Sheet 5

5.1 (Strategy for Engel's Theorem, one map case) Let V be an n-dimensional vector space, where $n \geq 1$. Suppose that $x \colon V \to V$ is a nilpotent linear map.

1. Find a non-zero vector $0 \neq v \in V$ such that $xv = 0$.
2. Let $U := \mathrm{Span}\{v\}$. Show that x induces a nilpotent linear map
$$\overline{x} \colon V/U \to V/U.$$
By induction deduce that there exists a basis $\{v_1 + U, \dots, v_{n-1} + U\}$ of V/U such that $\overline{x}$ has strictly upper triangular form.
3. Prove that $\{v, v_1, \dots, v_{n-1}\}$ is a basis for V, and that with respect to this basis, x is strictly upper triangular.

5.2 In the proof of Proposition 6.2, consider
$$\varphi \colon A \to \mathfrak{gl}(L/A)$$
where $\varphi(a) \colon L/A \to L/A$ sends $x + A \mapsto [a, x] + A$. Check that φ is well-defined, and is a Lie algebra homomorphism.

5.3 (Used in proof of Engel's Theorem 2nd version) Let L be a Lie subalgebra of $\mathfrak{gl}(V)$. Suppose that there exists a basis of V such that, with respect to this basis, every $x \in L$ has strictly upper triangular form. Show that L is isomorphic to a Lie subalgebra of $\mathfrak{n}_n$, and hence that L is nilpotent.

5.4 (The converse to Engel's Theorem is false) Consider I, the identity map in $\mathfrak{gl}(V)$. Show that the one-dimensional Lie subalgebra $\mathrm{Span}\{I\} \subseteq \mathfrak{gl}(V)$ is nilpotent, but that in any basis for V, the map I is not strictly upper triangular.

5.5 (Strategy for Lie's Theorem, one map case) Let V be an n-dimensional vector space, where $n \geq 1$. Suppose that $x \colon V \to V$ is a linear map.

1. Show that x has an eigenvector $0 \neq v \in V$.
2. Let $U := \mathrm{Span}\{v\}$. Show that x induces a linear map $\overline{x} \colon V/U \to V/U$. By induction deduce that there exists a basis $\{v_1 + U, \dots, v_{n-1} + U\}$ of V/U such that $\overline{x}$ has an upper triangular form.
3. Prove that $\{v, v_1, \dots, v_{n-1}\}$ is a basis for V, and that with respect to this basis, x has upper triangular form.

5.6 Fill in the details of the remaining parts of the proof of Lie's Theorem.