COMPLEX MANIFOLDS
EXAMPLE SHEET 1

The two questions marked $(*)$ can be handed in to be marked. For this, please leave the work in my DPMMS pigeon hole (labelled "Dervan") by 14:00 on January 28$^{\text{th}}$.

(1) Let U be an open subset of $\mathbb{C}^m = \mathbb{R}^{2m}$ and $f : U \to \mathbb{C}^n = \mathbb{R}^{2n}$ be smooth. The *complex Jacobian* of f is defined to be the matrix
$$J_{\mathbb{C}}(f) = \left(\frac{\partial f_i}{\partial z_j}\right)_{1\leq i\leq n, 1\leq j\leq m}$$
where $z_j = x_j + iy_j$ are the standard co-ordinates on U for $1 \leq j \leq n$. With standard co-ordinates $w_j = u_j + iv_j$ on $\mathbb{C}^n$, compute the matrix of
$$df : T\mathbb{R}^{2m} \to TR^{2n}.$$
Find also the matrix of the induced map
$$df_{\mathbb{C}} : T\mathbb{R}^{2m} \otimes \mathbb{C} \to T\mathbb{R}^{2n} \otimes \mathbb{C}$$
in terms of the frames $\{\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j}\}$ and $\{\frac{\partial}{\partial w_j}, \frac{\partial}{\partial \bar{w}_j}\}$, and relate this to $J_{\mathbb{C}}$ when f is holomorphic. Assuming $m = n$, conclude that
$$\det df = |J_{\mathbb{C}}(f)|^2.$$
Use this to prove that a complex structure induces a natural orientation on the underlying differentiable manifold, and hence that not all even dimensional smooth manifolds admit a complex structure.

(2) (*) Let U, V be open sets in $\mathbb{C}^n$. Prove that $f : U \to V$ is holomorphic if and only if $df_{\mathbb{R}}$ is $\mathbb{C}$-linear. Using this, prove the Inverse Function Theorem: if $z_0 \in U$ is such that
$$\det J_{\mathbb{C}}(f)(z_0) \neq 0,$$
then there is an open U' containing z_0 such that $f|_{U'} : U' \to f(U')$ is a biholomorphism. The smooth Inverse Function Theorem should be assumed.

(3) Suppose X is a complex manifold and G is a discrete group acting freely and properly discontinuously by biholomorphisms. Show that X/G admits the structure of a complex manifold such that the map $\pi : X \to X/G$ is holomorphic and locally biholomorphic. It follows that the Hopf surface, which is the quotient of $\mathbb{C}^2 - \{0\}/\sim$, where $(w_1, w_2) \sim (z_1, z_2)$ if $w_j = 2^s z_j$ for some fixed $s \in \mathbb{Z}$, is a complex manifold. We will later see that the Hopf surface is not projective.

(4) Show that a complex submanifold X of a complex manifold Y is naturally a complex manifold.

(5) Let $f : \mathbb{C}^n \to \mathbb{C}$ be holomorphic and $0 \in \mathbb{C}$ a regular value (i.e. the complex Jacobian $J_{\mathbb{C}}$ is surjective at any point in $f^{-1}(0)$). Show that $Z := f^{-1}(0)$ is a complex submanifold of $\mathbb{C}^n$.

(6) (*) Let F be a homogeneous polynomial in $n+1$ variables. Show that the set

$$Z = \{[z_0 : \ldots : z_n] \in \mathbb{P}^n : F(z_0, \ldots, z_n) = 0\}$$

is well defined. By considering the map $f : \mathbb{C}^{n+1} - \{0\} \to \mathbb{C}$ induced by F, show that if 0 is a regular value of f then Z is a complex submanifold of $\mathbb{P}^n$.

(7) Let X be a compact complex manifold. Show that any holomorphic function on X is constant. Deduce that a compact complex manifold cannot embed in $\mathbb{C}^n$ for any n.

(8) Let X be a compact complex manifold. Show that a smooth function $f : X \to \mathbb{C}$ is holomorphic if and only if $\bar{\partial} f = 0$.

(9) Let X be a compact complex manifold. Show that if a holomorphic function $f : X \to \mathbb{C}$ vanishes on an open subset of X, then it must be identically zero. Thus there are no holomorphic analogues of bump functions.

(10) Let $f : X \to Y$ be a holomorphic map between complex manifolds. Prove that if α is a (p,q)-form on Y then $f^*\alpha$ is a (p,q)-form on X. Give an example where this fails if f is not holomorphic. Using this show that f induces a homomorphism

$$f^* : H^{p,q}_{\bar{\partial}}(Y) \to H^{p,q}_{\bar{\partial}}(X)$$

given by

$$f^*[\alpha] = [f^*\alpha], \alpha \in \mathcal{A}^{p,q}(Y).$$