Solutions to 2AAC practice exam. (You only need to answer $\underline{4}$ questions, one from each section. Below are $\underline{\text{all}}$ the solutions)

Q1/(a) Let $z = x_1 + iy_1$ and $w = x_2 + iy_2$, where $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

(i) TRUE. $\overline{(z+w)} = \overline{(x_1+x_2) + i(y_1+y_2)} = (x_1+x_2) - i(y_1+y_2) = (x_1 - iy_1) + (x_2 - iy_2) = \bar{z} + \bar{w}$.

(ii) TRUE $\overline{(zw)} = \overline{(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)} = x_1x_2 - y_1y_2 - i(x_1y_2 + x_2y_1) = (x_1 - iy_1)(x_2 - iy_2) = \bar{z}\,\bar{w}$.

(iii) TRUE. $\frac{1}{w}$ is a complex number, so $\overline{\left(\frac{z}{w}\right)} = \overline{z \cdot \frac{1}{w}} \overset{(ii)}{=} \bar{z} \cdot \overline{\left(\frac{1}{w}\right)} = \bar{z} \cdot \frac{1}{\bar{w}} = \frac{\bar{z}}{\bar{w}}$.

(iv) FALSE. Pick $z \neq 0$ & $w = -z$, then $|z+w| = 0$ whereas $|z| + |w| = |z| + |-z| = 2|z| \neq 0$ since $z \neq 0$.

(v) TRUE. $|z^2| = |(x_1^2 - y_1^2) + i(2x_1y_1)| = \sqrt{(x_1^2 - y_1^2)^2 + (2x_1y_1)^2} = \sqrt{x_1^4 - 2x_1^2y_1^2 + y_1^4 + 4x_1^2y_1^2} = \sqrt{(x_1^2 + y_1^2)^2} = x_1^2 + y_1^2 = |z|^2$.

(b) (i) Let $X = \{1\} = Y$, then $X \cup Y = \{1\}$ and $X \cap Y = \{1\}$.

(ii) Let $X = \{1\}$, $Y = \{2\}$, then $X \setminus Y = \{x \in X \mid x \notin Y\} = \{1\} = X$

(iii) Let $X = \{1\}$, $Y = \{2\}$. Then there is only one function from X to Y, namely $1 \mapsto 2$. (call this f)
It is injective since $f(x_1) = f(x_2)$ with $x_1, x_2 \in X$ implies, (since $X = \{1\}$ has only one element) that $x_1 = x_2 (=1)$.

(c) ($\Rightarrow$) Suppose that f is injective. Pick $x_0 \in X$, then define $g: Y \to X$ by $g(y) := \begin{cases} x_0 & \text{if } y \notin \text{image of } f \\ \text{the unique } x \in X \text{ such that } y = f(x) & \text{if } y \in \text{image of } f. \end{cases}$

Then $(g \circ f)(x) = g(f(x)) = x \;\; \forall x \in X$, so $g \circ f = \mathrm{id}_X$.

($\Leftarrow$) Suppose that $g \circ f = \mathrm{id}_X$. If $f(x_1) = f(x_2)$, then $g(f(x_1)) = g(f(x_2))$ ie $g \circ f(x_1) = g \circ f(x_2)$
But $g \circ f = \mathrm{id}_X$, so this implies that $x_1 = x_2$. Hence f is injective.

Q2/ (a) [This is the Week 7 assignment; see solution online]

(b) By prime factorisation, every positive integer which divides $2^2 \times 3^2 \times 11^4$ has the form $2^{a_1} \cdot 3^{a_2} \cdot 11^{a_3}$ where $0 \le a_1 \le 2$, $0 \le a_2 \le 2$ and $0 \le a_3 \le 4$. Hence there are $3 \times 3 \times 5 = 40$ options.
Repeating the analysis for negative integers, each has the form $-2^{a_1} \cdot 3^{a_2} \cdot 11^{a_3}$, so 40 options.
Hence overall, there are 80 integers dividing $2^2 \times 3^2 \times 11^4$.

(c) If we pick $x \in X$, the equivalence class containing x is defined to be $\{y \in X \mid yRx\}$

Now let $n \in \mathbb{N}$, then define equivalence relation on $\mathbb{Z}$ by the rule $xRy \Leftrightarrow n \mid (x-y)$.
The set of equiv classes for this are $\{\bar{0}, \bar{1}, \bar{2}, \dots, \overline{n-1}\} =: \mathbb{Z}_n$ where $\bar{i}$ = equiv class containing i.
This is, by definition, the set of integers mod n.
We define addition & multiplication by $\bar{i} + \bar{j} := \overline{i+j}$
& $\bar{i}\,\bar{j} := \overline{ij}$.

Q2(a) Since $0^2 \equiv 0 \mod 6$
$1^2 \equiv 1 \mod 6$
$2^2 \equiv 4 \mod 6$
$3^2 = 9 \equiv 3 \mod 6$
$4^2 = 16 \equiv 4 \mod 6$
$5^2 = 25 \equiv 1 \mod 6$

, if $x^2 \equiv 4 \mod 6$, then $x \equiv 2$ or $4 \mod 6$.

Hence $x = 2 + 6k$ for some $k \in \mathbb{Z}$
or $x = 4 + 6l$ for some $l \in \mathbb{Z}$.

SECTION 2

Q3/(a) A sequence is a function $\mathbb{N} \xrightarrow{f} \mathbb{R}$. We usually denote $a_i := f(i)$, and write $(a_n)_{n\in\mathbb{N}}$ as the collection of its values. We say that $(a_n)_{n\in\mathbb{N}}$ converges to a limit $l \in \mathbb{R}$ if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ st $|a_n - l| < \varepsilon \ \forall n \geq N$.

(b) (i) $a_n := 1/n$ converges to 0.

Proof let $\varepsilon > 0$, then pick $N \in \mathbb{N}$ st $N > 1/\varepsilon$. Then $\forall n \geq N$,
$$|a_n - 0| = 1/n \leq 1/N < \varepsilon .$$

(ii) $a_n := \frac{n^3 - n^2}{n} = n^2 - n$. This sequence does not converge, since it is unbounded.

(iii) We first claim that $b_n := (0.7)^n$ converges to 0.

Proof: let $\varepsilon > 0$, then pick $N \in \mathbb{N}$ such that $N > \frac{\log \varepsilon}{|\log 0.7|}$. Then since
$$(0.7)^n < \varepsilon \iff \log (0.7)^n < \log \varepsilon \iff n \log 0.7 < \log \varepsilon \underset{\text{negative}}{\overset{\log 0.7}{\iff}} n > \frac{\log \varepsilon}{|\log 0.7|},$$
we see that $|b_n - 0| < \varepsilon \ \forall n \geq N$.

Now both $1/n$ and $(0.7)^n$ are sequences with limits, so by theorem in lectures limit of $(1/n + (0.7)^n)$ = limit of $1/n$ + limit of $(0.7)^n = 0 + 0 = 0$.

(iv) $a_n := \frac{\sqrt{n^2 - 1}}{n}$ converges to 1.

Proof let $\varepsilon > 0$, choose $N \in \mathbb{N}$ st $N > \sqrt{\varepsilon}$, then $\forall n \geq N$
$$|a_n - 1| = \left|\sqrt{1 - 1/n^2} - \sqrt{1}\right| = \left|\frac{1 - 1/n^2 - 1}{\sqrt{1 - 1/n^2} + \sqrt{1}}\right| \leq 1/n^2 \leq 1/N^2 < \varepsilon .$$

(c) (i) The harmonic series $\sum_{n=1}^{\infty} 1/n$ is such an example. $1/n \to 0$ as $n \to \infty$ by (b)(i). But $\sum_{n=1}^{\infty} 1/n$ does not converge, since
$$\sum_{n=1}^{\infty} 1/n = 1 + 1/2 + \underbrace{1/3 + 1/4}_{> 1/2} + \underbrace{1/5 + 1/6 + 1/7 + 1/8}_{> 4 \cdot \frac{1}{8} = 1/2} + \underbrace{1/9 + \_ + 1/16}_{> 8 \cdot 1/16 = 1/2} + \ldots$$
we keep adding 1/2, so it cannot converge.

(ii) The series given by $a_n := 0 \ \forall n \in \mathbb{N}$. Clearly this sequence a_n converges to 0. Further $S_n := \sum_{i=1}^{n} a_n = 0 \ \forall n \in \mathbb{N}$, so $S_n \to 0$ as $n \to \infty$, so the ~~sequence~~ series converges to 0.

a series is an infinite sum $\sum_{k=1} a_k$.

Q4/ (a) Consider $s_n := \sum_{k=1}^{n} a_k$. We say that the series $\sum_{k=1}^{\infty} a_k$ converges to a limit $l \in \mathbb{R}$ if the sequence $(s_n)_{n\in\mathbb{N}}$ converges to the limit l (in the sense of Q3(a)), that ~~if~~ is $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ st $|s_n - l| < \varepsilon \ \forall n \geq N$.

(b) (i) Here $s_n := \sum_{k=1}^{n} 1 = n$. Then the sequence $(s_n)_{n\in\mathbb{N}}$ is unbounded, hence it cannot converge. Hence the series does not converge.

(ii) Here $s_n := \sum_{k=1}^{n} a = an$. Unless $a=0$, the sequence $(s_n)_{n\in\mathbb{N}}$ is unbounded, hence it cannot converge. If $a=0$, $s_n = 0 \ \forall n \in \mathbb{N}$, so $s_n \to 0$ as $n \to \infty$. In this case, the series converges to 0.

(iii) Here $s_n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. The sequence $(s_n)_{n\in\mathbb{N}}$ is unbounded, hence it cannot converge. Hence the series does not converge.

(iv) This is the harmonic series. It does not converge since

$$\sum_{n=1}^{\infty} 1/n = 1 + 1/2 + \underbrace{1/3 + 1/4}_{> 2 \cdot 1/4 = 1/2} + \underbrace{1/5 + 1/6 + 1/7 + 1/8}_{> 4 \cdot 1/8 = 1/2} + \underbrace{1/9 + \dots + 1/16}_{> 8 \cdot 1/16 = 1/2} + \dots$$

we keep adding 1/2, so it cannot converge.

(c) The Taylor series of f around a is defined to be $\sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i$.

(d) (i)

$f(x) = \ln(1+x)$ $f(0) = 0$

$f^{(1)}(x) = \frac{1}{1+x}$ so $f^{(1)}(0) = 1$

$f^{(2)}(x) = \frac{-1}{(1+x)^2}$ $f^{(2)}(0) = -1$

$f^{(3)}(x) = \frac{2}{(1+x)^3}$ $f^{(3)}(0) = 2$

$f^{(4)}(x) = \frac{-2.3}{(1+x)^4}$ $f^{(4)}(0) = -2.3$

$\vdots$

Hence Maclaurin series $= 0 + \frac{1}{1}x + \frac{(-1)}{2}x^2 + \frac{2}{3!}x^3 + \frac{(-2.3)}{4!}x^4 + \dots$
$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$

(ii) By (i)

$f(1) = \ln(2)$

$f^{(1)}(1) = \frac{1}{2}$

$f^{(2)}(1) = -\frac{1}{2^2}$

$f^{(3)}(1) = \frac{2}{2^3}$

$f^{(4)}(1) = \frac{-2.3}{2^4}$

$\vdots$

Hence Taylor series around $a=1$ is $= \ln(2) + \frac{(\frac{1}{2})}{1}(x-1) + \frac{(-\frac{1}{2^2})}{2!}(x-1)^2 + \frac{(\frac{2}{2^3})}{3!}(x-1)^3 + \frac{(\frac{-2.3}{2^4})}{4!}(x-1)^4 + \dots$

$= \ln(2) + \frac{1}{2}(x-1) - \frac{1}{2^3}(x-1)^2 + \frac{1}{3.2^3}(x-1)^3 - \frac{1}{4.2^4}(x-1)^4 + \dots$

SECTION 3

Q5/ (a) (i) FALSE. $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ ~~is~~ has trace 0, but is not invertible.

(ii) FALSE $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ has determinant 0, but is not invertible.

(iii) TRUE. If $AB = BA$ then $(AB)^{-1} \underset{AB=BA}{=} (BA)^{-1} = A^{-1}B^{-1}$ uses $(CD)^{-1} = D^{-1}C^{-1}$

(iv) FALSE $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ satisfies $A^5 = 0$, but $A \neq 0$.

(v) TRUE $A.A^4 = I = A^4.A$, hence A^4 is the inverse matrix of A (so A is invertible).

(b) (i) The characteristic polynomial is $\det(A - t\mathbb{I})$.
We say that a ~~non-~~ non-zero vector v is an eigenvector of A if $Av = \lambda v$ for some ~~λ~~ scalar λ.
λ is called the eigenvalue.
(Note $Av = \lambda v \iff (A - \lambda\mathbb{I})v = 0 \iff \det(A - \lambda\mathbb{I}) = 0$, so could also define eigenvalues as roots of the characteristic polynomial)

(ii) ~~det(A-tI)~~ let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\det(A - t\mathbb{I}) = \det\begin{pmatrix} a-t & b \\ c & d-t \end{pmatrix} = (a-t)(d-t) - bc = ad - at - dt + t^2 - bc$$
$$= t^2 - (a+d)t + (ad - bc)$$
$$= t^2 - \mathrm{Tr}(A)t + \det(A).$$

(c) let $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 3 & 4 \end{pmatrix}$.

Then $\det(A - t\mathbb{I}) = \det\begin{pmatrix} 1-t & 0 & -1 \\ 0 & 2-t & 0 \\ 0 & 3 & 4-t \end{pmatrix} = (1-t)(2-t)(4-t)$ so eigenvalues are 1, 2 and 4.

for $\lambda = 1$: $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 3 & 4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \iff \begin{matrix} v_1 - v_3 = v_1 \\ 2v_2 = v_2 \\ 3v_2 + 4v_3 = v_3 \end{matrix} \iff \begin{matrix} v_2 = 0 \\ v_3 = 0 \end{matrix}$, so $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector with eigenvalue 1.

for $\lambda = 2$: $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 3 & 4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 2\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \iff \begin{matrix} v_1 - v_3 = 2v_1 \\ 2v_2 = 2v_2 \\ 3v_2 + 4v_3 = 2v_3 \end{matrix} \iff \begin{matrix} v_1 = -v_3 \\ \cancel{v_2 = -4v_3} \\ 3v_2 = -2v_3 \end{matrix}$ so $\begin{pmatrix} -3 \\ -2 \\ 3 \end{pmatrix}$ is an eigenvector with eigenvalue 2.

for $\lambda = 4$: $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 3 & 4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 4\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \iff \begin{matrix} v_1 - v_3 = 4v_1 \\ 2v_2 = 4v_2 \\ 3v_2 + 4v_3 = 4v_3 \end{matrix} \iff \begin{matrix} v_2 = 0 \\ v_3 = -3v_1 \end{matrix}$ so $\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$ is an eigenvector with eigenvalue 4.

Q6/(a) (i) for any $n\times n$ matrices X and Y, by lectures $\det(XY) = \det X \cdot \det Y$. — (*)

Hence $\det(P^{-1}AP) = \det(\underbrace{(P^{-1}A)}_{X}\underbrace{P}_{Y}) \underset{\text{by above}}{=} \det(\underbrace{P^{-1}}_{X}\underbrace{A}_{Y})\det(P) \underset{\text{by above}}{=} \det(P^{-1})\det(A)\det(P)$. — (+)

Now $P^{-1}P = \mathbb{I}$, so $\det(P^{-1}P) = 1$. By (*) this implies that $\det(P^{-1})\det(P) = 1$

Thus $\det(P^{-1}AP) \underset{(+)}{=} \det(P^{-1})\det(A)\det(P) \underset{\text{these are all just numbers, so commute}}{=} \det(A)\det(P^{-1})\det(P) = \det(A)$, as required.

(ii) Solution 1: from lectures, a matrix is invertible if and only if its determinant is non zero.
Hence by (i), ~~det~~ since $\det A = \det(P^{-1}AP)$, certainly A is invertible $\iff \det(A) \neq 0 \iff \det(P^{-1}AP) \neq 0 \iff P^{-1}AP$ is invertible.

<u>Solution 2</u>. ($\Rightarrow$) Suppose A is invertible, say $BA = \mathbb{I} = AB$.

$$\left.\begin{array}{l} \text{Then } (P^{-1}AP)(P^{-1}BP) = P^{-1}ABP = P^{-1}\mathbb{I}P = P^{-1}P = \mathbb{I} \\ \& \ (P^{-1}BP)(P^{-1}AP) = P^{-1}BAP = P^{-1}\mathbb{I}P = P^{-1}P = \mathbb{I} \end{array}\right\} \text{ so } P^{-1}AP \text{ is invertible.}$$

($\Leftarrow$) Suppose that $P^{-1}AP$ is invertible, say $(P^{-1}AP)C = \mathbb{I} = C(P^{-1}AP)$.

$$\left.\begin{array}{l} \text{Then } P^{-1}A(PCP^{-1}) = (P^{-1}APC)P^{-1} = P^{-1}, \text{ so multiplying by } P \text{ on left gives } A(PCP^{-1}) = \mathbb{I} \\ \text{Similarly } (PCP^{-1})AP = P(CP^{-1}AP) = P, \text{ so right multiplying by } P^{-1} \text{ gives } (PCP^{-1})A = \mathbb{I} \end{array}\right\} \text{ hence } A \text{ is invertible.}$$

(b) (i)

$$\begin{pmatrix} 0 & 3 & 6 & 9 & 33 \\ 1 & 2 & 7 & 0 & 11 \\ -2 & -2 & -10 & 7 & 2 \end{pmatrix} \underset{R_1 \leftrightarrow R_2}{\sim} \begin{pmatrix} 1 & 2 & 7 & 0 & 11 \\ 0 & 3 & 6 & 9 & 33 \\ -2 & -2 & -10 & 7 & 2 \end{pmatrix} \underset{R_3 \mapsto R_3 + 2R_1}{\sim} \begin{pmatrix} 1 & 2 & 7 & 0 & 11 \\ 0 & 3 & 6 & 9 & 33 \\ 0 & 2 & 4 & 7 & 24 \end{pmatrix} \underset{R_2 \mapsto \frac{1}{3}R_2}{\sim} \begin{pmatrix} 1 & 2 & 7 & 0 & 11 \\ 0 & 1 & 2 & 3 & 11 \\ 0 & 2 & 4 & 7 & 24 \end{pmatrix}$$

$$\underset{R_3 \mapsto R_3 - 2R_2}{\sim} \begin{pmatrix} 1 & 2 & 7 & 0 & 11 \\ 0 & 1 & 2 & 3 & 11 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

This is echelon form.

(ii)

$$\begin{pmatrix} 1 & 2 & 7 & 0 & 11 \\ 0 & 1 & 2 & 3 & 11 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \underset{R_2 \mapsto R_2 - 3R_3}{\sim} \begin{pmatrix} 1 & 2 & 7 & 0 & 11 \\ 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \underset{R_1 \mapsto R_1 - 2R_2}{\sim} \begin{pmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

This is reduced echelon form.

(iii) By Gaussian elimination

$$\left(\begin{array}{cccc|c} 0 & 3 & 6 & 9 & 33 \\ 1 & 2 & 7 & 0 & 11 \\ -2 & -2 & -10 & 7 & 2 \end{array}\right) \underset{\text{above}}{\overset{\text{by}}{\sim}} \left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

The third column has no leading 1, so we set $x_3 = t$. The 3rd ~~column~~ row says $x_4 = 2$.
The 2nd ~~column~~ row says $x_2 + 2t = 5$, ie $x_2 = 5 - 2t$
The 1st ~~column~~ row says $x_1 + 3t = 1$ ie $x_1 = 1 - 3t$.

Hence general solution is $\left\{ \begin{pmatrix} 1-3t \\ 5-2t \\ t \\ 2 \end{pmatrix} : t \in \mathbb{R} \right\}$.