A VERY ELEMENTARY PROOF THAT THE SOMOS 5 SEQUENCE IS INTEGER VALUED

Abstract. We give a short and elementary proof that every term in the Somos 5 sequence is integer valued, and is coprime to the proceeding two terms.

1. Introduction

Definition 1.1. *The Somos* 5 *sequence is the sequence* $(a_n)_{n\in\mathbb{N}}$ *defined by the rule*

$$a_n a_{n+5} = a_{n+1}a_{n+4} + a_{n+2}a_{n+3}$$

with $a_0 = a_1 = a_2 = a_3 = a_4 = 1$.

The sequence starts $1, 1, 1, 1, 1, 2, 3, 5, 11, \ldots$. There are various proofs of the fact that it is integer valued. One is via elliptic curves [2], another as a consequence of the Laurent phenomenon in cluster algebras [1]; presumably there are also many others that are unpublished or elsewhere in the literature. The purpose of this short note is to give a very elementary proof that furthermore establishes a stronger result, namely that each term in the Somos 5 sequence is integer valued, and coprime to the proceeding two terms.

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2. Proof

We require the following two well-known and easy lemmas. The first follows by inspection of the relevant prime decompositions.

Lemma 2.1. *For* $a, b, x, y \in \mathbb{N}$, $(a, x) = (a, y) = (b, x) = (b, y) = 1 \iff (ab, xy) = 1$.

Lemma 2.2. *For* $x, y \in \mathbb{N}$, $(x, y) = 1 \iff (x + y, y) = 1$.

Proof. ($\Rightarrow$) By the assumption there exists $p, q \in \mathbb{Z}$ such that $px + qy = 1$. Then $p(x + y) - py + qy = 1$ so that $p(x + y) + (q - p)y = 1$. Therefore $(x + y, y) = 1$.
($\Leftarrow$) By the assumption there exists $m, n \in \mathbb{Z}$ such that $m(x + y) + ny = 1$. Then $mx + (m + n)y = 1$, so $(x, y) = 1$. □

Recall that the Somos 5 sequence is denoted $(a_n)_{n\in\mathbb{N}}$.

Notation 2.3. For $n \geq 2$, we define $s_n := a_n^2 + a_{n-2}a_{n+2}$.

The following two results are elementary.

Lemma 2.4. *For* $n \geq 4$, $a_{n-3}s_n = a_{n+1}s_{n-2}$.

Proof. We compute

$$\begin{aligned}
a_{n-3}s_n &= a_{n-3}(a_n^2 + a_{n-2}a_{n+2}) \\
&= a_{n-3}a_n^2 + a_{n-2}a_{n-3}a_{n+2} \\
&= a_{n-3}a_n^2 + a_{n-2}(a_{n+1}a_{n-2} + a_n a_{n-1}) \\
&= a_{n-2}^2 a_{n+1} + a_n(a_n a_{n-3} + a_{n-1}a_{n-2}) \\
&= a_{n-2}^2 a_{n+1} + a_n a_{n+1} a_{n-4} \\
&= a_{n+1}(a_{n-2}^2 + a_n a_{n-4}) \\
&= a_{n+1}s_{n-2}.
\end{aligned}$$

□

Corollary 2.5. *For all $n \geq 2$ we have*

$$s_n = \begin{cases} 2a_{n+1}a_{n-1} & \text{for } n \text{ even} \\ 3a_{n+1}a_{n-1} & \text{for } n \text{ odd.} \end{cases}$$

Proof. We will show that this is true for n even. Certainly $s_2 = 1^2+1 = 2$ so the statement is true for $n = 2$. Now let $n \geq 4$ be even and suppose that the statement is true for all smaller even numbers. Then $a_{n-3}s_n = a_{n+1}s_{n-2}$ by Lemma 2.4, and so by inductive hypothesis $a_{n-3}s_n = a_{n+1}(2a_{n-1}a_{n-3})$. Cancelling terms, $s_n = 2a_{n+1}a_{n-1}$, proving the inductive step. The proof for n odd is identical. □

This leads to the main result.

Theorem 2.6. *In the Somos 5 sequence, each $a_n \in \mathbb{Z}$, and further we have $(a_n, a_{n-1}) = (a_n, a_{n-2}) = 1$.*

Proof. We prove the statement by induction, the statement being obvious by inspection for $n \leq 6$. Hence we consider $n \geq 7$, and suppose that the statement is true for smaller n.

We first show that $a_n \in \mathbb{Z}$. By Corollary 2.5 $s_{n-2} = ka_{n-1}a_{n-3}$, where either $k = 2$ or $k = 3$, depending on whether n is even or odd. Regardless, $k \in \mathbb{Z}$. Equating this expression with the definition of s_{n-2}, rearranging we obtain

$$a_{n-4}a_n = ka_{n-1}a_{n-3} - a_{n-2}^2.$$

By the inductive hypothesis $a_{n-1}, a_{n-3}, a_{n-2}$ and k are all integers, so it follows that $a_{n-4}a_n \in \mathbb{Z}$. Further, by the definition of the Somos-5 sequence,

$$a_na_{n-5} = a_{n-1}a_{n-4} + a_{n-2}a_{n-3}$$

and so since by inductive hypothesis $a_{n-1}, a_{n-4}, a_{n-2}$ and a_{n-3} are all integers, we also see that $a_{n-5}a_n \in \mathbb{Z}$. But $(a_{n-5}, a_{n-4}) = 1$ by inductive hypothesis, so there exists $p, q \in \mathbb{Z}$ such that $pa_{n-5} + qa_{n-4} = 1$. Simply multiplying this equation by a_n gives

$$a_n = pa_{n-5}a_n + qa_{n-4}a_n,$$

which shows that $a_n \in \mathbb{Z}$.

We next verify that $(a_n, a_{n-1}) = (a_n, a_{n-2}) = 1$. Certainly by inductive hypothesis we have

$$(a_{n-1}, a_{n-2}) = (a_{n-1}, a_{n-3}) = (a_{n-4}, a_{n-2}) = (a_{n-4}, a_{n-3}) = 1.$$

Hence, using Lemma 2.1,

$$(a_{n-1}a_{n-4}, a_{n-2}a_{n-3}) = 1.$$

Now using Lemma 2.2 with $x := a_{n-1}a_{n-4}$ and $y := a_{n-2}a_{n-3}$, we see that

$$(a_{n-1}a_{n-4} + a_{n-2}a_{n-3}, a_{n-2}a_{n-3}) = 1,$$

thus $(a_na_{n-5}, a_{n-2}a_{n-3}) = 1$ and so $(a_n, a_{n-2}) = 1$ by Lemma 2.1. By a very similar argument, using Lemma 2.2 with $y := a_{n-1}a_{n-4}$ and $x := a_{n-2}a_{n-3}$ we obtain $(a_na_{n-5}, a_{n-1}a_{n-4}) = 1$ and so again by Lemma 2.1 $(a_n, a_{n-1}) = 1$, as required. □

References

[1] S. Fomin and A. Zelevinsky, *The Laurent phenomenon*, Adv. in Appl. Math. **28** (2002), no. 2, 119–144.

[2] A. Hone and C. Swart, *Integrality and the Laurent phenomenon for Somos 4 and Somos 5 sequences*, Math. Proc. Cambridge Philos. Soc. **145** (2008), no. 1, 65–85.