RECONSTRUCTION ALGEBRAS OF TYPE D (II)

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Abstract. This is the third in a series of papers which give an explicit description of the reconstruction algebra as a quiver with relations; these algebras arise naturally as geometric generalizations of preprojective algebras of extended Dynkin quivers. This paper is the companion to [Wem09] and deals with dihedral groups $G = \mathbb{D}_{n,q}$ which have rank 2 special CM modules. We show that such reconstruction algebras are described by combining a preprojective algebra of type $\widetilde{D}$ with some reconstruction algebra of type A.

Contents

1. Introduction

Everybody loves the preprojective algebra of an extended Dynkin diagram. Algebraically they provide us with a rich source of non-trivial but manageable examples on which to test the latest theory, whilst geometrically they encode lots of information about the singularities $\mathbb{C}^2/G$ when G is a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$. However for quotients by finite subgroups of $\mathrm{GL}(2,\mathbb{C})$ which are not inside $\mathrm{SL}(2,\mathbb{C})$, preprojective algebras are not quite so useful.

Fortunately in this case the role of the preprojective algebra is played by a new algebra (the reconstruction algebra) which is built in a similar but slightly modified way. This new algebra is by definition the endomorphism ring of the special CM modules in the sense of Wunram [Wun88], and the recent main result of [Wem08] states that for any finite group $G \leq \mathrm{GL}(2,\mathbb{C})$ the quiver of the reconstruction algebra determines and is determined by the dual graph of the minimal resolution of $\mathbb{C}^2/G$, labelled with self-intersection numbers. Note that when the group G is inside $\mathrm{SL}(2,\mathbb{C})$ the reconstruction algebra is precisely the preprojective algebra of the associated extended Dynkin diagram.

However it should also be noted that in general the above statement is known only on the level of quivers; to give an algebra we need to add in the extra information of the relations and it is this problem which is addressed in this paper. Although technical in nature, knowledge of the relations of the reconstruction algebra unveils relationships between quotients arising from vastly different group structures and consequently reveals explicit geometric structure that has until now remained unnoticed. In this paper, which should be viewed as a companion to [Wem09], we deal with dihedral groups $\mathbb{D}_{n,q}$ inside $GL(2,\mathbb{C})$ for which $n < 2q$. The companion deals with the case $n > 2q$. The main difference between the two is that here rank two special CM modules enter the picture whereas in the $n > 2q$ case all special CM modules have rank one.

Using knowledge of the specials from both [IW08] and [Wem09] we are able to associate to the dual graph of the minimal commutative resolution of $\mathbb{C}^2/\mathbb{D}_{n,q}$ a quiver with relations and then prove that it is isomorphic to the endomorphism ring of the special CM modules.

Like most things involving explicit presentations of algebras the proof is rather technical, however we are able to side-step many issues by using both knowledge of the AR theory and knowledge of the geometry; in particular the non-explicit methods from [Wem08] are used extensively.

We note that the reconstruction algebras in this paper naturally split into two subfamilies but it should perhaps be emphasized that the two families are fairly similar, particularly when compared to [Wem09]. The $\nu = N-1$ family (for notation see Section 2) should be viewed as being very geometrically similar to the dihedral ordinary double points, whereas the examples in *loc. cit.* should be viewed as being very toric. The remaining $0 < \nu < N-1$ family sits geometrically somewhere between these two extremes.

This paper is organized as follows - in Section 2 we define the groups $\mathbb{D}_{n,q}$ and recap some combinatorics and results from [Wem09] and [IW08] which are needed for this paper. Section 3 deals with the case when less than the maximal number of rank 2 specials occurs, whereas Section 4 deals with the case involving the maximal number of rank 2 specials. In Section 5 we illustrate the correspondence between the algebra and the geometry by comparing two examples, making precise some of the above remarks which are of a more philosophical nature.

Throughout when working with quivers we shall write ab to mean a **followed by** b**.** We work over the ground field $\mathbb{C}$ but any algebraically closed field of characteristic zero will suffice.

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2. Dihedral groups and special CM modules

In this paper, as in [Wem09], we follow the notation of Riemenschneider [Rie77].

Definition 2.1. *For* $1 < q < n$ *with* $(n, q) = 1$ *define the group* $\mathbb{D}_{n,q}$ *to be*

$$\mathbb{D}_{n,q} = \begin{cases} \langle \psi_{2q}, \tau, \varphi_{2(n-q)} \rangle & \text{if } n-q \equiv 1 \text{ mod } 2 \\ \langle \psi_{2q}, \tau\varphi_{4(n-q)} \rangle & \text{if } n-q \equiv 0 \text{ mod } 2 \end{cases}$$

with the matrices

$$\psi_k = \begin{pmatrix} \varepsilon_k & 0 \\ 0 & \varepsilon_k^{-1} \end{pmatrix} \quad \tau = \begin{pmatrix} 0 & \varepsilon_4 \\ \varepsilon_4 & 0 \end{pmatrix} \quad \varphi_k = \begin{pmatrix} \varepsilon_k & 0 \\ 0 & \varepsilon_k \end{pmatrix}$$

where ε_t *is a primitive* t^{th} *root of unity.*

The order of the group $\mathbb{D}_{n,q}$ is $4(n-q)q$. By [Bri68, 2.11] the dual graph of the minimal resolution of $\mathbb{C}^2/\mathbb{D}_{n,q}$ is

where the α's come from the Jung-Hirzebruch continued fraction expansion

$$\frac{n}{q} = [\alpha_1, \ldots, \alpha_N].$$

By definition ν records the number of rank 2 special CM modules. Denoting the dual continued fraction expansion by

$$\frac{n}{n-q} = [a_2, \ldots, a_{e-1}],$$

we now briefly recap some combinatorics. To the above data we define the series c, d, r, i, l, b, Δ and Γ as follows:

1. The c series, defined as $c_2 = 1$, $c_3 = 0$, $c_4 = 1$ and $c_j = a_{j-1}c_{j-1} - c_{j-2}$ for all $5 \leq j \leq e$.
2. The d series, defined as $d_2 = 0$, $d_3 = 1$, $d_4 = a_3 - 1$ and $d_j = a_{j-1}d_{j-1} - d_{j-2}$ for all $5 \leq j \leq e$.

3. The r series, defined as $r_2 = a_2(n-q) - q$, $r_3 = r_2 - (n-q)$, $r_4 = (a_3+1)r_3 - r_2$ and $r_j = a_{j-1}r_{j-1} - r_{j-2}$ for all $5 \leq j \leq e$.
4. the i-series, defined as $i_0 = n$, $i_1 = q$ and $i_t = \alpha_{t-1}i_{t-1} - i_{t-2}$ for all $2 \leq t \leq N+1$.
5. The l-series, defined as $l_j = 2 + \sum_{p=1}^{j}(\alpha_p - 2)$ for $1 \leq j \leq N$.
6. The b-series. Define $b_0 := 1$, $b_{l_N - 1} := N$, and further for all $1 \leq t \leq l_N - 2$ (if such t exists), define b_t to be the smallest integer $1 \leq b_t \leq N$ such that $t \leq \sum_{p=1}^{b_t}(\alpha_p - 2)$.
7. Δ_k, defined as $\Delta_k = 1 + \sum_{t=\nu+1}^{k-1} c_{l_t}$ for all $\nu + 1 \leq k \leq N+1$.
8. Γ_k defined as $\Gamma_k = \sum_{t=\nu+1}^{k-1} d_{l_t}$ for all $\nu+1 \leq k \leq N+1$.

Note that in 7 and 8 we use the convention that for $k = \nu + 1$ the sum is empty and so equals zero.

Now after setting $w_1 = xy$ and

$$\begin{array}{ll} w_2 = (x^q + y^q)(x^q + (-1)^{a_2}y^q) & w_3 = (x^q - y^q)(x^q + (-1)^{a_2}y^q) \\ v_2 = x^{2q} + (-1)^{a_2}y^{2q} & v_3 = x^{2q} + (-1)^{a_2 - 1}y^{2q} \end{array}$$

we have the following result:

Theorem 2.2. [Rie77, Satz 2] *The polynomials $w_1^{2(n-q)}$ and $w_1^{r_t}v_2^{c_t}v_3^{d_t}$ for $2 \leq t \leq e$ generate the ring $\mathbb{C}[x,y]^{\mathbb{D}_{n,q}}$. Alternatively we may take the polynomials $w_1^{2(n-q)}$ and $w_1^{r_t}w_2^{c_t}w_3^{d_t}$ for $2 \leq t \leq e$ as a generating set.*

The next two lemmas are the main combinatorial results which will be needed later to determine the relations on the reconstruction algebra:

Lemma 2.3. [Wem09, 2.12, 2.13] *Consider $\mathbb{D}_{n,q}$. Then for all $2 \leq t \leq e-2$, $r_{t+1} = r_{t+2} + i_{b_t}$. Further $r_{l_t} = i_t - i_{t+1}$ for all $\nu + 1 \leq t \leq N$.*

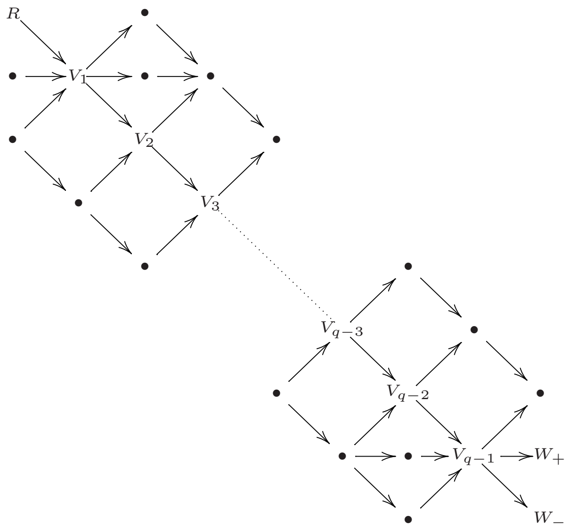
Lemma 2.4. [Wem09, 2.15] *Consider $\mathbb{D}_{n,q}$. Then for all $2 \leq t \leq e-2$,*

$$c_{t+2} = c_{t+1} + \Delta_{b_t} \quad \text{and} \quad d_{t+2} = d_{t+1} + \Gamma_{b_t}.$$

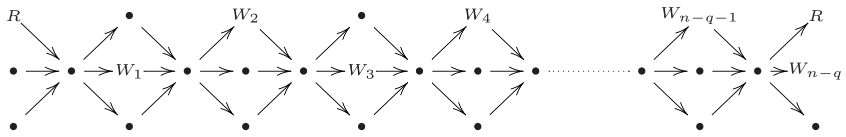
In particular if $\nu = N-1$ then for all $3 \leq t \leq e$

$$c_t = t - 3 \quad \text{and} \quad d_t = 1.$$

Define the rank 1 CM modules W_+, W_- and for each $1 \leq t \leq i_{\nu+1} + \nu(n-q) - 1 = q-1$ the rank 2 indecomposable CM module V_t by the following positions in the AR quiver of $\mathbb{C}[[x,y]]^{\mathbb{D}_{n,q}}$



i.e. all the V_t lie on the diagonal leaving the vertex R, whilst W_+ and W_- are the two rank 1 CM modules at the bottom of the diagonal. Furthermore for every $1 \leq t \leq n-q$ define the rank 1 CM module W_t by the following position in the AR quiver:



i.e. they all live on the non-zero zigzag leaving R. Note that when $n-q$ is even the picture changes slightly since the position of R on the right is twisted (for details we refer the reader to [IW08, §6]). It is worth pointing out that $V_t = (\mathbb{C}[x,y]\otimes_{\mathbb{C}} \rho_t)^{\mathbb{D}_{n,q}}$ where $\mathbb{D}_{n,q}$ acts on both sides of the tensor (on the polynomial ring it acts as inverses) where ρ_t is the two-dimensional irreducible representation

$n-q$ odd			$n-q$ even		
ψ_{2q}	$\mapsto$	$\begin{pmatrix} \varepsilon_{2q}^{t} & 0 \\ 0 & \varepsilon_{2q}^{-t}\end{pmatrix}$	ψ_{2q}	$\mapsto$	$\begin{pmatrix} \varepsilon_{2q}^{t} & 0 \\ 0 & \varepsilon_{2q}^{-t}\end{pmatrix}$
τ	$\mapsto$	$\begin{pmatrix} 0 & \varepsilon_{4}^{t} \\ \varepsilon_{4}^{t} & 0\end{pmatrix}$	$\tau\varphi_{4(n-q)}$	$\mapsto$	$\begin{pmatrix} 0 & \varepsilon_{4}^{t}\varepsilon_{4(n-q)}^{t} \\ \varepsilon_{4}^{t}\varepsilon_{4(n-q)}^{t} & 0\end{pmatrix}$
$\varphi_{2(n-q)}$	$\mapsto$	$\begin{pmatrix} \varepsilon_{2(n-q)}^{t} & 0 \\ 0 & \varepsilon_{2(n-q)}^{t}\end{pmatrix}$			

For a description of the W's in terms of representations, see [Wem09, §3].

The next two results summarize the classification of the specials for the groups $\mathbb{D}_{n,q}$.

Theorem 2.5. [Wem09, 3.11] *For any $\mathbb{D}_{n,q}$, the following CM modules are special and further they are 2-generated as $\mathbb{C}[x,y]^{\mathbb{D}_{n,q}}$-modules by the following elements:*

W_+	x^q+y^q	$(xy)^{n-q}(x^q-y^q)$	x^q+y^q	$(xy)^{n-q}(x^q-y^q)$
W_-	x^q-y^q	$(xy)^{n-q}(x^q+y^q)$	x^q-y^q	$(xy)^{n-q}(x^q+y^q)$
$W_{i_{\nu+1}}$	$(xy)^{i_{\nu+1}}$	$w_2=w_2^{\Delta_{\nu+1}}w_3^{\Gamma_{\nu+1}}$	$(xy)^{i_{\nu+1}}$	$v_2=v_2^{\Delta_{\nu+1}}v_3^{\Gamma_{\nu+1}}$
$W_{i_{\nu+2}}$	$(xy)^{i_{\nu+2}}$	$w_2^{\Delta_{\nu+2}}w_3^{\Gamma_{\nu+2}}$	$(xy)^{i_{\nu+2}}$	$v_2^{\Delta_{\nu+2}}v_3^{\Gamma_{\nu+2}}$
	$\vdots$			
W_{i_N}	$(xy)^{i_N}$	$w_2^{\Delta_N}w_3^{\Gamma_N}$	$(xy)^{i_N}$	$v_2^{\Delta_N}v_3^{\Gamma_N}$

where the left column is one such choice of generators, and the right hand column is another choice. Further there are no other non-free rank one specials.

Theorem 2.6. [IW08, 6.2] *Consider the group $\mathbb{D}_{n,q}$ with $n<2q$, then for all $0\leq s\leq \nu-1$ $V_{i_{\nu+1}+s(n-q)}$ is special. Furthermore these are all the rank 2 indecomposable special CM modules.*

For notational convenience denote $U_s := V_{i_{\nu+1}+(\nu-s)(n-q)}$ for all $1\leq s\leq\nu$, thus in this new notation the rank two indecomposable specials are simply $U_1,\ldots,U_\nu$. Since for dihedral groups $\mathbb{D}_{n,q}$ the maximal co-efficient of any curve in the fundamental cycle Z_f is two, by combining the previous two results we have full knowledge of all special CM modules. We assign to them their corresponding vertices in the dual graph of the minimal resolution in Lemma 3.10 and Lemma 4.7 below.

Before proceeding to study the reconstruction algebra, we must first slightly re-interpret the preprojective algebra since this is crucial to later arguments. Throughout the remainder of this section consider only the groups $G=\mathbb{D}_{n+1,n}\leq \mathrm{SL}(2,\mathbb{C})$ and denote their natural representation by V with basis e_1 and e_2. It is very well known that the endomorphism ring of the specials in this case (summed without multiplicity) is isomorphic to the preprojective algebra of the extended Dynkin diagram of type D, and furthermore that this is morita equivalent to $\mathbb{C}[x,y]\#G$. The relations for the preprojective algebra are obtained by choosing a G-equivariant basis for the maps in the McKay quiver; for details see [CB98] or [BSW08]. Since here for quivers we use the convention that ab is a followed by b, the number of arrows from ρ to σ in $\mathbb{C}[x,y]\#G$ is (see [BSW08, p6] for details and notations)

$$dim_{\mathbb{C}}e_\rho(V^*\otimes\mathbb{C}G)e_\sigma = dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}G}(\mathbb{C}Ge_\rho, V^*\otimes(\mathbb{C}Ge_\sigma))$$

so we interpret arrows from vertex ρ to vertex σ in the preprojective algebra as $\mathbb{C}G$-maps from ρ to $V^* \otimes \sigma$. Since G acts on $\mathbb{C}[x,y]$ in the same way as it acts on V^*, denote the basis of V^* by x and y. The point is that the $\mathbb{C}G$-maps can be written in matrix form in terms of x's and y's, and composition of arrows is given by matrix multiplication. To see this, for example denote the trivial representation of $\mathbb{D}_{n+1,n}$ by ρ_0 and denote its basis by v_0. Then we have a $\mathbb{C}G$-map

$$\begin{array}{rcl} \rho_0 & \rightarrow & V^* \otimes V \\ v_0 & \mapsto & x \otimes e_1 + y \otimes e_2 \end{array}$$

which we write as the matrix $\begin{pmatrix} x & y \end{pmatrix}$. Similarly we have a $\mathbb{C}G$-map

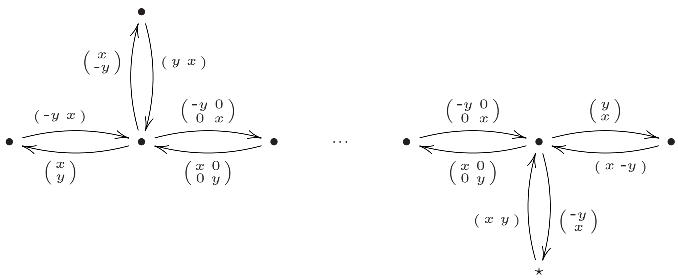
$$\begin{array}{rcl} V & \rightarrow & V^* \otimes \rho_0 \\ e_1 & \mapsto & (-y) \otimes v_0 \\ e_2 & \mapsto & x \otimes v_0 \end{array}$$

which we write as the matrix $\begin{pmatrix} -y \\ x \end{pmatrix}$. The composition of these two maps is (see [BSW08, p7])

$$v_0 \mapsto (x \otimes (-y) + y \otimes x) \otimes v_0$$

which is zero in the skew group ring since $x \otimes y = y \otimes x$. Thus we view the composition as being matrix multiplication $\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} -y \\ x \end{pmatrix}$, subject to the relation that x and y commute.

We can choose the following G-basis for all the maps in the preprojective algebra



and thus we can view the preprojective algebra as being the above algebra subject to the rules of matrix multiplication and the relations induced by the fact that x and y commute. From this picture we may also read off the generators of the CM modules corresponding to the vertices.

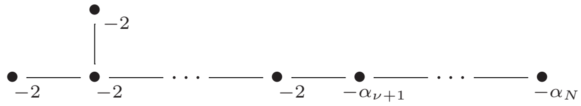
Remark 2.7. Note that the preprojective relations in the above picture at the two branching points are slightly different than the standard ones found elsewhere in the literature.

Remark 2.8. The above picture is good for another reason, namely it gives us an explicit description of the Azumaya locus. More precisely for any orbit containing the point $0 \neq (\lambda_x, \lambda_y) \in \mathbb{C}^2$, denote by $M_{(\lambda_x,\lambda_y)}$ the representation of the preprojective algebra obtained by placing $\mathbb{C}$ at the outside vertices and $\mathbb{C}^2$ at all other vertices, and further replacing every x in the above picture by λ_x and every y by λ_y. Then $M_{(\lambda_x,\lambda_y)}$ is the unique simple above the point corresponding to the orbit containing (λ_x, λ_y).

3. The reconstruction algebra for $0 < \nu < N-1$

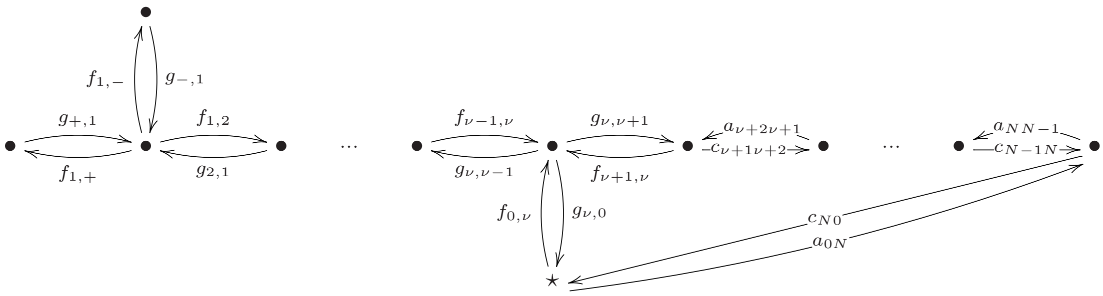
In this section we define the reconstruction algebra $D_{n,q}$ as a quiver with relations in the case $0 < \nu < N-1$ and prove that it is isomorphic to the endomorphism ring of the special CM modules.

Consider, for $N \in \mathbb{N}$ with $N \geq 2$ and for positive integers $\alpha_{\nu+1} \geq 3$, $\alpha_t \geq 2$ for $\nu + 2 \leq t \leq N$, the labelled Dynkin diagram of type D:



Note that the hypothesis $0 < \nu < N-1$ translates into the condition $\alpha_1 = \ldots = \alpha_\nu = 2$ with $\alpha_{\nu+1} \geq 3$. We call the left hand vertex the + vertex, the top vertex the − vertex and the remaining vertices $1, \ldots, N$ reading from left to right.

To this picture we add an extended vertex called $\star$ and 'double-up' as follows:

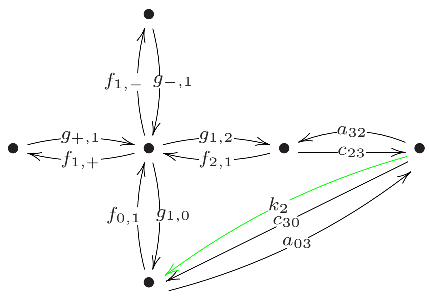


where we have attached $\star$ to the ν^{th} and N^{th} vertices. Now if $\sum_{i=1}^{N}(\alpha_i - 2) \geq 2$ add extra arrows in the following way:

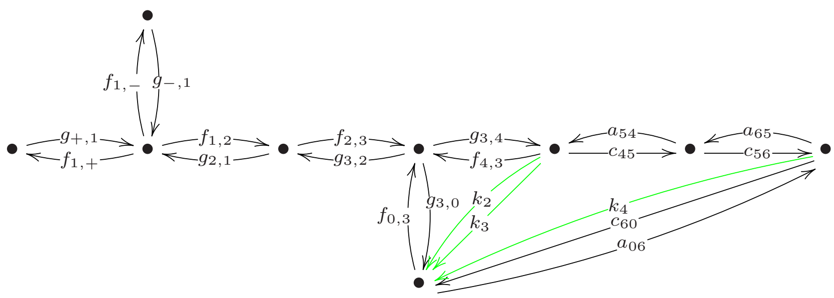
- if $\alpha_{\nu+1} > 3$ then add an extra $\alpha_{\nu+1} - 3$ arrows from the $(\nu+1)^{st}$ vertex to $\star$.
- If $\alpha_i > 2$ with $i \geq \nu + 2$ then add an extra $\alpha_i - 2$ arrows from the i^{th} vertex to $\star$.

Label the new arrows (if they exist) by $k_2, \ldots, k_{\sum(\alpha_i - 2)}$ reading left to right. Name this new quiver Q.

Example 3.1. Consider $\mathbb{D}_{13,8}$ then $\frac{13}{8} = [2,3,3]$, $\nu = 1$ and so the quiver Q is



Example 3.2. Consider $\mathbb{D}_{73,56}$ then $\frac{73}{56} = [2,2,2,5,2,3]$, $\nu = 3$ and so the quiver Q is



To define the reconstruction algebra we need to first set up some notations, then specify relations. Firstly we denote $k_1 := g_{\nu,0}$ and $k_{1+\sum_{i=1}^{N}(\alpha_i-2)} := c_{N0}$.

Definition 3.3. *For all $1 \leq r \leq 1 + \sum_{i=1}^{N}(\alpha_i - 2)$ define $\mathtt{B}_r$ ("the butt") to be the number of the vertex associated to the tail of the arrow k_r.*

Notice for all $2 \leq r \leq 1 + \sum_{i=1}^{N}(\alpha_i - 2)$ it is true that $\mathtt{B}_r = b_r$ where b_r is the b-series of $\frac{n}{q}$ defined in Section 2, however $\mathtt{B}_1 \neq b_1$ since $b_1 = \nu + 1$ whilst $\mathtt{B}_1 = \nu$ by the special definition of k_1.

Now define $u_\nu = 1$ and further for $\nu+1 \leq i \leq N$ denote

$$\begin{array}{c} u_i := \max\{j : 2 \leq j \leq 1+\sum_{i=1}^N(\alpha_i-2) \text{ with } b_j = i\} \\ v_i := \min\{j : 2 \leq j \leq 1+\sum_{i=1}^N(\alpha_i-2) \text{ with } b_j = i\} \end{array}$$

if such things exist (i.e. vertex i has an extra arrow leaving it). Also define $W_{\nu+1} := \nu$ and for every $\nu+2 \leq i \leq N$ define

$$W_i = \left\{ \begin{array}{cl} \nu & \text{if } \mathcal{S}_i \text{ is empty} \\ \text{the maximal number in } \mathcal{S}_i & \text{else} \end{array} \right.$$

where

$$\mathcal{S}_i = \{\text{vertex } j : 1 \leq j < i \text{ and } j \text{ has an extra arrow leaving it}\}.$$

Thus W_i is defined for $\nu+1 \leq i \leq N$. The idea behind it is that W_i records the closest vertex to the left of vertex i which has a k leaving it; since we have defined $k_1 := g_{\nu,0}$ this is always possible to find. Now define, for all $\nu+1 \leq i \leq N$, $V_i := u_{W_i}$. Thus V_i records the number of the largest k to the left of the vertex i, where since $k_1 := g_{\nu,0}$ and $u_\nu = 1$ it always exists.

Denote also

$$F_{1,\nu} := \left\{ \begin{array}{cl} e_1 & \text{if } \nu = 1 \\ f_{1,2}\dots f_{\nu-1,\nu} & \text{if } \nu > 1 \end{array} \right. \qquad G_{\nu,1} := \left\{ \begin{array}{cl} e_1 & \text{if } \nu = 1 \\ g_{\nu,\nu-1}\dots g_{2,1} & \text{if } \nu > 1 \end{array} \right.$$

where e_1 is the trivial path at vertex 1, and define $A_{0t} := a_{0N}\dots a_{t+1t}$ for every $\nu+1 \leq t \leq N$.

Definition 3.4. *For $\frac{n}{q} = [\alpha_1,\dots,\alpha_N]$ with $0 < \nu < N-1$ define $D_{n,q}$, the reconstruction algebra of type D, to be the path algebra of the quiver Q defined above subject to the relations*

$$f_{0,\nu}g_{\nu,\nu+1} = 2A_{0\nu}$$

$$\text{If } \nu = 1 \quad \left\{ \begin{array}{l} g_{+,1}f_{1,+} = 0 \quad f_{0,1}g_{1,0} = 0 \\ g_{-,1}f_{1,-} = 0 \quad f_{2,1}g_{1,2} = 0 \\ g_{1,0}f_{0,1} - g_{1,2}f_{2,1} = f_{1,+}g_{+,1} - f_{1,-}g_{-,1} \end{array} \right.$$

$$\text{If } \nu > 1 \quad \left\{ \begin{array}{l} g_{+,1}f_{1,+} = 0 \quad f_{0,\nu}g_{\nu,0} = 0 \\ g_{-,1}f_{1,-} = 0 \quad f_{\nu+1,\nu}g_{\nu,\nu+1} = 0 \\ f_{1,+}g_{+,1} - f_{1,-}g_{-,1} = 2f_{1,2}g_{2,1} \\ g_{t,t-1}f_{t-1,t} = f_{t,t+1}g_{t+1,t} \text{ for all } 2 \leq t \leq \nu-1 \\ g_{\nu,0}f_{0,\nu} - g_{\nu,\nu+1}f_{\nu+1,\nu} = 2g_{\nu,\nu-1}f_{\nu-1,\nu} \end{array} \right.$$

together with the relations defined algorithmically as

$$\begin{array}{lll}
\textit{Step } \nu+1 & \text{If } \alpha_{\nu+1} = 3 & c_{\nu+1\nu+2}a_{\nu+2\nu+1} = p_{\nu+1\nu+1} \\
& \text{If } \alpha_{\nu+1} > 3 & k_2A_{0\nu+1} = p_{\nu+1\nu+1}, A_{0\nu+1}k_2 = C_{0\nu}k_1 \\
& & k_tC_{0\nu+1} = k_{t+1}A_{0\nu+1}, C_{0\nu+1}k_t = A_{0\nu+1}k_{t+1} \ \forall\ 2 \leq t < u_{\nu+1} \\
& & k_{u_{\nu+1}}C_{0\nu+1} = c_{\nu+1\nu+2}a_{\nu+2\nu+1} \\
& & \vdots \\
\textit{Step } i & \text{If } \alpha_i = 2 & c_{ii+1}a_{i+1i} = a_{ii-1}c_{i-1i} \\
& \text{If } \alpha_i > 2 & k_{v_i}A_{0i} = a_{ii-1}c_{i-1i}, A_{0i}k_{v_i} = C_{0\mathrm{B}_{V_i}}k_{V_i} \\
& & k_tC_{0i} = k_{t+1}A_{0i}, C_{0i}k_t = A_{0i}k_{t+1} \ \forall\ v_i \leq t < u_i \\
& & k_{u_i}C_{0i} = c_{ii+1}a_{i+1i} \\
& & \vdots \\
\textit{Step } N & \text{If } \alpha_N = 2 & c_{N0}a_{0N} = a_{NN-1}c_{N-1N}, a_{0N}c_{N0} = C_{0\mathrm{B}_{V_N}}k_{V_N} \\
& \text{If } \alpha_N > 2 & k_{v_N}a_{0N} = a_{NN-1}c_{N-1N}, a_{0N}k_{v_N} = C_{0\mathrm{B}_{V_N}}k_{V_N} \\
& & k_tC_{0N} = k_{t+1}a_{0N}, C_{0N}k_t = a_{0N}k_{t+1} \ \forall\ v_N \leq t < u_N
\end{array}$$

The only things that remain to be defined are the p and the C's, and it is these which change according to the presentation, namely

$$p_{\nu+1\nu+1} = \begin{cases} f_{\nu+1,\nu}G_{\nu,1}f_{1,+}g_{+,1}F_{1,\nu}g_{\nu,\nu+1} \\ \frac{1}{2}f_{\nu+1,\nu}G_{\nu,1}(f_{1,+}g_{+,1}+f_{1,-}g_{-,1})F_{1,\nu}g_{\nu,\nu+1} \end{cases}$$
$$C_{0\nu} = \begin{cases} f_{0,\nu}G_{\nu,1}f_{1,-}g_{-,1}F_{1,\nu} \\ \frac{1}{2}f_{0,\nu}G_{\nu,1}(f_{1,+}g_{+,1}+f_{1,-}g_{-,1})F_{1,\nu} \end{cases}$$
$$C_{0\nu+1} = \begin{cases} f_{0,\nu}G_{\nu,1}f_{1,+}g_{+,1}F_{1,\nu}g_{\nu,\nu+1} \\ \frac{1}{2}f_{0,\nu}G_{\nu,1}(f_{1,+}g_{+,1}+f_{1,-}g_{-,1})F_{1,\nu}g_{\nu,\nu+1} \end{cases}$$

where for the moduli presentation we take the top choices and for the symmetric presentation we take the bottom choices. We also define, for all $\nu+2\leq t\leq N$, $C_{0t}=C_{0\nu+1}c_{\nu+1\nu+2}\dots c_{t-1t}$.

Remark 3.5. As in [Wem09] we should explain why we give two presentations. The symmetric case is pleasing since it treats the two (-2) horns equally, so that the algebra produced is independent of how we view the dual graph. On the other hand the relations in the moduli presentation are more binomial and so this makes the explicit geometry easier to write down in Section 5. We show in Theorem 3.11 that the two presentations yield isomorphic algebras, but for the moment denote $D_{n,q}$ for the moduli presentation and $D'_{n,q}$ for the symmetric presentation.

Remark 3.6. The algorithmic relations are just the relations from the reconstruction algebra of Type A. Thus the reconstruction algebra here should be viewed as a preprojective algebra of type D with a reconstruction algebra of type A stuck onto the side.

Remark 3.7. The $\nu=1$ and $\nu>1$ split in the above definition is not natural in the sense that in both cases they are just the preprojective relations. We only split them for notational ease. In both presentations notice the existence of a relation which is not a cycle and further that at any vertex in the reconstruction algebra corresponding to a (-2) curve there is only one relation, a preprojective relation.

Example 3.8. For the group $\mathbb{D}_{13,8}$ the symmetric presentation of the reconstruction algebra is the quiver in Example 3.1 subject to the relations

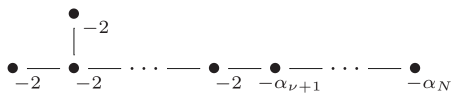
$$\begin{gathered}
f_{0,1}g_{1,2}=2a_{03}a_{32}\\
g_{+,1}f_{1,+}=0\\
g_{-,1}f_{1,-}=0\\
f_{0,1}g_{1,0}=0\\
f_{2,1}g_{1,2}=0\\
g_{1,0}f_{0,1}-g_{1,2}f_{2,1}=f_{1,+}g_{+,1}-f_{1,-}g_{-,1}\\
\tfrac{1}{2}f_{2,1}(f_{1,+}g_{+,1}+f_{1,-}g_{-,1})g_{1,2}=c_{23}a_{32}\\
a_{32}c_{23}=k_2(a_{03}) \qquad \tfrac{1}{2}f_{0,1}(f_{1,+}g_{+,1}+f_{1,-}g_{-,1})g_{1,0}=(a_{03})k_2\\
k_2(\tfrac{1}{2}f_{0,1}(f_{1,+}g_{+,1}+f_{1,-}g_{-,1})g_{1,2}c_{23})=c_{30}(a_{03}) \qquad (\tfrac{1}{2}f_{0,1}(f_{1,+}g_{+,1}+f_{1,-}g_{-,1})g_{1,2}c_{23})k_2=(a_{03})c_{30}
\end{gathered}$$

Example 3.9. For the group $\mathbb{D}_{73,56}$ the moduli presentation of the reconstruction algebra is the quiver in Example 3.2 subject to the relations

$$\begin{gathered}
f_{0,3}g_{3,4}=2a_{06}a_{65}a_{54}\\
g_{+,1}f_{1,+}=0\\
g_{-,1}f_{1,-}=0\\
f_{0,3}g_{3,0}=0\\
f_{4,3}g_{4,3}=0\\
f_{1,+}g_{+,1}-f_{1,-}g_{-,1}=2f_{1,2}g_{2,1}\\
g_{2,1}f_{1,2}=f_{2,3}g_{3,2}\\
g_{3,0}f_{0,3}-g_{3,4}f_{4,3}=2g_{3,2}f_{2,3}\\
f_{4,3}g_{3,2}g_{2,1}f_{1,+}g_{+,1}f_{1,2}f_{2,3}g_{3,4}=k_2(a_{06}a_{65}a_{54}) \qquad f_{0,3}g_{3,2}g_{2,1}f_{1,-}g_{-,1}f_{1,2}f_{2,3}g_{3,0}=(a_{06}a_{65}a_{54})k_2\\
k_2(f_{0,3}g_{3,2}g_{2,1}f_{1,+}g_{+,1}f_{1,2}f_{2,3}g_{3,4})=k_3(a_{06}a_{65}a_{54}) \qquad (f_{0,3}g_{3,2}g_{2,1}f_{1,+}g_{+,1}f_{1,2}f_{2,3}g_{3,4})k_2=(a_{06}a_{65}a_{54})k_3\\
k_3(f_{0,3}g_{3,2}g_{2,1}f_{1,+}g_{+,1}f_{1,2}f_{2,3}g_{3,4})=c_{45}a_{54}\\
a_{54}c_{45}=c_{56}a_{65}\\
a_{65}c_{56}=k_4(a_{06}) \qquad (a_{06})k_4=(f_{0,3}g_{3,2}g_{2,1}f_{1,+}g_{+,1}f_{1,2}f_{2,3}g_{3,4})k_3\\
k_4(f_{0,3}g_{3,2}g_{2,1}f_{1,+}g_{+,1}f_{1,2}f_{2,3}g_{3,4}c_{45}c_{56})=c_{60}(a_{06}) \qquad (a_{06})c_{60}=(f_{0,3}g_{3,2}g_{2,1}f_{1,+}g_{+,1}f_{1,2}f_{2,3}g_{3,4}c_{45}c_{56})k_4
\end{gathered}$$

Before we show that the endomorphism ring of the specials is isomorphic to this quiver with relations we must first justify the positions of the specials relative to the dual graph:

Lemma 3.10. *Consider $\mathbb{D}_{n,q}$ with $0<\nu<N-1$. Then the specials correspond to the dual graph of the minimal resolution in the following way*

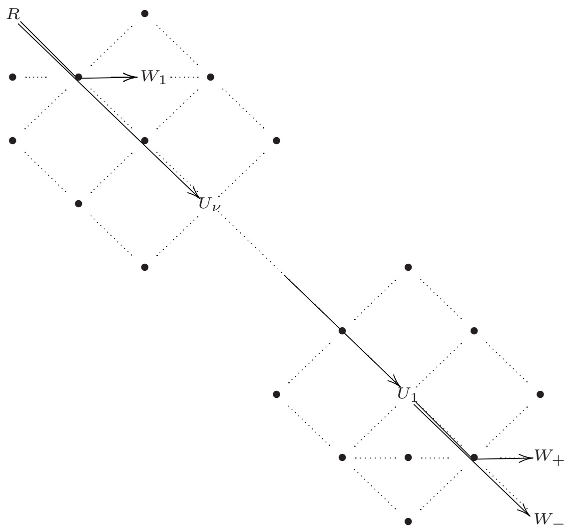


$$\begin{array}{ccccccccccccc} & & W_- & & & & & & & & & & \\ & & | & & & & & & & & & & \\ W_+ & — & U_1 & — & \cdots & — & U_\nu & — & W_{i_{\nu+1}} & — & \cdots & — & W_{i_N} \end{array}$$

Proof. Denote by E_t the curve corresponding to vertex t for $t\in\{+,-,1,\ldots,N\}$. The assumption $0<\nu<N-1$ translates into the condition $\alpha_{\nu+1}\geq 3$ which then forces $Z_f=E_+ + E_- + \sum_{p=1}^{\nu} 2E_p + \sum_{t=\nu+1}^{N} E_t$. Hence

$-Z_f\cdot E_t$	$(Z_K-Z_f)\cdot E_t$	
0	0	$t\in\{+,-,1,\ldots,\nu-1\}$
1	1	$t=\nu$
$\alpha_{\nu+1}-3$	-1	$t=\nu+1$
α_t-2	0	$\nu+1<t<N$
α_N-1	1	$t=N$

and so by [Wem08, 3.3] the quiver of the endomorphism ring of the specials coincides with Q as defined above. But it is clear that the following composition of maps in the AR quiver do not factor through other specials



where in the above picture the positions of R and W_1 are fixed, however since $U_\nu=V_{i_{\nu+1}}$ in general the length of the distance between R and U_ν depends on $i_{\nu+1}$. Also the distance between the other specials changes depending on $n-q$, however none of these changes in distance effect the irreducibility of the highlighted arrows.

The above irreducible maps thus fix the positions of W_1, W_+, W_- and the U's, and the rest follow by noting the factorization of $(xy)^{i_{\nu+1}}$ as

$$W_{i_{\nu+1}} \xleftarrow{(xy)^{i_{\nu+1}-i_{\nu+2}}} W_{i_{\nu+2}} \qquad \cdots \qquad W_{i_{N-1}} \xleftarrow{(xy)^{i_{N-1}-i_N}} W_1 \xleftarrow{xy} R$$

$\square$

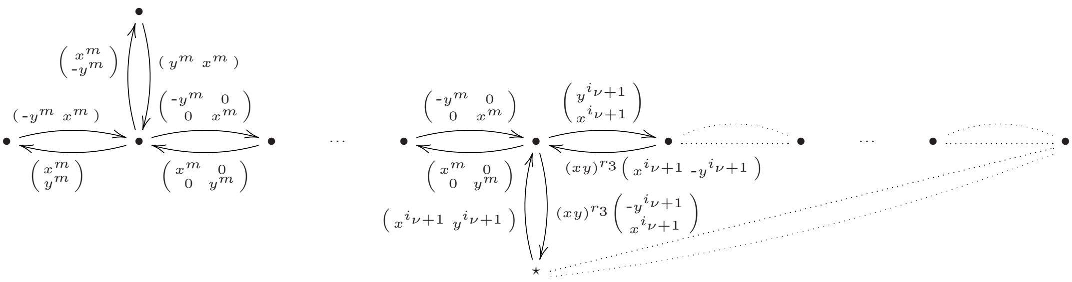
The following is the main result of this section:

Theorem 3.11. *For a group $\mathbb{D}_{n,q}$ with parameter $0<\nu<N-1$, denote $R=\mathbb{C}[x,y]^{\mathbb{D}_{n,q}}$ and let $T_{n,q}=R\oplus W_+\oplus W_-\oplus_{s=1}^{\nu} U_s \oplus_{t=\nu+1}^{N} W_{i_t}$ be the sum of the special CM modules. Then*

$$D_{n,q}\cong \mathrm{End}_R(T_{n,q})\cong D'_{n,q}.$$

Proof. We prove both statements at the same time by making different choices for the generators of the specials. As stated in the proof of Lemma 3.10 the quiver of the endomorphism ring of the specials is precisely that of the quiver Q defined above. We first find representatives for the known number of arrows:

Set $m := n-q$. By choosing a G-equivariant basis as explained in Section 2, the following are always homomorphisms between the specials



The proof that these maps are actually homomorphisms (equivalently G-equivariant) splits into two cases depending on the parity of $n-q$ since by definition both G and the representations depend on $n-q$; we suppress the details. By inspection they do not factor through any of the other specials and so can be chosen as representatives. Note that with these choices

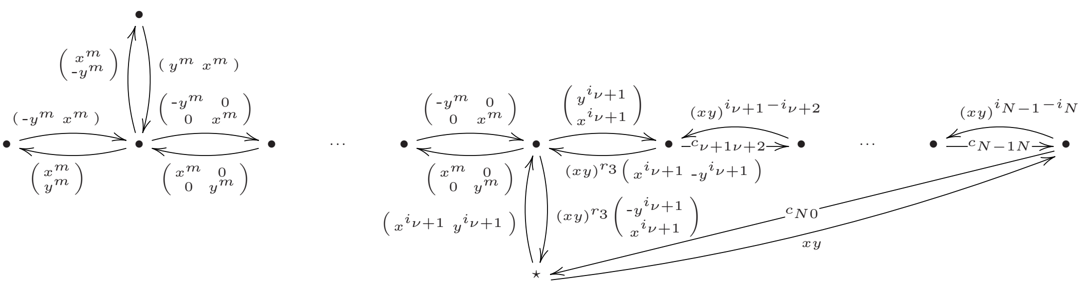
$$p_{\nu+1\nu+1} = \begin{cases} f_{\nu+1,\nu}G_{\nu,1}f_{1,+}g_{+,1}F_{1,\nu}g_{\nu,\nu+1} = (xy)^{r_3}w_2^{c_3}w_3^{d_3} = (xy)^{r_3}w_3 \\ \frac{1}{2}f_{\nu+1,\nu}G_{\nu,1}(f_{1,+}g_{+,1}+f_{1,-}g_{-,1})F_{1,\nu}g_{\nu,\nu+1} = (xy)^{r_3}v_2^{c_3}v_3^{d_3} = (xy)^{r_3}v_3 \end{cases}$$

$$C_{0\nu}k_1 = \begin{cases} f_{0,\nu}G_{\nu,1}f_{1,-}g_{-,1}F_{1,\nu}g_{\nu,0} = (xy)^{r_3}w_2^{c_3}w_3^{d_3} = (xy)^{r_3}w_3 \\ \frac{1}{2}f_{0,\nu}G_{\nu,1}(f_{1,+}g_{+,1}+f_{1,-}g_{-,1})F_{1,\nu}g_{\nu,0} = (xy)^{r_3}v_2^{c_3}v_3^{d_3} = (xy)^{r_3}v_3 \end{cases}$$

$$C_{0\nu+1} = \begin{cases} f_{0,\nu}G_{\nu,1}f_{1,+}g_{+,1}F_{1,\nu}g_{\nu,\nu+1} = w_2 \\ \frac{1}{2}f_{0,\nu}G_{\nu,1}(f_{1,+}g_{+,1}+f_{1,-}g_{-,1})F_{1,\nu}g_{\nu,\nu+1} = v_2 \end{cases}$$

We must now choose representatives of the remaining arrows, and it is different choices of these which give the two different presentations. As in the proof of the above lemma it is clear that the anti-clockwise arrows are $a_{N0} = xy$ and $a_{tt-1} = (xy)^{i_{t-1}-i_t}$. Also, since we must reach the generators of the specials as paths from $\star$ (i.e. the vertex corresponding to R), by Lemma 2.5 we can choose $c_{t-1t} = w_2^{c_{l_{t-1}}}w_3^{d_{l_{t-1}}}$ for all $\nu+2 \le t \le N$. Alternatively, also by Lemma 2.5, we can choose $c_{t-1t} = v_2^{c_{l_{t-1}}}v_3^{d_{l_{t-1}}}$ for all $\nu+2 \le t \le N$ (these choices will give the symmetric presentation). As explained in [Wem09, 4.11] we may choose $c_{N0} = w_2^{c_{l_N}}w_3^{d_{l_N}} = w_2^{c_{e-1}}w_3^{d_{e-1}}$ (respectively $v_2^{c_{e-1}}v_3^{d_{e-1}}$).

Thus we have proved that we may take as representatives of the arrows



with $c_{t-1t} = w_2^{c_{l_{t-1}}}w_3^{d_{l_{t-1}}}$ for all $\nu+2 \le t \le N$ and $c_{N0} = w_2^{c_{e-1}}w_3^{d_{e-1}}$. Now if $\sum_{i=1}^{N}(\alpha_i - 2) \ge 2$ then for the moduli presentation we take

$$k_t = (xy)^{r_{t+2}}w_2^{c_{t+1}}w_3^{d_{t+1}} \text{ for all } 2 \le t \le \sum(\alpha_i - 2)$$

as representatives. To see why we can do this, for example if $\alpha_{\nu+1} \ge 4$ consider $(xy)^{r_4}w_2^{c_3}w_3^{d_3} : W_{i_{\nu+1}} \to R$. Firstly it does not factor through maps we have already chosen, since if it does then since $f_{\nu+1,\nu}$ has a $(xy)^{r_3}$ factor and $c_{\nu+1\nu+2} = w_2^{c_{l_{\nu+1}}} w_3^{d_{l_{\nu+1}}}$ we may write $(xy)^{r_4} w_2^{c_3} w_3^{d_3} = (xy)^{r_3} F + w_2^{c_{l_{\nu+1}}} w_3^{d_{l_{\nu+1}}} h$ for some polynomials F and h. But by looking at xy powers we know $(xy)^{r_4}$ divides h and so after cancelling factors we may write $w_2^{c_3} w_3^{d_3} = (xy)^{r_3-r_4} F + w_2^{c_{l_{\nu+1}}} w_3^{d_{l_{\nu+1}}} h_1$. After cancelling $w_2^{c_3} w_3^{d_3}$ (which F must be divisible by) $1 = (xy)^{r_3-r_4} F' + w_2^{c_{l_{\nu+1}}-c_3} w_3^{d_{l_{\nu+1}}-d_3} h_1$ which is impossible since the assumption $\alpha_{\nu+1} \geq 4$ means that the right hand side cannot have degree zero terms. Secondly, $(xy)^{r_4} w_2^{c_3} w_3^{d_3}$ does not factor as a map $W_{i_{\nu+1}} \to R$ followed by a non-scalar invariant since this would contradict the embedding dimension. The proof for the remainder of the arrows is similar; in fact the proof is identical to [Wem09, 4.11].

The symmetric presentation is the same, but we replace everywhere w_2 by v_2 and w_3 by v_3.

The relations for the reconstruction algebra are now induced by the relations of matrix multiplication such that x and y commute. Using Lemma 2.3, Lemma 2.4 and the polynomial expressions for $p_{\nu+1\nu+1}$, $C_{0\nu}k_1$, and $C_{0\nu+1}$ above, it is straightforward to verify that the relations in Definition 3.4 (which we denote by $\mathcal{S}'$) are satisfied by these choices of polynomials, since the algorithmic relations are just the pattern in the cycles in $\mathtt{D}_1$ (respectively $\mathtt{D}_2$) from [Wem09, 3.5].

The remainder of the proof is now identical to [Wem09, 4.11]: we first work in the completed case (so we can use [Wem08, 3.3] and [BIRS, 3.4]) and we prove that the completion of the endomorphism ring of the specials is given as the completion of $\mathbb{C}Q$ (denoted $\mathbb{C}\hat{Q}$) modulo the closure of the ideal $\langle \mathcal{S}' \rangle$ (denoted $\overline{\langle \mathcal{S}' \rangle}$). The non-completed version of the theorem then follows by simply taking the associated graded ring of both sides of the isomorphism.

Denote the kernel of the surjection $\mathbb{C}\hat{Q} \to \mathrm{End}_{\mathbb{C}[[x,y]]^G}(T_{n,q}) := \Lambda$ by I, denote the radical of $\mathbb{C}\hat{Q}$ by J and further for $t \in \{\star, +, -, 1, \dots, N\}$ denote by S_t the simple corresponding to the vertex t of Q. In Lemma 3.12 below we show that the elements of $\mathcal{S}'$ are linearly independent in $I/(IJ + JI)$. Thus we may extend $\mathcal{S}'$ to a basis $\mathcal{S}$ of $I/(IJ + JI)$ and so by [BIRS, 3.4(a)] $I = \overline{\langle \mathcal{S} \rangle}$. But by combining [BIRS, 3.4(b)], [Wem08, 3.3] and inspecting our set $\mathcal{S}'$ we see that

$$\#(e_a \mathbb{C}\hat{Q} e_b) \cap \mathcal{S} = \#(e_a \mathbb{C}\hat{Q} e_b) \cap \mathcal{S}'$$

for all $a, b \in \{\star, +, -, 1, \dots, N\}$, proving that the number of elements in $\mathcal{S}$ and $\mathcal{S}'$ are the same. Hence $\mathcal{S}' = \mathcal{S}$ and so $I = \overline{\langle \mathcal{S}' \rangle}$, as required. $\square$

The proof of the following lemma is only subtly different to that of [Wem09, 4.11] and so we only prove the parts in which the subtle differences arise.

Lemma 3.12. *With notation from the above proof, the members of $\mathcal{S}'$ are linearly independent in $I/(IJ + JI)$.*

Proof. Following [Wem09, 4.11] we say that a word w in the path algebra $\mathbb{C}\hat{Q}$ satisfies condition (A) if

(i) It does not contain some proper subword which is a cycle.
(ii) It does not contain some proper subword which is a path from $\star$ to $\nu + 1$.

We know from the intersection theory and [Wem08, 3.3] that the ideal I is generated by one relation from $\star$ to $\nu + 1$, whereas all other generators are cycles. Consequently if a word w satisfies (A) then $w \notin IJ + JI$. It is also clear that $f_{0,\nu} g_{\nu,\nu+1} - 2A_{0\nu+1} \notin IJ + JI$.

All members of $\mathcal{S}'$ are either cycles at some vertex or paths from $\star$ to $\nu + 1$, so to prove that the members of $\mathcal{S}'$ are linearly independent in $I/(IJ + JI)$ we just need to show that

1. the elements of $\mathcal{S}'$ which are paths from $\star$ to $\nu + 1$ are linearly independent in $e_\star(I/(IJ + JI))e_{\nu+1}$.
2. for all $t \in \{\star, +, -, 1, \dots, N\}$, the elements of $\mathcal{S}'$ that are cycles at t are linearly independent in $e_t(I/(IJ + JI))e_t$.

The first condition is easy, since the only relation in $\mathcal{S}'$ from $\star$ to $\nu+1$ is $f_{0,\nu}g_{\nu,\nu+1}-2A_{0\nu+1}$ and we have already noted that it does not belong to $IJ+JI$, thus it is non-zero and so linearly independent in $e_\star(I/(IJ+JI))e_{\nu+1}$. Note also that there are only two paths of minimal grade from $\star$ to $\nu+1$ (namely $f_{0,\nu}g_{\nu,\nu+1}$ and $A_{0\nu+1}$) and by inspection of the polynomials they represent we do not have any other relation from $\star$ to $\nu+1$ of this grade.

For the second condition, we must check t case by case:
Case $t=+$ and $t=-$. There is only one relation in $\mathcal{S}'$ which is a cycle at that vertex and so to prove linearly independence requires only that the relation is non zero in $I/(IJ+JI)$. But clearly $g_{+,1}f_{1,+}\notin IJ+JI$ and $g_{-,1}f_{1,-}\notin IJ+JI$ since they both satisfy condition (A).
Case $t=1,\dots\nu$. Again there is only one relation, and since each word satisfies condition (A) it does not belong to $IJ+JI$ and so the relation is non-zero. For example if $f_{1,+}g_{+,1}-f_{1,-}g_{-,1}-2f_{1,2}g_{2,1}\in IJ+JI$ then $f_{1,+}g_{+,1}=f_{1,-}g_{-,1}+2f_{1,2}g_{2,1}+u$ for some $u\in IJ+JI$ in the free algebra $\mathbb{C}\hat{Q}$ which is impossible since $f_{1,+}g_{+,1}$ cannot appear in the right hand side.
Case $t=\nu+1$. If $\alpha_{\nu+1}=3$ then there are only two relations in $\mathcal{S}'$ from $\nu+1$ to $\nu+1$. To prove linear independence suppose that

$$\lambda_1(c_{\nu+1\nu+2}a_{\nu+2\nu+1}-p_{\nu+1\nu+1})+\lambda_2(f_{\nu+1,\nu}g_{\nu,\nu+1})=0$$

in $e_{\nu+1}(I/(IJ+JI))e_{\nu+1}$. Then

$$\lambda_1c_{\nu+1\nu+2}a_{\nu+2\nu+1}+\lambda_2f_{\nu+1,\nu}g_{\nu,\nu+1}=\lambda_1p_{\nu+1\nu+1}+u$$

in the free algebra $\mathbb{C}\hat{Q}$ for some $u\in IJ+JI$. But $c_{\nu+1\nu+2}a_{\nu+2\nu+1}$ satisfies condition (A) and so cannot appear on the right hand side, forcing $\lambda_1=0$. Similarly since $f_{\nu+1,\nu}g_{\nu,\nu+1}$ satisfies condition (A) $\lambda_2=0$. This proves the case $\alpha_{\nu+1}=3$ and so we can assume that $\alpha_{\nu+1}>3$. Now there are $\alpha_{\nu+1}-1$ relations in $\mathcal{S}'$ from $\nu+1$ to $\nu+1$, and suppose that

$$\begin{aligned}&\lambda_1(f_{\nu+1,\nu}g_{\nu,\nu+1})+\lambda_2(k_2A_{0\nu+1}-p_{\nu+1\nu+1})\\&\quad+\sum_{p=2}^{u_{\nu+1}-1}\lambda_{p+1}(k_{p+1}A_{0\nu+1}-k_pC_{0\nu+1})+\lambda_{u_{\nu+1}+1}(c_{\nu+1\nu+2}a_{\nu+2\nu+1}-k_{u_{\nu+1}}C_{0\nu+1})=0\end{aligned}$$

in $e_{\nu+1}(I/(IJ+JI))e_{\nu+1}$ (where the sum may be empty). Then

$$\begin{aligned}&\lambda_1f_{\nu+1,\nu}g_{\nu,\nu+1}+\sum_{p=1}^{u_{\nu+1}-1}\lambda_{p+1}k_{p+1}A_{0\nu+1}+\lambda_{u_{\nu+1}+1}c_{\nu+1\nu+2}a_{\nu+2\nu+1}\\&\qquad\qquad=\lambda_2p_{\nu+1\nu+1}+\sum_{p=2}^{u_{\nu+1}-1}\lambda_{p+1}k_pC_{0\nu+1}+\lambda_{u_{\nu+1}+1}k_{u_{\nu+1}}C_{0\nu+1}+u\end{aligned}$$

in the free algebra $\mathbb{C}\hat{Q}$ for some $u\in IJ+JI$. Now $f_{\nu+1,\nu}g_{\nu,\nu+1}$ and $c_{\nu+1\nu+2}a_{\nu+2\nu+1}$ satisfy condition (A) forcing $\lambda_1=0$ and $\lambda_{u_{\nu+1}+1}=0$. Thus

$$\lambda_{u_{\nu+1}}k_{u_{\nu+1}}A_{0\nu+1}\equiv\lambda_2p_{\nu+1\nu+1}+\text{terms starting with k of strictly smaller index}$$

mod $IJ+JI$. But since $k_{u_{\nu+1}}A_{0\nu+1}$ does not have any subwords which are cycles, the only way we can change it mod $IJ+JI$ is to bracket as $k_{u_{\nu+1}}(A_{0\nu+1})$ and use the relation in I from $\star$ to $\nu+1$. Doing this we get $k_{u_{\nu+1}}A_{0\nu+1}\equiv\frac{1}{2}k_{u_{\nu+1}}f_{0,\nu}g_{\nu,\nu+1}$. This still does not start with $p_{\nu+1\nu+1}$ or k of strictly lower index, so we must again use relations in $IJ+JI$ to change the terms. But $k_{u_{\nu+1}}f_{0,\nu}g_{\nu,\nu+1}$ does not contain any subwords which are cycles, which means the only way to change it mod $IJ+JI$ is to use a relation for $f_{0,\nu}g_{\nu,\nu+1}$. But there is only one relation for $f_{0,\nu}g_{\nu,\nu+1}$ in I and so using it we arrive back at

$$k_{u_{\nu+1}}A_{0\nu+1}\equiv\frac{1}{2}k_{u_{\nu+1}}f_{0,\nu}g_{\nu,\nu+1}\equiv k_{u_{\nu+1}}A_{0\nu+1}$$

mod $IJ+JI$. Thus mod $IJ+JI$ it is impossible to transform $k_{u_{\nu+1}}A_{0\nu+1}$ into an expression involving $p_{\nu+1\nu+1}$ or k terms with strictly lower index, thus we must have $\lambda_{u_{\nu+1}}=0$. Now re-arranging we get

$$\lambda_{u_{\nu+1}-1}k_{u_{\nu+1}-1}A_{0\nu+1} \equiv \lambda_2 p_{\nu+1\nu+1} + \text{terms starting with k of strictly smaller index}$$

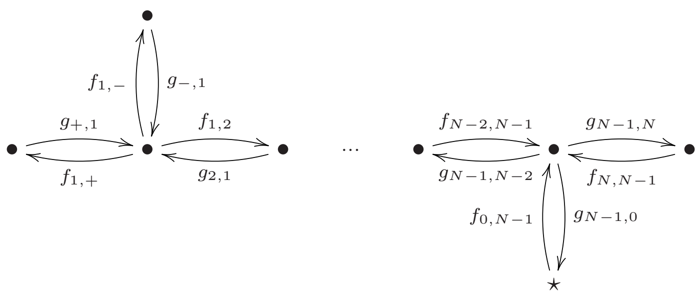
mod $IJ + JI$ and so by induction all the λ's are zero, as required.
Case t with $\nu+2 \leq t \leq N$. The proof here is identical to the $2 \leq t \leq N$ case in the proof of [Wem09, 4.11].
Case $t = \star$. This is proved using a very similar argument as in Case $t = \nu+1$ above. □

4. The reconstruction algebra for $\nu = N - 1$

In this section we define the reconstruction algebra $D_{n,q}$ when $\nu = N-1$ and prove that it is isomorphic to the endomorphism ring of the special CM modules. This is in fact very similar to the previous section, but here the vertex $\star$ is connected in a slightly different way and also the number of extra arrows out of the vertex $\nu+1$ is $\alpha_{\nu+1}-2$ compared to $\alpha_{\nu+1}-3$ previously.

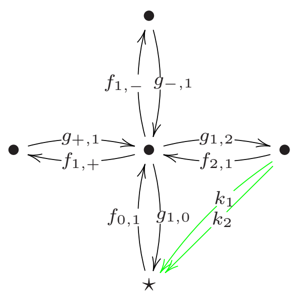
Consider, for $N \in \mathbb{N}$ with $N \geq 2$ and for a positive integer $\alpha_N \geq 2$, the labelled Dynkin diagram of type D:

We call the left hand vertex the $+$ vertex, the top vertex the $-$ vertex and the remaining vertices $1, \ldots, N$ reading from left to right. To this picture we add an extended vertex $\star$ and 'double-up' as follows:

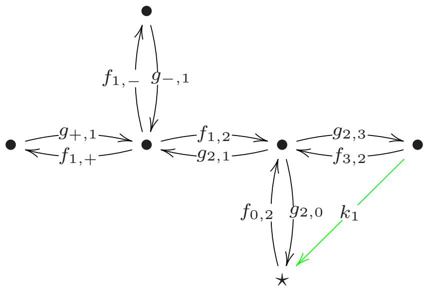


This is precisely the underlying quiver of the preprojective algebra of type D. Now if $\alpha_N > 2$ then add an extra $\alpha_N - 2$ arrows from the N^{th} vertex to $\star$ and label them $k_1, \ldots, k_{\alpha_N-2}$ reading from left to right. Call this quiver Q.

Example 4.1. Consider $\mathbb{D}_{7,4}$ then $\frac{7}{4} = [2,4]$ and so Q is



Example 4.2. Consider $\mathbb{D}_{7,5}$ then $\frac{7}{5} = [2,2,3]$ and so Q is



Definition 4.3. *For $\frac{n}{q} = [\alpha_1, \dots, \alpha_N]$ with $\nu = N-1$ define $D_{n,q}$, the reconstruction algebra of type D, to be the path algebra of the quiver Q defined above subject to the relations*

$$\text{If } N=2 \quad \begin{cases} g_{+,1}f_{1,+} = 0 \quad f_{0,1}g_{1,0} = 0 \\ g_{-,1}f_{1,-} = 0 \quad f_{2,1}g_{1,2} = 0 \\ g_{1,0}f_{0,1} - g_{1,2}f_{2,1} = f_{1,+}g_{+,1} - f_{1,-}g_{-,1} \end{cases}$$

$$\text{If } N>2 \quad \begin{cases} g_{+,1}f_{1,+} = 0 \quad f_{0,N-1}g_{N-1,0} = 0 \\ g_{-,1}f_{1,-} = 0 \quad f_{N,N-1}g_{N-1,N} = 0 \\ f_{1,+}g_{+,1} - f_{1,-}g_{-,1} = 2f_{1,2}g_{2,1} \\ g_{t,t-1}f_{t-1,t} = f_{t,t+1}g_{t+1,t} \text{ for all } 2 \leq t \leq N-2 \\ g_{N-1,0}f_{0,N-1} - g_{N-1,N}f_{N,N-1} = 2g_{N-1,N-2}f_{N-2,N-1} \end{cases}$$

Further if $\alpha_N > 2$ we add, in each case, the relations

$$\begin{array}{l} k_1 A_{0N} = p_{NN}, A_{0N}k_1 = p_{00} \\ k_t C_{0N} = k_{t+1}A_{0N}, A_{0N}k_{t+1} = C_{0N}k_t \text{ for all } 1 \leq t \leq \alpha_N - 3 \end{array}$$

where $A_{0N} := \frac{1}{2}f_{0,N-1}g_{N-1,N}$. The only thing that remains to be defined is the p's and C's, and it is these which change according to the presentation, namely

$$\begin{array}{rcl} p_{00} & = & \begin{cases} f_{0,N-1}G_{N-1,1}f_{1,-}g_{-,1}F_{1,N-1}g_{N-1,0} \\ \frac{1}{2}f_{0,N-1}G_{N-1,1}(f_{1,+}g_{+,1} + f_{1,-}g_{-,1})F_{1,N-1}g_{N-1,0} \end{cases} \\ p_{NN} & = & \begin{cases} f_{N,N-1}G_{N-1,1}f_{1,+}g_{+,1}F_{1,N-1}g_{N-1,N} \\ \frac{1}{2}f_{N,N-1}G_{N-1,1}(f_{1,+}g_{+,1} + f_{1,-}g_{-,1})F_{1,N-1}g_{N-1,N} \end{cases} \\ C_{0N} & = & \begin{cases} f_{0,N-1}G_{N-1,1}f_{1,+}g_{+,1}F_{1,N-1}g_{N-1,N} \\ \frac{1}{2}f_{0,N-1}G_{N-1,1}(f_{1,+}g_{+,1} + f_{1,-}g_{-,1})F_{1,N-1}g_{N-1,N} \end{cases} \end{array}$$

where for the moduli presentation we take the top choices and for the symmetric presentation we take the bottom choices.

Remark 4.4. The relations are very similar to before, but now all relations are cycles. As in the previous section for the moment denote $D_{n,q}$ for the moduli presentation and $D'_{n,q}$ for the symmetric presentation; we shall see in Theorem 4.9 that they are isomorphic.

Example 4.5. For the group $\mathbb{D}_{7,4}$ the moduli presentation of the reconstruction algebra is the quiver in Example 4.1 subject to the relations

$$\begin{gathered} g_{+,1}f_{1,+} = 0 \\ g_{-,1}f_{1,-} = 0 \\ f_{0,1}g_{1,0} = 0 \\ f_{2,1}g_{1,2} = 0 \\ g_{1,0}f_{0,1} - g_{1,2}f_{2,1} = f_{1,+}g_{+,1} - f_{1,-}g_{-,1} \end{gathered}$$

$$\begin{array}{rcrcl} f_{2,1}f_{1,+}g_{+,1}g_{1,2} = k_1(\frac{1}{2}f_{0,1}g_{1,2}) & \qquad & (\frac{1}{2}f_{0,1}g_{1,2})k_1 = f_{0,1}f_{1,-}g_{-,1}g_{1,0} \\ k_1(f_{0,1}f_{1,+}g_{+,1}g_{1,2}) = k_2(\frac{1}{2}f_{0,1}g_{1,2}) & \qquad & (\frac{1}{2}f_{0,1}g_{1,2})k_2 = (f_{0,1}f_{1,+}g_{+,1}g_{1,2})k_1 \end{array}$$

Example 4.6. For the group $\mathbb{D}_{7,5}$ the symmetric presentation of the reconstruction algebra is the quiver in Example 4.2 subject to the relations

$$\begin{gathered}
g_{+,1}f_{1,+} = 0\\
g_{-,1}f_{1,-} = 0\\
f_{0,2}g_{2,0} = 0\\
f_{3,2}g_{2,3} = 0\\
f_{1,+}g_{+,1} - f_{1,-}g_{-,1} = 2f_{1,2}g_{2,1}\\
g_{2,0}f_{0,2} - g_{2,3}f_{3,2} = 2g_{2,1}f_{1,2}\\
\tfrac{1}{2}f_{3,2}g_{2,1}(f_{1,+}g_{+,1} + f_{1,-}g_{-,1})f_{1,2}g_{2,3} = k_1(\tfrac{1}{2}f_{0,2}g_{2,3})\\
\tfrac{1}{2}f_{0,2}g_{2,1}(f_{1,+}g_{+,1} + f_{1,-}g_{-,1})f_{1,2}g_{2,0} = (\tfrac{1}{2}f_{0,2}g_{2,3})k_1
\end{gathered}$$

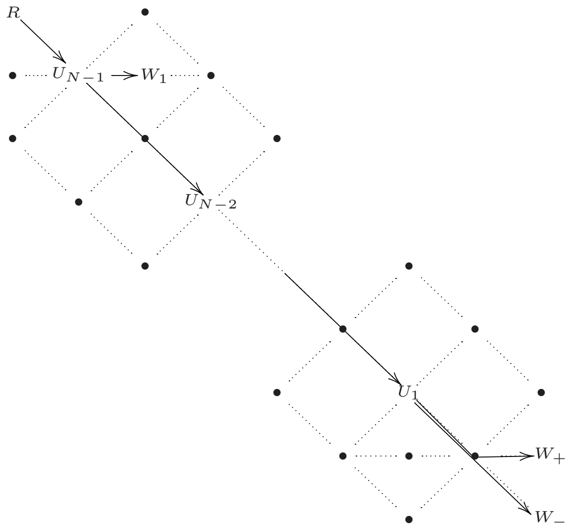
We now justify the positions of the specials relative to the dual graph. The proof is very similar to that of Lemma 3.10 but is sufficiently illuminating to be worthy of inclusion.

Lemma 4.7. *Consider $\mathbb{D}_{n,q}$ with $\nu = N-1$. Then the specials correspond to the dual graph of the minimal resolution in the following way*

$$\begin{tikzcd}
 & W_- \arrow[d, dash] & & & \\
W_+ \arrow[r, dash] & U_1 \arrow[r, dash] & \cdots \arrow[r, dash] & U_{N-1} \arrow[r, dash] & W_{i_N}
\end{tikzcd}$$

Proof. Now $Z_f = E_+ + E_- + \sum_{p=1}^{N-1} 2E_p + E_N$ as in the $\mathrm{SL}(2,\mathbb{C})$ case. Thus by [Wem08, 3.6] the quiver of the endomorphism ring of the specials is precisely the Q defined above in this section.

It is clear that the following composition of maps in the AR quiver do not factor through other specials



where the difference now (compared to Lemma 3.10) is that in the above picture the positions of R, W_1 *and* U_{N-1} are fixed; again the length of the distance between the positions of the other specials changes depending on $n-q$ (in the picture we've drawn $n-q=2$) but this does not effect the fact that the maps are irreducible. $\square$

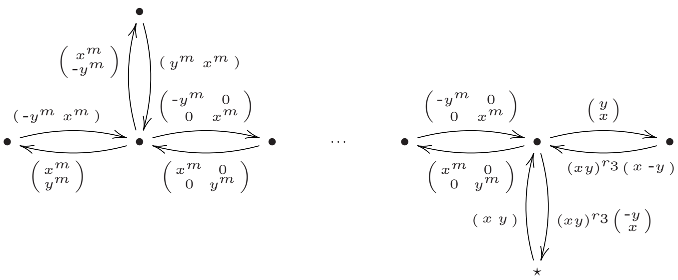
Remark 4.8. The picture in the above proof explains algebraically why in the case $\nu = N-1$ we don't have to directly connect R to the vertex $W_{i_N} = W_1$, since any such map must necessarily factor through U_{N-1}.

The following result is the main result of this section:

Theorem 4.9. *For a group $\mathbb{D}_{n,q}$ with parameter $\nu = N-1$, denote $R = \mathbb{C}[x,y]^{\mathbb{D}_{n,q}}$ and let $T_{n,q} = R \oplus W_+ \oplus W_- \oplus_{s=1}^{N-1} U_s \oplus W_{i_N}$ be the sum of the special CM modules. Then*

$$D_{n,q} \cong \mathrm{End}_R(T_{n,q}) \cong D'_{n,q}.$$

Proof. The proof is very similar to that of Theorem 3.11. Setting $m := n - q$ then as before we may choose the following as representatives of the irreducible maps between the specials



We note that with these choices we have

$$
\begin{aligned}
p_{NN} &= \begin{cases} f_{N,N-1}G_{N-1,1}f_{1,+}g_{+,1}F_{1,N-1}g_{N-1,N} = (xy)^{r_3}w_2^{c_3}w_3^{d_3} = (xy)^{r_3}w_3 \\ \frac{1}{2}f_{N,N-1}G_{N-1,1}(f_{1,+}g_{+,1}+f_{1,-}g_{-,1})F_{1,N-1}g_{N-1,N} = (xy)^{r_3}v_2^{c_3}v_3^{d_3} = (xy)^{r_3}v_3 \end{cases} \\
p_{00} &= \begin{cases} f_{0,N-1}G_{N-1,1}f_{1,-}g_{-,1}F_{1,N-1}g_{N-1,0} = (xy)^{r_3}w_2^{c_3}w_3^{d_3} = (xy)^{r_3}w_3 \\ \frac{1}{2}f_{0,N-1}G_{N-1,1}(f_{1,+}g_{+,1}+f_{1,-}g_{-,1})F_{1,N-1}g_{N-1,0} = (xy)^{r_3}v_2^{c_3}v_3^{d_3} = (xy)^{r_3}v_3 \end{cases} \\
C_{0N} &= \begin{cases} f_{0,N-1}G_{N-1,1}f_{1,+}g_{+,1}F_{1,N-1}g_{N-1,N} = w_2 \\ \frac{1}{2}f_{0,N-1}G_{N-1,1}(f_{1,+}g_{+,1}+f_{1,-}g_{-,1})F_{1,N-1}g_{N-1,N} = v_2 \end{cases}
\end{aligned}
$$

and $A_{0N} = xy$. Now if $\alpha_N > 2$ there are extra k arrows, and here in the moduli presentation we choose for representatives

$$k_t = (xy)^{r_{t+3}}w_2^{c_{t+2}}w_3^{d_{t+2}} \text{ for all } 1 \leq t \leq \alpha_N - 2.$$

The proof that we can actually choose these is identical to before. Again the symmetric presentation is obtained by everywhere replacing w_2 by v_2 and w_3 by v_3.

The relations for the reconstruction algebra are again induced by matrix multiplication such that x and y commute. By Lemma 2.3 and Lemma 2.4 it is very easy to verify that the algorithmic relations in Definition 4.3 are satisfied with these choices of polynomials. From here the proof is identical to that of Theorem 3.11 (actually all relations are now cycles, so in fact the proof of linear independence becomes a little easier) hence we suppress the details. □

5. Moduli Examples

In this section we justify some of the philosophy given in the introduction and also in the introduction of [Wem09]. Geometrically the key change in viewpoint is that we should not view the minimal resolution as G-Hilb (which we can do via a result of Ishii [Ish02]), but rather we should instead view the minimal resolution as being very similar to a space that we already understand. It is the reconstruction algebra which tells us which space to compare to, and it is the reconstruction algebra which encodes the difference.

In this section (to ease notation) we shall use reconstruction algebras to compare the geometry of the following two examples

from which the pattern is clear. The left example corresponds to the group $\mathbb{D}_{3,2}$ whereas the right example corresponds to $\mathbb{D}_{5,2}$. The first is an example inside $\mathrm{SL}(2,\mathbb{C})$ and so has been extensively studied by many people.

Now by inspection of the quiver and relations we have an obvious ring homomorphism $D_{3,2} \to D_{5,2}$, and denoting by $e_\star$ the idempotent corresponding to the vertex $\star$ this map restricts to a ring homomorphism $e_\star D_{3,2} e_\star \to e_\star D_{5,2} e_\star$. But these are just $\mathbb{C}[x,y]^{\mathbb{D}_{3,2}}$ and $\mathbb{C}[x,y]^{\mathbb{D}_{5,2}}$ respectively, so we have

$$\begin{array}{ccc} D_{3,2} & \to & D_{5,2} \\ \cup\!\!\!\uparrow & & \cup\!\!\!\uparrow \\ \mathbb{C}[x,y]^{\mathbb{D}_{3,2}} & \to & \mathbb{C}[x,y]^{\mathbb{D}_{5,2}} \end{array}$$

We now choose the dimension vector and stability

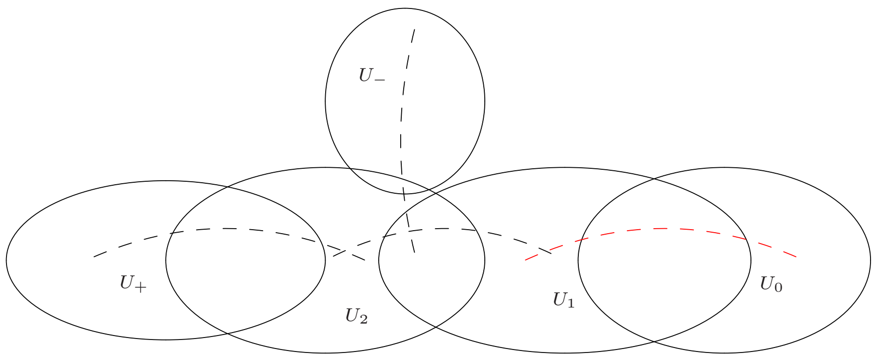
$$\alpha = \begin{array}{ccc} & 1 & \\ 1 & 2 & 1 \\ & 1 & \end{array} \qquad \theta = \begin{array}{ccc} & 1 & \\ 1 & 1 & 1 \\ & \text{-5} & \end{array}$$

for both reconstruction algebras and consider the resulting moduli spaces, denoted $\mathcal{M}_{3,2}$ and $\mathcal{M}_{5,2}$ respectively. Viewing this stability in the representation theoretic view of being $\star$-generated (see below) it is obvious that any θ-stable representation M of $D_{5,2}$ remains θ-stable if it is viewed as a $D_{3,2}$-module, thus we get the following commutative diagram

$$\begin{array}{ccc} \mathcal{M}_{3,2} & \leftarrow & \mathcal{M}_{5,2} \\ \downarrow & & \downarrow \\ \mathbb{C}^2/\mathbb{D}_{3,2} & \leftarrow & \mathbb{C}^2/\mathbb{D}_{5,2} \end{array}$$

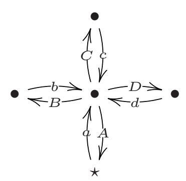
which allows us to compare the geometry of minimal resolutions which a priori have nothing to do with each other.

It is instructive to view this explicitly, where we are able to see that $\mathcal{M}_{5,2}$ is really built from $\mathcal{M}_{3,2}$ by only a very minor modification. Below in Example 5.1 is an open cover of $\mathcal{M}_{3,2}$ which we draw as follows:



The curves are all (-2)-curves. In Example 5.4 we see that by changing the reconstruction algebra from $D_{3,2}$ to $D_{5,2}$ the red (-2)-curve in $\mathcal{M}_{3,2}$ changes into a (-3) curve in $\mathcal{M}_{5,2}$ in a very efficient way. The open set U_0 changes, as does the glue between U_0 and U_1 - it is the change in glue which gives the change in self-intersection number. The remainder of the open sets (and their glues) remain the same and so we have the configuration of $\mathbb{P}^1$s which forms the minimal resolution of $\mathbb{C}^2/\mathbb{D}_{5,2}$. This is geometrically the nicest solution since we are changing the least amount of information to obtain one space from the other. Note that if we use the G-Hilb description of the minimal resolution (or the McKay quiver) it is not obvious that this should be true.

Example 5.1. Consider the group $\mathbb{D}_{3,2}$ of order 8. Since this is inside $\mathrm{SL}(2,\mathbb{C})$ the reconstruction algebra $D_{3,2}$ is just the preprojective algebra



$$aA = 0 \quad cC = 0$$
$$bB = 0 \quad dD = 0$$
$$Aa - Dd = Bb - Cc.$$

Choosing dimension vector and stability as above, to specify an open set in $\mathcal{M}_{3,2}$ is equivalent to

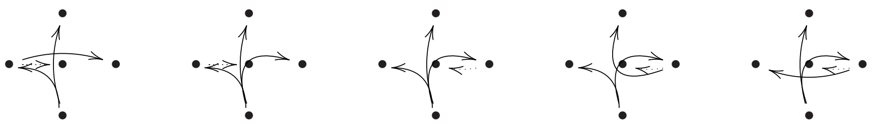
- specifying, for each of the three vertices which are not either $\star$ or the middle vertex, a non-zero path (which we can change basis to assume to be the identity) from $\star$ to that vertex.
- specifying paths (0 1) and (1 0) from $\star$ to the middle vertex.

Different choices in the above lead to different open sets. Note that we must be able to make such choices for any θ-stable module M since by definition M is $\star$-generated and so paths leaving the trivial vertex must generate the vector spaces at all other vertices. For a stable M, it must be true that $a \neq 0$ and so after changing basis we can (and will) always assume that $a = (1\ 0)$.

Define the open sets U_0, U_1, U_2, U_+ and U_- by the following conditions:

U_0	$aB = 1$	$aC = 1$	$aBbD = 1$	$a = (1\ 0)$	$b = (0\ 1)$
U_1	$aB = 1$	$aC = 1$	$aD = 1$	$a = (1\ 0)$	$b = (0\ 1)$
U_2	$aB = 1$	$aC = 1$	$aD = 1$	$a = (1\ 0)$	$d = (0\ 1)$
U_+	$aB = 1$	$aDdC = 1$	$aD = 1$	$a = (1\ 0)$	$d = (0\ 1)$
U_-	$aDdB = 1$	$aC = 1$	$aD = 1$	$a = (1\ 0)$	$d = (0\ 1)$

Pictorially we draw this as follows:



U_0 U_1 U_2 U_+ U_-

where the solid black lines correspond to the identity, and the dotted arrow corresponds to the choice of vector (0 1). To prove that these actually cover the moduli is perhaps the hardest part:

Lemma 5.2. *The open sets U_0, U_1, U_2, U_+ and U_- cover the moduli space.*

Proof. Take an arbitrary stable module

$$M = \begin{tikzcd} & \mathbb{C} \arrow[d, shift left, "(c_1\ c_2)"] & \\ \mathbb{C} \arrow[r, shift left, "(b_1\ b_2)"] & \mathbb{C}^2 \arrow[l, shift left, "\binom{B_1}{B_2}"] \arrow[u, shift left, "\binom{C_1}{C_2}"] \arrow[r, shift left, "\binom{D_1}{D_2}"] \arrow[d, shift left, "\binom{A_1}{A_2}"] & \mathbb{C} \arrow[l, shift left, "(d_1\ d_2)"] \\ & \mathbb{C} \arrow[u, shift left, "(1\ 0)"] & \end{tikzcd}$$

We must show that we can change basis so that M belongs to one of the above open sets. We firstly prove the following claim: if d is a linear multiple of (1 0) then we may choose $b = (0\ 1)$. To see this, since we must generate the module M from $\star$, for every vertex $+$, $-$ and 2 one of the paths corresponding to a generator of the corresponding CM module must be non-zero. By assumption on d we must be able to choose either $b = (0\ 1)$ or $c = (0\ 1)$ so that we generate the middle vertex. If we can choose $b = (0\ 1)$ we are done, hence suppose b is a linear multiple of (1 0). In this case $b = (\mu\ 0)$ and $d = (\lambda\ 0)$ for some $\lambda, \nu \in \mathbb{C}$ and so the preprojective relations force $\mu B_1 = 0$ and $\lambda D_1 = 0$. Consequently the paths aBb and aDd are zero. By inspection of the generators of the CM modules at vertices $+$, $-$ and 2 this forces $aB \neq 0$, $aC \neq 0$ and $aD \neq 0$. In particular $D_1 \neq 0$ and $B_1 \neq 0$, but this in turn forces $\lambda = \mu = 0$ which is a contradiction since the preprojective relation at the middle vertex cannot hold.

With this claim, the proof is now easy: Suppose first d can be chosen as (0 1). If $aB = 0$ then by inspecting the generators of the CM module at $+$ we must have $aDdB \neq 0$, which forces $aCcB \neq 0$ by the preprojective relations. Thus M is in U_-. Hence we may suppose that $aB \neq 0$. Now if $aC = 0$ then we must reach vertex $-$ by the other generator, so

$aDdC \neq 0$ and so M is in U_+. Thus we may also assume that $aC \neq 0$. Now if $aD = 0$ then $D_1 = 0$ so $D_2 \neq 0$ so that we generate. But $D_2 = dD = 0$, a contradiction. Hence necessarily $aD \neq 0$ and so M is in U_2.

The above covers the case when we can choose $d = (0\ 1)$ so suppose now that this is not the case, i.e. $d = (\lambda\ 0)$ for some $\lambda \in \mathbb{C}$. By the above claim we know we may choose $b = (0\ 1)$. If $aD \neq 0$ then $D_1 \neq 0$ from which $dD = 0$ gives us that $\lambda = 0$ and so $d = 0$. Consequently to generate at vertex $+$ we need $aB \neq 0$ and to generate at vertex $-$ we need $aC \neq 0$. Thus M is in U_1. Hence we may assume that $aD = 0$, and so to generate at vertex 2 requires that $aBbD \neq 0$ which forces $aCcD \neq 0$. Consequently M is in U_0. $\square$

Note that there are lots of other open covers we could take. We now compute the open sets above - we do the U_0 calculation in full and just summarize the others. Any stable module in U_0 looks like

$$\begin{tikzcd} & \mathbb{C} \arrow[d, bend left, "{(c_1\ c_2)}"] & \\ \mathbb{C} \arrow[r, bend left, "{(0\ 1)}"] & \mathbb{C}^2 \arrow[u, bend left, "{\left(\begin{smallmatrix}1\\ C_2\end{smallmatrix}\right)}"] \arrow[l, bend left, "{\left(\begin{smallmatrix}1\\ B_2\end{smallmatrix}\right)}"] \arrow[r, bend left, "{\left(\begin{smallmatrix}D_1\\ 1\end{smallmatrix}\right)}"] \arrow[d, bend left, "{\left(\begin{smallmatrix}A_1\\ A_2\end{smallmatrix}\right)}"] & \mathbb{C} \arrow[l, bend left, "{(d_1\ d_2)}"] \\ & \mathbb{C} \arrow[u, bend left, "{(1\ 0)}"] & \end{tikzcd}$$

where the variables are scalars, subject only to the quiver relations. Now

$$\begin{aligned} &aA = 0 \text{ implies } A_1 = 0\\ &bB = 0 \text{ implies } B_2 = 0\\ &cC = 0 \text{ implies } c_1 = -c_2C_2\\ &dD = 0 \text{ implies } d_2 = -d_1D_1 \end{aligned}$$

and so plugging this in our module becomes

$$\begin{tikzcd} & \mathbb{C} \arrow[d, bend left, "{(-c_2C_2\ c_2)}"] & \\ \mathbb{C} \arrow[r, bend left, "{(0\ 1)}"] & \mathbb{C}^2 \arrow[u, bend left, "{\left(\begin{smallmatrix}1\\ C_2\end{smallmatrix}\right)}"] \arrow[l, bend left, "{\left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right)}"] \arrow[r, bend left, "{\left(\begin{smallmatrix}D_1\\ 1\end{smallmatrix}\right)}"] \arrow[d, bend left, "{\left(\begin{smallmatrix}0\\ A_2\end{smallmatrix}\right)}"] & \mathbb{C} \arrow[l, bend left, "{(d_1\ -d_1D_1)}"] \\ & \mathbb{C} \arrow[u, bend left, "{(1\ 0)}"] & \end{tikzcd}$$

But now there is only one relation left, namely $Aa - Dd = Bb - Cc$. This gives

$$\begin{pmatrix} 0 & 0\\ A_2 & 0 \end{pmatrix} - \begin{pmatrix} d_1D_1 & -d_1D_1^2\\ d_1 & -d_1D_1 \end{pmatrix} = \begin{pmatrix} 0 & 1\\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -c_2C_2 & c_2\\ -c_2C_2^2 & c_2C_2 \end{pmatrix}$$

which yields the four conditions

$$\begin{aligned} d_1D_1 &= c_2C_2\\ d_1D_1^2 &= 1 + c_2\\ A_2 &= d_1 - c_2C_2^2\\ d_1D_1 &= c_2C_2 \end{aligned}$$

The second and third conditions eliminate the variables c_2 and A_2, whereas the first and last conditions are the same. Substituting the second condition into the first we see that this open set is completely parameterized by d_1, D_1 and C_2 subject to the one relation $d_1D_1 = (d_1D_1^2 - 1)C_2$, so U_0 is a smooth hypersurface in $\mathbb{C}^3$. Similarly we have

U_1

$$\begin{tikzcd} & \mathbb{C} \arrow[d, bend left, "{(-c_2C_2\ c_2)}"] & \\ \mathbb{C} \arrow[r, bend left, "{(0\ 1)}"] & \mathbb{C}^2 \arrow[u, bend left, "{\left(\begin{smallmatrix}1\\ C_2\end{smallmatrix}\right)}"] \arrow[l, bend left, "{\left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right)}"] \arrow[r, bend left, "{\left(\begin{smallmatrix}1\\ D_2\end{smallmatrix}\right)}"] \arrow[d, bend left, "{\left(\begin{smallmatrix}0\\ A_2\end{smallmatrix}\right)}"] & \mathbb{C} \arrow[l, bend left, "{(-d_2D_2\ d_2)}"] \\ & \mathbb{C} \arrow[u, bend left, "{(1\ 0)}"] & \end{tikzcd}$$

$$\begin{aligned} c_2C_2 &= d_2D_2\\ c_2 &= 1 + d_2\\ A_2 &= c_2C_2^2 - d_2D_2^2\\ c_2C_2 &= d_1D_1 \end{aligned}$$

$$\mathbb{C}^3_{d_2, D_2, C_2}/(1 + d_2)C_2 = d_2D_2$$

U_2

$$\begin{tikzcd} & \mathbb{C} \arrow[d, bend left, "(-c_2C_2\ c_2)"] & \\ \mathbb{C} \arrow[r, bend left, "(-b_2B_2\ b_2)"] & \mathbb{C}^2 \arrow[u, bend left, "\binom{1}{C_2}"] \arrow[l, bend left, "\binom{1}{B_2}"] \arrow[r, bend left, "\binom{1}{0}"] \arrow[d, bend left, "\binom{0}{A_2}"] & \mathbb{C} \arrow[l, bend left, "(0\ 1)"] \\ & \mathbb{C} \arrow[u, bend left, "(1\ 0)"] & \end{tikzcd}$$

$$\begin{gathered} b_2B_2 = c_2C_2 \\ b_2 = c_2 - 1 \\ A_2 = c_2C_2^2 - b_2B_2^2 \\ b_2B_2 = c_2C_2 \end{gathered} \qquad \mathbb{C}^3_{b_2,B_2,C_2}/(1+b_2)C_2 = b_2B_2$$

U_+

$$\begin{tikzcd} & \mathbb{C} \arrow[d, bend left, "(c_1\ -c_1C_1)"] & \\ \mathbb{C} \arrow[r, bend left, "(-b_2B_2\ b_2)"] & \mathbb{C}^2 \arrow[u, bend left, "\binom{C_1}{1}"] \arrow[l, bend left, "\binom{1}{B_2}"] \arrow[r, bend left, "\binom{1}{0}"] \arrow[d, bend left, "\binom{0}{A_2}"] & \mathbb{C} \arrow[l, bend left, "(0\ 1)"] \\ & \mathbb{C} \arrow[u, bend left, "(1\ 0)"] & \end{tikzcd}$$

$$\begin{gathered} b_2B_2 = -c_1C_1 \\ b_2 = -c_1C_1^2 - 1 \\ A_2 = -b_2B_2^2 - c_1 \\ b_2B_2 = -c_1C_1 \end{gathered} \qquad \mathbb{C}^3_{c_1,B_2,C_1}/(1+c_1C_1^2)B_2 = c_1C_1$$

U_-

$$\begin{tikzcd} & \mathbb{C} \arrow[d, bend left, "(-c_2C_2\ c_2)"] & \\ \mathbb{C} \arrow[r, bend left, "(b_1\ -b_1B_1)"] & \mathbb{C}^2 \arrow[u, bend left, "\binom{1}{C_2}"] \arrow[l, bend left, "\binom{B_1}{1}"] \arrow[r, bend left, "\binom{1}{0}"] \arrow[d, bend left, "\binom{0}{A_2}"] & \mathbb{C} \arrow[l, bend left, "(0\ 1)"] \\ & \mathbb{C} \arrow[u, bend left, "(1\ 0)"] & \end{tikzcd}$$

$$\begin{gathered} b_1B_1 = -c_2C_2 \\ c_2 = 1 - b_1B_1^2 \\ A_2 = c_2C_2^2 + b_1 \\ b_1B_1 = -c_2C_2 \end{gathered} \qquad \mathbb{C}^3_{b_1,B_1,C_2}/(b_1B_1^2 - 1)C_2 = b_1B_1$$

Note in U_2 above the equation $b_2 = c_2 - 1$ really means that we have a choice of co-ordinate between b_2 and c_2; thus we could equally well parameterize U_2 as $\mathbb{C}^3_{c_2,B_2,C_2}/c_2C_2 = (c_2-1)B_2$.

Hence we see that the space is covered by 5 open sets, each a smooth hypersurface in $\mathbb{C}^3$. By changing basis on the quiver it is also quite easy to write down the glues:

$$\begin{array}{rcl} U_0 \ni (d_1, D_1, C_2) & \longleftrightarrow & (-d_1D_1^2, D_1^{-1}, C_2) \in U_1 \\ U_1 \ni (d_2, D_2, C_2) & \longleftrightarrow & (d_2^{-1}, -d_2D_2, -C_2) \in U_2 \\ U_2 \ni (c_2, B_2, C_2) & \longleftrightarrow & (-c_2C_2^2, B_2, C_2^{-1}) \in U_+ \\ U_2 \ni (b_2, B_2, C_2) & \longleftrightarrow & (-b_2B_2^2, B_2^{-1}, C_2) \in U_- \end{array}$$

from which we can just see the configuration of $\mathbb{P}^1$'s. The picture of the glues should (roughly) coincide with the picture drawn earlier.

Remark 5.3. Similar calculations to the above can be found in [Len02] and [NdC09].

We shall now illustrate how the reconstruction algebra changes this moduli picture:

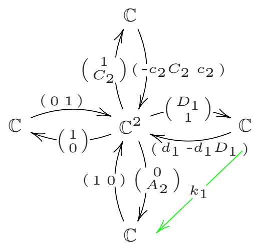
Example 5.4. Consider the group $\mathbb{D}_{5,3}$. By Theorem 4.9 the moduli presentation of the reconstruction algebra $D_{5,2}$ is

$$\begin{tikzcd} & \bullet \arrow[d, bend left, "c"] & \\ \bullet \arrow[r, bend left, "b"] & \bullet \arrow[u, bend left, "C"] \arrow[l, bend left, "B"] \arrow[r, bend left, "D"] \arrow[d, bend left, "A"] & \bullet \arrow[l, bend left, "d"] \arrow[dl, "k_1"] \\ & \star \arrow[u, bend left, "a"] & \end{tikzcd} \qquad \begin{gathered} aA = 0 \quad cC = 0 \\ bB = 0 \quad dD = 0 \\ Aa - Dd = Bb - Cc \\ k_1aD = dBbD \\ aDk_1 = aCcA \end{gathered}$$

Choose dimension vector and stability as in the previous example. Notice that exactly the same conditions that defined an open cover in the previous example give an open cover here, since arrows pointing to $\star$ don't add more choices.

Now the calculation from the previous example tells us almost everything, except now we have a new variable k_1 inside every open set. The point is that the only open set which changes is U_0. The reason for this is quite simple: in the relations of the reconstruction algebra $k_1aD = dBbD$ and notice that $aD = 1$ in every open set except U_0. Thus $k_1 = dBbD$ in every open set except U_0 and consequently we can put k_1 in terms of the other variables. Hence k_1 isn't really an extra variable in these open sets, so they do not change.

Thus the only open set that changes is U_0, and by the previous calculation (the relations above the line) and our new relations (shown below the line) we see that U_0 is given by

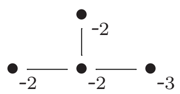


$$\begin{array}{c} d_1D_1 = c_2C_2 \\ d_1D_1^2 = 1 + c_2 \\ A_2 = d_1 - c_2C_2^2 \\ d_1D_1 = c_2C_2 \\ \hline k_1D_1 = d_1 \\ D_1k_1 = c_2A_2 \end{array}$$

Since $d_1 = k_1D_1$, instead of being given by d_1, D_1, C_2 subject to $d_1D_1 = (d_1D_1^2 - 1)C_2$ the open set U_0 is now given by k_1, D_1, C_2 subject to $k_1D_1^2 = (k_1D_1^3 - 1)C_2$. Also, the gluing between U_0 and U_1 has changed to

$$U_0 \ni (k_1, D_1, C_2) \xleftrightarrow{D_1 \neq 0} (\text{-}(k_1D_1)D_1^2, D_1^{-1}, C_2) = (\text{-}k_1D_1^3, D_1^{-1}, C_2) \in U_1 \ .$$

Thus we see that the red curve has changed into a (-3)-curve, nothing else in the open cover has changed and so the dual graph is now



References

[BR78] K. Behnke and O. Riemenschneider, *Diedersingularitäten*, Abh. Math. Sem. Univ. Hamburg **47** (1978), 210–227.

[BSW08] R. Bocklandt, T. Schedler and M. Wemyss, *Superpotentials and Higher Order Derivations*, arXiv:0802.0162 (2008)

[Bri02] T. Bridgeland, *Flops and derived categories*, Invent. Math. **147** (2002), no. 3, 613–632.

[Bri68] E. Brieskorn, *Rationale singularitäten komplexer flächen*, Invent. Math. **4** (1968), 336–358.

[BIRS] A. Buan, O. Iyama, I. Reiten and D. Smith, *Mutation of cluster-tilting objects and potentials*, arXiv:0804.3813 (2008)

[CB98] W. Crawley-Boevey and M. P. Holland, *Noncommutative deformations of Kleinian singularities*, Duke Math. J. **92** (1998), no. 3, 605–635.

[Ish02] A. Ishii, *On the McKay correspondence for a finite small subgroup of* $\mathrm{GL}(2, \mathbb{C})$, J. Reine Angew. Math. **549** (2002), 221–233

[IW08] O. Iyama and M. Wemyss, *The classification of special Cohen Macaulay modules*, arXiv:0809.1958 (2008)

[Len02] R. Leng, *The McKay Correspondence and Orbifold Riemann-Roch*, Warwick PhD thesis (2002).

[NdC09] A. Nolla de Celis, *Dihedral G-Hilb via representations of the McKay quiver*, arXiv:0905.1195 (2009).

[Rie77] O. Riemenschneider, *Invarianten endlicher Untergruppen*, Math. Z **153** (1977), 37–50.

[Wem07] M. Wemyss, *Reconstruction algebras of type A*, arXiv:0704.3693 (2007).

[Wem08] ———, *The GL(2) McKay correspondence*, arXiv:0809.1973 (2008).

[Wem09] ———, *Reconstruction algebras of type D (I)*, arXiv:0905.1154 (2009).

[Wun88] J. Wunram, *Reflexive modules on quotient surface singularities*, Mathematische Annalen **279** (1988), no. 4, 583–598.

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