STABILITY CONDITIONS FOR 3-FOLD FLOPS

YUKI HIRANO AND MICHAEL WEMYSS

Abstract. Let $f\colon X \to \operatorname{Spec} R$ be a 3-fold flopping contraction, where X has at worst Gorenstein terminal singularities and R is complete local. We describe the space of Bridgeland stability conditions on the null subcategory $\mathcal{C}$ of $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$, which consists of those complexes that derive pushforward to zero, and also on the affine subcategory $\mathcal{D}$, which consists of complexes supported on the exceptional locus. We show that a connected component $\operatorname{Stab}^{\circ}\mathcal{C}$ of $\operatorname{Stab}\mathcal{C}$ is the universal cover of the complexified complement of the real hyperplane arrangement associated to X via the Homological MMP, and more generally that $\operatorname{Stab}^{\circ}_{n}\mathcal{D}$ is a regular covering space of the infinite hyperplane arrangement constructed in [IW2]. Neither arrangement is Coxeter in general. As a consequence, we give the first description of the Stringy Kähler Moduli Space (SKMS) for all smooth irreducible 3-fold flops. The answer is surprising: we prove that the SKMS is always a sphere, minus either $3, 4, 6, 8, 12$ or 14 points, depending on the length of the curve.

1. Introduction

Our setting is 3-fold flopping contractions, namely $f\colon X \to \operatorname{Spec} R$, where $(R, \mathfrak{m})$ is a three-dimensional complete local Gorenstein $\mathbb{C}$-algebra with at worst terminal singularities. We allow X to be singular, with X having at worst terminal singularities. Consider the fibre $\mathrm{C} := f^{-1}(\mathfrak{m})$, which with its reduced scheme structure is well-known to decompose into a union of n irreducible curves, each isomorphic to $\mathbb{P}^1$.

Given this setup, consider the following two subcategories of $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$

$$\mathcal{C} := \{\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) \mid \mathbf{R}f_*\mathcal{F} = 0\}$$
$$\mathcal{D} := \{\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) \mid \operatorname{Supp}\mathcal{F} \subseteq \mathrm{C}\}.$$

It is a fundamental question to describe the spaces of stability conditions on $\mathcal{C}$ and $\mathcal{D}$, and to use this to help describe the autoequivalence group of $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$. Both $\mathcal{C}$ and $\mathcal{D}$ have finite length hearts, and it is well-known from surfaces [B6] that stability conditions on $\mathcal{C}$ should exhibit 'finite-type' ADE behaviour, whilst $\mathcal{D}$ should be the 'affine' version.

One of the problems is that traditional finite and affine Coxeter groups do not suffice in this setting. On one hand, it is possible that $\mathcal{C}$ is controlled by a Coxeter arrangement that does not have an associated affine Coxeter group. On the other hand, the category $\mathcal{D}$ predicts that such an affine object 'exists'. Even worse, it is possible that $\mathcal{C}$ is controlled by a simplicial hyperplane arrangement that is not Coxeter. In that case, the affine object that controls $\mathcal{D}$ is even less clear. Making precise statements about both $\mathcal{C}$ and $\mathcal{D}$ is in fact one of the main outcomes of this paper.

1.1. **Hidden t-structures.** Our approach to this problem is noncommutative, and necessarily so. One of our new insights is that many of the t-structures that arise in the stability manifold of $\mathcal{D}$ are 'hidden', in the sense that they are not obviously part of the birational geometry, nor are they translations of the birational geometry by line bundle twists. However, they do have very conceptual noncommutative interpretations. To describe this in more detail, it is helpful to first briefly review the known cases.

The first partial solution to describing stability conditions on $\mathcal{C}$ and $\mathcal{D}$ in this 3-fold setting is due to Toda [T], who worked under two additional assumptions: (1) X is smooth, and (2) for a generic hyperplane section $H \hookrightarrow \operatorname{Spec} R$, the pullback $X \times_R H$ is smooth. Both conditions are restrictive for different reasons, with the second being the least natural,

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and by far the most problematic to remove. The crucial point is that, under these additional assumptions, the dual graph is an ADE Dynkin diagram. When this happens, the traditional language of finite and affine Weyl groups suffice, and the relevant t-structures are all described by perverse sheaves and their tensors by line bundles. Toda [T] packages this together to describe a component of normalised stability conditions on $\mathcal{D}$ as a regular covering of the complexified complement of the associated affine root hyperplane arrangement. Furthermore, the Galois group has a very satisfying geometric description, as those compositions of flop functors and line bundle twists that act trivially on K-theory.

Alas, these satisfyingly geometric statements all fail without assumption (2). Perhaps counter-intuitively, the hardest case turns out to be the most elementary one: that of a single-curve flop. In this case, the flopping curve has an associated *length* invariant ℓ, which is some number between one and six. The assumption (2) holds if and only if the curve has length one. Evidently, this is quite restrictive.

One of our key observations is that, in the general situation of a 3-fold flop $X \to \operatorname{Spec} R$, tracking under flop functors and line bundle twists does not suffice. To illustrate this visually in the case of a two curve flop, we will show below that stability conditions on $\mathcal{D}$ are controlled by infinite hyperplane arrangements $\mathcal{H}^{\mathsf{aff}} \subseteq \mathbb{R}^n$ such as that shown in Figure 1. In general

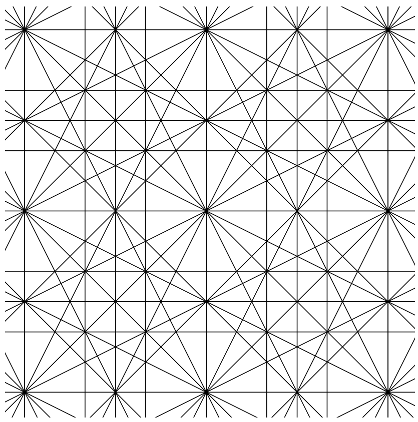


FIGURE 1. Example of hyperplane arrangement $\mathcal{H}^{\mathsf{aff}}$

the hyperplane arrangements are quite complicated, and there are many more chambers than one might naively expect. The above example has an obvious $\mathbb{Z}^2$ action, given by tensoring by the line bundles corresponding to the two curves. However, this action jumps the central chamber over many intermediate t-structures. These all turn out to be hearts of noncommutative resolutions, and their variants.

To circumvent this problem, which occurs even in the case of a single curve flop, we appeal to recent advances in noncommutative resolutions and their mutation theory. In the process, we will recover a conceptual understanding of the hidden t-structures, give a full description of (a component of) normalised stability conditions on $\mathcal{D}$, and for the first time compute the Stringy Kähler Moduli Space.

1.2. **Main Stability Results.** Describing stability conditions on $\mathcal{C}$ turns out to be quite easy. Working in our most general setup $f\colon X \to \operatorname{Spec} R$, we first prove the following, which is entirely parallel to [T, Section 6, ArXiv v2]. Below the hyperplane arrangement $\mathcal{H} \subset \mathbb{R}^n$ need not be ADE, or even Coxeter, but nevertheless tracking under the Flop functors still produces the chambers of the stability manifold for the category $\mathcal{C}$. As notation, consider the set $\operatorname{Flop}(X)$ consisting of all those pairs (F, Y) where $Y \to \operatorname{Spec} R$ is obtained from X through an iterated chain of simple flops, and F is a composition of flop functors and their inverses, from $\mathrm{D^b}(\operatorname{coh} Y)$ to $\mathrm{D^b}(\operatorname{coh} X)$.

Theorem 1.1 (6.4, 6.9)**.** *There is a union of chambers*

$$\operatorname{Stab}^{\circ}\mathcal{C} = \bigcup_{(Y,F)\in \operatorname{Flop}(X)} F(U),$$

where U is defined in Notation 6.1, and furthermore the natural map

$$\mathcal{Z}\colon \operatorname{Stab}^{\circ}\mathcal{C} \to \mathbb{C}^n\backslash\mathcal{H}_{\mathbb{C}}$$

is the universal cover of the complexified complement of $\mathcal{H}$.

The hyperplane arrangement $\mathcal{H}$ can be described in various ways, and this is explained in Section 3. The main content of the above theorem is to establish that the map is a regular covering map; our previous work [HW] on the faithfulness of the action then establishes that the cover is universal. From this, the seminal work of Deligne [D1] on the $K(\pi,1)$ conjecture for simplicial hyperplane arrangements immediately confirms the following.

Corollary 1.2 (6.10)**.** $\operatorname{Stab}^{\circ}\mathcal{C}$ *is contractible.*

However, the main content in this paper is our description of stability conditions on the category $\mathcal{D}$, which is much harder. Passing to a more noncommutative viewpoint, by the HomMMP [W1, 4.2] we first observe that the union in Theorem 1.1 can be reindexed using instead those pairs (Φ, L) where Φ is a chain of mutation functors and their inverses, and L belongs to the mutation class $\operatorname{Mut}_0(N)$ of N described in (2.C), where mutation at the submodule R is *not* allowed. Disregarding this last restriction, and thus allowing mutation at all summands, gives an infinite set $\operatorname{Mut}(N)$. Via [IW2], this turns out to index the chambers in a corresponding infinite hyperplane arrangement $\mathcal{H}^{\mathsf{aff}}$.

The following is our main result. The Galois group $\mathsf{PBr}\,\mathcal{D}$ is by definition all compositions of mutation functors and their inverses that start and finish at our fixed $\operatorname{End}_R(N)$.

Theorem 1.3 (6.4, 6.9)**.** *There is a union of chambers*

$$\operatorname{Stab}^{\circ}_n\mathcal{D} = \bigcup_{(\Phi,L)} \Phi(\mathbb{N}_L).$$

where $\mathbb{N}_L$ is defined in Notation 6.1, and Φ are compositions of mutation functors and their inverses. Furthermore, the natural forgetful map

$$\mathcal{Z}\colon \operatorname{Stab}^{\circ}_n\mathcal{D} \to \mathbb{C}^n\backslash\mathcal{H}^{\mathsf{aff}}_{\mathbb{C}}$$

is a regular covering map, with Galois group $\mathsf{PBr}\,\mathcal{D}$*.*

Whilst passing to noncommutative resolutions (and their variants) provides the conceptual framework to tackle the above problem, and to understand the extra t-structures, their appearance comes at a significant cost. Namely, it becomes much harder to argue when functors are the identity, and thus to establish that $\mathcal{Z}$ is a regular covering map. The following is one of our main technical results, which heavily uses the isolated cDV assumption.

Theorem 1.4 (5.11)**.** *Suppose that Γ is an arbitrary modifying algebra (or noncommutative crepant resolution) of R, where R is isolated cDV. Consider an equivalence*

$$G\colon\ \mathrm{D^b}(\operatorname{mod}\Gamma) \to \mathrm{D^b}(\operatorname{mod}\Gamma)$$

obtained as an arbitrarily long sequence of mutation functors and their inverses. If G restricts to an equivalence $\operatorname{mod}\Gamma \to \operatorname{mod}\Gamma$*, then* $G \cong \operatorname{Id}$*.*

There are additional variants to the above, summarised in Theorem 5.6, which may be of independent interest.

1.3. **Autoequivalence and SKMS results.** Aside from producing new invariants for 3-fold flops, linking to classification problems, autoequivalences, noncommutative resolutions and deformation theory, one of our main motivations for establishing Theorem 1.3 is that it provides the first mechanism to compute the fabled stringy Kähler moduli space (SKMS). To do this requires some additional work, since following and generalising [T] we view the SKMS as the quotient of $\operatorname{Stab}^{\circ}_n\mathcal{D}$ by a certain group $\operatorname{Aut}^{\circ}\mathcal{D}$.

The definition of the group $\operatorname{Aut}^{\circ}\mathcal{D}$ is a rather subtle point, since in this local model everything is relative to the base $\operatorname{Spec} R$, and so everything should respect this structure.

In particular, $\mathrm{Aut}^{\circ}\mathcal{D}$ should not contain isomorphisms between flopping contractions unless they preserve the R-scheme structure. We achieve this by defining $\mathrm{Aut}^{\circ}\mathcal{D}$ to be the group of *R-linear* Fourier–Mukai equivalences $\mathcal{D} \to \mathcal{D}$ that preserve $\mathrm{Stab}_n^{\circ}\mathcal{D}$. Even for a single curve flop with $\ell = 1$, the restriction to R-linear functors is necessary for the mathematically defined SKMS [T, p6169] to coincide with the physical version [A, Figure 1].

The intrinsically defined group $\mathrm{Aut}^{\circ}\mathcal{D}$ has the following more concrete description.

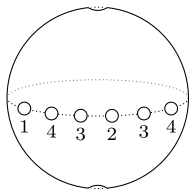
Theorem 1.5 (7.10)**.** $\mathrm{Aut}^{\circ}\mathcal{D} \cong \mathsf{PBr}\,\mathcal{D} \rtimes \mathsf{Pic}\,X$.

Combining Theorem 1.3 with Proposition 1.5 and an elementary hyperplane calculation allows us to finally compute the SKMS, as $\mathrm{Stab}_n^{\circ}\mathcal{D}/\mathrm{Aut}^{\circ}\mathcal{D}$, for smooth irreducible flops, generalising [T, p6169] and [A, Figure 1]. There does not appear to be any predictions or conjectures in the literature for what the SKMS should be for higher lengths. Perhaps this is just as well, since the result is quite surprising.

Corollary 1.6 (7.12)**.** *For a smooth irreducible flop $X \to \mathrm{Spec}\,R$ of length ℓ, the SKMS is always a 2-sphere, with holes removed at both the north and south pole, together with the following number of holes removed from the equator.*

ℓ	1	2	3	4	5	6
Holes	1	2	4	6	10	12

For example, when $\ell = 4$, it follows that the SKMS is



where we refer the reader to Theorem 7.12 for more details, including an explanation of the numerics, and the labelling of the holes on the equator.

List of Notation. A list of notation is provided in Appendix B.

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2. Flops via Noncommutative Methods

In this section, we recall the basics of modification algebras, tilting, mutations of modifying modules, and the relationship to flops, mainly to set notation. Throughout R is a three dimensional complete local Gorenstein normal $\mathbb{C}$-algebra, and $f\colon X \to \mathrm{Spec}\,R$ is a flopping contraction as in the introduction. Furthermore, write C for $f^{-1}(\mathfrak{m})$ endowed with reduced scheme structure. It is well known that $\mathrm{C} = \bigcup_{i=1}^{n} \mathrm{C}_i$ is a union of n $\mathbb{P}^1$s.

2.1. Tilting and Modification Modules.

Since R is complete local, there exist line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_n \in \mathrm{Pic}(X)$ such that $\mathcal{L}_i \cdot \mathrm{C}_j = \delta_{ij}$. Set $\mathcal{V}_0 := \mathcal{O}_X$, and write $\mathcal{V}_i$ for the vector bundle arising as the universal extension

$$0 \to \mathcal{O}_X^{\oplus r_i - 1} \to \mathcal{V}_i \to \mathcal{L}_i \to 0, \tag{2.A}$$

associated to a minimal set of $r_i - 1$ generators of the R-module $\mathrm{H}^1(X, \mathcal{L}_i^*)$. Then by [VdB, 3.5.5] the vector bundle $\mathcal{V}_X := \bigoplus_{i=0}^{n} \mathcal{V}_i^*$ is tilting, and so after setting

$$\Lambda := \mathrm{End}_X(\mathcal{V}_X) \cong \mathrm{End}_R(f_*\mathcal{V}_X),$$

the functor

$$\Psi := \mathbf{R}\mathrm{Hom}_X(\mathcal{V}_X, -)\colon \mathrm{D^b}(\mathrm{coh}\,X) \to \mathrm{D^b}(\mathrm{mod}\,\Lambda) \tag{2.B}$$

is an equivalence. Our approach to stability conditions will be through noncommutative methods. Recall that $\mathrm{CM}\,R$ denotes the category of (maximal) Cohen–Macaulay R-modules, and $\operatorname{ref} R$ denotes the category of reflexive R-modules. A reflexive R-module $L \in \operatorname{ref} R$ is called *modifying* if $\operatorname{End}_R(L) \in \operatorname{CM} R$.

In the flops setting, for the fixed $f\colon X \to \operatorname{Spec} R$, consider the underived direct image $N_i := f_*(\mathcal{V}_i^*) \in \operatorname{mod} R$. Note that $N_0 \cong R$. Throughout, we set

$$N := f_*(\mathcal{V}_X) \cong \bigoplus_{i=0}^{n} N_i. \tag{2.C}$$

It is known that $N \in \operatorname{CM} R$, and N is a modifying R-module [VdB, 3.2.10].

2.2. **Mutations and Equivalences.** Given any modifying R-module $L = \bigoplus_{j=0}^{n} L_j$ with each L_j indecomposable, there is an operation, called *mutation at* L_i, that gives a new modifying R-module written $\nu_i L$. We briefly recall the construction here. Set

$$L_i^c := \bigoplus_{j \neq i} L_j,$$

so that $L = L_i \oplus L_i^c$, and consider the *minimal right* $\operatorname{add}(L_i^c)^*$-*approximation*

$$a_i\colon U_i \to L_i^* \tag{2.D}$$

of L_i^*, which by definition means that

(1) $U_i \in \operatorname{add}(L_i^c)^*$ and $a_i \circ (-)\colon \operatorname{Hom}_R((L_i^c)^*, U_i) \to \operatorname{Hom}_R((L_i^c)^*, L_i^*)$ is surjective,
(2) If $b \in \operatorname{End}_R(U_i)$ satisfies $a_i = a_i \circ b$, then b is an isomorphism.

Since R is complete, such an a_i exists and is unique up to isomorphism. The (left) mutation of L at L_i is then defined to be

$$\nu_i L := (\operatorname{Ker} a_i)^* \oplus L_i^c.$$

The following properties are known.

Proposition 2.1. *With notation as above, in particular R is isolated cDV, the following statements hold.*

(1) *The mutation $\nu_i L$ is a modifying R-module.*
(2) *There is an isomorphism $\nu_i \nu_i L \cong L$.*

Proof. The first part is general; see e.g. [IW1, §6]. The second part is specific to isolated cDV singularities [IW2, 9.28]. □

Definition 2.2. Fix the modifying module N from (2.C). Write $\operatorname{Mut}(N)$ for the set of isomorphism classes of all iterated mutations of N, and $\operatorname{Mut}_0(N)$ for the subset consisting of those iterated mutations of N at only the $(i \neq 0)$-th summands. The exchange graph $\mathsf{EG}(N)$ is the graph whose vertices are the elements of $\operatorname{Mut}(N)$, and two vertices in $\mathsf{EG}(N)$ are joined by an edge if and only if the corresponding modifying modules are related by a mutation at an indecomposable summand. The exchange graph $\mathsf{EG}_0(N)$ is the full subgraph whose vertices are the elements of $\operatorname{Mut}_0(N)$.

Alternatively, the exchange graph $\mathsf{EG}_0(N)$ is the full subgraph whose vertices correspond to CM modules. We once and for all fix a decomposition $N = R \oplus N_1 \oplus \ldots \oplus N_n$, where $N_0 = R$. Via the Coxeter-style combinatorics in Section 3, this fixed decomposition induces an ordering on the summands of all other elements L of $\operatorname{Mut}_0(N)$, such that locally crossing a wall locally labelled s_i always corresponds to replacing the ith summand. In this way, there is a global labelling on the edges of both $\mathsf{EG}_0(N)$ and $\mathsf{EG}(N)$ using the sets $\{s_1, \ldots, s_n\}$ and $\{s_0, s_1, \ldots, s_n\}$ respectively.

The mutation of a modifying R-module L gives rise to a derived equivalence between $\Gamma := \operatorname{End}_R(L)$ and $\nu_i\Gamma := \operatorname{End}_R(\nu_i L)$, induced by a tilting bimodule T_i. Since R is isolated, in fact $T_i = \operatorname{Hom}_R(L, \nu_i L)$ by [IW1, 6.14], and the following functor is an equivalence:

$$\Phi_i := \mathbf{R}\operatorname{Hom}_\Gamma(T_i, -)\colon \operatorname{D^b}(\operatorname{mod}\Gamma) \xrightarrow{\sim} \operatorname{D^b}(\operatorname{mod}\nu_i\Gamma). \tag{2.E}$$

The functor Φ_i is called the *mutation functor* at the summand i.

2.3. **Flops and Mutation.** Recall that the exceptional locus of f, given reduced scheme structure, decomposes into n copies of $\mathbb{P}^1$, namely $\mathrm{C} = \bigcup_{i=1}^{n} \mathrm{C}_i$. For each C_i there exists a flopping contraction $g_i \colon X \to Y_i$ which contracts only C_i, and the flopping contraction $f \colon X \to \operatorname{Spec} R$ factors through g_i. Furthermore, there exists a flop $g_i^+ \colon X_i^+ \to Y_i$ of g_i such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[rr, dashed] \arrow[dr, "g_i"] \arrow[ddr, "f"'] & & X_i^+ \arrow[dl, "g_i^+"'] \arrow[ddl, "f_i^+"] \\
& Y_i \arrow[d, "h_i"] & \\
& \operatorname{Spec} R &
\end{tikzcd}$$

where $f_i^+ := h_i \circ g_i^+$. Then $(f_i^+)^{-1}(\mathfrak{m})$, with reduced scheme structure, is the union of n irreducible curves $\bigcup_{j=1}^{n} \mathrm{C}_j^+$, where for $j \neq i$ each C_j^+ is the proper transformation of C_j, and if $j = i$ then C_i^+ is the flopped curve.

Theorem 2.3 ([W1, 4.2])**.** *With notation as above, the following hold.*

(1) *There is an isomorphism of R-modules $\mathrm{H}^0(X_i^+, \mathcal{V}_{X_i^+}) \cong \nu_i N$.*
(2) *The following diagram of equivalences is functorially commutative*

$$\begin{tikzcd}
\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) \arrow[r, "\mathsf{Flop}_i"] \arrow[d, "\mathbf{R}\mathrm{Hom}_X(\mathcal{V}_X, -)"'] & \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X_i^+) \arrow[d, "\mathbf{R}\mathrm{Hom}_{X_i^+}(\mathcal{V}_{X_i^+}, -)"] \\
\mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda) \arrow[r, "\Phi_i"] & \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \nu_i \Lambda)
\end{tikzcd}$$

where $\mathsf{Flop}_i \colon \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) \to \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X_i^+)$ is the quasi-inverse of the Bridgeland–Chen flop functor [B3, C].

2.4. ***R*-linear equivalences.** In our flops setup $f \colon X \to \operatorname{Spec} R$, the category $\operatorname{coh} X$ is R-linear, and thus so too is $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$. Autoequivalences that preserve this structure will be particularly important later. Here we briefly recall the R-linear structure, and give some preliminary results.

Since $\operatorname{Spec} R$ is an affine scheme, there is a bijection

$$\operatorname{Mor}(X, \operatorname{Spec} R) \longleftrightarrow \operatorname{Hom}(R, \mathcal{O}_X(X)). \tag{2.F}$$

Given $g \colon X \to \operatorname{Spec} R$, we will write $\mathsf{g} \colon R \to \mathcal{O}_X(X)$ for the corresponding morphism.

For $a \in \operatorname{Hom}_{\operatorname{coh} X}(\mathcal{F}, \mathcal{G})$ and $\lambda \in \mathcal{O}_X(X)$, consider $\lambda \cdot a \in \operatorname{Hom}_{\operatorname{coh} X}(\mathcal{F}, \mathcal{G})$ defined by

$$(\lambda \cdot a)(x) := \lambda|_U \cdot a(x) \in \mathcal{G}(U)$$

for all $x \in \mathcal{F}(U)$. Under this action, $\operatorname{coh} X$ is an $\mathcal{O}_X(X)$-linear category. The morphism $f \colon X \to \operatorname{Spec} R$ then gives $\operatorname{coh} X$ the structure of an R-linear category, via $\mathsf{f} \colon R \to \mathcal{O}_X(X)$.

The following two results are general, and are not specific to our flops setup. Both are well-known, but for lack of reference we provide the proof.

Proposition 2.4. *Consider R-schemes $f \colon X \to \operatorname{Spec} R$, $g \colon Y \to \operatorname{Spec} R$, and a morphism $h \colon X \to Y$. Writing $\mathsf{h} \colon \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ for the corresponding morphism, then the following are equivalent.*

(1) $g \circ h = f$.
(2) $\mathsf{h} \circ \mathsf{g} = \mathsf{f}$.
(3) *$h^* \colon \operatorname{coh} Y \to \operatorname{coh} X$ is an R-linear functor.*

If h is an isomorphism, the last condition is equivalent to $h_* \colon \operatorname{coh} X \to \operatorname{coh} Y$ being an R-linear functor.

Proof. Note first that for $y \in \mathcal{O}_Y(Y)$, $a \in \mathrm{Hom}_Y(\mathcal{F},\mathcal{G})$ and $x \otimes \lambda \in h^*(\mathcal{F}) = \mathcal{F} \otimes_Y \mathcal{O}_X$,

$$\begin{aligned} h^*(y \cdot a)(x \otimes \lambda) &= (y \cdot a \otimes 1_X)(x \otimes \lambda) \\ &= (y \cdot a(x)) \otimes \lambda \\ &= a(x) \otimes \mathsf{h}(y)\lambda \\ &= \mathsf{h}(y) \cdot (a(x) \otimes \lambda) \\ &= \mathsf{h}(y) \cdot \big((a \otimes 1_X)(x \otimes \lambda)\big). \end{aligned}$$

Hence by linearity $h^*(y \cdot a) = \mathsf{h}(y) \cdot h^*(a)$ for all $y \in \mathcal{O}_Y(Y)$ and all $a \in \mathrm{Hom}_Y(\mathcal{F},\mathcal{G})$.
(1)⇔(2) This is an immediate consequence of the bijection (2.F).
(2)⇒(3) Assuming (2), then for any $r \in R$ and $a \in \mathrm{Hom}_Y(\mathcal{F},\mathcal{G})$,

$$h^*(r \cdot a) = h^*(\mathsf{g}(r) \cdot a) = \mathsf{h}(\mathsf{g}(r)) \cdot h^*(g) = \mathsf{f}(r) \cdot h^*(g) = r \cdot h^*(g),$$

and so (3) holds.
(3)⇒(2) There is a commutative diagram

$$\begin{tikzcd} \mathcal{O}_Y(Y) \arrow[r, "\mathsf{h}"] \arrow[d, "\sim"] & \mathcal{O}_X(X) \arrow[d, "\sim"] \\ \mathrm{Hom}_Y(\mathcal{O}_Y,\mathcal{O}_Y) \arrow[r, "h^*"] & \mathrm{Hom}_X(\mathcal{O}_X,\mathcal{O}_X), \end{tikzcd}$$

where the vertical arrows are R-linear isomorphisms. Since h^* is R-linear by assumption, it follows that so too is h. But then

$$\mathsf{h}(\mathsf{g}(r)) = \mathsf{h}(r \cdot 1) = r \cdot \mathsf{h}(1) = r \cdot 1 = \mathsf{f}(r),$$

for all $r \in R$, and thus $\mathsf{h} \circ \mathsf{g} = \mathsf{f}$.
The last statement holds since the inverse of an R-linear functor is R-linear. □

Lemma 2.5. *Consider an R-scheme $X \to \mathrm{Spec}\, R$, and a line bundle $\mathcal{L}$ on X. Then the functor $- \otimes \mathcal{L}\colon \mathrm{coh}\, X \to \mathrm{coh}\, X$ is $\mathcal{O}_X(X)$-linear, and in particular, is R-linear.*

Proof. For any $\lambda \in \mathcal{O}_X(X)$, $a \in \mathrm{Hom}_X(\mathcal{F},\mathcal{G})$ and $x \otimes \ell \in \mathcal{F} \otimes \mathcal{L}$, we have

$$\big((\lambda \cdot a) \otimes \mathcal{L}\big)\big(x \otimes \ell\big) = \lambda \cdot a(x) \otimes \ell = (\lambda \cdot (a \otimes \mathcal{L}))(x \otimes \ell)$$

and so by linearity the result follows. □

3. Hyperplane Arrangements via K-theory

When the generic hyperplane section of X is not smooth, it will turn out that stability conditions on $\mathcal{C}$ and $\mathcal{D}$ will not, in general, be the regular covering of a space constructed using global rules. The space will instead be constructed using iterated local rules, which we outline here. This section is largely a summary of [IW2] suitable for our needs, with the exception of some new results in Sections 3.4 and 3.5.

3.1. **General Elephants.** Slicing the flopping contraction $X \to \mathrm{Spec}\, R$ gives rise to combinatorial data, in the form of a labelled ADE Dynkin diagram Δ, vertices Δ_0, and a subset $\mathcal{J} \subseteq \Delta_0$. We briefly recall this here.

Pulling back $X \to \mathrm{Spec}\, R$ along the map $\mathrm{Spec}\, R/g \to \mathrm{Spec}\, R$ for a generic element $g \in R$, gives a morphism $S \to \mathrm{Spec}\, R/g$, say. By [R], R/g is an ADE surface singularity, and S is a partial crepant resolution. As such, S is obtained by blowing down curves in the minimal resolution $\widetilde{S}$, and so by McKay correspondence we can describe S combinatorially via a Dynkin diagram Δ, together with the subset $\mathcal{J} \subseteq \Delta_0$ of those vertices that are blown down to obtain S. Thus, by convention, $\mathcal{J}$ corresponds to the curves that have been contracted by $\widetilde{S} \to S$.

This data can be extended into the affine setting as follows. Consider the corresponding extended Dynkin diagram Δ_{aff}, and denote the extending vertex by $\star$. Set $\mathcal{J}_{\mathsf{aff}} := \mathcal{J}$, considered as a subset of the vertices of Δ_{aff}.

From this data, consider $\mathbb{R}^{|\Delta|}$ and $\mathbb{R}^{|\Delta_{\mathsf{aff}}|}$ based by the duals α_i^*, where the i are indexed over the vertices of Δ (respectively, Δ_{aff}). Inside these spaces, consider the Weyl chamber C_+, where all coordinates are positive, and set

$$\mathsf{TCone}(\Delta) = \bigcup_{w\in W_\Delta} w(C_+)$$
$$\mathsf{TCone}(\Delta_{\mathsf{aff}}) = \bigcup_{w\in W_{\Delta_{\mathsf{aff}}}} w(C_+),$$

where W_Δ is the finite Weyl group, and $W_{\Delta_{\mathsf{aff}}}$ the affine Weyl group.

There are subspaces $D_{\mathcal{J}} \subset \mathbb{R}^{|\Delta|}$ and $D_{\mathcal{J}_{\mathsf{aff}}} \subset \mathbb{R}^{|\Delta_{\mathsf{aff}}|}$ defined as

$$D_{\mathcal{J}} := \{\vartheta \in \mathbb{R}^{|\Delta|} \mid \vartheta_i = 0 \text{ if } i \in \mathcal{J}\},$$
$$D_{\mathcal{J}_{\mathsf{aff}}} := \{\vartheta \in \mathbb{R}^{|\Delta_{\mathsf{aff}}|} \mid \vartheta_i = 0 \text{ if } i \in \mathcal{J}_{\mathsf{aff}}\}.$$

These are based by α_i^* for $i \in \Delta - \mathcal{J}$, respectively $i \in \Delta_{\mathsf{aff}} - \mathcal{J}_{\mathsf{aff}}$. As such, $\dim D_{\mathcal{J}} = n$, the number of curves in the flopping contraction, and $\dim D_{\mathcal{J}_{\mathsf{aff}}} = n+1$.

Definition 3.1 ([IW2, §1])**.** For $\mathcal{J} \subseteq \Delta_0$ as above,

(1) $\mathsf{TCone}(\mathcal{J}) := \mathsf{TCone}(\Delta) \cap D_{\mathcal{J}}$ is called the *$\mathcal{J}$-finite hyperplane arrangement.*
(2) $\mathsf{TCone}(\mathcal{J}_{\mathsf{aff}}) := \mathsf{TCone}(\Delta_{\mathsf{aff}}) \cap D_{\mathcal{J}_{\mathsf{aff}}}$ is called the *$\mathcal{J}$-affine arrangement.*

3.2. **Affine Hyperplanes via K-theory.** The combinatorics of the previous section can also be constructed via K-theory, which is more useful for stability conditions later. Recall from (2.C) that the flopping contraction $f\colon X \to \operatorname{Spec} R$ associates a modifying R-module N, with summands $R = N_0, N_1, \dots, N_n$. Set $\Lambda := \operatorname{End}_R(N)$ and $\mathcal{P}_i := \operatorname{Hom}_R(N, N_i)$, so that $\{\mathcal{P}_i\}_{0\le i\le n}$ is the set of all indecomposable projective Λ-modules. It is well-known that

$$\mathcal{K}_N := K_0(\operatorname{Perf}\Lambda) \cong \bigoplus_{i=0}^{n} \mathbb{Z}[\mathcal{P}_i] \cong \mathbb{Z}^{n+1}.$$

For every $L \in \operatorname{Mut} N$, this process can be repeated: indeed each $\Lambda_L := \operatorname{End}_R(L)$ has K-theory of the same rank as above, and to avoid confusion write $\mathcal{K}_L := K_0(\operatorname{Perf}\Lambda_L)$. Since the given flopping contraction f, and its associated modification algebra Λ is *fixed*, throughout we will refer to the distinguished object

$$\mathcal{K} := \mathcal{K}_N.$$

Every mutation functor $\Phi_i\colon \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda_L) \to \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda_{\nu_i L})$ restricts to an equivalence on perfect complexes, and so write

$$\phi_i\colon \mathcal{K}_L \xrightarrow{\sim} \mathcal{K}_{\nu_i L}$$

for the induced isomorphism of K_0-groups. This map can be represented by an invertible $(n+1)\times(n+1)$ matrix over $\mathbb{Z}$, which is described as follows.

Lemma 3.2. *Suppose that L is modifying, and consider the exchange sequence* [IW1, (6.I)] *obtained as the dual of* (2.D)*, namely*

$$0 \to L_i \to \bigoplus_{j\neq i} L_j^{\oplus b_{ij}} \to K_i^*. \tag{3.A}$$

Write $\mathcal{P}_j = \operatorname{Hom}_R(L, L_j)$ for the projectives in Λ_L, and $\mathcal{Q}_j$ for the correspondingly ordered projectives in $\Lambda_{\nu_i L}$. Then $\phi_i^{-1}\colon \mathcal{K}_{\nu_i L} \to \mathcal{K}_L$ sends

$$[\mathcal{Q}_t] \mapsto \begin{cases} [\mathcal{P}_t] & \text{if } t \neq i, \\ -[\mathcal{P}_i] + \sum_{j\neq i} b_{ij}[\mathcal{P}_j] & \text{if } t = i. \end{cases} \tag{3.B}$$

Proof. Being induced by the tilting bimodule T_i from (2.E), it is clear that Φ_i sends T_i to $\Lambda_{\nu_i L}$. Since T_i only differs from Λ_L at the summand $\mathcal{P}_i$, it is obvious that Φ_i sends $\mathcal{P}_j$ to $\mathcal{Q}_j$ whenever $j \neq i$; see e.g. [W1, 4.15(1)]. Hence Φ_i^{-1} sends $\mathcal{Q}_j$ to $\mathcal{P}_j$.

When $j = i$, under Φ_i^{-1}, the projective $\mathcal{Q}_i$ gets mapped to the ith summand of T_i, which by definition is $\mathrm{Hom}_R(L, K_i)$. But by [IW1, (6.Q)], applying $\mathrm{Hom}_R(L, -)$ to (3.A) gives an exact sequence

$$0 \to \mathrm{Hom}_R(L, L_i) \to \bigoplus_{j \neq i} \mathrm{Hom}_R(L, L_j)^{\oplus b_{ij}} \to \mathrm{Hom}_R(L, K_i) \to 0.$$

From this, the identification in K-theory clearly follows. □

For $L \in \mathrm{Mut}(N)$, consider the shortest sequence of mutations

$$L \xrightarrow{i_1} \nu_{i_1} L \to \ldots \xrightarrow{i_n} N,$$

and define Φ_L to be the composition of the corresponding mutation functors

$$\Phi_L \colon \mathrm{D^b}(\mathrm{mod}\,\Lambda_L) \xrightarrow{\Phi_{i_n} \circ \ldots \circ \Phi_{i_1}} \mathrm{D^b}(\mathrm{mod}\,\Lambda). \tag{3.C}$$

We write $\phi_L \colon \mathcal{K}_L \to \mathcal{K}$ for the induced map on K-theory. Throughout, whenever $\mathbb{Z} \subseteq \Bbbk$, we will abuse notation and also write $\phi_L \colon \mathcal{K}_L \otimes \Bbbk \to \mathcal{K} \otimes \Bbbk$. The following result is mainly combinatorial, and it mirrors the corresponding Coxeter statement. We identify the basis element $\alpha_i^* \in \mathsf{TCone}(\mathcal{J}_{\mathsf{aff}})$ with $[\mathcal{P}_i] \in \mathcal{K}$ to allow for the comparison.

Theorem 3.3. [IW2, 9.8] *Suppose that $f \colon X \to \mathrm{Spec}\, R$ is a 3-fold flopping contraction, such that X has only terminal singularities. Then there is a decomposition*

$$\mathsf{TCone}(\mathcal{J}_{\mathsf{aff}}) = \bigcup_{L \in \mathrm{Mut}\, N} \phi_L(C_+) \subseteq \mathcal{K} \otimes \mathbb{R}.$$

In particular, the following statements hold.

(1) *The open decomposition $\bigcup \phi_L(C_+)$ gives the chambers of the $\mathcal{J}$-affine arrangement.*
(2) *If $L \ncong M$, then $\phi_L(C_+)$ and $\phi_M(C_+)$ do not intersect.*
(3) *$\phi_L(C_+)$ and $\phi_M(C_+)$ share a codimension one wall $\iff$ L and M differ by the mutation of an indecomposable summand.*

We remark that $\mathsf{TCone}(\mathcal{J}_{\mathsf{aff}})$ does not fill $\mathbb{R}^{|\Delta_{\mathsf{aff}}|}$, as can be seen in Example 3.5 below. Because of this, all information is contained in a slice.

Definition 3.4. The *real level* is defined to be

$$\mathsf{Level}_L := \left\{ z \in \mathcal{K}_L \otimes \mathbb{R} \;\middle|\; \sum_{j=0}^{n} (\mathsf{rk}_R\, L_j) z_j = 1 \right\}.$$

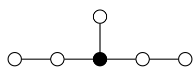
The walls of the open decomposition $\bigcup \phi_L(C_+)$ partition $\mathsf{Level} = \mathsf{Level}_N$ into open regions

$$\mathsf{Alcove}_L := \phi_L(C_+) \cap \mathsf{Level}_N,$$

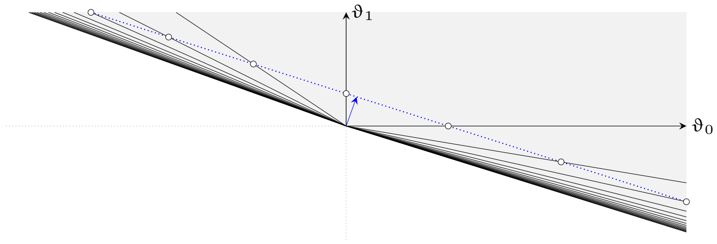
which by Theorem 3.3 are still in bijection with $\mathrm{Mut}(N)$. We call these open regions the $\mathcal{J}$-alcoves, and consider the infinite hyperplane arrangement

$$\mathcal{H}^{\mathsf{aff}} := \mathsf{Level} \setminus \bigcup_{L \in \mathrm{Mut}(N)} \mathsf{Alcove}_L\,. \tag{3.D}$$

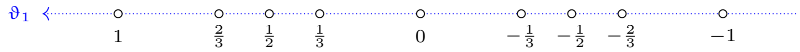
Example 3.5. Consider $\Delta = E_6$, and $\mathcal{J}$ the following choice of unshaded vertices:



Then, either by tracking the C_+ region directly over the labelling set in [IW2, 1.12], or by iterating the wall crossing rule in [IW2, 1.20(1)(2)], or by intersecting the full Tits cone of affine E_6 with the subspace $\mathbb{R}^2$ based by the shaded vertex and the extended vertex, it follows that $\mathsf{TCone}(\mathcal{J}_{\mathsf{aff}})$ is the shaded region in the following picture. Further, Level is illustrated by the dotted blue line $\vartheta_0 + 3\vartheta_1 = 1$.



The circles on the blue line are, reading top left to bottom right, at $\vartheta_1 = 1, \frac{2}{3}, \frac{1}{2}, \frac{1}{3}, 0, -\frac{1}{3}, -\frac{1}{2}$. Thus basing Level by $[\mathcal{P}_1]$, the level is the infinite hyperplane arrangement



The $\mathcal{J}$-alcoves are the open intervals on the blue line between two adjacent dots, and $\mathcal{H}^{\mathsf{aff}}$ is the infinite collection of dots.

3.3. **Finite Hyperplanes by K-theory.** For the finite version of the above combinatorics, with notation as in Lemma 3.2 consider

$$\Theta_L := \mathcal{K}_L/[\mathcal{P}_0] \cong \mathbb{Z}^n.$$

Again, since $X \to \operatorname{Spec} R$ and N are fixed from (2.C), there is a distinguished object $\Theta := \Theta_N$. If $i \neq 0$, then since ϕ_i sends $\mathcal{P}_0$ to $\mathcal{Q}_0$ by Lemma 3.2, ϕ_i induces an isomorphism

$$\varphi_i \colon \Theta_L \to \Theta_{\nu_i L}.$$

For $L \in \operatorname{Mut}_0(N)$, consider the shortest sequence of mutations

$$L \xrightarrow{i_1} \nu_{i_1} L \to \ldots \xrightarrow{i_n} N,$$

where each step does not mutate the vertex R. As before, write Φ_L for the composition of the corresponding mutation functors, but now write $\varphi_L \colon \Theta_L \to \Theta$ for the induced map on K-theory. Again, whenever $\mathbb{Z} \subseteq \Bbbk$, we will abuse notation and also write $\varphi_L \colon \Theta_L \otimes \Bbbk \to \Theta \otimes \Bbbk$.

The following was established first in [W1] when X is $\mathbb{Q}$-factorial, using King stability. The $\mathbb{Q}$-factorial can now be dropped, following [IW2].

Theorem 3.6. [W1, IW2] *Suppose that $f \colon X \to \operatorname{Spec} R$ is a 3-fold flopping contraction, such that X has only terminal singularities. Then there is a finite decomposition*

$$\mathsf{TCone}(\mathcal{J}) = \bigcup_{L \in \operatorname{Mut}_0(N)} \varphi_L(C_+) \subseteq \Theta \otimes \mathbb{R}$$

In particular, the following statements hold.

(1) *The open decomposition $\bigcup \varphi_L(C_+)$ gives the chambers of the $\mathcal{J}$-finite hyperplane arrangement.*
(2) *If $L \ncong M$, then $\varphi_L(C_+)$ and $\varphi_M(C_+)$ do not intersect.*
(3) *$\varphi_L(C_+)$ and $\varphi_M(C_+)$ share a codimension one wall $\iff$ L and M differ by the mutation of an indecomposable summand.*

For $L \in \operatorname{Mut}_0(N)$, write $C_L := \varphi_L(C_+)$ and set

$$\mathcal{H} := (\Theta \otimes \mathbb{R}) \backslash \bigcup_{L \in \operatorname{Mut}_0(N)} C_L. \tag{3.E}$$

By Theorem 3.6(1), $\mathcal{H}$ is a finite simplicial hyperplane arrangement, which by definition means that $\bigcap_{H \in \mathcal{H}} H = \{0\}$ and all chambers in $\mathbb{R}^n \backslash \mathcal{H}$ are open simplicial cones.

3.4. **The Tracking Rules of Mutation.** Given a modifying R-module L and any summand L_i, $\nu_i\nu_i L \cong L$ by Proposition 2.1(2). We will abuse notation and write

$$\mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda_L)) \quad \underset{\Phi_i}{\overset{\Phi_i}{\rightleftarrows}} \quad \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda_{\nu_i L}).$$

These, and their inverses, induce the following isomorphisms on K-theory

$$\mathcal{K}_L \underset{\phi_i}{\overset{\phi_i}{\rightleftarrows}} \mathcal{K}_{\nu_i L} \qquad\qquad \mathcal{K}_L \underset{\phi_i^{-1}}{\overset{\phi_i^{-1}}{\leftrightarrows}} \mathcal{K}_{\nu_i L} \tag{3.F}$$

Lemma 3.7. *All four isomorphisms in (3.F) are given by the same matrix, and this matrix squares to the identity. If $i \neq 0$, then the same statement holds for $\varphi_i, \varphi_i^{-1}$ and $\Theta_L, \Theta_{\nu_i L}$.*

Proof. By (3.B), the matrices for the inverses are controlled by numbers appearing in the relevant approximation sequences. Suppose that the top ϕ_i^{-1} is controlled by numbers b_{ij}, and the bottom ϕ_i^{-1} is controlled by numbers c_{ij}. That the two matrices labelled ϕ_i^{-1} are the same is simply the statement that $b_{ij} = c_{ij}$, which is precisely the proof of [W1, 5.22] when X is $\mathbb{Q}$-factorial, or [IW2, 9.28] generally. Given the fact that $b_{ij} = c_{ij}$, the statement that $\phi_i^{-1}\phi_i^{-1} = \mathrm{Id}$ can then simply be seen directly. Applying ϕ_i to each side then gives $\phi_i^{-1} = \phi_i$. All the statements on φ_i follow. $\square$

3.5. **Complexified Actions.** Via (3.D) and (3.E), associated to $X \to \operatorname{Spec} R$ is an infinite real hyperplane arrangement $\mathcal{H}^{\mathsf{aff}}$, and also a finite simplicial real hyperplane arrangement $\mathcal{H}$. Stability conditions will require the complexified versions of these.

By a slight abuse of notation, consider

$$\begin{aligned}\mathbb{H}_+ &:= \{x + \mathrm{i}y \in (\Theta_L)_{\mathbb{C}} \mid x_j + \mathrm{i}y_j \in \mathbb{H} \text{ for all } 1 \leq j \leq n\} \cong \mathbb{H}^n\\ \mathbb{H}'_+ &:= \{x + \mathrm{i}y \in (\mathcal{K}_L)_{\mathbb{C}} \mid x_j + \mathrm{i}y_j \in \mathbb{H} \text{ for all } 0 \leq j \leq n\} \cong \mathbb{H}^{n+1}\end{aligned}$$

where $\mathbb{H} = \{re^{\mathrm{i}\pi\vartheta} \in \mathbb{C} \mid r \in \mathbb{R}_{>0},\ 0 < \vartheta \leq 1\} \subset \mathbb{C}$ is the semi-closed upper half plane. The regions $\mathbb{H}_+$ and $\mathbb{H}'_+$ technically depend on L, since they are subsets of $(\Theta_L)_{\mathbb{C}}$ and $(\mathcal{K}_L)_{\mathbb{C}}$ respectively, but we drop this from the notation.

For $L \in \operatorname{Mut}_0(N)$, recall from §3.3 that after choosing a mutation path $L \to \ldots \to N$ that does not involve mutating R, there is a corresponding linear map $\varphi_L \colon (\Theta_L)_{\mathbb{C}} \to \Theta_{\mathbb{C}}$. We require the following result, where as usual $\mathcal{H}_{\mathbb{C}}$ denotes the complexification of the real hyperplane arrangement $\mathcal{H}$. The result is folklore when the arrangement $\mathcal{H}$ is Coxeter. Given our setting here is just mildly more general, and the proof is combinatorial in nature, we give a self-contained proof in Appendix A.

Proposition 3.8. *There is an equality*

$$(\Theta \otimes \mathbb{C})\backslash\mathcal{H}_{\mathbb{C}} = \bigcup_{L \in \operatorname{Mut}_0(N)} \varphi_L(\mathbb{H}_+)$$

where the union on the right hand side is disjoint.

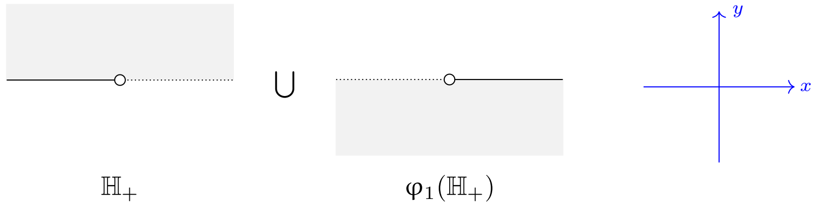
On the other hand, the affine version of Proposition 3.8 is a little bit more involved. We first pass to the *complexified level*, defined to be

$$(\mathsf{Level}_L)_{\mathbb{C}} := \left\{ z \in (\mathcal{K}_L)_{\mathbb{C}} \;\middle|\; \sum_{j=0}^{n} (\mathsf{rk}_R\, L_j) z_j = \mathrm{i} \right\},$$

and inside $(\mathsf{Level}_L)_{\mathbb{C}}$ consider the region

$$\mathbb{E}_+ := \left\{ z \in \mathbb{H}'_+ \;\middle|\; \sum_{j=0}^{n} (\mathsf{rk}_R\, L_j) z_j = \mathrm{i} \right\}.$$

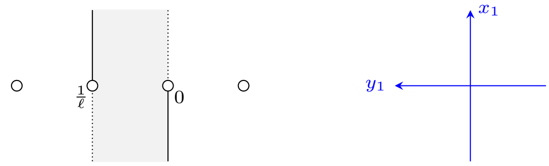
Example 3.9. In the case of any one-curve flop, writing $z = x + \mathrm{i}y$, then $\mathbb{C}\backslash\mathcal{H}_{\mathbb{C}} = \mathbb{C}\backslash\{0\}$ decomposes into the disjoint union



On the other hand, as in Example 3.5, for any one-curve flop $\mathcal{H}^{\mathsf{aff}}_{\mathbb{C}}$ consists of infinitely many points on the real axis. To exhibit the region $\mathbb{E}_+$, note first that $(z_0, z_1) \in \mathsf{Level}_{\mathbb{C}}$ if and only if we can write

$$(z_0, z_1) = ((-\ell x_1, 1 - \ell y_1), (x_1, y_1)),$$

where ℓ is the length of the curve. To belong to $\mathbb{E}_+$ is equivalent to both factors being in $\mathbb{H}$. If the second factor is in $\mathbb{H}$ then $y_1 \geq 0$, so the first factor being in $\mathbb{H}$ implies that $0 \leq y_1 \leq \frac{1}{\ell}$. In fact, it is elementary to check that $\mathbb{E}_+$ forms the following region:



The non-standard way of drawing the x and y axis is justified by Example 3.5. The co-ordinate y_1 should be viewed as the original Level seen in Example 3.5, which naturally points to the left, and x_1 should be viewed as the 'complexified co-ordinate'. The other regions $\varphi_L(\mathbb{E}_+)$ have the same shape as the above, sandwiched between the two adjacent dots, and so give a disjoint union that covers $(\mathsf{Level}_L)_{\mathbb{C}}$.

Set $\mathcal{W}$ to be the set of full hyperplanes in $\mathcal{K}\otimes\mathbb{R}$ that separate the open chambers $\phi_L(C_+)$ of $\mathsf{TCone}(\mathcal{J}_{\mathsf{aff}})$ (see e.g. Example A.1). We then consider the complexification of $\mathcal{H}^{\mathsf{aff}}$ in $\mathsf{Level}_{\mathbb{C}}$, defined to be

$$\mathcal{H}^{\mathsf{aff}}_{\mathbb{C}} := \mathcal{W}_{\mathbb{C}} \cap \mathsf{Level}_{\mathbb{C}} = \bigcup_{W\in\mathcal{W}} (W_{\mathbb{C}} \cap \mathsf{Level}_{\mathbb{C}}), \tag{3.G}$$

where $W_{\mathbb{C}} := W \oplus \mathrm{i}W$. As in Example 3.9, $\mathcal{H}^{\mathsf{aff}}_{\mathbb{C}}$ can be viewed as the complexification of hyperplanes in the real level, provided that we swap the roles of x and y. Indeed, if we set $H_W := W \cap \mathsf{Level}$, then $\mathcal{H}^{\mathsf{aff}} = \bigcup_{W\in\mathcal{W}} H_W$, and the linear bijection $\mathsf{Level}_{\mathbb{C}} \to \mathsf{Level} \oplus \mathrm{i}\,\mathsf{Level}$ defined by $x + \mathrm{i}y \mapsto (x+y) + \mathrm{i}y$ maps $\mathcal{H}^{\mathsf{aff}}_{\mathbb{C}}$ to $\bigcup_{W\in\mathcal{W}}(H_W \oplus \mathrm{i}H_W)$.

The following two results are evident, by inspection, for any one-curve flops using Example 3.9 above. The more general case requires a more involved combinatorial argument, so again the proofs are postponed until Appendix A.

Lemma 3.10 (A.4)**.** *The subspace* $\mathbb{E}_+ \subset (\mathcal{K}_L)_{\mathbb{C}}$ *is path connected.*

Proposition 3.11. *There is an equality*

$$\mathsf{Level}_{\mathbb{C}}\backslash\mathcal{H}^{\mathsf{aff}}_{\mathbb{C}} = \bigcup_{L\in\mathrm{Mut}(N)} \phi_L(\mathbb{E}_+),$$

where the union on the right hand side is disjoint.

4. Arrangement Groupoids

In this subsection we briefly recall the basics of the arrangement (=Deligne) groupoid of a locally finite real hyperplane arrangement, mainly to set notation. Some first results specific to the flops setting are presented in Subsection 4.2.

4.1. **Arrangements groupoids.** Throughout, let $\mathcal{H}$ be either the finite real hyperplane arrangement $\mathcal{H}$ from (3.E), or the infinite version $\mathcal{H}^{\mathsf{aff}}$ from (3.D). Both are *locally finite* arrangements, i.e. every point of $\mathbb{R}^n$ is contained in at most finitely many hyperplanes, and *essential arrangements*, i.e. the minimal intersections of hyperplanes are points.

Definition 4.1. The graph $\Gamma_{\mathcal{H}}$ of oriented arrows is defined as follows. The vertices of $\Gamma_{\mathcal{H}}$ are the chambers (i.e. the connected components) of $\mathbb{R}^n \backslash \mathcal{H}$. There is a unique arrow $a\colon v_1 \to v_2$ from chamber v_1 to chamber v_2 if the chambers are adjacent, otherwise there is no arrow. For an arrow $a\colon v_1 \to v_2$, we set $s(a) := v_1$ and $t(a) := v_2$. By definition, if there is an arrow $a\colon v_1 \to v_2$, there is a unique arrow $b\colon v_2 \to v_1$ with the opposite direction of a.

A *positive path of length* n in $\Gamma_{\mathcal{H}}$ is defined to be a formal symbol

$$p = a_n \circ \ldots \circ a_2 \circ a_1,$$

whenever there exists a sequence of vertices $v_0, \ldots, v_n$ of $\Gamma_{\mathcal{H}}$ and exist arrows $a_i\colon v_{i-1} \to v_i$ in $\Gamma_{\mathcal{H}}$. Set $s(p) := v_0$, $t(p) := v_n$, and $\ell(p) := n$, and write $p\colon s(p) \to t(p)$. The notation $\circ$ should remind us of composition, but we will often drop the $\circ$'s in future. If $q = b_m \circ \ldots \circ b_2 \circ b_1$ is another positive path with $t(p) = s(q)$, we consider the formal symbol

$$q \circ p := b_m \circ \ldots \circ b_2 \circ b_1 \circ a_n \circ \ldots \circ a_2 \circ a_1,$$

and call it the *composition* of p and q.

Definition 4.2. A positive path is called *reduced* if it does not cross any hyperplane twice.

In our setting where $\mathcal{H}$ is $\mathcal{H}$ or $\mathcal{H}^{\mathsf{aff}}$, reduced positive paths coincide with shortest positive paths. In the finite setting this can be found in e.g. [P1, 4.2], and in the infinite case see e.g. [S, Lemma 2] or [IM, §I.5].

Following [D2, p7], let $\sim$ denote the smallest equivalence relation, compatible with morphism composition, that identifies all morphisms that arise as positive reduced paths with same source and target. Then consider the free category $\mathrm{Free}(\Gamma_{\mathcal{H}})$ on the graph $\Gamma_{\mathcal{H}}$, where morphisms are directed paths, and the quotient category

$$\mathcal{G}_{\mathcal{H}}^{+} := \mathrm{Free}(\Gamma_{\mathcal{H}})/\sim,$$

called the category of positive paths.

Definition 4.3. The *arrangement (=Deligne) groupoid* $\mathcal{G}_{\mathcal{H}}$ is the groupoid defined as the groupoid completion of $\mathcal{G}_{\mathcal{H}}^{+}$, that is, a formal inverse is added for every morphism in $\mathcal{G}_{\mathcal{H}}^{+}$.

Notation 4.4. *When $\mathcal{H} = \mathcal{H}$ from (3.E), we denote the arrangement groupoid by $\mathbb{G}$, and when $\mathcal{H} = \mathcal{H}^{\mathsf{aff}}$ from (3.D), we denote the arrangement groupoid by $\mathbb{G}^{\mathsf{aff}}$.*

The following is well-known [D1, P1, P2, S]; in the level of generality here with $\mathcal{H}$ locally finite and essential, the statement below is [D2, p9].

Theorem 4.5. *For any vertex v in the arrangement groupoid, $\mathrm{End}_{\mathbb{G}}(v) \cong \pi_1(\mathbb{C}^n \backslash \mathcal{H}_{\mathbb{C}})$, and $\mathrm{End}_{\mathbb{G}^{\mathsf{aff}}}(v) \cong \pi_1(\mathsf{Level}_{\mathbb{C}} \backslash \mathcal{H}_{\mathbb{C}}^{\mathsf{aff}})$.*

4.2. **First Results for Flops.** This section proves that in our flops setting, pure braids act as the identity on K-theory. This will be crucial in showing that they act as deck transformations later in Section 6. Throughout this subsection, $\mathcal{H}$ is either $\mathcal{H}$ or $\mathcal{H}^{\mathsf{aff}}$, and C_L denotes the chamber in the complement of $\mathcal{H}$ corresponding to either $\varphi_L(C_+)$ or Alcove_L, when $\mathcal{H} = \mathcal{H}$ or $\mathcal{H} = \mathcal{H}^{\mathsf{aff}}$ respectively. The following two lemmas are elementary.

Lemma 4.6. *Suppose that $\alpha\colon A \to B$ is a reduced positive path for $\mathcal{H}$, and that s_i is a simple wall crossing separating B and C, so by definition there are morphisms $s_i\colon B \to C$ and $s_i\colon C \to B$ in $\Gamma_{\mathcal{H}}$. If $s_i \circ \alpha\colon A \to C$ is not reduced, then there exists some reduced positive path $\gamma\colon A \to C$ such that $s_i \circ \gamma\colon A \to B$ is reduced.*

Proof. This is very similar to [HW, 5.1]. ☐

Lemma 4.7. *Suppose that $\alpha = s_{i_t} \circ \ldots \circ s_{i_1}$ is a reduced positive path for $\mathcal{H}$, namely*

$$\alpha = (C_1 \xrightarrow{s_{i_1}} C_2 \xrightarrow{s_{i_2}} \ldots C_{t-1} \xrightarrow{s_{i_t}} C_t)$$

where each s_{i_j} crosses a hyperplane H_j, say. Then for all $j = 1, \ldots, t-1$, the chambers C_1 and C_j are on the same side of H_j.

Proof. There is nothing to prove in the case $j = 1$. If C_1 and C_j are on opposite sides of H_j for some $j > 1$, then clearly $\beta := s_{i_{j-1}} \circ \ldots \circ s_{i_1} : C_1 \to C_j$ must at some point cross H_j. But then $s_{i_j} \circ \beta$, and hence α, must cross H_j twice, which is a contradiction. $\square$

With respect to our applications, part (2) of the following proposition is crucial. For any positive $\alpha \in \Gamma_{\mathcal{H}^{\mathsf{aff}}}$, say $\alpha = s_{i_t} \circ \ldots \circ s_{i_1}$ we associate the functor

$$\Phi_\alpha := \Phi_{i_t} \circ \ldots \circ \Phi_{i_1}.$$

Now there is an order $\geq$ defined on tilting modules (see e.g. [HW, §3]). By Lemma 4.7, this order decreases along reduced paths, and so exactly as in [HW, 4.6] (see Remark 4.10 below), we see that $\Phi_\alpha \cong \Phi_\beta$ for any two reduced positive paths with the same start and end points. Hence the association $\alpha \mapsto \Phi_\alpha$ descends to a functor from $(\mathbb{G}^{\mathsf{aff}})^+$. As Φ_α is already invertible, this in turn formally descends to a functor from $\mathbb{G}^{\mathsf{aff}}$.

The same analysis holds for the finite situation $\mathbb{G}$. In both cases, for any α in the arrangement groupoid, we thus have an associated functor Φ_α, and its image ϕ_α on K-theory $\mathcal{K}$, respectively φ_α on Θ.

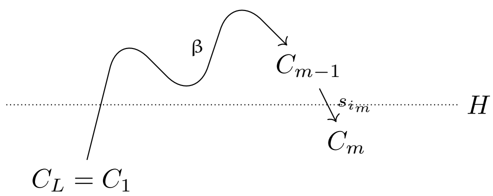
Proposition 4.8. *Choose a reduced positive path $\beta\colon C_L \to C_M$ for $\mathcal{H}^{\mathsf{aff}}$.*

(1) *If $\alpha\colon C_L \to C_M$ is any positive path in $\Gamma_{\mathcal{H}^{\mathsf{aff}}}$, then $\phi_\alpha = \phi_\beta$.*
(2) *If $p \in \operatorname{End}_{\mathbb{G}^{\mathsf{aff}}}(C_L)$, then $\phi_p = \operatorname{Id}_{\mathcal{K}_L}$.*
(3) *If $p, q \in \operatorname{Hom}_{\mathbb{G}^{\mathsf{aff}}}(C_L, C_M)$, then $\phi_p = \phi_q$.*

The same statements hold for $\mathcal{H}$, replacing $\mathbb{G}^{\mathsf{aff}}$ by $\mathbb{G}$, and ϕ by φ.

Proof. We will establish all statements over $\mathbb{Z}$, as then all the statements over $\Bbbk$ follow.
(1) By the discussion above the Proposition, we know that if α is furthermore reduced, then $\Phi_\alpha \cong \Phi_\beta$, and so in particular $\phi_\alpha = \phi_\beta$ holds. Hence we can assume that α is not reduced.

Consider the first time that $\alpha = s_{i_t} \circ \ldots \circ s_{i_1}$ crosses a hyperplane twice. So, say $\beta := s_{i_{m-1}} \circ \ldots \circ s_{i_1}$ is reduced, but $s_{i_m} \circ \ldots \circ s_{i_1}$ is not. Pictorially



By Lemma 4.6 we can find a positive reduced path $\gamma\colon C_1 \to C_m$ such that the composition $C_1 \xrightarrow{\gamma} C_m \xrightarrow{s_{i_m}} C_{m-1}$ is reduced. As β and $s_{i_m} \circ \gamma$ are reduced paths with the same start and end points, functorially they are the same, so

$$\begin{aligned}\Phi_\alpha &= \Phi_{i_t} \circ \ldots \circ \Phi_{i_m} \circ (\Phi_{i_{m-1}} \circ \ldots \circ \Phi_{i_1})\\ &\cong \Phi_{i_t} \circ \ldots \circ \Phi_{i_m} \circ (\Phi_{i_m} \circ \Phi_\gamma)\end{aligned}$$

Passing to K-theory, using the fact that $\phi_{i_m}\phi_{i_m} = \operatorname{Id}$ by Lemma 3.7, we see

$$\phi_\alpha = \phi_{i_t} \circ \ldots \circ \phi_{i_{m+1}} \circ \phi_\gamma.$$

Consider next the first time that $s_{i_t} \circ \ldots \circ s_{i_{m+1}} \circ \gamma$ crosses a hyperplane twice. Since γ is reduced, we move further to the left. Applying the above argument repeatedly, by induction we end up in the case of a reduced path, and hence $\phi_\alpha = \phi_\beta$.
(2) Say $\phi_p = \phi_{i_n}^{\pm 1} \circ \ldots \circ \phi_{i_1}^{\pm 1}$ for some choice of superscripts ± 1. By Lemma 3.7, $\phi_p = \phi_q$, where $q := s_{i_n} \circ \ldots \circ s_{i_1}$. This is a positive path, with start and end C_L, so by part (1) it follows that $\phi_p = \phi_q = \operatorname{Id}$.
(3) This follows by applying (2) to $q^{-1}p \in \operatorname{End}_{\mathbb{G}_{\mathcal{H}}}(C_L)$. $\square$

Remark 4.9. Proposition 4.8 also implies that Theorems 3.3 and 3.6, and also Propositions 3.8 and 3.11, can be indexed over reduced positive paths terminating at C_+.

Remark 4.10. Implicit in the above analysis is the fact, proved in [HW, 4.6] in the finite case and [IW2, 9.34] in the infinite case, that if $\alpha\colon L \to M$ is a positive minimal path, then $\Phi_\alpha \cong \mathbf{R}\mathrm{Hom}_{\Lambda_L}(\mathrm{Hom}_R(L,M),-)$. Hence any two positive minimal paths give rise to isomorphic functors.

5. Stability Conditions, Tilting and t-structure Transfer

5.1. **Generalities on stability conditions.** We will not give a full summary of stability conditions here; see for example [B4] or the survey [B1]. For our purposes, the following suffices. Throughout this subsection, $\mathcal{T}$ denotes a triangulated category for which the Grothendieck group $K_0(\mathcal{T})$ is a finitely generated free $\mathbb{Z}$-module.

Proposition 5.1 ([B4, 5.3])**.** *To specify a stability condition on $\mathcal{T}$ is equivalent to specifying a bounded t-structure on $\mathcal{T}$ with heart $\mathcal{A}$, together with a stability function Z on $\mathcal{A}$ that satisfies the Harder-Narasimhan property.*

As usual, in fact we will only study *locally finite* stability conditions, and we let $\mathrm{Stab}\,\mathcal{T}$ denote the set of locally finite stability conditions on $\mathcal{T}$. There is a topology on $\mathrm{Stab}\,\mathcal{T}$, induced by a natural metric.

Theorem 5.2 ([B4, 1.2])**.** *The space* $\mathrm{Stab}\,\mathcal{T}$ *has the structure of a complex manifold, and the forgetful map*
$$\pi\colon \mathrm{Stab}\,\mathcal{T} \to \mathrm{Hom}_{\mathbb{Z}}(K_0(\mathcal{T}),\mathbb{C})$$
is a local isomorphism onto an open subspace of $\mathrm{Hom}_{\mathbb{Z}}(K_0(\mathcal{T}),\mathbb{C})$.

Remark 5.3. In the components $\mathrm{Stab}^\circ\mathcal{C}$ and $\mathrm{Stab}^\circ\mathcal{D}$ we study in the flops setting below, all stability conditions will automatically be *full*, in the sense that they are always modelled on the whole of $\mathrm{Hom}_{\mathbb{Z}}(K_0(\mathcal{T}),\mathbb{C})$. In particular, our stability conditions will automatically satisfy the *support property*, see e.g. [BM, Appendix B]. We will freely use this throughout.

An exact equivalence of triangulated categories $\Phi\colon \mathcal{T} \to \mathcal{T}'$ induces a natural map
$$\Phi_*\colon \mathrm{Stab}\,\mathcal{T} \to \mathrm{Stab}\,\mathcal{T}'$$
defined by $\Phi_*(Z,\mathcal{A}) := (Z\circ\phi^{-1},\Phi(\mathcal{A}))$, where as before ϕ^{-1} denotes the isomorphism on K-theory $K_0(\mathcal{T}') \xrightarrow{\sim} K_0(\mathcal{T})$ induced by the functor Φ^{-1}, and $\Phi(\mathcal{A})$ denotes its essential image. As usual, if two exact equivalences $\Phi\colon \mathcal{T} \to \mathcal{T}'$ and $\Psi\colon \mathcal{T} \to \mathcal{T}'$ are naturally isomorphic, then $\Phi_*(Z,\mathcal{A}) = \Psi_*(Z,\mathcal{A})$ for any $(Z,\mathcal{A}) \in \mathrm{Stab}\,\mathcal{T}$, and thus the group $\mathrm{Auteq}(\mathcal{T})$ of isomorphism classes of autoequivalences of $\mathcal{T}$ acts on $\mathrm{Stab}\,\mathcal{T}$.

5.2. **Stability, Normalisation and Mutations.** We return to the setting where $f\colon X \to \mathrm{Spec}\,R$ is the flopping contraction as in the introduction, with distinguished R-module N from (2.C), $\Lambda := \mathrm{End}_R(N)$, and K-theory $\mathcal{K}$ and Θ from Sections 3.2 and 3.3 respectively.

For any $L = \bigoplus_{i=0}^n L_i \in \mathrm{Mut}(N)$, with the ordering on summands induced from N as explained under Definition 2.2, let $\mathcal{S}_i$ be the simple Λ_L-module corresponding to the projective $\mathcal{P}_i = \mathrm{Hom}(L,L_i)$. Write $\mathcal{B}_L$ for the subcategory of $\mathrm{mod}\,\Lambda_L$ consisting of finite-length modules. If $L \in \mathrm{Mut}_0(N)$, then we further write $\mathcal{A}_L$ for the full subcategory of $\mathcal{B}_L$ of those finite-length modules whose simple factors are not isomorphic to $\mathcal{S}_0$.

Consider the triangulated subcategories
$$\mathcal{C}_L := \{a \in \mathrm{D^b}(\mathrm{mod}\,\Lambda_L) \mid \mathrm{H}^i(a) \in \mathcal{A}_L \text{ for all } i\},$$
$$\mathcal{D}_L := \{b \in \mathrm{D^b}(\mathrm{mod}\,\Lambda_L) \mid \mathrm{H}^i(b) \in \mathcal{B}_L \text{ for all } i\}.$$
Since $\mathcal{A}_L$ and $\mathcal{B}_L$ are extension closed abelian subcategories of $\mathrm{mod}\,\Lambda_L$, the standard t-structure on $\mathrm{D^b}(\mathrm{mod}\,\Lambda_L)$ restricts to a bounded t-structure on $\mathcal{D}_L$ with heart $\mathcal{B}_L$, and a bounded t-structure on $\mathcal{C}_L$ with heart $\mathcal{A}_L$.

The categories $\mathcal{A}_L$ and $\mathcal{B}_L$ have finitely many simple objects, and so
$$K_0(\mathcal{C}_L) \cong \bigoplus_{i=1}^n \mathbb{Z}[\mathcal{S}_i] \qquad K_0(\mathcal{D}_L) \cong \bigoplus_{i=0}^n \mathbb{Z}[\mathcal{S}_i].$$
There are canonical isomorphisms
$$\mathrm{Hom}_{\mathbb{Z}}(K_0(\mathcal{C}_L),\mathbb{C}) \xrightarrow{\sim} (\Theta_L)_{\mathbb{C}} \qquad \mathrm{Hom}_{\mathbb{Z}}(K_0(\mathcal{D}_L),\mathbb{C}) \xrightarrow{\sim} (\mathcal{K}_L)_{\mathbb{C}}$$

given by $\gamma \mapsto \sum \gamma([\mathcal{S}_i])[\mathcal{P}_i]$. Composing these with the local homeomorphism in Theorem 5.2 defines local homeomorphisms

$$\mathcal{Z}_L\colon \operatorname{Stab}\mathcal{C}_L \to (\Theta_L)_{\mathbb{C}} \qquad \mathcal{Z}_L\colon \operatorname{Stab}\mathcal{D}_L \to (\mathcal{K}_L)_{\mathbb{C}}.$$

Write $\operatorname{Stab}\mathcal{A}_L$ for the stability functions on $\mathcal{A}_L$ which satisfy the Harder–Narasimhan property, then by Proposition 5.1 $\operatorname{Stab}\mathcal{A}_L$ can be regarded as a subspace of $\operatorname{Stab}\mathcal{C}_L$. Similarly for $\operatorname{Stab}\mathcal{B}_L$, which is a subspace of $\operatorname{Stab}\mathcal{D}_L$. It follows from [B4, 5.2] that the above local homeomorphisms restrict to isomorphisms

$$\mathcal{Z}_L\colon \operatorname{Stab}\mathcal{A}_L \xrightarrow{\sim} \mathbb{H}_+ \qquad \mathcal{Z}_L\colon \operatorname{Stab}\mathcal{B}_L \xrightarrow{\sim} \mathbb{H}'_+. \tag{5.A}$$

Applying all of the above to $N \in \operatorname{Mut}(N)$, it will be convenient to suppress N from the notation, so write $\mathcal{D} := \mathcal{D}_N$, $\mathcal{B} := \mathcal{B}_N$, $\mathcal{Z} := \mathcal{Z}_N$, etc.

There is a $\mathbb{C}$-action on $\operatorname{Stab}\mathcal{D}$, which later we will avoid. As such, following [B6], for any $L = \bigoplus_{i=0}^n L_i \in \operatorname{Mut}(N)$, consider $\operatorname{Stab}_n\mathcal{D}_L$ to be those stability conditions in $\operatorname{Stab}\mathcal{D}_L$ for which the central charge Z satisfies

$$\sum_{j=0}^n (\mathsf{rk}_R\, L_j)\, Z[\mathcal{S}_j] = \mathrm{i}.$$

We call such stability conditions *normalised*. In particular

$$\mathcal{Z}_L\colon \operatorname{Stab}_n\mathcal{D}_L \to (\mathsf{Level}_L)_{\mathbb{C}},$$

where the complexified level is defined in §3.5. Set $\operatorname{Stab}_n\mathcal{B}_L := \operatorname{Stab}\mathcal{B}_L \cap \operatorname{Stab}_n\mathcal{D}_L$, then the latter isomorphism in (5.A) restricts to an isomorphism

$$\mathcal{Z}_L\colon \operatorname{Stab}_n\mathcal{B}_L \xrightarrow{\sim} \mathbb{E}_+.$$

Proposition 5.4. *Let $L \in \operatorname{Mut}_0(N)$, respectively $L \in \operatorname{Mut}(N)$. Then the following diagrams commute.*

$$\begin{tikzcd} \operatorname{Stab}\mathcal{A}_L \arrow[r, "(\Phi_L)_*"] \arrow[d, "\mathcal{Z}_L"', "\sim"] & \operatorname{Stab}\mathcal{C} \arrow[d, "\mathcal{Z}"] \\ \mathbb{H}_+ \arrow[r, "\varphi_L"] & \Theta_{\mathbb{C}} \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Stab}\mathcal{B}_L \arrow[r, "(\Phi_L)_*"] \arrow[d, "\mathcal{Z}_L"', "\sim"] & \operatorname{Stab}\mathcal{D} \arrow[d, "\mathcal{Z}"] \\ \mathbb{H}'_+ \arrow[r, "\phi_L"] & \mathcal{K}_{\mathbb{C}} \end{tikzcd}$$

The latter diagram restricts to a commutative diagram

$$\begin{tikzcd} \operatorname{Stab}_n\mathcal{B}_L \arrow[r, "(\Phi_L)_*"] \arrow[d, "\mathcal{Z}_L"', "\sim"] & \operatorname{Stab}_n\mathcal{D} \arrow[d, "\mathcal{Z}"] \\ \mathbb{E}_+ \arrow[r, "\phi_L"] & \mathsf{Level}_{\mathbb{C}} \end{tikzcd}$$

Proof. To ease notation, set $\mathcal{Q}_i := \operatorname{Hom}_R(L, L_i)$ and write $\mathcal{S}'_i$ for the simple Λ_L-module corresponding to the projective Λ_L-module $\mathcal{Q}_i$. Similarly, write $\mathcal{P}_i = \operatorname{Hom}_R(N, N_i)$, and $\mathcal{S}_i$ for the corresponding simple Λ-module.

Consider the perfect pairing

$$\chi(-,-)\colon \Theta \times K_0(\mathcal{C}) \to \mathbb{Z}$$

given by $\chi(a, b) := \sum_{i\in\mathbb{Z}}(-1)^i \dim_{\mathbb{C}} \operatorname{Hom}(a, b[i])$. Since the pairing is perfect, setting

$$a_{ij} := \chi(\Phi_L(\mathcal{Q}_i), \mathcal{S}_j) \in \mathbb{Z}$$

implies that $[\Phi_L(\mathcal{Q}_i)] = \sum_{j=1}^n a_{ij}[\mathcal{P}_j]$ in Θ. Furthermore, since Φ_L^{-1} is right adjoint to Φ_L,

$$a_{ij} = \chi(\mathcal{Q}_i, \Phi_L^{-1}(\mathcal{S}_j)),$$

which in turn implies that $[\Phi_L^{-1}(\mathcal{S}_j)] = \sum_{i=1}^n a_{ij}[\mathcal{S}'_i]$ in $K_0(\mathcal{C})$. Therefore, for any point $\sigma = (Z, \mathcal{P}) \in \operatorname{Stab}\mathcal{A}_L$, necessarily

$$\varphi_L(\mathcal{Z}_L(\sigma)) = \sum_{i=1}^n Z(\mathcal{S}'_i)[\Phi_L(\mathcal{Q}_i)] = \sum_{j=1}^n \Big(\sum_{i=1}^n a_{ij} Z(\mathcal{S}'_i)\Big)[\mathcal{P}_j] = \mathcal{Z}(\Phi_{L*}(\sigma)).$$

The last diagram follows immediately, as mutation functors in K-theory take $\sum(\mathsf{rk}_R\, L_i)[\mathcal{S}_i]$ to $\sum(\mathsf{rk}_R\, N_i)[\mathcal{S}_i]$ and thus preserve the normalisation. $\square$

5.3. **Tilting at Simples via Mutation.** By the above, $\mathcal{A}_L \subset \mathcal{C}_L$ and $\mathcal{B}_L \subset \mathcal{D}_L$ are the hearts of bounded t-structures, with finitely many simples. Each of these simple objects induces two torsion theories, $(\langle \mathcal{S}_i\rangle, \mathcal{F}_i)$ and $(\mathcal{T}_i, \langle \mathcal{S}_i\rangle)$, where $\langle \mathcal{S}_i\rangle$ is the full subcategory of objects whose simple factors are isomorphic to $\mathcal{S}_i$. In the case of $\mathcal{A}_L$, the subcategories $\mathcal{F}_i$ and $\mathcal{T}_i$ are defined by

$$\mathcal{F}_i := \{a \in \mathcal{A}_L \mid \mathrm{Hom}_{\mathcal{A}_L}(\mathcal{S}_i, a) = 0\}$$
$$\mathcal{T}_i := \{a \in \mathcal{A}_L \mid \mathrm{Hom}_{\mathcal{A}_L}(a, \mathcal{S}_i) = 0\},$$

and the corresponding tilted hearts are defined by

$$\mathrm{L}_i(\mathcal{A}_L) := \{c \in \mathcal{C}_L \mid \mathrm{H}^k(c) = 0 \text{ for } k \notin \{0,1\},\ \mathrm{H}^0(c) \in \mathcal{F}_i,\ \mathrm{H}^1(c) \in \langle \mathcal{S}_i\rangle\}$$
$$\mathrm{R}_i(\mathcal{A}_L) := \{c \in \mathcal{C}_L \mid \mathrm{H}^k(c) = 0 \text{ for } k \notin \{-1,0\},\ \mathrm{H}^{-1}(c) \in \langle \mathcal{S}_i\rangle,\ \mathrm{H}^0(c) \in \mathcal{T}_i\},$$

where $\mathrm{H}^i(-)$ is the cohomological functor associated to the standard t-structure on $\mathcal{C}_L$ defining $\mathcal{A}_L$. A similar picture applies in the case of $\mathcal{B}_L$.

Lemma 5.5. *We have* $\mathrm{L}_i(\mathcal{A}_{\nu_i N}) = \Phi_i(\mathcal{A})$ *and* $\mathrm{R}_i(\mathcal{A}) = \Phi_i^{-1}(\mathcal{A}_{\nu_i N})$ *for all* $i = 1, \dots, n$. *The same statements hold replacing* $\mathcal{A}$ *by* $\mathcal{B}$, *for all* $i = 0, 1, \dots, n$

Proof. We will only show that $\mathrm{L}_i(\mathcal{A}_{\nu_i N}) = \Phi_i(\mathcal{A})$, since the other proof is similar. Since both categories are hearts of bounded t-structures, it suffices to show that $\Phi_i(\mathcal{A}) \subseteq \mathrm{L}_i(\mathcal{A}_{\nu_i N})$. For this, since $\mathcal{A}$ is finite length, it is enough to show that $\Phi_i(\mathcal{S}_j) \in \mathrm{L}_i(\mathcal{A}_{\nu_i N})$ for all $1 \leq j \leq n$.

If $j = i$, since $\Phi_i(\mathcal{S}_i) = \mathcal{S}_i[-1]$ by [W1, 4.15(2)], it follows that $\Phi_i(\mathcal{S}_i) \in \mathrm{L}_i(\mathcal{A}_{\nu_i N})$. Hence we can assume that $j \neq i$. Then $\Phi_i(\mathcal{S}_j) \in \mathcal{A}_{\nu_i N}$ by [HW, 4.4]. Hence $\Phi_i(\mathcal{S}_j) \cong \mathrm{H}^0(\Phi_i(\mathcal{S}_j))$, and so $\Phi_i(\mathcal{S}_j)$ is only in degree zero, and further

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{D^b}(\nu_i\Lambda)}(\mathcal{S}_i, \mathrm{H}^0(\Phi_i(\mathcal{S}_j))) &\cong \mathrm{Hom}_{\mathrm{D^b}(\Lambda)}(\mathcal{S}_i, \Phi_i(\mathcal{S}_j)) \\
&\cong \mathrm{Hom}_{\mathrm{D^b}(\Lambda)}(\Phi_i^{-1}(\mathcal{S}_i), \mathcal{S}_j) \\
&\cong \mathrm{Hom}_{\mathrm{D^b}(\Lambda)}(\mathrm{Tor}_1^{\nu_i\Lambda}(\mathcal{S}_i, T_i)[1], \mathcal{S}_j) \qquad \text{(by [HW, (4.B)])} \\
&= \mathrm{Ext}_\Lambda^{-1}(\mathrm{Tor}_1^{\nu_i\Lambda}(\mathcal{S}_i, T_i), \mathcal{S}_j) = 0.
\end{aligned}$$

Combining, it follows that $\Phi_i(\mathcal{S}_j) \in \mathrm{L}_i(\mathcal{A}_{\nu_i N})$. $\square$

5.4. **t-structure transfer.** In moving to the mutation functors, which reveals many hidden t-structures, we lose control over Fourier–Mukai techniques. The following theorem is one of our main results, and is a crucial ingredient in the proof of Theorem 6.9 later.

Theorem 5.6. *Let* $L \in \mathrm{Mut}(N)$, $\alpha \in \mathrm{Hom}_{\mathbb{G}^{\mathsf{aff}}}(C_L, C_L)$, *and consider*

$$\Phi_\alpha \colon \mathrm{D^b}(\mathrm{mod}\, \Lambda_L) \to \mathrm{D^b}(\mathrm{mod}\, \Lambda_L).$$

Then the following conditions are equivalent.

(1) Φ_α *maps the simples* $\mathcal{S}_0, \dots, \mathcal{S}_n$ *to simples.*
(2) $\Phi_\alpha \colon \mathcal{D}_L \to \mathcal{D}_L$ *restricts to an equivalence* $\mathcal{B}_L \to \mathcal{B}_L$.
(3) $\Phi_\alpha \colon \mathrm{D^b}(\mathrm{mod}\, \Lambda_L) \to \mathrm{D^b}(\mathrm{mod}\, \Lambda_L)$ *restricts to an equivalence* $\mathrm{mod}\, \Lambda_L \to \mathrm{mod}\, \Lambda_L$.
(4) *There is a functorial isomorphism* $\Phi_\alpha \cong \mathrm{Id}$.

If further $L \in \mathrm{Mut}_0(N)$ *and* $\alpha \in \mathrm{Hom}_{\mathbb{G}}(C_L, C_L)$, *the above are equivalent to*

(5) Φ_α *maps the simples* $\mathcal{S}_1, \dots, \mathcal{S}_n$ *to simples.*
(6) Φ_α *restricts to an equivalence* $\mathcal{A}_L \to \mathcal{A}_L$.

When the last additional conditions are satisfied, it is already known that (5)⇒(2) by [HW, 5.5]. Furthermore, it is clear that (1)⇔(2), (5)⇔(6) and (4)⇒(1)(5). Hence to prove Theorem 5.6 it suffices to show that (1)⇒(3) and (3)⇒(4).

Lemma 5.7. *Let* Γ *be a noetherian ring, and* $x \in \mathrm{D^b}(\mathrm{mod}\, \Gamma)$. *Then the following hold.*

(1) $x \in \mathrm{D^b}(\mathrm{mod}\, \Gamma)^{\leq 0} \iff \mathrm{Ext}^i_\Gamma(x, S) = 0$ *for all* $i < 0$ *and all simple* Γ*-modules* S.
(2) *If a triangulated equivalence* $F\colon \mathrm{D^b}(\mathrm{mod}\, \Gamma) \to \mathrm{D^b}(\mathrm{mod}\, \Gamma)$ *satisfies* $F(\Gamma) \cong \Gamma$, *then* F *restricts to an equivalence* $F\colon \mathrm{mod}\, \Gamma \to \mathrm{mod}\, \Gamma$.

Proof. (1) The direction ($\Rightarrow$) is clear, by replacing x by its projective resolution P, and observing that $\mathrm{Ext}^i_\Gamma(x,S)=\mathrm{Hom}_{\mathrm{K}^-(\mathrm{mod}\,\Gamma)}(P,S[i])=0$ for all $i<0$, since there are no chain maps between the complexes P and $S[i]$.

For ($\Leftarrow$), since x is bounded, let t be maximum such that $\mathrm{H}^t(x)\neq 0$. Since Γ is noetherian, every finitely generated module has a map to a simple, so there exists some simple S such that $\mathrm{Hom}_\Gamma(\mathrm{H}^t(x),S)\neq 0$. But via the spectral sequence (see e.g. [H2, (2.8)])

$$E_2^{p,q}=\mathrm{Ext}^p_\Gamma(\mathrm{H}^{-q}(x),S)\Rightarrow \mathrm{Ext}^{p+q}_\Gamma(x,S)$$

the nonzero $E_2^{0,-t}$ term survives to give a non-zero element of $\mathrm{Ext}^{-t}_\Gamma(x,S)$. Hence $t\leq 0$.
(2) Via the isomorphism $\mathrm{H}^n(x)\cong\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Gamma)}(\Gamma,x[n])$, it follows that F and its inverse take modules to modules, and so they restrict to a Morita equivalence. $\square$

The following establishes (1)$\Rightarrow$(3). In fact, we prove a slightly more general version, as we will need this later.

Corollary 5.8. *Suppose that $G\colon \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda_L)\to\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda_L)$ is any equivalence that maps the simples $\mathcal{S}_0,\dots,\mathcal{S}_n$ to simples. Then G restricts to an equivalence $\mathrm{mod}\,\Lambda_L\to\mathrm{mod}\,\Lambda_L$.*

Proof. To ease notation, set $\mathcal{E}=\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda_L)$. By Lemma 5.7(1) it follows that both G and its inverse restrict to an equivalence $\mathcal{E}^{\leq 0}\to\mathcal{E}^{\leq 0}$. Since $\mathcal{E}^{\geq 1}$ can be characterised as the perpendicular to $\mathcal{E}^{\leq 0}$, it follows that both G and its inverse restrict to an equivalence $\mathcal{E}^{\geq 1}\to\mathcal{E}^{\geq 1}$. Since $\mathrm{mod}\,\Lambda_L=\mathcal{E}^{\leq 0}\cap\mathcal{E}^{\geq 1}[1]$, the result follows. $\square$

The implication (3)$\Rightarrow$(4) is by far the most subtle. It requires the following two technical results, both of which rely heavily on the fact that R is isolated cDV.

Proposition 5.9. *In the setting of Theorem 5.6, if Φ_α restricts to $\mathrm{mod}\,\Lambda_L\xrightarrow{\sim}\mathrm{mod}\,\Lambda_L$, then $\Phi_i^{-1}\circ\Phi_\alpha\circ\Phi_i$ restricts to $\mathrm{mod}\,\Lambda_{\nu_iL}\xrightarrow{\sim}\mathrm{mod}\,\Lambda_{\nu_iL}$.*

Proof. Consider the functor $G=\Phi_i^{-1}\circ\Phi_\alpha\circ\Phi_i$. For each $j\neq i$, as in Lemma 3.2 we have $\Phi_i(\mathcal{P}_j)\cong\mathcal{P}_j$. Since Φ_α restricts to an equivalence on $\mathrm{mod}\,\Lambda_L$, and is necessarily the identity on $\mathcal{K}_L$ by Proposition 4.8, furthermore $\Phi_\alpha(\mathcal{P}_j)\cong\mathcal{P}_j$. In conclusion, whenever $j\neq i$ we have $G(\mathcal{P}_j)\cong\mathcal{P}_j$. In a similar vein, by [W1, 4.15(2)], $\Phi_i(\mathcal{S}_i)\cong\mathcal{S}_i[-1]$. The functor Φ_α must send simples to simples, and since it is the identity on $\mathcal{K}_L$, by the pairing between projectives and simples it follows that $\Phi_\alpha(\mathcal{S}_i)\cong\mathcal{S}_i$. Thus $G(\mathcal{S}_i)\cong\mathcal{S}_i$.

By the assumptions, since Λ_L is basic, necessarily $\Phi_\alpha(\Lambda_L)\cong\Lambda_L$. Now consider $\mathcal{T}=\mathrm{Hom}_R(\nu_iL,L)$ defining the functor $\Phi_i\colon\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda_{\nu_iL})\to\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda_L)$. By construction $G(\mathcal{T})\cong\Phi_i^{-1}\circ\Phi_\alpha(\Lambda_L)\cong\Phi_i^{-1}(\Lambda_L)\cong\mathcal{T}$. Since $\mathcal{T}=\mathcal{T}_i\oplus\bigoplus_{j\neq i}\mathcal{P}_j$, by the above paragraph necessarily $G(\mathcal{T}_i)\cong\mathcal{T}_i$.

Now by e.g. [W1, A.2(2)] the exchange sequences give rise to an exact sequence of Λ_{ν_iL}-modules

$$0\to\mathcal{P}_i\xrightarrow{b}\bigoplus_{j\neq i}\mathcal{P}_j^{\oplus a_{ij}}\to\bigoplus_{j\neq i}\mathcal{P}_j^{\oplus a_{ij}}\to\mathcal{P}_i\to\frac{\Lambda_{\nu_iL}}{(1-e_i)}\to 0$$

with $\mathrm{Cok}\,b\cong\mathcal{T}_i$, where e_i is the idempotent corresponding to the ith summand of Λ_{ν_iL}, and $(1-e_i)$ is the two-sided ideal generated by $1-e_i$. Since R is isolated, necessarily $\Lambda_{\nu_iL}/(1-e_i)$ has finite length, and is filtered only by the simple $\mathcal{S}_i$. Splicing gives triangles

$$\mathcal{P}_i\to\bigoplus_{j\neq i}\mathcal{P}_j^{\oplus a_{ij}}\to\mathcal{T}_i\to$$
$$\mathcal{T}_i\to\bigoplus_{j\neq i}\mathcal{P}_j^{\oplus a_{ij}}\to K_i\to$$
$$K_i\to\mathcal{P}_i\to\frac{\Lambda_{\nu_iL}}{(1-e_i)}\to$$

Applying G to each, the first triangle shows that $\mathrm{H}^t(G(\mathcal{P}_i))=0$ unless $t=0,1$. On the other hand, the second triangle shows that $\mathrm{H}^t(G(K_i))=0$ unless $t=-1,0$. But G must take $\Lambda_{\nu_iL}/(1-e_i)$ to degree zero, since $G(\mathcal{S}_i)\cong\mathcal{S}_i$ and $\Lambda_{\nu_iL}/(1-e_i)$ is filtered by $\mathcal{S}_i$. Hence the last triangle implies that $\mathrm{H}^t(G(\mathcal{P}_i))=0$ unless $t=-1,0$.

Combining, we see that $\mathrm{H}^t(G(\mathcal{P}_i)) = 0$ unless $t = 0$, thus $G(\mathcal{P}_i)$ is a module. Applying G to the first triangle give a triangle

$$G(\mathcal{P}_i) \to \bigoplus_{j \neq i} \mathcal{P}_j^{\oplus a_{ij}} \to \mathcal{T}_i \to$$

in which all terms are modules, so this is necessarily induced by a short exact sequence. Hence by the depth lemma, $G(\mathcal{P}_i)$ has depth 3. On the other hand, since $\mathcal{P}_i$ is perfect as a complex, so is $G(\mathcal{P}_i)$, thus $G(\mathcal{P}_i)$ has finite projective dimension as a $\Lambda_{\mathsf{v}_i L}$-module. By Auslander–Buchsbaum [IW1, 2.16], it follows that $G(\mathcal{P}_i)$ is projective. Since G is the identity on $\mathcal{K}_{\mathsf{v}_i L}$, necessarily $G(\mathcal{P}_i) \cong \mathcal{P}_i$ and hence $G(\Lambda_{\mathsf{v}_i L}) \cong \Lambda_{\mathsf{v}_i L}$. By Lemma 5.7(2), $G = \Phi_i^{-1} \circ \Phi_\alpha \circ \Phi_i$ restricts to an equivalence $\operatorname{mod} \Lambda_{\mathsf{v}_i L} \to \operatorname{mod} \Lambda_{\mathsf{v}_i L}$. □

Proposition 5.10. *Suppose that $G\colon \mathrm{D^b}(\operatorname{mod} \Lambda) \to \mathrm{D^b}(\operatorname{mod} \Lambda)$ is an R-linear equivalence such that $G(\mathcal{S}_0) \cong \mathcal{S}_0$, and suppose that there exists a permutation $\iota \in S_n$ such that $G(\mathcal{S}_i) \cong \mathcal{S}_{\iota(i)}$ and $\mathsf{rk}_R N_i \cong \mathsf{rk}_R N_{\iota(i)}$ for all $i = 1, \dots, n$. Then functorially $G \cong \mathrm{Id}$.*

Proof. By Corollary 5.8, G maps projectives to projectives. Since $G(\mathcal{S}_0) \cong \mathcal{S}_0$, by the pairing between simples and projectives, necessarily $G(\mathcal{P}_0) \cong \mathcal{P}_0$. Consider the R-linear composition F given by

$$\begin{tikzcd} \mathrm{D^b}(\operatorname{coh} X) \arrow[r, dotted, "F"] \arrow[d, "\mathbf{R}\mathrm{Hom}_X(\mathcal{V}_X, -)"'] & \mathrm{D^b}(\operatorname{coh} X) \\ \mathrm{D^b}(\operatorname{mod} \Lambda) \arrow[r, "G"] & \mathrm{D^b}(\operatorname{mod} \Lambda) \arrow[u, "- \otimes^{\mathbf{L}}_\Lambda \mathcal{V}_X"'] \end{tikzcd}$$

By [K1, 5.2.4] the functor $\mathbf{R}\mathrm{Hom}_X(\mathcal{V}_X, -)$ maps skyscrapers of closed points to modules of dimension vector $\mathsf{rk}_N = (\mathsf{rk}_R N_i)_{i=0}^n$ which satisfy the $\star$-generated condition, which by definition consists of those Λ-modules A of dimension vector rk_N such that $\mathrm{Hom}_\Lambda(A, \mathcal{S}_i) = 0$ for all $i = 1, \dots, n$ [SY, 6.11]. For any such A, by the assumptions on where G takes each simple, GA has dimension vector $(\mathsf{rk}_R N_{\iota(i)})_{i=0}^n$, which by the last assumption is precisely rk_N. Furthermore, for any such A, since G fixes $\mathcal{S}_0$ and permutes the other simples, GA is also $\star$-generated. Hence again appealing to [K1, 5.2.4] the functor $- \otimes^{\mathbf{L}}_\Lambda \mathcal{V}_X$ takes the module GA to a skyscraper. Combining, we see that skyscrapers of closed points get sent to skyscrapers of closed points, under the above R-linear composition F.

It follows from general Fourier–Mukai theory [BM2, §3.3] that $F \cong \varphi_* \circ (- \otimes \mathcal{L})$ where $\varphi\colon X \to X$ is an automorphism and $\mathcal{L}$ is some line bundle. Since $\mathbf{R}\mathrm{Hom}_X(\mathcal{V}_X, -)$ sends $\mathcal{O}_X$ to $\mathcal{P}_0$, and G sends $\mathcal{P}_0$ to $\mathcal{P}_0$, it follows that F sends $\mathcal{O}_X$ to $\mathcal{O}_X$, which in turn implies that $\mathcal{L}$ is trivial. Lastly, since $F \cong \varphi_*$ is R-linear, by Proposition 2.4 φ commutes with the map to the base. In particular, the restriction of φ to the dense open subset $U = X \backslash C$ is the identity. Hence $\varphi = \mathrm{Id}_X$, and as a result, $F \cong \mathrm{Id}$. From this, it follows that $G \cong \mathrm{Id}$. □

Finally, we prove (3)⇒(4), completing the proof of Theorem 5.6. The key is that Proposition 5.9 allows us to pull everything back to $\mathrm{D^b}(\operatorname{mod} \Lambda)$, where we can use the geometric Fourier–Mukai techniques of Proposition 5.10.

Corollary 5.11. *In the setting of Theorem 5.6, if Φ_α restricts to $\operatorname{mod} \Lambda_L \xrightarrow{\sim} \operatorname{mod} \Lambda_L$, then there is a functorial isomorphism $\Phi_\alpha \cong \mathrm{Id}$.*

Proof. Choose a positive path $\gamma\colon C_+ \to C_L$, and consider the composition

$$G = \Phi_\gamma^{-1} \circ \Phi_\alpha \circ \Phi_\gamma\colon \mathrm{D^b}(\operatorname{mod} \Lambda) \to \mathrm{D^b}(\operatorname{mod} \Lambda)$$

Since γ is a composition $s_{i_t} \circ \dots \circ s_{i_1}$, we may rewrite the above as

$$G = \Phi_{i_1}^{-1} \circ \dots \circ \Phi_{i_{t-1}}^{-1} \circ (\Phi_{i_t}^{-1} \circ \Phi_\alpha \circ \Phi_{i_t}) \circ \Phi_{i_{t-1}} \circ \dots \circ \Phi_{i_1}$$

By induction, using Proposition 5.9 repeatedly, we see that G restricts to an equivalence on $\operatorname{mod} \Lambda$. Since Morita equivalences preserve projectives, and G is the identity on K-theory $\mathcal{K} = \mathcal{K}_N$ by Proposition 4.8, G maps each projective to itself. Since Morita equivalences also preserve simples, by the pairing between projectives and simples, G maps each simple to itself. By Proposition 5.10 $G \cong \mathrm{Id}$ and hence $\Phi_\alpha \cong \mathrm{Id}$. □

6. Stability Conditions on $\mathcal{C}$ and $\mathcal{D}$

Consider $\mathrm{Stab}^\circ\mathcal{C}$, the connected component of $\mathrm{Stab}\,\mathcal{C}$ containing $\mathrm{Stab}\,\mathcal{A}$, and similarly $\mathrm{Stab}_n^\circ\mathcal{D}$, the connected component of $\mathrm{Stab}_n\mathcal{D}$ containing $\mathrm{Stab}_n\mathcal{B}$. In this section we describe both $\mathrm{Stab}^\circ\mathcal{C}$ and $\mathrm{Stab}_n^\circ\mathcal{D}$ as regular covers of the hyperplane arrangements in Section 3.

6.1. **Chamber Decomposition.** For the fixed R-module N from (2.C), consider the set of morphisms in $\mathbb{G}$ which terminate at C_+, namely

$$\mathsf{Term}_0(C_+) := \bigcup_{L\in\mathrm{Mut}_0(N)} \mathrm{Hom}_{\mathbb{G}}(C_L, C_+).$$

The set $\mathsf{Term}(C_+)$ is defined similarly, taking the union instead over $L \in \mathrm{Mut}(N)$ and replacing $\mathbb{G}$ by $\mathbb{G}^{\mathsf{aff}}$.

Notation 6.1. *For $L \in \mathrm{Mut}_0(N)$, respectively $L \in \mathrm{Mut}(N)$, consider the open subsets*

$$\begin{aligned}
\mathrm{U}_L &:= \{(Z,\mathcal{A}_L) \in \mathrm{Stab}\,\mathcal{A}_L \mid \mathrm{Im}(Z[\mathcal{S}_i]) > 0 \text{ for all } i = 1,\dots,n\}\\
\mathbb{U}_L &:= \{(Z,\mathcal{B}_L) \in \mathrm{Stab}\,\mathcal{B}_L \mid \mathrm{Im}(Z[\mathcal{S}_i]) > 0 \text{ for all } i = 0,1,\dots,n\}\\
\mathbb{N}_L &:= \mathbb{U}_L \cap \mathrm{Stab}_n\mathcal{B}_L = \mathbb{U}_L \cap \mathrm{Stab}_n\mathcal{D}_L
\end{aligned}$$

of $\mathrm{Stab}^\circ\mathcal{C}_L$, $\mathrm{Stab}^\circ\mathcal{D}_L$ and $\mathrm{Stab}_n^\circ\mathcal{D}_L$ respectively. For $\alpha \in \mathsf{Term}_0(C_+)$, $\beta \in \mathsf{Term}(C_+)$, set

$$\begin{aligned}
\mathrm{Stab}\,\mathcal{A}_\alpha &:= (\Phi_\alpha)_*(\mathrm{Stab}\,\mathcal{A}_{s(\alpha)}) & \mathrm{Stab}\,\mathcal{B}_\beta &:= (\Phi_\beta)_*(\mathrm{Stab}\,\mathcal{B}_{s(\beta)})\\
\mathrm{U}_\alpha &:= (\Phi_\alpha)_*(\mathrm{U}_{s(\alpha)}) & \mathbb{U}_\beta &:= (\Phi_\beta)_*(\mathbb{U}_{s(\beta)}),
\end{aligned}$$

where $s(\alpha)$ and $s(\beta)$ denote the modules corresponding to the chambers which are the sources of α and β respectively. Similarly,

$$\begin{aligned}
\mathrm{Stab}_n\mathcal{B}_\beta &:= (\Phi_\beta)_*(\mathrm{Stab}_n\mathcal{B}_{s(\beta)})\\
\mathbb{N}_\beta &:= (\Phi_\beta)_*(\mathbb{N}_{s(\beta)}).
\end{aligned}$$

As usual, write $\mathrm{U} = \mathrm{U}_N$, $\mathbb{U} = \mathbb{U}_N$ and $\mathbb{N} = \mathbb{N}_N$.

Lemma 6.2. *Given $\alpha,\beta \in \mathsf{Term}_0(C_+)$, respectively $\alpha,\beta \in \mathsf{Term}(C_+)$, write $M_{s(\alpha)}$ and $M_{s(\beta)}$ for the modules corresponding to the chambers which are the sources of α and β respectively. Then*

1. $\mathrm{U}_\alpha \cap \mathrm{U}_\beta \neq \emptyset \iff M_{s(\alpha)} \cong M_{s(\beta)}$ *and* $\Phi_\alpha \cong \Phi_\beta$.
2. $\mathbb{U}_\alpha \cap \mathbb{U}_\beta \neq \emptyset \iff M_{s(\alpha)} \cong M_{s(\beta)}$ *and* $\Phi_\alpha \cong \Phi_\beta \iff \mathbb{N}_\alpha \cap \mathbb{N}_\beta \neq \emptyset$.

Proof. (1) It suffices to show that, for $\gamma \in \mathsf{Term}_0(C_+)$, $\mathrm{U} \cap \mathrm{U}_\gamma \neq \emptyset$ if and only if $M_{s(\gamma)} \cong N$ and $\Phi_\gamma \cong \mathrm{id}_{\mathcal{C}}$. The implication $(\Leftarrow)$ is obvious, since the isomorphisms $M_{s(\gamma)} \cong N$ and $\Phi_\gamma \cong \mathrm{id}_{\mathcal{C}}$ implies that $\mathrm{U} = \mathrm{U}_\gamma$.

Conversely, suppose that $\mathrm{U} \cap \mathrm{U}_\gamma \neq \emptyset$, and write $\mathcal{Y}\colon \mathrm{Stab}\,\mathcal{C} \to \Theta_{\mathbb{R}}$ for the composition

$$\mathrm{Stab}\,\mathcal{C} \xrightarrow{\mathcal{Z}} \Theta_{\mathbb{C}} \xrightarrow{\mathrm{Im}} \Theta_{\mathbb{R}}$$

where the last map is the projection defined by taking the imaginary parts. By definition $\mathcal{Y}(\mathrm{U}) = C_+$ and $\mathcal{Y}(\mathrm{U}_\gamma) = \varphi_{M_{s(\gamma)}}(C_+)$. Since $\mathrm{U}\cap\mathrm{U}_\gamma \neq \emptyset$, necessarily $\mathcal{Y}(\mathrm{U})\cap\mathcal{Y}(\mathrm{U}_\gamma) \neq \emptyset$, thus $M_{s(\gamma)} \cong N$ by Theorem 3.6, and Φ_γ is an autoequivalence of $\mathcal{C}$. It is clear that $\mathrm{U} \cap \mathrm{U}_\gamma \neq \emptyset$ implies that $\Phi_\gamma\colon \mathcal{C} \to \mathcal{C}$ maps $\mathcal{A}$ to $\mathcal{A}$. By Theorem 5.6, this implies that $\Phi_\gamma \cong \mathrm{Id}$.
(2) The proof of the first $\iff$ is identical, appealing to Theorem 3.3 instead of Theorem 3.6 to deduce that Φ_γ maps $\mathcal{B}$ to $\mathcal{B}$. Theorem 5.6 again implies that $\Phi_\gamma \cong \mathrm{Id}$. The second $\iff$ follows immediately from the first. $\square$

Lemma 6.3. *For $\alpha,\beta \in \mathsf{Term}_0(C_+)$ with $l(\alpha) > l(\beta)$, the chambers $\mathrm{Stab}\,\mathcal{A}_\alpha$ and $\mathrm{Stab}\,\mathcal{A}_\beta$ share a codimension one boundary if and only if there exists a length one path $\gamma \in \mathrm{Mor}(\mathbb{G}^+)$ such that $\alpha = \beta \circ \gamma$ or $\alpha = \beta \circ \gamma^{-1}$ in $\mathrm{Mor}(\mathbb{G})$. A similar statement holds replacing $\mathcal{A}$ by $\mathcal{B}$, $\mathsf{Term}_0(C_+)$ by $\mathsf{Term}(C_+)$ and $\mathbb{G}$ by $\mathbb{G}^{\mathsf{aff}}$ respectively.*

Proof. It is enough to prove that, for $\gamma \in \mathsf{Term}_0(C_+)$, U and U_γ share a codimension one boundary if and only if there is a length one path δ such that $\gamma = \delta$ or $\gamma = \delta^{-1}$. This follows from [B5, 5.5] and Lemma 5.5. $\square$

The following is the analogue of [T, 4.11].

Theorem 6.4. *With notation as above, the following statements hold.*

(1) *There is a disjoint union of open chambers*

$$\mathcal{M} := \bigcup_{\alpha \in \mathsf{Term}_0(C_+)} \mathrm{U}_\alpha \quad \subset \operatorname{Stab}^\circ \mathcal{C}.$$

Furthermore, $\overline{\mathcal{M}} = \bigcup \overline{\mathrm{U}}_\alpha = \operatorname{Stab}^\circ \mathcal{C}$*, where* $\overline{\mathrm{U}}_\alpha$ *is the closure of* U_α *in* $\operatorname{Stab}^\circ \mathcal{C}$*.*

(2) *There is a disjoint union of open chambers*

$$\mathcal{N} := \bigcup_{\beta \in \mathsf{Term}(C_+)} \mathbb{U}_\beta \quad \subset \operatorname{Stab}^\circ \mathcal{D}.$$

Furthermore, $\overline{\mathcal{N}} = \bigcup \overline{\mathbb{U}}_\beta = \operatorname{Stab}^\circ \mathcal{D}$*.*

In particular, as $\operatorname{Stab}^\circ \mathcal{D} \cap \operatorname{Stab}_n \mathcal{D} = \operatorname{Stab}_n^\circ \mathcal{D}$*, there is a disjoint union of open chambers*

$$\mathcal{N}_n := \bigcup_{\beta \in \mathsf{Term}(C_+)} \mathbb{N}_\beta \quad \subset \operatorname{Stab}_n^\circ \mathcal{D}$$

such that $\overline{\mathcal{N}_n} = \bigcup \overline{\mathbb{N}}_\beta = \operatorname{Stab}_n^\circ \mathcal{D}$*.*

Proof. (1) By Lemma 6.2, $\mathcal{M}$ is a disjoint union, and by Lemma 6.3 $\overline{\mathcal{M}}$ is connected. Since $\overline{\mathcal{M}}$ contains $\overline{\mathrm{U}}$ and thus $\operatorname{Stab} \mathcal{A}$, there is an inclusion $\overline{\mathcal{M}} \subseteq \operatorname{Stab}^\circ \mathcal{C}$.

Let $\sigma \in \operatorname{Stab}^\circ \mathcal{C}$ be a point, and choose a point $\sigma_0 \in \mathrm{U}$ and a path

$$p \colon [0,1] \to \operatorname{Stab}^\circ \mathcal{C}$$

such that $p(0) = \sigma_0$ and $p(1) = \sigma$. Since $\mathcal{Z} \colon \operatorname{Stab}^\circ \mathcal{C} \to \Theta_{\mathbb{C}}$ is a local homeomorphism, by deforming p if necessary, we may assume that the path $\mathcal{Z} \circ p \colon [0,1] \to \Theta_{\mathbb{C}}$ passes through only finitely many codimension one boundaries of chambers $\varphi_L(\mathbb{H}_+)$. Thus there exists a sequence $0 < t_1 < t_2 < \ldots < t_{\ell-1} < t_\ell := 1$ of real numbers such that:

(a) for all $i \neq \ell$, every $\mathcal{Z}(p(t_i))$ is in a codimension one boundary of some chamber,
(b) for all i, each open interval $\mathcal{Z}(p(t_i, t_{i+1}))$ is contained in the interior of some chamber.

Since $p((0, t_1)) \subset \mathrm{U}$, by Lemma 6.3 there is a length one path $\gamma \in \mathsf{Term}_0(C_+)$ such that $p(t_1, t_2)$ is in U_γ. By iterating this argument, we see that $p((t_{l-1}, 1))$ is in some open chamber U_α, and hence its end point, σ, belongs to $\overline{\mathrm{U}}_\alpha$.
(2) This follows using an identical argument to (1).

For the last statements, we first prove that $\operatorname{Stab}^\circ \mathcal{D} \cap \operatorname{Stab}_n \mathcal{D} = \operatorname{Stab}_n^\circ \mathcal{D}$. Let $\sigma \in \operatorname{Stab}_n^\circ \mathcal{D}$. Since $\operatorname{Stab}_n^\circ \mathcal{D}$ is a connected and locally Euclidian space, it is path connected. Hence there is a path from σ to a point $\sigma_0 \in \operatorname{Stab}_n \mathcal{B} \subset \operatorname{Stab}^\circ \mathcal{D}$. Thus σ also lies in $\operatorname{Stab}^\circ \mathcal{D}$, proving $\operatorname{Stab}_n^\circ \mathcal{D} \subseteq \operatorname{Stab}^\circ \mathcal{D} \cap \operatorname{Stab}_n \mathcal{D}$.

For the opposite inclusion, it is enough to show that $\operatorname{Stab}^\circ \mathcal{D} \cap \operatorname{Stab}_n \mathcal{D}$ is connected, since $\operatorname{Stab}^\circ \mathcal{D} \cap \operatorname{Stab}_n \mathcal{D}$ contains $\operatorname{Stab}_n \mathcal{B}$. But by (2),

$$\operatorname{Stab}^\circ \mathcal{D} \cap \operatorname{Stab}_n \mathcal{D} = \big(\bigcup \overline{\mathbb{U}}_\beta\big) \cap \operatorname{Stab}_n \mathcal{D} = \bigcup \overline{\mathbb{N}}_\beta.$$

Since $\overline{\mathbb{N}}_\beta$ is the closure of $\mathbb{N}_\beta$ in $\operatorname{Stab}_n \mathcal{D}$, we have $\overline{\mathbb{N}}_\beta = \overline{\mathbb{U}}_\beta \cap \operatorname{Stab}_n \mathcal{D}$, and by Lemma 3.10 and Proposition 5.4, all $\overline{\mathbb{N}}_\beta$ are path connected. Thus again by Proposition 5.4, it suffices to show that $\overline{\mathbb{N}} \cap \overline{\mathbb{N}}_\gamma = \overline{\mathbb{U}} \cap \overline{\mathbb{U}}_\gamma \cap \operatorname{Stab}_n \mathcal{D} \neq \emptyset$ for any length one path γ. If $\gamma = s_i$, then consider the point $\sigma = (Z, \mathcal{B}) \in \operatorname{Stab} \mathcal{B}$ defined by $Z[\mathcal{S}_i] = -1/\lambda_i$, and $Z[\mathcal{S}_j] = (1 + \mathrm{i})/n\lambda_j$ for all $j \neq i$, where $\lambda_k := \mathsf{rk}_R\, N_k$. Then σ lies in $\operatorname{Stab}_n \mathcal{B} \subset \overline{\mathbb{U}} \cap \operatorname{Stab}_n \mathcal{D}$ and in the codimension one boundary of $\operatorname{Stab} \mathrm{L}_i(\mathcal{B})$ by [B5, Lemma 5.5]. But since $\operatorname{Stab} \mathrm{L}_i(\mathcal{B}) = (\Phi_i)_*(\operatorname{Stab} \mathcal{B}_{\nu_i N}) = \operatorname{Stab} \mathcal{B}_\gamma$ by Lemma 5.5, $\sigma \in \overline{\operatorname{Stab} \mathcal{B}_\gamma} = \overline{\mathbb{U}}_\gamma$. This implies that $\overline{\mathbb{U}} \cap \overline{\mathbb{U}}_\gamma \cap \operatorname{Stab}_n \mathcal{D} \neq \emptyset$. Similarly, we see that $\overline{\mathbb{U}} \cap \overline{\mathbb{U}}_\gamma \cap \operatorname{Stab}_n \mathcal{D} \neq \emptyset$ when $\gamma = s_i^{-1}$. Hence $\operatorname{Stab}^\circ \mathcal{D} \cap \operatorname{Stab}_n \mathcal{D}$ is connected, and thus $\operatorname{Stab}^\circ \mathcal{D} \cap \operatorname{Stab}_n \mathcal{D} = \operatorname{Stab}_n^\circ \mathcal{D}$ follows. The remaining statements are then immediate from (2). ∎

6.2. **Regular Covering Structure.**

Lemma 6.5. *If $L \in \mathrm{Mut}_0(N)$, respectively $L \in \mathrm{Mut}(N)$, then the following statements hold.*

(1) *If a point $\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}^\circ \mathcal{C}_L$ is in $\overline{\mathrm{U}}_\alpha$ for some $\alpha \in \mathrm{End}_{\mathbb{G}}(C_+)$, then $\mathcal{Z}_L(\sigma)$ is not on any complexified coordinate axis in $(\Theta_L)_\mathbb{C}$.*
(2) *If a point $\sigma = (Z, \mathcal{B}) \in \mathrm{Stab}^\circ \mathcal{D}_L$ is in $\overline{\mathbb{U}}_\beta$ for some $\beta \in \mathrm{End}_{\mathbb{G}^{\mathsf{aff}}}(C_+)$, then $\mathcal{Z}_L(\sigma)$ is not on any complexified coordinate axis in $(\mathcal{K}_L)_\mathbb{C}$.*

Proof. (1) First, for any $\rho \in \mathrm{U}_1 = \mathrm{U}$, every simple module $\mathcal{S}_i \in \mathcal{A}_L$ is ρ-semistable. Hence, by [BS, 7.6], $\mathcal{S}_i$ is ρ-semistable for all $\rho \in \overline{\mathrm{U}}_1$, and in particular $Z[\mathcal{S}_i] \neq 0$ for all $(Z, \mathcal{A}) \in \overline{\mathrm{U}}_1$.

Now, for $\sigma \in \overline{\mathrm{U}}_\alpha$, by Proposition 5.4 and Proposition 4.8(2), $\mathcal{Z}_L(\sigma) = \mathcal{Z}_L((\Phi_\alpha)_*^{-1}(\sigma))$, and by definition $(\Phi_\alpha)_*^{-1}(\sigma) \in \overline{\mathrm{U}}_1$. Hence, by the first paragraph, $\mathcal{Z}_L(\sigma)$ is not on any complexified coordinate axis.
(2) is identical to (1). □

Lemma 6.6. *With notation as above, the following statements hold.*

(1) *The map $\mathcal{Z}\colon \mathrm{Stab}^\circ \mathcal{C} \to \Theta_\mathbb{C}$ restricts to a surjective map*
$$\mathcal{Z}\colon \mathrm{Stab}^\circ \mathcal{C} \to \Theta_\mathbb{C} \backslash \mathcal{H}_\mathbb{C}.$$
(2) *The map $\mathcal{Z}\colon \mathrm{Stab}_n^\circ \mathcal{D} \to \mathsf{Level}_\mathbb{C}$ restricts to a surjective map*
$$\mathcal{Z}\colon \mathrm{Stab}_n^\circ \mathcal{D} \to \mathsf{Level}_\mathbb{C} \backslash \mathcal{H}_\mathbb{C}^{\mathsf{aff}}.$$

Proof. (1) First, we show that $\mathrm{Im}(\mathcal{Z}) \subseteq \Theta_\mathbb{C} \backslash \mathcal{H}_\mathbb{C}$. By Theorem 6.4 and Proposition 5.4, it is enough to show that $\mathcal{Z}(\sigma) = x + \mathrm{i}y \in \Theta_\mathbb{C} \backslash \mathcal{H}_\mathbb{C}$ for each point $\sigma = (Z, \mathcal{A}) \in \overline{\mathrm{U}}$ in the closure of U. Assume that $\mathcal{Z}(\sigma) \in H_\mathbb{C}$ for some $H \in \mathcal{H}$.

Let $\mathcal{I} := \{1 \leq i \leq n \mid y_i = 0\}$ and $\mathrm{B} := \{\vartheta \in \Theta_\mathbb{R} \mid \vartheta_j = 0 \text{ for all } j \in \mathcal{I}\}$. If $\mathcal{I} = \emptyset$, then the point σ necessarily lies in U, and so $\mathcal{Z}(\sigma) \in \mathbb{H}_+ \subset \Theta \backslash \mathcal{H}_\mathbb{C}$. Hence we may assume $\mathcal{I} \neq \emptyset$. Since by Lemma A.2(1) the hyperplane H has the form $\lambda_1 \vartheta_{i_1} + \ldots + \lambda_s \vartheta_{i_s} = 0$ for some $\lambda_1, \ldots, \lambda_s > 0$, the fact that $y_j > 0$ if $j \notin \mathcal{I}$ implies $i_1, \ldots, i_s \in \mathcal{I}$. Since $\mathrm{B} \subseteq H$, by Lemma A.8 there exist $k \in \mathcal{I}$ and a minimal mutation sequence
$$\alpha\colon L \to \ldots \to N \in \mathsf{MutTo}_\mathcal{I}(N)$$
such that $H = \varphi_\alpha(H_k)$, where $H_k = \{\vartheta_k = 0\} \subset (\Theta_L)_\mathbb{R}$ is the kth coordinate axis in $(\Theta_L)_\mathbb{R}$. If we set $\sigma' := (\Phi_\alpha^{-1})_*(\sigma) \in \mathrm{Stab}^\circ \mathcal{C}_L$, then $\mathcal{Z}_L(\sigma') \in (H_k)_\mathbb{C}$. Since $y \in \overline{C}_+$ and $\alpha \in \mathsf{MutTo}_\mathcal{I}(N)$, we see that $\varphi_\alpha^{-1}(y) = y$ by the rule (3.B), and so $\mathrm{Im}(\mathcal{Z}_L(\sigma')) \in \overline{C}_+$. But this implies that $\sigma' \in \overline{\mathrm{U}}_\beta \subset \mathrm{Stab}^\circ \mathcal{C}_L$ for some $\beta \in \mathrm{End}_{\mathbb{G}}(C_+)$, which is a contradiction by Lemma 6.5(1). Hence $\mathrm{Im}(\mathcal{Z}) \subseteq \Theta_\mathbb{C} \backslash \mathcal{H}_\mathbb{C}$.

Next, we show the map $\mathcal{Z}\colon \mathrm{Stab}^\circ \mathcal{C} \to \Theta_\mathbb{C} \backslash \mathcal{H}_\mathbb{C}$ is surjective. Pick $z \in \Theta_\mathbb{C} \backslash \mathcal{H}_\mathbb{C}$, then by Proposition 3.8 there exists some $L \in \mathrm{Mut}_0(N)$ such that $\varphi_L(h) = z$ for some $h \in \mathbb{H}_+$. The left hand side of the commutative diagram in Proposition 5.4 shows that we can find $\sigma \in \mathrm{Stab}\, \mathcal{A}_L$ such that $\mathcal{Z}_L(\sigma) = h$. The commutativity then shows that $\sigma' := (\Phi_L)_*(\sigma) \in \mathrm{Stab}\, \mathcal{C}$ maps, via $\mathcal{Z}$, to z. Since $\sigma' \in \mathrm{Stab}^\circ \mathcal{C}$ by Theorem 6.4, it follows that $\mathcal{Z}$ is surjective.
(2) By Theorem 6.4 and Proposition 5.4, for $\mathrm{Im}(\mathcal{Z}) \subseteq \mathsf{Level}_\mathbb{C} \backslash \mathcal{H}_\mathbb{C}^{\mathsf{aff}}$, it suffices to prove that $\mathcal{Z}(\sigma) = x + \mathrm{i}y \in \mathsf{Level}_\mathbb{C} \backslash \mathcal{H}_\mathbb{C}^{\mathsf{aff}}$ for any $\sigma = (Z', \mathcal{B}) \in \overline{\mathbb{N}} = \overline{\mathbb{U}} \cap \mathrm{Stab}_n \mathcal{D}$. To see this, let $\mathcal{I}' := \{0 \leq i \leq n \mid y_i = 0\}$, and note that $\mathcal{I}' \subsetneq \{0, \ldots, n\}$ since $x + y\mathrm{i} \in \mathsf{Level}_\mathbb{C}$. If $\mathcal{Z}(\sigma) \in \mathcal{H}_\mathbb{C}^{\mathsf{aff}}$, then by a similar argument to (1), now using Lemma A.2(2) and Lemma A.8 (with $\mathcal{I}'$), there exists a minimal mutation sequence $\alpha\colon L \to N \in \mathsf{MutTo}_{\mathcal{I}'}(N)$ such that $\sigma' := (\Phi_\alpha^{-1})_*(\sigma)$ lies in $\mathbb{U}_\beta$ for some $\beta \in \mathrm{End}_{\mathbb{G}^{\mathsf{aff}}}(C_+)$ and $\mathcal{Z}_L(\sigma') \in (H_k)_\mathbb{C} \cap (\mathsf{Level}_L)_\mathbb{C}$ for some coordinate axis H_k in $(\mathcal{K}_L)_\mathbb{R}$. This contradicts Lemma 6.5(2), and thus $\mathrm{Im}(\mathcal{Z}) \subseteq \mathsf{Level}_\mathbb{C} \backslash \mathcal{H}_\mathbb{C}^{\mathsf{aff}}$. The surjectivity of the map follows by a similar argument to (1), using Proposition 3.11, Proposition 5.4 and Theorem 6.4. □

Notation 6.7. *Consider the subgroups of* $\mathrm{Auteq}\, \mathcal{C}$ *and* $\mathrm{Auteq}\, \mathcal{D}$ *defined by*
$$\mathsf{PBr}\, \mathcal{C} := \{\Phi_\alpha|_\mathcal{C} \mid \alpha \in \mathrm{End}_{\mathbb{G}}(C_+)\},$$
$$\mathsf{PBr}\, \mathcal{D} := \{\Phi_\beta|_\mathcal{D} \mid \beta \in \mathrm{End}_{\mathbb{G}^{\mathsf{aff}}}(C_+)\}.$$

Theorem 6.8. *With notation as above, the following statements hold.*

(1) *The surjective map* $\mathcal{Z}\colon \mathrm{Stab}^{\circ}\mathcal{C} \to \Theta_{\mathbb{C}}\backslash\mathcal{H}_{\mathbb{C}}$ *induces a homeomorphism*

$$\mathrm{Stab}^{\circ}\mathcal{C}/\,\mathsf{PBr}\,\mathcal{C} \xrightarrow{\sim} \Theta_{\mathbb{C}}\backslash\mathcal{H}_{\mathbb{C}}.$$

(2) *The surjective map* $\mathcal{Z}\colon \mathrm{Stab}^{\circ}_{n}\mathcal{D} \to \mathsf{Level}_{\mathbb{C}}\backslash\mathcal{H}^{\mathsf{aff}}_{\mathbb{C}}$ *induces a homeomorphism*

$$\mathrm{Stab}^{\circ}_{n}\mathcal{D}/\,\mathsf{PBr}\,\mathcal{D} \xrightarrow{\sim} \mathsf{Level}_{\mathbb{C}}\backslash\mathcal{H}^{\mathsf{aff}}_{\mathbb{C}}.$$

Proof. We only prove (2), since the proof of (1) is identical. Let $\sigma \in \mathrm{Stab}^{\circ}_{n}\mathcal{D}$ and $\Phi \in \mathsf{PBr}\,\mathcal{D}$. Then $\mathcal{Z}(\Phi_*(\sigma)) = \mathcal{Z}(\sigma)$ by Proposition 4.8(2) and Proposition 5.4, and so $\mathcal{Z}$ induces a map $\mathrm{Stab}^{\circ}_{n}\mathcal{D}/\,\mathsf{PBr}\,\mathcal{D} \to \mathsf{Level}_{\mathbb{C}}\backslash\mathcal{H}^{\mathsf{aff}}_{\mathbb{C}}$ which is surjective by Lemma 6.6.

We show that this induced map is injective. Let $\sigma, \sigma' \in \mathrm{Stab}^{\circ}_{n}\mathcal{D}$ be two points such that $x := \mathcal{Z}(\sigma) = \mathcal{Z}(\sigma')$. By Theorem 6.4, $\sigma \in \overline{\mathbb{N}}_{\beta}$ and $\sigma' \in \overline{\mathbb{N}}_{\beta'}$ for some paths $\beta, \beta' \in \mathsf{Term}(C_+)$. But by Proposition 3.11, there is a unique $L \in \mathrm{Mut}(N)$ such that $x \in \phi_L(\mathbb{E}_+)$, and so by Proposition 5.4 we see that $\beta, \beta' \in \mathrm{Hom}_{\mathbb{G}^{\mathsf{aff}}}(C_L, C_+)$. Set $\gamma := \beta' \circ \beta^{-1} \in \mathrm{End}_{\mathbb{G}^{\mathsf{aff}}}(C_+)$, then by definition $(\Phi_\gamma)_*(\mathbb{N}_\beta) = \mathbb{N}_{\beta'}$.

Since the surjective map $\mathcal{Z}$ is a local homeomorphism, there exists an open neighbourhood $\mathcal{U}$ of σ such that the restrictions $\mathcal{Z}|_{\mathcal{U}}$ and $\mathcal{Z}|_{(\Phi_\gamma)_*(\mathcal{U})}$ are homeomorphisms. Choose a sequence $\{\sigma_i\}_{i=1}^{\infty} \subset \mathbb{N}_\beta \cap \mathcal{U}$ that converges to σ, and set $\sigma'_i := (\Phi_\gamma)_*(\sigma_i) \in \mathbb{N}_{\beta'} \cap (\Phi_\gamma)_*(\mathcal{U})$. Then again by Proposition 4.8(2) and Proposition 5.4, we have $x_i := \mathcal{Z}(\sigma_i) = \mathcal{Z}(\sigma'_i)$. The sequence $\{x_i\}_{i=1}^{\infty}$ converges to x since $\mathcal{Z}|_{\mathcal{U}}$ is a homeomorphism. Moreover, since $\mathcal{Z}|_{(\Phi_\beta)_*(\mathcal{U})}$ is also a homeomorphism, the sequence $\{\sigma'_i\}_{i=1}^{\infty}$ converges to σ'. Hence

$$\sigma' = \lim_{i\to\infty} \sigma'_i = \lim_{i\to\infty} (\Phi_\gamma)_*(\sigma_i) = (\Phi_\gamma)_*(\lim_{i\to\infty} \sigma_i) = (\Phi_\gamma)_*(\sigma).$$

This implies that $\sigma = \sigma'$ in $\mathrm{Stab}^{\circ}_{n}\mathcal{D}/\,\mathsf{PBr}\,\mathcal{D}$. $\square$

Given a group G acting on a topological space T, consider the following condition.

($*$) For each $x \in T$, there is an open neighbourhood $\mathcal{U}$ of x such that $\mathcal{U} \cap g\mathcal{U} = \emptyset$ for all $1 \neq g \in G$.

Theorem 6.9. *With notation as above, the following statements hold.*

(1) $\mathcal{Z}\colon \mathrm{Stab}^{\circ}\mathcal{C} \to \Theta_{\mathbb{C}}\backslash\mathcal{H}_{\mathbb{C}}$ *is a regular covering map, with Galois group* $\mathsf{PBr}\,\mathcal{C}$.

(2) $\mathcal{Z}\colon \mathrm{Stab}^{\circ}_{n}\mathcal{D} \to \mathsf{Level}_{\mathbb{C}}\backslash\mathcal{H}^{\mathsf{aff}}_{\mathbb{C}}$ *is a regular covering map, with Galois group* $\mathsf{PBr}\,\mathcal{D}$.

Moreover, the covering map in (1) is universal.

Proof. (2) We first show that the action of $\mathsf{PBr}\,\mathcal{D}$ on $\mathrm{Stab}^{\circ}_{n}\mathcal{D}$ satisfies the condition ($*$). For this, take a point $\sigma \in \mathrm{Stab}^{\circ}_{n}\mathcal{D}$ and consider the open neighbourhood $\sigma \in \mathcal{U}$ defined by

$$\mathcal{U} := \{\sigma' \in \mathrm{Stab}\,\mathcal{D} \mid d(\sigma, \sigma') < 1/4\} \cap \mathrm{Stab}^{\circ}_{n}\mathcal{D},$$

where $d(-,-)$ is the metric introduced in [B4, §6].

Assume that $\mathcal{U} \cap (\Phi_\beta)_*(\mathcal{U}) \neq \emptyset$ for some $\Phi_\beta \in \mathsf{PBr}\,\mathcal{D}$. Then, every point $\sigma' \in \mathcal{U}$ must satisfy $d\big(\sigma', (\Phi_\beta)_*(\sigma')\big) < 1$. Furthermore, the central charges of σ' and $(\Phi_\beta)_*(\sigma')$ are equal by Proposition 4.8(2) and Proposition 5.4. Therefore, it follows that $\sigma' = (\Phi_\beta)_*(\sigma')$ by [B4, 6.4], for every $\sigma' \in \mathcal{U}$.

By Theorem 6.4, there is some $\mathbb{N}_\gamma$ such that $\mathbb{N}_\gamma \cap \mathcal{U} \neq \emptyset$, so choose $\tau \in \mathbb{N}_\gamma \cap \mathcal{U}$. Then since $\tau \in \mathbb{N}_\gamma$, the heart of τ is $(\Phi_\gamma)_*(\mathcal{B}_{s(\gamma)})$. But on the other hand, since $\tau \in \mathcal{U}$, by the previous paragraph $\tau = (\Phi_\beta)_*(\tau)$. Thus the composition

$$\Phi_{\gamma^{-1}\beta\gamma} = \Phi_\gamma^{-1} \circ \Phi_\beta \circ \Phi_\gamma\colon\ \mathrm{D^b}(\mathrm{mod}\,\Lambda_{s(\beta)}) \to \mathrm{D^b}(\mathrm{mod}\,\Lambda_{s(\beta)})$$

restricts to an equivalence on $\mathcal{B}_{s(\gamma)}$. This implies $\Phi_\gamma^{-1} \circ \Phi_\beta \circ \Phi_\gamma \cong \mathrm{Id}$ by Theorem 5.6 applied to $\gamma^{-1}\beta\gamma$. Thus $\Phi_\beta \cong \mathrm{Id}$, and so the action satisfies the condition ($*$).

Since $\mathrm{Stab}^{\circ}_{n}\mathcal{D}$ is path connected, as is standard [H1, 1.40(a)(b)] it follows that $\mathsf{PBr}\,\mathcal{D}$ is the group of deck transformations for the regular cover

$$\mathrm{Stab}^{\circ}_{n}\mathcal{D} \to \mathrm{Stab}^{\circ}_{n}\mathcal{D}/\,\mathsf{PBr}\,\mathcal{D}.$$

Hence by Theorem 6.8, the map $\mathcal{Z}\colon \mathrm{Stab}^{\circ}_{n}\mathcal{D} \to \mathsf{Level}_{\mathbb{C}}\backslash\mathcal{H}^{\mathsf{aff}}_{\mathbb{C}}$ is a regular covering map, with Galois group $\mathsf{PBr}\,\mathcal{D}$. This completes the proof of (2).

(1) This follows using an identical argument to the above.

For the final statement, since $\mathrm{Stab}^{\circ}\mathcal{C}$ is a manifold, it is locally path connected. Hence as is standard (see e.g. [H1, 1.40(c)]) the cover is universal if and only if the natural map

$$\pi_1(\Theta_{\mathbb{C}}\backslash\mathcal{H}_{\mathbb{C}}) \twoheadrightarrow \mathsf{PBr}\,\mathcal{C}$$

is injective. But this is [HW], which works word-for-word in the more general terminal singularities setting here, as explained in [IW2, §10.3]. $\square$

Corollary 6.10. $\mathrm{Stab}^{\circ}\mathcal{C}$ *is contractible.*

Proof. The universal cover of the complexified complement simplicial hyperplane arrangement is contractible, due to Deligne's work on the $K(\pi, 1)$ conjecture [D1]. $\square$

7. Autoequivalence and SKMS Corollaries

The above description of stability conditions has consequences for autoequivalences, which in turn allows us to compute the SKMS.

7.1. **Autoequivalences of $\mathcal{C}$.** Consider the subgroup $\mathrm{Aut}^{\circ}\mathcal{C}$ of $\mathrm{Auteq}\,\mathcal{C}$, consisting of those $\Phi|_{\mathcal{C}}$ where Φ is a Fourier–Mukai equivalence $\mathrm{D^b}(\mathrm{coh}\,X) \to \mathrm{D^b}(\mathrm{coh}\,X)$ that commutes with $\mathbf{R}f_*$ and preserves $\mathrm{Stab}^{\circ}\mathcal{C}$. Since Φ commutes with $\mathbf{R}f_*$, automatically $\Phi|_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$.

Theorem 7.1. *Suppose that $X \to \mathrm{Spec}\,R$ is a 3-fold flop, where X has at worst terminal singularities. Then* $\mathrm{Aut}^{\circ}\mathcal{C} = \mathsf{PBr}\,\mathcal{C}$.

Proof. The inclusion $\mathsf{PBr}\,\mathcal{C} \subset \mathrm{Aut}^{\circ}\mathcal{C}$ follows since the Bridgeland–Chen flop functors are Fourier–Mukai equivalences that commute with $\mathbf{R}f_*$, and $\mathsf{PBr}\,\mathcal{C}$ acts on $\mathrm{Stab}^{\circ}\mathcal{C}$ by Theorem 6.9. For the reverse inclusion, consider $g \in \mathrm{Aut}^{\circ}\mathcal{C}$. Since $g \colon \mathrm{D^b}(\mathrm{coh}\,X) \to \mathrm{D^b}(\mathrm{coh}\,X)$ commutes with $\mathbf{R}f_*$, passing through $\Psi := \mathbf{R}\mathrm{Hom}_X(\mathcal{V}, -)$ to obtain $\Psi\circ g\circ\Psi^{-1} \colon \mathrm{D^b}(\mathrm{mod}\,\Lambda) \to \mathrm{D^b}(\mathrm{mod}\,\Lambda)$, necessarily by [W1, 2.14] $\Psi \circ g \circ \Psi^{-1}$ commutes with the exact functor $e(-)$, where e is the idempotent of Λ corresponding to R.

Since g preserves $\mathrm{Stab}^{\circ}\mathcal{C}$, $\mathrm{V} := \Psi \circ g \circ \Psi^{-1}(\mathrm{U}_1)$ is open and by Theorem 6.4 $\mathcal{M}$ is dense in $\mathrm{Stab}^{\circ}\mathcal{C}$, necessarily $V \cap \mathcal{M} \neq \emptyset$. Thus we must have $\Psi \circ g \circ \Psi^{-1}(\mathcal{A}) = \mathcal{A}_{\alpha}$ for some $\alpha \in \mathrm{Hom}_{\mathbb{G}}(C_B, C_+)$. Consider the composition

$$G = \Phi_{\alpha}^{-1} \circ \Psi \circ g \circ \Psi^{-1} \colon \mathrm{D^b}(\mathrm{mod}\,\Lambda) \to \mathrm{D^b}(\mathrm{mod}\,\Lambda_B),$$

which takes $\mathcal{A}$ to the standard heart on $\mathcal{C}_B$. Now $\Phi_{\alpha}^{-1} = \Phi_{\alpha^{-1}}$ is a composition of mutation functors and their inverses, where we do not mutate the vertex R. By [W1, 4.2] these are functorially isomorphic to flop functors and their inverses, which commute with $\mathbf{R}f_*$. Again by [W1, 2.14], this translates into $\Phi_{\alpha^{-1}}$ commuting with $e(-)$. Consequently the composition G commutes with $e(-)$.

Since G takes $\mathcal{A}$ to a standard algebraic heart, necessarily G takes the simples $\mathcal{S}_1, \ldots, \mathcal{S}_n$ to simples, a priori with a permutation. Hence (1) $\Psi^{-1} \circ G \circ \Psi$ commutes with $\mathbf{R}f_*$, and (2) it sends $\mathcal{O}_{\mathrm{C}_1}(-1), \ldots, \mathcal{O}_{\mathrm{C}_n}(-1)$ to themselves, a priori up to permutation. But exactly as in [DW, 7.17], property (1) implies that $\Psi^{-1} \circ G \circ \Psi$ preserves $\mathcal{C}$, and property (2) implies that $\Psi^{-1} \circ G \circ \Psi$ preserves the null category $\{a \in \mathrm{coh}\,X \mid \mathbf{R}f_*a = 0\}$. This implies that it necessarily preserves $\mathcal{C}^{>0}$ and $\mathcal{C}^{<0}$, and hence preserves zero perverse sheaves ${}^0\mathrm{Per}\,X$. In particular, $G(\mathcal{S}_0) \cong \mathcal{S}_0$, as the other simples are permuted.

But since $\Psi^{-1} \circ G \circ \Psi$ preserves ${}^0\mathrm{Per}\,X$, G restricts to a Morita equivalence $\mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda_B$. In particular projectives map to projectives, so since $G(\mathcal{S}_0) \cong \mathcal{S}_0$, under the pairing we have $G(\mathcal{P}_0) \cong \mathcal{P}_0$. Furthermore, since Λ and Λ_B are basic, the Morita equivalence sends $\Lambda \mapsto \Lambda_B$. Since G commutes with $e(-)$, it follows that $N \cong B$ in $\mathrm{D^b}(\mathrm{mod}\,R)$, and so $\Phi_{\alpha} \in \mathsf{PBr}\,\mathcal{C}$.

But now $\Psi^{-1} \circ G \circ \Psi$ sends $\mathcal{O}_X \mapsto \mathcal{O}_X$, since $G(\mathcal{P}_0) \cong \mathcal{P}_0$, and it preserves ${}^0\mathrm{Per}\,X$. By the standard Toda argument (see e.g. [DW, 7.18]), $\Psi^{-1} \circ G \circ \Psi \cong \varphi_* \circ (- \otimes \mathcal{L})$ for some isomorphism $\varphi \colon X \to X$ and some line bundle $\mathcal{L}$. The line bundle $\mathcal{L}$ is trivial since $\mathcal{O}_X \mapsto \mathcal{O}_X$. The isomorphism φ commutes with $\mathbf{R}f_*$ since $\Psi^{-1} \circ G \circ \Psi$ does, and hence φ is the identity, given it must be the identity on the dense open set obtained by removing the flopping curve. It follows that $G \cong \mathrm{Id}$, and so $g = \Psi^{-1} \circ \Phi_{\alpha} \circ \Psi \in \mathsf{PBr}\,\mathcal{C}$, as required. $\square$

7.2. **Identifying Line Bundle Twists.** To describe $\mathrm{Aut}^{\circ}\mathcal{D}$ requires us to first realise twists by line bundles as compositions of mutation functors. Set $\mathsf{L}_i := f_*\mathcal{L}_i$ and consider the subgroup $\mathbb{Z}^n \cong \langle \mathsf{L}_1, \dots, \mathsf{L}_n \rangle \leq \mathrm{Cl}(R)$. For any L in this subgroup and any $M \in \mathrm{Mut}(N)$, $\mathsf{L} \cdot M := (\mathsf{L} \otimes M)^{**}$ belongs to the mutation class of N, in the following predictable way.

Lemma 7.2. [IW2, 9.10(2)(3)] *The arrangement $\mathcal{H}^{\mathsf{aff}}$ has a $\mathbb{Z}^n$-action, where its generators take the chamber corresponding to N to the next chamber along the primitive vectors of the $\mathbb{Z}^n$-lattice given by the chamber N.*

To illustrate this, consider the two-curve example in Figure 2. Then the $\mathbb{Z}^2$-lattice is the black dots, and the primitive vectors of the $\mathbb{Z}^2$-lattice given by the chamber N are the red arrows. For any such L and $M \in \mathrm{Mut}(N)$, consider the isomorphism $\upepsilon\colon \Lambda_M \to \Lambda_{\mathsf{L}\cdot M}$ defined

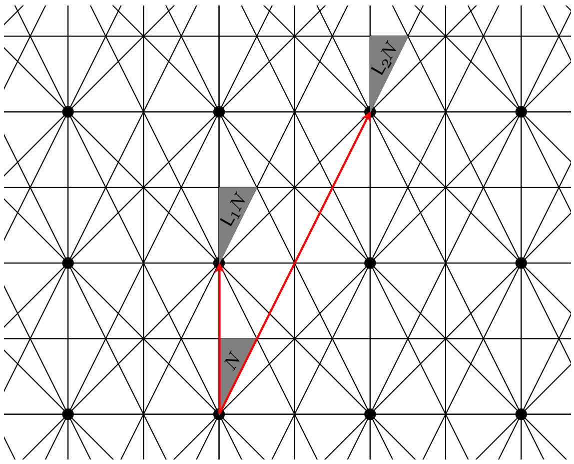


FIGURE 2. Example of $\mathbb{Z}^2$ action on $\mathcal{H}^{\mathsf{aff}}$

to be

$$\Lambda_M = \mathrm{End}_R(M) \xrightarrow{(-\otimes \mathsf{L})^{**}} \mathrm{End}_R((M \otimes \mathsf{L})^{**}).$$

Being an isomorphism of algebras, $\upepsilon$ induces an isomorphism of categories $\mathrm{D^b}(\mathrm{mod}\,\Lambda_M) \to \mathrm{D^b}(\mathrm{mod}\,\Lambda_{\mathsf{L}\cdot M})$, which we will also denote by $\upepsilon$.

For $\mathsf{L} \in \langle \mathsf{L}_1, \dots, \mathsf{L}_n \rangle$ and a path $\upalpha \in \mathrm{Hom}_{\mathbb{G}^{\mathsf{aff}}}(C_{M_1}, C_{M_2})$, the translation by L defines a path from $C_{\mathsf{L}\cdot M_1}$ to $C_{\mathsf{L}\cdot M_2}$, which we denote by $\mathsf{L} \cdot \upalpha$.

Lemma 7.3. *Let $\mathsf{g}, \mathsf{h} \in \langle \mathsf{L}_1, \dots, \mathsf{L}_n \rangle$, $M \in \mathrm{Mut}(N)$, and $\upalpha\colon C_M \to C_{\mathsf{g}\cdot M}$ be a positive minimal path. Then the following diagram commutes.*

$$\begin{tikzcd}
\mathrm{D^b}(\mathrm{mod}\,\Lambda_M) \arrow[r, "\Phi_\upalpha"] \arrow[d, "\upepsilon"'] & \mathrm{D^b}(\mathrm{mod}\,\Lambda_{\mathsf{g}\cdot M}) \arrow[d, "\upepsilon"] \\
\mathrm{D^b}(\mathrm{mod}\,\Lambda_{\mathsf{h}\cdot M}) \arrow[r, "\Phi_{\mathsf{h}\cdot\upalpha}"] & \mathrm{D^b}(\mathrm{mod}\,\Lambda_{\mathsf{h}\cdot\mathsf{g}\cdot M})
\end{tikzcd}$$

Proof. By Remark 4.10 the top functor composed with the right hand functor is given by the tilting bimodule $\mathrm{Hom}_R(M, \mathsf{g}\cdot M)$, with the natural action of Λ_M, and the action of $\Lambda_{\mathsf{h}\cdot\mathsf{g}\cdot M}$ twisted by the isomorphism $\upepsilon$. Applying $\mathsf{h}\cdot$, this is clearly isomorphic, as bimodules, to $\mathrm{Hom}_R(\mathsf{h}\cdot M, \mathsf{h}\cdot\mathsf{g}\cdot M)$, with the action of Λ_M twisted by $\upepsilon$, and the natural action of $\Lambda_{\mathsf{h}\cdot\mathsf{g}\cdot M}$. But this is the bimodule that realises the left hand functor composed with the bottom functor, and so the diagram commutes. $\square$

Theorem 7.4. *Suppose that $X \to \mathrm{Spec}\,R$ a 3-fold flop, where X has at worst terminal singularities. Writing $\upkappa$ for a minimal chain of mutation functors from one algebra to the* *other, for all $i = 1, \dots, n$ the following diagram commutes*

$$\begin{tikzcd} \mathrm{D^b}(\operatorname{coh} X) \arrow[rr, "-\otimes \mathcal{L}_i^*"] \arrow[d, "\Psi"'] & & \mathrm{D^b}(\operatorname{coh} X) \arrow[d, "\Psi"'] \\ \mathrm{D^b}(\operatorname{mod} \Lambda) \arrow[r, "\kappa"] & \mathrm{D^b}(\operatorname{mod} \Lambda_{\mathsf{L}_i \cdot N}) \arrow[r, "\varepsilon^{-1}"] & \mathrm{D^b}(\operatorname{mod} \Lambda) \end{tikzcd}$$

Proof. By Remark 4.10 the minimal chain of mutations κ is functorially isomorphic to the $\mathbf{R}$Hom functor given by the tilting bimodule $\operatorname{Hom}_R(N, \mathsf{L}_i \cdot N)$. So, setting $\mathcal{V} = \mathcal{V}_X$ and taking inverses, it suffices to prove that the diagram

$$\begin{tikzcd} \mathrm{D^b}(\operatorname{coh} X) & & \mathrm{D^b}(\operatorname{coh} X) \arrow[ll, "-\otimes \mathcal{L}_i"'] \\ \mathrm{D^b}(\Lambda) \arrow[u, "-\otimes^{\mathbf{L}} \mathcal{V}"] & \mathrm{D^b}(\operatorname{End}_X(\mathcal{V} \otimes \mathcal{L}_i)) \arrow[l, "-\otimes^{\mathbf{L}} \operatorname{Hom}_X(\mathcal{V}, \mathcal{V}\otimes\mathcal{L}_i)"'] & \mathrm{D^b}(\operatorname{End}_X(\mathcal{V})) \arrow[l, "\varepsilon"'] \arrow[u, "-\otimes^{\mathbf{L}} \mathcal{V}"'] \end{tikzcd} \tag{7.A}$$

commutes, where derived tensors are over the relevant endomorphism ring.

The bottom composition is isomorphic to the functor

$$- \otimes^{\mathbf{L}}_{\Lambda} {}_{\varepsilon} \operatorname{Hom}_X(\mathcal{V}, \mathcal{V} \otimes \mathcal{L}_i)_{\mathrm{Id}}$$

by which we mean the $\Lambda \cong \operatorname{End}_Y(\mathcal{V})$-action on $\operatorname{Hom}_X(\mathcal{V}, \mathcal{V}\otimes\mathcal{L}_i)$ on the left is the composition action via ε, and the Λ-action on the right is the standard composition action. Hence the composition from bottom right to top left, along first the bottom and then the left, is functorially isomorphic to

$$- \otimes^{\mathbf{L}}_{\Lambda} ({}_{\varepsilon} \operatorname{Hom}_X(\mathcal{V}, \mathcal{V} \otimes \mathcal{L}_i)_{\mathrm{Id}} \otimes^{\mathbf{L}}_{\Lambda} \mathcal{V}) \tag{7.B}$$

We first claim that this bimodule has cohomology only in degree zero, and for this we can ignore the left action of Λ completely, and consider $\operatorname{Hom}_X(\mathcal{V}, \mathcal{V} \otimes \mathcal{L}_i)_{\mathrm{Id}} \otimes^{\mathbf{L}}_{\Lambda} \mathcal{V}$. Since $\mathcal{L}_i$ is generated by global sections, there exists a surjection $\mathcal{O}^N \twoheadrightarrow \mathcal{L}_i$, for some N, and so tensoring by $\mathcal{V}$ gives a surjection $\mathcal{V}^N \twoheadrightarrow \mathcal{V} \otimes \mathcal{L}_i$. Applying $\operatorname{Hom}_X(\mathcal{V}, -)$ gives an exact sequence

$$\operatorname{Ext}^1_X(\mathcal{V}, \mathcal{V})^N \to \operatorname{Ext}^1_X(\mathcal{V}, \mathcal{V} \otimes \mathcal{L}_i) \to 0$$

where Ext^2 is zero since flopping contractions have fibre dimension one. Since $\mathcal{V}$ is tilting, $\operatorname{Ext}^1_X(\mathcal{V}, \mathcal{V}) = 0$, thus $\operatorname{Ext}^1_X(\mathcal{V}, \mathcal{V} \otimes \mathcal{L}_i) = 0$. It follows that

$$\operatorname{Hom}_X(\mathcal{V}, \mathcal{V} \otimes \mathcal{L}_i)_{\mathrm{Id}} \otimes^{\mathbf{L}}_{\Lambda} \mathcal{V} \cong \mathbf{R}\operatorname{Hom}_X(\mathcal{V}, \mathcal{V} \otimes \mathcal{L}_i)_{\mathrm{Id}} \otimes^{\mathbf{L}}_{\Lambda} \mathcal{V} \cong \mathcal{V} \otimes \mathcal{L}_i, \tag{7.C}$$

which has degree zero, as claimed.

By the claim, after truncating in the category of bimodules, (7.B) is functorially isomorphic to $- \otimes^{\mathbf{L}}_{\Lambda} ({}_{\varepsilon} \operatorname{Hom}_X(\mathcal{V}, \mathcal{V} \otimes \mathcal{L}_i)_{\mathrm{Id}} \otimes_{\Lambda} \mathcal{V})$. Thus (7.A) commutes provided that

$${}_{\varepsilon} \operatorname{Hom}_X(\mathcal{V}, \mathcal{V} \otimes \mathcal{L}_i)_{\mathrm{Id}} \otimes_{\Lambda} \mathcal{V} \cong \mathcal{V} \otimes \mathcal{L}_i \tag{7.D}$$

as Λ-$\mathcal{O}_X$-bimodules. But the derived counit map is an isomorphism, so by (7.C) it follows that the non-derived counit map

$$\operatorname{Hom}_X(\mathcal{V}, \mathcal{V} \otimes \mathcal{L}_i)_{\mathrm{Id}} \otimes_{\Lambda} \mathcal{V} \to \mathcal{V} \otimes \mathcal{L}_i$$

is also an isomorphism. But this map is the obvious one. By inspection it respects the bimodule structure in (7.D), and thus it gives the required bimodule isomorphism. $\square$

Remark 7.5. The position of ε at the end of the chain of mutations κ in Theorem 7.4 is irrelevant. Indeed, given any minimal chain from N and $\mathsf{L}_i \cdot N$, consider any chamber A through which this minimal path passes. Abusing notation and writing ε for any isomorphism induced by the class group action, and κ for a minimal composition of mutation functors, the following top diagram also commutes

$$\begin{tikzcd}
\mathrm{D^b}(\mathrm{coh}\, X) \arrow[rrr, "-\otimes \mathcal{L}_i^*"] \arrow[d, "\Psi"'] & & & \mathrm{D^b}(\mathrm{coh}\, X) \arrow[d, "\Psi"] \\
\mathrm{D^b}(\mathrm{mod}\, \Lambda) \arrow[r, "\kappa"] & \mathrm{D^b}(\mathrm{mod}\, \Lambda_A) \arrow[r, "\varepsilon^{-1}"] \arrow[drr, "\kappa"'] & \mathrm{D^b}(\mathrm{mod}\, \Lambda_{\mathsf{L}_i^{-1}\cdot A}) \arrow[r, "\kappa"] & \mathrm{D^b}(\mathrm{mod}\, \Lambda) \\
 & & & \mathrm{D^b}(\mathrm{mod}\, \Lambda_{\mathsf{L}_i^{-1}\cdot N}) \arrow[u, "\varepsilon^{-1}"']
\end{tikzcd}$$

since the outer diagram commutes by Theorem 7.4, and the commutativity of the bottom diagram is Lemma 7.3. Hence, when composing line bundle twists, we can move all the ε^{-1}s to the right, or to the left, whichever is the most convenient.

7.3. $\operatorname{Pic} X$ **action.** Recall that perfect complexes on X are equivalent to $\mathrm{K^b}(\mathrm{proj}\, \Lambda)$, and $\operatorname{Pic} X \cong \mathbb{Z}^n$ with basis $\mathcal{L}_1, \ldots, \mathcal{L}_n$, where $\mathcal{L}_i$ is the line bundle satisfying $\deg(\mathcal{L}_i|_{\mathrm{C}_j}) = \delta_{ij}$. For $\mathcal{L} \in \operatorname{Pic} X$, by abuse of notation, set

$$- \otimes \mathcal{L} := \Psi \circ (- \otimes \mathcal{L}) \circ \Psi^{-1} \colon \mathrm{D^b}(\mathrm{mod}\, \Lambda) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{mod}\, \Lambda).$$

Lemma 7.6. *Write $z = (z_j)_{j=0}^n \in \mathsf{Level}_{\mathbb{C}}$ with $z_j = x_j + \mathrm{i} y_j$. Then for $j > 0$, and any $1 \leq i \leq n$, the autoequivalence $- \otimes \mathcal{L}_i$ acts on $\mathsf{Level}_{\mathbb{C}}$ by*

$$x_j + \mathrm{i} y_j \mapsto \begin{cases} x_j + \mathrm{i}(y_j + 1) & j = i \\ x_j + \mathrm{i} y_j & j \neq i. \end{cases}$$

Proof. To ease notation, consider only the action of $\mathcal{L}_1$, since all other cases are identical. We first consider the action on the dual vector space $K_0(\mathcal{D})$ of $\mathcal{K}$, with basis $[\mathcal{S}_0], \ldots, [\mathcal{S}_n]$. Recall that $\Psi(\omega_{\mathrm{C}}[1]) \cong \mathcal{S}_0$ and $\Psi(\mathcal{O}_{\mathrm{C}_i}(-1)) \cong \mathcal{S}_i$ for $1 \leq i \leq n$. Since $\deg(\mathcal{L}_j|_{\mathrm{C}_i}) = \delta_{ij}$,

$$\mathcal{O}_{\mathrm{C}_i}(-1) \otimes \mathcal{L}_1 \cong \begin{cases} \mathcal{O}_{\mathrm{C}_1} & i = 1 \\ \mathcal{O}_{\mathrm{C}_i}(-1) & i \neq 1. \end{cases}$$

Let $\mathcal{O}_x$ be the skyscraper sheaf associated to a closed point $x \in \mathrm{C}$. Then $\mathbf{R}\mathrm{Hom}(\mathcal{V}_i^*, \mathcal{O}_x) \cong \mathrm{Hom}(\mathcal{V}_i^*, \mathcal{O}_x) = \mathbb{C}^{\mathsf{rk} N_i}$, so setting $\ell_i := \mathsf{rk}_R\, N_i$ gives

$$[\mathcal{O}_x] = -[\omega_{\mathrm{C}}] + \sum_{i=1}^{n} \ell_i [\mathcal{O}_{\mathrm{C}_i}(-1)]. \tag{7.E}$$

If $x \in \mathrm{C}_i$, then the exact sequence $0 \to \mathcal{O}_{\mathrm{C}_i}(-1) \to \mathcal{O}_{\mathrm{C}_i} \to \mathcal{O}_x \to 0$ implies that $[\mathcal{O}_{\mathrm{C}_i}] = [\mathcal{O}_{\mathrm{C}_i}(-1)] + [\mathcal{O}_x]$. Combining this with (7.E), it follow that

$$[\mathcal{O}_{\mathrm{C}_i}] = -[\omega_{\mathrm{C}}] + (\ell_i + 1)[\mathcal{O}_{\mathrm{C}_i}(-1)] + \sum_{k \neq i} \ell_k [\mathcal{O}_{\mathrm{C}_k}(-1)], \tag{7.F}$$

and so

$$[\mathcal{O}_{\mathrm{C}_i}(-1) \otimes \mathcal{L}_1] = \begin{cases} [\omega_{\mathrm{C}}[1]] + (\ell_1 + 1)[\mathcal{O}_{\mathrm{C}_1}(-1)] + \sum_{k \neq 1} \ell_k [\mathcal{O}_{\mathrm{C}_k}(-1)] & i = 1 \\ [\mathcal{O}_{\mathrm{C}_i}(-1)] & i \neq j. \end{cases} \tag{7.G}$$

Next we determine the action of $\mathcal{L}_1$ on $[\omega_{\mathrm{C}}[1]] = -[\omega_{\mathrm{C}}]$. Since $\mathcal{O}_x \otimes \mathcal{L}_1 \cong \mathcal{O}_x$, we have

$$[\mathcal{O}_x] = -[\omega_{\mathrm{C}} \otimes \mathcal{L}_1] + \sum_{i \neq 1} \ell_i [\mathcal{O}_{\mathrm{C}_i}(-1)] + \ell_1 [\mathcal{O}_{\mathrm{C}_1}].$$

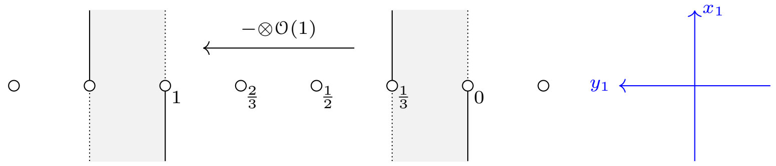
Substituting (7.E) to the left hand side and (7.F) to the third term of the right hand side,

$$[\omega_{\mathrm{C}}[1] \otimes \mathcal{L}_1] = (1 - \ell_1)[\omega_{\mathrm{C}}[1]] - \sum_{i=1}^{n} \ell_i \ell_1 [\mathcal{O}_{\mathrm{C}_i}(-1)]. \tag{7.H}$$

Combining (7.G) and (7.H), the action of $\mathcal{L}_1$ on $K_0(\mathcal{D})$ is given by the $(n+1) \times (n+1)$ matrix whose first row is $(1 - \ell_1, 1, 0, \ldots, 0)$, whose second row is $(-\ell_1^2, \ell_1 + 1, 0, \ldots, 0)$, and $(-\ell_1 \ell_j, \ell_j, 0, \ldots, 1, \ldots, 0)$ is the jth row for any $j > 1$, where 1 is in the $(j+1)$st entry.

The transpose of this matrix gives the required action on the dual $\mathcal{K}$. But for $j > 0$, the transpose has $(j+1)$st row $(0, \ldots, 1, \ldots, 0)$ where the only non-zero entry is in the $(j+1)$st position, and the first row of the transpose matrix is $(1, \ell_1 + 1, \ell_2, \ldots, \ell_n)$. Since $z \in \mathsf{Level}_{\mathbb{C}}$, by definition $x_0 = -\sum_{i=1}^{n} \ell_i x_i$ and $y_0 = 1 - \sum_{i=1}^{n} \ell_i y_i$. Using this, together with the rows identified, it is easy to check that the action is as stated. $\square$

It will be convenient to visualise this in the case of an irreducible flopping contraction. Recall from Example 3.9 that $(z_0, z_1) \in \mathsf{Level}_{\mathbb{C}}$ is determined by (x_1, y_1), and that by Lemma 7.6, the autoequivalence $- \otimes \mathcal{O}(1)$ acts by sending $(x_1, y_1) \mapsto (x_1, y_1 + 1)$. As in Example 3.9, we draw the y_1 axis horizontally pointing to the left, and the x_1 axis vertically. Thus, with this convention, tensoring by $\mathcal{O}(1)$ translates to the left. The case $\ell = 3$ is illustrated below.



(7.I)

7.4. **Autoequivalences of $\mathcal{D}$.** As in the introduction, consider $\mathrm{Aut}^{\circ}\mathcal{D}$, defined to be the subgroup of $\mathrm{Auteq}\,\mathcal{D}$ consisting of those $\Phi|_{\mathcal{D}} \colon \mathcal{D} \to \mathcal{D}$ where Φ is an R-linear Fourier–Mukai equivalence $\Phi \colon \mathrm{D^b}(\mathrm{coh}\, X) \to \mathrm{D^b}(\mathrm{coh}\, X)$ that preserves $\mathrm{Stab}_n^{\circ}\mathcal{D}$.

In order to control elements in $\mathrm{Aut}^{\circ}\mathcal{D}$, we first control isomorphisms of algebras that preserves the space of normalised stability conditions. Suppose that Λ_L and Λ_M arise from $L = \bigoplus_{i=0}^{n} L_i$ and $M = \bigoplus_{i=0}^{n} M_i$ in $\mathrm{Mut}(N)$, and
$$\uprho \colon \Lambda_L \to \Lambda_M$$
is an isomorphism of rings. Let $\mathcal{S}_i$ be the simple Λ_L-module corresponding to the projective module $\mathrm{Hom}_R(L, L_i)$. Being an isomorphism, simples get sent to simples, and so there is a bijection $\upiota \colon \{0, \ldots, n\} \to \{0, \ldots, n\}$ such that $\uprho_* \mathcal{S}_i \cong \mathcal{S}'_{\upiota(i)}$, where $\mathcal{S}'_j$ is the simple Λ_M-module corresponding to the summand M_j.

Lemma 7.7. *$\uprho_*$ preserves $\mathrm{Stab}_n^{\circ}\mathcal{D}$ if and only if $\mathsf{rk}_R\, L_i = \mathsf{rk}_R\, M_{\upiota(i)}$ for all $0 \leq i \leq n$.*

Proof. ($\Leftarrow$) It suffices to show that $\uprho_*(Z, \mathcal{B}_{\Lambda_L}) = (Z \circ \uprho^{-1}, \mathcal{B}_{\Lambda_M}) \in \mathrm{Stab}_n^{\circ}\mathcal{D}_{\Lambda_M}$ for all $(Z, \mathcal{B}_{\Lambda_L}) \in \mathrm{Stab}_n^{\circ}\mathcal{D}_{\Lambda_L}$. It is clear that $\uprho_*$ preserves the stability component, since $Z \circ \uprho^{-1}[\mathcal{S}'_{\upiota(i)}] = Z[\mathcal{S}_i] \in \mathbb{H}$ for each $0 \leq i \leq n$, and, furthermore, $\uprho_*$ preserves the normalisation since
$$\sum_{i=0}^{n} (\mathsf{rk}_R\, M_{\upiota(i)}) \cdot Z \circ \uprho^{-1}[\mathcal{S}'_{\upiota(i)}] = \sum_{i=0}^{n} (\mathsf{rk}_R\, L_i) \cdot Z[\mathcal{S}_i] = \mathrm{i}.$$
($\Rightarrow$) To ease notation, set $a_i := \mathsf{rk}_R\, L_i$, $b_j = \mathsf{rk}_R\, M_j$. Furthermore, for each $i \in \{0, \ldots, n\}$ consider $Z_i \colon K_0(\mathcal{D}) \to \mathbb{C}$ defined by
$$Z_i([\mathcal{S}_j]) := \begin{cases} -1 & j \neq i \\ \frac{a - a_i}{a_i} + \mathrm{i}\,\frac{1}{a_i} & j = i \end{cases}$$
where $a := \sum_{i=0}^{n} a_i$. Now each $Z_i([\mathcal{S}_j])$ belongs to $\mathbb{H}$, and $\sum_{i=0}^{n} a_j Z_i([\mathcal{S}_j]) = \mathrm{i}$, hence $(Z_i, \mathcal{B}_{\Lambda_L}) \in \mathrm{Stab}_n^{\circ}\mathcal{D}_{\Lambda_L}$. Since $\uprho_*$ preserves normalised stability conditions by assumption,
$$\sum_{j=0}^{n} b_{\upiota(j)} Z_i \circ \uprho^{-1}([\mathcal{S}'_{\upiota(j)}]) = \sum_{j=0}^{n} b_{\upiota(j)} Z_i([\mathcal{S}_j]) = \mathrm{i}$$
Considering the imaginary parts of both sides, we see that $a_i = b_{\upiota(i)}$. $\square$

Our next main result, Theorem 7.10, requires two technical lemmas.

Lemma 7.8. *For any $\mathsf{g} \in \langle \mathsf{L}_1, \ldots, \mathsf{L}_n \rangle$, consider the isomorphism $\upvarepsilon \colon \Lambda \to \Lambda_{\mathsf{g} \cdot N}$. Then there is an isomorphism $\upvarepsilon \cong \Phi_{\upgamma} \circ (- \otimes \mathcal{L})$ for some $\mathcal{L} \in \mathrm{Pic}(X)$ and some $\upgamma \in \mathrm{Hom}_{\mathbb{G}^{\mathsf{aff}}}(N, \mathsf{g} \cdot N)$.*

Proof. By assumption we may write $\mathsf{g} = \mathsf{L}_{i_1}^{\sigma_1}\cdot\mathsf{L}_{i_2}^{\sigma_2}\cdot\ldots\cdot\mathsf{L}_{i_k}^{\sigma_k}$ for some $k\geq 0$, $1\leq i_j\leq n$ and $\sigma_i\in\{\pm1\}$. If $k=0$, we have nothing to prove, and so we assume $k>0$.

We prove the result by induction on k, so first assume that $k=1$. There are two cases, depending on the parity of σ_1. In the case $\sigma_1=1$, we are considering the isomorphism $\upepsilon\colon\Lambda\to\Lambda_{\mathsf{L}_i\cdot N}$. By Theorem 7.4, the functor

$$(-\otimes\mathcal{L}_i^{-1})\colon\ \mathrm{D^b}(\operatorname{mod}\Lambda)\to\mathrm{D^b}(\operatorname{mod}\Lambda)\tag{7.J}$$

is isomorphic to the composition

$$\mathrm{D^b}(\operatorname{mod}\Lambda)\xrightarrow{\Phi_\upalpha}\mathrm{D^b}(\operatorname{mod}\Lambda_{\mathsf{L}_i\cdot N})\xrightarrow{\upepsilon^{-1}}\mathrm{D^b}(\operatorname{mod}\Lambda)$$

from which it follows that $\upepsilon\cong\Phi_\upalpha\circ(-\otimes\mathcal{L}_i)$. On the other hand, in the case $\sigma_1=-1$, we are considering the isomorphism $\upepsilon\colon\Lambda\to\Lambda_{\mathsf{L}_i^{-1}\cdot N}$. However, by Theorem 7.4 and Lemma 7.3 applied to $\mathsf{g}=\mathsf{L}_i$ and $\mathsf{h}=\mathsf{g}^{-1}$, the functor (7.J) is isomorphic to

$$\mathrm{D^b}(\operatorname{mod}\Lambda)\xrightarrow{\upepsilon}\mathrm{D^b}(\operatorname{mod}\Lambda_{\mathsf{h}\cdot N})\xrightarrow{\Phi_{\mathsf{h}\cdot\upalpha}}\mathrm{D^b}(\operatorname{mod}\Lambda),$$

from which it follows that $\upepsilon\cong\Phi_{(\mathsf{h}\cdot\upalpha)^{-1}}\circ(-\otimes\mathcal{L}_i^{-1})$. Either way, the result holds for $k=1$.

For the induction step, assume that $k>1$ and set $\mathsf{h}:=\mathsf{g}\cdot\mathsf{L}_{i_k}^{-\sigma_k}$ and $\mathsf{k}:=\mathsf{L}_{i_k}^{\sigma_k}$, so $\mathsf{g}=\mathsf{h}\cdot\mathsf{k}$. By the inductive hypothesis, there exist $\mathcal{M}\in\operatorname{Pic}X$ and $\upbeta\in\operatorname{Hom}_{\mathbb{G}^{\mathsf{aff}}}(N,\mathsf{h}\cdot N)$ such that the isomorphism $\upepsilon\colon\Lambda\to\Lambda_{\mathsf{h}\cdot N}$ is isomorphic to $\Phi_\upbeta\circ(-\otimes\mathcal{M})$. Combining this with Lemma 7.3, the following diagram commutes

$$\begin{tikzcd}
 & \mathrm{D^b}(\operatorname{mod}\Lambda) \arrow[dl, "-\otimes\mathcal{M}"'] \arrow[d, "\upepsilon"] \arrow[dd, bend left, "\upepsilon"] \\
\mathrm{D^b}(\operatorname{mod}\Lambda) \arrow[r, "\Phi_\upbeta"] \arrow[d, "\upepsilon"] & \mathrm{D^b}(\operatorname{mod}\Lambda_{\mathsf{h}\cdot N}) \arrow[d, "\upepsilon"] \\
\mathrm{D^b}(\operatorname{mod}\Lambda_{\mathsf{k}\cdot N}) \arrow[r, "\Phi_{\mathsf{k}\cdot\upbeta}"] & \mathrm{D^b}(\operatorname{mod}\Lambda_{\mathsf{g}\cdot N})
\end{tikzcd}$$

where the right hand composition is our desired isomorphism $\upepsilon\colon\Lambda\to\Lambda_{\mathsf{g}\cdot N}$. Now by length one case proved above applied to k, the bottom left $\upepsilon$ is isomorphic to $\Phi_{\upgamma_k}\circ(-\otimes\mathcal{L}_{i_k}^{\sigma_k})$ for some $\upgamma_k\in\operatorname{Hom}_{\mathbb{G}^{\mathsf{aff}}}(N,\mathsf{k}\cdot N)$. Thus if we set $\upgamma:=(\mathsf{k}\cdot\upbeta)\circ\upgamma_k\in\operatorname{Hom}_{\mathbb{G}^{\mathsf{aff}}}(N,\mathsf{g}\cdot N)$ and $\mathcal{L}:=\mathcal{M}\otimes\mathcal{L}_{i_k}^{\sigma_k}$, then $\upepsilon\colon\Lambda\to\Lambda_{\mathsf{g}\cdot N}$ is isomorphic to the composition $\Phi_\upgamma\circ(-\otimes\mathcal{L})$. $\square$

The following is an easy extension of results in [IW2, §9].

Lemma 7.9. *Suppose that $M\in\operatorname{Mut}(N)$, and that there is an R-linear isomorphism $\Lambda\to\Lambda_M$ such that $\mathcal{S}_0\mapsto\mathcal{S}_i$, where $\mathsf{rk}_R M_i=1$. Then $M\cong\mathsf{L}\cdot N$ for some $\mathsf{L}\in\langle\mathsf{L}_1,\ldots,\mathsf{L}_n\rangle$.*

Proof. Since $\mathcal{S}_0\mapsto\mathcal{S}_i$, by the pairing between simples and projectives, it follows that there is a chain of R-module isomorphisms

$$\operatorname{Hom}_R(N,R)=\mathcal{P}_0\cong\mathcal{P}_i=\operatorname{Hom}_R(M,M_i).$$

Since $\mathsf{rk}_R M_i=1$, the right hand side is isomorphic to $\operatorname{Hom}_R(M\cdot M_i^{-1},R)$. Hence applying $\operatorname{Hom}_R(-,R)$ shows that $M\cong M_i\cdot N$ as R-modules. Since M_i is a summand of M of rank one, and $M\in\operatorname{Mut}(N)$, by the bijections in 3.3 M_i must appear as a $\mathbb{Z}^n$-lattice point in the hyperplane arrangement (see [IW2, 9.10(2)(3)]). $\square$

The following is one of our key results.

Theorem 7.10. $\operatorname{Aut}^\circ\mathcal{D}\cong\mathsf{PBr}\,\mathcal{D}\rtimes\operatorname{Pic}X$.

Proof. Set $G=\operatorname{Aut}^\circ\mathcal{D}$. Then $K=\mathsf{PBr}\,\mathcal{D}$ is a clearly a subgroup of G, as is $H=\operatorname{Pic}X$ since elements of $\operatorname{Pic}X$ are R-linear by Lemma 2.5, and preserve $\operatorname{Stab}_n^\circ\mathcal{D}$ by Theorem 7.4.

The remainder of the proof splits into five steps.

Step 1: We first claim that, as sets, $G=KH$. Let $g\in G$. Since g preserves $\operatorname{Stab}_n^\circ\mathcal{D}$, arguing as in Theorem 7.1, there exists $\upbeta\in\operatorname{Hom}_{\mathbb{G}^{\mathsf{aff}}}(C_M,C_+)$ such that $g(\mathcal{B})=\mathcal{B}_\upbeta$. The composition

$$\mathrm{D^b}(\operatorname{mod}\Lambda) \xrightarrow{g} \mathrm{D^b}(\operatorname{mod}\Lambda) \xrightarrow{\Phi_\beta^{-1}} \mathrm{D^b}(\operatorname{mod}\Lambda_\beta)$$

is R-linear, and restricts to an equivalence between finite length Λ-modules and finite length Λ_β-modules. In particular, by Lemma 5.7(1) and Corollary 5.8 the composition restricts to an R-linear Morita equivalence

$$\operatorname{mod}\Lambda \xrightarrow{\Phi_\beta^{-1}\circ g} \operatorname{mod}\Lambda_\beta.$$

Since both algebras are basic, necessarily this is induced by an R-algebra isomorphism $\varphi\colon \Lambda \to \Lambda_\beta$. But by Lemma 7.7, it follows that M has rank one summand. By Lemma 7.9, there is $\mathsf{L} \in \langle \mathsf{L}_1, \dots, \mathsf{L}_n\rangle$ such that $M \cong \mathsf{L}\cdot N$.

Consider the composition autoequivalence

$$\Upsilon\colon \qquad \mathrm{D^b}(\operatorname{mod}\Lambda) \xrightarrow{g} \mathrm{D^b}(\operatorname{mod}\Lambda) \xrightarrow{\Phi_\beta^{-1}} \mathrm{D^b}(\operatorname{mod}\Lambda_\beta) \xrightarrow{\varepsilon^{-1}} \mathrm{D^b}(\operatorname{mod}\Lambda).$$

where $\varepsilon\colon \Lambda \xrightarrow{\sim} \Lambda_{\mathsf{L}\cdot N} = \Lambda_\beta$. By a similar argument as above, Υ is induced by an R-linear isomorphism $\psi\colon \Lambda \xrightarrow{\sim} \Lambda$. Then $\psi_*\mathcal{S}_0 \cong \mathcal{S}_i$ for some $0 \le i \le n$, and by the pairing between projectives and simples, ψ restricts to an isomorphism $\mathcal{P}_0 \cong \mathcal{P}_i$ of R-modules. By Lemma 7.7 we see that $\mathsf{rk}_R\, N_i = 1$. It follows that

$$\operatorname{Hom}_R(N,R) = \mathcal{P}_0 \cong \mathcal{P}_i \cong \operatorname{Hom}_R(N\cdot N_i^{-1}, R)$$

as R-modules, and so dualizing gives $N \cong N\cdot N_i^{-1}$. Then by Lemma 7.9, N_i necessarily lies in $\langle \mathsf{L}_1, \dots, \mathsf{L}_n\rangle$. By Lemma 7.2 we thus have $N_i \cong R$, since non-trivial elements of $\langle \mathsf{L}_1, \dots, \mathsf{L}_n\rangle$ non-trivially translate the hyperplane arrangement. Thus $\Upsilon(\mathcal{S}_0) \cong \mathcal{S}_0$. As an Morita equivalence, it follows that there exists a permutation $\iota \in S_n$ such that $\Upsilon(\mathcal{S}_i) \cong \mathcal{S}_{\iota(i)}$ for all $1 \le i \le n$. Since Υ preserves normalised stability conditions, by Lemma 7.7 we see $\mathsf{rk}_R\, N_i = \mathsf{rk}_R\, N_{\iota(i)}$. Hence by Proposition 5.10, there is a functorial isomorphism $\Upsilon \cong \mathrm{Id}$, and so $g \cong \Phi_\beta \circ \varepsilon$. Using Lemma 7.8 it follows that g is functorially isomorphic to the composition

$$\mathrm{D^b}(\operatorname{mod}\Lambda) \xrightarrow{F} \mathrm{D^b}(\operatorname{mod}\Lambda_N) \xrightarrow{\Phi_\gamma} \mathrm{D^b}(\operatorname{mod}\Lambda_{\mathsf{L}\cdot N}) \xrightarrow{\Phi_\beta} \mathrm{D^b}(\operatorname{mod}\Lambda_N).$$

for some $F \in \operatorname{Pic}(X)$. Hence $g \cong \Phi_{\beta\circ\gamma}\circ F \in KH$.

Step 2: Consider the subgroup $\mathsf{Tr}\,\mathcal{D}$ of G consisting of those elements that are the identity on K-theory $\mathcal{K}$. We know that $\mathsf{PBr}\,\mathcal{D} \subseteq \mathsf{Tr}\,\mathcal{D}$ by Proposition 4.8. We claim that $\mathsf{PBr}\,\mathcal{D} \supseteq \mathsf{Tr}\,\mathcal{D}$, so equality holds. To see this, consider $t \in \mathsf{Tr}\,\mathcal{D}$. By Step 1, since $t \in G$ we can write $t = kh$ for some $k \in K$ and some $h \in H$. Thus $h = k^{-1}t$, and so h is trivial on K-theory. But by Lemma 7.6 the only line bundle twist that satisfies this is the identity. Hence $h = 1$, so $k = t$ and thus $t \in K = \mathsf{PBr}\,\mathcal{D}$.

Step 3: $K \trianglelefteq G$. This follows immediately from Step 2, since being the identity on K-theory is clearly closed under conjugation.

Step 4: $K \cap H = \{1_G\}$. Again, this holds by Lemma 7.6, since the only line bundle twist that is the identity on K-theory is the identity.

Combining Steps 1, 3 and 4 we see that $G \cong K \rtimes H$, as required. □

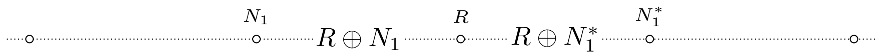
In particular, as is standard for semidirect products, there is an induced exact sequence

$$1 \to \mathsf{PBr}\,\mathcal{D} \to \operatorname{Aut}^\circ\mathcal{D} \to \operatorname{Pic} X \to 1. \tag{7.K}$$

This sequence is the generalisation of [T, 5.4(ii)] to higher length flops.

7.5. **Application: Stringy Kähler Moduli Spaces.** In this subsection we compute the stringy Kähler moduli space $\mathrm{Stab}_n^{\circ}\mathcal{D}/\mathrm{Aut}^{\circ}\mathcal{D}$ for all smooth single curve flops. In this case, Theorem 3.3 reduces to the statement that the mutation class containing N is in bijection with the chambers of an infinite hyperplane arrangement in $\mathbb{R}^1$, which we draw as

extended in both directions to infinity. The walls are labelled by the indecomposable R-modules that are summands of elements in the mutation class of N, and the chambers are labelled by their direct sums. Wall crossing corresponds to mutation. Since $N = R \oplus N_1$ from (2.C) must appear in its mutation class, the centre of the hyperplane arrangement has the following form

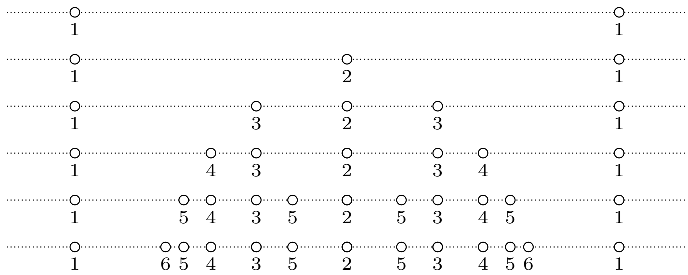


(7.L)

Under this convention, with $R \oplus N_1$ on the left and $R \oplus N_1^*$ on the right, $\mathsf{L} = f_*\mathcal{L} = f_*\mathcal{O}(1)$ generates a subgroup of $\mathrm{Cl}(R)$ which acts on (7.L), taking the wall labelled R to the next wall to the right for which the R-module labelling it has rank one.

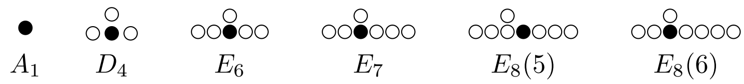
The following computes the numerics of the hyperplane arrangements that can arise.

Proposition 7.11. *Suppose that $X \to \operatorname{Spec} R$ is an irreducible length ℓ flop, where X is smooth. Then the corresponding affine hyperplane arrangement, together with the ranks of the modules labelling each wall, are, for $\ell = 1, \dots 6$ respectively:*



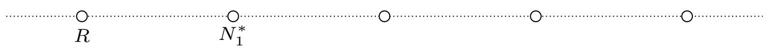
In each case the hyperplane arrangement is infinite, and the labels repeat.

Proof. By Katz–Morrison [KM, K2] it is known that for a smooth single-curve flop, the $\mathcal{J} \subset \Delta_0$ is one of the following cases:

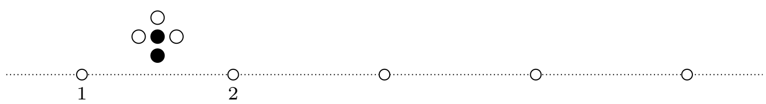


We analyse each individually. In each case, by [IW2, §1] the $\mathcal{J}$-affine arrangement $\mathsf{TCone}(\mathcal{J}_{\mathsf{aff}})$ can be calculated by using local wall crossing rules. Combinatorially, this is very elementary, and is explained in detail in [W2, 1.1]. We sketch the D_4 case here.

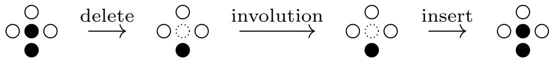
As in (7.L), consider the chamber



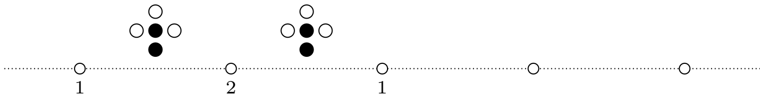
We first replace the modules by their ranks, and we label the chamber via McKay correspondence. The fact we will repeatedly use below is that the rank of a summand equals the number δ_i, where i is the vertex in the Dynkin diagram that the summand corresponds to, and $\delta = (\delta_i)_{i \in \Delta_{\mathsf{aff}}}$ is the null root [IW2, 9.3]. Doing this, we obtain



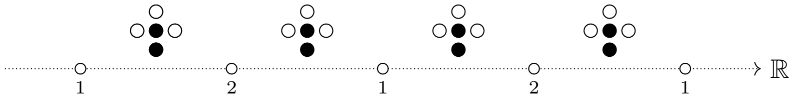
To obtain wall crossing over the wall labelled 2, temporarily delete the vertex corresponding to 2, apply the Dynkin involution to the *remainder* (which is trivial for $A_1 \times A_1 \times A_1 \times A_1$), then insert back in the vertex labelled 2.



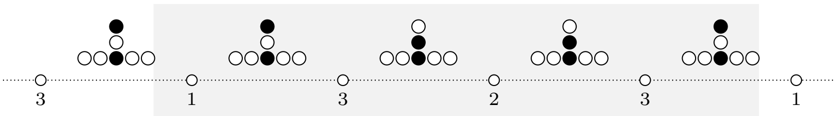
The wall crossing is thus described by



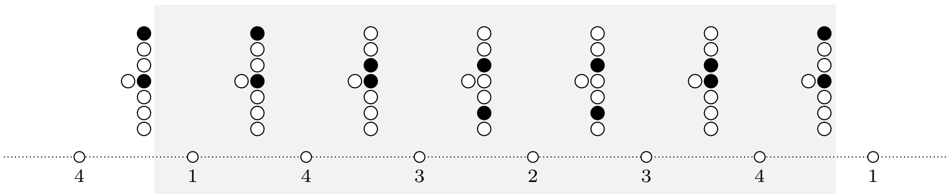
Applying the same local rule but instead at the other vertex, and repeating, gives

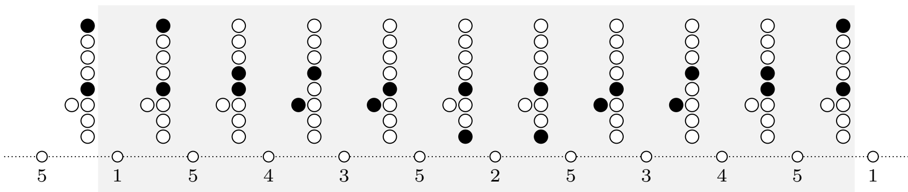


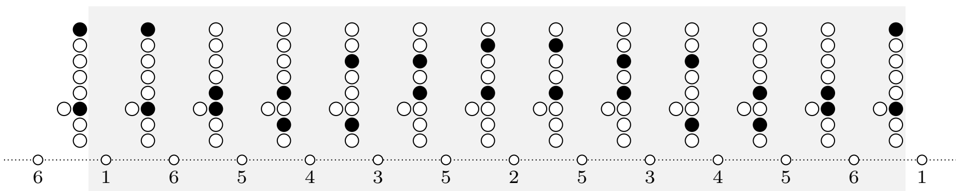
The E_6 case is explained in detail in [W2, 1.1], and is summarised by the following. The shaded region can be ignored for now, but will be used later in Theorem 7.12.



For E_7, $E_8(5)$ and $E_8(6)$, the calculations are, respectively,



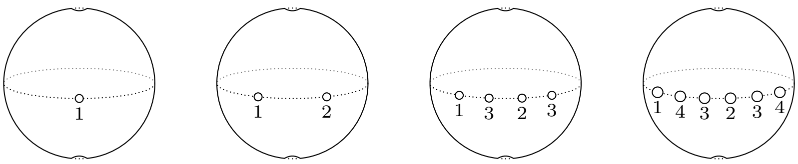




□

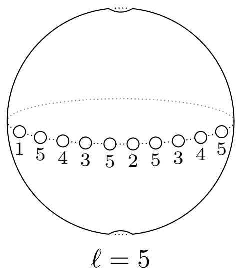
The following extends the pictures in [T, p6169] and [A, Figure 1], which describe the case $\ell = 1$, to all higher lengths.

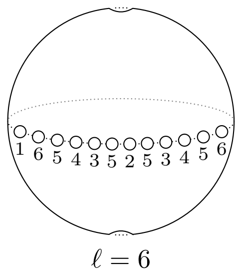
Theorem 7.12. *Suppose that $X \to \operatorname{Spec} R$ is a length ℓ irreducible flop, where X is smooth. Then the SKMS is one of the following:*



$\ell = 1$ $\qquad$ $\ell = 2$ $\qquad$ $\ell = 3$ $\qquad$ $\ell = 4$

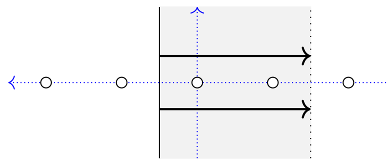
The cases $\ell = 5, 6$ behave slightly differently; respectively they are:



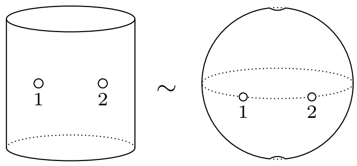


Proof. By Theorem 7.10 and the resulting exact sequence (7.K), we first quotient by $\mathsf{PBr}\,\mathcal{D}$, then quotient by $\operatorname{Pic} X$. By Theorem 6.9, it suffices to identify $(\mathbb{C}^n \backslash \mathcal{H}^{\mathsf{aff}}_{\mathbb{C}})/\operatorname{Pic} X$.

But this is §7.3, see (7.I). Indeed, the action of $\mathcal{O}(1)$ moves chambers to the left, by either $1, 2, 4, 6, 10$, or 12 steps, depending on the length of the flopping curve. Thus, for example in the case $\ell = 2$, the generator $\mathcal{O}(-1)$ of $\operatorname{Pic} X$ acts via



where we have shaded the fundamental domain. Thus, identifying edges to form a cylinder, the quotient space is



All other cases are similar, by identifying the left and right hand sides of the fundamental regions shaded in the proof of Proposition 7.11. $\square$

Appendix A. Combinatorial Tracking Results

In this appendix, which is independent of the rest of the paper, we give the proof of Propositions 3.8 and 3.11, and Lemma 3.10. The Propositions are proved in §A.2, whilst Lemma 3.10 appears as Lemma A.4. Throughout, we use the notation from Section 3.

A.1. Preliminary Results.

First, for $L \in \operatorname{Mut}_0(N)$ write $z \in (\Theta_L)_{\mathbb{C}}$ as

$$z = (x_1, \dots, x_n) + (y_1, \dots, y_n)\mathrm{i} \in \mathbb{R}^n_x + \mathbb{R}^n_y\,\mathrm{i}.$$

The action of φ_L on K-theory is induced from $\mathbb{Z}$, so it independently acts on both factors $\mathbb{R}^n_x$ and $\mathbb{R}^n_y$. We will write $\mathcal{H}_x$ for the hyperplanes $\mathcal{H}$ viewed in $\mathbb{R}^n_x$, and $\mathcal{H}_y$ for the hyperplanes $\mathcal{H}$ viewed in $\mathbb{R}^n_y$. Likewise for $H \in \mathcal{H}$, we will write $H_x \in \mathcal{H}_x$ and $H_y \in \mathcal{H}_y$ accordingly. Since $\mathcal{H}_{\mathbb{C}} := \{H \times H \mid H \in \mathcal{H}\}$, by definition

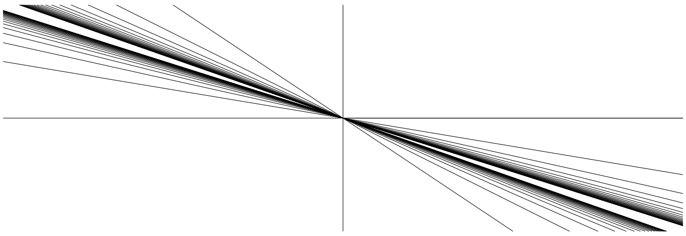
$$z = x + y\mathrm{i} \in \mathcal{H}_{\mathbb{C}} \iff \exists\, H \in \mathcal{H} \text{ such that } x \in H_x \text{ and } y \in H_y. \tag{A.A}$$

On the other hand, for $L \in \operatorname{Mut}(N)$ write $z \in (\mathcal{K}_L)_{\mathbb{C}}$ as

$$z = (x_0, x_1, \dots, x_n) + (y_0, y_1, \dots, y_n)\mathrm{i} \in \mathbb{R}^{n+1}_x + \mathbb{R}^{n+1}_y\,\mathrm{i}.$$

The action of ϕ_L again acts independently on both factors. As in §3.5, consider $\mathcal{W}$, the set of full hyperplanes in $\mathcal{K}_L \otimes \mathbb{R}$ that separate the open chambers of $\mathsf{TCone}(\mathcal{J}_{\mathsf{aff}})$.

Example A.1. In Example 3.5, $\mathcal{W}$ is the following infinite collection of hyperplanes in $\mathbb{R}^2$



The hyperplanes converge on the line $\vartheta_0 + 3\vartheta_1 = 0$, but $\mathcal{W}$ does not contain this line.

Write $\mathcal{W}_x$ for the hyperplanes $\mathcal{W}$ viewed in $\mathbb{R}^{n+1}_x$, and $\mathcal{W}_y$ for the hyperplanes $\mathcal{W}$ viewed in $\mathbb{R}^{n+1}_y$. Mirroring the above notation, for $W \in \mathcal{W}$, similarly consider $W_x \in \mathcal{W}_x$ and $W_y \in \mathcal{W}_y$. Again, by definition

$$z = x + y\mathrm{i} \in \mathcal{W}_{\mathbb{C}} \iff \exists\, W \in \mathcal{W} \text{ such that } x \in W_x \text{ and } y \in W_y. \tag{A.B}$$

The following is clear.

Lemma A.2. *With notation as above, the following statements hold.*

(1) *The hyperplanes in $\mathcal{H}$ contain the coordinate axes, and are all of the form $\lambda_1 x_{i_1} + \ldots + \lambda_s x_{i_s} = 0$ where each $i_j \in \{1, \ldots, n\}$ and each $\lambda_j > 0$.*
(2) *The hyperplanes in $\mathcal{W}$ contain the coordinate axes, and are all of the form $\lambda_1 x_{i_1} + \ldots + \lambda_s x_{i_s} = 0$ where each $i_j \in \{0, 1, \ldots, n\}$ and each $\lambda_j > 0$.*

Proof. By definition, in both cases C_+ is a chamber. Since the coordinate axes bound this, both first statements follow. The second statements follow from the fact that C_+ is a chamber, together with the observation that if some $\lambda_j < 0$, then the hyperplane would pass through C_+. $\square$

The following is then immediate, and establishes $\supseteq$ in Proposition 3.8.

Corollary A.3. $\bigcup \varphi_L(\mathbb{H}_+) \subseteq \Theta_{\mathbb{C}} \backslash \mathcal{H}_{\mathbb{C}}$

Proof. By (A.A) it is clear that φ_L restricts to a bijection between $(\mathcal{H}_L)_{\mathbb{C}}$ and $\mathcal{H}_{\mathbb{C}}$, since we already know that it does this on both factors. Hence it suffices to show that $\mathbb{H}_+ \subseteq \mathbb{C}^n \backslash \mathcal{H}_{\mathbb{C}}$.

For this, consider $z = x + y\mathrm{i} \in \mathbb{H}_+$. If all $y_i > 0$, then by Lemma A.2(1), all $\lambda_1 y_{i_1} + \ldots + \lambda_s y_{i_s} > 0$, so $y \notin \mathcal{H}_y$, and hence $z \notin \mathcal{H}_{\mathbb{C}}$ by (A.A). By permuting the numbering if necessary, we can thus assume that $y_1 = \ldots = y_t = 0$ for some $1 \leq t \leq n$, and that $y_{t+1}, \ldots, y_n > 0$. In this case, by the positivity of $y_{t+1}, \ldots, y_n$, and the fact the rest are zero, again using Lemma A.2(1) it follows that y avoids all members of $\mathcal{H}_y$ except those hyperplanes of the form

$$\lambda_1 y_{i_1} + \ldots + \lambda_s y_{i_s} = 0$$

where $i_1, \ldots, i_s \in \{1, \ldots, t\}$. But since $z \in \mathbb{H}_+$, the fact that $y_1 = \ldots = y_t = 0$ forces $x_1, \ldots, x_t < 0$. Hence x avoids all the corresponding members

$$\lambda_1 x_{i_1} + \ldots + \lambda_s x_{i_s} = 0$$

of $\mathcal{H}_x$. Thus, overall, y avoids some hyperplanes, and the hyperplanes that it does not avoid are avoided by x. Again by (A.A) it follows that $z \notin \mathcal{H}_{\mathbb{C}}$. $\square$

For the affine version of the above, recall that

$$\mathbb{E}_+ := \left\{ z \in \mathbb{H}'_+ \;\middle|\; \sum_{j=0}^{n} (\mathsf{rk}_R\, L_j) z_j = \mathrm{i} \right\} = (\mathsf{Level}_L)_{\mathbb{C}} \cap \mathbb{H}'_+.$$

The following is elementary.

Lemma A.4 (3.10)**.** *The subspace $\mathbb{E}_+ \subset (\mathcal{K}_L)_{\mathbb{C}}$ is path connected.*

Proof. Set $\lambda_i := \mathsf{rk}_R\, L_i$, and consider

$$\mathbb{E}_+^\circ := \left\{ z \in (\mathcal{K}_L)_\mathbb{C} \;\middle|\; \sum_{j=0}^{n} \lambda_j z_j = \mathrm{i} \text{ and } \mathrm{Im}(z_j) > 0 \text{ for all } j \right\} = \mathbb{H}_+'^\circ \cap (\mathsf{Level}_L)_\mathbb{C}.$$

This is visibly path connected. To prove the statement, it suffices to show that for every boundary point $z \in \mathbb{E}_+ \backslash \mathbb{E}_+^\circ$, there is a path in $\mathbb{E}_+$ from z to a point w in $\mathbb{E}_+^\circ$.

Write $z = x + y\mathrm{i}$, then since $z \in \mathbb{E}_+$, not all y_j can be zero. Further, since $z \in \mathbb{E}_+ \backslash \mathbb{E}_+^\circ$, after reordering if necessary we can write

$$y = (\underbrace{0, \ldots, 0}_{k}, \underbrace{y_k, \ldots, y_n}_{>0}) \in \mathbb{H}_+^\circ \cap (\mathsf{Level}_L)_\mathbb{C}.$$

for some k such that $0 < k < n$. Set $\gamma := \sum_{i=0}^{k} \lambda_i$ and fix s such that $0 < s < \frac{\lambda_n y_n}{\gamma}$. Then for any $t \in [0, s]$, consider the point

$$y(t) := (\underbrace{t, \ldots, t}_{k}, y_k, \ldots, y_{n-1}, y_n - \tfrac{\gamma t}{\lambda_n}) \in \mathcal{K}_L.$$

This satisfies $\sum_{i=0}^{n} \lambda_i y(t)_i = \sum_{i=k}^{n} \lambda_i y_i$, which equals 1 since $z \in \mathbb{E}_+$. Setting $z(t) = x + y(t)\mathrm{i}$, it is then clear that $z(t) \in (\mathsf{Level}_L)_\mathbb{C}$ for all $t \in [0, s]$. On the other hand, by inspection $z(t) \in \mathbb{H}'_+$ for all $t \in [0, s]$. It follows that $z(t) \in \mathbb{E}_+$ for all $t \in [0, s]$, so setting $w := x + y(s)\mathrm{i} \in \mathbb{E}_+^\circ$, the path $p\colon [0, s] \to \mathbb{E}_+$ sending $t \mapsto z(t)$ connects z and w. ☐

Recall that $\mathcal{H}_\mathbb{C}^{\mathsf{aff}} = \mathcal{W}_\mathbb{C} \cap \mathsf{Level}_\mathbb{C}$. The following establishes $\supseteq$ in Proposition 3.11.

Corollary A.5. $\bigcup \phi_L(\mathbb{E}_+) \subseteq \mathsf{Level}_\mathbb{C} \backslash \mathcal{H}_\mathbb{C}^{\mathsf{aff}}$

Proof. By (A.B) ϕ_L restricts to a bijection between $(\mathcal{W}_L)_\mathbb{C}$ and $\mathcal{W}_\mathbb{C}$. Since mutation functors also preserve the level, and $\mathcal{H}_\mathbb{C}^{\mathsf{aff}} = \mathcal{W}_\mathbb{C} \cap \mathsf{Level}_\mathbb{C}$, it suffices to show that $\mathbb{E}_+ \subseteq \mathsf{Level}_\mathbb{C} \backslash \mathcal{H}_\mathbb{C}^{\mathsf{aff}}$.

For this, consider $z = x + y\mathrm{i} \in \mathbb{E}_+$. The key is to view this in $\mathbb{H}'_+$, then follow the proof of Corollary A.3. We appeal to Lemma A.2(2) instead of Lemma A.2(1), and (A.B) instead of (A.A), then it follows that $z \notin \mathcal{W}_\mathbb{C}$. Hence $z \in \mathsf{Level}_\mathbb{C} \backslash \mathcal{H}_\mathbb{C}^{\mathsf{aff}}$. ☐

To obtain the converse direction in both Propositions 3.8 and 3.11 is slightly more tricky. As preparation, recall that if $\mathcal{H}$ is a real hyperplane arrangement, then the intersection poset $\mathcal{L}(\mathcal{H})$ of $\mathcal{H}$ is the set of all possible intersections of subsets of hyperplanes from $\mathcal{H}$. For $X, Y \in \mathcal{L}(\mathcal{H})$, consider

$$\begin{aligned} \mathcal{H}_X &:= \{H \mid X \subseteq H\}, \\ \mathcal{H}^Y &:= \{H \cap Y \mid Y \nsubseteq H\}, \end{aligned}$$

called the localization and restriction arrangements respectively. The localization $\mathcal{H}_X$ is a subarrangement of $\mathcal{H}$, whilst $\mathcal{H}^Y$ is an arrangement in Y. The following is elementary.

Lemma A.6. *If $X \subseteq Y$, then $(\mathcal{H}_X)^Y = (\mathcal{H}^Y)_{X \cap Y}$.*

Proof. On one hand $(\mathcal{H}_X)^Y = \{H \cap Y \mid X \subseteq H \text{ and } Y \nsubseteq H\}$. On the other hand

$$(\mathcal{H}^Y)_{X \cap Y} = \{H \cap Y \mid Y \nsubseteq H \text{ and } X \cap Y \subseteq H \cap Y\}.$$

These two sets are clearly the same, using the assumption that $X \subseteq Y$. ☐

Returning to our flops setting, recall from Lemma A.2 that the coordinate axes belong to $\mathcal{H}$ and $\mathcal{W}$. As notation, for a subset $\mathcal{J} \subseteq \{1, \ldots, n\}$, consider

$$\mathrm{B} = \{\vartheta_j = 0 \text{ for all } j \in \mathcal{J}\} = \bigcap_{j \in \mathcal{J}} \{\vartheta_i = 0\} \in \mathcal{L}(\mathcal{H}).$$

Similarly, for $\mathcal{J}' \subseteq \{0, 1, \ldots, n\}$, consider $\mathbb{B} = \{x_j = 0 \text{ for all } j \in \mathcal{J}'\} \in \mathcal{L}(\mathcal{W})$. In the affine setting, the following result will be crucial.

Lemma A.7. *If $\mathcal{J}' \subsetneq \{0, \ldots, n\}$, then $\mathcal{W}_\mathbb{B}$ is finite*

Proof. Say the Tits cone associated to the extended ADE graph Δ_{aff} lives inside the vector space $V \cong \mathbb{R}^{|\Delta_{\mathsf{aff}}|}$, with coordinates $(x_0, \dots, x_m)$. Write $\mathcal{T}$ for the set of full hyperplanes in V that separate the chambers in the Tits cone. The arrangement $\mathcal{W}$ is the set of full hyperplanes that separate chambers in $\mathsf{TCone}(\mathcal{J}_{\mathsf{aff}})$, so by Definition 3.1 we can write $\mathcal{W} = \mathcal{T}^Y$ for some Y obtained by intersecting coordinate axes. By the commutative diagram in Lemma A.6, to show that $\mathcal{W}_{\mathbb{B}} = (\mathcal{T}^Y)_{\mathbb{B}}$ is finite, it suffices to show that $\mathcal{W} = \mathcal{T}_X$ is finite for any $X = \bigcap_{k \in \mathcal{K}} \{x_k = 0\}$ with $\mathcal{K} \subsetneq \{0, \dots, m\}$. In turn, it suffices to show that the quotient

$$\mathcal{T}/X := \{H/X \mid H \in \mathcal{T}_X\},$$

is a finite arrangement. This lives in V/X, which has lower dimension.

For $k \in \mathcal{K}$, applying the Coxeter element s_k to the basis $\{x_j + X \mid j \in \mathcal{K}\}$ of V/X negates the x_k entry, and adds some multiple of x_k to its neighbours in $\mathcal{K}$, via the standard Coxeter rule. By inspection, this is the same as the Coxeter rule for Γ, where Γ is obtained from Δ_{aff} by deleting the vertices that are not in $\mathcal{K}$. It follows that $\mathcal{T}/X$ is the Tits cone associated to the diagram Γ. But deleting a non-empty set of vertices in an extended ADE Dynkin diagram gives a finite ADE Dynkin diagram, or a disjoint union thereof. Hence the Tits cone for Γ has finitely many hyperplanes, hence so too does $\mathcal{T}/X$, and thus $\mathcal{T}_X$. □

Given $\mathcal{J} \subseteq \{1, \dots, n\}$, $\mathcal{J}' \subsetneq \{0, 1, \dots, n\}$, define

$$\mathrm{D}_- := \{\vartheta \in \Theta_{\mathbb{R}} \mid \vartheta_j < 0 \text{ for all } j \in \mathcal{J}\}$$
$$\mathbb{D}_- := \{x \in \mathcal{K}_{\mathbb{R}} \mid x_j < 0 \text{ for all } j \in \mathcal{J}'\},$$

and let $\mathsf{MutTo}_{\mathcal{J}}(N)$, respectively $\mathsf{MutTo}_{\mathcal{J}'}(N)$, be the set of all mutations $L \to \dots \to N$ whose constituent length one paths all have labels in the set $\mathcal{J}$, respectively $\mathcal{J}'$. The following is one of the main technical results of this Appendix.

Proposition A.8. *Consider subsets $\mathcal{J} \subseteq \{1, \dots, n\}$, $\mathcal{J}' \subsetneq \{0, 1, \dots, n\}$, with associated* B *and* $\mathbb{B}$*. Then the following statements hold.*

(1) *The hyperplanes in $\mathcal{H}_{\mathrm{B}}$ are precisely those hyperplanes $\lambda_1\vartheta_{i_1} + \dots + \lambda_s\vartheta_{i_s} = 0$ from $\mathcal{H}$ such that every $i_j \in \mathcal{J}$. Necessarily each $\lambda_j > 0$.*
(2) *The hyperplanes in $\mathcal{W}_{\mathbb{B}}$ are precisely those hyperplanes $\lambda_1 x_{i_1} + \dots + \lambda_s x_{i_s} = 0$ from $\mathcal{W}$ such that every $i_j \in \mathcal{J}'$. Necessarily each $\lambda_j > 0$.*
(3) *There are decompositions*

$$\bigcup_{\alpha \in \mathsf{MutTo}_{\mathcal{J}}(N)} \varphi_\alpha(\mathrm{D}_-) = \Theta_{\mathbb{R}} \backslash \mathcal{H}_{\mathrm{B}} \quad \text{and} \quad \bigcup_{\beta \in \mathsf{MutTo}_{\mathcal{J}'}(N)} \phi_\alpha(\mathbb{D}_-) = \mathcal{K}_{\mathbb{R}} \backslash \mathcal{W}_{\mathbb{B}}.$$

Proof. (1) The first statement is elementary. The second is immediate from Lemma A.2(1). Part (2) is identical, using instead Lemma A.2(2).
(3) Tracking across the mutation $\nu_i N \to N$ for $i \in \mathcal{J}$, by (3.B) D_- gets sent to the region

$$\vartheta_i > 0, \quad \vartheta_j + b_{ij}\vartheta_i < 0 \text{ for } j \in \mathcal{J} - \{i\} \tag{A.C}$$

of $\Theta_{\mathbb{R}} = \mathbb{R}^n$, where $b_{ij} \in \mathbb{Z}_{\geq 0}$. In contrast, C_- gets sent to the region

$$\vartheta_i > 0, \quad \vartheta_j + b_{ij}\vartheta_i < 0 \text{ for } j \in \{1, \dots, n\} - \{i\}. \tag{A.D}$$

Thus, by part (1) we see that the walls of (A.C) are precisely those hyperplanes in $\mathcal{H}$ that belong to $\mathcal{H}_{\mathrm{B}}$ and also bound the walls of (A.D). Hence the region (A.C) is bounded by elements of $\mathcal{H}_{\mathrm{B}}$. There are no further walls inside this region, since otherwise there would be further walls within (A.D), which is not the case, using the C_- version of Theorem 3.6.

Repeating the above argument, all chambers adjacent to D_- in $\mathcal{H}_{\mathrm{B}}$ can be obtained by tracking D_- through some mutation $\nu_i N \to N$. The proof then just proceeds by induction. Consider $\nu_j\nu_i N \to \nu_i N \to N$, track D_- through both mutations, and just appeal to the C_- version of Theorem 3.6. This process finishes since the numbers of chambers in $\mathcal{H}_{\mathrm{B}}$ is finite, since $\mathcal{H}$ is, and at each stage mutation at the labels in $\mathcal{J}$ describes the $|\mathcal{J}|$ possible wall-crossings in each chamber.

The last statement is similar, replacing Δ by Δ_{aff}, and using the C_- version of Theorem 3.3 in place of Theorem 3.6. The key point is that, since $\mathcal{J}'$ is a proper subset, by Lemma A.7 the hyperplane arrangement $\mathcal{W}_{\mathbb{B}}$ is still finite, and so the conclusion of the last sentence in the above paragraph still holds. □

Corollary A.9. *There is an inclusion* $\mathbb{C}^n\backslash\mathcal{H}_{\mathbb{C}} \subseteq \bigcup_{L\in\mathrm{Mut}_0(N)} \varphi_L(\mathbb{H}_+)$*, and furthermore* $\mathsf{Level}_{\mathbb{C}}\backslash\mathcal{H}^{\mathsf{aff}}_{\mathbb{C}} \subseteq \bigcup_{L\in\mathrm{Mut}(N)} \phi_L(\mathbb{E}_+)$.

Proof. Pick $z \in \mathbb{C}^n\backslash\mathcal{H}_{\mathbb{C}}$, and write $z = x + y\mathrm{i}$. By Theorem 3.6 applied to $\mathcal{H}_y$, we can find some $L \in \mathrm{Mut}_0(N)$ such that $\varphi_L^{-1}(y)$ has all coordinates ≥ 0. If all coordinates are positive, then $\varphi_L^{-1}(z) \in \mathbb{H}_+$, thus $z \in \varphi_L(\mathbb{H}_+)$, as required.

Hence we can assume that some coordinate of $y' := \varphi_L^{-1}(y) \in \Theta_L$ is zero. Thus there exists some non-empty subset $\mathcal{I} \subseteq \{1,\ldots,n\}$ such that $y'_i = 0$ for all $i \in \mathcal{I}$, and $y'_j > 0$ for $j \notin \mathcal{I}$. Note that $\mathcal{I} = \{1,\ldots,n\}$ is possible, in which case all $y'_i = 0$.

Recall that $\varphi_L := \varphi_\alpha$, where $\alpha\colon L \to \ldots \to N$. Write

$$\varphi_\alpha^{-1}(z) = \varphi_L^{-1}(z) = x' + y'\mathrm{i}.$$

Since x' belongs to the closure of some chamber in the x-version of Θ_L, applying Proposition A.8(3) to $(\mathcal{H}_L)_x \subset \Theta_L$ we can find a sequence of mutations

$$M \xrightarrow{\nu_{j_1}} \ldots \xrightarrow{\nu_{j_s}} L$$

with each $j_k \in \mathcal{I}$, such that $\varphi_{j_1}^{-1}\ldots\varphi_{j_s}^{-1}(x) \in \overline{D}_-$. Since each $j_k \in \mathcal{I}$, $y'_{j_k} = 0$, and so this path has no effect on y'. In particular,

$$\varphi_{j_1}^{-1}\ldots\varphi_{j_s}^{-1}\varphi_\alpha^{-1}(z) = x'' + y'\mathrm{i}, \tag{A.E}$$

where $x''_i \leq 0$ provided that $i \in \mathcal{I}$. Since $z \notin \mathcal{H}_{\mathbb{C}}$, we must have $x''_i \neq 0$ for all $i \in \mathcal{I}$, else $\varphi_{j_1}^{-1}\ldots\varphi_{j_s}^{-1}\varphi_\alpha^{-1}(z)$ and thus z belongs to a complexified hyperplane. Hence (A.E) belongs to $\mathbb{H}_+$, and so $z \in \varphi_\alpha\varphi_{j_s}\ldots\varphi_{j_1}(\mathbb{H}_+)$. This is the tracking of $\mathbb{H}_+$ through the chain

$$M \xrightarrow{\nu_{j_1}} \ldots \xrightarrow{\nu_{j_s}} \overbrace{L \to \ldots \to}^{\alpha} N.$$

Appealing to Proposition 4.8, $\varphi_\alpha\varphi_{j_s}\ldots\varphi_{j_1} = \varphi_M$, and so $z \in \varphi_M(\mathbb{H}_+)$.

For the second statement, let $z \in \mathsf{Level}_{\mathbb{C}}\backslash\mathcal{H}^{\mathsf{aff}}_{\mathbb{C}}$. The proof proceeds as above, replacing φ by ϕ at all stages. We obtain a subset $\mathcal{I}' \subseteq \{0,1,\ldots,n\}$, but crucially now $\mathcal{I}' \neq \{0,1,\ldots,n\}$ since $\varphi_L^{-1}(z) \in (\mathsf{Level}_L)_{\mathbb{C}}$. This is due to the fact that mutation functors preserve the level, and so multiples of the y coordinates must sum to one, hence not all the y can be zero. Hence $\mathcal{I}'$ is a proper subset, which still allows us to still appeal to Proposition A.8(3). We thus still deduce that $\phi_M^{-1}(z) \in \mathbb{H}'_+$. Since mutation functors preserve the level, automatically it follows that $\phi_M^{-1}(z) \in \mathbb{E}_+$, and so $z \in \phi_M(\mathbb{E}_+)$. $\square$

We lastly show that the unions are disjoint, which requires the following.

Lemma A.10. *Let* $L, M \in \mathrm{Mut}_0(N)$*, respectively* $\mathrm{Mut}(N)$*, and let* $\alpha\colon L \to M$ *be a minimal path. Suppose that* $\varphi_\alpha(a'_1,\ldots,a'_n) = (a_1,\ldots,a_n)$ *in* $(\Theta_M)_{\mathbb{R}}$*, respectively* $\phi_\alpha(a'_0,\ldots,a'_n) = (a_0,\ldots,a_n)$ *in* $(\mathcal{K}_M)_{\mathbb{R}}$*, with all* $a_i, a'_i \in \mathbb{R}_{\geq 0}$*, and write* $\mathcal{I} = \{i \mid a'_i = 0\}$.

(1) *If* $\mathcal{I} = \emptyset$*, then* $L \cong M$.
(2) *If* $\mathcal{I} \neq \emptyset$*, then the following statements hold.*
  (a) *The simple mutations giving* α *all have labels in the set* $\mathcal{I}$.
  (b) $a'_i = a_i$ *for all* i.
  (c) $\varphi_\alpha[\mathcal{P}'_i] = [\mathcal{P}_i]$*, respectively* $\phi_\alpha[\mathcal{P}'_i] = [\mathcal{P}_i]$*, for all* $i \notin \mathcal{I}$.

Proof. We prove the affine case $\mathcal{K}_M$, with the case Θ_M being similar.

Set $p = \sum_{i=0}^n a'_i[\mathcal{P}_i]$. For a general subset J of $\{0,1,\ldots,n\}$ and $N' \in \mathrm{Mut}(N)$, set

$$C_J := \left\{ \sum_{i=0}^n c_i[\mathcal{P}_i] \in (\mathcal{K}_{N'})_{\mathbb{R}} \;\middle|\; \begin{array}{ll} c_i = 0 & \text{if } i \in J \\ c_i > 0 & \text{if } i \notin J \end{array} \right\}.$$

As calibration, note that $C_+ = C_\emptyset$.
(1) If all $a_i > 0$, then since chambers map to chambers, $\phi_\alpha(p) \in C_+$. Thus $C_+ \cap \phi_\alpha(C_+) \neq \emptyset$. By Theorem 3.3 (respectively 3.6 for Θ), $L \cong M$.
(2) By assumption $\mathcal{I} \neq \emptyset$, in which case its complement $\mathcal{I}^c$ in $\{0,1,\ldots,n\}$ is a proper subset. Since $p \in C_{\mathcal{I}}$, which is a codimension $|\mathcal{I}^c|$ wall of C_+, and ϕ_α maps walls to walls (maintaining codimension), it follows that $\phi_\alpha(p)$ lies in a codimension $|\mathcal{I}^c|$ wall of C_+. These all have

the form $C_{\mathcal{J}'}$ for some $|\mathcal{J}'| = |\mathcal{J}|$, and so it follows that $\phi_\alpha(p) \in C_{\mathcal{J}'}$ for some such $\mathcal{J}'$ with $|\mathcal{J}'| = |\mathcal{J}|$.

We first argue that (a) implies (b) and (c). Indeed, since ϕ_j, with $j \in \mathcal{J}$, negates the entry j (which is zero) and adds zero to the neighbours, evidently this has no effect on elements in $C_{\mathcal{J}}$, and so $(a_0, \dots, a_n) = \phi_\alpha(a'_0, \dots, a'_n) = (a'_0, \dots, a'_n)$. Consequently, we have $a'_i = a_i$ for all i, proving (b). Furthermore, (c) follows immediately from (a), using Lemma 3.2.

Now we prove (a). Write $\alpha\colon L_0 := L \to L_1 \to \dots \to L_m := M$ for a minimal path from L to M. By [IW2, 1.12], in every $\mathcal{K}_{L_i}$, all chambers and all walls of all codimension are labelled by Coxeter information. Namely, by wC_J for some w in the affine Weyl group $W_{\Delta_{\mathsf{aff}}}$, and some subset J of the vertices of the affine Dynkin diagram Δ_{aff}. The codimension $|\mathcal{J}^c|$ walls have the form wC_J for certain J and $w \in W_{\Delta_{\mathsf{aff}}}$, with $|J| = |\mathcal{J}^c|$. Consequently, tracking $C_{\mathcal{J}}$ in $\mathcal{K}_L \otimes \mathbb{R}$ through $\phi_\alpha = \phi_{i_n} \dots \phi_{i_1}$ gives a sequence of labelled walls

$$C_{\mathcal{J}} \mapsto w_1 C_{\mathcal{J}_1} \mapsto w_2 C_{\mathcal{J}_2} \mapsto \dots \mapsto w_m C_{\mathcal{J}_m} = \phi_\alpha(C_{\mathcal{J}}),$$

where the last term is in $\mathcal{K}_M \otimes \mathbb{R}$. At each step, since atoms follow the weak order, the length of the smallest coset representative w_i cannot decrease. This holds just since the statement is true for chambers, and the labels on the walls are induced from these.

Since $\phi_\alpha(p) \in C_{\mathcal{J}'}$, and $\phi_\alpha(p) \in \phi_\alpha(C_{\mathcal{J}}) = w_m C_{\mathcal{J}_m}$, it follows that $C_{\mathcal{J}'} \cap w_m C_{\mathcal{J}_m} \neq \emptyset$. As is standard [B2, V.4.6, Proposition 5], we deduce that $\mathcal{J}_m = \mathcal{J}'$, and $w_m \in W_{\mathcal{J}'}$, where $W_{\mathcal{J}'}$ is the subgroup of $W_{\Delta_{\mathsf{aff}}}$ generated by s_i with $i \in \mathcal{J}'$. In particular $w_m C_{\mathcal{J}_m} = C_{\mathcal{J}'}$, and so the above chain is

$$C_{\mathcal{J}} \mapsto w_1 C_{\mathcal{J}_1} \mapsto w_2 C_{\mathcal{J}_2} \mapsto \dots \mapsto C_{\mathcal{J}'}.$$

As the minimal length of the coset representative w_i cannot decrease throughout the chain, and the chain starts and finishes with length zero, it follows that at each step $w_i C_{\mathcal{J}_i} = C_{\mathcal{J}_i}$. Thus, at step one, $\phi_{i_1}\colon C_{\mathcal{J}} \mapsto C_{\mathcal{J}_1}$. By inspection, this occurs if and only if the label i_1 is in the set $\mathcal{J}$, and $\mathcal{J} = \mathcal{J}_1$. Inducting along the chain, every label i_t is in the set $\mathcal{J}$, and (a) follows. □

Corollary A.11. *Suppose that $L, M \in \operatorname{Mut}_0(N)$, and let α be the minimal path from L to M. Then $\varphi_\alpha(\mathbb{H}_+) \cap \mathbb{H}_+ \neq \emptyset$ in $\Theta_M \iff L \cong M$. The same statement holds for $\operatorname{Mut}(N)$, using instead $\phi_\alpha(\mathbb{E}_+) \cap \mathbb{E}_+ \neq \emptyset$.*

Proof. ($\Leftarrow$) is clear. For ($\Rightarrow$), if the intersection is nonempty, then

$$\varphi_\alpha(z'_1, \dots, z'_n) = (z_1, \dots, z_n)$$

where all $z_i, z'_i \in \mathbb{H}$. Splitting into real and imaginary parts,

$$\varphi_\alpha(x'_1, \dots, x'_n) = (x_1, \dots, x_n) \quad \text{and} \quad \varphi_\alpha(y'_1, \dots, y'_n) = (y_1, \dots, y_n). \tag{A.F}$$

Since all $z_i, z'_i \in \mathbb{H}$, necessarily the right hand equation belongs to $\varphi_\alpha(\overline{C}_+) \cap \overline{C}_+$. As before, write $\mathcal{J}$ for the set of i for which $y'_i = 0$.

On one hand, if $\mathcal{J}$ is not a proper subset, then $y_i = y'_i = 0$ for all i. Since every $z_i, z'_i \in \mathbb{H}$, all $x_i, x'_i < 0$. Using the C_- version of Proposition 3.6 applied to the x-coordinate, $L \cong M$. On the other hand, if $\mathcal{J} = \emptyset$, then by Lemma A.10(1) we also have $L \cong M$.

Hence we can assume that $\mathcal{J} \neq \emptyset$ and $\mathcal{J}$ is a proper subset. Since $\mathcal{J} \neq \emptyset$, by Lemma A.10(2), α comprises of mutations with labels only from the set $\mathcal{J}$. Further, all $y'_i = y_i$ (and so in particular $y_i = 0$ if $i \in \mathcal{J}$), and $\varphi_\alpha[\mathcal{P}'_i] = [\mathcal{P}_i]$ if $i \notin \mathcal{J}$. Since $z_i, z'_i \in \mathbb{H}$, we then deduce that $x_i < 0$ and $x'_i < 0$ for all $i \in \mathcal{J}$, and that we can re-write the left hand equation in (A.F) to obtain

$$\sum_{i \in \mathcal{J}} x'_i \varphi_\alpha[\mathcal{P}'_i] + \sum_{i \notin \mathcal{J}} x'_i [\mathcal{P}_i] = \sum_{i \in \mathcal{J}} x_i [\mathcal{P}_i] + \sum_{i \notin \mathcal{J}} x_i [\mathcal{P}_i].$$

Set $I \subset \mathcal{J}^c$ to consist of those i such that $x'_i - x_i \geq 0$, and let $I^c = \mathcal{J}^c - I$. Re-arranging gives

$$\begin{aligned}\sum_{i \in I} (x'_i - x_i)[\mathcal{P}_i] + \sum_{i \in \mathcal{J}} (-x_i)[\mathcal{P}_i] &= \sum_{i \in I^c} (x_i - x'_i)[\mathcal{P}_i] - \sum_{i \in \mathcal{J}} x'_i \varphi_\alpha[\mathcal{P}'_i]. \\ &= \varphi_\alpha \left(\sum_{i \in I^c} (x_i - x'_i)[\mathcal{P}'_i] + \sum_{i \in \mathcal{J}} (-x'_i)[\mathcal{P}'_i] \right).\end{aligned}$$

The left hand side has non-negative coefficients in every entry $\{0, 1 \ldots, n\}$, and the right hand side is in $\varphi_\alpha(\overline{C}_+)$. If $I = \emptyset$, so that $I^c = \mathcal{I}^c$, then the right hand side is in $\varphi_\alpha(C_+)$ and so by Lemma A.10(1), $L \cong M$.

Hence our final case is when $\mathcal{I} \neq \emptyset$, $\mathcal{I}$ is proper, and $I \neq \emptyset$. We will show that this cannot occur, by exhibiting a contradiction. Again, the above displayed equation lies in $\varphi_\alpha(\overline{C}_+) \cap \overline{C}_+$ and so now by Lemma A.10(2) the coefficients on both sides must match. But coefficients in I^c do not appear on the left hand side, nor do coefficients in I on the right, so we deduce that $x_i - x_i' = 0$ for all $i \in \mathcal{I}^c$. But then

$$\varphi_\alpha\left(\sum_{i\in\mathcal{I}}(-x_i')[\mathcal{P}_i]\right) = \sum_{i\in\mathcal{I}}(-x_i)[\mathcal{P}_i],$$

and so by Lemma A.10(2) α comprises mutations only from the set $\mathcal{I}^c$. But as stated above, α can only comprise labels in the set $\mathcal{I}$. This is a contradiction, which shows that this final case cannot exist.

The proof of the second statement is similar. The set $\mathcal{I}$ must be proper since $z \in \mathbb{E}_+$, so all y coordinates are nonnegative, and after weighting by $\mathsf{rk}_R\, M_i$ they sum to one. Hence not all can be zero. Given this, the rest of the proof remains the same: since $\mathbb{E}_+ \subset \mathbb{H}'_+$, replacing φ by ϕ throughout, and starting the indices with 0, the logic above still holds, as we can still appeal to Lemma A.10 in the affine case. $\square$

Corollary A.12. *If $L, M \in \mathrm{Mut}_0(N)$, then $\varphi_L(\mathbb{H}_+) \cap \varphi_M(\mathbb{H}_+) \neq \emptyset$ in $\Theta_N \iff L \cong M$. The same statement holds for $\mathrm{Mut}(N)$, using instead $\phi_L(\mathbb{E}_+) \cap \phi_M(\mathbb{E}_+) \neq \emptyset$ in $\mathcal{K}_N$.*

Proof. ($\Leftarrow$) is clear. For ($\Rightarrow$), if $\varphi_L(\mathbb{H}_+) \cap \varphi_M(\mathbb{H}_+) \neq \emptyset$, then $\varphi_M^{-1} \circ \varphi_L(\mathbb{H}_+) \cap \mathbb{H}_+ \neq \emptyset$. By Proposition 4.8(3), $\varphi_\alpha(\mathbb{H}_+) \cap \mathbb{H}_+ \neq \emptyset$ in Θ_M, where α is the minimal path from L to M. By Corollary A.11, the atom α is the identity, and the result follows. The second statement is identical, as we can still appeal to Proposition 4.8(3) and Corollary A.11. $\square$

A.2. **Proof of Propositions 3.8 and 3.11.** Proposition 3.8 asserts that there is an equality

$$\Theta_{\mathbb{C}} \backslash \mathcal{H}_{\mathbb{C}} = \bigcup_{L \in \mathrm{Mut}_0(N)} \varphi_L(\mathbb{H}_+),$$

where the union on the right hand side is disjoint. This now follows by combining Corollaries A.3, A.9 and A.12. On the other hand, Proposition 3.11 asserts that

$$\mathsf{Level}_{\mathbb{C}} \backslash \mathcal{H}_{\mathbb{C}}^{\mathsf{aff}} = \bigcup_{L \in \mathrm{Mut}(N)} \phi_L(\mathbb{E}_+),$$

where the union on the right hand side is disjoint. This now follows by combining Corollaries A.5, A.9 and A.12.

Appendix B. List of Notation

C	$:= f^{-1}(\mathfrak{m})$ with reduced scheme structure	p4
n	number of irreducible components of C	p4
$\mathrm{C}_1, \ldots, \mathrm{C}_n$	irreducible components of C	p4
$\mathcal{L}_i$	line bundle on X such that $\mathcal{L}_i \cdot \mathrm{C}_j = \delta_{i,j}$	§2.1
$\mathcal{V}_i$	universal extension of $\mathcal{L}_i$	(2.A)
$\mathcal{V}_X$	tilting bundle $\oplus_{i=0}^n \mathcal{V}_i^*$ on X	(2.B)
N_i, N	$f_*(\mathcal{V}_i^*)$ and $f_*(\mathcal{V}_X)$ respectively	(2.C)
Λ	endomorphism algebra $\mathrm{End}_X(\mathcal{V}_X) \cong \mathrm{End}_R(N)$	§2.1
$\mathrm{ref}\, R$	category of reflexive R-modules	§2.1
$\mathrm{CM}\, R$	category of maximal CM R-modules	§2.1
$\mathrm{add}\, M$	summands of finite sums of module M	§2.2
$\nu_i L$	mutation of L at i-th summand	§2.2
$\nu_i \Lambda$	endomorphism algebra $\mathrm{End}_R(\nu_i N)$	§2.2
$\mathrm{Mut}(N)$	iterated mutations of N	2.2
$\mathrm{Mut}_0(N)$	subset mutating at only $(i \neq 0)$-th summands	2.2
$\mathsf{EG}(N)$	exchange graph of N	2.2

$\mathsf{EG}_0(N)$	subgraph with vertices CM modules	2.2
Φ_i	mutation functor at the i-th summand	(2.E)
Flop_i	quasi-inverse of the flop functor	2.3
Δ, Δ_{aff}	ADE Dynkin diagram, and extended version	§3.1
$\star$	extending vertex in Δ_{aff}	§3.1
$\mathcal{J}$	subset of vertices of Δ	§3.1
$\mathcal{J}_{\mathsf{aff}}$	$\mathcal{J}$ considered as a subset of Δ_{aff}	§3.1
$\mathsf{TCone}(\mathcal{J})$	$\mathcal{J}$-finite hyperplane arrangement	§3.1
$\mathsf{TCone}(\mathcal{J}_{\mathsf{aff}})$	$\mathcal{J}$-affine arrangement	§3.1
$\mathcal{P}_i$	projective module of the i-th summand	§3.2
Λ_L	endomorphism algebra $\mathrm{End}_R(L)$	§3.2
$\mathcal{K}_L$	K_0-group of perfect complexes over Λ_L	§3.2
ϕ_i	isomorphism of K_0-groups associated to Φ_i	§3.2
Φ_L	compositions of Φ_i along a shortest path	(3.C)
ϕ_L	isomorphism of K_0-groups associated to Φ_L	§3.2
C_+	positive cone in a real vector space	§3.2
Θ_L	$:= \mathcal{K}_L/[\mathcal{P}_0]$, set of stability parameters	§3.3
φ_L	isomorphism on Θ induced from ϕ_L	§3.3
$\mathsf{rk}_R\, M$	rank of R-module M	3.4
Level_L	$:= \{z \in \mathcal{K}_L \otimes \mathbb{R} \mid \sum(\mathsf{rk}_R\, L_j)z_j = 1\}$	3.4
Alcove_L	$:= \phi_L(C_+) \cap \mathsf{Level}_N$	§3.2
$\mathcal{H}^{\mathsf{aff}}$	complement of the union of all Alcove_L	(3.D)
$\mathcal{H}$	complement of the union of all $\varphi_L(C_+)$	(3.E)
$\mathbb{H}$	the semi-closed upper half plane in $\mathbb{C}$	§3.5
$\mathbb{H}_+$	subset of $(\Theta_L)_{\mathbb{C}} \cong \mathbb{C}^n$ corresponding to $\mathbb{H}^n$	§3.5
$\mathbb{H}'_+$	subset of $(\mathcal{K}_L)_{\mathbb{C}} \cong \mathbb{C}^{n+1}$ corresponding to $\mathbb{H}^{n+1}$	§3.5
i	imaginary number $\sqrt{-1}$	§3.5
$(\mathsf{Level}_L)_{\mathbb{C}}$	$:= \{z \in \mathcal{K}_L \otimes \mathbb{C} \mid \sum(\mathsf{rk}_R\, L_j)z_j = \mathrm{i}\}$	§3.5
$\mathbb{E}_+$	$:= \mathbb{H}'_+ \cap (\mathsf{Level}_L)_{\mathbb{C}}$	§3.5
$\mathcal{W}$	hyperplanes separating chambers of $\mathsf{TCone}(\mathcal{J}_{\mathsf{aff}})$	p12
$W_{\mathbb{C}}$	$W \oplus \mathrm{i}W$	p12
$\mathcal{W}_{\mathbb{C}}$	union of all $W_{\mathbb{C}}$	p12
$\mathcal{H}^{\mathsf{aff}}_{\mathbb{C}}$	$\mathcal{W}_{\mathbb{C}} \cap (\mathsf{Level}_N)_{\mathbb{C}}$	(3.G)
$\Gamma_{\mathcal{H}}$	graph of a hyperplane arrangement $\mathcal{H}$	4.1
$\mathcal{G}^+_{\mathcal{H}}$	category of positive paths in $\Gamma_{\mathcal{H}}$	4.1
$\mathcal{G}_{\mathcal{H}}$	groupoid completion of $\mathcal{G}^+_{\mathcal{H}}$	4.3
$\mathbb{G}^{\mathsf{aff}}$, $\mathbb{G}$	$\mathcal{G}_{\mathcal{H}^{\mathsf{aff}}}$ and $\mathcal{G}_{\mathcal{H}}$ respectively	4.4
Φ_α	compositions of Φ_i along a path α	§4.2
$\mathrm{Stab}\,\mathcal{T}$	stability conditions on triangulated category $\mathcal{T}$	§5.1
$\mathrm{Auteq}\,\mathcal{T}$	group of autoequivalences of $\mathcal{T}$	§5.1
Φ_*	map on stability conditions induced by Φ	§5.1
$\mathcal{S}_i$	simple module corresponding to projective $\mathcal{P}_i$	§5.2
$\mathcal{B}_L$	category of finite length Λ_L-modules	§5.2
$\mathcal{A}_L$	subcategory of $\mathcal{B}_L$ without $\mathcal{S}_0$	§5.2
$\mathcal{C}_L$, $\mathcal{D}_L$	subcategory of $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda_L)$ of complexes with cohomology in $\mathcal{A}_L$, respectively $\mathcal{B}_L$	§5.2
$\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$	corresponding categories when $L = N$	§5.2
$\mathrm{Stab}\,\mathcal{A}$	stability on heart $\mathcal{A}$ satisfying HN property	§5.2
$\mathrm{Stab}_n\mathcal{D}_L$	space of normalised stability conditions on $\mathcal{D}_L$	§5.2
$\mathrm{Stab}_n\mathcal{B}_L$	$:= \mathrm{Stab}\,\mathcal{B}_L \cap \mathrm{Stab}_n\mathcal{D}_L$	§5.2
$\mathrm{Stab}^\circ\mathcal{C}$	component of $\mathrm{Stab}\,\mathcal{C}$ containing $\mathrm{Stab}\,\mathcal{A}$	p20
$\mathrm{Stab}^\circ_n\mathcal{D}$	component of $\mathrm{Stab}_n\mathcal{D}$ containing $\mathrm{Stab}\,\mathcal{B}$	p20
$\mathsf{Term}_0(C_+)$	morphisms in $\mathbb{G}$ which terminate at C_+	§6.1
$\mathsf{Term}(C_+)$	morphisms in $\mathbb{G}^{\mathsf{aff}}$ which terminate at C_+	§6.1
U_L	certain open subset of $\mathrm{Stab}\,\mathcal{A}_L$	6.1
$\mathbb{U}_L$	certain open subset of $\mathrm{Stab}\,\mathcal{B}_L$	6.1

$\mathbb{N}_L$	$:= \mathbb{U}_L \cap \mathrm{Stab}_n \mathcal{B}_L$	6.1
$\mathrm{Stab}\,\mathcal{A}_\alpha$	$:= (\Phi_\alpha)_*(\mathrm{Stab}\,\mathcal{A}_{s(\alpha)})$	6.1
$\mathrm{Stab}\,\mathcal{B}_\beta$	$:= (\Phi_\beta)_*(\mathrm{Stab}\,\mathcal{B}_{s(\beta)})$	6.1
U_α	$:= (\Phi_\alpha)_*(\mathrm{U}_{s(\alpha)})$	6.1
$\mathbb{U}_\beta$	$:= (\Phi_\beta)_*(\mathbb{U}_{s(\beta)})$	6.1
$\mathrm{Stab}_n\mathcal{B}_\beta$	$:= (\Phi_\beta)_*(\mathrm{Stab}_n\mathcal{B}_{s(\beta)})$	6.1
$\mathbb{N}_\beta$	$:= (\Phi_\beta)_*(\mathbb{N}_{s(\beta)})$	6.1
$\mathsf{PBr}\,\mathcal{C}$	$:= \{\Phi_\alpha\|_{\mathcal{C}} \mid \alpha \in \mathrm{End}_{\mathbb{G}}(C_+)\}$	6.7
$\mathsf{PBr}\,\mathcal{D}$	$:= \{\Phi_\beta\|_{\mathcal{D}} \mid \beta \in \mathrm{End}_{\mathbb{G}^{\mathsf{aff}}}(C_+)\}$	6.7
$\mathrm{Aut}^\circ\mathcal{C}$	certain subgroup of $\mathrm{Auteq}\,\mathcal{C}$	§7.1
L_i	$:= f_*\mathcal{L}_i$	§7.2
$\mathsf{L}\cdot M$	$:= (\mathsf{L}\otimes M)^{**}$	§7.2
ε	natural isomorphism from Λ_M to $\Lambda_{\mathsf{L}\cdot M}$	§7.2
$\mathrm{Aut}^\circ\mathcal{D}$	certain subgroup of $\mathrm{Auteq}\,\mathcal{D}$	§7.4

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Y. Hirano, Department of Mathematics, Kyoto University, Kitashirakawa-Oiwake-cho, Sakyo-ku, Kyoto, 606-8502, Japan.

E-mail address: y.hirano@math.kyoto-u.ac.jp

M. Wemyss, The Mathematics and Statistics Building, University of Glasgow, University Place, Glasgow, G12 8QQ, UK.

E-mail address: michael.wemyss@glasgow.ac.uk