COMPLEX MANIFOLDS
EXAMPLE SHEET 2

The two questions marked $(*)$ can be handed in to be marked. For this, please leave the work in my DPMMS pigeon hole (labelled "Dervan") by 14:00 on February 11^{th}.

(1) (*) Prove that a submanifold $D \subset \mathbb{C}^n$ of complex dimension $n-1$ is given by $D = V(f)$ for some holomorphic $f : \mathbb{C}^n \to C$.

(2) Show that a sequence if sheaves

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

on X is exact if and only if the induced map on stalks

$$0 \to \mathcal{E}_x \to \mathcal{F}_x \to \mathcal{G}_x \to 0$$

is exact for all $x \in X$.

(3) With $\mathcal{A}^{p,q}_{\mathbb{C}}$ the sheaf of (p,q)-forms on a complex manifold X, show that $H^i(X, \mathcal{A}^{p,q}_{\mathbb{C}}) = 0$ for all $i > 0$.

(4) Suppose

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

is a short exacct sequence of sheaves on a topological space X. In lectures we constructed a connecting homomorphism $\delta : H^i(X, \mathcal{G}) \to H^{i+1}(X, \mathcal{E})$ for all $i \geq 0$. Suppose in addition that for all covers $\mathcal{U}$ of X there is a refinement $\mathcal{V}$ of $\mathcal{U}$ (i.e. $\mathcal{V} \leq \mathcal{U}$) such that

$$0 \to \mathcal{E}(V) \to \mathcal{F}(V) \to \mathcal{G}(V) \to 0$$

for all $V \in \mathcal{V}$. Show that there is a long exact sequence in cohomology of the form claimed in the lectures:

$$0 \to H^0(X, \mathcal{E}) \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^1(X, \mathcal{E}) \to H^1(X, \mathcal{F}) \to \dots$$

(5) Let $\mathcal{F}$ be a presheaf. The sheafification $\mathcal{F}^+$ of $\mathcal{F}$ is the sheaf for which

$$\mathcal{F}(U) = \{\text{maps } s : U \to \cup_{x \in U} \mathcal{F}_x \text{ with } s(x) \in \mathcal{F}_x\}$$

with and such that for all $x \in U$ there exists an open subset $x \in V \subset U$ and a section $t \in \mathcal{F}(V)$ with $s(y) = t(y)$ for all $y \in V$. Show that $\mathcal{F}^+$ is a sheaf. One then defines the cokernel and image sheaves as the sheafification of the natural presheaves.

(6) Show that the coboundary operator δ used in the definition of Čech cohomology satisfies $\delta \circ \delta = 0$.

(7) Let $X = \mathbb{P}^1$ and p, q distinct points on X. Let $\mathcal{O}(-p-q)$ denote the sheaf of holomorphic functions on X vanishing at both p and q. Show that there is a short exact sequence of sheaves

$$0 \to \mathcal{O}(-p-q) \to \mathcal{O} \to \mathbb{C}_p \oplus \mathbb{C}_q \to 0,$$

where the sheaf on the right should be carefully defined. Show that the map $H^0(X, \mathcal{O}) \to H^0(X, \mathbb{C}_p \oplus \mathbb{C}_q) \to 0$ is not surjective and conclude that $H^1(X, \mathcal{O}(-p-q)) \neq 0$.

(8) (*) Show that any holomorphic line bundle on a disc $\Delta \subset \mathbb{C}$ is trivial. Deduce that any holomorphic line bundle on $\mathbb{P}^1$ is of the form $\mathcal{O}(n)$ for some integer n. *[Actually the same is true for $\mathbb{P}^m$ as well].*

(9) (i) Let $X = \mathbb{C}^* = \mathbb{C} - \{0\}$ with open cover given by $U_0 = \mathbb{C} - \mathbb{R}^+ \times \{0\}$ and $U_1 = \mathbb{C} - \mathbb{R}^- \times \{0\}$. Compute $\check{H}^q(\mathcal{U}, \mathbb{Z})$.

(ii) Let $X = \mathbb{C}^2 - \{0\}$ with open cover given by $U_0 = \mathbb{C} \times \mathbb{C}^*$ and $U_1 = \mathbb{C}^* \times \mathbb{C}$. Show that $\check{H}^1(\mathcal{U}, \mathcal{O})$ is infinite dimensional and $\check{H}^q(\mathcal{U}, \mathcal{O})$ is trivial for all $q > 1$.

(10) Show that if $X = \mathbb{P}^n$ there is an exact sequence of holomorphic vector bundles

$$0 \to \mathcal{O} \to \oplus_{j=0}^n \mathcal{O}(1) \to TX^{1,0} \to 0.$$

This sequence is called the Euler sequence.