcal{S}_0 . Similarly, when $i = 1$, the projectives are

$$P_0 = \text{Hom}_R(R \oplus M, R) \quad \text{and} \quad P_1 = \text{Hom}_R(R \oplus M, M),$$

and as explained in 4.7 it is known that $\mathcal{Tilt}_1({}^0\mathcal{P}er) = {}^{-1}\mathcal{P}er$. Again, by [V, 3.5.8] the corresponding simples are $\mathcal{O}_{\ell\mathbb{C}}$ and $\mathcal{O}_{\mathbb{C}}(-1)[1]$ respectively, which are \mathcal{S}_1 and $\mathcal{S}_0[1]$.

We next observe that in Type A (i.e. $\ell = 1$), by 4.10, we are already done. All the simples appearing in $\mathcal{Tilt}_i({}^0\mathcal{P}er)$ are line bundle tensors of \mathcal{S}_0 and \mathcal{S}_1 , and the order is as claimed due to the permutation in 4.10. This proves all statements when $\ell = 1$.

Step 2: We can now assume that $\ell > 1$, in which case N is even. Observe next that we just need to prove the statements in (1) for $i \in \{0, \dots, N/2\}$ because, by Perverse-Tilt Duality 4.11, if $i \in \{-N/2 + 1, \dots, -1\}$ then $\mathcal{Tilt}_i({}^0\mathcal{P}er)$ is dual to $\mathcal{Tilt}_{1-i}({}^0\mathcal{P}er)$. If $|i|$ is even, then $1 - i$ is odd, so the ordered simples in $\mathcal{Tilt}_i({}^0\mathcal{P}er)$ are thus $\mathbb{D}(\mathcal{S}_{1-i})$ and $\mathbb{D}(\mathcal{S}_{-i}[1])$. By (4.C), these equal $\mathcal{S}_{i-1}[1]$ and \mathcal{S}_i respectively. A similar analysis holds if $|i|$ is odd, showing in both cases that the (unordered) simples are \mathcal{S}_i and $\mathcal{S}_{i-1}[1]$, and furthermore the ordered version matches the statement in (2). It then follows we have identified the ordered simples in both (1) and (2) within the region

$$\{-N/2 + 1, \dots, N/2\}.$$

By 4.10 this is a fundamental region, and we are done.

Step 3: We now prove the statements in (1) for $i \in \{0, \dots, N/2\}$. If $\ell = 2$ then $N = 2$, so we are done by the first paragraph of this proof. Thus we can assume that $\ell \geq 3$, in which case $N/2 \geq 2$. Since $i = 0, 1$ is known by the first paragraph, it suffices to prove all the statements in (1) for the region $i \in \{2, \dots, N/2\}$, for $\ell \geq 3$. Since all functors are equivalences, by 4.12 it suffices to show that

$$S_0 = \begin{cases} \kappa_i \Psi(\mathcal{S}_{i-1}[1]) & \text{if } i \text{ even} \\ \kappa_i \Psi(\mathcal{S}_i) & \text{if } i \text{ odd} \end{cases} \quad \text{and} \quad S_1 = \begin{cases} \kappa_i \Psi(\mathcal{S}_i) & \text{if } i \text{ even} \\ \kappa_i \Psi(\mathcal{S}_{i-1}[1]) & \text{if } i \text{ odd} \end{cases} \quad (4.D)$$

for all $2 \leq i \leq N/2$.

We first show the case $i = 2$ in (4.D), which is even, where the claim is that

$$S_0 = \Phi_1 \Phi_0 \Psi(\mathcal{O}_{\ell\mathbb{C}}[1]) \quad \text{and} \quad S_1 = \Phi_1 \Phi_0 \Psi(\mathcal{O}_{(\ell-1)\mathbb{C}}) \quad (4.E)$$

in mod Λ_2 . The case $i = 1$ implies that, in mod Λ_1 , there are equalities

$$S_0 = \Phi_0 \Psi(\mathcal{O}_{\ell\mathbb{C}}) \quad \text{and} \quad S_1 = \Phi_0 \Psi(\mathcal{O}_{\mathbb{C}}(-1)[1]).$$

Simply applying Φ_1 to the first of these, using $\Phi_1(S_0) = S_0[-1]$ by 3.1, gives the first required equality in (4.E). Now applying $\Phi_1 \Phi_0 \Psi$ to the non-split Katz triangle

$$\mathcal{O}_{(\ell-1)\mathbb{C}} \rightarrow \mathcal{O}_{\mathbb{C}}(-1)[1] \rightarrow \mathcal{O}_{\ell\mathbb{C}}[1]$$

from 2.2, where the last map is clearly non-zero, gives a triangle

$$\Phi_1 \Phi_0 \Psi(\mathcal{O}_{(\ell-1)\mathbb{C}}) \rightarrow \Phi_1(S_1) \rightarrow S_0 \rightarrow \quad (4.F)$$

in $D^b(\text{mod } \Lambda_2)$ where the last map is non-zero. As the last term is just $S_0 = \mathbb{C}$, this non-zero map must be surjective. But by 3.1, the middle term is a module of dimension vector $(n_1, 1)$, so the long exact sequence in cohomology then implies that $\Phi_1 \Phi_0 \Psi(\mathcal{O}_{(\ell-1)\mathbb{C}})$ is a module. By inspection of 2.6 we see that $n_1 = 1$ since $\ell \geq 3$, thus we conclude that $\Phi_1 \Phi_0 \Psi(\mathcal{O}_{(\ell-1)\mathbb{C}})$ has dimension vector $(1, 0)$, and hence is the simple S_0 .

Step 4: We have now established the case $i = 2$ in (4.D), so when $N = 4$ (equivalently, when $\ell = 3$), we are done. Thus we can assume that $\ell \geq 4$. The next step is to establish the case $i = 3$ in (4.D), where the claim is that

$$S_0 = \Phi_2 \Phi_1 \Phi_0 \Psi(\mathcal{O}_{(\ell-2)\mathbb{C}}) \quad \text{and} \quad S_1 = \Phi_2 \Phi_1 \Phi_0 \Psi(\mathcal{O}_{(\ell-1)\mathbb{C}}[1]). \quad (4.G)$$

Applying Φ_2 to the previous $i = 2$ equality $S_1 = \Phi_1 \Phi_0 \Psi(\mathcal{O}_{(\ell-1)\mathbb{C}})$, using 3.1, immediately gives the second claim in (4.G). Now note that (4.F) does not split, else the Katz triangle would split. Hence in the rotated triangle

$$\Phi_1(S_1) \rightarrow S_0 \rightarrow \Phi_1 \Phi_0 \Psi(\mathcal{O}_{(\ell-1)\mathbb{C}}[1]),$$

the last map is non-zero. Applying Φ_2 gives a triangle

$$\Phi_2\Phi_1(S_1) \rightarrow \Phi_2(S_0) \rightarrow S_1,$$

where the last map is non-zero, and thus is surjective. By 3.1 the middle term is a module of dimension vector $(1, n_2)$, and by inspection of 2.6 we see that $n_2 = 2$ since $\ell \geq 4$. From the long exact sequence in cohomology, we deduce that $\Phi_2\Phi_1(S_1)$ is a module, of dimension vector $(1, 1)$. Now apply $\Phi_2\Phi_1\Phi_0\Psi$ to the non-split Katz triangle

$$\mathcal{O}_{(\ell-2)C} \rightarrow \mathcal{O}_C(-1)[1] \rightarrow \mathcal{O}_{(\ell-1)C}[1],$$

where the last map is clearly non-zero, to obtain a triangle

$$\Phi_2\Phi_1\Phi_0\Psi(\mathcal{O}_{(\ell-2)C}) \rightarrow \Phi_2\Phi_1(S_1) \rightarrow S_1 \quad (4.H)$$

where the rightmost two terms are modules, of dimension vectors $(1, 1)$ and $(0, 1)$ respectively. The last map is non-zero and hence is surjective. We deduce that $\Phi_2\Phi_1\Phi_0\Psi(\mathcal{O}_{(\ell-2)C})$ is a module, of dimension vector $(1, 0)$, and so is isomorphic to S_0 .

Step 5: We have now established the case $i = 3$ in (4.D), so when $N = 6$ (equivalently, when $\ell = 4$), we are done. Thus we can assume that $\ell \geq 5$. From here, a unified proof, although possible, is notationally very heavy, so for notational ease, we now split the proof.

Case: $\ell = 5$. Since $N = 10$, we just need to verify (4.D) for $i = 4, 5$. For the case $i = 4$, we know that \mathcal{O}_{3C} and $\mathcal{O}_{4C}[1]$ are the two previous simples, and by inspection of 2.6 we know $n_3 = 3$. As always, by 3.1 $\Phi_3\Phi_2\Phi_1\Phi_0\Psi(\mathcal{O}_{3C}[1]) = S_0$, and so $\mathcal{O}_{3C}[1]$ is one of the simples. For the other, first apply Φ_3 to (4.H) and rotate to obtain

$$\Phi_3\Phi_2\Phi_1(S_1) \rightarrow \Phi_3(S_1) \rightarrow S_0.$$

The rightmost two terms are modules, of dimension vectors $(3, 1)$ and $(1, 0)$ respectively. The last map is non-zero, hence surjective, and so $\Phi_3\Phi_2\Phi_1(S_1)$ is a module of dimension vector $(2, 1)$. Applying $\Phi_3 \dots \Phi_0\Psi$ to the non-split Katz triangle

$$\mathcal{O}_{2C} \rightarrow \mathcal{O}_C(-1)[1] \rightarrow \mathcal{O}_{3C}[1],$$

then gives a triangle

$$\Phi_3 \dots \Phi_0\Psi(\mathcal{O}_{2C}) \rightarrow \Phi_3\Phi_2\Phi_1(S_1) \rightarrow S_0.$$

Applying the same logic as above, we deduce that the above is a non-split short exact sequence of modules, and $\Phi_3 \dots \Phi_0\Psi(\mathcal{O}_{2C})$ has dimension vector $(1, 1)$. In particular, it is not simple. It must have a filtration by the two simples, and since

$$\begin{aligned} \text{Hom}(S_0, \Phi_3 \dots \Phi_0\Psi(\mathcal{O}_{2C})) &= \text{Hom}(\Phi_3 \dots \Phi_0\Psi(\mathcal{O}_{3C}[1]), \Phi_3 \dots \Phi_0\Psi(\mathcal{O}_{2C})) \\ &= \text{Hom}(\mathcal{O}_{3C}[1], \mathcal{O}_{2C}) \\ &= 0, \end{aligned}$$

we first see that the filtration is necessarily of the form

$$0 \rightarrow S_1 \rightarrow \Phi_3 \dots \Phi_0\Psi(\mathcal{O}_{2C}) \rightarrow S_0 \rightarrow 0, \quad (4.I)$$

and second that this sequence cannot split. Applying the inverse of $\Phi_3 \dots \Phi_0\Psi$ and rotating, we obtain a non-split triangle

$$\mathcal{O}_{3C} \rightarrow \mathcal{Z} \rightarrow \mathcal{O}_{2C} \quad (4.J)$$

where

$$\mathcal{Z} = \Psi^{-1}\Phi_0^{-1} \dots \Phi_3^{-1}(S_1). \quad (4.K)$$

Necessarily the above triangle (4.J) arises from a non-split short exact sequence of sheaves. This proves that \mathcal{Z} is a sheaf, corresponding to the simple S_1 .

It thus remains to verify the case (4.D) for $i = 5$. By inspection of 2.6, $n_4 = 1$. As always, by 3.1 $\Phi_4 \dots \Phi_0\Psi(\mathcal{Z}[1]) = S_1$, and so $\mathcal{Z}[1]$ is one of the simples. For the other, simply apply Φ_4 to the rotation of (4.I) to obtain a triangle

$$\Phi_4 \dots \Phi_0\Psi(\mathcal{O}_{2C}) \rightarrow \Phi_4(S_0) \rightarrow S_1$$

where the last map is non-zero, and by 3.1 the rightmost two terms are modules, of dimension vector $(1, n_4) = (1, 1)$ and $(0, 1)$ respectively. The last map must thus be surjective, and so $\Phi_4 \dots \Phi_0 \Psi(\mathcal{O}_{2C})$ is a module of dimension vector $(1, 0)$. In particular, it is isomorphic to S_0 , as required.

Case: $\ell = 6$. Since $N = 12$, we just need to verify (4.D) for $i = 4, 5, 6$. Since $n_4 = 2$ by 2.6, the proof for $i = 4$ is very similar to the proof of $(i = 3, \ell \geq 4)$ above, using instead the Katz sequence

$$\mathcal{O}_{(\ell-3)C} \rightarrow \mathcal{O}_C(-1)[1] \rightarrow \mathcal{O}_{(\ell-2)C}[1].$$

This settles the case $i = 4$. For the case $i = 5$, since the previous simples are \mathcal{O}_{3C} and $\mathcal{O}_{4C}[1]$, and $n_4 = 3$, the proof for $i = 5$ is very similar to the proof for $(i = 4, \ell = 5)$ above. Similarly, since $n_5 = 1$, the proof for $i = 6$ is very similar to the proof for $(i = 5, \ell = 5)$ above. \square

The following is a consequence, which may be of independent interest.

Corollary 4.14. *With notation as above, suppose that $\ell \geq 3$, so that $\mathcal{O}_{2C}, \mathcal{O}_{3C}$ exist. Then*

$$\mathrm{Ext}_X^1(\mathcal{O}_{2C}, \mathcal{O}_{3C}) \neq 0 \iff \ell = 5, 6.$$

In this case, $\mathrm{Ext}_X^1(\mathcal{O}_{2C}, \mathcal{O}_{3C}) = \mathbb{C}$, and so the sheaf \mathcal{Z} defined in (4.K) is the unique non-split extension of \mathcal{O}_{2C} by \mathcal{O}_{3C} .

Proof. Note first that for consecutive entries \mathcal{S}_{i-1} and \mathcal{S}_i in the simples helix,

$$\mathrm{Ext}_X^1(\mathcal{S}_i, \mathcal{S}_{i-1}) = \mathrm{Hom}_{\mathrm{D}^b(\mathrm{coh} X)}(\mathcal{S}_i, \mathcal{S}_{i-1}[1]) = 0,$$

since by 4.13 $\mathcal{S}_{i-1}[1]$ and \mathcal{S}_i are the two distinct simples in the heart of some t-structure on $\mathrm{D}^b(\mathrm{coh} X)$, and so there can be no homomorphisms between them.

When $\ell = 3, 4$, by inspection the simples helix contains consecutive entries $\mathcal{O}_{3C}, \mathcal{O}_{2C}$ (in that order). By the above, $\mathrm{Ext}_X^1(\mathcal{O}_{2C}, \mathcal{O}_{3C}) = 0$.

In contrast, when $\ell = 5, 6$ the sheaf \mathcal{Z} separates the sheaves \mathcal{O}_{3C} and \mathcal{O}_{2C} in the simples helix, and by (4.J) there is a non-split short exact sequence

$$0 \rightarrow \mathcal{O}_{3C} \rightarrow \mathcal{Z} \rightarrow \mathcal{O}_{2C} \rightarrow 0.$$

In particular, $\mathrm{Ext}_X^1(\mathcal{O}_{2C}, \mathcal{O}_{3C}) \neq 0$. To compute the precise dimension, observe that $\mathrm{Ext}_X^1(\mathcal{Z}, \mathcal{O}_{3C}) = 0$ since $\mathcal{O}_{3C}, \mathcal{Z}$ are consecutive entries in the simples helix. Hence applying $\mathrm{Hom}_X(-, \mathcal{O}_{3C})$ to the above gives a long exact sequence

$$0 \rightarrow \mathrm{Hom}_X(\mathcal{O}_{2C}, \mathcal{O}_{3C}) \rightarrow \mathrm{Hom}_X(\mathcal{Z}, \mathcal{O}_{3C}) \rightarrow \mathrm{Hom}_X(\mathcal{O}_{3C}, \mathcal{O}_{3C}) \rightarrow \mathrm{Ext}_X^1(\mathcal{O}_{2C}, \mathcal{O}_{3C}) \rightarrow 0.$$

Note that $\mathrm{Hom}_X(\mathcal{O}_{3C}, \mathcal{O}_{3C}) = \mathbb{C}$ since \mathcal{O}_{3C} is a simple. Since $\mathrm{Ext}_X^1(\mathcal{O}_{2C}, \mathcal{O}_{3C}) \neq 0$, and a one-dimensional vector space surjects onto it, necessarily $\mathrm{Ext}_X^1(\mathcal{O}_{2C}, \mathcal{O}_{3C}) = \mathbb{C}$. \square

The following is also a straightforward consequence of the properties of the helix.

Corollary 4.15. *If $\ell \geq 2$, then $\omega_{2C} \cong \mathcal{O}_{2C}(-1)$.*

Proof. As $\ell \geq 2$, N is even. We compare the hearts $\mathrm{Filt}_{-N/2+1}({}^0\mathcal{P}\mathrm{er})$ and $\mathrm{Filt}_{N/2+1}({}^0\mathcal{P}\mathrm{er})$. By definition of the helix $\mathcal{S}_{N/2} = \mathcal{O}_{2C}$ and furthermore

$$\mathcal{S}_{-N/2} = \mathbb{D}(\mathcal{S}_{N/2})[-1] = \mathbb{D}(\mathcal{O}_{2C})[-1] \stackrel{2.3}{=} \omega_{2C}.$$

Then by 4.13 we know that the hearts $\mathrm{Filt}_{-N/2+1}({}^0\mathcal{P}\mathrm{er})$ and $\mathrm{Filt}_{N/2+1}({}^0\mathcal{P}\mathrm{er})$ have a simple $\mathcal{S}_{-N/2}[1]$ and $\mathcal{S}_{N/2}[1]$ respectively. Noting the ordering of simples, by 4.10 these are related by tensor by $\mathcal{O}(1)$, namely $\mathcal{S}_{-N/2}[1] \otimes \mathcal{O}(1) = \mathcal{S}_{N/2}[1]$, and the result follows. \square

5. TILTING SHEAVES AND PROGENERATORS

In this section we establish that all the tilted hearts $\mathcal{Tilt}_i({}^0\mathcal{P}er)$ have progenerators, and these are described by consecutive terms of a helix of vector bundles $\{\mathcal{V}_i\}_{i \in \mathbb{Z}}$ constructed in §5.1. When $\ell = 1$, the helix will just be $\{\mathcal{V}_i = \mathcal{O}(i)\}_{i \in \mathbb{Z}}$, but for higher length flops it is more complicated. We show in 5.9 that the helix gives rise to a \mathbb{Z} -indexed set of tilting bundles on X , and we prove in 5.12 that when X is smooth, these are all reflexive tilting sheaves on X .

5.1. The Vector Bundle Helix.

Definition 5.1. For $i \in \mathbb{Z}$, define $\mathcal{V}_i := \Psi_i^{-1} \text{Hom}_R(M_i, V_i)$, and set $\mathcal{P}_i := \mathcal{V}_{i-1} \oplus \mathcal{V}_i$.

It is clear that $\mathcal{V}_i \in \text{D}^b(\text{coh } X)$, but we will prove later that in fact \mathcal{V}_i is a vector bundle. The following two lemmas are elementary.

Lemma 5.2. With notation as above, the following statements hold.

- (1) $\mathcal{V}_{-1} = \mathcal{N}$, $\mathcal{V}_0 = \mathcal{O}$ and $\mathcal{V}_1 = \mathcal{M}$.
- (2) $\mathcal{V}_{i+kN} \cong \mathcal{V}_i \otimes \mathcal{O}(k)$ for all $i \in \mathbb{Z}$ and all $k \geq 0$.

Proof. (1) Note that $\mathcal{V}_{-1} = \Psi^{-1} \Phi_{-1} \text{Hom}_R(M_{-1}, V_{-1}) = \Psi^{-1} \text{Hom}_R(M_0, V_{-1}) \cong \mathcal{N}$. The statements for \mathcal{V}_0 and \mathcal{V}_1 are similar.

(2) If $k = 0$ there is nothing to prove, so suppose that $k > 0$. Consider the commutative diagram from 3.4, namely

$$\begin{array}{ccc} \text{D}^b(\text{coh } X) & \xrightarrow{-\otimes \mathcal{O}(-k)} & \text{D}^b(\text{coh } X) \\ \Psi_i \downarrow & & \Psi_i \downarrow \\ \text{D}^b(\text{mod } \Lambda_i) & \xrightarrow{\Phi_{i+kN-1} \circ \dots \circ \Phi_i} & \text{D}^b(\text{mod } \Lambda_{i+kN}) \xrightarrow{\beta^{-1}} \text{D}^b(\text{mod } \Lambda_i). \end{array}$$

We track $\text{Hom}_R(M_{i+kN}, V_{i+kN}) \in \text{D}^b(\text{mod } \Lambda_{i+kN})$ both ways back to $\text{D}^b(\text{coh } X)$ in the top left. On one hand, using (3.D), it tracks back to $\Psi_{i+kN}^{-1} \text{Hom}_R(M_{i+kN}, V_{i+kN})$, which by definition is \mathcal{V}_{i+kN} . On the other hand, under β^{-1} it tracks to $\text{Hom}_R(M_i, V_i)$, which then under Ψ_i^{-1} tracks to \mathcal{V}_i , which then under $(-\otimes \mathcal{O}(-k))^{-1}$ tracks to $\mathcal{V}_i \otimes \mathcal{O}(k)$. The isomorphism $\mathcal{V}_{i+kN} \cong \mathcal{V}_i \otimes \mathcal{O}(k)$ follows. \square

Lemma 5.3. For all $i \in \mathbb{Z}$, the following hold.

- (1) $\mathcal{P}_i \cong \Psi_i^{-1} \Lambda_i$.
- (2) \mathcal{P}_i progenerates $\mathcal{Tilt}_i({}^0\mathcal{P}er)$.

Proof. (1) The case $i = 0$ is 5.2(1). When $i > 0$, note that

$$\mathcal{V}_{i-1} = \Psi^{-1} \kappa_{i-1}^{-1} \text{Hom}_R(M_{i-1}, V_{i-1}) = \Psi^{-1} \kappa_i^{-1} \Phi_i \text{Hom}_R(M_{i-1}, V_{i-1}),$$

which by (3.B) equals $\Psi^{-1} \kappa_i^{-1} \text{Hom}_R(M_i, V_{i-1})$. It thus follows that $\mathcal{P}_i = (\kappa_i \Psi)^{-1} \Lambda_i$. Lastly, when $i < 0$, again using (3.B) we see that

$$\mathcal{V}_{i-1} = \Psi^{-1} \lambda_{i-1} \text{Hom}_R(M_{i-1}, V_{i-1}) = \Psi^{-1} \lambda_i \text{Hom}_R(M_i, V_{i-1}),$$

from which $\mathcal{P}_i = \Psi^{-1} \lambda_i \Lambda_i$ follows.

(2) This follows from (1), since by (4.B) $\Psi_i: \mathcal{Tilt}_i({}^0\mathcal{P}er) \rightarrow \text{mod } \Lambda_i$ is an equivalence. \square

Remark 5.4. We remark that, amongst other things, when $\ell = 2$ (and so $N = 2$) the above 5.2 establishes the non-obvious fact that $\mathcal{M} \cong \mathcal{N} \otimes \mathcal{O}(1)$.

The content in this section is to establish that for all $i \in \mathbb{Z}$, the \mathcal{V}_i are vector bundles, and that there are short exact sequences of sheaves

$$0 \rightarrow \mathcal{V}_{i-1} \rightarrow \mathcal{V}_i^{\oplus n_i} \rightarrow \mathcal{V}_{i+1} \rightarrow 0.$$

To do this requires the following preparatory lemma.

Lemma 5.5. *If $i \geq 0$, then with respect to the standard t -structure on $D^b(\text{coh } X)$, objects in $\mathcal{Tilt}_i({}^0\text{Per})$ have cohomology only in non-positive degrees.*

Proof. Since $i \geq 0$, by definition $\mathcal{Tilt}_i(\text{mod } \Lambda) = \kappa_i^{-1}(\text{mod } \Lambda_i)$. Since the composition κ_i is induced by a single tilting module of projective dimension one, it follows that with respect to the standard t -structure on $D^b(\text{mod } \Lambda)$, the heart $\mathcal{Tilt}_i(\text{mod } \Lambda)$ lives in homological degrees -1 and 0 . Tracking this over the equivalence Ψ^{-1} , which takes modules to homological degrees -1 and 0 , it is easy to see that $\mathcal{Tilt}_i({}^0\text{Per}) = \Psi^{-1}(\mathcal{Tilt}_i(\text{mod } \Lambda))$ lives in homological degrees -2 , -1 and 0 . \square

The following is also required, and becomes a key concept in §7.1.

Definition 5.6. *For each $i \in \mathbb{Z}$, the i -th helix deformation algebra is defined to be*

$$\Lambda_i^{\text{def}} := \Lambda_i/[V_i] \cong \text{End}_R(V_{i-1})/[V_i]$$

where $[V_i]$ is the two-sided ideal of $\Lambda_i \cong \text{End}_R(V_{i-1} \oplus V_i)$ consisting of those morphisms which factor through $\text{add } V_i$.

The connection with deformations will be explained in 7.2 later. For now, the following suffices.

Lemma 5.7. *For all $i \in \mathbb{Z}$, the algebra Λ_i^{def} is finite dimensional, and complete with respect to its augmentation ideal.*

Proof. Let \mathfrak{p} be a height two prime. Then since localisation is exact

$$(\Lambda_i^{\text{def}})_{\mathfrak{p}} \cong \text{End}_{R_{\mathfrak{p}}}\left(\frac{(V_{i-1})_{\mathfrak{p}}}{(V_i)_{\mathfrak{p}}}\right). \quad (5.A)$$

But $(V_j)_{\mathfrak{p}} \in \text{ref } R_{\mathfrak{p}} = \text{CM } R_{\mathfrak{p}}$ for all $j \in \mathbb{Z}$, since $\dim R_{\mathfrak{p}} = 2$. Since R has only an isolated singularity, we conclude that all $(V_j)_{\mathfrak{p}}$ are free, and so it follows from (5.A) that $(\Lambda_i^{\text{def}})_{\mathfrak{p}} = 0$. Thus Λ_i^{def} is only supported on the maximal ideal as an R -module, and thus is finite dimensional. The fact that it is complete follows exactly as in [DW2, 3.9(4)]: R is \mathfrak{m} -adically complete, hence so too is Λ_i , and thus by Nakayama both Λ_i and its factor Λ_i^{def} are complete with respect to their augmentation ideals. \square

When $\ell = 1$ all the $n_i = 2$, and the following are all just twists of the Euler sequence. We thus view the following, which is the main result of this subsection, as giving generalised Euler sequences for higher length flops.

Theorem 5.8. *For all $i \in \mathbb{Z}$, the \mathcal{V}_i are vector bundles, and there are short exact sequences*

$$0 \rightarrow \mathcal{V}_{i-1} \rightarrow \mathcal{V}_i^{\oplus n_i} \rightarrow \mathcal{V}_{i+1} \rightarrow 0$$

where the n_i are from 2.6.

Proof. By 5.2(2) it clearly suffices to prove the result for $i \geq 0$, as the result for $i < 0$ can be obtained by twisting by a line bundle.

We already know from 5.2(1) that $\mathcal{V}_{-1} = \mathcal{N}$, $\mathcal{V}_0 = \mathcal{O}$ and $\mathcal{V}_1 = \mathcal{M}$, and these are vector bundles. Furthermore, we have a pullback diagram with short exact sequences as follows, where the right hand and bottom rows are as in [V, 3.5.2].

$$\begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \downarrow & & \downarrow & \\ & & & \mathcal{N} & \longleftarrow & \mathcal{N} & \\ & & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & \mathcal{O}^{\oplus \ell-1} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{O}^{\oplus \ell+1} \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}^{\oplus \ell-1} & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{L} \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

Now $\text{Ext}_X^1(\mathcal{O}, \mathcal{O}) = 0$, so $\mathcal{F} \cong \mathcal{O}^{\oplus 2\ell}$, which gives a short exact sequence

$$0 \rightarrow \mathcal{N} \rightarrow \mathcal{O}^{\oplus 2\ell} \rightarrow \mathcal{M} \rightarrow 0.$$

Since $n_0 = 2\ell$ by 2.6, this establishes the short exact sequence in the case $i = 0$.

Next we assume that \mathcal{V}_i and \mathcal{V}_{i-1} are bundles, from this establish that so too is \mathcal{V}_{i+1} , and furthermore the exact sequence between \mathcal{V}_{i-1} , \mathcal{V}_i and \mathcal{V}_{i+1} holds. By induction, the result follows. To do this, consider the exact sequences

$$\begin{aligned} 0 \rightarrow \text{Hom}_R(M_i, V_{i-1}) \rightarrow \text{Hom}_R(M_i, V_i)^{\oplus n_i} \rightarrow \text{Hom}_R(M_i, V_{i+1}) \rightarrow 0 \\ 0 \rightarrow \text{Hom}_R(M_i, V_{i+1}) \rightarrow \text{Hom}_R(M_i, V_i)^{\oplus n_i} \rightarrow \text{Hom}_R(M_i, V_{i-1}) \rightarrow \Lambda_i^{\text{def}} \rightarrow 0 \end{aligned}$$

in $\text{mod } \Lambda_i$. By construction of the mutation functor, the tilting module gets sent to the projectives, and so $\text{Hom}_R(M_i, V_{i+1}) = \Phi_i^{-1} \text{Hom}_R(M_{i+1}, V_{i+1})$. Combining with 5.3(1), this implies that pulling back the first via $(\kappa_i \Psi)^{-1}$ gives an exact sequence

$$0 \rightarrow \mathcal{V}_{i-1} \rightarrow \mathcal{V}_i^{\oplus n_i} \rightarrow \mathcal{V}_{i+1} \rightarrow 0$$

in $\text{Tilt}_i({}^0\text{Per})$. Pulling back the lower sequence, and splicing, gives two exact sequences

$$\begin{aligned} 0 \rightarrow \mathcal{V}_{i+1} \rightarrow \mathcal{V}_i^{\oplus n_i} \rightarrow \mathcal{K}_i \rightarrow 0 \\ 0 \rightarrow \mathcal{K}_i \rightarrow \mathcal{V}_{i-1} \rightarrow \mathcal{E}_i \rightarrow 0 \end{aligned}$$

in $\text{Tilt}_i({}^0\text{Per})$, and thus three exact sequences in total.

Since Λ_i^{def} is finite dimensional by 5.7, and is filtered by the simple S_0 (when i is odd) or S_1 (when i is even), using 4.13 we see that \mathcal{E}_i is a sheaf, filtered by the sheaf \mathcal{S}_i . Using the long exact sequence in usual cohomology applied to the lower exact sequence above, which is a triangle, then since $\mathcal{K}_i \in \text{Tilt}_i({}^0\text{Per})$ can only live in non-positive degrees by 5.5, it follows that \mathcal{K}_i is a sheaf. In turn, the middle exact sequence then implies that \mathcal{V}_{i+1} is a sheaf, and so all of the three exact sequences above are in fact exact sequences of sheaves. In particular, by the top sequence $\text{pd}_{\mathcal{O}_{X,x}}(\mathcal{V}_{i+1})_x \leq 1$ for all $x \in X$.

But by 2.2 it is easy to see that all \mathcal{S}_i have depth 1, so it follows that $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{E}_i)_x \geq 1$ for all $x \in X$. Applying the depth lemma to the bottom two exact sequences, we deduce that $\text{depth}_{\mathcal{O}_{X,x}}(\mathcal{V}_{i+1})_x = 3$. Since $\text{pd}_{\mathcal{O}_{X,x}}(\mathcal{V}_{i+1})_x < \infty$ by the above, Auslander–Buchsbaum then implies that

$$\text{pd}_{\mathcal{O}_{X,x}}(\mathcal{V}_{i+1})_x = \dim \mathcal{O}_{X,x} - \text{depth}_{\mathcal{O}_{X,x}}(\mathcal{V}_{i+1})_x = 3 - 3 = 0,$$

and so \mathcal{V}_{i+1} is locally free. \square

5.2. Functorial Properties.

Theorem 5.9. *For all $i \in \mathbb{Z}$, the following statements hold.*

- (1) \mathcal{P}_i is a progenerator of $\text{Tilt}_i({}^0\text{Per})$, and is a tilting bundle on X .
- (2) $f_*(\mathcal{V}_i) \cong V_i$, where the V_i are from §2.3.
- (3) There is a functorial isomorphism $\Psi_i \cong \mathbf{R}\text{Hom}_X(\mathcal{P}_i, -)$.

Proof. (1) The fact \mathcal{P}_i is a progenerator is 5.3(2), and the fact it is a vector bundle is 5.8. That it is a tilting bundle is clear using $\mathcal{P}_i \cong \Psi^{-1}\Lambda_i$ in 5.3(1).

(2) The statement is true for $i = 0$. For $i > 0$, we have that

$$\Psi(\mathcal{V}_i) = \kappa_i^{-1} \text{Hom}_R(M_i, V_i) \cong \text{Hom}_R(M_0, V_i), \quad (5.B)$$

where the last isomorphism holds since κ_i is given by the tilting module $\text{Hom}_R(M_0, M_i)$, and thus it sends summands of this tilting module to the projectives. It follows that $\Psi(\mathcal{V}_i)$ is in degree zero. But then

$$\begin{aligned} f_*\mathcal{V}_i &\cong \text{Hom}_R(f_*\mathcal{V}_0, f_*\mathcal{V}_i) && \text{(since } f_*\mathcal{V}_0 \cong R) \\ &\cong \text{Hom}_X(\mathcal{V}_0, \mathcal{V}_i) && \text{(by reflexive equivalence)} \\ &\cong \text{Hom}_{\Lambda_0}(\text{Hom}_R(M_0, V_0), \text{Hom}_R(M_0, V_i)) && \text{(apply } \Psi) \\ &\cong \text{Hom}_R(V_0, V_i), && \text{(by reflexive equivalence)} \end{aligned}$$

which, since $V_0 = R$, is isomorphic to V_i . This establishes the claim for all $i \geq 0$. But then for any $i < 0$, by 5.2 there exists some k such that $\mathcal{V}_i \cong \mathcal{V}_{i+kN} \otimes \mathcal{O}(-k)$ with $i+kN \geq 0$. Since f_* is a reflexive equivalence, it follows that

$$f_*(\mathcal{V}_i) \cong f_*(\mathcal{V}_{i+kN} \otimes \mathcal{O}(-k)) \cong f_*\mathcal{V}_{i+kN} \cdot f_*\mathcal{O}(-k) \cong V_{i+kN} \cdot L^{-k},$$

which by 2.5 is isomorphic to V_i . The claim follows.

(3) Suppose first that $i > 0$. As $\Psi(\mathcal{P}_i) = \mathbf{R}\mathrm{Hom}_X(\mathcal{P}_0, \mathcal{P}_i) \cong \mathrm{Hom}_X(\mathcal{P}_0, \mathcal{P}_i)$, given $\Psi(\mathcal{P}_i)$ is in degree zero by (5.B), the equivalence Ψ implies that the adjunction map

$$\mathcal{P}_0 \otimes_{\mathbf{L}\mathrm{End}_X(\mathcal{P}_0)}^{\mathbf{L}} \mathrm{Hom}_X(\mathcal{P}_0, \mathcal{P}_i) \rightarrow \mathcal{P}_i$$

is an isomorphism, both as sheaves and as right $\mathrm{End}_X(\mathcal{P}_i)$ -modules. Thus, setting $\mathcal{T} := \mathrm{Hom}_X(\mathcal{P}_0, \mathcal{P}_i)$ the top half of the following diagram commutes

$$\begin{array}{ccc} \mathrm{D}^b(\mathrm{coh} X) & \xrightarrow{\mathrm{Id}} & \mathrm{D}^b(\mathrm{coh} X) \\ \Psi = \mathbf{R}\mathrm{Hom}_X(\mathcal{P}_0, -) \downarrow & & \downarrow \mathbf{R}\mathrm{Hom}_X(\mathcal{P}_i, -) \\ \mathrm{D}^b(\mathrm{mod} \mathrm{End}_X(\mathcal{P}_0)) & \xrightarrow{\mathbf{R}\mathrm{Hom}(\mathcal{T}, -)} & \mathrm{D}^b(\mathrm{mod} \mathrm{End}_X(\mathcal{P}_i)) \\ \downarrow & & \downarrow \\ \mathrm{D}^b(\mathrm{mod} \Lambda_0) & \xrightarrow{\kappa_i} & \mathrm{D}^b(\mathrm{mod} \Lambda_i) \end{array}$$

The bottom half clearly commutes, since the outermost maps are isomorphisms induced by global sections, $f_*\mathcal{P}_0 \cong M_0$ and $f_*\mathcal{P}_i \cong M_i$ by (2), and κ_i is given by the tilting module $\mathrm{Hom}_R(M_0, M_i)$, which is global sections of \mathcal{T} . Part (3) follows, for all $i \geq 0$.

We finally assume that $i < 0$, and prove that (3) holds for i . This then finishes the proof. We first claim $\mathbf{R}\mathrm{Hom}_X(\mathcal{P}_j, \mathcal{P}_k)$ is in degree zero for all $k > j \geq 0$. By the above, the left hand side, and the large rectangle in the following diagram are commutative.

$$\begin{array}{ccccc} \mathrm{D}^b(\mathrm{coh} X) & \xrightarrow{\mathrm{Id}} & \mathrm{D}^b(\mathrm{coh} X) & \xrightarrow{\mathrm{Id}} & \mathrm{D}^b(\mathrm{coh} X) \\ \Psi = \mathbf{R}\mathrm{Hom}_X(\mathcal{P}_0, -) \downarrow & & \mathbf{R}\mathrm{Hom}_X(\mathcal{P}_j, -) \downarrow & & \mathbf{R}\mathrm{Hom}_X(\mathcal{P}_k, -) \downarrow \\ \mathrm{D}^b(\mathrm{mod} \Lambda_0) & \xrightarrow{\kappa_j} & \mathrm{D}^b(\mathrm{mod} \Lambda_j) & \xrightarrow{\Phi_{k-1} \circ \dots \circ \Phi_j} & \mathrm{D}^b(\mathrm{mod} \Lambda_k) \end{array}$$

Hence it follows that the right hand side is commutative. Tracking Λ_k back both ways round the right hand diagram, we see that $\mathbf{R}\mathrm{Hom}_X(\mathcal{P}_j, \mathcal{P}_k) \cong \mathrm{Hom}_R(M_j, M_k)$, and hence in particular is in degree zero.

This being the case, after tensoring by a line bundle, it follows that $\mathbf{R}\mathrm{Hom}_X(\mathcal{P}_i, \mathcal{P}_0)$ is only in degree zero. This, together with the fact that \mathcal{P}_i is a tilting bundle by (1), implies that the adjunction map

$$\mathcal{P}_i \otimes_{\mathbf{L}\mathrm{End}_X(\mathcal{P}_i)}^{\mathbf{L}} \mathrm{Hom}_X(\mathcal{P}_i, \mathcal{P}_0) \rightarrow \mathcal{P}_0$$

is an isomorphism. As above, abusing notation slightly we see that

$$\lambda_i \circ \mathbf{R}\mathrm{Hom}_X(\mathcal{P}_i, -) \cong \mathbf{R}\mathrm{Hom}_X(\mathcal{P}_i \otimes_{\mathbf{L}\mathrm{End}_X(\mathcal{P}_i)}^{\mathbf{L}} \mathrm{Hom}_X(\mathcal{P}_i, \mathcal{P}_0), -) \cong \mathbf{R}\mathrm{Hom}_X(\mathcal{P}_0, -) = \Psi.$$

Hence since $\Psi_i = \lambda_i^{-1} \circ \Psi$, applying λ_i^{-1} to the above line gives $\Psi_i \cong \mathbf{R}\mathrm{Hom}_X(\mathcal{P}_i, -)$. \square

Remark 5.10. By 5.9(2), the rank of the vector bundle \mathcal{V}_i equals the rank of the R -module V_i . This is given explicitly by the table in 2.6.

For reference later, we also record the following, which generalises (2.C).

Corollary 5.11. *For all $i \in \mathbb{Z}$, there is a commutative diagram as follows.*

$$\begin{array}{ccc} \mathrm{D}^{\mathrm{b}}(\mathrm{coh} X) & \xrightarrow[\sim]{\mathrm{RHom}_X(\mathcal{P}_i, -)} & \mathrm{D}^{\mathrm{b}}(\mathrm{mod} \Lambda_i) \\ \uparrow & & \uparrow \\ \mathrm{Tilt}_i({}^0\mathrm{Per}) & \xrightarrow[\sim]{} & \mathrm{mod} \Lambda_i \end{array}$$

Proof. By (4.B) we already know that the diagram commutes if the top functor is replaced by Ψ_i . The result is simply then 5.9(3). \square

Theorem 5.9 also allows us to classify all possible tilting bundles on X , when X is smooth. Recall that a sheaf is called basic if there are no repetitions in its Krull–Schmidt decomposition into indecomposables.

Corollary 5.12. *Suppose that \mathcal{P} is a basic reflexive tilting sheaf on X , and that X is smooth. Then $\mathcal{P} \cong \mathcal{P}_i$ for some $i \in \mathbb{Z}$. In particular,*

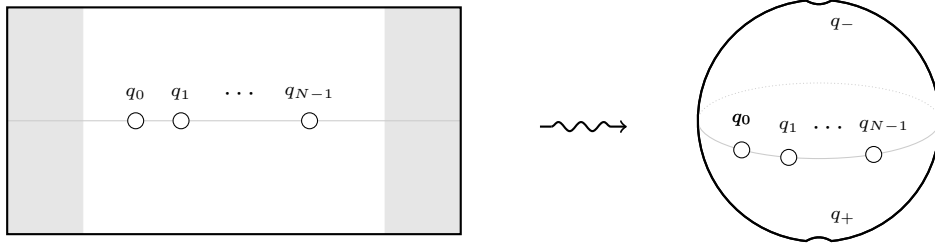
- (1) *The set of all basic tilting bundles on X equals $\{\mathcal{P}_i = \mathcal{V}_{i-1} \oplus \mathcal{V}_i\}_{i \in \mathbb{Z}}$.*
- (2) *All reflexive tilting sheaves on X are vector bundles.*

Proof. Certainly $f_*\mathcal{P}$ is a basic reflexive module giving an NCCR. By 2.4 all such basic reflexive modules are isomorphic to $V_{i-1} \oplus V_i$ for some $i \in \mathbb{Z}$. Hence $f_*\mathcal{P} \cong V_{i-1} \oplus V_i$, say. Since $f_*\mathcal{P}_i \cong V_{i-1} \oplus V_i$ by 5.9(2), we see that $f_*\mathcal{P} \cong f_*\mathcal{P}_i$. Then reflexive equivalence, see e.g. [V, 4.2.1], implies that $\mathcal{P} \cong \mathcal{P}_i$. The final two statements follow immediately, since by 5.8 all the \mathcal{V}_i are vector bundles. \square

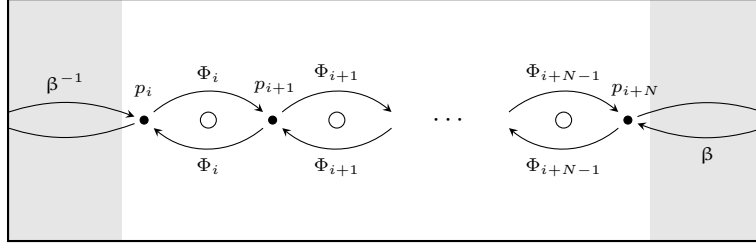
6. MONODROMY ON $\mathcal{M}_{\mathcal{S}\mathcal{X}}$

This section applies the theory developed to construct actions on the derived categories of the 3-fold X , and the sequence of algebras Λ_i associated to it. Section 6.1 first describes the monodromy action on the algebraic side, namely on the categories $\mathrm{D}^{\mathrm{b}}(\mathrm{mod} \Lambda_i)$. In §6.2 we construct local twist functors for the simples helix following [DW1], and in §6.3 we construct the monodromy action on $\mathrm{D}^{\mathrm{b}}(\mathrm{coh} X)$ in terms of these twist functors. In particular, this allows us to prove the main results in §1.3.

6.1. Algebraic Actions. Write $\mathcal{M}_{\mathcal{S}\mathcal{X}}$ for the punctured sphere $S^2 \setminus \{N + 2 \text{ points}\}$ as before. It is convenient to think of this via an orientation-preserving identification of the grey regions in the punctured rectangle below. As notation, we write $q_- \in S^2$ for the top pole, and q_+ for the bottom pole, and we refer to $q_i \in S^2$ as the *equatorial punctures*.



Proposition 6.1. *For all $i \in \mathbb{Z}$, the fundamental groupoid $\pi_1(\mathcal{M}_{\mathcal{S}\mathcal{X}}, \{p_j\})$ acts on the categories $\mathrm{D}^{\mathrm{b}}(\mathrm{mod} \Lambda_j)$ as follows: assign the category $\mathrm{D}^{\mathrm{b}}(\mathrm{mod} \Lambda_j)$ to each point p_j shown in the diagram below, and functors to homotopy classes of paths as indicated:*



Recall that Φ_j are the mutation functors from §3.2, and that β is the isomorphism induced by line bundle twist from §3.3.

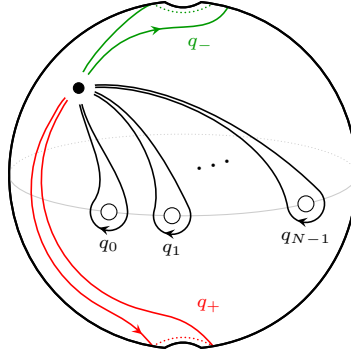
Proof. This follows just since the fundamental groupoid is generated by the arrows shown, subject to the relation that arrows marked β^{-1} and β are inverse. \square

The above action induces a similar action with basepoints a subset of $\{p_j\}$. In the following, we just consider the case $i = 0$. Recall from §3.3 that κ_i is the composition of mutation functors.

Corollary 6.2. *There is a group homomorphism*

$$\begin{aligned} \pi_1(\mathcal{M}_{S\mathcal{X}}) &\rightarrow \text{Auteq } D^b(\text{mod } \Lambda_0) \\ q_i &\mapsto \kappa_i^{-1} \circ (\Phi_i \circ \Phi_i) \circ \kappa_i \\ q_- &\mapsto \beta^{-1} \circ (\Phi_{N-1} \circ \dots \circ \Phi_0) \\ q_+ &\mapsto (\Phi_0 \circ \dots \circ \Phi_{N-1}) \circ \beta \end{aligned}$$

where we reuse the notation q for monodromy around each hole, and take loop orientations as follows.



Proof. This is immediate from 6.1, simply by composing paths there. \square

6.2. Local Twist Functors. Let \mathcal{E}_k be the object corresponding to Λ_k^{def} across the equivalence in 5.11. Note that Λ_k^{def} has finite projective dimension, since by (2.E) and (2.F) mutation twice returns us to our original module, so we can use the analogue of [DW1, (5.B)] to conclude that $\mathcal{E}_k \in \text{Perf}(X)$. Further, since Λ_k^{def} is filtered by the corresponding simple Λ_k -module, \mathcal{E}_k is filtered by \mathcal{S}_k , and thus is itself also a sheaf.

Theorem 6.3. *For all $k \in \mathbb{Z}$, there is an equivalence $\text{Twist}_{\mathcal{S}_k} : D^b(\text{coh } X) \rightarrow D^b(\text{coh } X)$, which fits into a functorial triangle as follows.*

$$\mathbf{R}\text{Hom}_X(\mathcal{E}_k, -) \otimes_{\Lambda_k^{\text{def}}}^{\mathbf{L}} \mathcal{E}_k \rightarrow \text{Id} \rightarrow \text{Twist}_{\mathcal{S}_k} \rightarrow$$

Proof. The natural ring surjection $\Lambda_k \rightarrow \Lambda_k^{\text{def}}$ gives rise to a short exact sequence of Λ_k -bimodules $0 \rightarrow I_k \rightarrow \Lambda_k \rightarrow \Lambda_k^{\text{def}} \rightarrow 0$. As in [DW1, 6.13], but now simply replacing \mathcal{P}_0 with the new tilting bundle \mathcal{P}_k , we define

$$\text{Twist}_{\mathcal{S}_k} := \mathbf{R}\text{Hom}_X(\mathcal{P}_k, -)^{-1} \circ \mathbf{R}\text{Hom}_{\Lambda_k}(I_k, -) \circ \mathbf{R}\text{Hom}_X(\mathcal{P}_k, -). \quad (6.A)$$

Using the analogue of [DW1, 6.16], $\text{Twist}_{\mathcal{S}_k}$ is then a Fourier–Mukai functor that, being a composition of equivalences, is an equivalence. The functorial triangle follows exactly as in [DW1, 6.10, 6.11] (see also [DW3, end of §5.1]), with the bundle \mathcal{P}_k replacing \mathcal{P}_0 . \square

Combining with 5.9(3), the following will be used extensively below.

Corollary 6.4. *For all $k \in \mathbb{Z}$ there is a functorial isomorphism*

$$\Psi_k^{-1} \circ (\Phi_k \circ \Phi_k) \circ \Psi_k \cong \text{Twist}_{\mathcal{S}_k}.$$

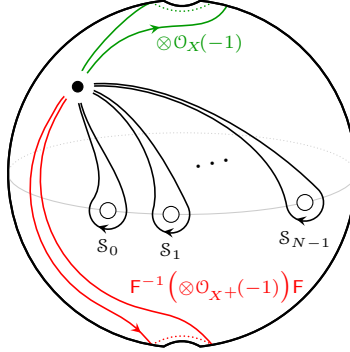
Proof. Since $\mathbf{RHom}_{\Lambda_k}(I_k, -) \cong \Phi_k \circ \Phi_k$ by [DW3, 4.3], and $\mathbf{RHom}_X(\mathcal{P}_k, -) \cong \Psi_k$ by 5.9(3), the result follows directly from the definition (6.A). \square

6.3. Geometric Monodromy. The following is one of our main results. In particular, it shows that the monodromies around equatorial punctures correspond to noncommutative twists around the members of our simples helix.

Theorem 6.5. *There is a group homomorphism*

$$\begin{aligned} \pi_1(\mathcal{M}_{\mathcal{S}X}) &\rightarrow \text{Auteq } D^b(\text{coh } X) \\ q_i &\mapsto \text{Twist}_{\mathcal{S}_i} \\ q_- &\mapsto - \otimes \mathcal{O}(-1) \\ q_+ &\mapsto F^{-1} \circ (- \otimes \mathcal{O}(-1)) \circ F \end{aligned}$$

where $\mathcal{S}_i \in \text{coh } X$ is as in §4.4, the functors $\text{Twist}_{\mathcal{S}_i}$ are defined in 6.3, and with orientations as below.



Proof. We apply $\Psi^{-1} \circ (-) \circ \Psi$ to the representation from 6.2. The action of the loops q_i follows using 6.4, and the action of q_- follows from 3.4. For the action of the loop q_+ , consider the following diagram

$$\begin{array}{ccccccc} D^b(\text{coh } X) & \xrightarrow{F} & D^b(\text{coh } X^+) & \xrightarrow{- \otimes \mathcal{O}_{X^+}(-1)} & D^b(\text{coh } X^+) & \xrightarrow{F^{-1}} & D^b(\text{coh } X) \\ \Psi \downarrow & & \Psi^+ \downarrow & & \Psi^+ \downarrow & & \Psi \downarrow \\ D^b(\text{mod } \Lambda_0) & \xrightarrow{\Phi_0^{-1}} & D^b(\text{mod } \Lambda_1) & \xrightarrow{\Phi_{1-N} \circ \dots \circ \Phi_0} & D^b(\text{mod } \Lambda_{1-N}) & \xrightarrow{\beta} & D^b(\text{mod } \Lambda_1) & \xrightarrow{\Phi_0} & D^b(\text{mod } \Lambda_0) \end{array}$$

where the outer squares commute by [W2, 4.2], and the middle commutes by [HW2, 7.4] applied to X^+ . Then, exactly as in 3.4, the bottom row is isomorphic to

$$D^b(\text{mod } \Lambda_0) \xrightarrow{\beta} D^b(\text{mod } \Lambda_N) \xrightarrow{\Phi_0 \circ \dots \circ \Phi_{N-1}} D^b(\text{mod } \Lambda_0). \quad \square$$

Remark 6.6. For the case $\ell = 1$ there is a single equatorial puncture, and the fundamental group is given by the relation $q_0^{-1} \circ q_+ \circ q_- = \text{Id}$. In this setting Toda showed that $\text{Twist}_{\mathcal{S}_0}^{-1} = F \circ F$ [T1, 3.1], so the result of 6.5 is equivalent to the functorial isomorphism

$$F \circ (- \otimes \mathcal{O}_{X^+}(-1)) \circ F \circ (- \otimes \mathcal{O}_X(-1)) = \text{Id}.$$

This was verified in the case of the Atiyah flop by the first author in [D, 7.12]. It follows for other length one flops from work of Toda: see [T2, end of §5.2, Example] where $\mathcal{M}_{S\mathcal{X}}$ is realised as a quotient of the normalized Bridgeland stability manifold for X .

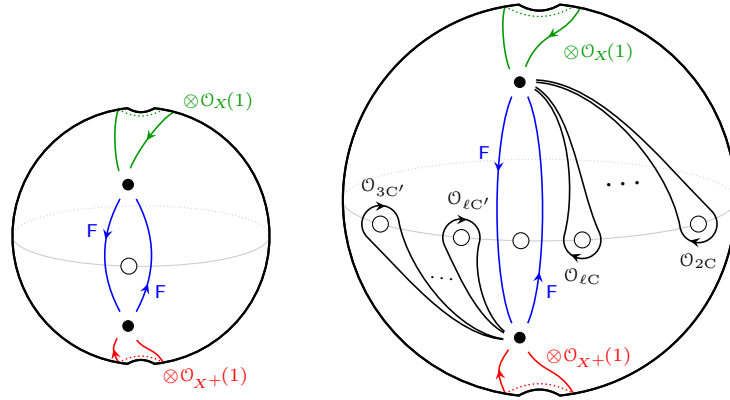
Remark 6.7. In all Dynkin types, the simples helix always contains the sheaves

$$\mathcal{O}_C, \mathcal{O}_{2C}, \dots, \mathcal{O}_{\ell C},$$

albeit not in that order. Consequently, by 6.3 all of the above give rise to a twist autoequivalence (see also 7.10 later). The more surprising fact is that the relation between them in 6.5 requires other sheaves, including \mathcal{Z} and their Grothendieck duals. This answers a question of Kawamata [K2, 6.8].

6.4. Symmetric Geometric Monodromy. It is possible to obtain a more symmetric version of the above. Write $p_{\pm} \in \mathcal{M}_{S\mathcal{X}}$ for a choice of basepoints near the poles $q_{\pm} \in S^2$. We think of these points p_{\pm} as large radius limits in $\mathcal{M}_{S\mathcal{X}}$.

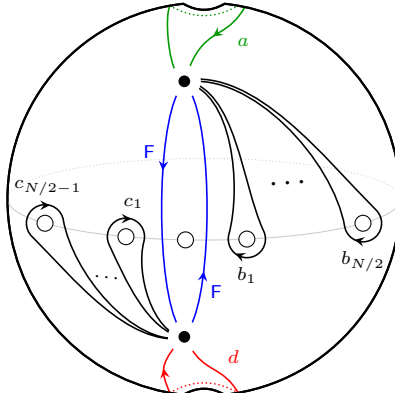
Theorem 6.8. *The fundamental groupoid $\pi_1(\mathcal{M}_{S\mathcal{X}}, \{p_{\pm}\})$ acts on the pair of categories $D^b(\text{coh } X)$ and $D^b(\text{coh } X^+)$ as follows: assign categories to points p_{\pm} , and functors to paths as below, where the left hand side is $\ell = 1$, and the right hand side $\ell > 1$.*



Proof. By [W2, 4.2], the Bridgeland–Chen flop functors are inverse to the mutation functors Φ_0 under the tilting equivalences Ψ and $\Psi_+ = \mathbf{R}\text{Hom}_{X^+}(\mathcal{O}_{X^+} \oplus \mathcal{M}_{X^+}, -)$, so we can functorially replace (3.C) with the following strip.

$$\begin{array}{ccccccc} \Phi_{-2} & & \Psi^{-1}\Phi_{-1} & & F^{-1} & & \Phi_1\Psi_+ \\ \leftarrow & D^b(\text{mod } \Lambda_{-1}) & \leftarrow & D^b(\text{coh } X) & \leftarrow & D^b(\text{coh } X^+) & \leftarrow & D^b(\text{mod } \Lambda_2) & \leftarrow & \Phi_2 \\ \Phi_{-2} & & \Phi_{-1}\Psi & & F^{-1} & & \Psi_+^{-1}\Phi_1 & & \Phi_2 & \end{array}$$

Bring the category $D^b(\text{coh } X)$ to near the top pole, the category $D^b(\text{coh } X_+)$ to near the bottom, and notate monodromy and orientation as in the following diagram.



Then, simply composing arrows and their inverses in the above strip to travel above and below holes as appropriate, we tautologically obtain a representation via

$$\begin{aligned} b_i &\mapsto \mathbf{F} \circ (\Phi_1 \Psi_+)^{-1} \circ \dots \circ \Phi_{i-1}^{-1} \circ \Phi_i \circ \Phi_i \circ \Phi_{i-1} \circ \dots \circ (\Phi_1 \Psi_+) \circ \mathbf{F}^{-1} \\ c_i &\mapsto \mathbf{F} \circ (\Phi_{-1} \Psi)^{-1} \circ \dots \circ \Phi_{-i+1}^{-1} \circ \Phi_{-i} \circ \Phi_{-i} \circ \Phi_{-i+1} \circ \dots \circ (\Phi_{-1} \Psi) \circ \mathbf{F}^{-1} \end{aligned}$$

and

$$\begin{aligned} a &\mapsto (\Psi^{-1} \Phi_{-1}) \circ \dots \circ \Phi_{-N/2} \circ \beta^{-1} \circ \Phi_{N/2-1} \circ \dots \circ (\Phi_1 \Psi_+) \circ \mathbf{F}^{-1} \\ d &\mapsto (\Psi_+^{-1} \Phi_1) \circ \dots \circ \Phi_{N/2-1} \circ \beta \circ \Phi_{-N/2} \circ \dots \circ (\Phi_{-1} \Psi) \circ \mathbf{F}^{-1}. \end{aligned}$$

By [W2, 4.2] we have $\Psi_+ \mathbf{F}^{-1} \cong \Phi_0 \Psi$, so using 6.4 we see that the image of b_i is functorially isomorphic to $\text{Twist}_{\mathcal{S}_i}$. Similarly, observe that on X^+ the perverse zero tilted algebra is $\Lambda'_0 \cong \Lambda_1$, and that the hyperplane arrangement swaps direction for X^+ . Hence, by the symmetry of the situation, using [W2, 4.2] and the X^+ version of 6.4, it follows that the image of c_i is isomorphic to $\text{Twist}_{\mathcal{S}'_i}$, where $\{\mathcal{S}'_i\}_{i \in \mathbb{Z}}$ is the simples helix on X^+ . That the image of a is $-\otimes \mathcal{O}_X(1)$ and the image of d is $-\otimes \mathcal{O}_{X^+}(1)$ follows exactly as in 6.5. \square

7. APPLICATIONS TO DEFORMATION THEORY AND CURVE INVARIANTS

In this section, we show that the noncommutative deformation functor associated to every member of the simples helix is representable. We then control the representing object, describe precisely when it is not commutative, and give lower bounds on its dimension. As a consequence, we construct global autoequivalences and produce tight lower bounds on Gopakumar–Vafa (GV) invariants for higher length flops.

7.1. Deformation Theory. For background on noncommutative deformation theory, we refer the reader to [DW1] or [DW2]. For our purposes here, for any $E \in \text{coh } X$ there is a deformation functor

$$\text{Def}_E^X : \mathbf{Art}_1 \rightarrow \mathbf{Sets}$$

from the category \mathbf{Art}_1 of augmented \mathbb{C} -algebras to the category \mathbf{Sets} of sets, controlled by the DGA $\text{End}_X(\mathcal{J})$, where $0 \rightarrow E \rightarrow \mathcal{J}$ is some injective resolution. For any $M \in \text{mod } \Lambda$ there is a similar functor Def_M^Λ , again controlled by the endomorphism DGA of M .

Recall from 5.6 that $\Lambda_i^{\text{def}} := \Lambda_i / [\mathcal{V}_i]$. Since Λ_i^{def} is clearly a factor of Λ_i , the techniques in [DW1, DW2, DW3] still hold.

Remark 7.1. As calibration, $\Lambda_0^{\text{def}} = \text{End}_R(N) / [R]$. This is the contraction algebra Λ_{con} in [DW1, 2.12, 3.1], which represents noncommutative deformations of $\mathcal{O}_C(-1)$, the zeroth member \mathcal{S}_0 of the simples helix. Further, $\Lambda_1^{\text{def}} = \text{End}_R(R) / [M]$. This is the fibre algebra \mathcal{B}_{fib} in [DW2, 5.1, (5.B)], which represents noncommutative deformations of $\mathcal{O}_{\ell C}$, the first member \mathcal{S}_1 of the simples helix.

The following extends 7.1 to all $i \in \mathbb{Z}$, and is the main result of this subsection.

Theorem 7.2. *For all $i \in \mathbb{Z}$, write S for the simple Λ_i -module corresponding to the projective $\text{Hom}_X(\mathcal{P}_i, \mathcal{V}_{i-1})$. Then*

- (1) *There is a functorial isomorphism $\text{Def}_{\mathcal{S}_i}^X \cong \text{Def}_S^{\Lambda_i}$.*
- (2) *Λ_i^{def} represents the noncommutative deformation functor of \mathcal{S}_i in X .*

Proof. This follows immediately from our previous papers [DW1, DW2], with the key point being that 5.11 just directly replaces (2.C) in all proofs. Indeed, the simple S corresponding to the projective summand $\text{Hom}_X(\mathcal{P}_i, \mathcal{V}_{i-1})$ always corresponds to a sheaf (namely \mathcal{S}_i) across the equivalence 5.11.

Using the fact that S corresponds to a sheaf, just repeating word-for-word [DW2, 3.9, 5.3], or [DW1, §3], there is a functorial isomorphism

$$\text{Def}_{\mathcal{S}_i}^X \cong \text{Def}_S^{\Lambda_i}.$$

That the right hand side is prorepresented by Λ_i^{def} is just [DW1, 3.1]. That $\Lambda_i^{\text{def}} \in \mathbf{Art}_1$, and so the functor is representable, is 5.7. \square

When X is smooth, GV invariants $\text{GV}_1, \dots, \text{GV}_\ell$ of the flopping curve were defined in [BKL, K1]. The following shows how to obtain *all* of these from the NCCRs of R , via mutation and taking factors. We remark that the following is mildly awkward to state, due in part to the ordering in the simples helix, but also due to the existence of \mathcal{Z} in high length.

Corollary 7.3. *The commutative deformations of \mathcal{S}_i are represented by the abelianisation of Λ_i^{def} , written $(\Lambda_i^{\text{def}})_{\text{ab}}$. In particular, if X is smooth, consider the GV invariants $\text{GV}_1, \dots, \text{GV}_\ell$. Then*

$$\text{GV}_i = \dim_{\mathbb{C}}(\Lambda_j^{\text{def}})_{\text{ab}}$$

for appropriate j , depending on both i and the length ℓ .

Proof. By 7.2 noncommutative deformations of \mathcal{S}_i are represented by Λ_i^{def} and so, as is standard, its abelianisation represents the commutative deformations.

It is well-known [BKL, K1] that GV_i is the multiplicity of $\mathcal{O}_{i\mathbb{C}}$ in the Hilbert scheme, which is precisely the dimension of the representing object of the commutative deformations of the sheaf $\mathcal{O}_{i\mathbb{C}}$. Then simply observe that the simples helix begins either $\mathcal{O}_{\mathbb{C}}(-1), \mathcal{O}_{\ell\mathbb{C}}, \dots, \mathcal{O}_{2\mathbb{C}}$ when $\ell \leq 4$, or $\mathcal{O}_{\mathbb{C}}(-1), \mathcal{O}_{\ell\mathbb{C}}, \dots, \mathcal{O}_{3\mathbb{C}}, \mathcal{Z}, \mathcal{O}_{2\mathbb{C}}$ when $\ell = 5, 6$. In all cases, the sheaves $\mathcal{O}_{i\mathbb{C}}$ appear as some \mathcal{S}_j . \square

7.2. Commutativity of Deformation Algebras. We can go further, and determine when Λ_i^{def} is commutative, and thus establish when commutative deformations suffice. From this, we also determine the only sheaves in the simples helix that can possibly be genuinely spherical, and furthermore later give lower bounds on GV invariants.

The techniques in this subsection are applicable when X is Gorenstein terminal, however, as our main goal is controlling GV invariants, we restrict here to smooth X . To determine Λ_i and thus Λ_i^{def} requires first an understanding of its quiver. By the Perverse–Tilt duality, we just need to know the quivers for $i = 1, \dots, N/2$. To do this, we first slice by a generic element g .

Theorem 7.4. *Suppose that $X \rightarrow \text{Spec } R$ where X is smooth. Writing $\mathbb{F} = - \otimes_R R/g$, then the dimensions of $\mathbb{F}\Lambda_i^{\text{def}}$ and $(\mathbb{F}\Lambda_i^{\text{def}})_{\text{ab}}$, and whether Λ_i is commutative, is summarised in the following table*

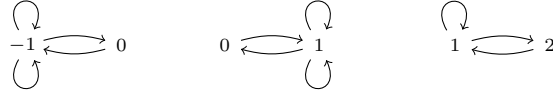
ℓ	# loops in $\mathbb{F}\Lambda_i^{\text{def}}$	$\dim \mathbb{F}\Lambda_i^{\text{def}}$	$\dim(\mathbb{F}\Lambda_i^{\text{def}})_{\text{ab}}$	Is Λ_i^{def} commutative?
1	0	1	1	✓
2	2, 0	4, 1	3, 1	✗, ✓
3	2, 0, 1	12, 1, 3	5, 1, 3	✗, ✓, ✓
4	2, 0, 1, 2	24, 1, 2, 6	6, 1, 2, 4	✗, ✓, ✓, ✗
5	2, 0, 1, 1, 0, 2	40, 1, 2, 4, 1, 10	7, 1, 2, 4, 1, 6	✗, ✓, ✓, ✓, ✓, ✗
6	2, 0, 1, 1, 2, 1, 2	60, 1, 2, 3, 6, 2, 15	6, 1, 2, 3, 4, 2, 6	✗, ✓, ✓, ✓, ✗, ✓, ✗

where in each row the sequence is over the range $i = 0, \dots, N/2$.

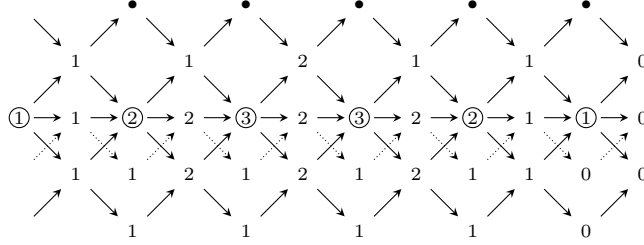
Proof. We prove the $\ell = 3$ row, with all other rows being similar. To ease notation, set $\mathbb{V}_i := \mathbb{F}\mathbb{V}_i$ and $S := \mathbb{V}_0$. It is well-known that $\mathbb{F}\Lambda_i \cong \text{End}_S(\mathbb{V}_{i-1} \oplus \mathbb{V}_i)$. By Katz–Morrison [KM], the CM modules \mathbb{V}_{-1} and \mathbb{V}_1 correspond to the middle vertex of the extended E_6 Dynkin diagram via McKay correspondence, and by the combinatorics in the proof of 2.6, \mathbb{V}_2 corresponds to the rank 2 vertex between the middle and extending vertices.

The quivers of the $\mathbb{F}\Lambda_i$, and the dimension and a presentation of the $\mathbb{F}\Lambda_i^{\text{def}}$ can be obtained via knitting, which we very briefly outline here, referring the reader to [W2, §5.4] and [W1, §4] for more details. Indeed, the number of arrows between the vertices in the quiver of $\mathbb{F}\Lambda_i$ is identical to [W1, §4], and we see that the quivers for $\mathbb{F}\Lambda_i \cong \text{End}_S(\mathbb{V}_{i-1} \oplus \mathbb{V}_i)$

for $i = 0, 1, 2$ are



where a vertex labelled i corresponds to the module V_i . As for the dimension of say $\mathbb{F}\Lambda_0^{\text{def}}$, exactly as in [DW1, 3.16] we can calculate it via knitting as follows:



We deduce that $\dim_{\mathbb{C}} \mathbb{F}\Lambda_0^{\text{def}} = 1 + 2 + 3 + 3 + 2 + 1 = 12$. A very similar calculation gives $\mathbb{F}\Lambda_1^{\text{def}} = 1$, and $\mathbb{F}\Lambda_2^{\text{def}} = 3$.

As is standard, from the dimension calculation above, it is possible to pick algebra generators of $\mathbb{F}\Lambda_i^{\text{def}}$. Then, directly verifying that certain relations hold, and comparing dimensions on both sides, we can furthermore deduce by direct calculation that $\mathbb{F}\Lambda_0^{\text{def}} \cong \mathbb{C}\langle x, y \rangle / (x^2, y^3, (x+y)^3)$, $\mathbb{F}\Lambda_1^{\text{def}} \cong \mathbb{C}$, and $\mathbb{F}\Lambda_2^{\text{def}} \cong \mathbb{C}[x]/x^3$. Taking the abelianisation of these, we see that $\dim_{\mathbb{C}} (\mathbb{F}\Lambda_0^{\text{def}})_{\text{ab}} = 5$, $\dim_{\mathbb{C}} (\mathbb{F}\Lambda_1^{\text{def}})_{\text{ab}} = 1$, and $\dim_{\mathbb{C}} (\mathbb{F}\Lambda_2^{\text{def}})_{\text{ab}} = 3$.

Repeating this calculation over all lengths ℓ gives all columns in the table except the last, and furthermore a presentation for each algebra $\mathbb{F}\Lambda_i^{\text{def}}$. We claim that Λ_i^{def} is not commutative if and only if the quiver of $\mathbb{F}\Lambda_i^{\text{def}}$ has two loops. The last column in the table then follows from the first column.

The (\Leftarrow) direction is a calculation. Indeed, by the above knitting calculation (repeated for all lengths), if the quiver of $\mathbb{F}\Lambda_i^{\text{def}}$ has two loops then, by its explicit presentation in each case, we observe that $\mathbb{F}\Lambda_i^{\text{def}}$ is not commutative. Hence since $\mathbb{F}\Lambda_i^{\text{def}}$ is a factor of Λ_i^{def} , certainly Λ_i^{def} is not commutative.

For the (\Rightarrow) direction, we show the contrapositive. Suppose that the quiver of $\mathbb{F}\Lambda_i^{\text{def}}$ does not have two loops. By the above table, it must have ≤ 1 loops, and so there is an isomorphism $\mathbb{F}\Lambda_i^{\text{def}} \cong \mathbb{C}[y]/y^n$ for some y and some n . But since Λ_i^{def} is local, and g is not a unit, g belongs to the radical J of Λ_i^{def} . Further, every element of A can be written in the form $1 + ya_1 + ga_2$ for some $a_1, a_2 \in A$, and $J = yA + gA$. But then y and g generate the finite dimensional algebra Λ_i^{def} , and so in particular Λ_i^{def} is commutative. \square

The following asserts the conditions under which noncommutative deformation theory is necessary, and yields additional information.

Corollary 7.5. *Suppose that X is smooth. Then the representing object of \mathcal{S}_i is not commutative if and only if \mathcal{S}_i is, up to line bundle twist, one of the following:*

- (1) $\mathcal{O}_{\mathbb{C}}$ when $\ell > 1$.
- (2) $\mathcal{O}_{2\mathbb{C}}$ when $\ell = 4, 5, 6$.
- (3) $\mathcal{O}_{3\mathbb{C}}$ and $\omega_{3\mathbb{C}}$ when $\ell = 6$.

Furthermore, \mathcal{S}_i can be a spherical object only if \mathcal{S}_i is, up to line bundle twist, one of the following:

- (4) $\mathcal{O}_{\ell\mathbb{C}}$ and $\omega_{\ell\mathbb{C}}$ in all types.
- (5) \mathcal{Z} and \mathcal{Z}^{ω} when $\ell = 5$.

The sheaves in (4) and (5) may or may not be spherical, depending on the example. Indeed, the sheaves in (4) are spherical if and only if the top GV invariant is one.

Proof. Accounting for line bundle twists and Perverse–Tilt duality, parts (1)–(3) are then a direct translation of the last column of the table in 7.4. The sheaf \mathcal{S}_i is spherical if and only if $\Lambda_i^{\text{def}} \cong \mathbb{C}$. Hence, if it is spherical, then certainly the factor $\mathbb{F}\Lambda_i^{\text{def}}$ also needs to be one-dimensional. Thus, referring to the table in 7.4, parts (4) and (5) follow. Finally, since the deformation algebra of $\mathcal{S}_1 = \mathcal{O}_{\ell C}$ is always commutative by 7.4, its dimension equals the dimension of its abelianisation. Hence the last statement is just 7.3. \square

Remark 7.6. The above gives more evidence that noncommutative deformations of \mathcal{O}_{aC} controls how both aC and all its higher multiples $n(aC) := (na)C$ move. Indeed, by 7.5 above, strictly noncommutative deformations of \mathcal{O}_{aC} exist if and only if $2a \leq \ell$, which is if and only if higher multiples $n(aC)$ of aC exist.

7.3. Gopakumar–Vafa corollaries. The table below gives the first non-trivial lower bounds on GV invariants. The final column also greatly improves on results from both [DW1, 3.17] and [T4, 1.4]; see also 7.8 below.

Corollary 7.7. *Lower bounds for GV invariants and the dimension of the contraction algebra are as follows.*

ℓ	GV lower bound	$\dim A_{\text{con}}$ lower bound
1	(1)	1
2	(4, 1)	8
3	(5, 3, 1)	26
4	(6, 4, 2, 1)	56
5	(7, 6, 4, 2, 1)	124
6	(6, 6, 4, 3, 2, 1)	200

Proof. First recall from 7.3 that the GV invariants GV_i can be obtained as $\dim_{\mathbb{C}}(\Lambda_j^{\text{def}})_{\text{ab}}$ for appropriate j .

The case $\ell = 1$ is clear. When $\ell = 2$, since the normal bundle of C is $(-3, 1)$, by [W2, 2.15] there are two loops in the quiver of Λ_0^{def} . Hence the abelianisation is at least 4-dimensional, being a factor of $\mathbb{C}[[x, y]]$ modulo two relations, both quadratic or higher. This proves that $GV_1 \geq 4$. The fact that $GV_2 \geq 1$ is obvious, given that $\mathcal{S}_1 = \mathcal{O}_{2C}$ is a non-zero object. For all other cases, the dimension of a factor of the abelianisation is the column labelled $\dim \mathbb{F}(\Lambda_i^{\text{def}})_{\text{ab}}$ in 7.4. Obviously this gives a lower bound for the dimension of the abelianisation. The result then follows, just by matching the first $N/2$ members of the simples helix with the data in 7.4, where recall that the simples helix begins either $\mathcal{O}_C(-1), \mathcal{O}_{\ell C}, \dots, \mathcal{O}_{2C}$ when $\ell \leq 4$, or $\mathcal{O}_C(-1), \mathcal{O}_{\ell C}, \dots, \mathcal{O}_{3C}, \mathcal{Z}, \mathcal{O}_{2C}$ when $\ell = 5, 6$. \square

Remark 7.8. It is known that the lower bounds in 7.7 are obtained when $\ell = 1$ or $\ell = 2$. From [BW] it is known that the lower bound is also achieved for $\ell = 6$. The current lowest known GV invariants from [BW] when $\ell = 3, 4, 5$ are (6, 3, 1), (6, 5, 2, 1), and (8, 6, 4, 2, 1), which have contraction algebras of dimension 27, 60, and 125 respectively. Thus, the bounds in 7.7 are very close to the current minimum.

7.4. Global Equivalences. Deformation theory is analytic, but our results in the previous sections have global consequences. Suppose that $f: Y \rightarrow Y_{\text{con}}$ is a flopping contraction of quasi-projective 3-folds, where Y has at worst Gorenstein terminal singularities in a neighbourhood of the flopping curves. Consider the exceptional fibre, which with its reduced scheme structure consists of a finite collection of curves, each isomorphic to \mathbb{P}^1 . Choose such a curve C , and as notation, consider an affine open subset $U_{\text{con}} = \text{Spec } R$ around the point $f(C)$, and set $U := f^{-1}(U_{\text{con}})$. Taking the formal fibre then gives the

following commutative diagram.

$$\begin{array}{ccccc}
 \mathfrak{U} & \longrightarrow & U & \hookrightarrow & Y \\
 \downarrow & & \downarrow f|_U & & \downarrow f \\
 \mathrm{Spec} \mathfrak{R} & \longrightarrow & \mathrm{Spec} R & \hookrightarrow & Y_{\mathrm{con}}
 \end{array}$$

The curve C can be contracted individually in \mathfrak{U} , and as such individually it has an associated length ℓ , and thus has an associated simples helix $\{\mathcal{S}_i\}_{i \in \mathbb{Z}}$ in \mathfrak{U} . We then view this simples helix on both U and Y via pushforward, but suppress this from the notation.

Lemma 7.9. *With notation as above, there is a chain of isomorphisms*

$$\mathrm{Def}_Y^{\mathcal{S}_i} \cong \mathrm{Def}_U^{\mathcal{S}_i} \cong \mathrm{Def}_{\mathfrak{U}}^{\mathcal{S}_i} \cong \mathrm{Hom}_{\mathrm{Art}_1}(\Lambda_i^{\mathrm{def}}, -).$$

Proof. This is standard; see e.g. [DW3, (2.G)]. \square

Thus by 7.9 the deformation functor associated to each sheaf $\mathcal{S}_i \in \mathrm{coh} Y$ is representable, and exactly as in [DW1, 3.3] the resulting universal sheaf is given as the pushforward of the universal sheaf on \mathfrak{U} . Again, we suppress this from the notation, and simply write $\mathcal{E}_i \in \mathrm{coh} Y$ for the universal sheaf on Y . Exactly as in [DW1, 3.9], $\mathrm{End}_Y(\mathcal{E}_i) \cong \Lambda_i^{\mathrm{def}}$.

Theorem 7.10. *Suppose that $Y \rightarrow Y_{\mathrm{con}}$ is a flopping contraction of quasi-projective 3-folds, where Y has at worst Gorenstein terminal singularities. For every contracted curve C with reduced scheme structure, there is an associated length ℓ , and simples helix $\{\mathcal{S}_i\}_{i \in \mathbb{Z}}$ viewed on Y . Then for all $k \in \mathbb{Z}$, there is a global autoequivalence*

$$\mathrm{Twist}_{\mathcal{S}_k} : \mathrm{D}^b(\mathrm{coh} Y) \rightarrow \mathrm{D}^b(\mathrm{coh} Y),$$

and the following statements hold.

- (1) $\mathcal{E}_k \in \mathrm{Perf}(Y)$.
- (2) There is a functorial triangle

$$\mathrm{RHom}_Y(\mathcal{E}_k, -) \otimes_{\Lambda_k^{\mathrm{def}}}^{\mathbf{L}} \mathcal{E}_k \rightarrow \mathrm{Id} \rightarrow \mathrm{Twist}_{\mathcal{S}_k} \rightarrow .$$

Proof. Part (1) follows exactly as in [DW1, 7.1]. Indeed, complete locally Λ_k^{def} has finite projective dimension, since by (2.E) and (2.F) mutation twice returns us to our original module, so we can simply use the analogue of [DW1, (5.B)]. Hence Λ_k^{def} also has finite projective dimension Zariski locally, since projective dimension can be calculated complete locally, and Λ_k^{def} is supported only at one point. This translates into \mathcal{E}_k being perfect on U , and thus its pushforward is perfect on Y .

There is a functor $\mathrm{Twist}_{\mathcal{S}_k}^*$ which is constructed in a word-for-word identical manner to [DW3, §5], except that the tilting bundle \mathcal{P}_k at all stages replaces the tilting bundle used in [DW3], which is \mathcal{P}_0 in the notation here. The key point is that \mathcal{E}_k is still perfect, by (1). Exactly the same spanning class argument in [DW3, 5.22] shows that a functor $\mathrm{Twist}_{\mathcal{S}_k}^*$ is an equivalence, and $\mathrm{Twist}_{\mathcal{S}_k}$ is then defined to be its inverse. The functorial triangle in part (2) is again very similar: the bundle \mathcal{P}_k just replaces \mathcal{P}_0 , and the analogue of [DW3, 5.23] holds. \square

Remark 7.11. The above establishes that all members \mathcal{S}_i of the simples helix, although they need not be perfect since X is singular, do deform to give a universal sheaf which is a perfect complex.

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